Exercises

Part one： Choice questions

1. The order of the differential equation $\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^3+3ye^x=2x$ is (B)

A. 1　　B. 2　　C. 3　　D. 0

2. Given $f(x,y)=\begin{cases}\frac{xy}{\sqrt{x^2+y^2}}, & x^2+y^2\neq 0\\ 0, & x^2+y^2=0\end{cases}$, then at $(0,0)$, $f(x,y)$ is (B)

A. continuous and differentiable.　　B. continuous and non-differentiable.
C. discontinuous and differentiable.　　D. discontinuous and non-differentiable.

3. Given $I_i=\iint\limits_{D_i}e^{-(x^2+y^2)}dxdy,\ i=1,\ 2,\ 3,$ where $D_1=\{(x,y)\,|\,x^2+y^2\le R^2\},$

$D_2=\{(x,y)\,|\,x^2+y^2\le 2R^2\},\quad D_3=\{(x,y)\,\|\,x|+|\,y\,|\le 2R\}$, then（ A ）

A. $I_3>I_2>I_1$　　B. $I_1>I_2>I_3$　　C. $I_1>I_3>I_2$　　D. $I_3>I_1>I_2$

4. If $z=f(ax^2+by^2)$，where f is differentiable, a, b are constants，then z satisfies the following equation（ C ）

A. $ax\frac{\partial z}{\partial x}+by\frac{\partial z}{\partial y}=0$　　B. $ax\frac{\partial z}{\partial x}-by\frac{\partial z}{\partial y}=0$

C. $by\frac{\partial z}{\partial x}-ax\frac{\partial z}{\partial y}=0$　　D. $by\frac{\partial z}{\partial x}+ax\frac{\partial z}{\partial y}=0$

5. If $u(x,y)$ has continuous second order partial derivatives on the closed bounded region D, and satisfies $\frac{\partial^2 u}{\partial x\partial y}\neq 0$ and $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$, then (B)

A. both the absolute maximum and minimum occur in the interior of D.

B. both the absolute maximum and minimum occur on the boundary of D.

C. the absolute maximum occurs in the interior of D, and the absolute minimum occurs on the boundary of D

D. the absolute minimum occurs in the interior of D, and the absolute maximum occurs on the boundary of D

6. If $I_1 = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \ln(\sin x)dx$, $I_2 = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \ln(\cos x)dx$, then (A).

A. $I_2 < I_1 < 0$; B. $I_1 < I_2 < 0$; C. $0 < I_1 < I_2$; D. $0 < I_2 < I_1$

7. At $(0,0)$, $f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$ is (C).

A. continuous and the partial derivatives exist;

B. continuous and the partial derivatives don't exist;

C. Discontinuous but the partial derivatives exist;

D. discontinuous and the partial derivatives don't exist.

8. If $y_1(x), y_2(x), y_3(x)$ are linearly independent and they are the solutions of 2nd order ODE $y'' + P(x)y' + Q(x)y = f(x)$, C_1, C_2 are arbitrary constants, then the general solution of this differential equation is (D).

A. $C_1y_1 + C_2y_2 + y_3$ B. $C_1y_1 + C_2y_2 - (C_1 + C_2)y_3$

C. $C_1y_1 + C_2y_2 - (1 - C_1 - C_2)y_3$ D. $C_1y_1 + C_2y_2 + (1 - C_1 - C_2)y_3$

9. If $I_1 = \iint\limits_D \cos\sqrt{x^2+y^2}\,d\sigma, I_2 = \iint\limits_D \cos(x^2+y^2)\,d\sigma, I_3 = \iint\limits_D \cos(x^2+y^2)^2\,d\sigma$, where $D = \{(x,y) \mid x^2+y^2 \le 1\}$, then (A).

A. $I_3 > I_2 > I_1$ B. $I_1 > I_2 > I_3$ C. $I_2 > I_1 > I_3$ D. $I_3 > I_1 > I_2$

10. Considering the following properties of $f(x,y)$:

① $f(x,y)$ is continuous at (x_0, y_0),

②The partial derivatives of $f(x,y)$ is continuous at (x_0, y_0),

③ $f(x,y)$ is differentialble at (x_0, y_0),

④ The partial derivatives of $f(x,y)$ exist at (x_0, y_0),

The notation "$P \Rightarrow Q$" represents that property Q can be obtained by property P, then (A).

A. ②⇒③⇒① B. ③⇒②⇒① C. ③⇒④⇒① D. ③⇒①⇒④

11. If $f(x,y)$ is continuous on a neighborhood of (0,0),and

$$\lim_{(x,y)\to(0,0)} \frac{f(x,y)-xy}{(x^2+y^2)^2}=1$$

then the correct one of the following four statements is（ A ）

A. $f(0,0)$ is neither a local maximum nor a local minimum;

B. $f(0,0)$ is a local maximum;

C. $f(0,0)$ is a local minimum;

D. Unable to determine whether $f(0,0)$ is a local maximum or a local minimum from the given information;

12. If $y=\frac{1}{2}e^{2x}+(x-\frac{1}{3})e^{x}$ is a particular solution of 2nd order constant coefficients linear differential equation $y''+ay'+by=ce^{x}$, then (A).

A. $a=-3,\ b=2,\ c=-1$ B. $a=3,\ b=2,\ c=-1$

C. $a=-3,\ b=2,\ c=1$ D. $a=3,\ b=2,\ c=1$

13. If the partial derivatives of $f(x,y)$ at (x_0,y_0) exist，then (C).

A. $f(x,y)$ is continuous at (x_0,y_0); B. $f(x,y)$ is differentiable at (x_0,y_0);

C. Both $\lim\limits_{x\to x_0} f(x,y_0),\ \lim\limits_{y\to y_0} f(x_0,y)$ exist D. $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y)$ exists

14. If the curve $y=f(x)$ passes through the origin，and the normal line at the origin is perpendicular to the line $y-3x=5$, the function $y=f(x)$ satisfies $y''+y'-2y=0$，then $f(x)=$（ B ）

A. $e^{-2x}-e^{x}$， B. $e^{x}-e^{-2x}$ C. $\frac{1}{9}e^{-2x}-\frac{1}{9}e^{x}$ D. $\frac{1}{9}e^{x}-\frac{1}{9}e^{-2x}$

15. If $I_1=\iint\limits_D yx^3 d\sigma, I_2=\iint\limits_D y^2x^3 d\sigma, I_3=\iint\limits_D \sqrt{y}x^3 d\sigma$， where D is a closed region in the second quadrant，and $0<y<1$，then (C).

A. $I_3>I_2>I_1$ B. $I_1>I_2>I_3$ C. $I_2>I_1>I_3$ D. $I_3>I_1>I_2$

16. The particular solution of $y''-2y'+10y=e^{x}\cos 3x$ is in the form of (D)

A. $Ae^{x}\cos 3x$, B. $e^{x}(A\cos 3x+B\sin 3x)$, C. $Axe^{x}\cos 3x$, D. $e^{x}(Ax\cos 3x+Bx\sin 3x)$

17. For $f(x,y)=x|x|+|y|$, then (A)

A. $f_x(0,0)$ exists, but $f_y(0,0)$ doesn't exist;

B. $f_x(0,0)$ doesn't exist, but $f_y(0,0)$ exists;

C. Both $f_x(0,0)$ and $f_y(0,0)$ exist;

D. Neither $f_x(0,0)$ nor $f_y(0,0)$ exists.

18. For the following statements, (B) is right.

A. If the function $f(x,y)$ is continuous at $P(x_0, y_0)$, then its partial derivatives exist at $P(x_0, y_0)$;

B. If the function $f(x,y)$ is differentiable at $P(x_0, y_0)$, then it is continuous at $P(x_0, y_0)$;

C. If the mixed second partial derivatives $f_{xy}(x,y), f_{yx}(x,y)$ exist at $P(x_0, y_0)$, then $f_{xy}(x_0, y_0)=f_{yx}(x_0, y_0)$;

D. If the partial derivatives $f_x(x,y), f_y(x,y)$ exist at $P(x_0, y_0)$, then $f(x,y)$ is differentiable at $P(x_0, y_0)$.

