Delta rules

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Introduction

Introduction

Lecture topics

- Make it pretty: delta rules
- Booleans, boolean logic operators, if-then-else
- Naturals, arithmetic operators, comparison operators

Encoding boolean logic

Encoding boolean logic

Introduction

- We can decide that some specific lambda terms have special meanings
- For example, we could decide that a given lambda term means `TRUE`, another `FALSE`, etc.
- The important thing is that we choose terms that behave as we wish

Encoding boolean logic

As we wish?

- Suppose we define some lambda terms for `TRUE`, `FALSE`, and `AND`
- We expect these terms to reduce[a] following our expectations of boolean logic
- We can use truth tables to encode our expectations

---

[a] That is, computed according to $\rightarrow_{\beta}$

Encoding boolean logic

We want to formulate `TRUE`, `FALSE`, and `AND` so that

- $\texttt{TRUE} \wedge \texttt{TRUE} \rightarrow_\beta \texttt{TRUE}$
- $\texttt{TRUE} \wedge \texttt{FALSE} \rightarrow_\beta \texttt{FALSE}$
- $\texttt{FALSE} \wedge \texttt{TRUE} \rightarrow_\beta \texttt{FALSE}$
- $\texttt{FALSE} \wedge \texttt{FALSE} \rightarrow_\beta \texttt{FALSE}$

Defining terms with special meaning

Defining terms with special meaning

Choice terms

- Terms with special meaning essentially make a choice when given parameters
- The choice is expressed by either returning, or applying, the parameters

Defining terms with special meaning

Delta rules

- We wish to use special symbols to these terms with special meaning
- We define a series of delta rules, which are transformation from pretty symbols into lambda terms (and vice-versa)

Defining terms with special meaning

Delta rules

This means that we will be able to write lambda programs such as 5+3, that will then be translated into the appropriate lambda terms

Booleans

Booleans

Idea

- Boolean operators such as `TRUE` and `FALSE` must be defined so as to identify themselves
- The choice is expressed by returning their identity from a choice of two options

Booleans

`TRUE` is defined as a selector of the representative for true, that is the first argument[a]

---

[a] by arbitrary convention

```
(λt→f→t)
```

`FALSE` is defined as a selector of the representative for false, that is the second argument[a]

---

[a] by arbitrary convention, as long as different from the previous

```
(λt→f→f)
```

Booleans

```
((TRUE bit1) bit0)
```

Booleans

```
((TRUE bit1) bit0)
```

```
((TRUE bit1) bit0)
```

Booleans

```
((TRUE bit1) bit0)
```

Booleans

$$\texttt{((}\underline{\texttt{TRUE}}\ \texttt{bit1)}\ \texttt{bit0)}$$

$$\texttt{(((}\lambda\texttt{t}\rightarrow\texttt{f}\rightarrow\texttt{t)}\ \texttt{bit1)}\ \texttt{bit0)}$$

Booleans

```
(((λt→f→t) bit1) bit0)
```

Booleans

$$\texttt{(((}\lambda\texttt{t}\rightarrow\texttt{f}\rightarrow\texttt{t) bit1) bit0)}$$

$$\texttt{(}\underline{\texttt{((}\lambda\texttt{t}\rightarrow\texttt{f}\rightarrow\texttt{t) bit1)}}\texttt{ bit0)}$$

Booleans

$$\texttt{(}\underline{\texttt{((}\lambda\texttt{t}\rightarrow\texttt{f}\rightarrow\texttt{t) bit1)}}\ \texttt{bit0)}$$

Booleans

$$(\underline{((\lambda \texttt{t}\rightarrow\texttt{f}\rightarrow\texttt{t})\ \texttt{bit1})}\ \texttt{bit0})$$

$$((\lambda \texttt{f}\rightarrow\texttt{bit1})\ \texttt{bit0})$$

Booleans

```
((λf→bit1) bit0)
```

Booleans

```
((λf→bit1) bit0)
```

$$\underline{\texttt{((}\lambda\texttt{f}\rightarrow\texttt{bit1) bit0)}}$$

Booleans

$\underline{\texttt{((}\lambda\texttt{f}\rightarrow\texttt{bit1) bit0)}}$

Booleans

$\underline{\texttt{((}\lambda\texttt{f}\rightarrow\texttt{bit1) bit0)}}$

$\texttt{bit1}$

Booleans

```
((FALSE bit1) bit0)
```

Booleans

```
((FALSE bit1) bit0)
```

```
((FALSE bit1) bit0)
```

Booleans

```
((FALSE bit1) bit0)
```

Booleans

```
((FALSE bit1) bit0)
```

```
(((λt→f→f) bit1) bit0)
```

Booleans

```
(((λt→f→f) bit1) bit0)
```

Booleans

$$(((\lambda \mathtt{t} \rightarrow \mathtt{f} \rightarrow \mathtt{f})\ \mathtt{bit1})\ \mathtt{bit0})$$

$$(\underline{((\lambda \mathtt{t} \rightarrow \mathtt{f} \rightarrow \mathtt{f})\ \mathtt{bit1})}\ \mathtt{bit0})$$

Booleans

$$\boxed{\texttt{(}\underline{\texttt{((}\lambda\texttt{t}\rightarrow\texttt{f}\rightarrow\texttt{f)\ bit1)}}\texttt{\ bit0)}}$$

Booleans

$$(\underline{((\lambda t \rightarrow f \rightarrow f)\ \mathtt{bit1})}\ \mathtt{bit0})$$

$$((\lambda f \rightarrow f)\ \mathtt{bit0})$$

Booleans

```
((λf→f) bit0)
```

Booleans

$\texttt{((}\lambda\texttt{f}\rightarrow\texttt{f) bit0)}$

$\underline{\texttt{((}\lambda\texttt{f}\rightarrow\texttt{f) bit0)}}$

Booleans

$$\underline{((\lambda\texttt{f}\rightarrow\texttt{f})\ \ \texttt{bit0})}$$

Booleans

$\underline{\texttt{((}\lambda\texttt{f}\rightarrow\texttt{f)~~bit0)}}$

`bit0`

Booleans

`AND`

- The conjunction[a] of two terms is a function that takes as input two booleans and returns a boolean
- Since we just defined booleans to be two-parameter functions, we know that the two input booleans can be applied to each other
- Given two booleans `a` and `b`, their conjunction is `b` if `a` was `true`, or `false` otherwise
-
```
(λa→b→((a b) a))
```

---

[a] `AND`, or $\wedge$

Booleans

AND

Let us begin to with $\texttt{TRUE} \wedge \texttt{TRUE} \rightarrow_\beta \texttt{TRUE}$

Booleans

```
(TRUE ∧ TRUE)
```

Booleans

$(\texttt{TRUE}\ \wedge\ \texttt{TRUE})$

$((\underline{\wedge}\ \texttt{TRUE})\ \texttt{TRUE})$

Booleans

```
((∧ TRUE) TRUE)
```

Booleans

```
((△ TRUE) TRUE)
```

```
(((λa→b→((a b) a)) TRUE) TRUE)
```

Booleans

```
(((λa→b→((a b) a)) TRUE) TRUE)
```

Booleans

$$\texttt{(((}\lambda\texttt{a}\rightarrow\texttt{b}\rightarrow\texttt{((a b) a)) TRUE) TRUE)}$$

$$\texttt{(((}\lambda\texttt{a}\rightarrow\texttt{b}\rightarrow\texttt{((a b) a)) }\underline{\texttt{TRUE}}\texttt{) TRUE)}$$

Booleans

$$\texttt{(((}\lambda\texttt{a}\rightarrow\texttt{b}\rightarrow\texttt{((a b) a)) }\underline{\texttt{TRUE}}\texttt{) TRUE)}$$

Booleans

$$\texttt{(((}\lambda\texttt{a}\rightarrow\texttt{b}\rightarrow\texttt{((a b) a)) }\underline{\texttt{TRUE}}\texttt{) TRUE)}$$

$$\texttt{(((}\lambda\texttt{a}\rightarrow\texttt{b}\rightarrow\texttt{((a b) a)) (}\lambda\texttt{t}\rightarrow\texttt{f}\rightarrow\texttt{t)) TRUE)}$$

Booleans

```
(((λa→b→((a b) a)) (λt→f→t)) TRUE)
```

Booleans

$$\texttt{(((}\lambda\texttt{a}\rightarrow\texttt{b}\rightarrow\texttt{((a b) a)) (}\lambda\texttt{t}\rightarrow\texttt{f}\rightarrow\texttt{t)) TRUE)}$$

$$\texttt{(((}\lambda\texttt{a}\rightarrow\texttt{b}\rightarrow\texttt{((a b) a)) (}\lambda\texttt{t}\rightarrow\texttt{f}\rightarrow\texttt{t)) }\underline{\texttt{TRUE}}\texttt{)}$$

Booleans

$$\texttt{(((}\lambda\texttt{a}\rightarrow\texttt{b}\rightarrow\texttt{((a b) a)) (}\lambda\texttt{t}\rightarrow\texttt{f}\rightarrow\texttt{t)) }\underline{\texttt{TRUE}}\texttt{)}$$

Booleans

$$\texttt{(((}\lambda\texttt{a}\rightarrow\texttt{b}\rightarrow\texttt{((a b) a)) (}\lambda\texttt{t}\rightarrow\texttt{f}\rightarrow\texttt{t)) }\underline{\texttt{TRUE}}\texttt{)}$$

$$\texttt{(((}\lambda\texttt{a}\rightarrow\texttt{b}\rightarrow\texttt{((a b) a)) (}\lambda\texttt{t}\rightarrow\texttt{f}\rightarrow\texttt{t)) (}\lambda\texttt{t}\rightarrow\texttt{f}\rightarrow\texttt{t))}$$

Booleans

```
(((λa→b→((a b) a)) (λt→f→t)) (λt→f→t))
```

Booleans

$$\boxed{(((\lambda \mathtt{a}\rightarrow \mathtt{b}\rightarrow ((\mathtt{a}\ \mathtt{b})\ \mathtt{a}))\ (\lambda \mathtt{t}\rightarrow \mathtt{f}\rightarrow \mathtt{t}))\ (\lambda \mathtt{t}\rightarrow \mathtt{f}\rightarrow \mathtt{t}))}$$

$$\boxed{(\underline{((\lambda \mathtt{a}\rightarrow \mathtt{b}\rightarrow ((\mathtt{a}\ \mathtt{b})\ \mathtt{a}))\ (\lambda \mathtt{t}\rightarrow \mathtt{f}\rightarrow \mathtt{t}))}\ (\lambda \mathtt{t}\rightarrow \mathtt{f}\rightarrow \mathtt{t}))}$$

Booleans

$$\boxed{(\underline{((\lambda a \rightarrow b \rightarrow ((a\ b)\ a))\ (\lambda t \rightarrow f \rightarrow t))}\ (\lambda t \rightarrow f \rightarrow t))}$$

Booleans

$$(\underline{((\lambda a \to b \to ((a\ b)\ a))\ (\lambda t \to f \to t))}\ (\lambda t \to f \to t))$$

$$((\lambda b \to (((\lambda t \to f \to t)\ b)\ (\lambda t \to f \to t)))\ (\lambda t \to f \to t))$$

Booleans

```
((λb→(((λt→f→t) b) (λt→f→t))) (λt→f→t))
```

Booleans

$$\boxed{\texttt{((}\lambda\texttt{b}\rightarrow\texttt{(((}\lambda\texttt{t}\rightarrow\texttt{f}\rightarrow\texttt{t) b) (}\lambda\texttt{t}\rightarrow\texttt{f}\rightarrow\texttt{t))) (}\lambda\texttt{t}\rightarrow\texttt{f}\rightarrow\texttt{t))}}$$

$$\boxed{\underline{\texttt{((}\lambda\texttt{b}\rightarrow\texttt{(((}\lambda\texttt{t}\rightarrow\texttt{f}\rightarrow\texttt{t) b) (}\lambda\texttt{t}\rightarrow\texttt{f}\rightarrow\texttt{t))) (}\lambda\texttt{t}\rightarrow\texttt{f}\rightarrow\texttt{t))}}}$$

Delta rules

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Conclusion

Booleans

$$\boxed{\underline{\texttt{((}\lambda\texttt{b}\rightarrow\texttt{(((}\lambda\texttt{t}\rightarrow\texttt{f}\rightarrow\texttt{t) b) (}\lambda\texttt{t}\rightarrow\texttt{f}\rightarrow\texttt{t))) (}\lambda\texttt{t}\rightarrow\texttt{f}\rightarrow\texttt{t))}}}$$

Booleans

$$\underline{\texttt{((}\lambda\texttt{b}\rightarrow\texttt{(((}\lambda\texttt{t}\rightarrow\texttt{f}\rightarrow\texttt{t) b) (}\lambda\texttt{t}\rightarrow\texttt{f}\rightarrow\texttt{t))) (}\lambda\texttt{t}\rightarrow\texttt{f}\rightarrow\texttt{t))}}$$

$$\texttt{(((}\lambda\texttt{t}\rightarrow\texttt{f}\rightarrow\texttt{t) (}\lambda\texttt{t}\rightarrow\texttt{f}\rightarrow\texttt{t)) (}\lambda\texttt{t}\rightarrow\texttt{f}\rightarrow\texttt{t))}$$

Booleans

```
(((λt→f→t) (λt→f→t)) (λt→f→t))
```

Booleans

$$(((\lambda t \to f \to t)\ (\lambda t \to f \to t))\ (\lambda t \to f \to t))$$

$$(\underline{((\lambda t \to f \to t)\ (\lambda t \to f \to t))}\ (\lambda t \to f \to t))$$

Booleans

$$(\underline{((\lambda \mathtt{t} \rightarrow \mathtt{f} \rightarrow \mathtt{t})\ (\lambda \mathtt{t} \rightarrow \mathtt{f} \rightarrow \mathtt{t}))}\ (\lambda \mathtt{t} \rightarrow \mathtt{f} \rightarrow \mathtt{t}))$$

Booleans

$$(\underline{((\lambda \texttt{t} \rightarrow \texttt{f} \rightarrow \texttt{t})\ (\lambda \texttt{t} \rightarrow \texttt{f} \rightarrow \texttt{t}))}\ (\lambda \texttt{t} \rightarrow \texttt{f} \rightarrow \texttt{t}))$$

$$((\lambda \texttt{f} \rightarrow \texttt{t} \rightarrow \texttt{f} \rightarrow \texttt{t})\ (\lambda \texttt{t} \rightarrow \texttt{f} \rightarrow \texttt{t}))$$

Booleans

```
((λf→t→f→t) (λt→f→t))
```

Booleans

$$\texttt{((}\lambda\texttt{f}\rightarrow\texttt{t}\rightarrow\texttt{f}\rightarrow\texttt{t)}\ \texttt{(}\lambda\texttt{t}\rightarrow\texttt{f}\rightarrow\texttt{t))}$$

$$\underline{\texttt{((}\lambda\texttt{f}\rightarrow\texttt{t}\rightarrow\texttt{f}\rightarrow\texttt{t)}\ \texttt{(}\lambda\texttt{t}\rightarrow\texttt{f}\rightarrow\texttt{t))}}$$

Booleans

$$\boxed{\underline{\texttt{((}\lambda\texttt{f}\rightarrow\texttt{t}\rightarrow\texttt{f}\rightarrow\texttt{t) (}\lambda\texttt{t}\rightarrow\texttt{f}\rightarrow\texttt{t))}}}$$

Booleans

$$\underline{\texttt{((}\lambda\texttt{f}\rightarrow\texttt{t}\rightarrow\texttt{f}\rightarrow\texttt{t) (}\lambda\texttt{t}\rightarrow\texttt{f}\rightarrow\texttt{t))}}$$

$$\texttt{(}\lambda\texttt{t}\rightarrow\texttt{f}\rightarrow\texttt{t)}$$

Booleans

```
(λt→f→t)
```

Booleans

$(\lambda \texttt{t} \rightarrow \texttt{f} \rightarrow \texttt{t})$

$\underline{(\lambda \texttt{t} \rightarrow \texttt{f} \rightarrow \texttt{t})}$

Booleans

$\underline{(\lambda \texttt{t} \rightarrow \texttt{f} \rightarrow \texttt{t})}$

Booleans

$\underline{(\lambda \texttt{t} \rightarrow \texttt{f} \rightarrow \texttt{t})}$

`TRUE`

Booleans

`AND`

Let us move to $\texttt{TRUE} \wedge \texttt{FALSE} \rightarrow_{\beta} \texttt{FALSE}$

Booleans

```
(TRUE ∧ FALSE)
```

Booleans

$$(\mathtt{TRUE}\ \wedge\ \mathtt{FALSE})$$

$$((\underline{\wedge}\ \mathtt{TRUE})\ \mathtt{FALSE})$$

Booleans

$((\underline{\wedge}\ \texttt{TRUE})\ \texttt{FALSE})$

Booleans

```
((△ TRUE) FALSE)
```

```
(((λa→b→((a b) a)) TRUE) FALSE)
```

Booleans

```
(((λa→b→((a b) a)) TRUE) FALSE)
```

Booleans

$$(((\lambda \texttt{a} \to \texttt{b} \to ((\texttt{a}\ \ \texttt{b})\ \ \texttt{a}))\ \ \texttt{TRUE})\ \ \texttt{FALSE})$$

$$(((\lambda \texttt{a} \to \texttt{b} \to ((\texttt{a}\ \ \texttt{b})\ \ \texttt{a}))\ \ \underline{\texttt{TRUE}})\ \ \texttt{FALSE})$$

Booleans

$$\texttt{(((}\lambda\texttt{a}\rightarrow\texttt{b}\rightarrow\texttt{((a b) a))}\ \underline{\texttt{TRUE}}\texttt{) FALSE)}$$

