INFSEN02-1 Sample exam

The INFDEV@HR Team

1 Question 1

Given the following lambda program, and a series of relevant delta rules, show the beta reductions for this program.

```
let apply = (λx f→(f x)) in ((apply 3) (λx→(3 + x)))
```

1.1 Relevant delta rules

Integer addition

```
(λm n→ (λs z→((m s) ((n s) z))))
```

Integer three (3)

```
(λs z→(s (s (s z))))
```

Integer size (6)

```
(λs z→(s (s (s (s (s (s z)))))))
```

1.2 Answer 1 (note: you do not need to write all this detail yourself, it is only included for completeness)

```
let apply = (λx f→(f x)) in ((apply 3) (λx→(3 + x)))
```

```
let apply = (λx f→(f x)) in ((apply 3) (λx→(3 + x)))
```

```
((λapply→((apply 3) (λx→(3 + x)))) (λx f→(f x)))
```

```
((λapply→((apply 3) (λx→(3 + x)))) (λx f→(f x)))
```

```
((( (λx f→(f x))  3) (λx→(3 + x)))
```

```
(((λx f→(f x)) 3) (λx→(3 + x)))
```

```
(((λx f→(f x))  (λs z→(s (s (s z)))) ) (λx→(3 + x)))

(((λx f→(f x)) (λs z→(s (s (s z))))) (λx→(3 + x)))

((λf→(f  (λs z→(s (s (s z)))) )) (λx→(3 + x)))

((λf→(f (λs z→(s (s (s z)))))) (λx→(3 + x)))

( (λx→(3 + x))  (λs z→(s (s (s z)))))

((λx→(3 + x)) (λs z→(s (s (s z)))))

(3 +  (λs z→(s (s (s z))) )

((+ 3) (λs z→(s (s (s z)))))

(( (λm n→ (λs z→((m s) ((n s) z)))  3) (λs z→(s (s (s z)))))

(((λm n→ (λs z→((m s) ((n s) z)))) 3) (λs z→(s (s (s z)))))

(((λm n→ (λs z→((m s) ((n s) z))))  (λs z→(s (s (s z)))) ) (λs
    z→(s (s (s z)))))

((λm→ n→(λs z→((m s) ((n s) z))) (λs z→(s (s (s z))))) (λs z→(s
    (s (s z)))))

((λn s z→(( (λs z→(s (s (s z))))  s) ((n s) z))) (λs z→(s (s (s
    z)))))

(λn→ sz→(((λs z→(s (s (s z)))) s) ((n s) z)) (λs z→(s (s (s z)))))

(λs z→(((λs z→(s (s (s z)))) s) (( (λs z→(s (s (s z))))  s) z))
    )

(λs z→(((λs z→(s (s (s z)))) s) (((λs z→(s (s (s z)))) s) z)))

(λs z→((λz→( s  ( s  ( s  z)))) (((λs z→(s (s (s z)))) s) z)))

(λs z→((λz→(s (s (s z)))) (((λs z→(s (s (s z)))) s) z)))
```

2

```
(λs z→((λz→(s (s (s z)))) ((λz→(s (s (s z)))) z)))
```

```
(λs z→((λz→(s (s (s z)))) ((λz→(s (s (s z)))) z)))
```

```
(λs z→((λz→(s (s (s z)))) (s (s (s z)))))
```

```
(λs z→((λz→(s (s (s z)))) (s (s (s z)))))
```

```
(λs z→(s (s (s (s (s (s z)))))))
```

```
(λs z→(s (s (s (s (s (s z)))))))
```

```
6
```

2 Question 2

Given the following lambda calculus program, and a series of relevant delta rules, give the full typing derivation for the program.

```
(λ(m:Nat) (n:Nat)→Λα⇒ (λ(s:(α→α)) (z:α)→(((m α) s) (((n α)
    s) z))))
```

2.1 Relevant delta rules

Boolean type:

```
(∀α ⇒((α→α)→α→α))
```

2.2 Answer 2 (note: you do not need to write all this detail yourself, it is only included for completeness)

```
(λ(m:Nat) (n:Nat)→Λα⇒ (λ(s:(α→α)) (z:α)→(((m α) s) (((n α)
    s) z))))
```

```
λ(m:Nat)→ (n:Nat)→Λα⇒(λ(s:(α→α)) (z:α)→(((m α) s) (((n α) s) z)))
```

```
(λ(m:Nat) (n:Nat)→Λα⇒ (λ(s:(α→α)) (z:α)→(((Nat α) s) (((n
    α) s) z))))
```

```
(λ(m:Nat)(n:Nat)→Λα⇒
    (λ(s:(α→α)) (z:α)→(((Nat α) s) (((n α) s) z))))
```