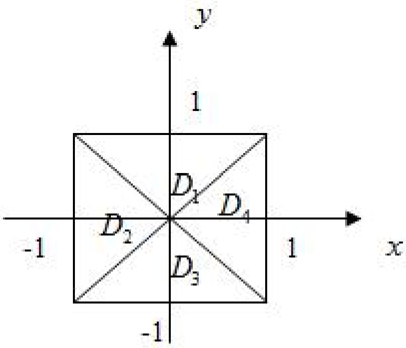
19. If $D=\left\{(x,y)\,|\,|x|+|y|\le\frac{\pi}{2}\right\}$,and $I_1=\iint_D\sqrt{x^2+y^2}\,dxdy$, $I_2=\iint_D\sin\sqrt{x^2+y^2}\,dxdy$

$I_3=\iint_D\left(1-\cos\sqrt{x^2+y^2}\right)dxdy$, then (A)

A. $I_3<I_2<I_1$; B. $I_1<I_2<I_3$; C. $I_2<I_1<I_3$; D. $I_2<I_3<I_1$

20. Like as the right hand, The square $\{(x,y)\,|\,|x|\le 1,|y|\le 1\}$ is partitioned into $D_k\,(k=1,2,3,4)$ by the diagonals , $I_k=\iint\limits_{D_k} y\cos x\,dxdy$, then $\max\limits_{1\le k\le 4}\{I_k\}=$ (A)

A. I_1 B. I_2 C. I_3 D. I_4



21. If $z=z(x,y)$ is defined by $F(\frac{y}{x},\frac{z}{x})=0$, where F is a differentiable function, and $F_2'\neq 0$, then $x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=$ (B)

A. x　　B. z　　C. $-x$　　D. $-z$

22. $\lim\limits_{x\to\infty}\sum\limits_{i=1}^{n}\sum\limits_{j=1}^{n}\frac{n}{(n+i)(n^2+j^2)}=$ (D)

A. $\int_0^1 dx\int_0^x \frac{1}{(1+x)(1+y^2)}dy$　　B. $\int_0^1 dx\int_0^x \frac{1}{(1+x)(1+y)}dy$

C. $\int_0^1 dx\int_0^1 \frac{1}{(1+x)(1+y)}dy$　　D. $\int_0^1 dx\int_0^1 \frac{1}{(1+x)(1+y^2)}dy$

23. If $f(x,y)$ is continuous at $(0,0)$, then the correct one of the following statements is （ B ）

A. If $\lim\limits_{\substack{x\to 0\\ y\to 0}}\frac{f(x,y)}{|x|+|y|}$ exists，then $f(x,y)$ is differentiable at $(0,0)$;

B. If $\lim\limits_{\substack{x\to 0\\ y\to 0}}\frac{f(x,y)}{x^2+y^2}$ exists，then $f(x,y)$ is differentiable at $(0,0)$;

C. If $f(x,y)$ is differentiable at $(0,0)$, then $\lim\limits_{\substack{x\to 0\\ y\to 0}}\frac{f(x,y)}{|x|+|y|}$ exists;

D. If $f(x,y)$ is differentiable at $(0,0)$, then $\lim\limits_{\substack{x\to 0\\ y\to 0}}\frac{f(x,y)}{x^2+y^2}$ exists;

24. For the differential equation $y''-4y'+8y=e^{2x}(1+\cos 2x)$, the form of particular solution y^* is (C)

A. $Ae^{2x}+e^{2x}(B\cos 2x+C\sin 2x)$ B. $Axe^{2x}+e^{2x}(B\cos 2x+C\sin 2x)$

C. $Ae^{2x}+xe^{2x}(B\cos 2x+C\sin 2x)$ D. $Axe^{2x}+xe^{2x}(B\cos 2x+C\sin 2x)$

25. If $f(x,y)$ has partial derivatives, $\dfrac{\partial f(x,y)}{\partial x}>0, \dfrac{\partial f(x,y)}{\partial y}<0$ at any point (x, y), then (D)

A. $f(0,0)>f(1,1)$; B. $f(0,0)<f(1,1)$; C. $f(0,1)>f(1,0)$; D. $f(0,1)<f(1,0)$

26. If $I_k=\int_0^{k\pi} e^{x^2}\sin x\mathrm{d}x, (k=1,2,3)$, then (D)

A. $I_1<I_2<I_3$ B. $I_3<I_2<I_1$ C. $I_2<I_3<I_1$ D. $I_2<I_1<I_3$

27. If $f(x,y)$ is continuous, then $\int_{\frac{\pi}{2}}^{\pi}dx\int_{\sin x}^{1}f(x,y)dy$ =(B).

A. $\int_0^1 dy\int_{\pi+\arcsin y}^{\pi}f(x,y)dy$; B. $\int_0^1 dy\int_{\pi-\arcsin y}^{\pi}f(x,y)dy$

C. $\int_0^1 dy\int_{\frac{\pi}{2}}^{\pi+\arcsin y}f(x,y)dy$; D. $\int_0^1 dy\int_{\frac{\pi}{2}}^{\pi-\arcsin y}f(x,y)dy$

28. If $f(x)$ is a continuous function, $F(t)=\int_1^t dy\int_y^t f(x)dx$, then $F'(2)=$ (A)

A. $2f(2)$ B. $f(2)$ C. $-f(2)$ D. 0

29. If $D=\{(x,y)\big|x^2+y^2\le 4, x\ge 0, y\ge 0\}$, $f(x)$ is a positive continuous function on D, a, b are constants, then $\iint\limits_D \dfrac{a\sqrt{f(x)}+b\sqrt{f(y)}}{\sqrt{f(x)}+\sqrt{f(y)}}d\sigma=$ (D)

$ab\pi$. B. $\dfrac{ab}{2}\pi$. C. $(a+b)\pi$. D. $\dfrac{a+b}{2}\pi$.

30. If $I_1=\int_0^{\frac{\pi}{4}}\dfrac{\tan x}{x}dx, I_2=\int_0^{\frac{\pi}{4}}\dfrac{x}{\tan x}dx$, then (B)

A. $I_1>I_2>1$. B. $1>I_1>I_2$. C. $I_2>I_1>1$. D. $1>I_2>I_1$.

31. If $f(x,y)$ is a continuous function, $\int_0^1 dy\int_{-\sqrt{1-y^2}}^{1-y}f(x,y)dx=$ (D)

A. $\int_0^1 dx\int_1^{x-1}f(x,y)dy+\int_{-1}^0 dx\int_0^{\sqrt{1-x^2}}f(x,y)dy$

B. $\int_0^1 dx\int_1^{1-x}f(x,y)dy+\int_{-1}^0 dx\int_{-\sqrt{1-x^2}}^{0}f(x,y)dy$

C. $\int_0^{\frac{\pi}{2}} d\theta \int_0^{\frac{1}{\cos\theta+\sin\theta}} f(r\cos\theta, r\sin\theta)dr + \int_{\frac{\pi}{2}}^{\pi} dx \int_0^1 f(r\cos\theta, r\sin\theta)dr$

D. $\int_0^{\frac{\pi}{2}} d\theta \int_0^{\frac{1}{\cos\theta+\sin\theta}} f(r\cos\theta, r\sin\theta)rdr + \int_{\frac{\pi}{2}}^{\pi} dx \int_0^1 f(r\cos\theta, r\sin\theta)rdr$

Part Two: Filling blanks

1. The domain of $z = \arcsin\frac{y}{x} + \sqrt{xy}$ is

$\underline{\{(x,y) \mid x \ge y \ge 0, x \ne 0\} \cup \{(x,y) \mid x \le y \le 0, x \ne 0\}}$

2. The general solution of $y'' + 4y' + 4y = 0$ is $\underline{y = (C_1 + C_2 x)e^{-2x}}$.

3. $\lim\limits_{(x,y)\to(0,0)} \frac{\sqrt{x^2+y^2} - \sin\sqrt{x^2+y^2}}{(\sqrt{x^2+y^2})^3} = \underline{\frac{1}{6}}$.

4. Reverse the order of iterated integrals $\int_0^2 dx \int_{x^2}^{2x} f(x,y)dy = \underline{\int_0^4 dy \int_{\frac{y}{2}}^{\sqrt{y}} f(x,y)dx}$.

5. Let $D = \{(x,y), x^2 + y^2 \le 1\}$, then $\iint\limits_D e^{(x^2+y^2)} dxdy = \underline{\pi(e-1)}$.

6. If $f(u)$ is differentiable, and $f'(0) = \frac{1}{2}$, then the total differential of $z = f(4x^2 - y^2)$ at $(1,2)$ is $dz\big|_{(1,2)} = \underline{4dx - 2dy}$.

7. If $\overline{Y} + y_1^*$ is the general solution of $y'' + P(x)y' + Q(x)y = f_1(x)$, y_2^* is a particular solution of $y'' + P(x)y' + Q(x)y = f_2(x)$, then the general solution of $y'' + P(x)y' + Q(x)y = f_1(x) + f_2(x)$ is $\underline{y = Y + y_1^* + y_2^*}$.

