การอบรมคอมพิวเตอร์โอลิมปิก สอวน. ค่ายที่ 2 ปีการศึกษา 2563

ศูนย์สอวน. โรงเรียนสามเสนวิทยาลัย โรงเรียนสตรีศรีสุริโยทัย
และ โรงเรียนสตรีวัดมหาพฤฒาราม
โดย ภาควิชาวิทยาการคอมพิวเตอร์
คณะวิทยาศาสตร์และเทคโนโลยี ม.ธรรมศาสตร์

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ช่วงที่ 3: อัลกอริทึม (Algorithms)

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เค้าโครงการอบรม

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- บทนำ (Introduction)
- การวิเคราะห์ความซับซ้อนของอัลกอริทึม (Algorithmic Complexity)
- กลวิธีทางอัลกอริทึม
  - Brute Force
  - Divide-and-Conquer
  - Decrease-and-Conquer
  - Transform-and-Conquer
  - String Matching Algorithms
  - Backtracking & Branch-and-Bound
  - Greedy Algorithms
  - Graph Algorithms
  - Dynamic Programming

บทนำ (Introduction)

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Introduction: What Is an Algorithm?

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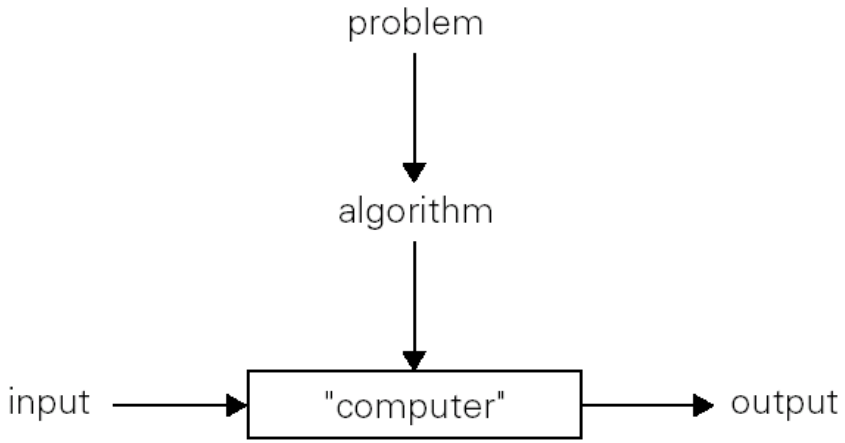
- an **algorithm** is a sequence of unambiguous instructions for solving a *well-specified computational problem*
- an **algorithm** is a sequence of computational steps that transforms the *legitimate input* into the *desired output* in a *finite amount of time*



Introduction: ตัวอย่าง Computational Problems: The Sorting Problem

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Sorting (การเรียงลำดับข้อมูล)

- **Problem Statement:**
  - *Input:* a sequence of n numbers $\langle a_1, a_2, \ldots, a_n \rangle$
  - *Output:* a reordering of the input sequence $\langle a_1, a_2, \ldots, a_n \rangle$ such that $a_i \leq a_j$ whenever $i < j$
- a problem Instance: the sequence <31, 41, 59, 26, 41, 58>
- *Algorithms:* Insertion Sort, Selection Sort, Bubble Sort, Merge Sort, Quick Sort

Introduction: ตัวอย่าง Computational Problems: The Sorting Problem (2)

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- given a problem instance: the *input* sequence <31, 41, 59, 26, 41, 58>
- any sorting algorithm that works correctly would yield the desired *output*, that is, the sequence <26, 31, 41, 41, 58, 59>

Introduction: ตัวอย่าง Computational Problems:

The Sorting Problem: -- in action

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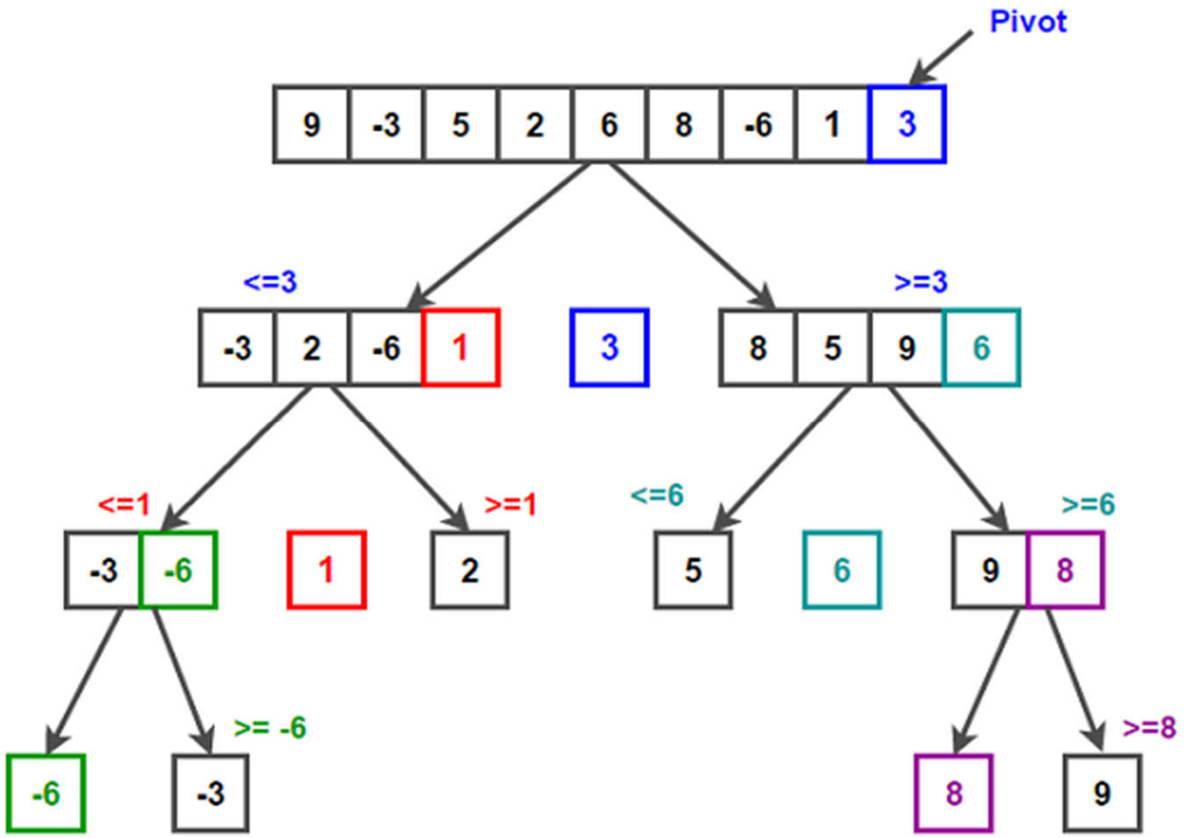
- Bubble Sort
- Insertion Sort
- Selection Sort

See https://visualgo.net/en/sorting

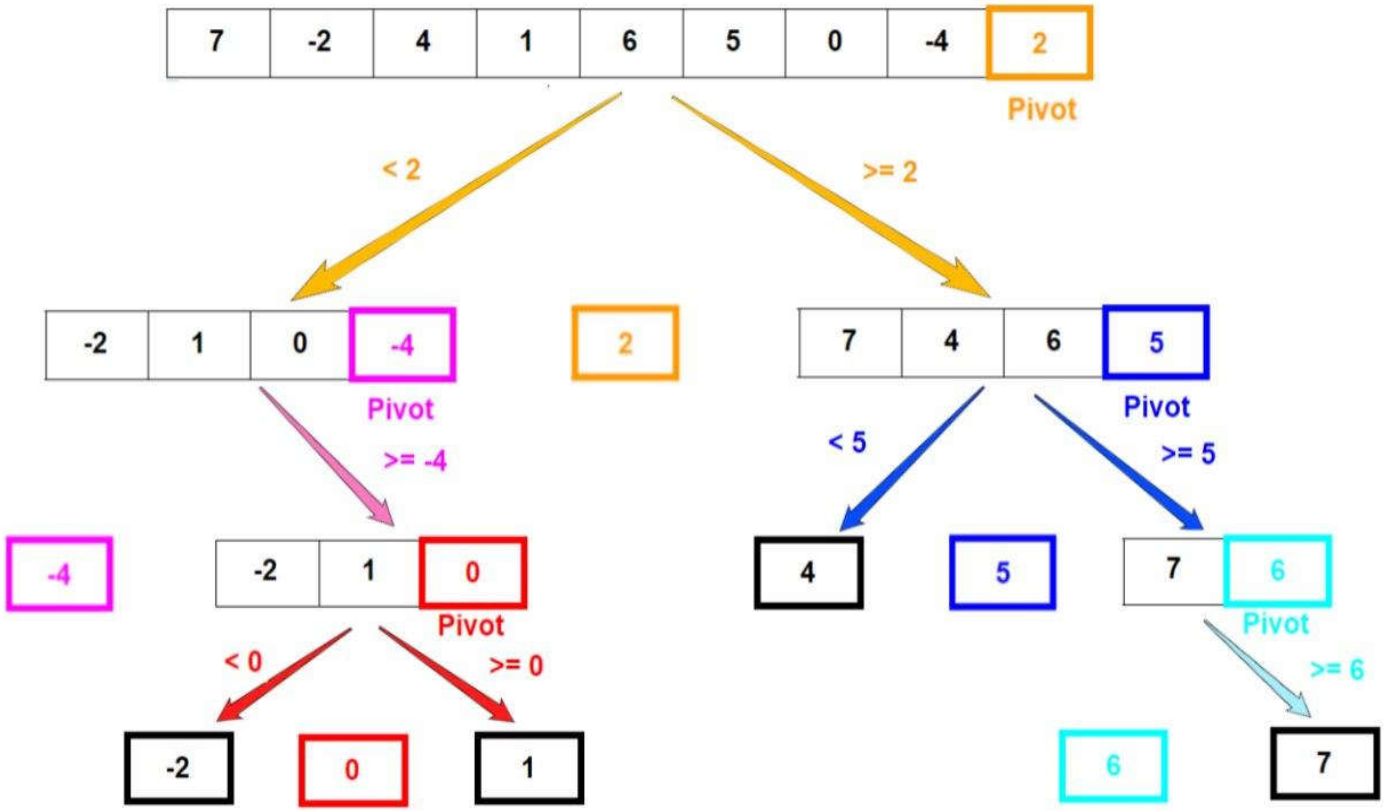
Introduction: ตัวอย่าง Computational Problems:

The Sorting Problem: Quick Sort -- in action



https://www.techiedelight.com/quicksort/

Introduction: ตัวอย่าง Computational Problems:
The Sorting Problem: Quick Sort -- in action (2)



https://morioh.com/p/b0deaa623ac4

Introduction: ตัวอย่าง Computational Problems:

The Sorting Problem: Quick Sort -- in action (3)

- **Worst-Case Scenario:** จากตัวอย่าง ลิสต์มีข้อมูลจำนวน 7 ตัว จะพบว่า

  รอบที่ 1: ทำการเปรียบเทียบ 6 ครั้ง

  รอบที่ 2: ทำการเปรียบเทียบ 5 ครั้ง

  รอบที่ 3: ทำการเปรียบเทียบ 4 ครั้ง

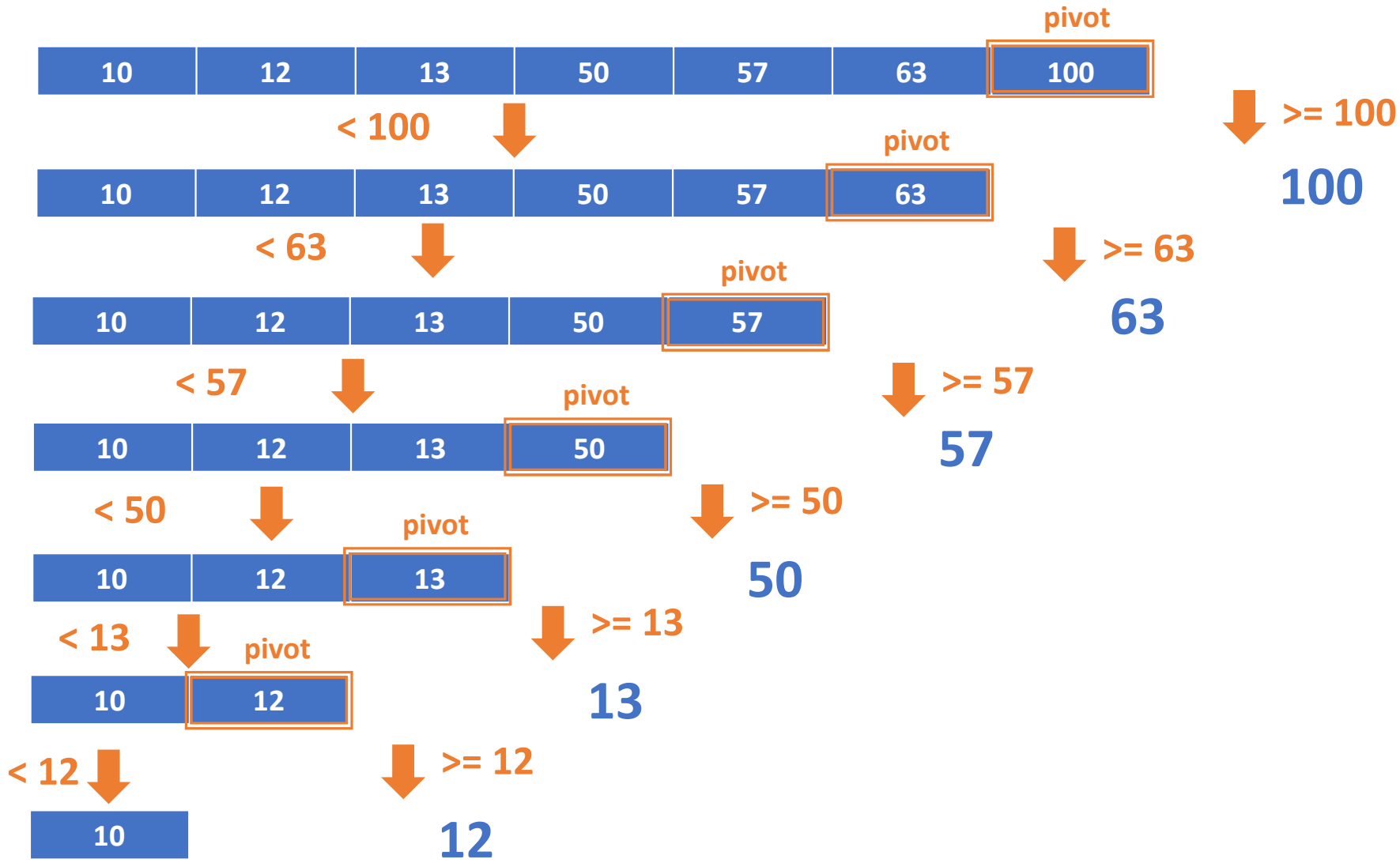
  . . .

  รอบที่ 7: ทำการเปรียบเทียบ 1 ครั้ง

  เมื่อ n = 7, $\sum_{i=1}^{7}(7-i) = \sum_{i=1}^{n}(n-i) = \sum_{i=1}^{n} n - \sum_{i=1}^{n} i$

$$= \underbrace{(n+n+\cdots+n)}_{\text{n ครั้ง}} - \frac{n\times(n+1)}{2}$$

$$= (n \times n) - \frac{n^2+n}{2} = \frac{n^2-n}{2}$$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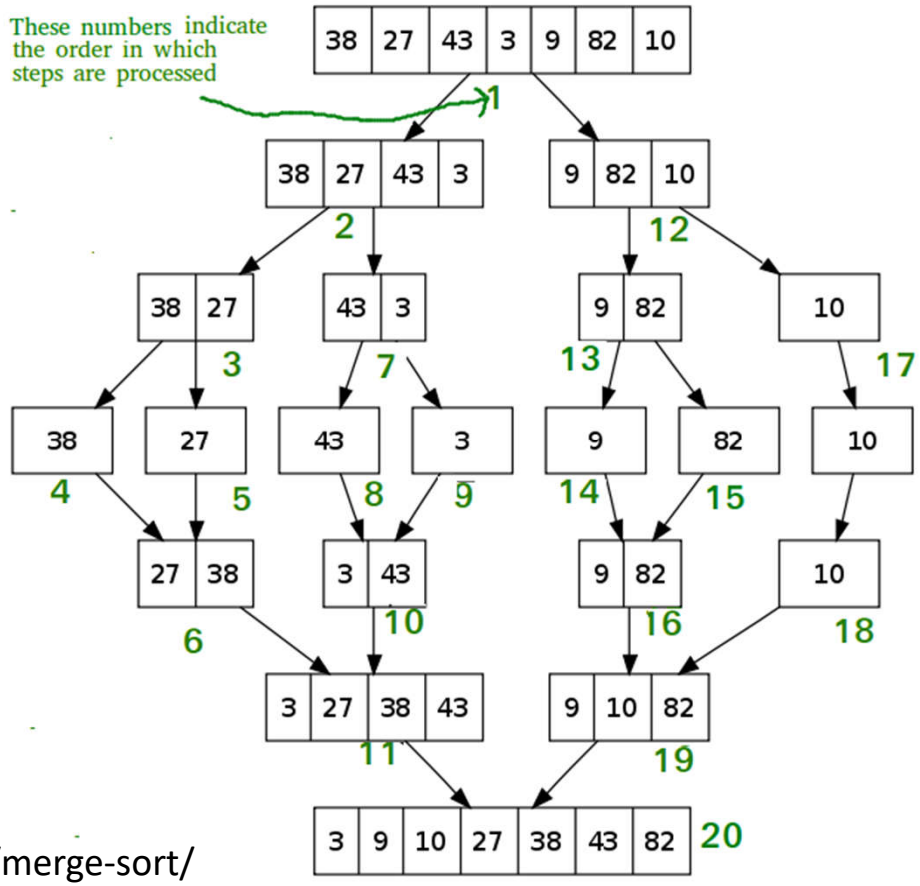
Introduction: ตัวอย่าง Computational Problems:

The Sorting Problem: Merge Sort -- in action



https://www.geeksforgeeks.org/merge-sort/

Introduction: ตัวอย่าง Computational Problems : gcd(m, n)

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Finding the greatest common divisor (gcd) of two non-negative, not both zero integers

- **Problem Statement:**
  - *Input:* two non-negative, not both zero integers called m and n , $m \geq n$
  - *Output:* the greatest common divisor of m and n
- a problem Instance: (18, 12), (60, 24), (12, 0)
- *Algorithms:* the intuitive algorithm, Euclid's Algorithm

Introduction: ตัวอย่าง Computational Problems: gcd(m, n) : the Consecutive Integers Checking algorithm

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- Intuitively based on the definition of gcd

Algorithm gcd(***m***, ***n***)

If ***n*** = 0 Return ***m*** and stop.

