1.

(1) 由题目:

$$\left|\overrightarrow{AB}\times\overrightarrow{AC}\right| = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 2 & a & 0 \\ 0 & -1 & 2 \end{vmatrix} = (2a,-4,-2) = \sqrt{4a^2+16+4} = 2\sqrt{6} \Leftrightarrow a=1,$$

$$V_{ABCD} = \begin{vmatrix} 2 & a & 0 \\ 0 & -1 & 2 \\ 2 & b & 1 \end{vmatrix} = -2+4a-4b = 2 \Leftrightarrow a-b=1,$$

故

$$a=1, \qquad b=0.$$

(2) 平面过$\overrightarrow{r_0}=(2,3,4)$点，且一个法向量为

$$\overrightarrow{n} = \overrightarrow{AB}\times\overrightarrow{AD} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 2 & 1 & 0 \\ 2 & 0 & 1 \end{vmatrix} = (1,-2,-2),$$

故平面方程可由$\overrightarrow{n}\cdot(\overrightarrow{r}-\overrightarrow{r_0})=0$给出，即

$$(x-2)-2(y-3)-2(z-4)=0 \Leftrightarrow x-2y-3z+12=0.$$

(3)当$\boldsymbol{x}=(1,2,1)$时，由

$$\frac{\partial f}{\partial x}=2x+yz=4, \qquad \frac{\partial f}{\partial y}=2y+xz=5, \qquad \frac{\partial f}{\partial x}=2z+xy=4,$$

故

$$\nabla f|_{\boldsymbol{x}=(1,2,1)} = \left(\frac{\partial f}{\partial x},\frac{\partial f}{\partial y},\frac{\partial f}{\partial z}\right)_{\boldsymbol{x}=(1,2,1)} = (4,5,4).$$

因为与$\overrightarrow{AB}$同向的方向向量为$\overrightarrow{n_1}=\left(\frac{2}{\sqrt{5}},\frac{1}{\sqrt{5}},0\right)$，所以所求方向导数为

$$\left.\frac{\partial f}{\partial n}\right|_{\boldsymbol{x}=(1,2,1)} = \overrightarrow{n_1}\cdot\nabla f|_{\boldsymbol{x}=(1,2,1)} = \frac{13}{\sqrt{5}}.$$

2.

(1) 由于$\lim\limits_{x\to2}\dfrac{f(2x-2)}{x-2}$存在，$\lim\limits_{x\to2}(x-2)=2$，且$f(2x-2)$在2处有定义，$f(2)=0$一定成立.

又因为$\forall x\in(2,4), f'(x)>0$，所以$f(x)$在$(2,4)$上单调递增，故$\forall x\in[2,4], f(x)\geqslant f(2)=0$，且$f(x)=0, x\in[2,4]\Leftrightarrow x=2$

(2) 对定义于$[2,4]$的函数$g(y)=6f(y)-y\int_2^4 f(x)\mathrm{d}x$在$[2,4]$上积分:

$$\int_2^4 g(y)\mathrm{d}y=6\int_2^4 f(y)\mathrm{d}y-\int_2^4 y\mathrm{d}y\int_2^4 f(x)\mathrm{d}x=(6-6)\int_2^4 f(x)\mathrm{d}x=0.$$

由于$g(y)$是定义于闭区间上的连续函数，故必存在$\xi\in(2,4)$，使得$g(\xi)=0\Leftrightarrow \dfrac{6f(\xi)}{\xi}=\int_2^4 f(x)\mathrm{d}x.$

(3) 对(2)所证结论等价变形得

$$6\cdot\frac{f(\xi)-f(2)}{\xi-2}=\frac{\xi}{\xi-2}\int_2^4 f(x)\mathrm{d}x.$$

因为$f(x)$在$[2,4]$上连续，在$(2,4)$上可导，故由Lagrange中值定理，对(2)中的ξ，存在$\eta\in(2,\xi)\subset(2,4)$（因此自然有$\eta\neq\xi$），使得

$$6f'(\eta)=\frac{\xi}{\xi-2}\int_2^4 f(x)\mathrm{d}x.$$

3.