9. If $f(u)$ is differentiable, and $f(0) = 0, f'(0) = 1$, then

$\lim\limits_{t\to 0^+} \frac{1}{\pi\, t^3} \iint\limits_{x^2+y^2 \le t^2} f(\sqrt{x^2+y^2}) d\sigma = \underline{\frac{2}{3}}$.

8. $\dfrac{\mathrm{d}}{\mathrm{d}x}\int_0^x \sin^{100}(x-t)\ \mathrm{d}t = \underline{\sin^{100} x}$

9. $\int_{-1}^{1}(x^2\sin^3 x + x\tan^2 x)dx = \underline{\quad 0 \quad}$

10. If $f(x)$ is continuous, and $f(x)=3x^2-\int_0^2 f(x)dx-2$, then $f(x)=\underline{3x^2-\dfrac{10}{3}}$

11. $\iint\limits_{x^2+y^2\le a^2}(4-5\sin x+3y)d\sigma = \underline{4\pi a^2}$

12. If the single variable function $f(u)$ has continuous derivative, and $z=f(e^x+e^y)$

satisfies $\dfrac{\partial z}{\partial x}+\dfrac{\partial z}{\partial y}=2$, and $f(1)=0$, then $f(u)=\underline{\ln u^2}$

13. $\lim\limits_{(x,y)\to(0,0)}\dfrac{xy}{\sqrt{2-e^{xy}}-1} = \underline{-2}$.

14. If $z=\dfrac{y}{x}$,then when $x=2, y=1, \Delta x=0.1, \Delta y=-0.2$,the total differential is $dz\big|_{(2,1)} = \underline{-0.125}$.

15. If $f(x)$ is a continuous function on $[0, 1]$, and $\int_0^1 f(x)\mathrm{d}x = A$, then

$\int_0^1 \mathrm{d}x\int_x^1 f(x)f(y)\mathrm{d}y = \underline{\dfrac{A^2}{2}}$.

16. The area of region enclosed by the curve $y=\dfrac{4}{x}$ and the lines $y=x$, $y=4x$ in the first quadrant is $\underline{4\ln 2}$

17. $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} t^{20}\sin^5 t dt = \underline{\quad 0 \quad}$.

18. $\lim\limits_{x\to 0}\dfrac{\int_0^x \cos t^2 dt}{x} = \underline{\quad 1 \quad}$

19. The total differential of the function $z=y\sin(2x-y)$ at the point (1, 2) is $\underline{dz = 4dx-2dy}$

20. $\iint\limits_{x^2+y^2\le 2}(4+5\sin x-3y)d\sigma=$ $\underline{\quad 8\pi \quad}$

21. The domain of $z=\ln(xy)+\sqrt{4-x^2-y^2}$ is $\underline{\left\{(x,y)\mid xy>0,\text{ and } x^2+y^2\le 4\right\}}$

22. $\lim\limits_{n\to\infty}\frac{1}{n}(\sin\frac{1}{n}+\sin\frac{2}{n}+\cdots+\sin\frac{n}{n})=$ $\underline{\quad 1-\cos 1 \quad}$

23. The general solution of the differential equation $xy''+3y'=0$ is $\underline{y=\frac{C_1}{x^2}+C_2}$

24. If $y_1=e^{-x}$, $y=e^{3x}$ are two particular solutions of the homogeneous linear equation $y''+py'+qy=0$ (p, q are two constants), then the general solution of $y''+py'+qy=x$ is $\underline{y=C_1e^{-x}+C_2e^{3x}-\frac{1}{3}x+\frac{2}{9}}$

25. If the function $f(x,y)$ has continuous partial derivatives,

$$f(1,\ 1)=1,\quad f_x'(1,\ 1)=a \ \text{ and } \ f_y'(1,\ 1)=b,$$

Then the differential of $u(x)=f(x,f(x,x))$ at $x=1$ is $\underline{du=(a+ab+b^2)dx}$

26. Interchange the iterated integral order of $\int_2^3 dy\int_y^3 f(x,y)dx=\underline{\int_2^3 dx\int_2^x f(x,y)dy}$

27. If $D=\{(x,y)\mid |x|+|y|\le 1\}$, then $\iint_D(\sqrt{5}+1)dxdy=\underline{2(\sqrt{5}+1)}$.

28. Transform the iterated integral in rectangular system into iterated integral in polar system: $I=\int_0^1 dx\int_{1-x}^{\sqrt{1-x^2}}\frac{1}{\sqrt{(x^2+y^2)^3}}dy=\underline{\int_0^{\frac{\pi}{2}}d\theta\int_{\frac{1}{\sin\theta+\cos\theta}}^1\frac{1}{r^2}dr}$ and $I=\underline{2-\frac{\pi}{2}}$

29. If $f(u,v)$ has continuous partial derivative, and $f_u(u,v)+f_v(u,v)=uv$, then $y(x)=e^{-2x}f(x,x)$ is the solution of the differential equation $\underline{y'+2y=x^2e^{-2x}}$ and the general solution of this differential equation is $\underline{y=e^{-2x}\left(C+\frac{x^3}{3}\right)}$.

30. If $f(u)$ is differentiable, $z=f(\sin y-\sin x)+xy$, then $\frac{1}{\cos x}\frac{\partial z}{\partial x}+\frac{1}{\cos y}\frac{\partial z}{\partial y}=$

$\frac{y}{\cos x}+\frac{x}{\cos y}$

31. If $f(u)$ is differentiable, $z = yf\left(\frac{y^2}{x}\right)$, then $2x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = \underline{z}$

32. Reverse the order of iterated integrals $\int_0^2 \int_{x^3}^8 f(x,y)dydx = \underline{\int_0^8 \int_0^{\sqrt[3]{y}} f(x,y)dxdy}$.

33. $\frac{d}{dx}\int_1^{\sin x} 3t^2 dt = \cancel{3\cos x \sin^2 x}$

34. $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1 + t^{20}\sin^5 t)dt = \underline{\quad\quad \pi \quad\quad}$.

Part 3: Calculation:

1. If the function $y = y(x)$ satisfies the integral equation $y + \int_0^x t\,y(t)dt = e^{x^2}$, write the corresponding differential equation, and find its solution.

Solution： From the equation, we know $y = y(x)$ is the solution of initial value problem $\begin{cases} y' + xy = 2xe^{x^2} \\ y|_{x=0} = 1 \end{cases}$,

Then the general solution is $y = e^{-\frac{1}{2}x^2}(C + \frac{2}{3}e^{\frac{3}{2}x^2}) = Ce^{-\frac{1}{2}x^2} + \frac{2}{3}e^{x^2}$

From the initial value condition, we can find $C = \frac{1}{3}$, therefore,

$$y = \frac{1}{3}e^{-\frac{1}{2}x^2} + \frac{2}{3}e^{x^2}.$$

2. Find $I = \iint_D \sqrt{1 - \sin^2(x+y)}dxdy$， where $D: 0 \le x \le \frac{\pi}{2}, 0 \le y \le \frac{\pi}{2}$.

Solution:

$$\iint_D \sqrt{1-\sin^2(x+y)}dxdy = \iint_D |\cos(x+y)|\,dxdy$$

$$= \int_0^{\frac{\pi}{2}} dx \int_0^{\frac{\pi}{2}-x} \cos(x+y)dy + \int_0^{\frac{\pi}{2}} dx \int_{\frac{\pi}{2}-x}^{\frac{\pi}{2}} -\cos(x+y)dy$$

$$= \frac{\pi}{2} - 1 + \frac{\pi}{2} - 1 = \pi - 2$$

3. The region enclosed by $2x=y^2$ and $x=\dfrac{1}{2}$ is revolved about the line $y=1$ to generate a solid, find its volume.

Solution：

$$V=\int_0^{\frac{1}{2}}\pi(\sqrt{2x}+1)^2dx-\int_0^{\frac{1}{2}}\pi(-\sqrt{2x}+1)^2dx$$

$$=\int_0^{\frac{1}{2}}4\pi\sqrt{2x}dx=\frac{4}{3}\pi(2x)^{\frac{3}{2}}\Big|_0^{\frac{1}{2}}=\frac{4}{3}\pi$$

4. $\int_0^{\frac{\pi}{2}}\sqrt{1-\sin 2x}\,\mathrm{d}x$

Solution： $\int_0^{\frac{\pi}{2}}\sqrt{1-\sin 2x}\,\mathrm{d}x=\int_0^{\frac{\pi}{2}}|\sin x-\cos x|\mathrm{d}x$

$$=\int_0^{\frac{\pi}{4}}(\cos x-\sin x)\,\mathrm{d}x+\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}(\sin x-\cos x)\,\mathrm{d}x$$

$$=(\sin x+\cos x)\Big|_0^{\frac{\pi}{4}}+(-\cos x-\sin x)\Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}}=2(\sqrt{2}-1)$$

5. $\int_0^1\frac{\ln(1+x)}{(2-x)^2}dx$

Solution： $\int_0^1\frac{\ln(1+x)}{(2-x)^2}dx=\frac{\ln(1+x)}{2-x}\Big|_0^1-\int_0^1\frac{1}{(2-x)(1+x)}dx$

$$=\ln 2-\frac{1}{3}\ln\left|\frac{1+x}{x-2}\right|\Bigg|_0^1=\ln 2-\frac{2}{3}\ln 2=\frac{1}{3}\ln 2$$

6. Let $z=f(x^2-y^2,\mathrm{e}^{xy})$，where f has continuous second order partial derivatives，find $\dfrac{\partial z}{\partial x},\dfrac{\partial^2 z}{\partial x\partial y}$.

