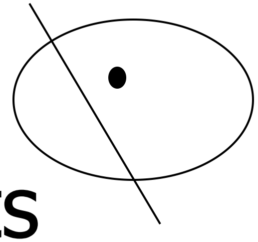
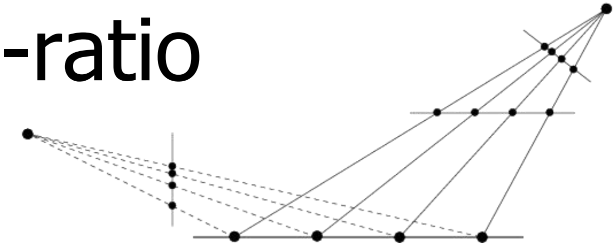
Projective 2D Geometry

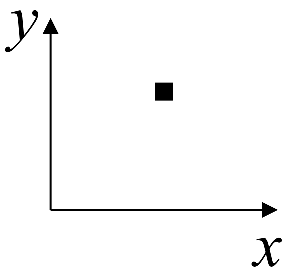
- Points, lines & conics
- Transformations & invariants



- 1D projective geometry and the Cross-ratio



Homogeneous(齐次) Coordinate



A point in Euclidean 2D-space

$$\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix} \longleftrightarrow \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \sim \begin{pmatrix} 2x \\ 2y \\ 2 \end{pmatrix} \sim \cdots \sim \begin{pmatrix} kx \\ ky \\ k \end{pmatrix}$$

Points are represented by ***equivalence classes*** of coordinate triples.

For the homogeneous coordinate $(x_1, x_2, x_3)^T$

, *corresponding inhomogeneous* coordinates is

$$\left(\frac{x_1}{x_3}, \frac{x_2}{x_3}\right)$$

Ideal point

$$\begin{pmatrix} x \\ y \\ 0 \end{pmatrix} \sim \begin{pmatrix} x/0 \\ y/0 \end{pmatrix}$$

Point at infinity (ideal point):

points represented by homogeneous coordinates in which the last coordinate is zero

2D projective space $\mathbf{P}^2$

Set of all equivalence classes of vectors in $\mathbf{R}^3-(0,0,0)^T$ forms 2D projective space $\mathbf{P}^2$

Homogeneous representation of lines

Homogeneous representation of lines:

$$ax + by + c = 0 \longrightarrow (a,b,c)^\mathsf{T}$$

$$(ka)x + (kb)y + kc = 0, \forall k \neq 0 \quad (a,b,c)^\mathsf{T} \sim k(a,b,c)^\mathsf{T}$$

equivalence class of vectors, any vector is representative

The point $\boldsymbol{x}$ lies on the line $\boldsymbol{l}$ if and only if

$$\mathbf{x}^T\mathbf{l} = \mathbf{l}^T\mathbf{x} = \mathbf{x}\cdot\mathbf{l} = 0$$

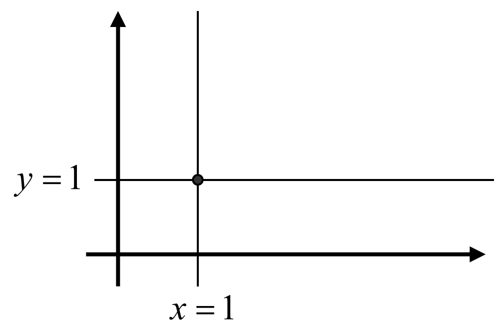
Point from lines and vice versa

Intersections of lines *l* and *l'*

The intersection of two lines *l* and *l'* is $x = l \times l'$

$$\mathbf{l} \cdot (\mathbf{l} \times \mathbf{l}') = \mathbf{l}' \cdot (\mathbf{l} \times \mathbf{l}') = 0$$

Example



Line joining two points

The line through two points x and x' is $l = x \times x'$

Ideal points and the line at infinity

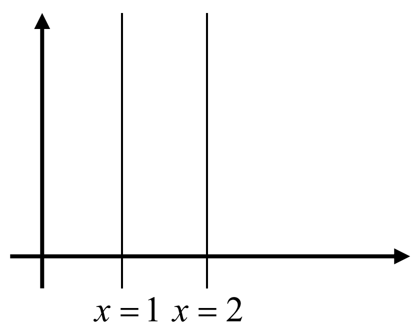
Intersections of parallel lines:

$\mathbf{l} = (a, b, c)^{\mathsf{T}}$ and $\mathbf{l}' = (a, b, c')^{\mathsf{T}}$ $\quad$ $\mathbf{l} \times \mathbf{l}' = (b, -a, 0)^{\mathsf{T}}$