Booleans

$$\texttt{(((}\lambda\texttt{a}\to\texttt{b}\to\texttt{((a b) a)) }\underline{\texttt{TRUE}}\texttt{) FALSE)}$$

$$\texttt{(((}\lambda\texttt{a}\to\texttt{b}\to\texttt{((a b) a)) (}\lambda\texttt{t}\to\texttt{f}\to\texttt{t)) FALSE)}$$

Booleans

```
(((λa→b→((a b) a)) (λt→f→t)) FALSE)
```

Booleans

$$\boxed{\texttt{(((}\lambda\texttt{a}\rightarrow\texttt{b}\rightarrow\texttt{((a b) a)) (}\lambda\texttt{t}\rightarrow\texttt{f}\rightarrow\texttt{t)) FALSE)}}$$

$$\boxed{\texttt{(((}\lambda\texttt{a}\rightarrow\texttt{b}\rightarrow\texttt{((a b) a)) (}\lambda\texttt{t}\rightarrow\texttt{f}\rightarrow\texttt{t)) }\underline{\texttt{FALSE}}\texttt{)}}$$

Booleans

`(((λa→b→((a b) a)) (λt→f→t)) FALSE)`

Booleans

$$\boxed{\texttt{(((}\lambda\texttt{a}\rightarrow\texttt{b}\rightarrow\texttt{((a b) a)) (}\lambda\texttt{t}\rightarrow\texttt{f}\rightarrow\texttt{t)) }\underline{\texttt{FALSE}}\texttt{)}}$$

$$\boxed{\texttt{(((}\lambda\texttt{a}\rightarrow\texttt{b}\rightarrow\texttt{((a b) a)) (}\lambda\texttt{t}\rightarrow\texttt{f}\rightarrow\texttt{t)) (}\lambda\texttt{t}\rightarrow\texttt{f}\rightarrow\texttt{f))}}$$

Booleans

```
(((λa→b→((a b) a)) (λt→f→t)) (λt→f→f))
```

Booleans

$$(((\lambda a \rightarrow b \rightarrow ((a\ b)\ a))\ (\lambda t \rightarrow f \rightarrow t))\ (\lambda t \rightarrow f \rightarrow f))$$

$$(\underline{((\lambda a \rightarrow b \rightarrow ((a\ b)\ a))\ (\lambda t \rightarrow f \rightarrow t))}\ (\lambda t \rightarrow f \rightarrow f))$$

Booleans

$$\boxed{(\underline{((\lambda a \to b \to ((a\ b)\ a))\ (\lambda t \to f \to t))}\ (\lambda t \to f \to f))}$$

Booleans

$$(\underline{((\lambda a \rightarrow b \rightarrow ((a\ b)\ a))\ (\lambda t \rightarrow f \rightarrow t))}\ (\lambda t \rightarrow f \rightarrow f))$$

$$((\lambda b \rightarrow (((\lambda t \rightarrow f \rightarrow t)\ b)\ (\lambda t \rightarrow f \rightarrow t)))\ (\lambda t \rightarrow f \rightarrow f))$$

Booleans

```
((λb→(((λt→f→t) b) (λt→f→t))) (λt→f→f))
```

Booleans

$$((\lambda b \rightarrow (((\lambda t \rightarrow f \rightarrow t)\ b)\ (\lambda t \rightarrow f \rightarrow t)))\ (\lambda t \rightarrow f \rightarrow f))$$

$$\underline{((\lambda b \rightarrow (((\lambda t \rightarrow f \rightarrow t)\ b)\ (\lambda t \rightarrow f \rightarrow t)))\ (\lambda t \rightarrow f \rightarrow f))}$$

Booleans

$\boxed{\underline{\texttt{((}\lambda\texttt{b}\rightarrow\texttt{(((}\lambda\texttt{t}\rightarrow\texttt{f}\rightarrow\texttt{t) b) (}\lambda\texttt{t}\rightarrow\texttt{f}\rightarrow\texttt{t))) (}\lambda\texttt{t}\rightarrow\texttt{f}\rightarrow\texttt{f))}}}$

Booleans

$$\boxed{\underline{\texttt{((}\lambda\texttt{b}\rightarrow\texttt{(((}\lambda\texttt{t}\rightarrow\texttt{f}\rightarrow\texttt{t) b) (}\lambda\texttt{t}\rightarrow\texttt{f}\rightarrow\texttt{t))) (}\lambda\texttt{t}\rightarrow\texttt{f}\rightarrow\texttt{f))}}}$$

$$\boxed{\texttt{(((}\lambda\texttt{t}\rightarrow\texttt{f}\rightarrow\texttt{t) (}\lambda\texttt{t}\rightarrow\texttt{f}\rightarrow\texttt{f)) (}\lambda\texttt{t}\rightarrow\texttt{f}\rightarrow\texttt{t))}}$$

Booleans

```
(((λt→f→t) (λt→f→f)) (λt→f→t))
```

Booleans

$$\texttt{(((}\lambda\texttt{t}\rightarrow\texttt{f}\rightarrow\texttt{t)}\ \texttt{(}\lambda\texttt{t}\rightarrow\texttt{f}\rightarrow\texttt{f))}\ \texttt{(}\lambda\texttt{t}\rightarrow\texttt{f}\rightarrow\texttt{t))}$$

$$\texttt{(}\underline{\texttt{((}\lambda\texttt{t}\rightarrow\texttt{f}\rightarrow\texttt{t)}\ \texttt{(}\lambda\texttt{t}\rightarrow\texttt{f}\rightarrow\texttt{f))}}\ \texttt{(}\lambda\texttt{t}\rightarrow\texttt{f}\rightarrow\texttt{t))}$$

Booleans

$$\boxed{(\underline{((\lambda t\rightarrow f\rightarrow t)\ (\lambda t\rightarrow f\rightarrow f))}\ (\lambda t\rightarrow f\rightarrow t))}$$

Booleans

$$(\underline{((\lambda t \rightarrow f \rightarrow t)\ (\lambda t \rightarrow f \rightarrow f))}\ (\lambda t \rightarrow f \rightarrow t))$$

$$((\lambda f \rightarrow t \rightarrow f \rightarrow f)\ (\lambda t \rightarrow f \rightarrow t))$$

Booleans

```
((λf→t→f→f) (λt→f→t))
```

Booleans

$\texttt{((}\lambda\texttt{f}\rightarrow\texttt{t}\rightarrow\texttt{f}\rightarrow\texttt{f) (}\lambda\texttt{t}\rightarrow\texttt{f}\rightarrow\texttt{t))}$

$\underline{\texttt{((}\lambda\texttt{f}\rightarrow\texttt{t}\rightarrow\texttt{f}\rightarrow\texttt{f) (}\lambda\texttt{t}\rightarrow\texttt{f}\rightarrow\texttt{t))}}$

Booleans

$$\boxed{\underline{\texttt{((}\lambda\texttt{f}\rightarrow\texttt{t}\rightarrow\texttt{f}\rightarrow\texttt{f) (}\lambda\texttt{t}\rightarrow\texttt{f}\rightarrow\texttt{t))}}}$$

Booleans

$\underline{((\lambda \mathtt{f} \rightarrow \mathtt{t} \rightarrow \mathtt{f} \rightarrow \mathtt{f})\ (\lambda \mathtt{t} \rightarrow \mathtt{f} \rightarrow \mathtt{t}))}$

$(\lambda \mathtt{t} \rightarrow \mathtt{f} \rightarrow \mathtt{f})$

Booleans

$(\lambda t \rightarrow f \rightarrow f)$

Booleans

$(\lambda t \rightarrow f \rightarrow f)$

$\underline{(\lambda t \rightarrow f \rightarrow f)}$

Booleans

$\underline{\texttt{(}\lambda\texttt{t}\rightarrow\texttt{f}\rightarrow\texttt{f)}}$

Booleans

`(λt→f→f)`

`FALSE`

Booleans

AND

Let us move to $\texttt{FALSE} \wedge \texttt{TRUE} \rightarrow_\beta \texttt{FALSE}$

Booleans

```
(FALSE ∧ TRUE)
```

Booleans

$\texttt{(FALSE}\ \wedge\ \texttt{TRUE)}$

$\texttt{((}\underline{\wedge}\ \texttt{FALSE)}\ \texttt{TRUE)}$

Booleans

```
((△̲ FALSE) TRUE)
```

Booleans

(($\underline{\triangle}$ FALSE) TRUE)

(((λa→b→((a b) a)) FALSE) TRUE)

Booleans

```
(((λa→b→((a b) a)) FALSE) TRUE)
```

Booleans

$$\texttt{(((}\lambda\texttt{a}\rightarrow\texttt{b}\rightarrow\texttt{((a b) a)) FALSE) TRUE)}$$

$$\texttt{(((}\lambda\texttt{a}\rightarrow\texttt{b}\rightarrow\texttt{((a b) a)) }\underline{\texttt{FALSE}}\texttt{) TRUE)}$$

Booleans

$$\boxed{\texttt{(((}\lambda\texttt{a}\rightarrow\texttt{b}\rightarrow\texttt{((a b) a)) }\underline{\texttt{FALSE}}\texttt{) TRUE)}}$$

Booleans

$$\texttt{(((}\lambda\texttt{a}\rightarrow\texttt{b}\rightarrow\texttt{((a b) a)) }\underline{\texttt{FALSE}}\texttt{) TRUE)}$$

$$\texttt{(((}\lambda\texttt{a}\rightarrow\texttt{b}\rightarrow\texttt{((a b) a)) (}\lambda\texttt{t}\rightarrow\texttt{f}\rightarrow\texttt{f)) TRUE)}$$

Booleans

```
(((λa→b→((a b) a)) (λt→f→f)) TRUE)
```

Booleans

$(((\lambda \texttt{a}\to \texttt{b}\to((\texttt{a}\ \ \texttt{b})\ \ \texttt{a}))\ \ (\lambda \texttt{t}\to \texttt{f}\to \texttt{f}))\ \ \texttt{TRUE})$

$(((\lambda \texttt{a}\to \texttt{b}\to((\texttt{a}\ \ \texttt{b})\ \ \texttt{a}))\ \ (\lambda \texttt{t}\to \texttt{f}\to \texttt{f}))\ \ \underline{\texttt{TRUE}})$

Booleans

$\boxed{\texttt{(((}\lambda\texttt{a}\rightarrow\texttt{b}\rightarrow\texttt{((a b) a)) (}\lambda\texttt{t}\rightarrow\texttt{f}\rightarrow\texttt{f)) }\underline{\texttt{TRUE}}\texttt{)}}$

Booleans

```
(((λa→b→((a b) a)) (λt→f→f)) TRUE)
```

```
(((λa→b→((a b) a)) (λt→f→f)) (λt→f→t))
```

Booleans

```
(((λa→b→((a b) a)) (λt→f→f)) (λt→f→t))
```

Booleans

$$(((\lambda a \to b \to ((a\ b)\ a))\ (\lambda t \to f \to f))\ (\lambda t \to f \to t))$$

$$(\underline{((\lambda a \to b \to ((a\ b)\ a))\ (\lambda t \to f \to f))}\ (\lambda t \to f \to t))$$

Booleans

$$\boxed{\texttt{(}\underline{\texttt{((}\lambda\texttt{a}\to\texttt{b}\to\texttt{((a b) a)) (}\lambda\texttt{t}\to\texttt{f}\to\texttt{f))}}\texttt{ (}\lambda\texttt{t}\to\texttt{f}\to\texttt{t))}}$$

Booleans

$$\boxed{(\underline{((\lambda a\rightarrow b\rightarrow((a\ b)\ a))\ (\lambda t\rightarrow f\rightarrow f))}\ (\lambda t\rightarrow f\rightarrow t))}$$

$$\boxed{((\lambda b\rightarrow(((\lambda t\rightarrow f\rightarrow f)\ b)\ (\lambda t\rightarrow f\rightarrow f)))\ (\lambda t\rightarrow f\rightarrow t))}$$

Booleans

```
((λb→(((λt→f→f) b) (λt→f→f))) (λt→f→t))
```

Booleans

$((\lambda b \rightarrow (((\lambda t \rightarrow f \rightarrow f)\ b)\ (\lambda t \rightarrow f \rightarrow f)))\ (\lambda t \rightarrow f \rightarrow t))$

$\underline{((\lambda b \rightarrow (((\lambda t \rightarrow f \rightarrow f)\ b)\ (\lambda t \rightarrow f \rightarrow f)))\ (\lambda t \rightarrow f \rightarrow t))}$

Booleans

$\boxed{\underline{\texttt{((}\lambda\texttt{b}\rightarrow\texttt{((((}\lambda\texttt{t}\rightarrow\texttt{f}\rightarrow\texttt{f) b) (}\lambda\texttt{t}\rightarrow\texttt{f}\rightarrow\texttt{f))) (}\lambda\texttt{t}\rightarrow\texttt{f}\rightarrow\texttt{t))}}}$

Booleans

$$\underline{((\lambda b \rightarrow (((\lambda t \rightarrow f \rightarrow f)\ b)\ (\lambda t \rightarrow f \rightarrow f)))\ (\lambda t \rightarrow f \rightarrow t))}$$

$$(((\lambda t \rightarrow f \rightarrow f)\ (\lambda t \rightarrow f \rightarrow t))\ (\lambda t \rightarrow f \rightarrow f))$$

Booleans

```
(((λt→f→f) (λt→f→t)) (λt→f→f))
```

Booleans

$$\texttt{(((}\lambda\texttt{t}\to\texttt{f}\to\texttt{f)}\ \ \texttt{(}\lambda\texttt{t}\to\texttt{f}\to\texttt{t))}\ \ \texttt{(}\lambda\texttt{t}\to\texttt{f}\to\texttt{f))}$$

$$\texttt{(}\underline{\texttt{((}\lambda\texttt{t}\to\texttt{f}\to\texttt{f)}\ \ \texttt{(}\lambda\texttt{t}\to\texttt{f}\to\texttt{t))}}\ \ \texttt{(}\lambda\texttt{t}\to\texttt{f}\to\texttt{f))}$$

Booleans

$$(\underline{((\lambda t \to f \to f)\ (\lambda t \to f \to t))}\ (\lambda t \to f \to f))$$

Booleans

$$(\underline{((\lambda \mathtt{t} \to \mathtt{f} \to \mathtt{f})\ (\lambda \mathtt{t} \to \mathtt{f} \to \mathtt{t}))}\ (\lambda \mathtt{t} \to \mathtt{f} \to \mathtt{f}))$$

$$((\lambda \mathtt{f} \to \mathtt{f})\ (\lambda \mathtt{t} \to \mathtt{f} \to \mathtt{f}))$$

Booleans

```
((λf→f) (λt→f→f))
```

Booleans

$$((\lambda f \rightarrow f)\ \ (\lambda t \rightarrow f \rightarrow f))$$

$$\underline{((\lambda f \rightarrow f)\ \ (\lambda t \rightarrow f \rightarrow f))}$$

Booleans

$$\boxed{\underline{((\lambda f \to f)\ \ (\lambda t \to f \to f))}}$$

Booleans

$\underline{((\lambda \mathtt{f} \rightarrow \mathtt{f})\ \ (\lambda \mathtt{t} \rightarrow \mathtt{f} \rightarrow \mathtt{f}))}$

$(\lambda \mathtt{t} \rightarrow \mathtt{f} \rightarrow \mathtt{f})$

Booleans

```
(λt→f→f)
```

Booleans

$(\lambda \texttt{t} \rightarrow \texttt{f} \rightarrow \texttt{f})$

$\underline{(\lambda \texttt{t} \rightarrow \texttt{f} \rightarrow \texttt{f})}$

Booleans

$\underline{(\lambda \texttt{t} \rightarrow \texttt{f} \rightarrow \texttt{f})}$

Booleans

$\underline{(\lambda \mathtt{t} \rightarrow \mathtt{f} \rightarrow \mathtt{f})}$

`FALSE`

Booleans

AND

Let us move to $\texttt{FALSE} \wedge \texttt{FALSE} \rightarrow_{\beta} \texttt{FALSE}$

Booleans

```
(FALSE ∧ FALSE)
```

Booleans

$(\texttt{FALSE}\ \land\ \texttt{FALSE})$

$((\underline{\land}\ \texttt{FALSE})\ \texttt{FALSE})$

Booleans

```
((∧ FALSE) FALSE)
```

Booleans

$((\underline{\triangle}\ \texttt{FALSE})\ \texttt{FALSE})$

```
(((λa→b→((a b) a)) FALSE) FALSE)
```

Booleans

```
(((λa→b→((a b) a)) FALSE) FALSE)
```

Booleans

$$\texttt{(((}\lambda\texttt{a}\rightarrow\texttt{b}\rightarrow\texttt{((a b) a)) FALSE) FALSE)}$$

$$\texttt{(((}\lambda\texttt{a}\rightarrow\texttt{b}\rightarrow\texttt{((a b) a)) }\underline{\texttt{FALSE}}\texttt{) FALSE)}$$

Booleans

$$\boxed{\texttt{(((}\lambda\texttt{a}\rightarrow\texttt{b}\rightarrow\texttt{((a b) a)) }\underline{\texttt{FALSE}}\texttt{) FALSE)}}$$

Booleans

$$\boxed{\mathtt{(((\lambda a \rightarrow b \rightarrow ((a\ \ b)\ \ a))\ \ \underline{FALSE})\ \ FALSE)}}$$

$$\boxed{\mathtt{(((\lambda a \rightarrow b \rightarrow ((a\ \ b)\ \ a))\ \ (\lambda t \rightarrow f \rightarrow f))\ \ FALSE)}}$$

Booleans

```
(((λa→b→((a b) a)) (λt→f→f)) FALSE)
```

Booleans

$$\boxed{\texttt{(((}\lambda\texttt{a}\rightarrow\texttt{b}\rightarrow\texttt{((a b) a)) (}\lambda\texttt{t}\rightarrow\texttt{f}\rightarrow\texttt{f)) FALSE)}}$$

$$\boxed{\texttt{(((}\lambda\texttt{a}\rightarrow\texttt{b}\rightarrow\texttt{((a b) a)) (}\lambda\texttt{t}\rightarrow\texttt{f}\rightarrow\texttt{f)) }\underline{\texttt{FALSE}}\texttt{)}}$$