(1) 易知

$$f(x)=\sum_{n=1}^{N}(-1)^{n-1}\frac{x^n}{n}+o(x^n)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots+o(x^n).$$

(2) 用Taylor展开式重写待求极限的表达式:

$$
\begin{aligned}
\frac{1}{x^2}\left[2(1+x)^{\frac{1}{x}}+\mathrm{e}(x-2)\right] &= \frac{1}{x^2}\left\{2\exp\left[\frac{\ln(1+x)}{x}\right]+\mathrm{e}(x-2)\right\} \\
&= \frac{1}{x^2}\left\{2\exp\left[\frac{x-\frac{x^2}{2}+\frac{x^3}{3}+o(x^3)}{x}\right]+\mathrm{e}(x-2)\right\} \\
&= \frac{1}{x^2}\left\{2\exp\left[1-\frac{x}{2}+\frac{x^2}{3}+o(x^2)\right]+\mathrm{e}(x-2)\right\} \\
&= \frac{1}{x^2}\left\{2\mathrm{e}\exp\left[-\frac{x}{2}+\frac{x^2}{3}+o(x^2)\right]+\mathrm{e}(x-2)\right\} \\
&= \frac{1}{x^2}\left\{2\mathrm{e}\left[1+\left(-\frac{x}{2}+\frac{x^2}{3}\right)+\frac{1}{2}\left(-\frac{x}{2}+\frac{x^2}{3}\right)^2+o(x^2)\right]+\mathrm{e}\right. \\
&= \frac{1}{x^2}\left\{2\mathrm{e}\left[1-\frac{x}{2}+\frac{x^2}{3}-\frac{x^2}{8}+o(x^2)\right]+\mathrm{e}(x-2)\right\} \\
&= \frac{5\mathrm{e}}{12}+o(1),
\end{aligned}
$$

故所求极限$I=\frac{5\mathrm{e}}{12}$.

4.

一个关于齐次函数的题目.

(1) 对确定的x和y，在实数域上定义函数$g(t)=f(tx,ty)-t^2f(x,y)$，则事实上$g(t)$就是常函数0，则

$$
0=g'(t)=x\left.\frac{\partial f}{\partial x}\right|_{(tx,ty)}+y\left.\frac{\partial f}{\partial y}\right|_{(tx,ty)}-2tf(x,y).
$$

令上式中$x=1,y=2,t=1$，则

$$
0=\left.\frac{\partial f}{\partial x}\right|_{(1,2)}+2\left.\frac{\partial f}{\partial y}\right|_{(1,2)}-2f(1,2)=5+2\left.\frac{\partial f}{\partial y}\right|_{(1,2)}\Leftrightarrow\left.\frac{\partial f}{\partial y}\right|_{(1,2)}=-\frac{5}{2}.
$$

(2) 对于函数$h(t)=\left\{1+f\left(2t-2\sin t+1,\sqrt[3]{1+t^3}+1\right)\right\}^{\frac{1}{\ln(1+t^3)}}$，先计算$\lim\limits_{t\to 0}h(t)$. 然后，考虑L'Hospital法则.

对此，依次考虑下面函数在自变量取零附近的Taylor展开式:

$$
\begin{aligned}
\delta X(t) = 2t - 2\sin t = 2t - 2\left(t - \frac{1}{6}t^3\right) + o(t^3) &= \frac{1}{3}t^3 + \\
\delta Y(t) = \sqrt[3]{1+t^3} - 1 = 1 + \frac{1}{3}t^3 - 1 + o(t^3) &= \frac{1}{3}t^3 + o(t^3), \\
f(2t - 2\sin t + 1, \sqrt[3]{1+t^3} + 1) &= f(1 + \delta X, 2 + \delta Y) \\
&= f(1,2) + f_x(1,2)\delta X + f_y(1,2)\delta Y - \\
&= -\frac{5}{2}(\delta Y - 2\delta X) + o(t^3) \\
&= -\frac{5}{2}\left(\frac{1}{3}t^3 - \frac{2}{3}t^3\right) + o(t^3) \\
&= \frac{5}{6}t^3 + o(t^3),
\end{aligned}
$$

$$
\ln h(t) = \frac{\ln\left[1 + f\left(2t - 2\sin t + 1, \sqrt[3]{1+t^3} + 1\right)\right]}{\ln(1+t^3)} = \frac{\ln\left[1 + \frac{5}{6}t^3 + o(t^3)\right]}{\ln(1+t^3)} = \frac{5}{6} + o(1),
$$

故$\lim\limits_{t\to 0} h(t) = \mathrm{e}^{\frac{5}{6}}$.考虑到原式的分子和分母在$x \to 0$时都趋于0，由L'Hospital法则，原式等于$\mathrm{e}^{\frac{5}{6}}$.