$(b, -a)$ tangent vector

(a, b) normal direction

Example



$$\mathbf{x} = \mathbf{l} \times \mathbf{l}' = \begin{vmatrix} i & j & k \\ -1 & 0 & 1 \\ -1 & 0 & 2 \end{vmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

Ideal points $\quad (x_1, x_2, 0)^{\mathsf{T}}$

Line at infinity $\quad \mathbf{l}_\infty = (0, 0, 1)^{\mathsf{T}}$

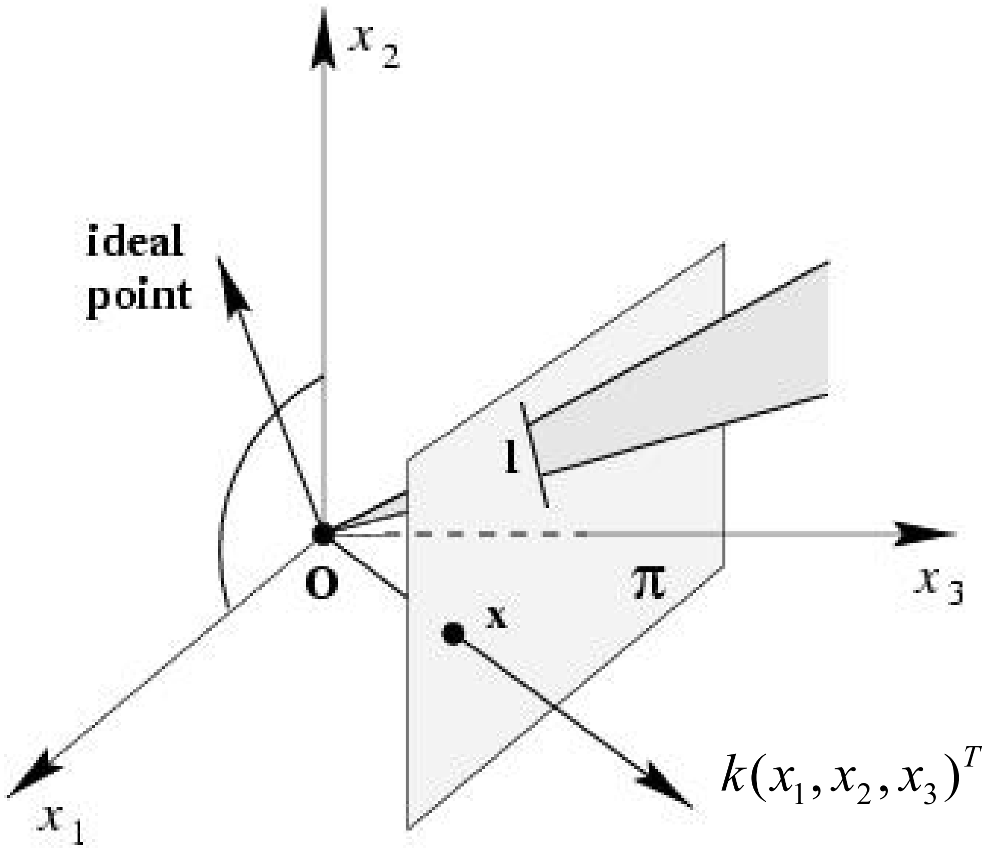
2D projective space

$$\mathbf{P}^2 = \mathbf{R}^2 \cup \mathbf{l}_\infty$$

Note that in **P**^{2} there is no distinction between ideal points and others

- Two distinct lines meet in a single point
- Two distinct points lie on a single line

A model for the projective plane



exactly one line through two points

exaclty one point at intersection of two lines

Duality (二元性)

$$\mathbf{x} \longleftrightarrow \mathbf{l}$$

$$\mathbf{x}^{\mathsf{T}}\mathbf{l} = 0 \longleftrightarrow \mathbf{l}^{\mathsf{T}}\mathbf{x} = 0$$

$$\mathbf{x} = \mathbf{l} \times \mathbf{l}' \longleftrightarrow \mathbf{l} = \mathbf{x} \times \mathbf{x}'$$

Duality principle:

To any theorem of 2-dimensional projective geometry there corresponds a dual theorem, which may be derived by interchanging the role of points and lines in the original theorem

Conics （圆锥曲线）

In Euclidean geometry conics include:

hyperbola (双曲线), ellipse, parabola(抛物线).