3

$(\lambda(\texttt{m}:\texttt{Nat})\ (\texttt{n}:\texttt{Nat})\to\Lambda\alpha\Rightarrow\ (\lambda(\texttt{s}:(\alpha\to\alpha))\ (\texttt{z}:\alpha)\to(((\texttt{Nat}\ \alpha)\ \texttt{s})\ ((($ <mark>Nat</mark> $\alpha)\ \texttt{s})\ \texttt{z})))$

$(\lambda(\texttt{m}:\texttt{Nat})\ (\texttt{n}:\texttt{Nat})\to\Lambda\alpha\Rightarrow\ \underline{\lambda(\texttt{s}:(\alpha\to\alpha))\to}\ \underline{(\texttt{z}:\alpha)\to}$
$\underline{(((\texttt{Nat}\ \alpha)\ \texttt{s})\ (((\texttt{Nat}\ \alpha)\ \texttt{s})\ \texttt{z}))}$

$(\lambda(\texttt{m}:\texttt{Nat})\ (\texttt{n}:\texttt{Nat})\to\Lambda\alpha\Rightarrow\ (\lambda(\texttt{s}:(\alpha\to\alpha))\ (\texttt{z}:\alpha)\to(((\texttt{Nat}\ \alpha)\ $<mark>$(\alpha\to\alpha)$</mark>$)$
$(((\texttt{Nat}\ \alpha)\ $<mark>$(\alpha\to\alpha)$</mark>$)\ \texttt{z})))$

$(\lambda(\texttt{m}:\texttt{Nat})\ (\texttt{n}:\texttt{Nat})\to\Lambda\alpha\Rightarrow\ (\lambda(\texttt{s}:(\alpha\to\alpha))\underline{(\texttt{z}:\alpha)\to}(\underline{((\texttt{Nat}\ \alpha)\ (\alpha\to\alpha))}$
$\underline{(((\texttt{Nat}\ \alpha)\ (\alpha\to\alpha))\ \texttt{z})))$

$(\lambda(\texttt{m}:\texttt{Nat})\ (\texttt{n}:\texttt{Nat})\to\Lambda\alpha\Rightarrow\ (\lambda(\texttt{s}:(\alpha\to\alpha))\ (\texttt{z}:\alpha)\to(((\texttt{Nat}\ \alpha)\ (\alpha\to\alpha))$
$(((\texttt{Nat}\ \alpha)\ (\alpha\to\alpha))\ $<mark>$\alpha$</mark>$)))$

$(\lambda(\texttt{m}:\texttt{Nat})\ (\texttt{n}:\texttt{Nat})\to\Lambda\alpha\Rightarrow\ (\lambda(\texttt{s}:(\alpha\to\alpha))\ (\texttt{z}:\alpha)\to(((\underline{\texttt{Nat}}\ \alpha)\ (\alpha\to\alpha))$
$(((\texttt{Nat}\ \alpha)\ (\alpha\to\alpha))\ \alpha)))$

$(\lambda(\texttt{m}:\texttt{Nat})\ (\texttt{n}:\texttt{Nat})\to\Lambda\alpha\Rightarrow\ (\lambda(\texttt{s}:(\alpha\to\alpha))\ (\texttt{z}:\alpha)\to(((\ $
$\ $<mark>$(\forall\alpha\Rightarrow((\alpha\to\alpha)\to\alpha\to\alpha))$</mark>$\ \alpha)\ (\alpha\to\alpha))\ (((\texttt{Nat}\ \alpha)\ (\alpha\to\alpha))\ \alpha)))$

$(\lambda(\texttt{m}:\texttt{Nat})\ (\texttt{n}:\texttt{Nat})\to\Lambda\alpha\Rightarrow\ (\lambda(\texttt{s}:(\alpha\to\alpha))\ (\texttt{z}:\alpha)\to((\ $
$\underline{((\forall\alpha\Rightarrow((\alpha\to\alpha)\to\alpha\to\alpha))\ \alpha)}\ (\alpha\to\alpha))\ (((\texttt{Nat}\ \alpha)\ (\alpha\to\alpha))\ \alpha)))$