While ***n*** > 1 do

Begin

    If ***m*** is divisible by ***n*** Return ***n*** and stop.

    Otherwise, decrease ***n*** by one.

End

Return ***n***.

Introduction: ตัวอย่าง Computational Problems: gcd(m, n): Euclid's algorithm

- gcd (***m***, ***n***) = gcd (***n***, ***m*** mod ***n***)

  gcd (***m***, ***0***) = ***m***

- Ex. gcd (***18***, ***12***)

  = gcd (***12***, ***6***)

  = gcd (***6***, ***0***)

  = ***6***

Algorithm gcd(***m***, ***n***)

While ***n*** ≠ 0 do

Begin

r = ***m*** mod ***n***

m = ***n***

n = ***r***

End.

Return ***m***.

Introduction (13)

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- several algorithms for solving the same problem may exist
  - different people have different ideas when solving a problem
  - different ***algorithm design techniques***
- one algorithm can be implemented in several different ways using different ***data structures***
- as a result, *different algorithms* that are based on *different ideas* and employed different data structures could produce the desired result with largely ***different speeds*** (different ***execution time*** or ***run time***)

Introduction (14)

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- given an algorithm implemented with one algorithm design technique using one type of data structure
- there are many factors that still play a role in the *running time* of the algorithm
- Let's look at the Sorting problem as our example
  - **the number of items to be sorted (input size)**
  - **the extent to which the items are already somewhat sorted**
  - the architecture of the computer
  - the kind of storage devices to be used: M/M, disks, or tapes

Introduction:
Fundamentals of Algorithmic Problem Solving

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1. Understand the problem
2. Choose between exact and approximate problem solving
3. Decide on algorithm design techniques and appropriate data structures
4. Specify the algorithm: natural language, pseudocode, flowcharts
5. Prove the algorithm's correctness
    1. Exact algorithms: the algorithm yields a required result for every legitimate input
    2. Approximate algorithms: the error produced by the algorithm does not exceed a pre-defined limit
6. Analyze the algorithm
7. Coding the algorithm

Introduction: วัตถุประสงค์การเรียนรู้ (Our Goals)

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- learn several **algorithm design techniques** by studying a standard set of important algorithms from different areas of computing (different problem types)
- discuss the general framework for efficiency analysis
- so that, students
  - can develop algorithms on their own when given a computational problem
  - can formally argue the correctness of their algorithms
  - can analyze the efficiency of their algorithms

Introduction: Important Problem Types

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- Sorting
- Searching
- String Processing e.g. string matching - searching for a given word in a text
- Graph problems
- Combinatorial problems
- Geometric problems
- Numerical problems

พื้นฐานการวิเคราะห์ความซับซ้อนของอัลกอริทึม

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What is an Analysis of Algorithms?

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- "Analysis of algorithms" means an investigation of an algorithm's efficiency with respect to two resources: running time and memory space
  - การประเมินประสิทธิภาพของอัลกอริทึม
- time efficiency ประสิทธิภาพเชิงเวลา (time complexity) indicates how fast an algorithm in question runs
- space efficiency ประสิทธิภาพเชิงพื้นที่ (space complexity) refers to the amount of memory units the algorithm requires

Analysis of Algorithms: the Analysis Framework

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- it is logical to investigate an algorithm's efficiency as "a function of some parameter *n* indicating **the algorithm's input size**"

Analysis of Algorithms: the Analysis Framework (2)

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- Example: the problem of sorting elements in a list
  - the **algorithm's input size** is indicated by **the list's size**
- Example: the problem of searching for the maximum element in a list
  - the **algorithm's input size** is indicated by **the list's size**
- Example: problems of matrix addition, matrix subtraction
  - the **algorithm's input size** is indicated by **the matrix's size**
- Example: problem of evaluating a polynomial degree k

$$p(x) = a_k x^k + a_{k-1} x^{k-1} + \ldots + a_1 x + a_0$$

  - the **algorithm's input size** is indicated by **the degree of polynomial**

Analysis of Algorithms: the Analysis Framework: Time Efficiency

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- **Time efficiency** is analyzed by counting the number of times an algorithm's basic operation is executed on input of size *n*
  - basic operation: the operation contributing the most to the total running time

Rule for finding algorithm's basic operation: it is the most time-consuming operation in the algorithm's innermost loop

Analysis of Algorithms: the Analysis Framework

Time Efficiency (2)

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Example: the problem of searching for the maximum element in a list

```
Algorithm FindMax(A[0..n-1])
max = A[0];
For (i = 1; i < n; i++) do
        if (max < A[i]) max = A[i];
```

- basic operation: key comparison inside loop for

Analysis of Algorithms: the Analysis Framework

Time Efficiency (3)

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Example: the problem of sorting all elements in a list

```
Algorithm BubbleSort(A[0..n-1])
For (i = 0; i < n-1; i++)
        For (j = 0; j < n – i – 1; j++)
                if (A[j] > A[j+1])
                        Swap(A[j], A[j+1]);
```

- basic operations: key comparison inside the inner loop for
  swap operation inside the inner loop for

Analysis of Algorithms: the Analysis Framework: Time Efficiency (4)

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- Let **n** be the algorithm's input size
- let c_{op} be the execution time of an algorithm's basic operation on a particular computer
- let **C(n)** be the number of times this basic operation needs to be executed for this algorithm Ex. **C(n) = 0.5n(n-1)**
- then, the **running time T(n)** of a program implementing this algorithm on this particular computer can be estimated by the formula

$$T(n) \approx c_{op} \times C(n)$$

Analysis of Algorithms: the Analysis Framework: Time Efficiency (5)

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- We usually focus on how much the running time changes as the input size increases, not the running time for a particular input size

How much longer will the algorithm run if we double its input size?

$$\frac{T(2n)}{T(n)} = \frac{c_{op} \times C(2n)}{c_{op} \times C(n)}$$

note that we were able to answer the question w/o actually knowing the value of c_{op}: the value was neatly **cancelled out** in the ratio

Analysis of Algorithms: the Analysis Framework: Time Efficiency (6)

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- it is for these reasons that **the time efficiency analysis framework ignores multiplicative constants** and **concentrates on the count**'s **order of growth** to within a constant multiple for **large-size inputs**

$$T(n) \approx C(n)$$

Analysis of Algorithms: the Analysis Framework: Time Efficiency (7)

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- When discussing **the count**'s **order of growth** to within a constant multiple for **large-size inputs**, we do not need to be exact
- We usually use **asymptotic notations** to express the time efficiency of an algorithm (**the count**'s **order of growth**)

Analysis of Algorithms: the Analysis Framework:
Time Efficiency: Asymptotic Notations

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- to **compare** and **rank** the **orders of growth** of an algorithm's **basic operation counts**, computer scientists use three notations:

1. ***O*** (big Oh)
2. **Ω** (big Omega)
3. **Θ** (big Theta)

Analysis of Algorithms: the Analysis Framework: Time Efficiency: Asymptotic Notations: big-Oh

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- **O(g(n))** is the set of all functions with a **smaller or same order of growth** as **g(n)** (to within a constant multiple, as **n** goes to infinity).
- **g(n)** is an "upper-bound".
- for example,

$$n \in O(n^2), \qquad 100n + 5 \in O(n^2), \qquad \frac{1}{2}n(n-1) \in O(n^2)$$

$$n^3 \notin O(n^2), \qquad 0.00001n^3 \notin O(n^2), \qquad n^4 + n + 1 \notin O(n^2)$$

Analysis of Algorithms: the Analysis Framework: Time Efficiency: Asymptotic Notations: big-Omega

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- **Ω(g(n))**, stands for the set of all functions with a **larger or same order of growth** as **g(n)** (to within a constant multiple, as **n** goes to infinity).
- **g(n)** is a "lower-bound".
- for example,

$$n^3 \in \Omega(n^2), \qquad \frac{1}{2}n(n-1) \in \Omega(n^2), \qquad \text{but } 100n + 5 \notin \Omega(n^2).$$

Analysis of Algorithms: the Analysis Framework: Time Efficiency: Asymptotic Notations: big-Theta