Conic: 2D curve described by 2nd-order equation.

$$ax^2 + bxy + cy^2 + dx + ey + f = 0$$

or *homogenized* $x \mapsto x_1/x_3,\ y \mapsto x_2/x_3$

$$a{x_1}^2 + bx_1x_2 + c{x_2}^2 + dx_1x_3 + ex_2x_3 + fx_3^2 = 0$$

or in matrix form

$$\mathrm{x}^{\mathsf{T}}\mathbf{C}\,\mathrm{x} = 0 \quad \text{with} \quad \mathbf{C} = \begin{bmatrix} a & b/2 & d/2 \\ b/2 & c & e/2 \\ d/2 & e/2 & f \end{bmatrix}$$

5DOF: $\{a : b : c : d : e : f\}$

Five points define a conic

For each point the conic passes through

$$ax_i^2 + bx_iy_i + cy_i^2 + dx_i + ey_i + f = 0$$

or

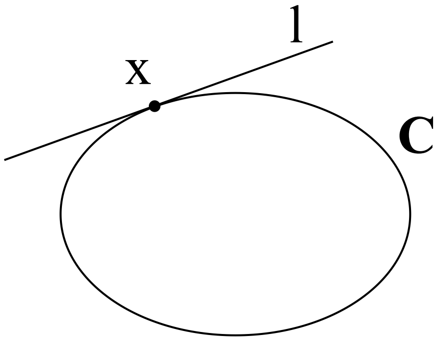
$$\left(x_i^2, x_iy_i, y_i^2, x_i, y_i, 1\right)\mathbf{c} = 0 \quad \mathbf{c} = \left(a, b, c, d, e, f\right)^\mathsf{T}$$

stacking constraints yields

$$\begin{bmatrix} x_1^2 & x_1y_1 & y_1^2 & x_1 & y_1 & 1 \\ x_2^2 & x_2y_2 & y_2^2 & x_2 & y_2 & 1 \\ x_3^2 & x_3y_3 & y_3^2 & x_3 & y_3 & 1 \\ x_4^2 & x_4y_4 & y_4^2 & x_4 & y_4 & 1 \\ x_5^2 & x_5y_5 & y_5^2 & x_5 & y_5 & 1 \end{bmatrix}\mathbf{c} = 0$$

Tangent lines to conics

The line $\boldsymbol{l}$ tangent to $\mathbf{C}$ at point $\boldsymbol{x}$ on $\mathbf{C}$ is given by $\boldsymbol{l}=\mathbf{C}\boldsymbol{x}$

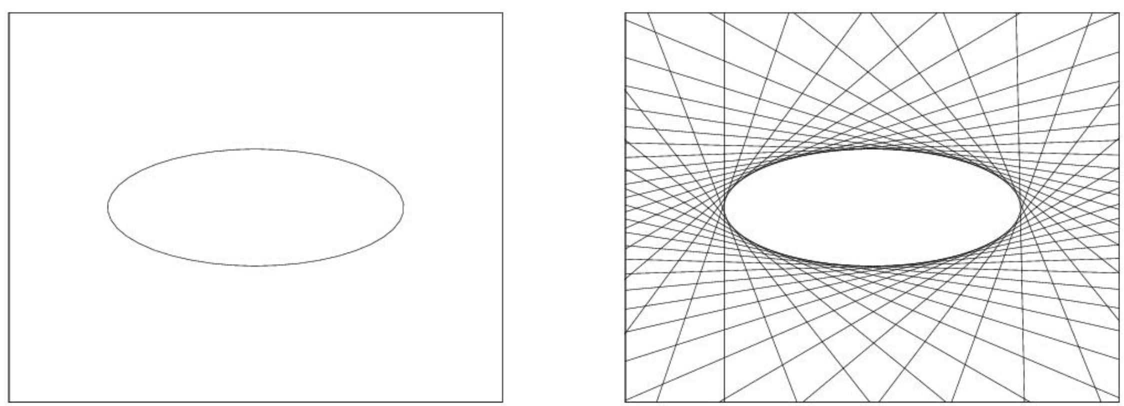


Dual conics

A line tangent to the conic $\mathbf{C}$ satisfies $\mathbf{l}^{\mathsf{T}}\mathbf{C}^{*}\mathbf{l}=0$

In general ($\mathbf{C}$ full rank): $\mathbf{C}^{*}=\mathbf{C}^{-1}$

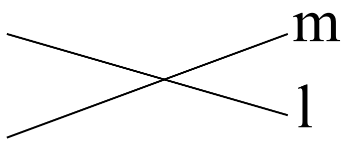
Dual conics = line conics = conic envelopes



Degenerate conics

A conic is degenerate if matrix $\mathbf{C}$ is not of full rank

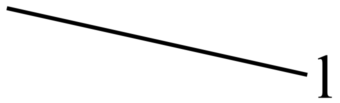
e.g. two lines (rank 2)



$$\mathbf{C} = \mathrm{l}\mathrm{m}^{\mathsf{T}} + \mathrm{m}\mathrm{l}^{\mathsf{T}}$$

e.g. repeated line (rank 1)

$$\mathbf{C} = \mathrm{l}\mathrm{l}^{\mathsf{T}}$$



Degenerate line conics: 2 points (rank 2), double point (rank1)