$(\lambda(\texttt{m}:\texttt{Nat})\ (\texttt{n}:\texttt{Nat})\to\Lambda\alpha\Rightarrow\ (\lambda(\texttt{s}:(\alpha\to\alpha))\ (\texttt{z}:\alpha)\to((\ $<mark>$((\alpha\to\alpha)\to\alpha\to\alpha)$</mark>$\ ($
$\alpha\to\alpha))\ (((\texttt{Nat}\ \alpha)\ (\alpha\to\alpha))\ \alpha)))$

$(\lambda(\texttt{m}:\texttt{Nat})\ (\texttt{n}:\texttt{Nat})\to\Lambda\alpha\Rightarrow\ (\lambda(\texttt{s}:(\alpha\to\alpha))\ (\texttt{z}:\alpha)\to(\ $
$\underline{(((\alpha\to\alpha)\to\alpha\to\alpha)\ (\alpha\to\alpha))}\ (((\texttt{Nat}\ \alpha)\ (\alpha\to\alpha))\ \alpha)))$

$(\lambda(\texttt{m}:\texttt{Nat})\ (\texttt{n}:\texttt{Nat})\to\Lambda\alpha\Rightarrow\ (\lambda(\texttt{s}:(\alpha\to\alpha))\ (\texttt{z}:\alpha)\to(\ $<mark>$(\alpha\to\alpha)$</mark>$\ (((\texttt{Nat}\ \alpha)$
$(\alpha\to\alpha))\ \alpha)))$

$(\lambda(\texttt{m}:\texttt{Nat})\ (\texttt{n}:\texttt{Nat})\to\Lambda\alpha\Rightarrow\ (\lambda(\texttt{s}:(\alpha\to\alpha))\ (\texttt{z}:\alpha)\to((\alpha\to\alpha)\ (((\underline{\texttt{Nat}}\ \alpha)$
$(\alpha\to\alpha))\ \alpha)))$

$(\lambda(\texttt{m}:\texttt{Nat})\ (\texttt{n}:\texttt{Nat})\to\Lambda\alpha\Rightarrow\ (\lambda(\texttt{s}:(\alpha\to\alpha))\ (\texttt{z}:\alpha)\to((\alpha\to\alpha)\ (((\ $
$\ $<mark>$(\forall\alpha\Rightarrow((\alpha\to\alpha)\to\alpha\to\alpha))$</mark>$\ \alpha)\ (\alpha\to\alpha))\ \alpha)))$

$(\lambda(\texttt{m}:\texttt{Nat})\ (\texttt{n}:\texttt{Nat})\to\Lambda\alpha\Rightarrow\ (\lambda(\texttt{s}:(\alpha\to\alpha))\ (\texttt{z}:\alpha)\to((\alpha\to\alpha)\ ((\ $
$\underline{((\forall\alpha\Rightarrow((\alpha\to\alpha)\to\alpha\to\alpha))\ \alpha)}\ (\alpha\to\alpha))\ \alpha)))$

$(\lambda(\texttt{m:Nat})\ (\texttt{n:Nat})\rightarrow\Lambda\alpha\Rightarrow\ (\lambda(\texttt{s}:(\alpha\rightarrow\alpha))\ (\texttt{z}:\alpha)\rightarrow((\alpha\rightarrow\alpha)\ ((\ \underline{((\alpha\rightarrow\alpha)\rightarrow\alpha\rightarrow\alpha)}\ (\alpha\rightarrow\alpha))\ \alpha))))$

$(\lambda(\texttt{m:Nat})\ (\texttt{n:Nat})\rightarrow\Lambda\alpha\Rightarrow\ (\lambda(\texttt{s}:(\alpha\rightarrow\alpha))\ (\texttt{z}:\alpha)\rightarrow((\alpha\rightarrow\alpha)\ (\ \underline{(((\alpha\rightarrow\alpha)\rightarrow\alpha\rightarrow\alpha)\ (\alpha\rightarrow\alpha))}\ \alpha))))$