5.

解偏微分方程？有点意思.

(1) 将$u_{xy}(x, y) = 0$看作对x固定而以y为自变量的一元函数，对y积分，则存在函数$f_0(x)$使得$u_x(x, y) = f_0(x)$；再将$u_x(x, y)$看作对y固定而以x为自变量的一元函数，对x积分，则存在函数$f(x)$和$g(y)$，使得$f'(x) = f_0(x)$且$u(x, y) = f(x) + g(y)$.

将$u(x, y) = f(x) + g(y)$中的x或y固定为x_0或y_0，可据此定义以y或x为自变量的一元函数$u(x_0, y) = f(x_0) + g(y)$或$u(x, y_0) = f(x) + g(y_0)$.因为$u(x, y)$是二阶可导的，所以$f(x)$和$g(y)$也是二阶可导的.

(2) 因为

$$
\begin{aligned}
\frac{\partial z}{\partial x} &= \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} = \frac{1}{u}\frac{\partial u}{\partial x} \\
\frac{\partial^2 z}{\partial x \partial y} &= \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial y}\left(\frac{1}{u}\frac{\partial u}{\partial x}\right) = -\frac{1}{u^2}\frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + \frac{1}{u}\frac{\partial^2 u}{\partial x \partial y} \\
&= -\frac{1}{u^2}\frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + \frac{1}{u}\left(\frac{1}{u}\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}\right) = 0
\end{aligned}
$$

所以若$z = \ln u(x, y)$，则$\frac{\partial^2 z}{\partial x \partial y} = 0$.

(3) 由(2)，若作变量替换$z = \ln u(x, y)$，则

$$\begin{cases} \frac{\partial^2 z}{\partial x \partial y} = 0, & (x,y) \in D \\ \varepsilon(x,0) = -\frac{x^2}{2} + \ln x, & x \in (0,+\infty), \\ \varepsilon(1,y) = -\frac{y^2}{2} - \frac{1}{2}, & y \in (-\infty,+\infty). \end{cases}$$

由(1)，存在二阶可导的函数$f(x)$和$g(y)$，使得

$$z(x,y) = f(x) + g(y).$$

令$y=0$，则

$$z(x,0) = f(x) + g(0) = -\frac{x^2}{2} + \ln x;$$

令$x=1$，则

$$\begin{aligned} z(1,y) &= f(1) + g(y) = -\frac{1}{2} - \frac{y^2}{2}, \\ z(1,0) &= -\frac{1}{2} = f(1) + g(0). \end{aligned}$$

据上面四式有

$$z(x,y) = -\frac{x^2+y^2+1}{2} + \ln x - f(1) - g(0) = -\frac{x^2+y^2}{2} + \ln x,$$

继而

$$u(x,y) = \exp z(x,y) = x \exp\left(\frac{x^2+y^2}{2}\right).$$

6.

(1) 证明$g(x,y)$在$(0,0)$处取到极值.

只需证明$\nabla g|_{(0,0)} = (0,0)$即可，为此需对$g(x,y) = f(u(x,y), v(x,y)) = f(\mathrm{e}^{xy}, x^2+y^2)$求导. 因为$f(x,y) = 1 - x - y + o(|\Delta \boldsymbol{x}|)$，所以在$(x,y) = (1,0)$处，$f_x = f_y = -1$.注意到此时$u(0,0) = 1, v(0,0) = 0$，因此$(u,v) = (1,0), (x,y) = (0,0)$；并注意区分$\partial_x f$和$\partial_x u$（其他需要区分的偏导类似）：