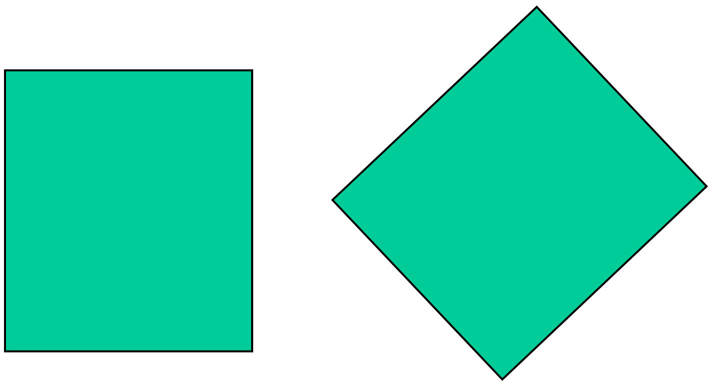
2D Projective transformations

- ***Euclidean*** transformation (3 DOF)
  - Rotation, translation
- ***Metric*** (***similarity***) transformation (4 DOF)
  - Rotation, translation, isotromic scaling
- ***Affine*** transformation (6 DOF)
- ***Projective*** transformation (8 DOF)

Class I: Euclidean

- Combination of rotation and translation

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & t_x \\ \sin\theta & \cos\theta & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$



$$\mathbf{p}' = \mathbf{H}_E \mathbf{p} = \begin{bmatrix} \mathbf{R} & \mathbf{t} \\ 0^\mathsf{T} & 1 \end{bmatrix} \mathbf{p} \qquad \mathbf{R}^\mathsf{T}\mathbf{R} = \mathbf{I}$$

3DOF (1 rotation, 2 translation)

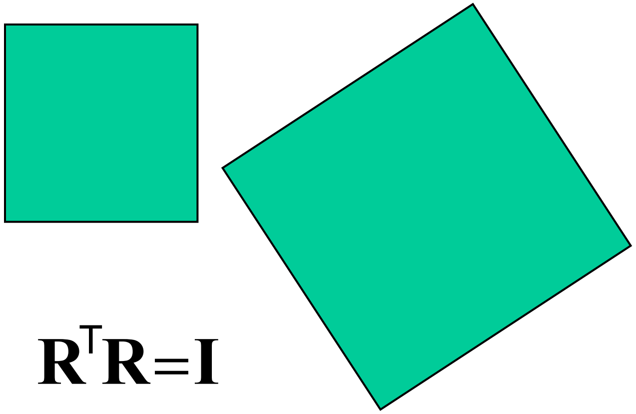
special cases: pure rotation, pure translation

Invariants: length, angle, area

Class II: Similarities

- Combination of rotation, translation and overall scalling

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{bmatrix} s\cos\theta & -s\sin\theta & t_x \\ s\sin\theta & s\cos\theta & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$



$$\mathrm{x}' = \mathbf{H}_S\, \mathrm{x} = \begin{bmatrix} s\mathbf{R} & \mathrm{t} \\ 0^{\mathsf{T}} & 1 \end{bmatrix} \mathrm{x}$$

$$\mathbf{R}^{\mathsf{T}}\mathbf{R} = \mathbf{I}$$

4DOF (1 scale, 1 rotation, 2 translation)

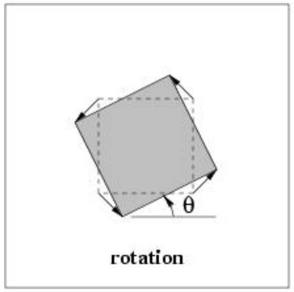
also know as *equi-form* (shape preserving)

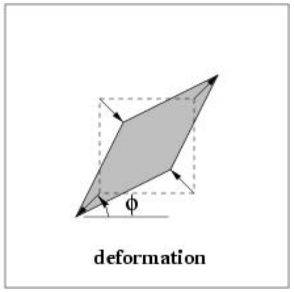
metric structure = structure up to similarity

Invariants: ratios of length, angle, ratios of areas, parallel lines

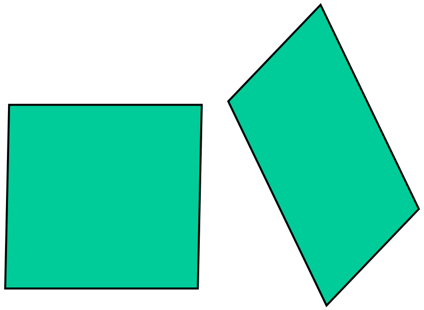
Class III: Affine transformations

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{bmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$





$$\mathbf{x}' = \mathbf{H}_A\,\mathbf{x} = \begin{bmatrix} \mathbf{A} & \mathbf{t} \\ \mathbf{0}^\mathsf{T} & 1 \end{bmatrix} \mathbf{x}$$



$$\mathbf{A} = \mathbf{R}(\theta)\mathbf{R}(-\phi)\mathbf{D}\mathbf{R}(\phi) \qquad \mathbf{D} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