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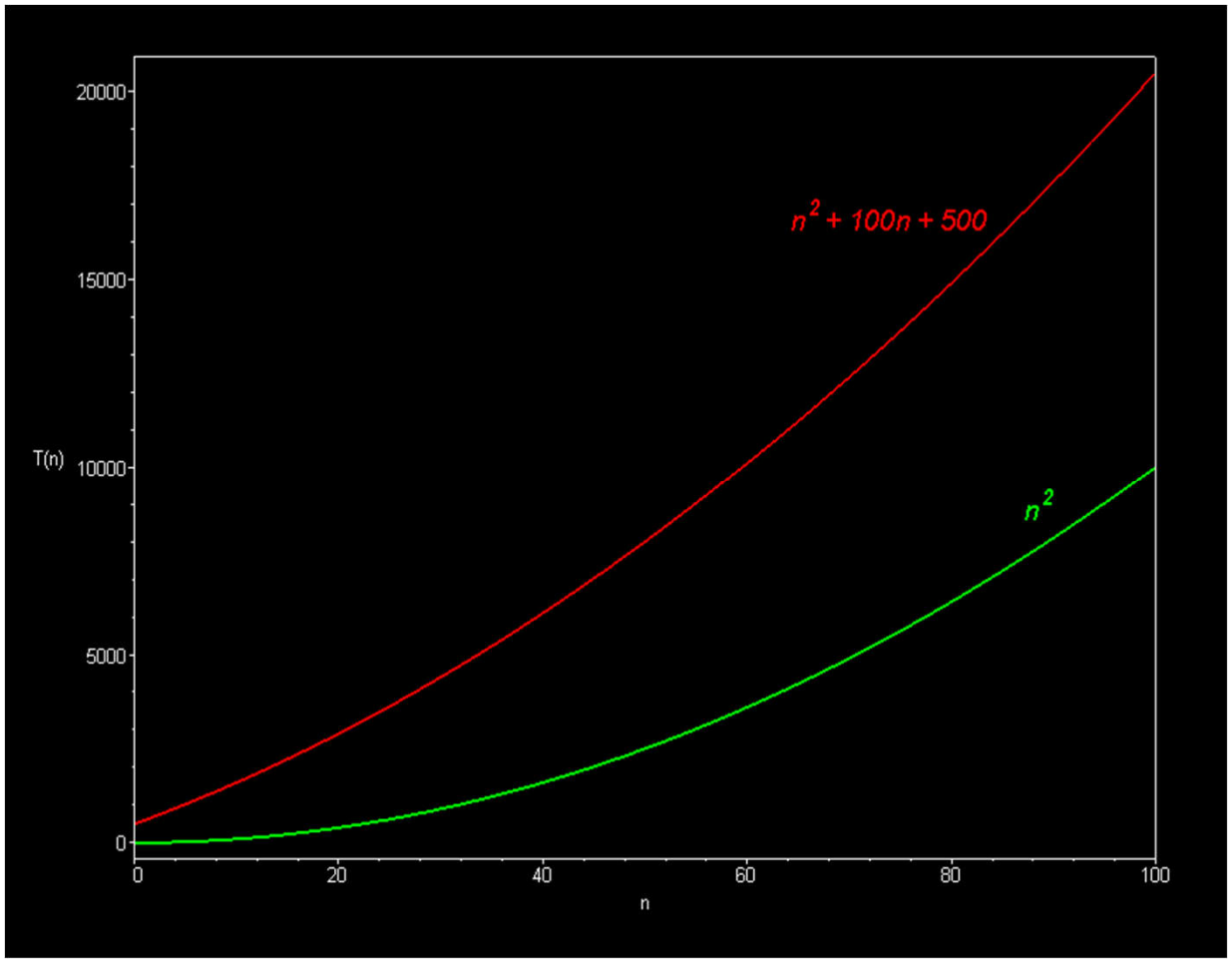
- $\Theta(g(n))$ is the set of all functions that have the **same order of growth** as $g(n)$ (to within a constant multiple, as n goes to infinity).
- thus, every quadratic function $an^2 + bn + c$ with $a > 0$ is in $\Theta(n^2)$

source: http://science.slc.edu/~jmarshall/courses/2002/spring/cs50/BigO/

Asymptotic Notations: Informal Introduction: big-Theta (2)



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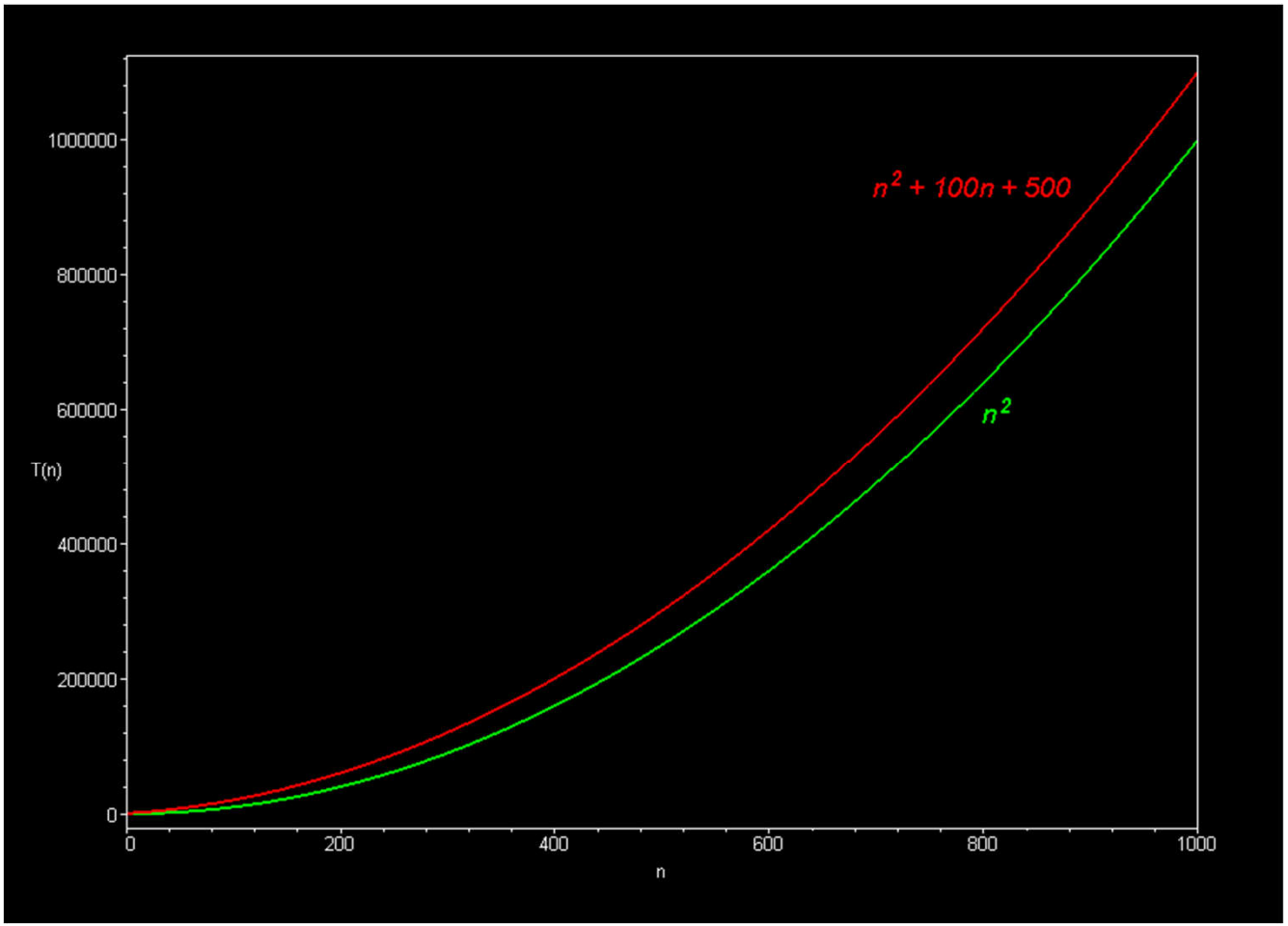
Asymptotic Notations: Informal Introduction: big-Theta (3)



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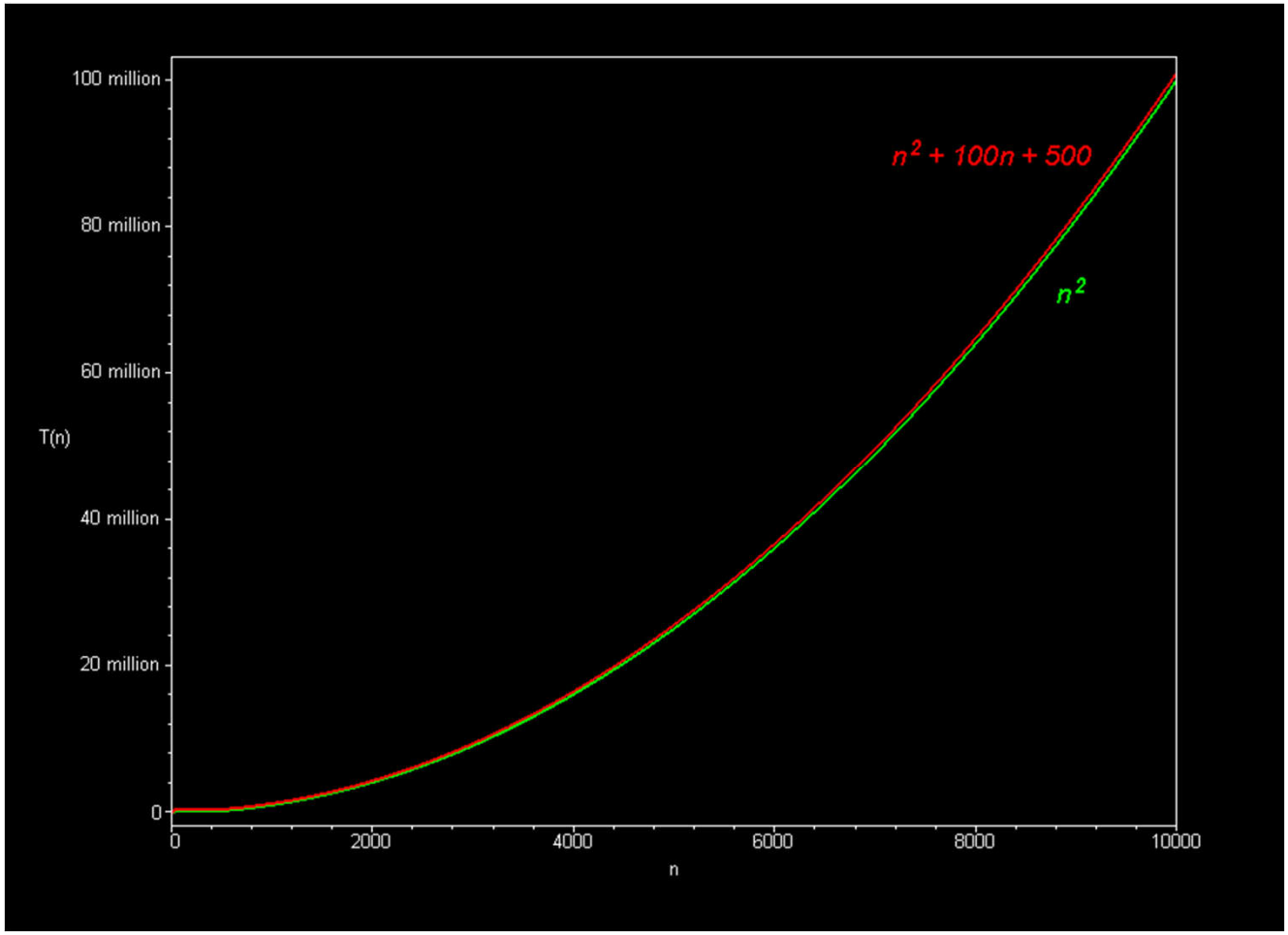
Asymptotic Notations: Informal Introduction: big-Theta (4)