Booleans

$\texttt{(((}\lambda\texttt{a}\rightarrow\texttt{b}\rightarrow\texttt{((a b) a)) (}\lambda\texttt{t}\rightarrow\texttt{f}\rightarrow\texttt{f)) }\underline{\texttt{FALSE}}\texttt{)}$

Booleans

$$\boxed{\texttt{(((}\lambda\texttt{a}\rightarrow\texttt{b}\rightarrow\texttt{((a b) a)) (}\lambda\texttt{t}\rightarrow\texttt{f}\rightarrow\texttt{f)) }\underline{\texttt{FALSE}}\texttt{)}}$$

$$\boxed{\texttt{(((}\lambda\texttt{a}\rightarrow\texttt{b}\rightarrow\texttt{((a b) a)) (}\lambda\texttt{t}\rightarrow\texttt{f}\rightarrow\texttt{f)) (}\lambda\texttt{t}\rightarrow\texttt{f}\rightarrow\texttt{f))}}$$

Booleans

```
(((λa→b→((a b) a)) (λt→f→f)) (λt→f→f))
```

Booleans

$$(((\lambda \mathtt{a} \rightarrow \mathtt{b} \rightarrow ((\mathtt{a}\ \ \mathtt{b})\ \ \mathtt{a}))\ \ (\lambda \mathtt{t} \rightarrow \mathtt{f} \rightarrow \mathtt{f}))\ \ (\lambda \mathtt{t} \rightarrow \mathtt{f} \rightarrow \mathtt{f}))$$

$$(\underline{((\lambda \mathtt{a} \rightarrow \mathtt{b} \rightarrow ((\mathtt{a}\ \ \mathtt{b})\ \ \mathtt{a}))\ \ (\lambda \mathtt{t} \rightarrow \mathtt{f} \rightarrow \mathtt{f}))}\ \ (\lambda \mathtt{t} \rightarrow \mathtt{f} \rightarrow \mathtt{f}))$$

Booleans

$$(\underline{((\lambda \mathtt{a} \to \mathtt{b} \to ((\mathtt{a}\ \mathtt{b})\ \mathtt{a}))\ (\lambda \mathtt{t} \to \mathtt{f} \to \mathtt{f}))}\ (\lambda \mathtt{t} \to \mathtt{f} \to \mathtt{f}))$$

Booleans

$$\boxed{(\underline{((\lambda a \rightarrow b \rightarrow ((a\ b)\ a))\ (\lambda t \rightarrow f \rightarrow f))}\ (\lambda t \rightarrow f \rightarrow f))}$$

$$\boxed{((\lambda b \rightarrow (((\lambda t \rightarrow f \rightarrow f)\ b)\ (\lambda t \rightarrow f \rightarrow f)))\ (\lambda t \rightarrow f \rightarrow f))}$$

Booleans

$$\boxed{\texttt{((}\lambda\texttt{b}\rightarrow\texttt{(((}\lambda\texttt{t}\rightarrow\texttt{f}\rightarrow\texttt{f) b) (}\lambda\texttt{t}\rightarrow\texttt{f}\rightarrow\texttt{f))) (}\lambda\texttt{t}\rightarrow\texttt{f}\rightarrow\texttt{f))}}$$

Booleans

$$\texttt{((}\lambda\texttt{b}\rightarrow\texttt{(((}\lambda\texttt{t}\rightarrow\texttt{f}\rightarrow\texttt{f) b) (}\lambda\texttt{t}\rightarrow\texttt{f}\rightarrow\texttt{f))) (}\lambda\texttt{t}\rightarrow\texttt{f}\rightarrow\texttt{f))}$$

$$\underline{\texttt{((}\lambda\texttt{b}\rightarrow\texttt{(((}\lambda\texttt{t}\rightarrow\texttt{f}\rightarrow\texttt{f) b) (}\lambda\texttt{t}\rightarrow\texttt{f}\rightarrow\texttt{f))) (}\lambda\texttt{t}\rightarrow\texttt{f}\rightarrow\texttt{f))}}$$

Booleans

$$\underline{((\lambda b\rightarrow(((\lambda t\rightarrow f\rightarrow f)\ b)\ (\lambda t\rightarrow f\rightarrow f)))\ (\lambda t\rightarrow f\rightarrow f))}$$

Booleans

```
((λb→(((λt→f→f) b) (λt→f→f))) (λt→f→f))
```

```
(((λt→f→f) (λt→f→f)) (λt→f→f))
```

Booleans

```
(((λt→f→f) (λt→f→f)) (λt→f→f))
```

Booleans

$$(((\lambda \mathtt{t} \rightarrow \mathtt{f} \rightarrow \mathtt{f})\ (\lambda \mathtt{t} \rightarrow \mathtt{f} \rightarrow \mathtt{f}))\ (\lambda \mathtt{t} \rightarrow \mathtt{f} \rightarrow \mathtt{f}))$$

$$(\underline{((\lambda \mathtt{t} \rightarrow \mathtt{f} \rightarrow \mathtt{f})\ (\lambda \mathtt{t} \rightarrow \mathtt{f} \rightarrow \mathtt{f}))}\ (\lambda \mathtt{t} \rightarrow \mathtt{f} \rightarrow \mathtt{f}))$$

Booleans

$$\boxed{(\underline{((\lambda t \rightarrow f \rightarrow f)\ (\lambda t \rightarrow f \rightarrow f))}\ (\lambda t \rightarrow f \rightarrow f))}$$

Booleans

$$(\underline{((\lambda \mathtt{t} \to \mathtt{f} \to \mathtt{f})\ (\lambda \mathtt{t} \to \mathtt{f} \to \mathtt{f}))}\ (\lambda \mathtt{t} \to \mathtt{f} \to \mathtt{f}))$$

$$((\lambda \mathtt{f} \to \mathtt{f})\ (\lambda \mathtt{t} \to \mathtt{f} \to \mathtt{f}))$$

Booleans

```
((λf→f) (λt→f→f))
```

Booleans

$((\lambda \texttt{f} \rightarrow \texttt{f})\ \ (\lambda \texttt{t} \rightarrow \texttt{f} \rightarrow \texttt{f}))$

$\underline{((\lambda \texttt{f} \rightarrow \texttt{f})\ \ (\lambda \texttt{t} \rightarrow \texttt{f} \rightarrow \texttt{f}))}$

Booleans

$\underline{\texttt{((}\lambda\texttt{f}\rightarrow\texttt{f)\ (}\lambda\texttt{t}\rightarrow\texttt{f}\rightarrow\texttt{f))}}$

Booleans

$$\underline{((\lambda f \rightarrow f)\ \ (\lambda t \rightarrow f \rightarrow f))}$$

$$(\lambda t \rightarrow f \rightarrow f)$$

Booleans

$$\boxed{(\lambda \mathtt{t} \rightarrow \mathtt{f} \rightarrow \mathtt{f})}$$

Booleans

$(\lambda \mathtt{t} \rightarrow \mathtt{f} \rightarrow \mathtt{f})$

$\underline{(\lambda \mathtt{t} \rightarrow \mathtt{f} \rightarrow \mathtt{f})}$

Booleans

$\underline{(\lambda \texttt{t} \rightarrow \texttt{f} \rightarrow \texttt{f})}$

Booleans

```
(λt→f→f)
```

```
FALSE
```

Booleans

It works, but it is probably only because of black magic.

Booleans

It works, but it is probably only because of black magic.

Or is it? Let's see if we can get lucky again...

Booleans

`OR`

- The disjunction[a] of two terms is a function that takes as input two booleans and returns a boolean
- Like with conjunction, remember that the two input booleans can be applied to one another
- Given two booleans `a` and `b`, their disjunction is `true` if `a` was `true`, or `b` otherwise
-
```
(λa→b→((a a) b))
```

---

[a] `OR`, or $\vee$

Booleans

OR

Let us begin to with $\texttt{TRUE} \vee \texttt{TRUE} \rightarrow_{\beta} \texttt{TRUE}$

Booleans

```
(TRUE ∨ TRUE)
```

Booleans

```
(TRUE ∨ TRUE)
```

```
((∨ TRUE) TRUE)
```

Booleans

```
((∨ TRUE) TRUE)
```

Booleans

$$((\underline{\vee}\ \texttt{TRUE})\ \texttt{TRUE})$$

$$(((\lambda \texttt{a} \rightarrow \texttt{b} \rightarrow ((\texttt{a}\ \texttt{a})\ \texttt{b}))\ \texttt{TRUE})\ \texttt{TRUE})$$

Booleans

```
(((λa→b→((a a) b)) TRUE) TRUE)
```

Booleans

$$\texttt{(((}\lambda\texttt{a}\rightarrow\texttt{b}\rightarrow\texttt{((a a) b)) TRUE) TRUE)}$$

$$\texttt{(((}\lambda\texttt{a}\rightarrow\texttt{b}\rightarrow\texttt{((a a) b)) }\underline{\texttt{TRUE}}\texttt{) TRUE)}$$

Booleans

$$\boxed{\texttt{(((}\lambda\texttt{a}\rightarrow\texttt{b}\rightarrow\texttt{((a a) b)) }\underline{\texttt{TRUE}}\texttt{) TRUE)}}$$

Booleans

$$\texttt{(((}\lambda\texttt{a}\rightarrow\texttt{b}\rightarrow\texttt{((a a) b))}\ \underline{\texttt{TRUE}}\texttt{) TRUE)}$$

$$\texttt{(((}\lambda\texttt{a}\rightarrow\texttt{b}\rightarrow\texttt{((a a) b))}\ \texttt{(}\lambda\texttt{t}\rightarrow\texttt{f}\rightarrow\texttt{t)) TRUE)}$$

Booleans

```
(((λa→b→((a a) b)) (λt→f→t)) TRUE)
```

Booleans

`(((λa→b→((a a) b)) (λt→f→t)) TRUE)`

`(((λa→b→((a a) b)) (λt→f→t))` $\underline{\texttt{TRUE}}$`)`

Booleans

$$\texttt{(((}\lambda\texttt{a}\rightarrow\texttt{b}\rightarrow\texttt{((a a) b)) (}\lambda\texttt{t}\rightarrow\texttt{f}\rightarrow\texttt{t)) }\underline{\texttt{TRUE}}\texttt{)}$$

Booleans

```
(((λa→b→((a a) b)) (λt→f→t)) TRUE)
```

```
(((λa→b→((a a) b)) (λt→f→t)) (λt→f→t))
```

Booleans

```
(((λa→b→((a a) b)) (λt→f→t)) (λt→f→t))
```

Booleans

$$(((\lambda a \to b \to ((a\ a)\ b))\ (\lambda t \to f \to t))\ (\lambda t \to f \to t))$$

$$(\underline{((\lambda a \to b \to ((a\ a)\ b))\ (\lambda t \to f \to t))}\ (\lambda t \to f \to t))$$

Booleans

$$\boxed{(\underline{((\lambda a\rightarrow b\rightarrow((a\ a)\ b))\ (\lambda t\rightarrow f\rightarrow t))}\ (\lambda t\rightarrow f\rightarrow t))}$$

Booleans

$$(\underline{((\lambda a \rightarrow b \rightarrow ((a\ a)\ b))\ (\lambda t \rightarrow f \rightarrow t))}\ (\lambda t \rightarrow f \rightarrow t))$$

$$((\lambda b \rightarrow (((\lambda t \rightarrow f \rightarrow t)\ (\lambda t \rightarrow f \rightarrow t))\ b))\ (\lambda t \rightarrow f \rightarrow t))$$

Booleans

```
((λb→(((λt→f→t) (λt→f→t)) b)) (λt→f→t))
```

Booleans

$$((\lambda b \to (((\lambda t \to f \to t)\ (\lambda t \to f \to t))\ b))\ (\lambda t \to f \to t))$$

$$\underline{((\lambda b \to (((\lambda t \to f \to t)\ (\lambda t \to f \to t))\ b))\ (\lambda t \to f \to t))}$$

Booleans

$$\boxed{\underline{\texttt{((}\lambda\texttt{b}\rightarrow\texttt{(((}\lambda\texttt{t}\rightarrow\texttt{f}\rightarrow\texttt{t) (}\lambda\texttt{t}\rightarrow\texttt{f}\rightarrow\texttt{t)) b)) (}\lambda\texttt{t}\rightarrow\texttt{f}\rightarrow\texttt{t))}}}$$

Booleans

$$\underline{((\lambda b \rightarrow (((\lambda t \rightarrow f \rightarrow t)\ (\lambda t \rightarrow f \rightarrow t))\ b))\ (\lambda t \rightarrow f \rightarrow t))}$$

$$(((\lambda t \rightarrow f \rightarrow t)\ (\lambda t \rightarrow f \rightarrow t))\ (\lambda t \rightarrow f \rightarrow t))$$

Booleans

```
(((λt→f→t) (λt→f→t)) (λt→f→t))
```

Booleans

$$(((\lambda t \to f \to t)\ (\lambda t \to f \to t))\ (\lambda t \to f \to t))$$

$$(\underline{((\lambda t \to f \to t)\ (\lambda t \to f \to t))}\ (\lambda t \to f \to t))$$

Booleans

$$\boxed{(\underline{((\lambda t \rightarrow f \rightarrow t)\ (\lambda t \rightarrow f \rightarrow t))}\ (\lambda t \rightarrow f \rightarrow t))}$$

Booleans

$$(\underline{((\lambda t \rightarrow f \rightarrow t)\ \ (\lambda t \rightarrow f \rightarrow t))}\ \ (\lambda t \rightarrow f \rightarrow t))$$

$$((\lambda f \rightarrow t \rightarrow f \rightarrow t)\ \ (\lambda t \rightarrow f \rightarrow t))$$

Booleans

```
((λf→t→f→t) (λt→f→t))
```

Booleans

$$((\lambda \mathtt{f} \rightarrow \mathtt{t} \rightarrow \mathtt{f} \rightarrow \mathtt{t})\ \ (\lambda \mathtt{t} \rightarrow \mathtt{f} \rightarrow \mathtt{t}))$$

$$\underline{((\lambda \mathtt{f} \rightarrow \mathtt{t} \rightarrow \mathtt{f} \rightarrow \mathtt{t})\ \ (\lambda \mathtt{t} \rightarrow \mathtt{f} \rightarrow \mathtt{t}))}$$

Booleans

$\underline{\texttt{((}\lambda\texttt{f}\rightarrow\texttt{t}\rightarrow\texttt{f}\rightarrow\texttt{t) (}\lambda\texttt{t}\rightarrow\texttt{f}\rightarrow\texttt{t))}}$

Booleans

$\underline{\texttt{((}\lambda\texttt{f}\rightarrow\texttt{t}\rightarrow\texttt{f}\rightarrow\texttt{t)\ (}\lambda\texttt{t}\rightarrow\texttt{f}\rightarrow\texttt{t))}}$

$\texttt{(}\lambda\texttt{t}\rightarrow\texttt{f}\rightarrow\texttt{t)}$

Booleans

```
(λt→f→t)
```

Booleans

$(\lambda \texttt{t} \rightarrow \texttt{f} \rightarrow \texttt{t})$

$\underline{(\lambda \texttt{t} \rightarrow \texttt{f} \rightarrow \texttt{t})}$

Booleans

$$\underline{(\lambda \mathtt{t} \rightarrow \mathtt{f} \rightarrow \mathtt{t})}$$

Booleans

$\underline{(\lambda t \rightarrow f \rightarrow t)}$

`TRUE`

Booleans

OR

Let us begin to with `TRUE` $\vee$ `FALSE` $\rightarrow_\beta$ `TRUE`

Booleans

```
(TRUE ∨ FALSE)
```

Booleans

$(\texttt{TRUE}\ \lor\ \texttt{FALSE})$

$((\underline{\lor}\ \texttt{TRUE})\ \texttt{FALSE})$

Booleans

```
((∨ TRUE) FALSE)
```

Booleans

$$((\underline{\vee}\ \mathtt{TRUE})\ \mathtt{FALSE})$$

$$(((\lambda \mathtt{a} \rightarrow \mathtt{b} \rightarrow ((\mathtt{a}\ \mathtt{a})\ \mathtt{b}))\ \mathtt{TRUE})\ \mathtt{FALSE})$$

Booleans

```
(((λa→b→((a a) b)) TRUE) FALSE)
```

Booleans

$(((\lambda a \rightarrow b \rightarrow ((a\ \ a)\ \ b))\ \ TRUE)\ \ FALSE)$

$(((\lambda a \rightarrow b \rightarrow ((a\ \ a)\ \ b))\ \ \underline{TRUE})\ \ FALSE)$

Booleans

$$\boxed{\texttt{(((}\lambda\texttt{a}\rightarrow\texttt{b}\rightarrow\texttt{((a a) b))}\ \underline{\texttt{TRUE}}\texttt{) FALSE)}}$$

Booleans

```
(((λa→b→((a a) b)) TRUE) FALSE)
```

```
(((λa→b→((a a) b)) (λt→f→t)) FALSE)
```

Booleans

```
(((λa→b→((a a) b)) (λt→f→t)) FALSE)
```

Booleans

$$\texttt{(((}\lambda\texttt{a}\rightarrow\texttt{b}\rightarrow\texttt{((a a) b)) (}\lambda\texttt{t}\rightarrow\texttt{f}\rightarrow\texttt{t)) FALSE)}$$

$$\texttt{(((}\lambda\texttt{a}\rightarrow\texttt{b}\rightarrow\texttt{((a a) b)) (}\lambda\texttt{t}\rightarrow\texttt{f}\rightarrow\texttt{t)) }\underline{\texttt{FALSE}}\texttt{)}$$