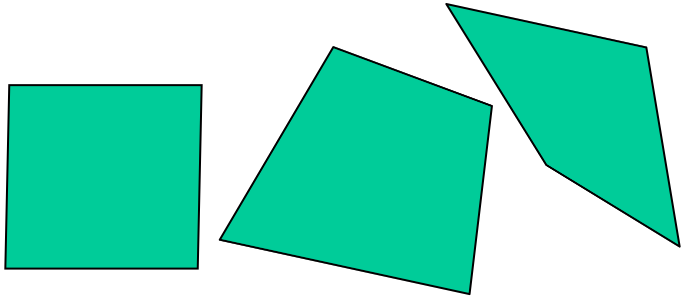
6DOF (2 scale, 2 rotation, 2 translation)

non-isotropic scaling!

Invariants: parallel lines, ratios of parallel lengths,
ratios of areas

Class VI: Projective transformations

$$\mathrm{x}' = \mathbf{H}_P\,\mathrm{x} = \begin{bmatrix} \mathbf{A} & \mathbf{t} \\ \mathrm{v}^{\mathsf{T}} & v \end{bmatrix} \mathrm{x}$$



$$\mathrm{v} = \left(v_1, v_2\right)^{\mathsf{T}}$$

8DOF (2 scale, 2 rotation, 2 translation, 2 line at infinity)

Action non-homogeneous over the plane

Invariants: cross-ratio of four points on a line (ratio of ratio)

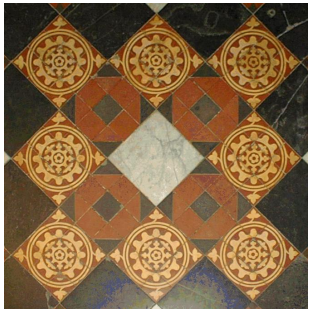
Actually any **3×3 non-singular matrix** represents a 2D projective transformation

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} \cong \mathbf{H}_{2D} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \cong \left[\begin{array}{cc|c} a & b & p \\ c & d & q \\ \hline 0 & 0 & s \end{array}\right] \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

A hierarchy of transformations

- Oriented Euclidean group (upper left 2x2 det 1)
- Similarity group (upper left 2x2 orthogonal)
- Affine group (last row (0,0,1))
- Projective linear group

Alternative, characterize transformation in terms of elements or quantities that are preserved or *invariant*



Action of affinities and projectivities on line at infinity

$$\begin{bmatrix} \mathbf{A} & \mathrm{t} \\ 0^{\mathsf{T}} & v \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix} = \begin{pmatrix} \mathbf{A} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ 0 \end{pmatrix}$$

Line at infinity stays at infinity, but points move along the line

$$\begin{bmatrix} \mathbf{A} & \mathrm{t} \\ \mathrm{v}^{\mathsf{T}} & v \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix} = \begin{pmatrix} \mathbf{A} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ v_1 x_1 + v_2 x_2 \end{pmatrix}$$

Line at infinity becomes finite, allows to observe vanishing points, horizon,

Projective transformations

<u>Definition:</u>

A ***projectivity*** is an invertible mapping h from P^2 to itself such that three points x_1,x_2,x_3 lie on the same line *if and only if* $h(x_1),h(x_2),h(x_3)$ do.

projectivity=collineation

Projective transformations

<u>Theorem:</u>

A mapping $h:\mathrm{P}^2 \rightarrow \mathrm{P}^2$ is a projectivity *if and only if* there exist a non-singular 3x3 matrix $\mathbf{H}$ such that for any point in P^2 represented by a vector x it is true that $h(\mathrm{x})=\mathbf{H}\mathrm{x}$