Solution：

$$\frac{\partial z}{\partial x}=2xf_1+ye^{xy}f_2$$

$$\frac{\partial^2 z}{\partial x\partial y}=\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)=-4xyf_{11}+2x^2e^{xy}f_{12}+e^{xy}f_2+xye^{xy}f_2-2y^2e^{xy}f_{21}+xye^{2xy}f_{22}$$

$$=-4xyf_{11}+(1+xy)e^{xy}f_2+2(x^2-y^2)e^{xy}f_{12}+xye^{2xy}f_{22}$$

7. If $y(x)$ has continuous derivative on $[0,+\infty)$, and satisfied

$$y(x)=-1+x+2\int_0^x (x-t)y(t)y'(t)dt$$

find $y(x)$.

Solution: Differentiate both sides with respect to *x*, then

$$y'(x)=1+2\int_0^x y(t)y'(t)dt+2xy(x)y'(x)-2xy(x)y'(x)$$

$$=1+2\int_0^x y(t)y'(t)dt=1+y^2(x)-y^2(0)$$

From the integral equation, we know $y(0)=-1$, then $y'(x)=y^2(x)$

$$\frac{1}{y^2}dy=dx$$

Integrate both sides, we have $-\frac{1}{y}=x+C$

By the initial condition $y(0)=-1$, we can find $C=1$. Therefore, $y=-\frac{1}{x+1}$

8. The region enclosed by $2x=y^2$ and $x=2$ is revolved about the line $y=2$ to generate a solid, find the volume of the solid.

Solution: $V=\int_0^2 \pi(\sqrt{2x}+2)^2dx-\int_0^2 \pi(-\sqrt{2x}+2)^2dx$

$$=\int_0^2 8\pi\sqrt{2x}dx=\frac{8}{3}\pi(2x)^{\frac{3}{2}}\Big|_0^2=\frac{64}{3}\pi$$

9. Find $\iint_D |\sin(y-x)|\,d\sigma$, where D is the region enclosed by the lines $x=0$, $y=2\pi$, $y=x$.

Solution: Let $D_1=\{(x,y)\,|\,y-x\le\pi\}\cap D,\ D_2=\{(x,y)\,|\,y-x>\pi\}\cap D$, then

$$\iint_D |\sin(y-x)|\,d\sigma=\iint_{D_1}|\sin(y-x)|\,d\sigma+\iint_{D_2}|\sin(y-x)|\,d\sigma$$

$$=\int_0^\pi\int_x^{x+\pi}\sin(y-x)dydx+\int_\pi^{2\pi}\int_x^{2\pi}\sin(y-x)dydx-\int_0^\pi\int_{x+\pi}^{2\pi}\sin(y-x)dydx$$

$$=\int_0^\pi -\cos(y-x)\Big|_x^{x+\pi}dx+\int_\pi^{2\pi}-\cos(y-x)\Big|_x^{2\pi}dx+\int_0^\pi \cos(y-x)\Big|_{x+\pi}^{2\pi}dx$$

$$=\int_0^\pi 2dx+\int_\pi^{2\pi}\left(1-\cos(2\pi-x)\right)dx+\int_0^\pi\left(\cos(2\pi-x)+1\right)dx$$

$$=2\pi+\pi+\pi=4\pi$$

10. The region enclosed by $y = x^2$ and $x = y^2$ is revolved about y axis to generate a solid, find the volume of the solid.

Solution：It is easy to find the intersection points of two curves are (0, 0) and (1, 1)，and we can think of the solid is obtained by cutting out the solid revolving the curve $x = y^2,\ 0 \le y \le 1$ about y axis from the solid generated by revolving the curve $x = \sqrt{y},\ 0 \le y \le 1$ about y axis, then

$$\begin{aligned} V &= V_1 - V_2 \\ &= \int_0^1 \pi(\sqrt{y})^2 dy - \int_0^1 \pi(y^2)^2 dy \\ &= \pi\int_0^1 (y - y^4) dy \\ &= \pi(\frac{y^2}{2} - \frac{y^5}{5})\,|_0^1 = \frac{3\pi}{10} \end{aligned}$$

11. The function $\varphi(x)$ is continuous, and satisfies

$$\varphi(x) = e^x + \int_0^x t\varphi(t)dt - x\int_0^x \varphi(t)dt,$$

find $\varphi(x)$.

Solution：Differentiate both sides of the above integral equation twice，we have

$$\varphi'(x) = e^x + x\varphi(x) - \int_0^x \varphi(t)dt - x\varphi(x) = e^x - \int_0^x \varphi(t)dt$$

$$\varphi''(x) = e^x - \varphi(x)$$

Then $\varphi(x)$ is the solution of the initial value problem $\begin{cases} y'' + y = e^x \\ y|_{x=0} = 1,\ \ y'|_{x=0} = 1 \end{cases}$

The character equation is $r^2 + 1 = 0$.

The character solutions are $r_1 = i,\ r_2 = -i$, then the general solution of the corresponding homogeneous equation is

$$C_1 \cos x + C_2 \sin x$$

Let the particular solution is $y^* = ae^x$, then $y^*{''} + y^* = ae^x + ae^x = e^x$

We can find $a = \frac{1}{2}$, then $y^* = \frac{1}{2}e^x$, the general solution of the differential equation is

$$y = C_1 \cos x + C_2 \sin x + \frac{1}{2}e^x$$

Since $y|_{x=0} = 1$, $y'|_{x=0} = 1$, then $C_1 + \frac{1}{2} = 1$, $C_2 + \frac{1}{2} = 1$.

Therefore, $C_1 = \frac{1}{2}$, $C_2 = \frac{1}{2}$.

The solution of initial value problem is $y = \frac{1}{2}\cos x + \frac{1}{2}\sin x + \frac{1}{2}e^x$

12. Find $\iint\limits_D \ln(1+x^2+y^2)d\sigma$, where D is the region enclosed by the circle $x^2+y^2=1$ and axes in the first quadrant.

Solution: The region D is represented by polar coordinate

$$D = \{(r,\theta) \mid 0 \le r \le 1, 0 \le \theta \le \frac{\pi}{2}\},$$

then

$$\begin{aligned}\iint\limits_D \ln(1+x^2+y^2)d\sigma &= \iint\limits_D \ln(1+r^2)rdrd\theta \\ &= \int_0^{\frac{\pi}{2}} d\theta \int_0^1 \ln(1+r^2)rdr \\ &= \frac{\pi}{2} \times \frac{1}{2}\int_0^1 \ln(1+r^2)d(1+r^2) \\ &= \frac{\pi}{4}\int_1^2 \ln t dt \\ &= \frac{\pi}{4}(t\ln t\big|_1^2 - \int_1^2 1dt) = \frac{\pi}{4}(2\ln 2 - 1)\end{aligned}$$

13. Find the volume of the solid generated by revolving the circle $x^2+(y-5)^2=16$ about x axis.

Solution: The solid can be viewed as the the solid generated by revolving the curve $y = 5+\sqrt{16-x^2}$, $0 \le x \le 4$ about x axis is cut out by the solid generated by revolving the curve $y = 5-\sqrt{16-x^2}$, $0 \le x \le 4$ about x axis, then

$$
\begin{aligned}
V &= V_1 - V_2 \\
&= \int_0^4 \pi(5+\sqrt{16-x^2})^2 dx - \int_0^4 \pi(5-\sqrt{16-x^2})^2 dx \\
&= 20\pi \int_0^4 \sqrt{16-x^2} dx \\
&= 20\pi \int_0^{\frac{\pi}{2}} 4\cos\theta d4\sin\theta \\
&= 320\pi \int_0^{\frac{\pi}{2}} \cos^2\theta d\theta \\
&= 320\pi \left(\frac{1}{2}\theta + \frac{1}{4}\sin 2\theta \right)\Bigg|_0^{\frac{\pi}{2}} = 160\pi^2
\end{aligned}
$$

14. Find the solution of initial value problem $\begin{cases} y''+y=e^x \\ y|_{x=0}=1,\ \ y'|_{x=0}=1 \end{cases}$.

Solution： The character equation is $r^2+1=0$, then the character roots are $r_1=i,\ r_2=-i$, therefore, the general solution of corresponding homogeneous equation is

$$C_1\cos x + C_2\sin x$$

Let $y^*=ae^x$ be the particular solution of non-homogeneous equation, then

$$y^*{}''+y^*=ae^x+ae^x=e^x$$

We can find $a=\frac{1}{2}$, thus $y^*=\frac{1}{2}e^x$. Then the general solution of non-homogeneous differential is

$$y=C_1\cos x+C_2\sin x+\frac{1}{2}e^x$$

Since $y|_{x=0}=1,\ y'|_{x=0}=1$, then $C_1+\frac{1}{2}=1$， $C_2+\frac{1}{2}=1$.We can find $C_1=\frac{1}{2}$ $C_2=\frac{1}{2}$.