Booleans

```
(((λa→b→((a  a)  b))  (λt→f→t))  FALSE)
```

Booleans

$(((\lambda \texttt{a} \rightarrow \texttt{b} \rightarrow ((\texttt{a}\ \texttt{a})\ \texttt{b}))\ (\lambda \texttt{t} \rightarrow \texttt{f} \rightarrow \texttt{t}))\ \underline{\texttt{FALSE}})$

$(((\lambda \texttt{a} \rightarrow \texttt{b} \rightarrow ((\texttt{a}\ \texttt{a})\ \texttt{b}))\ (\lambda \texttt{t} \rightarrow \texttt{f} \rightarrow \texttt{t}))\ (\lambda \texttt{t} \rightarrow \texttt{f} \rightarrow \texttt{f}))$

Booleans

```
(((λa→b→((a a) b)) (λt→f→t)) (λt→f→f))
```

Booleans

$$(((\lambda \mathtt{a} \to \mathtt{b} \to ((\mathtt{a}\ \mathtt{a})\ \mathtt{b}))\ (\lambda \mathtt{t} \to \mathtt{f} \to \mathtt{t}))\ (\lambda \mathtt{t} \to \mathtt{f} \to \mathtt{f}))$$

$$(\underline{((\lambda \mathtt{a} \to \mathtt{b} \to ((\mathtt{a}\ \mathtt{a})\ \mathtt{b}))\ (\lambda \mathtt{t} \to \mathtt{f} \to \mathtt{t}))}\ (\lambda \mathtt{t} \to \mathtt{f} \to \mathtt{f}))$$

Booleans

$$\texttt{(}\underline{\texttt{((}\lambda\texttt{a}\rightarrow\texttt{b}\rightarrow\texttt{((a a) b)) (}\lambda\texttt{t}\rightarrow\texttt{f}\rightarrow\texttt{t))}}\texttt{ (}\lambda\texttt{t}\rightarrow\texttt{f}\rightarrow\texttt{f))}$$

Booleans

$$(\underline{((\lambda \mathtt{a} \to \mathtt{b} \to ((\mathtt{a}\ \mathtt{a})\ \mathtt{b}))\ (\lambda \mathtt{t} \to \mathtt{f} \to \mathtt{t}))}\ (\lambda \mathtt{t} \to \mathtt{f} \to \mathtt{f}))$$

$$((\lambda \mathtt{b} \to (((\lambda \mathtt{t} \to \mathtt{f} \to \mathtt{t})\ (\lambda \mathtt{t} \to \mathtt{f} \to \mathtt{t}))\ \mathtt{b}))\ (\lambda \mathtt{t} \to \mathtt{f} \to \mathtt{f}))$$

Booleans

```
((λb→(((λt→f→t) (λt→f→t)) b)) (λt→f→f))
```

Booleans

$$((\lambda \mathtt{b} \rightarrow (((\lambda \mathtt{t} \rightarrow \mathtt{f} \rightarrow \mathtt{t})\ (\lambda \mathtt{t} \rightarrow \mathtt{f} \rightarrow \mathtt{t}))\ \mathtt{b}))\ (\lambda \mathtt{t} \rightarrow \mathtt{f} \rightarrow \mathtt{f}))$$

$$\underline{((\lambda \mathtt{b} \rightarrow (((\lambda \mathtt{t} \rightarrow \mathtt{f} \rightarrow \mathtt{t})\ (\lambda \mathtt{t} \rightarrow \mathtt{f} \rightarrow \mathtt{t}))\ \mathtt{b}))\ (\lambda \mathtt{t} \rightarrow \mathtt{f} \rightarrow \mathtt{f}))}$$

Booleans

$$\boxed{\underline{\texttt{((}\lambda\texttt{b}\rightarrow\texttt{(((}\lambda\texttt{t}\rightarrow\texttt{f}\rightarrow\texttt{t)\ (}\lambda\texttt{t}\rightarrow\texttt{f}\rightarrow\texttt{t))\ b))\ (}\lambda\texttt{t}\rightarrow\texttt{f}\rightarrow\texttt{f))}}}$$

Booleans

$$\underline{((\lambda b \rightarrow (((\lambda t \rightarrow f \rightarrow t)\ (\lambda t \rightarrow f \rightarrow t))\ b))\ (\lambda t \rightarrow f \rightarrow f))}$$

$$(((\lambda t \rightarrow f \rightarrow t)\ (\lambda t \rightarrow f \rightarrow t))\ (\lambda t \rightarrow f \rightarrow f))$$

Booleans

```
(((λt→f→t) (λt→f→t)) (λt→f→f))
```

Booleans

$$(((\lambda \mathtt{t} \rightarrow \mathtt{f} \rightarrow \mathtt{t})\ (\lambda \mathtt{t} \rightarrow \mathtt{f} \rightarrow \mathtt{t}))\ (\lambda \mathtt{t} \rightarrow \mathtt{f} \rightarrow \mathtt{f}))$$

$$(\underline{((\lambda \mathtt{t} \rightarrow \mathtt{f} \rightarrow \mathtt{t})\ (\lambda \mathtt{t} \rightarrow \mathtt{f} \rightarrow \mathtt{t}))}\ (\lambda \mathtt{t} \rightarrow \mathtt{f} \rightarrow \mathtt{f}))$$

Booleans

$$(\underline{((\lambda t \rightarrow f \rightarrow t)\ (\lambda t \rightarrow f \rightarrow t))}\ (\lambda t \rightarrow f \rightarrow f))$$

Booleans

$$(\underline{((\lambda t \rightarrow f \rightarrow t)\ (\lambda t \rightarrow f \rightarrow t))}\ (\lambda t \rightarrow f \rightarrow f))$$

$$((\lambda f \rightarrow t \rightarrow f \rightarrow t)\ (\lambda t \rightarrow f \rightarrow f))$$

Booleans

```
((λf→t→f→t) (λt→f→f))
```

Booleans

$((\lambda f \to t \to f \to t)\ \ (\lambda t \to f \to f))$

$\underline{((\lambda f \to t \to f \to t)\ \ (\lambda t \to f \to f))}$

Booleans

$\underline{\texttt{((}\lambda\texttt{f}\rightarrow\texttt{t}\rightarrow\texttt{f}\rightarrow\texttt{t) (}\lambda\texttt{t}\rightarrow\texttt{f}\rightarrow\texttt{f))}}$

Booleans

$\underline{((\lambda \mathtt{f} \rightarrow \mathtt{t} \rightarrow \mathtt{f} \rightarrow \mathtt{t})\ (\lambda \mathtt{t} \rightarrow \mathtt{f} \rightarrow \mathtt{f}))}$

$(\lambda \mathtt{t} \rightarrow \mathtt{f} \rightarrow \mathtt{t})$

Booleans

```
(λt→f→t)
```

Booleans

$(\lambda \texttt{t} \to \texttt{f} \to \texttt{t})$

$\underline{(\lambda \texttt{t} \to \texttt{f} \to \texttt{t})}$

Booleans

$$\underline{(\lambda\texttt{t}\rightarrow\texttt{f}\rightarrow\texttt{t})}$$

Booleans

$\underline{\texttt{(}\lambda\texttt{t}\rightarrow\texttt{f}\rightarrow\texttt{t)}}$

```
TRUE
```

Booleans

OR

Let us begin to with `False` $\vee$ `TRUE` $\rightarrow_\beta$ `TRUE`

Booleans

```
(FALSE ∨ TRUE)
```

Booleans

```
(FALSE ∨ TRUE)
```

```
((∨ FALSE) TRUE)
```

Booleans

```
((∨ FALSE) TRUE)
```

Booleans

$$\texttt{((}\underline{\vee}\ \texttt{FALSE) TRUE)}$$

$$\texttt{(((}\lambda\texttt{a}\rightarrow\texttt{b}\rightarrow\texttt{((a a) b)) FALSE) TRUE)}$$

Booleans

```
(((λa→b→((a a) b)) FALSE) TRUE)
```

Booleans

```
(((λa→b→((a a) b)) FALSE) TRUE)
```

```
(((λa→b→((a a) b)) FALSE) TRUE)
```

Booleans

$$(((\lambda \texttt{a} \rightarrow \texttt{b} \rightarrow ((\texttt{a}\ \ \texttt{a})\ \ \texttt{b}))\ \ \underline{\texttt{FALSE}})\ \ \texttt{TRUE})$$

Booleans

```
(((λa→b→((a a) b)) FALSE) TRUE)
```

```
(((λa→b→((a a) b)) (λt→f→f)) TRUE)
```

Booleans

```
(((λa→b→((a a) b)) (λt→f→f)) TRUE)
```

Booleans

$$\texttt{(((}\lambda\texttt{a}\rightarrow\texttt{b}\rightarrow\texttt{((a a) b)) (}\lambda\texttt{t}\rightarrow\texttt{f}\rightarrow\texttt{f)) TRUE)}$$

$$\texttt{(((}\lambda\texttt{a}\rightarrow\texttt{b}\rightarrow\texttt{((a a) b)) (}\lambda\texttt{t}\rightarrow\texttt{f}\rightarrow\texttt{f)) }\underline{\texttt{TRUE}}\texttt{)}$$

Booleans

$$\texttt{(((}\lambda\texttt{a}\rightarrow\texttt{b}\rightarrow\texttt{((a a) b)) (}\lambda\texttt{t}\rightarrow\texttt{f}\rightarrow\texttt{f)) }\underline{\texttt{TRUE}}\texttt{)}$$

Booleans

$(((\lambda \texttt{a} \rightarrow \texttt{b} \rightarrow ((\texttt{a}\ \texttt{a})\ \texttt{b}))\ (\lambda \texttt{t} \rightarrow \texttt{f} \rightarrow \texttt{f}))\ \underline{\texttt{TRUE}})$

$(((\lambda \texttt{a} \rightarrow \texttt{b} \rightarrow ((\texttt{a}\ \texttt{a})\ \texttt{b}))\ (\lambda \texttt{t} \rightarrow \texttt{f} \rightarrow \texttt{f}))\ (\lambda \texttt{t} \rightarrow \texttt{f} \rightarrow \texttt{t}))$

Booleans

```
(((λa→b→((a a) b)) (λt→f→f)) (λt→f→t))
```

Booleans

$$(((\lambda a \rightarrow b \rightarrow ((a\ a)\ b))\ (\lambda t \rightarrow f \rightarrow f))\ (\lambda t \rightarrow f \rightarrow t))$$

$$(\underline{((\lambda a \rightarrow b \rightarrow ((a\ a)\ b))\ (\lambda t \rightarrow f \rightarrow f))}\ (\lambda t \rightarrow f \rightarrow t))$$

Booleans

$$\boxed{(\underline{((\lambda a \rightarrow b \rightarrow ((a\ a)\ b))\ (\lambda t \rightarrow f \rightarrow f))}\ (\lambda t \rightarrow f \rightarrow t))}$$

Booleans

$$(\underline{((\lambda a \rightarrow b \rightarrow ((a\ a)\ b))\ (\lambda t \rightarrow f \rightarrow f))}\ (\lambda t \rightarrow f \rightarrow t))$$

$$((\lambda b \rightarrow (((\lambda t \rightarrow f \rightarrow f)\ (\lambda t \rightarrow f \rightarrow f))\ b))\ (\lambda t \rightarrow f \rightarrow t))$$

Booleans

$$\boxed{\texttt{((}\lambda\texttt{b}\rightarrow\texttt{(((}\lambda\texttt{t}\rightarrow\texttt{f}\rightarrow\texttt{f) (}\lambda\texttt{t}\rightarrow\texttt{f}\rightarrow\texttt{f)) b)) (}\lambda\texttt{t}\rightarrow\texttt{f}\rightarrow\texttt{t))}}$$

Booleans

$$\texttt{((}\lambda\texttt{b}\rightarrow\texttt{((}(\lambda\texttt{t}\rightarrow\texttt{f}\rightarrow\texttt{f)\ (}\lambda\texttt{t}\rightarrow\texttt{f}\rightarrow\texttt{f))\ b))\ (}\lambda\texttt{t}\rightarrow\texttt{f}\rightarrow\texttt{t))}$$

$$\underline{\texttt{((}\lambda\texttt{b}\rightarrow\texttt{((}(\lambda\texttt{t}\rightarrow\texttt{f}\rightarrow\texttt{f)\ (}\lambda\texttt{t}\rightarrow\texttt{f}\rightarrow\texttt{f))\ b))\ (}\lambda\texttt{t}\rightarrow\texttt{f}\rightarrow\texttt{t))}}$$

Booleans

$$\boxed{\underline{\texttt{((}\lambda\texttt{b}\rightarrow\texttt{(((}\lambda\texttt{t}\rightarrow\texttt{f}\rightarrow\texttt{f) (}\lambda\texttt{t}\rightarrow\texttt{f}\rightarrow\texttt{f)) b)) (}\lambda\texttt{t}\rightarrow\texttt{f}\rightarrow\texttt{t))}}}$$

Booleans

$$\underline{((\lambda \mathtt{b} \rightarrow (((\lambda \mathtt{t} \rightarrow \mathtt{f} \rightarrow \mathtt{f})\ (\lambda \mathtt{t} \rightarrow \mathtt{f} \rightarrow \mathtt{f}))\ \mathtt{b}))\ (\lambda \mathtt{t} \rightarrow \mathtt{f} \rightarrow \mathtt{t}))}$$

$$(((\lambda \mathtt{t} \rightarrow \mathtt{f} \rightarrow \mathtt{f})\ (\lambda \mathtt{t} \rightarrow \mathtt{f} \rightarrow \mathtt{f}))\ (\lambda \mathtt{t} \rightarrow \mathtt{f} \rightarrow \mathtt{t}))$$

Booleans

```
(((λt→f→f) (λt→f→f)) (λt→f→t))
```

Booleans

$$(((\lambda t \rightarrow f \rightarrow f)\ (\lambda t \rightarrow f \rightarrow f))\ (\lambda t \rightarrow f \rightarrow t))$$

$$(\underline{((\lambda t \rightarrow f \rightarrow f)\ (\lambda t \rightarrow f \rightarrow f))}\ (\lambda t \rightarrow f \rightarrow t))$$

Booleans

$$\boxed{(\underline{((\lambda t \rightarrow f \rightarrow f)\ (\lambda t \rightarrow f \rightarrow f))}\ (\lambda t \rightarrow f \rightarrow t))}$$

Booleans

$$(\underline{((\lambda \texttt{t}\rightarrow\texttt{f}\rightarrow\texttt{f})\ (\lambda \texttt{t}\rightarrow\texttt{f}\rightarrow\texttt{f}))}\ (\lambda \texttt{t}\rightarrow\texttt{f}\rightarrow\texttt{t}))$$

$$((\lambda \texttt{f}\rightarrow\texttt{f})\ (\lambda \texttt{t}\rightarrow\texttt{f}\rightarrow\texttt{t}))$$

Booleans

```
((λf→f) (λt→f→t))
```

Booleans

```
((λf→f) (λt→f→t))
```

$\underline{\texttt{((}\lambda\texttt{f}\rightarrow\texttt{f) (}\lambda\texttt{t}\rightarrow\texttt{f}\rightarrow\texttt{t))}}$

Booleans

$$\boxed{\underline{\texttt{((}\lambda\texttt{f}\rightarrow\texttt{f) (}\lambda\texttt{t}\rightarrow\texttt{f}\rightarrow\texttt{t))}}}$$

Booleans

$\underline{\texttt{((}\lambda\texttt{f}\rightarrow\texttt{f) (}\lambda\texttt{t}\rightarrow\texttt{f}\rightarrow\texttt{t))}}$

$\texttt{(}\lambda\texttt{t}\rightarrow\texttt{f}\rightarrow\texttt{t)}$

Booleans

```
(λt→f→t)
```

Booleans

$(\lambda \texttt{t} \rightarrow \texttt{f} \rightarrow \texttt{t})$

$\underline{(\lambda \texttt{t} \rightarrow \texttt{f} \rightarrow \texttt{t})}$

Booleans

$\boxed{\underline{(\lambda t \to f \to t)}}$

Booleans

$\underline{\texttt{(}\lambda\texttt{t}\rightarrow\texttt{f}\rightarrow\texttt{t)}}$

`TRUE`

Booleans

OR

Let us begin to with `FALSE` $\vee$ `FALSE` $\rightarrow_\beta$ `FALSE`

Booleans

```
(FALSE ∨ FALSE)
```

Booleans

$(\texttt{FALSE}\ \lor\ \texttt{FALSE})$

$((\underline{\lor}\ \texttt{FALSE})\ \texttt{FALSE})$

Booleans

```
((∨ FALSE) FALSE)
```

Booleans

$$((\underline{\vee}\ \texttt{FALSE})\ \texttt{FALSE})$$

$$(((\lambda \texttt{a}\rightarrow \texttt{b}\rightarrow ((\texttt{a}\ \texttt{a})\ \texttt{b}))\ \texttt{FALSE})\ \texttt{FALSE})$$

Booleans

```
(((λa→b→((a a) b)) FALSE) FALSE)
```

Booleans

$\texttt{(((}\lambda\texttt{a}\rightarrow\texttt{b}\rightarrow\texttt{((a a) b)) FALSE) FALSE)}$

$\texttt{(((}\lambda\texttt{a}\rightarrow\texttt{b}\rightarrow\texttt{((a a) b))}\ \underline{\texttt{FALSE}}\texttt{) FALSE)}$

Booleans

$$\boxed{\texttt{(((}\lambda\texttt{a}\rightarrow\texttt{b}\rightarrow\texttt{((a a) b)) }\underline{\texttt{FALSE}}\texttt{) FALSE)}}$$

Booleans

$$\texttt{(((}\lambda\texttt{a}\rightarrow\texttt{b}\rightarrow\texttt{((a a) b)) }\underline{\texttt{FALSE}}\texttt{) FALSE)}$$

$$\texttt{(((}\lambda\texttt{a}\rightarrow\texttt{b}\rightarrow\texttt{((a a) b)) (}\lambda\texttt{t}\rightarrow\texttt{f}\rightarrow\texttt{f)) FALSE)}$$