Analysis of Algorithms: the Analysis Framework:

Time Efficiency: Basic Asymptotic Efficiency Classes

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Class	Name
1	Constant
log n	Logarithmic
n	linear
n log n	“n-log-n”
n^2	Quadratic
n^3	Cubic
2^n	exponential
n!	factorial

Analysis of Algorithms: the Analysis Framework: Time Efficiency: Order of Growth

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TABLE 2.1 Values (some approximate) of several functions important for analysis of algorithms

n	$\log_2 n$	n	$n \log_2 n$	n^2	n^3	2^n	$n!$
10	3.3	10^1	$3.3 \cdot 10^1$	10^2	10^3	10^3	$3.6 \cdot 10^6$
10^2	6.6	10^2	$6.6 \cdot 10^2$	10^4	10^6	$1.3 \cdot 10^{30}$	$9.3 \cdot 10^{157}$
10^3	10	10^3	$1.0 \cdot 10^4$	10^6	10^9		
10^4	13	10^4	$1.3 \cdot 10^5$	10^8	10^{12}		
10^5	17	10^5	$1.7 \cdot 10^6$	10^{10}	10^{15}		
10^6	20	10^6	$2.0 \cdot 10^7$	10^{12}	10^{18}		

Logarithmic
Linear
Linearithmetic
Quadratic
Cubic
Exponential
Factorial

Analysis of Algorithms: the Analysis Framework: Time Efficiency: Order of Growth (2)

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- the function ***growing the slowest*** among these is the **logarithmic function**
- the function ***growing the fastest*** among these is the **factorial function**
- a program implementing an algorithm with a **logarithmic basic-operation count** runs practically on inputs of all realistic sizes
- a program implementing an algorithm that requires an **exponential number of basic-operations** are practical for only problems of very small sizes

Analysis of Algorithms: the Analysis Framework: Time Efficiency: Order of Growth (3)

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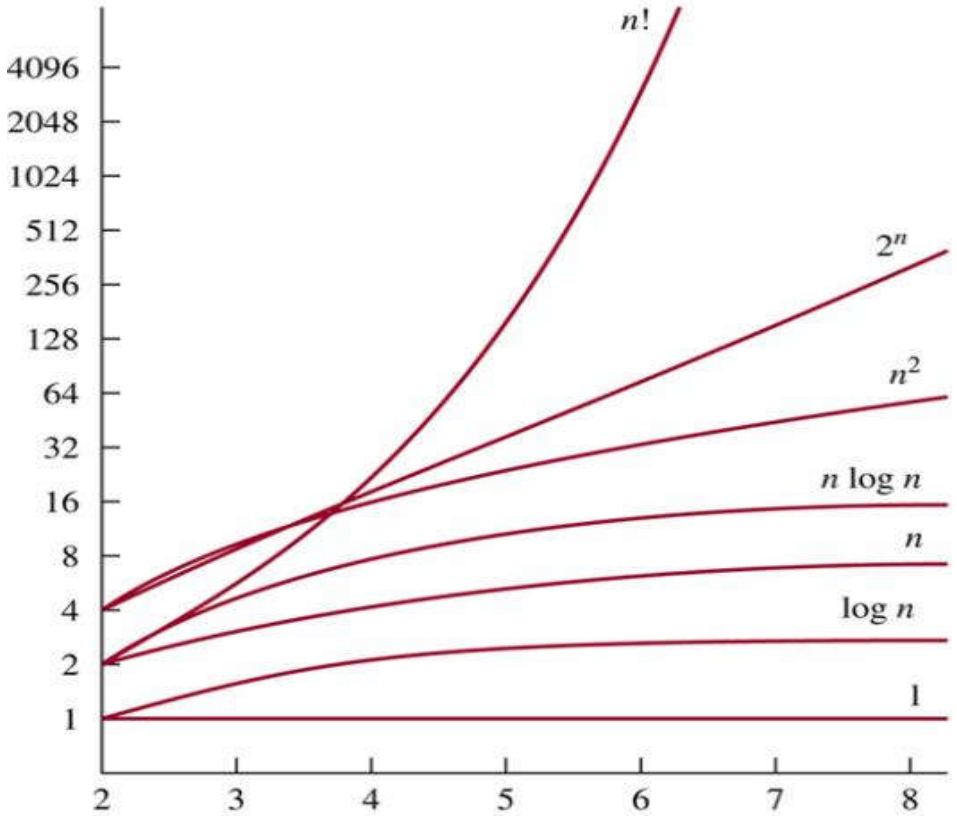
- Example. See how functions **C(n)** with different growths react to a twofold increase in the value of their arguments (**n**)

	C(n)	C(2n)
1. Logarithmic	$\log_2 n$	$\log_2 2n = \log_2 2 + \log_2 n = 1 + \log_2 n$ [i.e. **increase by just 1**]
2. Linear	n	$2n$ [i.e. **twofold**]
3. Linearithmic	$n \log_2 n$	$2n(1 + \log_2 n) = 2n + 2n \log_2 n$ [i.e. **slightly more than twofold**]
4. Quadratic	n^2	$(2n)^2 = 4n^2$ [i.e. **fourfold**]
5. Cubic	n^3	$(2n)^3 = 8n^3$ [i.e. **eightfold**]
6. Exponential	2^n	$2^{2n} = (2^n)^2$ [i.e. **2^n-fold**]
7. Factorial	$n!$	$(2n)!$

Analysis of Algorithms: the Analysis Framework: Time Efficiency: Order of Growth (4)



Analysis of Algorithms: the Analysis Framework

Time Efficiency: Example (1)

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For an algorithm with **$C(n) = 0.5n(n-1)$**

- $C(n) = 0.5n^2 - 0.5n$
- big-Oh for the time efficiency of the algorithm is **$O(n^2)$**

Analysis of Algorithms: the Analysis Framework

Time Efficiency: Example (2)

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Example: the problem of searching for the maximum element in a list

```
Algorithm FindMax(A[0..n-1])
max = A[0];
For (i = 1; i < n; i++) do
        if (max < A[i]) max = A[i];
```

- basic operation: key comparison inside loop for
- $C(n) = n$
- big-Oh for the time efficiency of the algorithm is $O(n)$

Analysis of Algorithms: the Analysis Framework

Time Efficiency: Example (3)

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Example: the problem of sorting all elements in a list

```
Algorithm BubbleSort(A[0..n-1])
For (i = 0; i < n-1; i++)
    For (j=0; j < n – i -1; j++)
        if (A[j] > A[j+1])
            Swap(A[j], A[j+1]);
```

$$C(n) = \sum_{i=0}^{n-2} \sum_{j=0}^{n-i-2} 2 = \sum_{i=0}^{n-2} 2(n-i-1)$$

$$= 2\left[\sum_{i=0}^{n-2} n - \sum_{i=0}^{n-2} i - \sum_{i=0}^{n-2} 1\right]$$

$$= n^2 - n$$

big-Oh for the time efficiency of the algorithm = $O(n^2)$

- basic operations: key comparison inside the inner loop for

  swap operation inside the inner loop for

Analysis of Algorithms: the Analysis Framework

Time Efficiency: Worst-Case, Best-Case, Average-Case

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- There are many algorithms for which running time depends *not only* on an ***input size*** *but also* on ***the specifics of a particular input***:
  - worst-case, best-case, and average-case running time

Analysis of Algorithms: the Analysis Framework

Time Efficiency: Worst-Case

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- the **worst-case efficiency** of an algorithm is its efficiency for **the worst-case input** of size **n**, which is an input of size **n** for which the algorithm **runs the longest** among all possible inputs of that size

Analysis of Algorithms: the Analysis Framework

Time Efficiency: Best-Case

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- the **best-case efficiency** of an algorithm is its efficiency for **the best-case input** of size **n**, which is an input of size **n** for which the algorithm runs **the fastest** among all possible inputs of that size.
- **best case does not mean the smallest input**

Analysis of Algorithms: the Analysis Framework

Time Efficiency: Average-Case

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- **the average-case efficiency seeks to provide the algorithm's behavior on a "typical" or "random" input.**
- to analyze the algorithm's **average-case efficiency**, we must make some **assumptions** about possible inputs **of size** `n`.