$$\begin{aligned} \left.\frac{\partial g}{\partial x}\right|_{(x,y)} &= \left.\frac{\partial f}{\partial x}\right|_{(u,v)} \left.\frac{\partial u}{\partial x}\right|_{(x,y)} + \left.\frac{\partial f}{\partial y}\right|_{(u,v)} \left.\frac{\partial v}{\partial x}\right|_{(x,y)} = (-1)(ye^{xy}) + (-1)(2x) = 0 \\ \left.\frac{\partial g}{\partial y}\right|_{(x,y)} &= \left.\frac{\partial f}{\partial x}\right|_{(u,v)} \left.\frac{\partial u}{\partial y}\right|_{(x,y)} + \left.\frac{\partial f}{\partial y}\right|_{(u,v)} \left.\frac{\partial v}{\partial y}\right|_{(x,y)} = (-1)(xe^{xy}) + (-1)(2y) = 0 \end{aligned}$$

因此完成证明，$g(x,y)$确实在$(0,0)$处取到极值.

(2) 求极值.

由于$f(1,0)=\lim\limits_{(x,y)\to(1,0)} f(x,y)=1-1-0=0$且$g(0,0)=f(1,0)=0$，故该极值为$0$.

(3) 判断极值点类型.

在对$g(x,y)$求二阶导之前，为清晰起见，需明确一些符号的具体含义.对于以(u,v)为自变量的诸函数$\frac{\partial f}{\partial x},\frac{\partial f}{\partial y}$，后面记作$f_1(u,v),f_2(u,v)$，这两个函数在$(u=1,v=0)$即$(x=0,y=0)$处的一阶导为

$$
\begin{aligned}
(f_1)_x &= f_{11}u_x+f_{12}v_x=0\\
(f_2)_x &= f_{21}u_x+f_{22}v_x=0\\
(f_1)_y &= f_{11}u_y+f_{12}v_y=0\\
(f_2)_y &= f_{21}u_y+f_{22}v_y=0
\end{aligned}
$$

于是，我们开始对$g(x,y)$求二阶导：

$$
\begin{aligned}
g_{xx} &= (f_1u_x+f_2v_x)_x=f_1u_{xx}+f_2v_{xx}=(-1)\left(y^2e^{xy}\right)+(-1)(2)=-2\\
g_{xy} &= (f_1u_x+f_2v_x)_y=f_1u_{xy}+f_2v_{xy}=(-1)\left[(1+xy)e^{xy}\right]+(-1)(0)=-1\\
g_{yy} &= (f_1u_y+f_2v_y)_y=f_1u_{yy}+f_2v_{yy}=(-1)\left(x^2e^{xy}\right)+(-1)(2)=-2
\end{aligned}
$$

据此可得$g(x,y)$在$(x,y)=(0,0)$处的Hessian为$\begin{pmatrix}-2 & -1\\ -1 & -2\end{pmatrix}$，它是特征值为$-1$和$-3$的实对称矩阵，因此也是负定矩阵，这说明这个极值点是极大值点.

7.

微分几何背景，所给条件和曲率有关.但是本题的求解主要运用隐函数存在定理，表示出隐函数的各阶导数.考虑到隐函数存在定理只能给出局部存在性，故考虑逐渐延拓解的存在范围.事实上，由于条件对称，这个过程容易进行，不过还是需要严格化.

看到$f_y(xy)\neq 0$就能想到隐函数定理，这个条件很有特色，应注意识别.

(1) 必要性显然，但稍加说明为妙.

对任意实数$C\in I$，若曲线$F(x,y)=C$为直线，则$F(x,y)$为一（经过平移的）线性函数，即$F(x,y)=Ax+By+C'$，且$B\neq 0$.因为

$$
\frac{\partial f}{\partial x}=A,\quad \frac{\partial f}{\partial y}=B,\quad \frac{\partial^2 f}{\partial x\partial y}=\frac{\partial^2 f}{\partial x^2}=\frac{\partial^2 f}{\partial y^2}=0
$$

所以

$$\left(\frac{\partial f}{\partial y}\right)^2\frac{\partial^2 f}{\partial x^2}-2\frac{\partial f}{\partial x}\frac{\partial f}{\partial y}\frac{\partial^2 f}{\partial x\partial y}+\left(\frac{\partial f}{\partial x}\right)^2\frac{\partial^2 f}{\partial y^2}=B^2\cdot 0-2AB\cdot 0+A^2\cdot 0=0$$

(2) 充分性分下面几步证明.