Booleans

```
(((λa→b→((a a) b)) (λt→f→f)) FALSE)
```

Booleans

$$\boxed{\texttt{(((}\lambda\texttt{a}\rightarrow\texttt{b}\rightarrow\texttt{((a a) b)) (}\lambda\texttt{t}\rightarrow\texttt{f}\rightarrow\texttt{f)) FALSE)}}$$

$$\boxed{\texttt{(((}\lambda\texttt{a}\rightarrow\texttt{b}\rightarrow\texttt{((a a) b)) (}\lambda\texttt{t}\rightarrow\texttt{f}\rightarrow\texttt{f)) }\underline{\texttt{FALSE}}\texttt{)}}$$

Booleans

$(((\lambda \texttt{a}\rightarrow\texttt{b}\rightarrow((\texttt{a}\ \texttt{a})\ \texttt{b}))\ (\lambda \texttt{t}\rightarrow\texttt{f}\rightarrow\texttt{f}))\ \underline{\texttt{FALSE}})$

Booleans

$$\texttt{(((}\lambda\texttt{a}\rightarrow\texttt{b}\rightarrow\texttt{((a a) b)) (}\lambda\texttt{t}\rightarrow\texttt{f}\rightarrow\texttt{f)) }\underline{\texttt{FALSE}}\texttt{)}$$

$$\texttt{(((}\lambda\texttt{a}\rightarrow\texttt{b}\rightarrow\texttt{((a a) b)) (}\lambda\texttt{t}\rightarrow\texttt{f}\rightarrow\texttt{f)) (}\lambda\texttt{t}\rightarrow\texttt{f}\rightarrow\texttt{f))}$$

Booleans

```
(((λa→b→((a a) b)) (λt→f→f)) (λt→f→f))
```

Booleans

$$\boxed{(((\lambda a \rightarrow b \rightarrow ((a\ a)\ b))\ (\lambda t \rightarrow f \rightarrow f))\ (\lambda t \rightarrow f \rightarrow f))}$$

$$\boxed{(\underline{((\lambda a \rightarrow b \rightarrow ((a\ a)\ b))\ (\lambda t \rightarrow f \rightarrow f))}\ (\lambda t \rightarrow f \rightarrow f))}$$

Booleans

$$\boxed{(\underline{((\lambda \mathtt{a} \rightarrow \mathtt{b} \rightarrow ((\mathtt{a}\ \mathtt{a})\ \mathtt{b}))\ (\lambda \mathtt{t} \rightarrow \mathtt{f} \rightarrow \mathtt{f}))}\ (\lambda \mathtt{t} \rightarrow \mathtt{f} \rightarrow \mathtt{f}))}$$

Booleans

$$(\underline{((\lambda a \to b \to ((a\ a)\ b))\ (\lambda t \to f \to f))}\ (\lambda t \to f \to f))$$

$$((\lambda b \to (((\lambda t \to f \to f)\ (\lambda t \to f \to f))\ b))\ (\lambda t \to f \to f))$$

Booleans

```
((λb→(((λt→f→f) (λt→f→f)) b)) (λt→f→f))
```

Booleans

$((\lambda \texttt{b}\to(((\lambda \texttt{t}\to \texttt{f}\to \texttt{f})\ \ (\lambda \texttt{t}\to \texttt{f}\to \texttt{f}))\ \ \texttt{b}))\ \ (\lambda \texttt{t}\to \texttt{f}\to \texttt{f}))$

$\underline{((\lambda \texttt{b}\to(((\lambda \texttt{t}\to \texttt{f}\to \texttt{f})\ (\lambda \texttt{t}\to \texttt{f}\to \texttt{f}))\ \texttt{b}))\ (\lambda \texttt{t}\to \texttt{f}\to \texttt{f}))}$

Booleans

$$\boxed{\underline{\texttt{((}\lambda\texttt{b}\rightarrow\texttt{(((}\lambda\texttt{t}\rightarrow\texttt{f}\rightarrow\texttt{f) (}\lambda\texttt{t}\rightarrow\texttt{f}\rightarrow\texttt{f)) b)) (}\lambda\texttt{t}\rightarrow\texttt{f}\rightarrow\texttt{f))}}}$$

Booleans

$$\underline{((\lambda b\rightarrow(((\lambda t\rightarrow f\rightarrow f)\ (\lambda t\rightarrow f\rightarrow f))\ b))\ (\lambda t\rightarrow f\rightarrow f))}$$

$$(((\lambda t\rightarrow f\rightarrow f)\ (\lambda t\rightarrow f\rightarrow f))\ (\lambda t\rightarrow f\rightarrow f))$$

Booleans

```
(((λt→f→f) (λt→f→f)) (λt→f→f))
```

Booleans

$$\boxed{(((\lambda t \rightarrow f \rightarrow f)\ (\lambda t \rightarrow f \rightarrow f))\ (\lambda t \rightarrow f \rightarrow f))}$$

$$\boxed{(\underline{((\lambda t \rightarrow f \rightarrow f)\ (\lambda t \rightarrow f \rightarrow f))}\ (\lambda t \rightarrow f \rightarrow f))}$$

Booleans

$$\boxed{(\underline{((\lambda \texttt{t} \rightarrow \texttt{f} \rightarrow \texttt{f})\ (\lambda \texttt{t} \rightarrow \texttt{f} \rightarrow \texttt{f}))}\ (\lambda \texttt{t} \rightarrow \texttt{f} \rightarrow \texttt{f}))}$$

Booleans

$$(\underline{((\lambda \mathtt{t} \rightarrow \mathtt{f} \rightarrow \mathtt{f})\ (\lambda \mathtt{t} \rightarrow \mathtt{f} \rightarrow \mathtt{f}))}\ (\lambda \mathtt{t} \rightarrow \mathtt{f} \rightarrow \mathtt{f}))$$

$$((\lambda \mathtt{f} \rightarrow \mathtt{f})\ (\lambda \mathtt{t} \rightarrow \mathtt{f} \rightarrow \mathtt{f}))$$

Booleans

```
((λf→f) (λt→f→f))
```

Booleans

$((\lambda f \rightarrow f)\ (\lambda t \rightarrow f \rightarrow f))$

$\underline{((\lambda f \rightarrow f)\ (\lambda t \rightarrow f \rightarrow f))}$

Booleans

$$\boxed{\underline{\texttt{((}\lambda\texttt{f}\rightarrow\texttt{f) (}\lambda\texttt{t}\rightarrow\texttt{f}\rightarrow\texttt{f))}}}$$

Booleans

$\underline{((\lambda f \to f)\ \ (\lambda t \to f \to f))}$

$(\lambda t \to f \to f)$

Booleans

$(\lambda t \rightarrow f \rightarrow f)$

Booleans

$(\lambda \mathtt{t} \rightarrow \mathtt{f} \rightarrow \mathtt{f})$

$\underline{(\lambda \mathtt{t} \rightarrow \mathtt{f} \rightarrow \mathtt{f})}$

Booleans

$\underline{\texttt{(}\lambda\texttt{t}\rightarrow\texttt{f}\rightarrow\texttt{f)}}$

Booleans

$\underline{(\lambda t \to f \to f)}$

FALSE

Booleans

`if-then-else`

- The conditional operator `if-then-else` chooses one of two parameters based on the value of the input condition
- Given a boolean `c` and two values `t` and `e`, the result is `t` if `c` was `true`, or `e` otherwise
- Since `c` is a boolean, it already performs this choice!

-
  ```
  (λc→t→e→((c t) e))
  ```

Booleans

`if-then-else`

Let us try with `if TRUE` $\vee$ `FALSE then A else B` $\rightarrow_\beta$ `A`

Booleans

```
if (TRUE ∨ FALSE) then A else B
```

Booleans

```
if (TRUE ∨ FALSE) then A else B
```

```
(((if-then-else (TRUE ∨ FALSE)) A) B)
```

Booleans

`(((if-then-else (TRUE ∨ FALSE)) A) B)`

Booleans

$$\texttt{(((}\underline{\texttt{if-then-else}}\texttt{ (TRUE } \lor \texttt{ FALSE)) A) B)}$$

$$\texttt{((((}\lambda\texttt{c}\rightarrow\texttt{t}\rightarrow\texttt{e}\rightarrow\texttt{((c t) e)) (TRUE } \lor \texttt{ FALSE)) A) B)}$$

Booleans

$$\boxed{\texttt{((((}\lambda\texttt{c}\rightarrow\texttt{t}\rightarrow\texttt{e}\rightarrow\texttt{((c t) e)) (TRUE } \vee \texttt{ FALSE)) A) B)}}$$

Booleans

$$\texttt{((((}\lambda\texttt{c}\rightarrow\texttt{t}\rightarrow\texttt{e}\rightarrow\texttt{((c t) e)) (TRUE }\vee\texttt{ FALSE)) A) B)}$$

$$\texttt{((((}\lambda\texttt{c}\rightarrow\texttt{t}\rightarrow\texttt{e}\rightarrow\texttt{((c t) e)) ((}\underline{\vee}\texttt{ TRUE) FALSE)) A) B)}$$

Booleans

((((λc→t→e→((c t) e)) (($\underline{\vee}$ TRUE) FALSE)) A) B
)

Booleans

```
((((λc→t→e→((c t) e)) ((∨ TRUE) FALSE)) A) B
    )
```

```
((((λc→t→e→((c t) e)) (((λa→b→((a a) b))
    TRUE) FALSE)) A) B)
```

Booleans

```
((((λc→t→e→((c t) e)) (((λa→b→((a a) b))
    TRUE) FALSE)) A) B)
```

Booleans

((((λc→t→e→((c t) e)) (((λa→b→((a a) b))
    TRUE) FALSE)) A) B)

((((λc→t→e→((c t) e)) (((λa→b→((a a) b))
    $\underline{\texttt{TRUE}}$) FALSE)) A) B)

Booleans

```
((((λc→t→e→((c t) e)) (((λa→b→((a a) b))
    TRUE) FALSE)) A) B)
```

Booleans

$$\texttt{((((}\lambda\texttt{c}\rightarrow\texttt{t}\rightarrow\texttt{e}\rightarrow\texttt{((c t) e)) (((}\lambda\texttt{a}\rightarrow\texttt{b}\rightarrow\texttt{((a a) b))}\ \underline{\texttt{TRUE}}\texttt{) FALSE)) A) B)}$$

$$\texttt{((((}\lambda\texttt{c}\rightarrow\texttt{t}\rightarrow\texttt{e}\rightarrow\texttt{((c t) e)) (((}\lambda\texttt{a}\rightarrow\texttt{b}\rightarrow\texttt{((a a) b)) (}\lambda\ \texttt{t}\rightarrow\texttt{f}\rightarrow\texttt{t)) FALSE)) A) B)}$$

Booleans

```
((((λc→t→e→((c t) e)) (((λa→b→((a a) b)) (λ
    t→f→t)) FALSE)) A) B)
```

Booleans

```
((((λc→t→e→((c t) e)) (((λa→b→((a a) b)) (λ
    t→f→t)) FALSE)) A) B)
```

$$((((\lambda c\to t\to e\to((c\ t)\ e))\ (((\lambda a\to b\to((a\ a)\ b))\ (\lambda t\to f\to t))\ \underline{\texttt{FALSE}}))\ A)\ B)$$

Booleans

$$\texttt{((((}\lambda\texttt{c}\rightarrow\texttt{t}\rightarrow\texttt{e}\rightarrow\texttt{((c t) e)) (((}\lambda\texttt{a}\rightarrow\texttt{b}\rightarrow\texttt{((a a) b)) (}\lambda\texttt{t}\rightarrow\texttt{f}\rightarrow\texttt{t)) }\underline{\texttt{FALSE}}\texttt{)) A) B)}$$

Booleans

((((λc→t→e→((c t) e)) (((λa→b→((a a) b)) (λ t→f→t)) FALSE)) A) B)

((((λc→t→e→((c t) e)) (((λa→b→((a a) b)) (λ t→f→t)) (λt→f→f))) A) B)

Booleans

```
((((λc→t→e→((c t) e)) (((λa→b→((a a) b)) (λ
    t→f→t)) (λt→f→f))) A) B)
```

Booleans

$$\mathtt{((((\lambda c \to t \to e \to ((c\ t)\ e))\ (((\lambda a \to b \to ((a\ a)\ b))\ (\lambda\ t \to f \to t))\ (\lambda t \to f \to f)))\ A)\ B)}$$

$$\mathtt{((((\lambda c \to t \to e \to ((c\ t)\ e))\ (\ \underline{((\lambda a \to b \to ((a\ a)\ b))\ (\lambda t \to f \to t))}\ (\lambda t \to f \to f)))\ A)\ B\ )}$$

Booleans

$$\texttt{((((}\lambda\texttt{c}\rightarrow\texttt{t}\rightarrow\texttt{e}\rightarrow\texttt{((c t) e)) (}$$

$$\underline{\texttt{((}\lambda\texttt{a}\rightarrow\texttt{b}\rightarrow\texttt{((a a) b)) (}\lambda\texttt{t}\rightarrow\texttt{f}\rightarrow\texttt{t))}}\ \texttt{(}\lambda\texttt{t}\rightarrow\texttt{f}\rightarrow\texttt{f))) A) B}$$

$$\texttt{)}$$

Booleans

$$((((\lambda c\rightarrow t\rightarrow e\rightarrow((c\ \ t)\ \ e))\ \ (\ \underline{((\lambda a\rightarrow b\rightarrow((a\ \ a)\ \ b))\ \ (\lambda t\rightarrow f\rightarrow t))}\ \ (\lambda t\rightarrow f\rightarrow f)))\ \ A)\ \ B\ )$$

$$((((\lambda c\rightarrow t\rightarrow e\rightarrow((c\ \ t)\ \ e))\ \ ((\lambda b\rightarrow(((\lambda t\rightarrow f\rightarrow t)\ \ (\lambda t\rightarrow f\rightarrow t))\ \ b))\ \ (\lambda t\rightarrow f\rightarrow f)))\ \ A)\ \ B)$$

Booleans

```
((((λc→t→e→((c t) e)) ((λb→(((λt→f→t) (λt→
    f→t)) b)) (λt→f→f))) A) B)
```

Booleans

```
((((λc→t→e→((c t) e)) ((λb→(((λt→f→t) (λt→
    f→t)) b)) (λt→f→f))) A) B)
```

```
((((λc→t→e→((c t) e)) ((λb→(((λt→f→t) (λt→
    f→t)) b)) (λt→f→f))) A) B)
```

Booleans

```
((((λc→t→e→((c t) e)) ((λb→(((λt→f→t) (λt→
   f→t)) b)) (λt→f→f))) A) B)
```

Booleans

```
((((λc→t→e→((c t) e)) ((λb→(((λt→f→t) (λt→f→t)) b)) (λt→f→f))) A) B)
```

$$\texttt{((}\ \underline{\texttt{((}\lambda\texttt{c}\to\texttt{t}\to\texttt{e}\to\texttt{((c t) e)) ((}\lambda\texttt{b}\to\texttt{(((}\lambda\texttt{t}\to\texttt{f}\to\texttt{t) (}\lambda\texttt{t}\to\texttt{f}\to\texttt{t)) b}}\ \texttt{A) B)}$$

Booleans

```
((
    ((λc→t→e→((c t) e)) ((λb→(((λt→f→t) (λt→f→t)) b
    A) B)
```

Booleans

```
((
    ((λc→t→e→((c t) e)) ((λb→(((λt→f→t) (λt→f→t)) b
    A) B)
```

```
(((λt→e→((((λb→(((λt→f→t) (λt→f→t)) b)) (λ
    t→f→f)) t) e)) A) B)
```

Booleans

```
(((λt→e→((((λb→(((λt→f→t) (λt→f→t)) b)) (λ
    t→f→f)) t) e)) A) B)
```

Booleans

```
(((λt→e→((((λb→(((λt→f→t) (λt→f→t)) b)) (λ
    t→f→f)) t) e)) A) B)
```

```
(
    ((λt→e→((((λb→(((λt→f→t) (λt→f→t)) b)) (λt→f→
    B)
```

Booleans

```
(
    ((λt→e→((((λb→(((λt→f→t) (λt→f→t)) b)) (λt→f→f
    B)
```

Booleans

```
(
    ((λt→e→((((λb→(((λt→f→t) (λt→f→t)) b)) (λt→f→f
    B)
```

```
((λe→((((λb→(((λt→f→t) (λt→f→t)) b)) (λt→f
    →f)) A) e)) B)
```

Booleans

```
((λe→((((λb→(((λt→f→t) (λt→f→t)) b)) (λt→f
    →f)) A) e)) B)
```

Booleans

```
((λe→((((λb→(((λt→f→t) (λt→f→t)) b)) (λt→f
   →f)) A) e)) B)
```

```
((λe→((((λb→(((λt→f→t) (λt→f→t)) b)) (λt→f→f)) A) e
```

Booleans

```
((λe→((((λb→(((λt→f→t) (λt→f→t)) b)) (λt→f→f)) A)
```

Booleans

```
((λe→((((λb→(((λt→f→t) (λt→f→t)) b)) (λt→f→f)) A) e
```

```
((((λb→(((λt→f→t) (λt→f→t)) b)) (λt→f→f))
   A) B)
```

Booleans

```
((((λb→(((λt→f→t) (λt→f→t)) b)) (λt→f→f))
    A) B)
```

Booleans

```
((((λb→(((λt→f→t) (λt→f→t)) b)) (λt→f→f))
    A) B)
```

$$((\underline{((\lambda b\rightarrow(((\lambda t\rightarrow f\rightarrow t)\ (\lambda t\rightarrow f\rightarrow t))\ b))\ (\lambda t\rightarrow f\rightarrow f))}\ A)\ B)$$