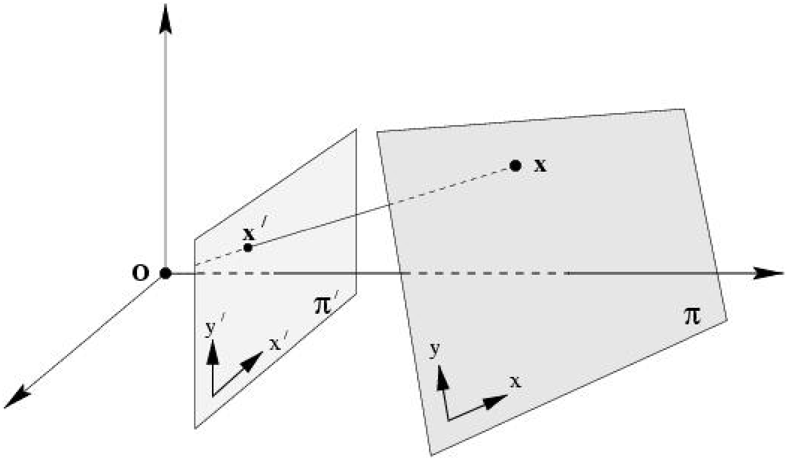
<u>Definition:</u> Projective transformation

$$\begin{pmatrix} x'_1 \\ x'_2 \\ x'_3 \end{pmatrix} = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad \text{or} \quad \mathrm{x}' = \mathbf{H}\,\mathrm{x}$$

8DOF

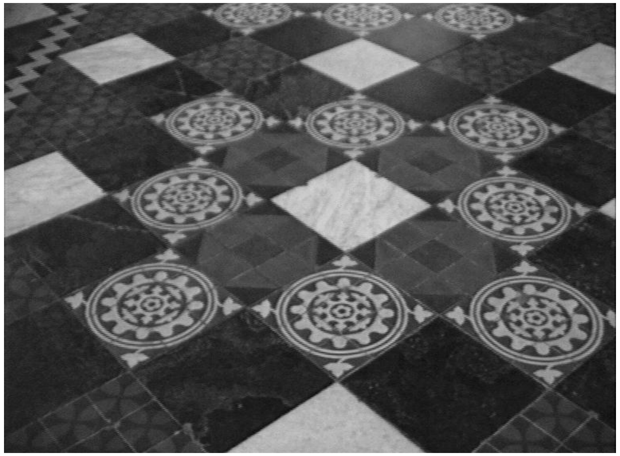
projectivity=collineation=projective transformation =homography

Mapping between planes



central projection may be expressed by x’=Hx

(application of theorem)