The solution of initial value problem is

$$y=\frac{1}{2}\cos x+\frac{1}{2}\sin x+\frac{1}{2}e^x$$

15. If $z=y^x$, find $\frac{\partial z}{\partial x}, \frac{\partial^2 z}{\partial x^2}, \frac{\partial^2 z}{\partial x \partial y}$.

Solution: $\frac{\partial z}{\partial x}=y^x\ln y$

$$\frac{\partial^2 z}{\partial x^2} = y^x \ln y \ln y = y^x \left(\ln y\right)^2$$

$$\frac{\partial^2 z}{\partial x \partial y} = xy^{x-1} \ln y + y^x \frac{1}{y} = xy^{x-1} \ln y + y^{x-1} = y^{x-1}(1 + x \ln y)$$

16. Find $\iint\limits_D \arctan \frac{y}{x} d\sigma$, where D is the region enclosed by circles $x^2 + y^2 = 4$, $x^2 + y^2 = 1$ and lines $y = 0,\ y = x$ in the first quadrant.

Solution：The D is represented in polar form $D = \{(r, \theta) \mid 1 \le r \le 2, 0 \le \theta \le \frac{\pi}{4}\}$, then

$$\iint\limits_D \arctan \frac{y}{x} d\sigma = \iint\limits_D \arctan \tan \theta r dr d\theta$$

$$= \int_0^{\frac{\pi}{4}} \theta d\theta \int_1^2 r dr$$

$$= \frac{\pi^2}{32} \frac{3}{2} = \frac{3\pi^2}{64}$$

17. $\int_1^e \frac{1}{x\sqrt{2 + \ln x}} \mathrm{d}x$

Solution：$\int_1^e \frac{1}{x\sqrt{2 + \ln x}} \mathrm{d}x = \int_1^e \frac{1}{\sqrt{2 + \ln x}} \mathrm{d}(2 + \ln x)$

$$\overset{t=2+\ln x}{=} \int_2^3 \frac{1}{\sqrt{t}} dt$$

$$= 2\sqrt{t}\Big|_{t=2}^{t=3} = 2(\sqrt{3} - \sqrt{2})$$

18. $\int_1^e x \ln x dx$

Solution: $\int_1^e x \ln x dx = \int_1^e \ln x d\,\frac{x^2}{2}$

$$= \frac{1}{2} x^2 \ln x \Big|_1^e - \int_1^e \frac{1}{2} x^2 \cdot \frac{1}{x} dx$$

$$=\left(\frac{1}{2}x^2\ln x-\frac{1}{4}x^2\right)\Bigg|_1^e=\frac{1}{4}(e^2+1)$$

19. Find $\int_0^1 \frac{x+1}{x^2-2x+5}dx$.

Solution:

$$\int_0^1 \frac{x-1+2}{x^2-2x+5}dx=\int_0^1 \frac{x-1}{x^2-2x+5}dx+2\int_0^1 \frac{1}{x^2-2x+5}dx \qquad \text{.... 2 marks}$$

$$=\frac{1}{2}\int_0^1 \frac{1}{x^2-2x+5}d(x^2-2x+5)+2\int_0^1 \frac{1}{(x-1)^2+4}dx$$

$$=\frac{1}{2}\ln(x^2-2x+5)\Big|_0^1+\arctan\frac{x-1}{2}\Bigg|_0^1 \qquad \text{..... 4 marks}$$

$$=\frac{1}{2}\ln\frac{4}{5}-\arctan\frac{1}{2} \qquad \text{..... 6 marks}$$

20. Find $\lim\limits_{n\to\infty}\sum_{k=1}^{n}\frac{k}{n^2}\ln\left(1+\frac{k}{n}\right)$

Solution：

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{k}{n^2}\ln\left(1+\frac{k}{n}\right)=\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\frac{k}{n}\ln\left(1+\frac{k}{n}\right)$$

$$=\int_0^1 x\ln(1+x)dx=\frac{1}{2}x^2\ln(1+x)\big|_0^1-\frac{1}{2}\int_0^1\frac{x^2}{1+x}dx$$

$$=\frac{1}{2}\ln 2-\frac{1}{2}\int_0^1\left(x-1+\frac{1}{1+x}\right)dx$$

$$=\frac{1}{2}\ln 2-\frac{1}{2}\left(\frac{x^2}{2}-x+\ln(1+x)\right)\Bigg|_0^1=\frac{1}{4}$$

21. Find the limit $\lim\limits_{x\to 0^+}\frac{\int_0^x\sqrt{x-t}\,e^t dt}{\sqrt{x^3}}$

Solution:

$$\lim_{x\to 0^+}\frac{\int_0^x \sqrt{x-t}e^t dt}{\sqrt{x^3}}=\lim_{x\to 0^+}\frac{\int_0^x \sqrt{y}e^{x-y} dy}{\sqrt{x^3}}=\lim_{x\to 0^+}\frac{e^x\int_0^x \sqrt{y}e^{-y} dy}{\sqrt{x^3}}$$

$$=\lim_{x\to 0^+}e^x\lim_{x\to 0^+}\frac{\int_0^x \sqrt{y}e^{-y} dy}{\sqrt{x^3}}$$

$$=\lim_{x\to 0^+}\frac{\int_0^x \sqrt{y}e^{-y} dy}{\sqrt{x^3}}=\lim_{x\to 0^+}\frac{\sqrt{x}e^{-x}}{\frac{3}{2}\sqrt{x}}=\frac{2}{3}$$

22. If $f(x)$ is continuous and increasing on $[a,\ b]$, show that

$$\int_a^b xf(x)dx \ge \frac{a+b}{2}\int_a^b f(x)dx$$

Proof: Since

$$\int_a^b xf(x)dx-\frac{a+b}{2}\int_a^b f(x)dx=\int_a^b\left(x-\frac{a+b}{2}\right)f(x)dx$$

$$=\int_a^{\frac{a+b}{2}}\left(x-\frac{a+b}{2}\right)f(x)dx+\int_{\frac{a+b}{2}}^b\left(x-\frac{a+b}{2}\right)f(x)dx$$

$$=f(\xi)\int_a^{\frac{a+b}{2}}\left(x-\frac{a+b}{2}\right)dx+f(\eta)\int_{\frac{a+b}{2}}^b\left(x-\frac{a+b}{2}\right)dx$$ By Mean Value Theorem， $\xi\in\left(a,\ \frac{a+b}{2}\right)$, $\eta\in\left(\frac{a+b}{2},\ b\right)$

$$=-\frac{1}{8}f(\xi)(b-a)^2+\frac{1}{8}f(\eta)(b-a)^2$$

$$=\frac{1}{8}(b-a)^2(f(\eta)-f(\xi))>0$$

then

$$\int_a^b xf(x)dx \ge \frac{a+b}{2}\int_a^b f(x)dx$$

23. Find the general solution of the following differential equation:

$$y''+2y'+y=x^2$$

Solution： The auxiliary equation of complementary equation is

$$r^2+2r+1=0$$

then the roots are $r_1=r_2=-1$, and the general solution of complementary equation is

$$y_c=(C_1+C_2x)e^{-x}$$

Since $\lambda=0$ is not the root of auxiliary equation, then we can let the particular solution be $y_p=ax^2+bx+c$,therefore

$$y_p' = 2ax + b, \qquad y_p'' = 2a$$

Substitute above in the original ODE, then

$$2a + 2(2ax + b) + ax^2 + bx + c = x^2$$

That is $(2a + 2b + c) + (4a + b)x + ax^2 = x^2$, then

$$\begin{cases} a = 1 \\ 4a + b = 0 \\ 2a + 2b + c = 0 \end{cases}$$

We can find

$$\begin{cases} a = 1 \\ b = -4 \\ c = 6 \end{cases}$$

Then the particular solution is $y_p = x^2 - 4x + 6$, and the general solution of original ODE is

$$y = (C_1 + C_2 x)e^{-x} + x^2 - 4x + 6$$

24. If $z = \arctan\dfrac{x}{y}$, $x = u\cos v, y = u\sin v$, find $\dfrac{\partial z}{\partial u}, \dfrac{\partial z}{\partial v}$