Booleans

$$\boxed{((\underline{((\lambda b\rightarrow(((\lambda t\rightarrow f\rightarrow t)\ (\lambda t\rightarrow f\rightarrow t))\ b))\ (\lambda t\rightarrow f\rightarrow f))}\ A)\ B)}$$

Booleans

$$\boxed{\texttt{((}\underline{\texttt{((}\lambda\texttt{b}\rightarrow\texttt{(((}\lambda\texttt{t}\rightarrow\texttt{f}\rightarrow\texttt{t) (}\lambda\texttt{t}\rightarrow\texttt{f}\rightarrow\texttt{t)) b)) (}\lambda\texttt{t}\rightarrow\texttt{f}\rightarrow\texttt{f))}}\texttt{ A) B)}}$$

$$\boxed{\texttt{(((((}\lambda\texttt{t}\rightarrow\texttt{f}\rightarrow\texttt{t) (}\lambda\texttt{t}\rightarrow\texttt{f}\rightarrow\texttt{t)) (}\lambda\texttt{t}\rightarrow\texttt{f}\rightarrow\texttt{f)) A) B)}}$$

Booleans

```
(((((λt→f→t) (λt→f→t)) (λt→f→f)) A) B)
```

Booleans

$$\boxed{(((((\lambda t \to f \to t)\ (\lambda t \to f \to t))\ (\lambda t \to f \to f))\ A)\ B)}$$

$$\boxed{(((\underline{((\lambda t \to f \to t)\ (\lambda t \to f \to t))}\ (\lambda t \to f \to f))\ A)\ B)}$$

Booleans

$$(((\underline{((\lambda \mathtt{t} \to \mathtt{f} \to \mathtt{t})\ (\lambda \mathtt{t} \to \mathtt{f} \to \mathtt{t}))}\ (\lambda \mathtt{t} \to \mathtt{f} \to \mathtt{f}))\ \mathtt{A})\ \mathtt{B})$$

Booleans

$$(((\underline{((\lambda \mathtt{t} \rightarrow \mathtt{f} \rightarrow \mathtt{t})\ (\lambda \mathtt{t} \rightarrow \mathtt{f} \rightarrow \mathtt{t}))}\ (\lambda \mathtt{t} \rightarrow \mathtt{f} \rightarrow \mathtt{f}))\ \mathtt{A})\ \mathtt{B})$$

$$((((\lambda \mathtt{f} \rightarrow \mathtt{t} \rightarrow \mathtt{f} \rightarrow \mathtt{t})\ (\lambda \mathtt{t} \rightarrow \mathtt{f} \rightarrow \mathtt{f}))\ \mathtt{A})\ \mathtt{B})$$

Booleans

```
((((λf→t→f→t) (λt→f→f)) A) B)
```

Booleans

$$((((\lambda f \rightarrow t \rightarrow f \rightarrow t)\ (\lambda t \rightarrow f \rightarrow f))\ A)\ B)$$

$$((\underline{((\lambda f \rightarrow t \rightarrow f \rightarrow t)\ (\lambda t \rightarrow f \rightarrow f))}\ A)\ B)$$

Booleans

$$((\underline{((\lambda \mathtt{f} \rightarrow \mathtt{t} \rightarrow \mathtt{f} \rightarrow \mathtt{t})\ (\lambda \mathtt{t} \rightarrow \mathtt{f} \rightarrow \mathtt{f}))}\ \mathtt{A})\ \mathtt{B})$$

Booleans

$$((\underline{((\lambda f \rightarrow t \rightarrow f \rightarrow t)\ (\lambda t \rightarrow f \rightarrow f))}\ A)\ B)$$

$$(((\lambda t \rightarrow f \rightarrow t)\ A)\ B)$$

Booleans

```
(((λt→f→t) A) B)
```

Booleans

$$(((\lambda \mathtt{t} \rightarrow \mathtt{f} \rightarrow \mathtt{t})\ \mathtt{A})\ \mathtt{B})$$

$$(\underline{((\lambda \mathtt{t} \rightarrow \mathtt{f} \rightarrow \mathtt{t})\ \mathtt{A})}\ \mathtt{B})$$

Booleans

$$(\underline{((\lambda \mathtt{t} \to \mathtt{f} \to \mathtt{t})\ \mathtt{A})}\ \mathtt{B})$$

Booleans

$$\boxed{(\underline{((\lambda t \rightarrow f \rightarrow t)\ \ A)}\ \ B)}$$

$$\boxed{((\lambda f \rightarrow A)\ \ B)}$$

Booleans

```
((λf→A) B)
```

Booleans

$((\lambda \mathtt{f} \rightarrow \mathtt{A})\ \mathtt{B})$

$\underline{((\lambda \mathtt{f} \rightarrow \mathtt{A})\ \mathtt{B})}$

Booleans

```
((λf→A) B)
```

Booleans

$\underline{((\lambda f \rightarrow A)\ \ B)}$

A

Natural numbers

Natural numbers

Idea

- Natural numbers such as 3 and 0 must be defined so as to identify themselves
- Their identity is determined by how many times they perform an action
- The only action we have available is applying a function to a term

Natural numbers

Idea

- We will use unary numbers
- A number is defined by how many times it applies a function to a given term
- Zero applications are also possible, in this case we default to the given term

Natural numbers

0, 1, etc.

A number is defined as an applicator of a term identifying as successor to another term identifying as zero[a]

---

[a] first and second arguments by arbitrary convention

Natural numbers

0 will thus look like

```
(λs→z→z)
```

1 will look like

```
(λs→z→(s z))
```

7 will look like

```
(λs→z→(s (s (s (s (s (s (s z))))))))
```

etc.

Natural numbers

Addition

- Adding numbers is a function that takes as input two numbers (say `m` and `n`), and returns a number
- The first number applies its first parameter `m` times to its second parameter
- The second number applies its first parameter `n` times to its second parameter
- We can use the second number as the second parameter to the first, therefore obtaining something that applies `m+n` times
-
```
(λm→n→s→z→((m s) ((n s) z)))
```

Natural numbers

Addition

Let us try it out to $2 + 1 \rightarrow_{\beta} 3$

Natural numbers

```
(2 + 1)
```

Natural numbers

```
(2 + 1)
```

```
((± 2) 1)
```

Natural numbers

$$((\underline{+}\ 2)\ \ 1)$$

Natural numbers

```
((+ 2) 1)
```

```
(((λm→n→s→z→((m s) ((n s) z))) 2) 1)
```

Natural numbers

```
(((λm→n→s→z→((m s) ((n s) z))) 2) 1)
```

Natural numbers

$$\texttt{(((}\lambda\texttt{m}\rightarrow\texttt{n}\rightarrow\texttt{s}\rightarrow\texttt{z}\rightarrow\texttt{((m s) ((n s) z))) 2) 1)}$$

$$\texttt{(((}\lambda\texttt{m}\rightarrow\texttt{n}\rightarrow\texttt{s}\rightarrow\texttt{z}\rightarrow\texttt{((m s) ((n s) z))) }\underline{\texttt{2}}\texttt{) 1)}$$

Natural numbers

$$\boxed{\texttt{(((}\lambda\texttt{m}\rightarrow\texttt{n}\rightarrow\texttt{s}\rightarrow\texttt{z}\rightarrow\texttt{((m s) ((n s) z))) }\underline{\texttt{2}}\texttt{) 1)}}$$

Natural numbers

```
(((λm→n→s→z→((m s) ((n s) z))) 2) 1)
```

```
(((λm→n→s→z→((m s) ((n s) z))) (λs→z→(s (s
     z)))) 1)
```

Natural numbers

```
(((λm→n→s→z→((m s) ((n s) z))) (λs→z→(s (s
    z)))) 1)
```

Natural numbers

$$\texttt{(((}\lambda\texttt{m}\rightarrow\texttt{n}\rightarrow\texttt{s}\rightarrow\texttt{z}\rightarrow\texttt{((m s) ((n s) z))) (}\lambda\texttt{s}\rightarrow\texttt{z}\rightarrow\texttt{(s (s z)))) 1)}$$

$$\texttt{(((}\lambda\texttt{m}\rightarrow\texttt{n}\rightarrow\texttt{s}\rightarrow\texttt{z}\rightarrow\texttt{((m s) ((n s) z))) (}\lambda\texttt{s}\rightarrow\texttt{z}\rightarrow\texttt{(s (s z)))) }\underline{\texttt{1}}\texttt{)}$$

Natural numbers

$$\boxed{\texttt{(((}\lambda\texttt{m}\rightarrow\texttt{n}\rightarrow\texttt{s}\rightarrow\texttt{z}\rightarrow\texttt{((m s) ((n s) z))) (}\lambda\texttt{s}\rightarrow\texttt{z}\rightarrow\texttt{(s (s z)))) }\underline{\texttt{1}}\texttt{)}}$$

Natural numbers

```
(((λm→n→s→z→((m s) ((n s) z))) (λs→z→(s (s
    z)))) 1)
```

```
(((λm→n→s→z→((m s) ((n s) z))) (λs→z→(s (s
    z)))) (λs→z→(s z)))
```

Natural numbers

```
(((λm→n→s→z→((m s) ((n s) z))) (λs→z→(s (s
    z)))) (λs→z→(s z)))
```

Natural numbers

$$(((\lambda m \rightarrow n \rightarrow s \rightarrow z \rightarrow ((m\ s)\ ((n\ s)\ z)))\ (\lambda s \rightarrow z \rightarrow (s\ (s\ z))))\ (\lambda s \rightarrow z \rightarrow (s\ z)))$$

$$(\underline{((\lambda m \rightarrow n \rightarrow s \rightarrow z \rightarrow ((m\ s)\ ((n\ s)\ z)))\ (\lambda s \rightarrow z \rightarrow (s\ (s\ z))))}\ (\lambda s \rightarrow z \rightarrow (s\ z)))$$

Natural numbers

$$(\underline{((\lambda \mathtt{m} \rightarrow \mathtt{n} \rightarrow \mathtt{s} \rightarrow \mathtt{z} \rightarrow ((\mathtt{m}\ \mathtt{s})\ ((\mathtt{n}\ \mathtt{s})\ \mathtt{z})))\ (\lambda \mathtt{s} \rightarrow \mathtt{z} \rightarrow (\mathtt{s}\ (\mathtt{s}\ \mathtt{z}))))}$$

$$(\lambda \mathtt{s} \rightarrow \mathtt{z} \rightarrow (\mathtt{s}\ \mathtt{z})))$$

Natural numbers

$$(\underline{((\lambda \mathtt{m} \rightarrow \mathtt{n} \rightarrow \mathtt{s} \rightarrow \mathtt{z} \rightarrow ((\mathtt{m}\ \mathtt{s})\ ((\mathtt{n}\ \mathtt{s})\ \mathtt{z})))\ (\lambda \mathtt{s} \rightarrow \mathtt{z} \rightarrow (\mathtt{s}\ (\mathtt{s}\ \mathtt{z}))))}\ (\lambda \mathtt{s} \rightarrow \mathtt{z} \rightarrow (\mathtt{s}\ \mathtt{z})))$$

$$((\lambda \mathtt{n} \rightarrow \mathtt{s} \rightarrow \mathtt{z} \rightarrow (((\lambda \mathtt{s} \rightarrow \mathtt{z} \rightarrow (\mathtt{s}\ (\mathtt{s}\ \mathtt{z})))\ \mathtt{s})\ ((\mathtt{n}\ \mathtt{s})\ \mathtt{z})))\ (\lambda \mathtt{s} \rightarrow \mathtt{z} \rightarrow (\mathtt{s}\ \mathtt{z})))$$

Natural numbers

```
((λn→s→z→(((λs→z→(s (s z))) s) ((n s) z)))
    (λs→z→(s z)))
```

Natural numbers

```
((λn→s→z→(((λs→z→(s (s z))) s) ((n s) z)))
    (λs→z→(s z)))
```

```
((λn→s→z→(((λs→z→(s (s z))) s) ((n s) z))) (λs→z→(s
```

Natural numbers

```
((λn→s→z→(((λs→z→(s (s z))) s) ((n s) z))) (λs→z→(s
```

Natural numbers

```
((λn→s→z→(((λs→z→(s (s z))) s) ((n s) z))) (λs→z→(s
```

```
(λs→z→(((λs→z→(s (s z))) s) (((λs→z→(s z))
     s) z)))
```

Natural numbers

```
(λs→z→(((λs→z→(s (s z))) s) (((λs→z→(s z))
    s) z)))
```

Natural numbers

$$(\lambda s \rightarrow z \rightarrow (((\lambda s \rightarrow z \rightarrow (s\ (s\ z)))\ s)\ (((\lambda s \rightarrow z \rightarrow (s\ z))\ s)\ z)))$$

$$(\lambda s \rightarrow z \rightarrow (\underline{((\lambda s \rightarrow z \rightarrow (s\ (s\ z)))\ s)}\ (((\lambda s \rightarrow z \rightarrow (s\ z))\ s)\ z)))$$

Natural numbers

$$\boxed{(\lambda s \rightarrow z \rightarrow (\underline{((\lambda s \rightarrow z \rightarrow (s\ (s\ z)))\ s)}\ (((\lambda s \rightarrow z \rightarrow (s\ z))\ s)\ z)))}$$

Natural numbers

$$(\lambda \texttt{s} \to \texttt{z} \to (\underline{((\lambda \texttt{s} \to \texttt{z} \to (\texttt{s}\ (\texttt{s}\ \texttt{z})))\ \texttt{s})}\ (((\lambda \texttt{s} \to \texttt{z} \to (\texttt{s}\ \texttt{z}))\ \texttt{s})\ \texttt{z})))$$

$$(\lambda \texttt{s} \to \texttt{z} \to ((\lambda \texttt{z} \to (\texttt{s}\ (\texttt{s}\ \texttt{z})))\ (((\lambda \texttt{s} \to \texttt{z} \to (\texttt{s}\ \texttt{z}))\ \texttt{s})\ \texttt{z})))$$

Natural numbers

```
(λs→z→((λz→(s (s z))) (((λs→z→(s z)) s) z))
    )
```

Natural numbers

$$\mathtt{(\lambda s{\rightarrow}z{\rightarrow}((\lambda z{\rightarrow}(s\ \ (s\ \ z)))\ \ (((\lambda s{\rightarrow}z{\rightarrow}(s\ \ z))\ \ s)\ \ z))\ )}$$

$$\mathtt{(\lambda s{\rightarrow}z{\rightarrow}((\lambda z{\rightarrow}(s\ \ (s\ \ z)))\ \ (\underline{((\lambda s{\rightarrow}z{\rightarrow}(s\ z))\ \ s)}\ \ z)))}$$

Natural numbers

$$\boxed{\texttt{(}\lambda\texttt{s}\rightarrow\texttt{z}\rightarrow\texttt{((}\lambda\texttt{z}\rightarrow\texttt{(s (s z))) (}\underline{\texttt{((}\lambda\texttt{s}\rightarrow\texttt{z}\rightarrow\texttt{(s z)) s)}}\texttt{ z)))}}$$

Natural numbers

$$\boxed{(\lambda s \rightarrow z \rightarrow ((\lambda z \rightarrow (s\ (s\ z)))\ (\underline{((\lambda s \rightarrow z \rightarrow (s\ z))\ s)}\ z)))}$$

$$\boxed{(\lambda s \rightarrow z \rightarrow ((\lambda z \rightarrow (s\ (s\ z)))\ ((\lambda z \rightarrow (s\ z))\ z)))}$$

Natural numbers

```
(λs→z→((λz→(s (s z))) ((λz→(s z)) z)))
```

Natural numbers

$$(\lambda \mathtt{s} \rightarrow \mathtt{z} \rightarrow ((\lambda \mathtt{z} \rightarrow (\mathtt{s}\ \ (\mathtt{s}\ \ \mathtt{z})))\ \ ((\lambda \mathtt{z} \rightarrow (\mathtt{s}\ \ \mathtt{z}))\ \ \mathtt{z})))$$

$$(\lambda \mathtt{s} \rightarrow \mathtt{z} \rightarrow ((\lambda \mathtt{z} \rightarrow (\mathtt{s}\ \ (\mathtt{s}\ \ \mathtt{z})))\ \ \underline{((\lambda \mathtt{z} \rightarrow (\mathtt{s}\ \mathtt{z}))\ \mathtt{z})}))$$

Natural numbers

$$(\lambda \texttt{s}\rightarrow\texttt{z}\rightarrow((\lambda\texttt{z}\rightarrow(\texttt{s}\ (\texttt{s}\ \texttt{z})))\ \underline{((\lambda\texttt{z}\rightarrow(\texttt{s}\ \texttt{z}))\ \texttt{z})}))$$

Natural numbers

$$(\lambda s \rightarrow z \rightarrow ((\lambda z \rightarrow (s\ \ (s\ \ z)))\ \ \underline{((\lambda z \rightarrow (s\ \ z))\ \ z)}))$$

$$(\lambda s \rightarrow z \rightarrow ((\lambda z \rightarrow (s\ \ (s\ \ z)))\ \ (s\ \ z)))$$

Natural numbers

```
(λs→z→((λz→(s (s z))) (s z)))
```

Natural numbers

```
(λs→z→((λz→(s (s z))) (s z)))
```

```
(λs→z→((λz→(s (s z))) (s z)))
```

Natural numbers

```
(λs→z→((λz→(s (s z))) (s z)))
```

Natural numbers

$$(\lambda \mathtt{s} \to \mathtt{z} \to ((\lambda \mathtt{z} \to (\mathtt{s}\ \ (\mathtt{s}\ \ \mathtt{z})))\ \ (\mathtt{s}\ \ \mathtt{z})))$$

$$(\lambda \mathtt{s} \to \mathtt{z} \to \underline{((\lambda \mathtt{z} \to (\mathtt{s}\ \ (\mathtt{s}\ \ \mathtt{z})))\ \ (\mathtt{s}\ \ \mathtt{z}))})$$