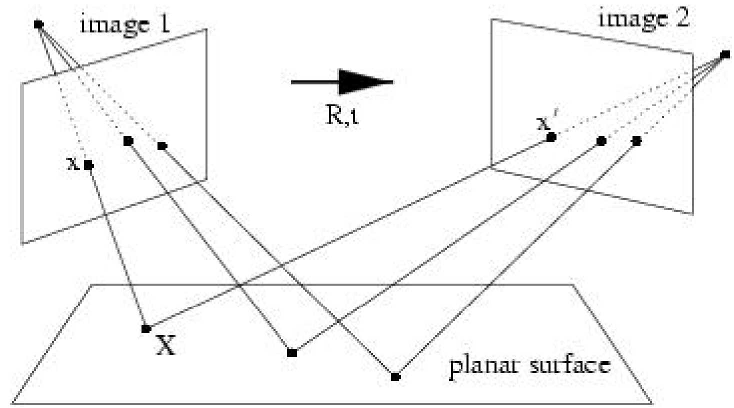
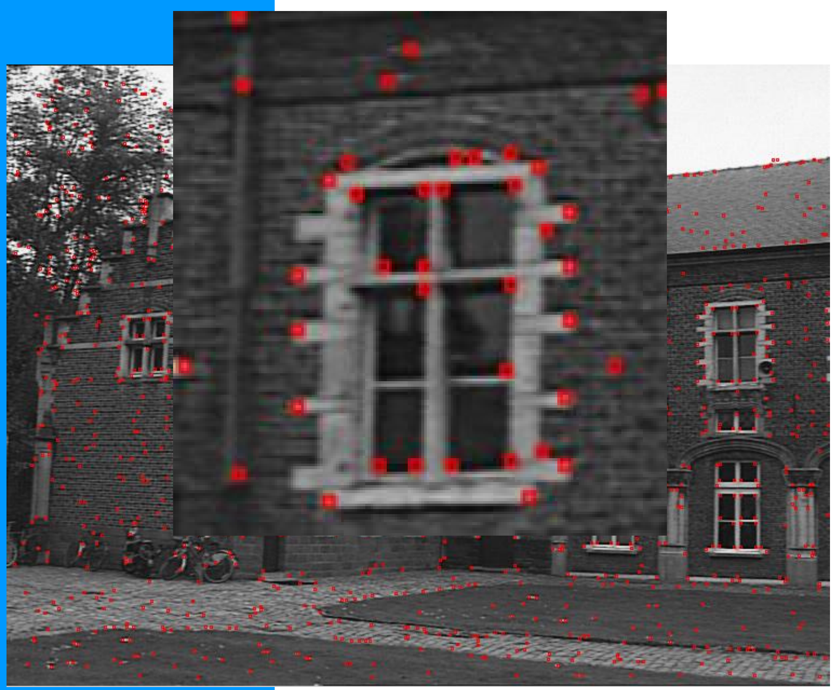
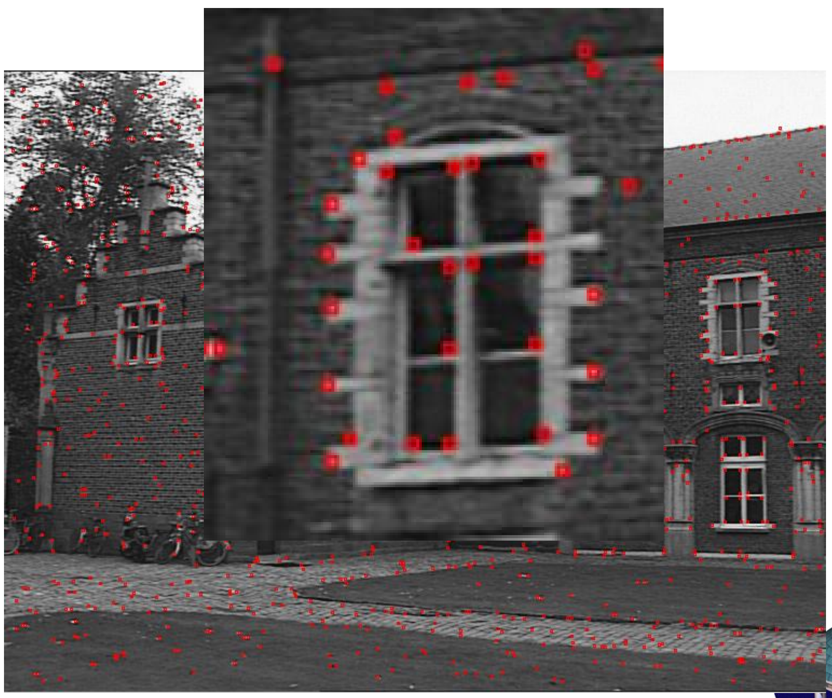
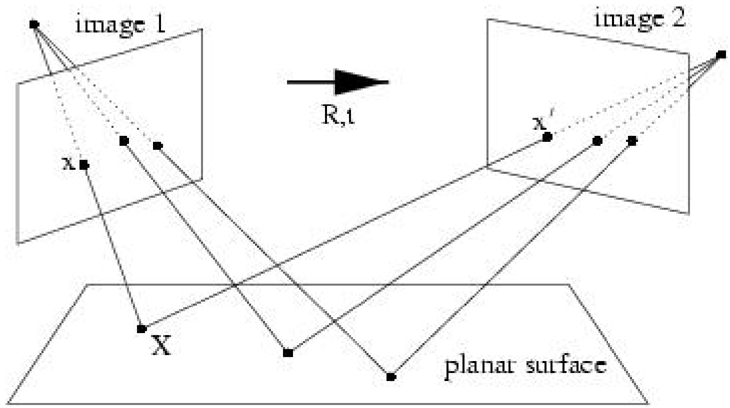


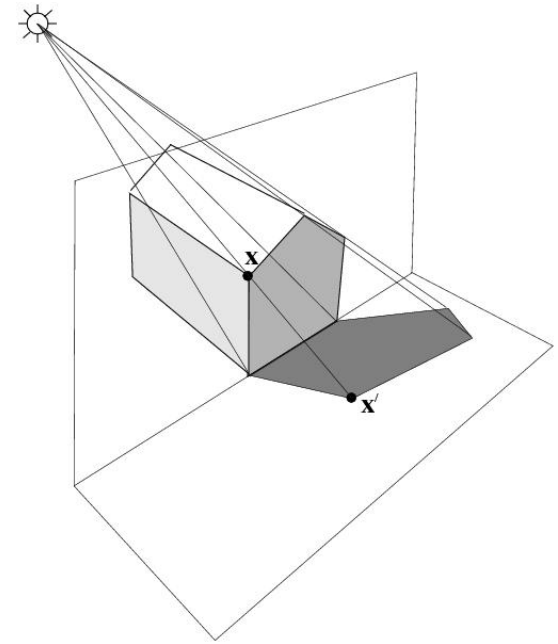
image 1
image 2
R,t
x
x'
X
planar surface