Solution:

$$\frac{\partial z}{\partial u}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial u}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial u}$$

$$=\frac{1}{1+\left(\frac{x}{y}\right)^2}\frac{1}{y}\cos v+\frac{1}{1+\left(\frac{x}{y}\right)^2}(-\frac{x}{y^2})\sin v$$

$$=\frac{y}{x^2+y^2}\cos v-\frac{x}{x^2+y^2}\sin v$$

$$=\frac{u\sin v}{u^2}\cos v-\frac{u\cos v}{u^2}\sin v$$

$$=0$$

$$\frac{\partial z}{\partial v}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial v}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial v}$$

$$=\frac{1}{1+\left(\frac{x}{y}\right)^2}\frac{1}{y}(-u\sin v)+\frac{1}{1+\left(\frac{x}{y}\right)^2}(-\frac{x}{y^2})u\cos v$$

$$=\frac{y}{x^2+y^2}(-u\sin v)-\frac{x}{x^2+y^2}u\cos v$$

$$=\frac{u\sin v}{u^2}(-u\sin v)-\frac{u\cos v}{u^2}u\cos v$$

$$=-1$$

25. Find the double integral: $\int_1^2 dy\int_y^2 \frac{x}{y\ln x}dx$.

Solution:

$$\int_1^2 dy\int_y^2 \frac{x}{y\ln x}dx=\int_1^2 dx\int_1^x \frac{x}{y\ln x}dy=\int_1^2 xdx=\frac{3}{2}$$

26. Find double integral: $\iint_D |y^2-x^2|\,d\sigma$, where

$$D=\{(x,\ y)\,|\,x\in[-1,\ 1],y\in[0,\ 2]\}$$

Solution:

$$\iint_D |y^2-x^2|\,d\sigma=\int_{-1}^0 dx\int_0^{-x}(x^2-y^2)dy+\int_0^1 dx\int_0^x(x^2-y^2)dy$$

$$+\int_{-1}^0 dx\int_{-x}^2(y^2-x^2)dy+\int_0^1 dx\int_x^2(y^2-x^2)dy$$

$$=\int_{-1}^0\left(-\frac{2}{3}x^3\right)dx+\int_0^1\left(\frac{2}{3}x^3\right)dx+\int_{-1}^0\left(\frac{8}{3}--2x^2-\frac{2}{3}x^3\right)dx+\int_0^1\left(\frac{8}{3}--2x^2+\frac{2}{3}x^3\right)dx$$

$$=\frac{1}{6}+\frac{1}{6}+\frac{13}{6}+\frac{13}{6}=\frac{14}{3}$$

27. If $D=\{(x,y)\,|\ x^2+y^2\le R^2\}$, find the volume of the cylinder with D as its base and with the top surface $z = e^{-x^2-y^2}$.

Solution:

$$V = \iint_D e^{-x^2-y^2}\,dxdyz = \int_0^R \int_0^{2\pi} e^{-r^2} r\,d\theta dr$$
$$= 2\pi \int_0^R e^{-r^2} r dr = -\pi e^{-r^2}\Big|_0^R = \pi(1-e^{-R^2})$$

28. Find all the local extrema of the function $f(x,y) = x^2 y + y^3 - y$.

Solution: Since

$$f_x(x,y) = 2xy, \quad f_y(x,y) = x^2 + 3y^2 - 1$$

then let $\begin{cases} f_x(x,y) = 2xy = 0 \\ f_y(x,y) = x^2 + 3y^2 - 1 = 0 \end{cases}$, we get the following critical points $\left(0,\ \frac{\sqrt{3}}{3}\right), \left(0,\ -\frac{\sqrt{3}}{3}\right), (1,\ 0), (-1,\ 0)$.

Since $f_{xx}(x,y) = 2y, \quad f_{yy}(x,y) = 6y, \quad f_{xy}(x,y) = 2x$, then at $\left(0,\ \frac{\sqrt{3}}{3}\right)$,

$$f_{xx}f_{yy} - f_{xy}^2\Big|_{\left(0,\frac{\sqrt{3}}{3}\right)} > 0, \quad f_{xx}\left(0,\frac{\sqrt{3}}{3}\right) > 0,$$

Then $\left(0,\ \frac{\sqrt{3}}{3}\right)$ is a local minimum point, and the corresponding local minimum is

$$f\left(0,\frac{\sqrt{3}}{3}\right) = -\frac{2\sqrt{3}}{9}.$$

At $\left(0,\ -\frac{\sqrt{3}}{3}\right)$,

$$f_{xx}f_{yy} - f_{xy}^2\Big|_{\left(0,-\frac{\sqrt{3}}{3}\right)} > 0, \quad f_{xx}\left(0,-\frac{\sqrt{3}}{3}\right) < 0,$$

Then $\left(0,\ -\frac{\sqrt{3}}{3}\right)$ is a local maximum point, and the corresponding local maximum is

$$f\left(0,-\frac{\sqrt{3}}{3}\right) = \frac{2\sqrt{3}}{9}.$$

At $(1,\ 0)$ and $(-1,\ 0)$,

$$f_{xx}f_{yy}-f_{xy}^2\Big|_{(1,0)}<0,\quad f_{xx}f_{yy}-f_{xy}^2\Big|_{(-1,0)}<0,$$

Then $(1,\ 0,\ 0)$ and $(-1,\ 0,\ 0)$ are saddle points.

29. If $f(u,v)$ has continuous second partial derivative, and $g(x,y)=xy-f(x+y,x-y)$, find $\dfrac{\partial^2 g}{\partial x^2}+\dfrac{\partial^2 g}{\partial x\partial y}+\dfrac{\partial^2 g}{\partial y^2}$.

Solution: Since

$$\frac{\partial g}{\partial x}=y-f_u(x+y,x-y)-f_v(x+y,x-y)$$
$$\frac{\partial g}{\partial y}=x-f_u(x+y,x-y)+f_v(x+y,x-y)$$

then

$$\frac{\partial^2 g}{\partial x^2}=-f_{uu}(x+y,x-y)-2f_{uv}(x+y,x-y)-f_{vv}(x+y,x-y)$$
$$\frac{\partial^2 g}{\partial x\partial y}=1-f_{uu}(x+y,x-y)+f_{uv}(x+y,x-y)$$
$$\frac{\partial^2 g}{\partial y^2}=-f_{uu}(x+y,x-y)+2f_{uv}(x+y,x-y)-f_{vv}(x+y,x-y)$$

Therefore

$$\frac{\partial^2 g}{\partial x^2}+\frac{\partial^2 g}{\partial x\partial y}+\frac{\partial^2 g}{\partial y^2}=1-3f_{uu}(x+y,x-y)-f_{vv}(x+y,x-y)$$

30. The function $\boldsymbol{f(u)}$ has continuous second derivatives, and $\boldsymbol{z=f(e^x\cos y)}$ satisfies $\boldsymbol{\dfrac{\partial^2 z}{\partial x^2}+\dfrac{\partial^2 z}{\partial y^2}=(4z+e^x\cos y)e^{2x}}$. If $\boldsymbol{f(0)=0, f'(0)=0}$, find $\boldsymbol{f(u)}$.

Solution: Since

$$\frac{\partial z}{\partial x}=e^x\cos yf'(e^x\cos y),\quad \frac{\partial^2 z}{\partial x^2}=e^x\cos yf'(e^x\cos y)+e^{2x}\cos^2 yf''(e^x\cos y)$$
$$\frac{\partial z}{\partial y}=-e^x\sin yf'(e^x\cos y),\quad \frac{\partial^2 z}{\partial y^2}=-\cos ye^x f'(e^x\cos y)+e^{2x}\sin^2 xf''(e^x\cos y)$$

Then

$$\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=e^{2x}f''(e^x\cos y)=(4z+e^x\cos y)e^{2x}$$

Therefore,

$$f''(u)=4f(u)+u$$

The auxiliary equation is

$$r^2-4=0$$

Then the roots are $r_1=2,\ r_2=-2$.

Let $y_p=au+b$ be the particular solution, then

$$y_p''=0=4au+4b+u$$

Therefore $a=-\frac{1}{4},\ b=0$.