Natural numbers

$$\boxed{(\lambda \mathtt{s} \rightarrow \mathtt{z} \rightarrow \underline{((\lambda \mathtt{z} \rightarrow (\mathtt{s}\ \ (\mathtt{s}\ \ \mathtt{z})))\ \ (\mathtt{s}\ \ \mathtt{z}))})}$$

Natural numbers

$$(\lambda \mathtt{s} \rightarrow \mathtt{z} \rightarrow \underline{((\lambda \mathtt{z} \rightarrow (\mathtt{s}\ \ (\mathtt{s}\ \ \mathtt{z})))\ \ (\mathtt{s}\ \ \mathtt{z}))})$$

$$(\lambda \mathtt{s} \rightarrow \mathtt{z} \rightarrow (\mathtt{s}\ \ (\mathtt{s}\ \ (\mathtt{s}\ \ \mathtt{z}))))$$

Natural numbers

```
(λs→z→(s (s (s z))))
```

Natural numbers

$$(\lambda \mathtt{s} \rightarrow \mathtt{z} \rightarrow (\mathtt{s}\ (\mathtt{s}\ (\mathtt{s}\ \mathtt{z}))))$$

$$\underline{(\lambda \mathtt{s} \rightarrow \mathtt{z} \rightarrow (\mathtt{s}\ (\mathtt{s}\ (\mathtt{s}\ \mathtt{z}))))}$$

Natural numbers

$$\underline{(\lambda \mathtt{s} \rightarrow \mathtt{z} \rightarrow (\mathtt{s}\ (\mathtt{s}\ (\mathtt{s}\ \mathtt{z}))))}$$

Natural numbers

```
(λs→z→(s (s (s z))))
```

```
3
```

Natural numbers

Multiplication

- Multiplying numbers is a function that takes as input two numbers (say `m` and `n`), and returns a number
- The first number applies its first parameter `m` times to its second parameter
- The second number applies its first parameter `n` times to its second parameter
- We can use the second number as the first parameter to the first, therefore obtaining something that applies `n+ m` times, starting from `z`
- $(\lambda m \rightarrow n \rightarrow s \rightarrow z \rightarrow ((m\ (n\ s))\ z))$

Natural numbers

Multiplication

Let us try it out to $2 \times 2 \rightarrow_{\beta} 4$

Natural numbers

```
(2  ×  2)
```

Natural numbers

$$(2 \ \times \ 2)$$

$$((\underline{\times} \ 2) \ \ 2)$$

Natural numbers

$$\boxed{((\underline{\times}\ \ 2)\ \ 2)}$$

Natural numbers

$$\texttt{((}\underline{\times}\texttt{ 2) 2)}$$

$$\texttt{(((}\lambda\texttt{m}\to\texttt{n}\to\texttt{s}\to\texttt{z}\to\texttt{((m (n s)) z)) 2) 2)}$$

Natural numbers

```
(((λm→n→s→z→((m (n s)) z)) 2) 2)
```

Natural numbers

$$(((\lambda m\rightarrow n\rightarrow s\rightarrow z\rightarrow((m\ (n\ s))\ z))\ 2)\ 2)$$

$$(((\lambda m\rightarrow n\rightarrow s\rightarrow z\rightarrow((m\ (n\ s))\ z))\ \underline{2})\ 2)$$

Natural numbers

$$\boxed{(((\lambda \texttt{m}\rightarrow\texttt{n}\rightarrow\texttt{s}\rightarrow\texttt{z}\rightarrow((\texttt{m}\ \ (\texttt{n}\ \ \texttt{s}))\ \ \texttt{z}))\ \ \underline{\texttt{2}})\ \ \texttt{2})}$$

Natural numbers

$$\texttt{(((}\lambda\texttt{m}\rightarrow\texttt{n}\rightarrow\texttt{s}\rightarrow\texttt{z}\rightarrow\texttt{((m (n s)) z)) }\underline{\texttt{2}}\texttt{) 2)}$$

$$\texttt{(((}\lambda\texttt{m}\rightarrow\texttt{n}\rightarrow\texttt{s}\rightarrow\texttt{z}\rightarrow\texttt{((m (n s)) z)) (}\lambda\texttt{s}\rightarrow\texttt{z}\rightarrow\texttt{(s (s z)))) 2)}$$

Natural numbers

```
(((λm→n→s→z→((m (n s)) z)) (λs→z→(s (s z))
   )) 2)
```

Natural numbers

Delta rules

The INFDEV@HR Team

Introduction

Conclusion

```
(((λm→n→s→z→((m (n s)) z)) (λs→z→(s (s z))
    )) 2)
```

$$\texttt{(((}\lambda\texttt{m}\rightarrow\texttt{n}\rightarrow\texttt{s}\rightarrow\texttt{z}\rightarrow\texttt{((m (n s)) z)) (}\lambda\texttt{s}\rightarrow\texttt{z}\rightarrow\texttt{(s (s z)))) }\underline{\texttt{2}}\texttt{)}$$

Natural numbers

```
(((λm→n→s→z→((m (n s)) z)) (λs→z→(s (s z))
   )) 2)
```

Natural numbers

```
(((λm→n→s→z→((m (n s)) z)) (λs→z→(s (s z))
    )) 2)
```

```
(((λm→n→s→z→((m (n s)) z)) (λs→z→(s (s z))
    )) (λs→z→(s (s z))))
```

Natural numbers

```
(((λm→n→s→z→((m (n s)) z)) (λs→z→(s (s z))
    )) (λs→z→(s (s z))))
```

Natural numbers

$$(((\lambda m \rightarrow n \rightarrow s \rightarrow z \rightarrow ((m\ (n\ s))\ z))\ (\lambda s \rightarrow z \rightarrow (s\ (s\ z))))\ (\lambda s \rightarrow z \rightarrow (s\ (s\ z))))$$

$$(\underline{((\lambda m \rightarrow n \rightarrow s \rightarrow z \rightarrow ((m\ (n\ s))\ z))\ (\lambda s \rightarrow z \rightarrow (s\ (s\ z))))}\ (\lambda s \rightarrow z \rightarrow (s\ (s\ z))))$$

Natural numbers

$$\boxed{\texttt{(}\underline{\texttt{((}\lambda\texttt{m}\rightarrow\texttt{n}\rightarrow\texttt{s}\rightarrow\texttt{z}\rightarrow\texttt{((m (n s)) z)) (}\lambda\texttt{s}\rightarrow\texttt{z}\rightarrow\texttt{(s (s z))))}}\ \texttt{(}\lambda\texttt{s}\rightarrow\texttt{z}\rightarrow\texttt{(s (s z))))}}$$

Natural numbers

$$(\underline{((\lambda m\to n\to s\to z\to((m\ (n\ s))\ z))\ (\lambda s\to z\to(s\ (s\ z))))}\ (\lambda s\to z\to(s\ (s\ z))))$$

$$((\lambda n\to s\to z\to(((\lambda s\to z\to(s\ (s\ z)))\ (n\ s))\ z))\ (\lambda s\to z\to(s\ (s\ z))))$$

Natural numbers

```
((λn→s→z→(((λs→z→(s (s z))) (n s)) z)) (λs
    →z→(s (s z))))
```

Natural numbers

```
((λn→s→z→(((λs→z→(s (s z))) (n s)) z)) (λs
    →z→(s (s z))))
```

```
((λn→s→z→(((λs→z→(s (s z))) (n s)) z)) (λs→z→(s (s
```

Natural numbers

```
((λn→s→z→(((λs→z→(s (s z))) (n s)) z)) (λs→z→(s (s
```

Natural numbers

```
((λn→s→z→(((λs→z→(s (s z))) (n s)) z)) (λs→z→(s (s
```

```
(λs→z→(((λs→z→(s (s z))) ((λs→z→(s (s z)))
    s)) z))
```

Natural numbers

```
(λs→z→(((λs→z→(s (s z))) ((λs→z→(s (s z)))
     s)) z))
```

Natural numbers

$$(\lambda s \rightarrow z \rightarrow (((\lambda s \rightarrow z \rightarrow (s\ (s\ z)))\ ((\lambda s \rightarrow z \rightarrow (s\ (s\ z)))\ s))\ z))$$

$$(\lambda s \rightarrow z \rightarrow (((\lambda s \rightarrow z \rightarrow (s\ (s\ z)))\ \underline{((\lambda s \rightarrow z \rightarrow (s\ (s\ z)))\ s)})\ z))$$

Natural numbers

$$\boxed{\begin{aligned}&(\lambda s \rightarrow z \rightarrow (((\lambda s \rightarrow z \rightarrow (s\ \ (s\ \ z)))\\ &\quad \underline{((\lambda s \rightarrow z \rightarrow (s\ \ (s\ \ z)))\ \ s)})\ \ z))\end{aligned}}$$

Natural numbers

$$
\begin{array}{l}
(\lambda s \rightarrow z \rightarrow (((\lambda s \rightarrow z \rightarrow (s\ \ (s\ \ z)))\\
\quad \underline{((\lambda s \rightarrow z \rightarrow (s\ \ (s\ \ z)))\ \ s)})\ \ z))
\end{array}
$$

$$
\begin{array}{l}
(\lambda s \rightarrow z \rightarrow (((\lambda s \rightarrow z \rightarrow (s\ \ (s\ \ z)))\ \ (\lambda z \rightarrow (s\ \ (s\ \ z))))\ \ z)\\
\quad)
\end{array}
$$

Natural numbers

```
(λs→z→(((λs→z→(s (s z))) (λz→(s (s z)))) z)
   )
```

Natural numbers

```
(λs→z→(((λs→z→(s (s z))) (λz→(s (s z)))) z)
    )
```

```
(λs→z→(((λs→z→(s (s z))) (λz→(s (s z)))) z)
    )
```

Natural numbers

```
(λs→z→(((λs→z→(s (s z))) (λz→(s (s z)))) z)
   )
```

Natural numbers

$$(\lambda \mathtt{s} \rightarrow \mathtt{z} \rightarrow (((\lambda \mathtt{s} \rightarrow \mathtt{z} \rightarrow (\mathtt{s}\ \ (\mathtt{s}\ \ \mathtt{z})))\ \ (\lambda \mathtt{z} \rightarrow (\mathtt{s}\ \ (\mathtt{s}\ \ \mathtt{z}))))\ \ \mathtt{z})\\ )$$

$$(\lambda \mathtt{s} \rightarrow \mathtt{z} \rightarrow (\underline{((\lambda \mathtt{s} \rightarrow \mathtt{z} \rightarrow (\mathtt{s}\ \ (\mathtt{s}\ \ \mathtt{z})))\ \ (\lambda \mathtt{z} \rightarrow (\mathtt{s}\ \ (\mathtt{s}\ \ \mathtt{z}))))}\ \ \mathtt{z}))$$

Natural numbers

$$\boxed{(\lambda s \rightarrow z \rightarrow (\underline{((\lambda s \rightarrow z \rightarrow (s\ (s\ z)))\ (\lambda z \rightarrow (s\ (s\ z))))}\ z))}$$

Natural numbers

$$\texttt{(}\lambda\texttt{s}\rightarrow\texttt{z}\rightarrow\texttt{(}\underline{\texttt{((}\lambda\texttt{s}\rightarrow\texttt{z}\rightarrow\texttt{(s (s z))) (}\lambda\texttt{z}\rightarrow\texttt{(s (s z))))}}\texttt{ z))}$$

$$\texttt{(}\lambda\texttt{s}\rightarrow\texttt{z}\rightarrow\texttt{((}\lambda\texttt{z}\rightarrow\texttt{((}\lambda\texttt{z}\rightarrow\texttt{(s (s z))) ((}\lambda\texttt{z}\rightarrow\texttt{(s (s z))) z))) z))}$$

Natural numbers

```
(λs→z→((λz→((λz→(s (s z))) ((λz→(s (s z)))
    z))) z))
```

Natural numbers

$$(\lambda \mathtt{s} \rightarrow \mathtt{z} \rightarrow ((\lambda \mathtt{z} \rightarrow ((\lambda \mathtt{z} \rightarrow (\mathtt{s}\ \ (\mathtt{s}\ \ \mathtt{z})))\ \ ((\lambda \mathtt{z} \rightarrow (\mathtt{s}\ \ (\mathtt{s}\ \ \mathtt{z})))\ \ \mathtt{z})))\ \ \mathtt{z}))$$

$$(\lambda \mathtt{s} \rightarrow \mathtt{z} \rightarrow \underline{((\lambda \mathtt{z} \rightarrow ((\lambda \mathtt{z} \rightarrow (\mathtt{s}\ \ (\mathtt{s}\ \ \mathtt{z})))\ \ ((\lambda \mathtt{z} \rightarrow (\mathtt{s}\ \ (\mathtt{s}\ \ \mathtt{z})))\ \ \mathtt{z})))\ \ \mathtt{z})})$$

Natural numbers

$$\begin{array}{l} \texttt{(}\lambda\texttt{s}\rightarrow\texttt{z}\rightarrow \\ \quad \underline{\texttt{((}\lambda\texttt{z}\rightarrow\texttt{((}\lambda\texttt{z}\rightarrow\texttt{(s (s z))) ((}\lambda\texttt{z}\rightarrow\texttt{(s (s z))) z))) z)}}\texttt{)} \end{array}$$

Natural numbers

$$\begin{array}{l}(\lambda \mathtt{s} \rightarrow \mathtt{z} \rightarrow \\ \quad \underline{\mathtt{((}\lambda \mathtt{z} \rightarrow \mathtt{((}\lambda \mathtt{z} \rightarrow \mathtt{(s\ (s\ z)))\ ((}\lambda \mathtt{z} \rightarrow \mathtt{(s\ (s\ z)))\ z)))\ z)}}\mathtt{)}\end{array}$$

$$(\lambda \mathtt{s} \rightarrow \mathtt{z} \rightarrow \mathtt{((}\lambda \mathtt{z} \rightarrow \mathtt{(s\ (s\ z)))\ ((}\lambda \mathtt{z} \rightarrow \mathtt{(s\ (s\ z)))\ z)))}$$

Natural numbers

```
(λs→z→((λz→(s (s z))) ((λz→(s (s z))) z)))
```

Natural numbers

$$(\lambda s \to z \to ((\lambda z \to (s\ (s\ z)))\ ((\lambda z \to (s\ (s\ z)))\ z)))$$

$$(\lambda s \to z \to ((\lambda z \to (s\ (s\ z)))\ \underline{((\lambda z \to (s\ (s\ z)))\ z)}))$$

Natural numbers

$$\boxed{\texttt{(}\lambda\texttt{s}\rightarrow\texttt{z}\rightarrow\texttt{((}\lambda\texttt{z}\rightarrow\texttt{(s (s z))) }\underline{\texttt{((}\lambda\texttt{z}\rightarrow\texttt{(s (s z))) z)}}\texttt{))}}$$

Natural numbers

$$\boxed{\texttt{(}\lambda\texttt{s}\rightarrow\texttt{z}\rightarrow\texttt{((}\lambda\texttt{z}\rightarrow\texttt{(s (s z))) }\underline{\texttt{((}\lambda\texttt{z}\rightarrow\texttt{(s (s z))) z)}}\texttt{))}}$$

$$\boxed{\texttt{(}\lambda\texttt{s}\rightarrow\texttt{z}\rightarrow\texttt{((}\lambda\texttt{z}\rightarrow\texttt{(s (s z))) (s (s z))))}}$$

Natural numbers

```
(λs→z→((λz→(s (s z))) (s (s z))))
```

Natural numbers

```
(λs→z→((λz→(s (s z))) (s (s z))))
```

```
(λs→z→((λz→(s (s z))) (s (s z))))
```

Natural numbers

```
(λs→z→((λz→(s (s z))) (s (s z))))
```

Natural numbers

$$(\lambda \mathtt{s}\to \mathtt{z}\to ((\lambda \mathtt{z}\to (\mathtt{s}\ (\mathtt{s}\ \mathtt{z})))\ (\mathtt{s}\ (\mathtt{s}\ \mathtt{z}))))$$

$$(\lambda \mathtt{s}\to \mathtt{z}\to \underline{((\lambda \mathtt{z}\to (\mathtt{s}\ (\mathtt{s}\ \mathtt{z})))\ (\mathtt{s}\ (\mathtt{s}\ \mathtt{z})))})$$

Natural numbers

$$\boxed{(\lambda \mathtt{s}\to \mathtt{z}\to \underline{((\lambda \mathtt{z}\to (\mathtt{s}\ \ (\mathtt{s}\ \ \mathtt{z})))\ \ (\mathtt{s}\ \ (\mathtt{s}\ \ \mathtt{z})))})}$$

Natural numbers

$$(\lambda s \to z \to \underline{((\lambda z \to (s\ (s\ z)))\ (s\ (s\ z)))})$$

$$(\lambda s \to z \to (s\ (s\ (s\ (s\ z)))))$$

Natural numbers

```
(λs→z→(s (s (s (s z)))))
```

Natural numbers

$$\texttt{(}\lambda\texttt{s}\to\texttt{z}\to\texttt{(s (s (s (s z)))))}$$

$$\underline{\texttt{(}\lambda\texttt{s}\to\texttt{z}\to\texttt{(s (s (s (s z)))))}}$$

Natural numbers

$\underline{(\lambda \mathtt{s} \rightarrow \mathtt{z} \rightarrow (\mathtt{s}\ (\mathtt{s}\ (\mathtt{s}\ (\mathtt{s}\ \mathtt{z})))))}$

Natural numbers

```
(λs→z→(s (s (s (s z)))))
```

```
4
```

Natural numbers

Zero checking

- We might wish to verify whether or not a number is zero
- We can simply pass the number parameters that fail the check (`s`) and pass it (`z`)

-
  ```
  (λm→((m (λx→FALSE)) TRUE))
  ```