Analysis of Algorithms: the Analysis Framework

Time Efficiency: Worst-Case, Best-Case, Average-Case: Example

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- Example: Sequential Search Algorithm

ALGORITHM SequentialSearch (A[0..(n-1)], K)

// Search for a given value in a given array by sequential search

// **Input:** an array `A[0..n-1]` and a search key `K`

// **Output:** return the index of the first element of `A` that matches `K`

// or -1 if there are no matching elements

```
i ← 0
while i < n and A[i] ≠ K do
    i ← i + 1
if i < n return i
else return -1
```

Analysis of Algorithms: the Analysis Framework

Time Efficiency: Worst-Case, Best-Case, Average-Case:

Example (2)

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- Sequential Search Algorithm: Worst Case:
  - when there are no matching elements or
  - the first matching element happens to be the last one on the list

$$C_{worst}(n) = n$$

- the **worst-case analysis** works as an **upper-bound for the running time of an algorithm**: it guarantees that for any instance of size n, the running time **will not exceed** $C_{worst}(n)$

Analysis of Algorithms: the Analysis Framework

Time Efficiency: Worst-Case, Best-Case, Average-Case:

Example (2)

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- Sequential Search Algorithm: Best Case:
  - when the first element in the list equals to the search key

$$C_{best}(n) = 1$$

Analysis of Algorithms: the Analysis Framework

Time Efficiency: Worst-Case, Best-Case, Average-Case:

Example (3)

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- Sequential Search Algorithm: Average Case:
- **Standard assumptions**

(1) the probability of a successful search is equal to `p`

(0 <= `p` <= 1)

and

(2) the probability of the first match occurring

in the i^{th} position of the list is the same for every `i`

Analysis of Algorithms: the Analysis Framework

Time Efficiency: Worst-Case, Best-Case, Average-Case:

Example (4)

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- in the case of a **successful search**, **the probability of the first match occurring in the *i*th position** of the list is **`p/n` for every `i`**, and **the number of comparisons** made by the algorithm in such a situation is obviously **`i`**
- in the case of an **unsuccessful search**, **the number of comparisons** is **`n`** with the **probability of such a search being** `(1-p)`.
- Therefore,

$$C_{avg}(n) = [1 \cdot \frac{p}{n} + 2 \cdot \frac{p}{n} + \cdots + i \cdot \frac{p}{n} + \cdots + n \cdot \frac{p}{n}] + n \cdot (1 - p)$$

$$= \frac{p}{n}[1 + 2 + \cdots + i + \cdots + n] + n(1 - p)$$

$$= \frac{p}{n}\frac{n(n+1)}{2} + n(1 - p) = \frac{p(n+1)}{2} + n(1 - p).$$

Brute Force

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Introduction

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- ***Brute force*** is a straightforward approach to solving a problem, usually directly based on the problem statement and definitions of the concepts involved.
- ***Brute force*** = "just do it"
- Often, the brute-force strategy is indeed the one that is easiest to apply.

Brute-Force Algorithm: Exponentiation

ALGORITHM Exponentiation (b, n)

// Compute b^n = b x b x .... x b (total n times)

// Input: a base b which is an integer and an exponent n which is a positive integer

// Output: return the result of b to the power n, b^n

```
result ← 1
for (i = 1; i <= n; i++) do
    result ← result * b
return result
```

consider the multiplication to be the algorithm's basic operation

ลูปนี้ทำงาน n รอบ

แต่ละรอบทำงาน 'คูณ' 1 ครั้ง

$C(n) = \sum_{i=1}^{n} 1 = n$

$\therefore$ big-Oh = O(n)

Brute-Force Algorithm: gcd(m, n) : the Consecutive Integers Checking algorithm

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ALGORITHM gcd (m, n)
// Compute the Greatest Common Divisor (GCD) for two non-negative, not both zero
// integers ***m*** and ***n***
// Input: two non-negative, not both zero integers called m and n AND $m \geq n$
// Output: return gcd(m, n)
If ***n*** = 0 Return ***m*** and stop.
While ***n*** > 1 do
Begin
    If ***m*** is divisible by ***n*** Return ***n*** and stop.
    Otherwise, decrease ***n*** by one.
End
Return ***n***.

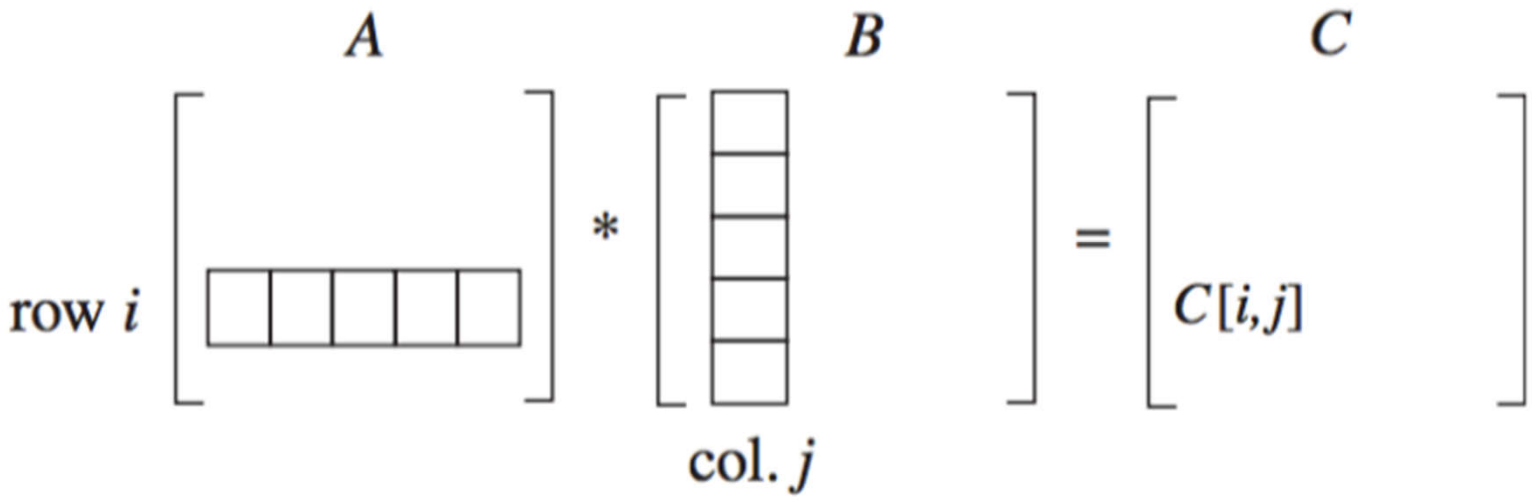
consider the division to be the algorithm's basic operation

ลูปนี้ทำงาน n รอบ (worst-case)
แต่ละรอบทำงาน 'หาร' 1 ครั้ง

$C(n) = \sum_{i=1}^{n} 1 = n$

$\therefore$ big-Oh = O(n)

Brute-Force Algorithm: Matrix Multiplication



$$C[i,j] = A[i,0]B[0,j] + \ldots + A[i,k]B[k,j] + \ldots + A[i,n-1]B[n-1,j]$$

where $0 \leq i, j \leq n-1$.

Brute-Force Algorithm: Matrix Multiplication (2)

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ALGORITHM *MatrixMultiplication*$(A[0..n-1, 0..n-1], B[0..n-1, 0..n-1])$

```
//Multiplies two square matrices of order n by the definition-based algorithm
//Input: Two n × n matrices A and B
//Output: Matrix C = AB
for i ← 0 to n − 1 do
    for j ← 0 to n − 1 do
        C[i, j] ← 0.0
        for k ← 0 to n − 1 do
            C[i, j] ← C[i, j] + A[i, k] * B[k, j]
return C
```

consider the multiplication to be the algorithm's basic operation

The measurement of the input size is 'n' (the matrix order)

Brute-Force Algorithm: Matrix Multiplication (3)

- just **one multiplication** is made **for each repetition of the innermost loop**, i.e., for each value of the loop's variable **`k`** between its limits **`0`** and **`n-1`**
- that is, the number of **multiplications** made for every specific pair of variables **`i`** and **`j`** is

$$\sum_{k=0}^{n-1} 1$$

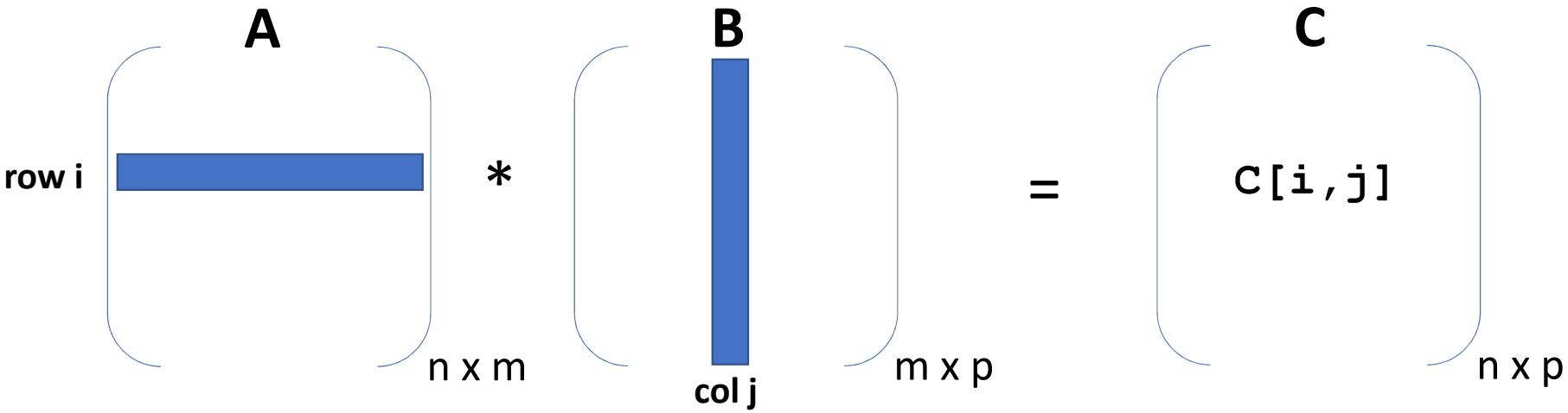
- the total number of **multiplications** **`M(n)`** is expressed by the following:

$$M(n) = \sum_{i=0}^{n-1}\sum_{j=0}^{n-1}\sum_{k=0}^{n-1} 1 = \sum_{i=0}^{n-1}\sum_{j=0}^{n-1} n = \sum_{i=0}^{n-1} n^2 = n^3 \in \Theta(n^3)$$