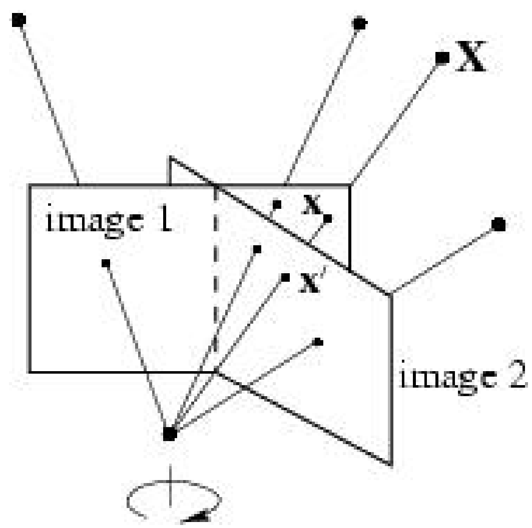




More examples







Removing projective distortion

select four points in a plane with know coordinates

$$x' = \frac{x'_1}{x'_3} = \frac{h_{11}x + h_{12}y + h_{13}}{h_{31}x + h_{32}y + h_{33}} \qquad y' = \frac{x'_2}{x'_3} = \frac{h_{21}x + h_{22}y + h_{23}}{h_{31}x + h_{32}y + h_{33}}$$

$$x'\left(h_{31}x + h_{32}y + h_{33}\right) = h_{11}x + h_{12}y + h_{13}$$
$$y'\left(h_{31}x + h_{32}y + h_{33}\right) = h_{21}x + h_{22}y + h_{23}$$

(linear in h_{ij})

(2 constraints/point, 8DOF $\Rightarrow$ 4 points needed)

Remark: no calibration at all necessary,
better ways to compute (see later)

Transformation of lines and conics

For a point transformation

$$\mathbf{x}' = \mathbf{H}\,\mathbf{x}$$

Transformation for lines

$$\mathbf{l}' = \mathbf{H}^{-\mathsf{T}}\,\mathbf{l}$$

Transformation for conics

$$\mathbf{C}' = \mathbf{H}^{-\mathsf{T}}\mathbf{C}\mathbf{H}^{-1}$$

Transformation for dual conics

$$\mathbf{C}'^{*} = \mathbf{H}\mathbf{C}^{*}\mathbf{H}^{\mathsf{T}}$$

Decomposition of projective transformations

$$\mathbf{H} = \mathbf{H}_S \mathbf{H}_A \mathbf{H}_P = \begin{bmatrix} s\mathbf{R} & \mathrm{t} \\ 0^\mathsf{T} & 1 \end{bmatrix} \begin{bmatrix} \mathbf{K} & 0 \\ 0^\mathsf{T} & 1 \end{bmatrix} \begin{bmatrix} \mathbf{I} & 0 \\ \mathrm{v}^\mathsf{T} & v \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathrm{t} \\ \mathrm{v}^\mathsf{T} & v \end{bmatrix}$$

$$\mathbf{A} = s\mathbf{R}\mathbf{K} + \mathrm{t}\mathrm{v}^\mathsf{T}$$

decomposition unique (if chosen s>0)

$\mathbf{K}$ upper-triangular, $\det \mathbf{K} = 1$

Example:

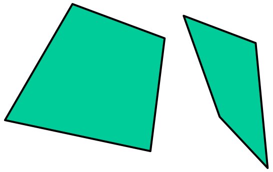
$$\mathbf{H} = \begin{bmatrix} 1.707 & 0.586 & 1.0 \\ 2.707 & 8.242 & 2.0 \\ 1.0 & 2.0 & 1.0 \end{bmatrix}$$

$$\mathbf{H} = \begin{bmatrix} 2\cos 45^\circ & -2\sin 45^\circ & 1.0 \\ 2\sin 45^\circ & 2\cos 45^\circ & 2.0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0.5 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix}$$

Overview transformations

Projective
8dof

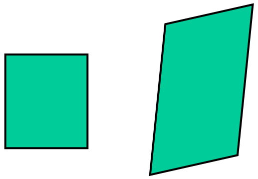
$$\begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix}$$



Concurrency, collinearity, order of contact (intersection, tangency, inflection, etc.), cross ratio

Affine
6dof

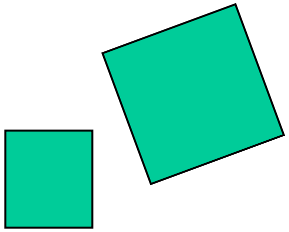
$$\begin{bmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$$



Parallellism, ratio of areas, ratio of lengths on parallel lines (e.g midpoints), linear combinations of vectors (centroids).
The line at infinity $\mathbf{l}_\infty$

Similarity
4dof

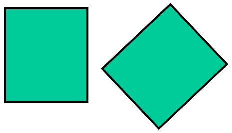
$$\begin{bmatrix} sr_{11} & sr_{12} & t_x \\ sr_{21} & sr_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$$



Ratios of lengths, angles.
The circular points I,J

Euclidean
3dof

$$\begin{bmatrix} r_{11} & r_{12} & t_x \\ r_{21} & r_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$$



lengths, areas.