Natural numbers

Zero checking

Let us try it out to $2 \times 2 \rightarrow_\beta 4$

Natural numbers

```
(2 = 0)
```

Natural numbers

$$\texttt{(2 = 0)}$$

$$\texttt{(}\underline{\texttt{0?}}\ \ \texttt{2)}$$

Natural numbers

$$\texttt{(}\underline{\texttt{0?}}\ \ \texttt{2)}$$

Natural numbers

```
(0? 2)
```

```
((λm→((m (λx→FALSE)) TRUE)) 2)
```

Natural numbers

```
((λm→((m (λx→FALSE)) TRUE)) 2)
```

Natural numbers

$$((\lambda \texttt{m} \rightarrow ((\texttt{m}\ \ (\lambda \texttt{x} \rightarrow \texttt{FALSE}))\ \ \texttt{TRUE}))\ \ \texttt{2})$$

$$((\lambda \texttt{m} \rightarrow ((\texttt{m}\ \ (\lambda \texttt{x} \rightarrow \texttt{FALSE}))\ \ \texttt{TRUE}))\ \ \underline{\texttt{2}})$$

Natural numbers

```
((λm→((m (λx→FALSE)) TRUE)) 2)
```

Natural numbers

```
((λm→((m (λx→FALSE)) TRUE)) 2)
```

```
((λm→((m (λx→FALSE)) TRUE)) (λs→z→(s (s z)))
   )
```

Natural numbers

```
((λm→((m (λx→FALSE)) TRUE)) (λs→z→(s (s z)))
  )
```

Natural numbers

```
((λm→((m (λx→FALSE)) TRUE)) (λs→z→(s (s z)))
    )
```

```
((λm→((m (λx→FALSE)) TRUE)) (λs→z→(s (s z))))
```

Natural numbers

$$\boxed{\underline{\texttt{((}\lambda\texttt{m}\rightarrow\texttt{((m (}\lambda\texttt{x}\rightarrow\texttt{FALSE)) TRUE)) (}\lambda\texttt{s}\rightarrow\texttt{z}\rightarrow\texttt{(s (s z))))}}}$$

Natural numbers

$$\boxed{\underline{\texttt{((}\lambda\texttt{m}\rightarrow\texttt{((m (}\lambda\texttt{x}\rightarrow\texttt{FALSE)) TRUE)) (}\lambda\texttt{s}\rightarrow\texttt{z}\rightarrow\texttt{(s (s z))))}}}$$

$$\boxed{\texttt{(((}\lambda\texttt{s}\rightarrow\texttt{z}\rightarrow\texttt{(s (s z))) (}\lambda\texttt{x}\rightarrow\texttt{FALSE)) TRUE)}}$$

Natural numbers

```
(((λs→z→(s (s z))) (λx→FALSE)) TRUE)
```

Natural numbers

$$\texttt{(((}\lambda\texttt{s}\rightarrow\texttt{z}\rightarrow\texttt{(s (s z))) (}\lambda\texttt{x}\rightarrow\texttt{FALSE)) TRUE)}$$

$$\texttt{(}\underline{\texttt{((}\lambda\texttt{s}\rightarrow\texttt{z}\rightarrow\texttt{(s (s z))) (}\lambda\texttt{x}\rightarrow\texttt{FALSE))}}\texttt{ TRUE)}$$

Natural numbers

$$\boxed{(\underline{((\lambda \mathtt{s} \rightarrow \mathtt{z} \rightarrow (\mathtt{s}\ (\mathtt{s}\ \mathtt{z})))\ (\lambda \mathtt{x} \rightarrow \mathtt{FALSE}))}\ \mathtt{TRUE})}$$

Natural numbers

$$\texttt{(}\underline{\texttt{((}\lambda\texttt{s}\rightarrow\texttt{z}\rightarrow\texttt{(s (s z))) (}\lambda\texttt{x}\rightarrow\texttt{FALSE))}}\texttt{ TRUE)}$$

$$\texttt{((}\lambda\texttt{z}\rightarrow\texttt{((}\lambda\texttt{x}\rightarrow\texttt{FALSE) ((}\lambda\texttt{x}\rightarrow\texttt{FALSE) z))) TRUE)}$$

Natural numbers

```
((λz→((λx→FALSE) ((λx→FALSE) z))) TRUE)
```

Natural numbers

$$((\lambda \texttt{z}\rightarrow((\lambda \texttt{x}\rightarrow\texttt{FALSE})\ ((\lambda \texttt{x}\rightarrow\texttt{FALSE})\ \texttt{z})))\ \texttt{TRUE})$$

$$((\lambda \texttt{z}\rightarrow((\lambda \texttt{x}\rightarrow\texttt{FALSE})\ ((\lambda \texttt{x}\rightarrow\texttt{FALSE})\ \texttt{z})))\ \underline{\texttt{TRUE}})$$

Natural numbers

$$\boxed{\texttt{((}\lambda\texttt{z}\rightarrow\texttt{((}\lambda\texttt{x}\rightarrow\texttt{FALSE) ((}\lambda\texttt{x}\rightarrow\texttt{FALSE) z))) }\underline{\texttt{TRUE}}\texttt{)}}$$

Natural numbers

$$\boxed{\texttt{((}\lambda\texttt{z}\rightarrow\texttt{((}\lambda\texttt{x}\rightarrow\texttt{FALSE) ((}\lambda\texttt{x}\rightarrow\texttt{FALSE) z))) }\underline{\texttt{TRUE}}\texttt{)}}$$

$$\boxed{\texttt{((}\lambda\texttt{z}\rightarrow\texttt{((}\lambda\texttt{x}\rightarrow\texttt{FALSE) ((}\lambda\texttt{x}\rightarrow\texttt{FALSE) z))) (}\lambda\texttt{t}\rightarrow\texttt{f}\rightarrow\texttt{t))}}$$

Natural numbers

```
((λz→((λx→FALSE) ((λx→FALSE) z))) (λt→f→t))
```

Natural numbers

$\texttt{((}\lambda\texttt{z}\rightarrow\texttt{((}\lambda\texttt{x}\rightarrow\texttt{FALSE) ((}\lambda\texttt{x}\rightarrow\texttt{FALSE) z))) (}\lambda\texttt{t}\rightarrow\texttt{f}\rightarrow\texttt{t))}$

$\underline{\texttt{((}\lambda\texttt{z}\rightarrow\texttt{((}\lambda\texttt{x}\rightarrow\texttt{FALSE) ((}\lambda\texttt{x}\rightarrow\texttt{FALSE) z))) (}\lambda\texttt{t}\rightarrow\texttt{f}\rightarrow\texttt{t))}}$

Natural numbers

$$\boxed{\underline{\texttt{((}\lambda\texttt{z}\rightarrow\texttt{((}\lambda\texttt{x}\rightarrow\texttt{FALSE) ((}\lambda\texttt{x}\rightarrow\texttt{FALSE) z))) (}\lambda\texttt{t}\rightarrow\texttt{f}\rightarrow\texttt{t))}}}}$$

Natural numbers

$$\underline{\texttt{((}\lambda\texttt{z}\rightarrow\texttt{((}\lambda\texttt{x}\rightarrow\texttt{FALSE) ((}\lambda\texttt{x}\rightarrow\texttt{FALSE) z))) (}\lambda\texttt{t}\rightarrow\texttt{f}\rightarrow\texttt{t))}}$$

$$\texttt{((}\lambda\texttt{x}\rightarrow\texttt{FALSE) ((}\lambda\texttt{x}\rightarrow\texttt{FALSE) (}\lambda\texttt{t}\rightarrow\texttt{f}\rightarrow\texttt{t)))}$$

Natural numbers

$$\boxed{((\lambda \texttt{x} \rightarrow \texttt{FALSE})\ ((\lambda \texttt{x} \rightarrow \texttt{FALSE})\ (\lambda \texttt{t} \rightarrow \texttt{f} \rightarrow \texttt{t})))}$$

Natural numbers

$$\boxed{\texttt{((}\lambda\texttt{x}\rightarrow\texttt{FALSE) ((}\lambda\texttt{x}\rightarrow\texttt{FALSE) (}\lambda\texttt{t}\rightarrow\texttt{f}\rightarrow\texttt{t)))}}$$

$$\boxed{\texttt{((}\lambda\texttt{x}\rightarrow\texttt{FALSE) }\underline{\texttt{((}\lambda\texttt{x}\rightarrow\texttt{FALSE) (}\lambda\texttt{t}\rightarrow\texttt{f}\rightarrow\texttt{t))}}\texttt{)}}$$

Natural numbers

$$\boxed{\mathtt{((}\lambda \mathtt{x}\rightarrow\mathtt{FALSE)}\ \underline{\mathtt{((}\lambda \mathtt{x}\rightarrow\mathtt{FALSE)}\ \mathtt{(}\lambda \mathtt{t}\rightarrow\mathtt{f}\rightarrow\mathtt{t)))}}\mathtt{)}}$$

Natural numbers

$$\texttt{((}\lambda\texttt{x}\rightarrow\texttt{FALSE)}\ \underline{\texttt{((}\lambda\texttt{x}\rightarrow\texttt{FALSE)}\ \texttt{(}\lambda\texttt{t}\rightarrow\texttt{f}\rightarrow\texttt{t))}}\texttt{)}$$

$$\texttt{((}\lambda\texttt{x}\rightarrow\texttt{FALSE)}\ \texttt{FALSE)}$$

Natural numbers

```
((λx→FALSE) FALSE)
```

Natural numbers

```
((λx→FALSE) FALSE)
```

```
((λx→FALSE) FALSE)
```

Natural numbers

```
((λx→FALSE) FALSE)
```

Natural numbers

$\texttt{((}\lambda\texttt{x}\rightarrow\texttt{FALSE) FALSE)}$

$\underline{\texttt{((}\lambda\texttt{x}\rightarrow\texttt{FALSE) FALSE)}}$

Natural numbers

$\underline{\texttt{((}\lambda\texttt{x}\rightarrow\texttt{FALSE) FALSE)}}$

Natural numbers

$\underline{((\lambda x \rightarrow \texttt{FALSE})\ \texttt{FALSE})}$

```
FALSE
```

Natural numbers

```
FALSE
```

Natural numbers

```
FALSE
```

```
FALSE
```

Natural numbers

```
FALSE
```

Natural numbers

<u>`FALSE`</u>

$(\lambda \mathtt{t} \rightarrow \mathtt{f} \rightarrow \mathtt{f})$

Natural numbers

$$\boxed{(\lambda \mathtt{t} \rightarrow \mathtt{f} \rightarrow \mathtt{f})}$$

Natural numbers

$(\lambda \texttt{t} \rightarrow \texttt{f} \rightarrow \texttt{f})$

$\underline{(\lambda \texttt{t} \rightarrow \texttt{f} \rightarrow \texttt{f})}$

Natural numbers

$\boxed{\underline{\texttt{(}\lambda\texttt{t}\rightarrow\texttt{f}\rightarrow\texttt{f)}}}$

Natural numbers

$\underline{\texttt{(}\lambda\texttt{t}\rightarrow\texttt{f}\rightarrow\texttt{f)}}$

`FALSE`

Natural numbers

Zero checking

Let us try it out to $2 \times 2 \rightarrow_{\beta} 4$

Natural numbers

```
(0  =  0)
```

Natural numbers

```
(0  =  0)
```

```
(0?  0)
```

Natural numbers

$(\underline{\texttt{0?}}\ \ \texttt{0})$

Natural numbers

```
(0? 0)
```

```
((λm→((m (λx→FALSE)) TRUE)) 0)
```

Natural numbers

```
((λm→((m (λx→FALSE)) TRUE)) 0)
```

Natural numbers

```
((λm→((m (λx→FALSE)) TRUE)) 0)
```

$$((\lambda m \rightarrow ((m\ (\lambda x \rightarrow FALSE))\ TRUE))\ \underline{0})$$

Natural numbers

$$\boxed{\texttt{((}\lambda\texttt{m}\rightarrow\texttt{((m }\ \texttt{(}\lambda\texttt{x}\rightarrow\texttt{FALSE))}\ \ \texttt{TRUE))}\ \ \underline{\texttt{0}}\texttt{)}}$$

Natural numbers

`((λm→((m (λx→FALSE)) TRUE)) `$\underline{\texttt{0}}$`)`

`((λm→((m (λx→FALSE)) TRUE)) (λs→z→z))`

Natural numbers

```
((λm→((m (λx→FALSE)) TRUE)) (λs→z→z))
```

Natural numbers

$$((\lambda \mathtt{m} \rightarrow ((\mathtt{m}\ \ (\lambda \mathtt{x} \rightarrow \mathtt{FALSE}))\ \ \mathtt{TRUE}))\ \ (\lambda \mathtt{s} \rightarrow \mathtt{z} \rightarrow \mathtt{z}))$$

$$\underline{((\lambda \mathtt{m} \rightarrow ((\mathtt{m}\ \ (\lambda \mathtt{x} \rightarrow \mathtt{FALSE}))\ \ \mathtt{TRUE}))\ \ (\lambda \mathtt{s} \rightarrow \mathtt{z} \rightarrow \mathtt{z}))}$$

Natural numbers

$$\boxed{\underline{\texttt{((}\lambda\texttt{m}\rightarrow\texttt{((m (}\lambda\texttt{x}\rightarrow\texttt{FALSE)) TRUE)) (}\lambda\texttt{s}\rightarrow\texttt{z}\rightarrow\texttt{z))}}}$$

Natural numbers

$$\boxed{\underline{\texttt{((}\lambda\texttt{m}\rightarrow\texttt{((m (}\lambda\texttt{x}\rightarrow\texttt{FALSE)) TRUE)) (}\lambda\texttt{s}\rightarrow\texttt{z}\rightarrow\texttt{z))}}}$$

$$\boxed{\texttt{(((}\lambda\texttt{s}\rightarrow\texttt{z}\rightarrow\texttt{z) (}\lambda\texttt{x}\rightarrow\texttt{FALSE)) TRUE)}}$$

Natural numbers

```
(((λs→z→z) (λx→FALSE)) TRUE)
```

Natural numbers

$$\texttt{(((}\lambda\texttt{s}\to\texttt{z}\to\texttt{z) (}\lambda\texttt{x}\to\texttt{FALSE)) TRUE)}$$

$$\texttt{(}\underline{\texttt{((}\lambda\texttt{s}\to\texttt{z}\to\texttt{z) (}\lambda\texttt{x}\to\texttt{FALSE))}}\texttt{ TRUE)}$$

Natural numbers

$$\boxed{(\underline{((\lambda \texttt{s} \rightarrow \texttt{z} \rightarrow \texttt{z})\ \ (\lambda \texttt{x} \rightarrow \texttt{FALSE}))}\ \ \texttt{TRUE})}$$

Natural numbers

$\texttt{(}\underline{\texttt{((}\lambda\texttt{s}\rightarrow\texttt{z}\rightarrow\texttt{z) (}\lambda\texttt{x}\rightarrow\texttt{FALSE))}}\texttt{ TRUE)}$

$\texttt{((}\lambda\texttt{z}\rightarrow\texttt{z) TRUE)}$

Natural numbers

```
((λz→z) TRUE)
```

Natural numbers

$$((\lambda \mathtt{z} \to \mathtt{z})\ \ \mathtt{TRUE})$$

$$((\lambda \mathtt{z} \to \mathtt{z})\ \ \underline{\mathtt{TRUE}})$$

Natural numbers

$$((\lambda \texttt{z} \rightarrow \texttt{z})\ \ \underline{\texttt{TRUE}})$$

Natural numbers

$((\lambda z \rightarrow z)\ \underline{\texttt{TRUE}})$

$((\lambda z \rightarrow z)\ (\lambda t \rightarrow f \rightarrow t))$

Natural numbers

```
((λz→z) (λt→f→t))
```

Natural numbers

$$((\lambda z \to z)\ (\lambda t \to f \to t))$$

$$\underline{((\lambda z \to z)\ (\lambda t \to f \to t))}$$

Natural numbers

$$\underline{\texttt{((}\lambda\texttt{z}\rightarrow\texttt{z)\ (}\lambda\texttt{t}\rightarrow\texttt{f}\rightarrow\texttt{t))}}$$

Natural numbers

$$\underline{((\lambda\mathtt{z}\rightarrow\mathtt{z})\ \ (\lambda\mathtt{t}\rightarrow\mathtt{f}\rightarrow\mathtt{t}))}$$

$$(\lambda\mathtt{t}\rightarrow\mathtt{f}\rightarrow\mathtt{t})$$

Natural numbers

```
(λt→f→t)
```

Natural numbers

$(\lambda \texttt{t} \rightarrow \texttt{f} \rightarrow \texttt{t})$

$\underline{(\lambda \texttt{t} \rightarrow \texttt{f} \rightarrow \texttt{t})}$

Natural numbers

$$\boxed{\underline{(\lambda \mathtt{t} \rightarrow \mathtt{f} \rightarrow \mathtt{t})}}$$

Natural numbers

```
(λt→f→t)
```

```
TRUE
```

Natural numbers

Other arithmetic operators

- Division, subtraction, and all manners of comparison operators can be defined similarly
- The level of detail of the specification can be compared to that of a very high level CPU
- This means that we are, to an extent, programming in a sort of assembly
- This is the reason why the traces have been so verbose so far

Natural numbers

Other arithmetic operators

- We could also define numbers in base two instead of base one
- This would save processing time, but would result in a slighter more complex specification
- We will just ignore these engineering details: we only focus on **what** can be done, not the best way to do it

Conclusion

Conclusion

Recap

- Lambda terms can be used to encode arbitrary basic data types
- The terms are always lambda expression which, when they get parameters passed in, identify themselves somehow
- Identification can be done by applying something (possibly even a given number of times), or returning one of the parameters

Conclusion

Recap

- There are many encodings of data types, but they all behave in the same way by producing the same outputs for the same inputs
- From now on we will start ignoring the reduction steps for simple terms such as 3+3
- We will instead focus on more complex data structures, such as tuples, discriminated unions, and even lists

This is it!

The best of luck, and thanks for the attention!