Brute-Force Algorithm: Exercise 1

Matrix Multiplication: given one `n` x `m` matrix (called `A`) and one `m` x `p` matrix (called `B`),

- write a brute-force version of MatrixMultiplication algorithm for computing their product `C` = `A` x `B` and
- find the time efficiency of such definition-based algorithm



Brute-Force and The Sorting Problem

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the application of the brute-force approach to the problem of sorting

The Sorting Problem: given a list of ***n*** orderable items, re-arrange them in non-decreasing order

- What would be the most straight-forward method for solving the sorting problem?

Brute-Force and The Sorting Problem:

Selection Sort

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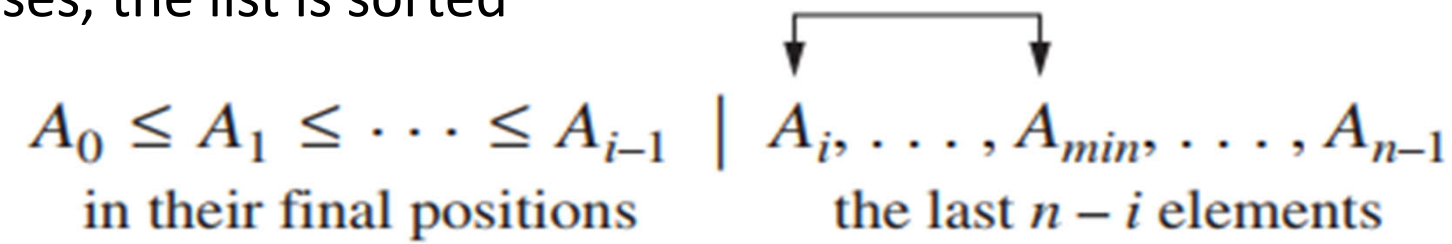
- The selection sort algorithm sorts an array by repeatedly finding the minimum element from unsorted part and putting it at the beginning.
- The algorithm maintains two subarrays in a given array.

1) The subarray which is already sorted.
2) Remaining subarray which is unsorted.

- In every iteration of selection sort, the minimum element from the unsorted subarray is picked and moved to the end of sorted subarray.
- After n-1 passes, the list is sorted



Brute-Force and The Sorting Problem:

Selection Sort : Example

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| 89	45	68	90	29	34	**17**
17 |	45	68	90	**29**	34	89
17	29 |	68	90	45	**34**	89
17	29	34 |	90	**45**	68	89
17	29	34	45 |	90	**68**	89
17	29	34	45	68 |	90	**89**
17	29	34	45	68	89 |	90

Brute-Force and The Sorting Problem: Selection Sort (3)

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ALGORITHM *SelectionSort*$(A[0..n-1])$

```
//Sorts a given array by selection sort
//Input: An array A[0..n − 1] of orderable elements
//Output: Array A[0..n − 1] sorted in nondecreasing order
for i ← 0 to n − 2 do
    min ← i
    for j ← i + 1 to n − 1 do
        if A[j] < A[min]  min ← j
    swap A[i] and A[min]
```

Brute-Force and The Sorting Problem:

Selection Sort (4)

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- the input size is given by the number of elements in the list ***n***
- the basic operation is the key comparison $A[j] < A[min]$

$$C(n) = \sum_{i=0}^{n-2} \sum_{j=i+1}^{n-1} 1 = \sum_{i=0}^{n-2} [(n-1) - (i+1) + 1] = \sum_{i=0}^{n-2} (n-1-i).$$

= (n-1) + (n-2) + (n-3) + … + 3 + 2 + 1 = $\sum_{i=1}^{n-1} i = \frac{(n-1)n}{2}$

$\in O(n^2)$

Brute-Force and The Sorting Problem: Bubble Sort

- Bubble Sort is the simplest sorting algorithm that works by repeatedly swapping the adjacent elements if they are in wrong order
- Pass *i (*$0 \le i \le n - 2$*)* of bubble sort can be represented by the following diagram:

$$A_0, \ldots, A_j \overset{?}{\leftrightarrow} A_{j+1}, \ldots, A_{n-i-1} \mid \underbrace{A_{n-i} \le \cdots \le A_{n-1}}_{\text{in their final positions}}$$

Brute-Force and The Sorting Problem:
Bubble Sort (2)

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- Example: sort the list (5, 1, 4, 2, 8) in non-decreasing order
  (เรียงจากน้อยไปมาก)

First Pass:

(**5 1** 4 2 8) –> (**1 5** 4 2 8),
(1 **5 4** 2 8) –> (1 **4 5** 2 8),
(1 4 **5 2** 8) –> (1 4 **2 5** 8),
(1 4 2 **5 8**) –> (1 4 2 **5 8**), I

Second Pass:

(**1 4** 2 5 8) –> (**1 4** 2 5 8)
(1 **4 2** 5 8) –> (1 **2 4** 5 8),
(1 2 **4 5** 8) –> (1 2 **4 5** 8)
(1 2 4 **5 8**) –> (1 2 4 **5 8**)

Third Pass:

(**1 2** 4 5 8) –> (**1 2** 4 5 8)
(1 **2 4** 5 8) –> (1 **2 4** 5 8)
(1 2 **4 5** 8) –> (1 2 **4 5** 8)
(1 2 4 **5 8**) –> (1 2 4 **5 8**)

Brute-Force and The Sorting Problem:

Bubble Sort (3)

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ALGORITHM *BubbleSort*$(A[0..n-1])$

//Sorts a given array by bubble sort
//Input: An array $A[0..n-1]$ of orderable elements
//Output: Array $A[0..n-1]$ sorted in nondecreasing order
for $i \leftarrow 0$ **to** $n-2$ **do**
    for $j \leftarrow 0$ **to** $n-2-i$ **do**
        if $A[j+1] < A[j]$ swap $A[j]$ and $A[j+1]$

- consider key comparison and key swap as the basic operations of the algorithm
- consider the worst-case where key swaps are done at every iteration

Brute-Force and The Sorting Problem:
Bubble Sort (4)

C(n) = $\sum_{i=0}^{n-2} \sum_{j=0}^{n-2-i} 1 = \sum_{i=0}^{n-2} (n-1-i)$

= (n-1) + (n-2) + (n-3) + … + 3 + 2 + 1 = $\sum_{i=1}^{n-1} i = \frac{(n-1)n}{2} \in O(n^2)$

$S_{worst}(n) = C(n) \in O(n^2)$

The overall time complexity of the algorithm is $O(n^2)$

Brute-Force String Matching

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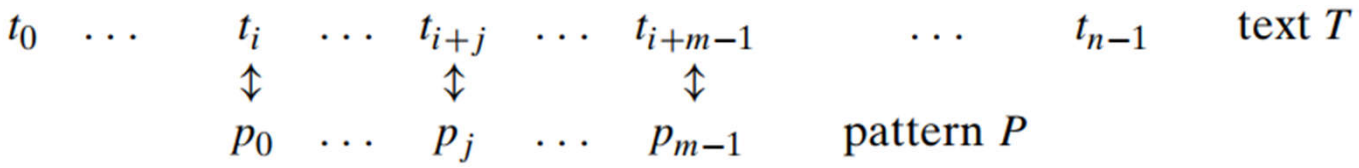
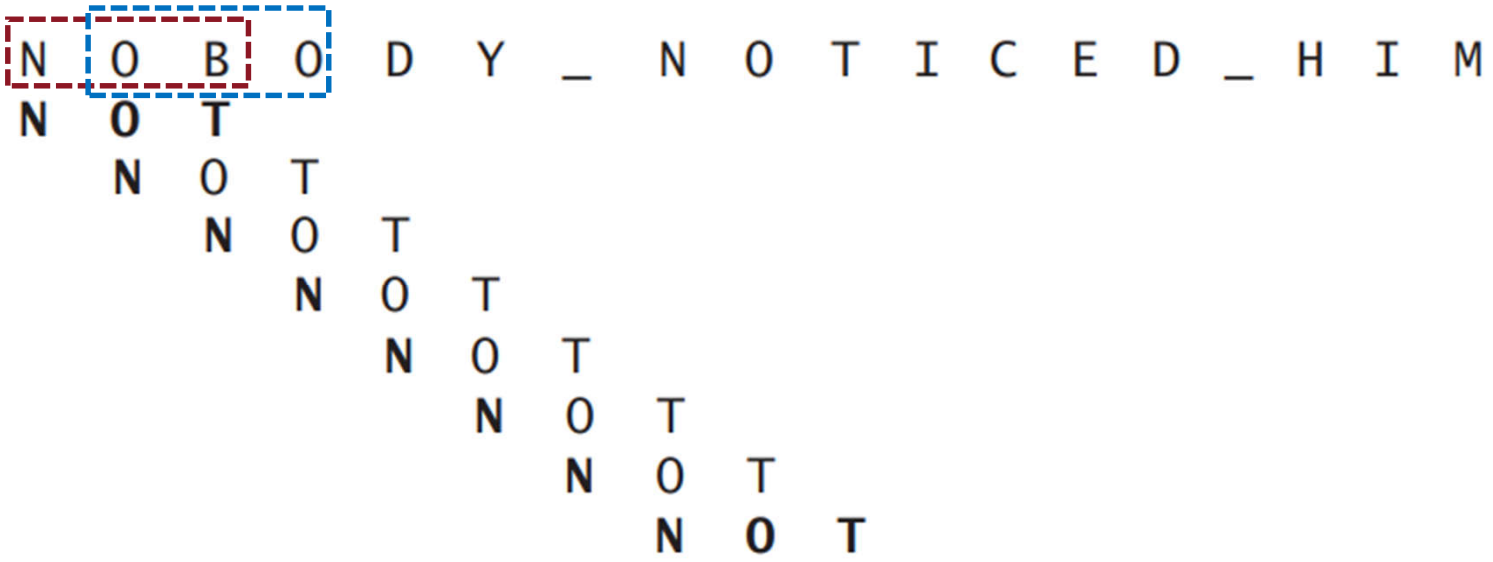
- given a string of *n* characters called the ***text***
  and a string of *m* characters *(m ≤ n)* called the ***pattern***,
  find a substring of the text that matches the pattern.