$(\lambda(\texttt{m:Nat})\ (\texttt{n:Nat})\rightarrow\Lambda\alpha\Rightarrow\ (\lambda(\texttt{s}:(\alpha\rightarrow\alpha))\ (\texttt{z}:\alpha)\rightarrow((\alpha\rightarrow\alpha)\ (\ \underline{(\alpha\rightarrow\alpha)}\ \alpha))))$

$(\lambda(\texttt{m:Nat})\ (\texttt{n:Nat})\rightarrow\Lambda\alpha\Rightarrow\ (\lambda(\texttt{s}:(\alpha\rightarrow\alpha))\ (\texttt{z}:\alpha)\rightarrow((\alpha\rightarrow\alpha)\ \underline{((\alpha\rightarrow\alpha)\ \alpha))}))$

$(\lambda(\texttt{m:Nat})\ (\texttt{n:Nat})\rightarrow\Lambda\alpha\Rightarrow\ (\lambda(\texttt{s}:(\alpha\rightarrow\alpha))\ (\texttt{z}:\alpha)\rightarrow((\alpha\rightarrow\alpha)\ \underline{\alpha})))$

$(\lambda(\texttt{m:Nat})\ (\texttt{n:Nat})\rightarrow\Lambda\alpha\Rightarrow\ (\lambda(\texttt{s}:(\alpha\rightarrow\alpha))\ (\texttt{z}:\alpha)\rightarrow\underline{((\alpha\rightarrow\alpha)\ \alpha)}))$

$(\lambda(\texttt{m:Nat})\ (\texttt{n:Nat})\rightarrow\Lambda\alpha\Rightarrow\ (\lambda(\texttt{s}:(\alpha\rightarrow\alpha))\ (\texttt{z}:\alpha)\rightarrow\underline{\alpha}))$

$(\lambda(\texttt{m:Nat})\ (\texttt{n:Nat})\rightarrow\Lambda\alpha\Rightarrow\ (\lambda(\texttt{s}:(\alpha\rightarrow\alpha))\underline{(\texttt{z}:\alpha)\rightarrow\underline{\alpha}}))$

$(\lambda(\texttt{m:Nat})\ (\texttt{n:Nat})\rightarrow\Lambda\alpha\Rightarrow\ (\lambda(\texttt{s}:(\alpha\rightarrow\alpha))\rightarrow\ \underline{(\alpha\rightarrow\alpha)}))$

$(\lambda(\texttt{m:Nat})\ (\texttt{n:Nat})\rightarrow\Lambda\alpha\Rightarrow\ \underline{(\lambda(\texttt{s}:(\alpha\rightarrow\alpha))\rightarrow(\alpha\rightarrow\alpha))})$

$(\lambda(\texttt{m:Nat})\ (\texttt{n:Nat})\rightarrow\Lambda\alpha\Rightarrow\ \underline{((\alpha\rightarrow\alpha)\rightarrow\alpha\rightarrow\alpha)})$

$(\lambda(\texttt{m:Nat})\ (\texttt{n:Nat})\rightarrow\underline{\Lambda\alpha\Rightarrow((\alpha\rightarrow\alpha)\rightarrow\alpha\rightarrow\alpha)})$

$(\lambda(\texttt{m:Nat})\ (\texttt{n:Nat})\rightarrow\underline{(\forall\alpha\ \Rightarrow((\alpha\rightarrow\alpha)\rightarrow\alpha\rightarrow\alpha))})$

$(\lambda(\texttt{m:Nat})\ (\texttt{n:Nat})\rightarrow\underline{(\forall\alpha\ \Rightarrow((\alpha\rightarrow\alpha)\rightarrow\alpha\rightarrow\alpha))})$

$(\lambda(\texttt{m:Nat})\ (\texttt{n:Nat})\rightarrow\underline{\texttt{Nat}})$

$(\lambda(\texttt{m:Nat})\underline{(\texttt{n:Nat})\rightarrow\underline{\texttt{Nat}}})$

$(\lambda(\texttt{m:Nat})\rightarrow\ \underline{(\texttt{Nat}\rightarrow\texttt{Nat})})$

$\underline{(\lambda(\texttt{m:Nat})\rightarrow(\texttt{Nat}\rightarrow\texttt{Nat}))}$

$\underline{(\texttt{Nat}\rightarrow\texttt{Nat}\rightarrow\texttt{Nat})}$