The general solution is $f(u)=C_1e^{2u}+C_2e^{-2u}-\frac{1}{4}u$.

From the initial condition $\boldsymbol{f(0)=0, f'(0)=0}$, we can find

$$C_1=\frac{1}{16},\ C_2=-\frac{1}{16}.$$

Thus

$$f(u)=\frac{1}{16}e^{2u}-\frac{1}{16}e^{-2u}-\frac{1}{4}u.$$

31. Find $\int_0^1 \frac{f(x)}{\sqrt{x}}dx$, where $f(x)=\int_1^x \frac{\ln(t+1)}{t}dt$.

Solution:

$$\int_0^1\frac{f(x)}{\sqrt{x}}dx=2\sqrt{x}f(x)\Big|_0^1-\int_0^1 2\sqrt{x}f'(x)dx$$

$$=-\int_0^1 2\sqrt{x}f'(x)dx=-\int_0^1 2\sqrt{x}\frac{\ln(1+x)}{x}dx$$

$$=-4\sqrt{x}\ln(1+x)\Big|_0^1+4\int_0^1\sqrt{x}\frac{1}{1+x}dx$$

$$=-4\ln 2+8\int_0^1\frac{t^2}{1+t^2}dt$$

$$=-4\ln 2+8-2\pi$$

32. If $f(x,y)$ has continuous second partial derivatives, and $f(1,y)=0$, $f(x,1)=0$, $\iint\limits_D f(x,y)dxdy=a$, where $D=\{(x,y)\,|\,0\le x\le 1, 0\le y\le 1\}$, find $I=\iint\limits_D xyf''_{xy}(x,y)dxdy$.

Solution:

$$I=\int_0^1 xdx\int_0^1 yf_{xy}(x,y)dy$$
$$=\int_0^1 xdx\left[yf_x(x,y)|_0^1-\int_0^1 f_x(x,y)dy\right]$$
$$=\int_0^1 x\left[f_x(x,1)-\int_0^1 f_x(x,y)dy\right]dx$$
$$=\int_0^1 xf_x(x,1)dx-\int_0^1 x\int_0^1 f_x(x,y)dydx$$
$$=xf(x,1)|_0^1-\int_0^1 f(x,1)dx-\int_0^1 dy\int_0^1 xf_x(x,y)dx$$
$$=-\int_0^1 dy\int_0^1 xf_x(x,y)dx=-\int_0^1 dy\left[xf(x,y)|_0^1-\int_0^1 f(x,y)dx\right]$$
$$=\int_0^1\int_0^1 f(x,y)dxdy=a$$

33. Find $\iint_R (x^2+y^2)dxdy$ where R is the triangular region with vertices (0, 0), (1, 0) and (0, 1).

Solution:

$$\iint_R (x^2+y^2)dA=\int_0^1\int_0^{1-x}(x^2+y^2)dydx \qquad \cdots\cdots\cdots\cdots 2\text{ marks}$$
$$=\int_0^1 (x^2y+\frac{y^3}{3})\Big|_{y=0}^{y=1-x}dx \qquad \cdots\cdots\cdots\cdots 3\text{ marks}$$
$$=\int_0^1 (x^2(1-x)+\frac{(1-x)^3}{3})dx$$
$$=\int_0^1 (2x^2-\frac{4}{3}x^3-x+\frac{1}{3})dx \qquad \cdots\cdots\cdots\cdots 5\text{ marks}$$
$$=\frac{2}{3}-\frac{1}{3}-\frac{1}{2}++\frac{1}{3}=\frac{1}{6} \qquad \cdots\cdots\cdots\cdots 6\text{ marks}$$

34. From the origin to draw the tangent line of the curve $y=\ln x$, the region enclosed by the tangent line, the curve $y=\ln x$ and the *x*-axis is *D*.

(1) Find the area of *D*.

(2) Find the volume of the solid generated by revolving *D* about the line $x=e$.

Solution: (1) Suppose that the tangent point is (x_0,y_0), then the slope of the tangent line satisfies

$$y'|_{x=x_0}=\frac{1}{x_0}=\frac{y_0}{x_0}$$

Therefore, $y_0 = 1,\ x_0 = e$, and the tangent line is $y = \frac{1}{e}x$. ...... 2 marks

The area of D is

$$\int_0^1 (e^y - ey)dy = \left(e^y - \frac{1}{2}ey^2\right)\Bigg|_0^1 = \frac{1}{2}e - 1 \quad ......4 marks$$

(3) The solid is obtained by cutting out the solid generated by revolving the curve $y = \ln x$ about the line $x = e$ from the solid generated by revolving the curve the line $y = \frac{1}{e}x$ about the line $x = e$. Then volume is

$$V = \int_0^1 \pi(e - ey)^2 dy - \int_0^1 \pi(e - e^y)^2 dy \quad .... \text{6marks}$$

$$= \pi\int_0^1 (e^2y^2 - 2e^2y + 2ee^y - e^{2y})dy$$

$$= \pi\left(\frac{1}{3}e^2y^3 - e^2y^2 + 2ee^y - \frac{1}{2}e^{2y}\right)\Bigg|_0^1 \quad .... \text{7 marks}$$

$$= \pi(\frac{5}{6}e^2 - 2e + \frac{1}{2}) \quad .... \text{8 marks}$$

35. Find the points on the curve $x^2 + xy + y^2 = 1$ in the xy-plane that are nearest to and farthest from the origin.

Solution: Since the distance of the point P(x, y) to the origin is

$$|PO| = \sqrt{x^2 + y^2}$$

then we want to find the maximum and minimum of $|PO|$ subject to

$$x^2 + xy + y^2 = 1$$

Since $|PO|$ has a minimum and maximum whenever the function

$$f(x, y) = x^2 + y^2$$

has a minimum and maximum, then we can find the points where $f(x, y)$ attains its maximum and minimum value subject to the constraint $x^2 + xy + y^2 = 1$.

Let $F(x, y, \lambda) = x^2 + y^2 + \lambda(x^2 + xy + y^2 - 1)$, then solve the system

$$\begin{cases} F_x = 2x + 2\lambda x + \lambda y = 0 \\ F_y = 2y + 2\lambda y + \lambda x = 0 \\ F_\lambda = x^2 + xy + y^2 - 1 = 0 \end{cases}$$

then we can find

$$\begin{cases} x_1 = \frac{\sqrt{3}}{3} \\ y_1 = \frac{\sqrt{3}}{3} \end{cases}, \quad \begin{cases} x_2 = -\frac{\sqrt{3}}{3} \\ y_2 = -\frac{\sqrt{3}}{3} \end{cases}, \quad \begin{cases} x_3 = 1 \\ y_3 = -1 \end{cases}, \quad \begin{cases} x_4 = -1 \\ y_3 = 1 \end{cases}$$

Since $f(x_1, y_1) = \frac{2}{3},\ f(x_2, y_2) = \frac{2}{3},\ f(x_3, y_3) = 2,\ f(x_4, y_4) = 2$, then the closest points on the curve to the origin are $(\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3})$ and $(-\frac{\sqrt{3}}{3}, -\frac{\sqrt{3}}{3})$, the farthest points on the curve to the origin are $(1,-1)$ and $(-1,1)$.

Part Four: Application Problem

1. Suppose that a company sales the same product in two different markets, the demands are

$$P_1 = 18 - 2Q_1, \qquad P_2 = 12 - Q_2$$

respectively, where P_1, P_2 are the prices (in \$thousand/ton), Q_1, Q_2 represents the sales (in ton) in two markets, and the cost of the company is $C = 2Q + 5$, where Q represents the total sales in two markets, that is, $Q = Q_1 + Q_2$.

(1) If the company performs the different prices strategy, try to determine the sales and prices in two markets that will maximize the largest profit.

(2) If the company performs the same price strategy, try to determine the sales and the same price in two markets that will maximize the largest profit.

(3) Determine which strategy is better.

Solution: The profit is

$$L = R - C = P_1Q_1 + P_2Q_2 - (2Q + 5) = -2Q_1^2 - Q_2^2 + 16Q_1 + 10Q_2 - 5$$

Solve the system

$$\begin{cases} L'_{Q_1} = -4Q_1 + 16 = 0 \\ L'_{Q_2} = -2Q_2 + 10 = 0 \end{cases} \Rightarrow \begin{cases} Q_1 = 4 & P_1 = 10 \\ Q_2 = 5 & P_2 = 7 \end{cases}$$

Since the critical point is unique, and from our experience, the absolute maximum exists, so at (4， 5), $L(Q_1, Q_2)$ attains its absolute maximum, and the corresponding maximum value is $L_{\max} = 52$.