1. 对任意实数$C\in I$，都存在(x_0,y_0)，使得$F(x,y)=f(x,y)-C=0$.在(x_0,y_0)处，由于$F(x,y)$满足
   i. F在包含(x_0,y_0)的区域D内连续,
   ii. $F(x_0,y_0)=0$,
   iii. $F_y(x,y)$在D内连续,
   iv. $F_y(x_0,y_0)\neq 0$,
   故$\exists\delta>0$，$\forall x\in\left(x_0-\frac{3\delta}{2},x_0+\frac{3\delta}{2}\right)=I_0$，存在唯一一个定义于$I_0$上的函数$y_0(x)$，使得$F(x,y_0(x))=0,\forall x\in I_0$.
2. 因为$f(x,y)$具有二阶连续偏导数，故具体说来$F_x(x,y)$也是连续函数，故$y_0(x)$在I_0上具有连续导函数$y_0'(x)=-\frac{F_x(x,y)}{F_y(x,y)}=-\frac{f_x(x,y)}{f_y(x,y)}$；因为分子和分母都连续可导，故$y_0$也二阶连续可导，且

$$\begin{aligned}
y_0''(x)&=-\frac{\mathrm{d}}{\mathrm{d}x}\frac{F_x(x,y)}{F_y(x,y)}=-\frac{\mathrm{d}}{\mathrm{d}x}\frac{f_x(x,y_0(x))}{f_y(x,y_0(x))}\\
&=-\frac{1}{f_y^2}\left[\left(\left.\frac{\partial f_x}{\partial z}\right|_{(x,y_0)}+\left.\frac{\partial f_x}{\partial y}\right|_{(x,y_0)}y_0'\right)f_y-f_x\left(\left.\frac{\partial f_y}{\partial z}\right|_{(x,y_0)}+\left.\frac{\partial f_y}{\partial y}\right|_{(x,y_0)}y_0'\right)\right]\\
&=\frac{1}{f_y^2}\left\{f_x\left[f_{xy}+f_{yy}\left(-\frac{f_x}{f_y}\right)\right]-f_y\left[f_{xx}+f_{xy}\left(-\frac{f_x}{f_y}\right)\right]\right\}\\
&=\frac{1}{f_y^3}\left(f_xf_yf_{xy}-f_x^2f_{yy}-f_y^2f_{xx}+f_xf_yf_{xy}\right)\\
&=0
\end{aligned}$$

   因此，在区间I_0上积两次分，即得$y_0(x)=a_0x+m_0$.
3. 下面证明：在区间$I_k=I_0+k\delta(k\in\mathbb{Z})$上，该条件均唯一定义函数$y_k(x)$，且该函数必为线性函数.
   i. 我们只需给出$(k\in\mathbb{Z}^+)$情形的证明过程.
   ii. $k=0$已经验证.
   iii. 现在假设对于所有的$k=n(n>0)$都得到验证，则以此为基础，取$x=x_0+n\delta$，因为由归纳假设，y_n得到定义，则可以取$y=y_n(x_0+n\delta)$，则(x,y)点处满足的一切条件都和第1、2步相同，故区间I_{n+1}上也唯一定义一个线性函数$y_{k+1}(x)$.
   因此，我们证明了在区间I_k上该函数都是唯一定义的，且是线性函数，即$y_k(x)=a_kx+m_k$.
4. 又因为对于函数y_k和y_{k+1}，两函数的定义域有公共区域$J_k=I_k\cup I_{k+1}=\left(x_0+\left(k+\frac{1}{2}\right)\delta,x_0+\left(k+\frac{3}{2}\right)\delta\right)$，所以由唯一性，两函数在公共区域的表达式相同，因此$a_k=a,m_k=m$.
5. 因此，在相应条件下，该条件隐式定义了一个

i. 唯一的
ii. 连续的
iii. 满足$f(x,y)=C$的

线性函数，即曲线$f(x,y)=C$为直线.