Brute-Force String Matching (2)



- Pattern = “NOT”

Brute-Force String Matching (3)

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ALGORITHM *BruteForceStringMatch*$(T[0..n-1],\ P[0..m-1])$

```
//Implements brute-force string matching
//Input: An array T[0..n − 1] of n characters representing a text and
//       an array P[0..m − 1] of m characters representing a pattern
//Output: The index of the first character in the text that starts a
//        matching substring or −1 if the search is unsuccessful
for i ← 0 to n − m do
    j ← 0
    while j < m and P[j] = T[i + j] do
        j ← j + 1
    if j = m return i
return −1
```

comparisons between corresponding characters in Pattern and current Text window

- The basic operation is the comparison between each character in Pattern and those in the current text window.
- worst-case scenario: we need to make all m comparisons prior to realizing that this is not a match for every text window

Brute-Force String Matching (4)

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- C(n) = $\sum_{i=0}^{n-m}(m) = m\ (n - m\ - 0 + 1)$

  = nm – m^2 + m
- Since $m \leq n$, C(n) $\in$ O($n\ m$)

Exercise 1: Prime Numbers

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- Given a positive integer `N` > 1, check whether `N` is a prime number.

Definitions:

I. A prime number is a whole number greater than 1 that **cannot** be made by multiplying other whole numbers.

II. A prime number is a whole number greater than 1, that has only two factors – 1 and the number itself.

III. A prime number is divisible only by the number 1 or itself.

Exercise 2: Nugget Numbers

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- ร้านฟาสต์ฟู้ดแห่งหนึ่งขายนักเก็ตเป็นกล่อง โดยมีกล่องนักเก็ตอยู่ 3 ขนาด ได้แก่ ขนาดเล็ก ขนาดกลาง และขนาดใหญ่ บรรจุนักเก็ต จำนวน 6 ชิ้น, 9 ชิ้น, และ 20 ชิ้น ตามลำดับ
- เลขนักเก็ต คือ เลขจำนวนเต็มบวกที่เกิดจากผลรวมของจำนวนชิ้นนักเก็ตในกล่องขนาดต่าง ๆ ยกตัวอย่างเช่น
  - เลข 6 เป็นเลขนักเก็ต เพราะเป็นจำนวนนักเก็ตในกล่องขนาดเล็กหนึ่งกล่อง
  - เลข 9 เป็นเลขนักเก็ต เพราะเป็นจำนวนนักเก็ตในกล่องขนาดกลางหนึ่งกล่อง
  - เลข 12 เป็นเลขนักเก็ต เพราะเกิดจากการรวมกันของจำนวนนักเก็ตในกล่องขนาดเล็ก 2 กล่อง
  - เลข 15 เป็นเลขนักเก็ต เพราะเกิดจากการรวมกันของจำนวนนักเก็ตในกล่องขนาดเล็ก และ ขนาดกลาง อย่างละหนึ่งกล่อง
  - เลข 20 เป็นเลขนักเก็ต เพราะเป็นจำนวนนักเก็ตในกล่องขนาดใหญ่หนึ่งกล่อง
  - เลข 4 และ 10 ไม่เป็นเลขนักเก็ต
- ให้นักศึกษาเขียนโปรแกรมเพื่อรับเลขจำนวนเต็มบวก N >= 6 และคำนวณพร้อมแสดงผลลัพธ์เป็นเลขนักเก็ตทั้งหมดที่มีค่าน้อยกว่าหรือเท่ากับ N

Brute-Force: Exhaustive Search

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Brute-Force: Exhaustive Search

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- ***Exhaustive search*** is simply a brute-force approach to combinatorial problems.
- It suggests generating each and every element of the problem domain, selecting those of them that satisfy all the constraints, and then finding a desired element (e.g., the one that optimizes some objective function).

- Traveling Salesman Problem, Knapsack Problem, Assignment Problem

Exhaustive Search: Traveling Salesman Problem

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- Given a set of ***n*** cities, find the shortest tour through the whole city set such that the salesman visits each city only once before returning to the city where the tour started

Exhaustive Search: Traveling Salesman Problem

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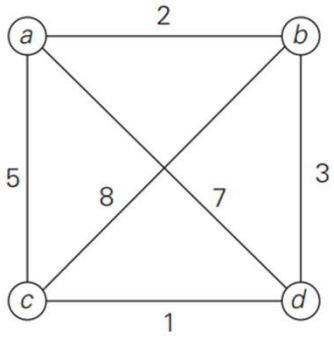
- The problem can be modeled a weighted undirected graph G=(V, E):
  - V represents the set of cities
  - E maintains the road information between cities
  - Each edge in E has an associated weight specifying distance between two cities
- Then, the problem can be stated as the problem of finding the shortest ***Hamiltonian circuit*** of the graph.
- a Hamiltonian circuit is defined as a cycle that passes through all the vertices of the graph exactly once.

Exhaustive Search: Traveling Salesman Problem (2)



Tour	Length	
$a \rightarrow b \rightarrow c \rightarrow d \rightarrow a$	$l = 2 + 8 + 1 + 7 = 18$	
$a \rightarrow b \rightarrow d \rightarrow c \rightarrow a$	$l = 2 + 3 + 1 + 5 = 11$	optimal
$a \rightarrow c \rightarrow b \rightarrow d \rightarrow a$	$l = 5 + 8 + 3 + 7 = 23$	
$a \rightarrow c \rightarrow d \rightarrow b \rightarrow a$	$l = 5 + 1 + 3 + 2 = 11$	optimal
$a \rightarrow d \rightarrow b \rightarrow c \rightarrow a$	$l = 7 + 3 + 8 + 5 = 23$	
$a \rightarrow d \rightarrow c \rightarrow b \rightarrow a$	$l = 7 + 1 + 8 + 2 = 18$	

Exhaustive Search: Knapsack Problem

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- Given *n* items of known weights $w_1, w_2, \ldots, w_n$ and values $v_1, v_2, \ldots, v_n$ and a knapsack of capacity *W*,
  find the most valuable subset of the items that fit into the knapsack.
- Example: W = 10

	weight	value
1	7	$42
2	3	$12
3	4	$40
4	5	$25

Exhaustive Search: Knapsack Problem (2)

	weight	value
1	7	$42
2	3	$12
3	4	$40
4	5	$25

- list all combinations:
  C(n,0) + C(n,1) + C(n,2) + … + C(n,n)
- Ex n = 4
  all combinations = 1 + 4 + 6 + 4 + 1
  = 16 possible patterns

Subset	Total weight	Total value
∅	0	$ 0
{1}	7	$42
{2}	3	$12
{3}	4	$40
{4}	5	$25
{1, 2}	10	$54
{1, 3}	11	not feasible
{1, 4}	12	not feasible
{2, 3}	7	$52
{2, 4}	8	$37
{3, 4}	**9**	**$65**
{1, 2, 3}	14	not feasible
{1, 2, 4}	15	not feasible
{1, 3, 4}	16	not feasible
{2, 3, 4}	12	not feasible
{1, 2, 3, 4}	19	not feasible

Exhaustive Search: Assignment Problem

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- There are *n* people who need to be assigned to execute *n* jobs, one person per job. (That is, each person is assigned to exactly one job and each job is assigned to exactly one person.)
- The cost that would accrue if the i^{th} person is assigned to the j^{th} job is a known quantity $C[i, j]$ for each pair $i, j = 1, 2, \ldots, n$.
- The problem is to find an assignment with the minimum total cost.

Exhaustive Search: Assignment Problem (2)

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Example n = 4

- the exhaustive-search approach to the assignment problem would require ***generating all the permutations*** of integers 1, 2, . . . , *n*,
- computing the total cost of each assignment by summing up the corresponding elements of the cost matrix,
- and finally selecting the one with the smallest sum.

	Job 1	**Job 2**	**Job 3**	**Job 4**
Person 1	9	2	7	8
Person 2	6	4	3	7
Person 3	5	8	1	8
Person 4	7	6	9	4