（2）If the company performs the same price strategy，that is $P_1 = P_2$， then there is a constraint $2Q_1 - Q_2 = 6$.

Let

$$F(Q_1, Q_2, \lambda) = -2Q_1^2 - Q_2^2 + 16Q_1 + 10Q_2 - 5 + \lambda(2Q_1 - Q_2 - 6)$$

$$\begin{cases} F'_{Q_1} = -4Q_1 + 16 + 2\lambda = 0 \\ F'_{Q_2} = -2Q_2 + 10 - \lambda = 0 \\ F'_{\lambda} = 2Q_1 - Q_2 - 6 \end{cases} \Rightarrow \begin{cases} Q_1 = 5 \\ Q_2 = 4 & P_1 = P_2 = 8 \\ \lambda = 2 \end{cases}$$

Since the critical point is unique, and from our experience, the absolute maximum exists, so at (5，4), $L(Q_1, Q_2)$ attains its absolute maximum, and the corresponding maximum value is $L_{\max} = 49$.

From the above analysis, we know that if the company performs the different prices strategy, it will receive more profit. Therefore, the different prices strategy is better.

2. If we want to design an open top rectangular container with volume *V*, what is the dimension of the container that will cost least material.

Solution：Let x, y, z be the length, width, height of the container respectively, then the surface area is

$S = 2(xz + yz) + xy$， and $xyz = V$.

Let $L(x, y, z, \lambda) = 2(xz + yz) + xy + \lambda(xyz - V)$， then

$$L_x(x, y, z, \lambda) = 2z + y + \lambda yz = 0$$

$$L_y(x, y, z, \lambda) = 2z + x + \lambda xz = 0$$

$$L_z(x, y, z, \lambda) = 2x + 2y + \lambda xy = 0$$

$$L_\lambda(x, y, z, \lambda) = xyz - V = 0$$

We find $x = y = z = \sqrt[3]{2V}$, $\lambda = -\dfrac{4}{\sqrt[3]{2V}}$.

From our experience, the absolute minimum of S exists, so when

$x = y = z = \sqrt[3]{2V}$, the container will cost least material.

3. When a space probe with the shape of ellipsoid $4x^2 + y^2 + 4z^2 \le 16$ enters the earth atmosphere, its surface is heated, 1 hour later, the temperature of the probe at (x, y, z) is $T = 8x^2 + 4yz - 16z + 600$, find the hottest point on the surface of the probe.

Solution: The problem is find the absolute maximum of $T = 8x^2 + 4yz - 16z + 600$ on the ellipsoid $4x^2 + y^2 + 4z^2 = 16$.

Let $F(x, y, z, \lambda)$ be the Lagrange function

$$F(x, y, z, \lambda) = 8x^2 + 4yz - 16z + 600 + \lambda(4x^2 + y^2 + 4z^2 - 16)$$

then

$$F_x(x, y, z, \lambda) = 16x + 8\lambda x$$

$$F_y(x, y, z, \lambda) = 4z + 2\lambda y$$

$$F_z(x, y, z, \lambda) = 4y - 16 + 8\lambda z$$

Let

$$\begin{cases} F_x(x, y, z, \lambda) = 16x + 8\lambda x = 0 \\ F_y(x, y, z, \lambda) = 4z + 2\lambda y = 0 \\ F_z(x, y, z, \lambda) = 4y - 16 + 8\lambda z = 0 \\ 4x^2 + y^2 + 4z^2 - 16 = 0 \end{cases}$$

We can find all suspicious points

$$\begin{cases} x_1 = \frac{4}{3} \\ y_1 = -\frac{4}{3} \\ z_1 = -\frac{4}{3} \end{cases}, \quad \begin{cases} x_2 = -\frac{4}{3} \\ y_2 = -\frac{4}{3} \\ z_2 = -\frac{4}{3} \end{cases}, \quad \begin{cases} x_3 = 0 \\ y_3 = 4 \\ z_3 = 0 \end{cases}, \quad \begin{cases} x_4 = 0 \\ y_4 = -2 \\ z_4 = \sqrt{3} \end{cases}, \quad \begin{cases} x_5 = 0 \\ y_5 = -2 \\ z_5 = -\sqrt{3} \end{cases}$$

Since $T(\frac{4}{3},-\frac{4}{3},-\frac{4}{3}) = \frac{176}{3}$,

$T(-\frac{4}{3},-\frac{4}{3},-\frac{4}{3}) = \frac{176}{3}$,

$T(0,4,0) = 600$,

$T(0,-2,\sqrt{3}) = 600 - 24\sqrt{3}$

$T(0,-2,-\sqrt{3}) = 600 + 24\sqrt{3}$

then the hottest point on the ellipsoid is $(0,-2,-\sqrt{3})$.

4. Find the shortest distance from the point $(1,0,-2)$ to the plane $x+2y+z=4$.

Solution:

The distance from any point (x, y, z) to the point $(1,0,-2)$ is

$$d = \sqrt{(x-1)^2 + y^2 + (z+2)^2},$$

but if (x, y, z) lies on the plane $x+2y+z=4$, then $z = 4-x-2y$ and so we have

$$d = \sqrt{(x-1)^2 + y^2 + (6-x-2y)^2}.$$

We can minimize d by minimizing the simpler expression

$$d^2 = f(x,y) = (x-1)^2 + y^2 + (6-x-2y)^2.$$

By solving the equations

$$f_x = 2(x-1) - 2(6-x-2y) = 4x + 4y - 14 = 0,$$

$$f_y = 2y - 4(6-x-2y) = 4x + 10y - 24 = 0,$$

we find that the only critical point is $(\frac{11}{6}, \frac{5}{3})$. Since $f_{xx} = 4, f_{xy} = 4,$ and $f_{yy} = 10,$ we have $D(x,y) = f_{xx}f_{xy} - (f_{xy})^2 = 24 > 0$ and $f_{xx} > 0$, so by the Second Derivatives Test f has a local minimum at $(\frac{11}{6}, \frac{5}{3})$. Intuitively, we can see that this local minimum is actually an absolute minimum because there must be a point on the given plane that is closet to $(1,0,-2)$. If $x=\frac{11}{6}$ and $y=\frac{5}{3}$, then

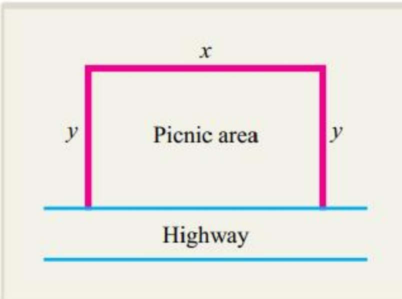
$$d=\sqrt{(x-1)^2+y^2+(6-x-2y)^2}=\sqrt{(\tfrac{5}{6})^2+(\tfrac{5}{3})^2+(\tfrac{5}{6})^2}=\tfrac{5}{6}\sqrt{6}.$$

The shortest distance from $(1,0,-2)$ to the plane $x+2y+z=4$ is $=\frac{5}{6}\sqrt{6}$....10 marks

5. The highway department is planning to build a picnic area for motorists along a major highway. It is to be rectangular with an area of 800 square yards and is to be fenced off on the three sides not adjacent to the highway. What is the least amount of fencing that will be needed to complete the job?

Solution:

Label the sides of the picnic area as indicated in the following Figure and let f denote the amount of fencing required.



Then, $f(x,y)=x+2y$. The goal is to minimize f given the requirement that the area must be 800 square yards; that is, subject to the constraint

$$g(x,y)=xy=800.$$

Find the partial derivatives

$$f_x=1 \qquad f_y=2 \qquad g_x=y \qquad \text{and} \qquad g_y=x$$

and obtain the three Lagrange equations

$$1=\lambda y \qquad 2=\lambda x \qquad \text{and} \qquad xy=800.$$

From the first and second equations you get

$$\lambda=\frac{1}{y} \quad \text{and} \quad \lambda=\frac{2}{x}$$

(since $y\neq 0$ and $x\neq 0$), which implies that

$$\frac{1}{y}=\frac{2}{x} \quad \text{and} \quad x=2y$$

Now substitute $x = 2y$ into the third Lagrange equation to get

$$2y^2 = 800 \text{ or } y = \pm 20$$

and use $y = 20$ in the equation $x = 2y$ to get $x = 40$. Thus, $x = 40$ and $y = 20$ are the values that minimize the function $f(x, y) = x + 2y$ subject to the constraint $xy = 800$. The optimal picnic area is 40 yards wide (along the highway), extends 20 yards back from the road, and requires 40+20+20=80 yards of fencing.

Part Five: Proof Questions

1. If $z = f[e^{xy}, \cos(xy)]$ and f is differentiable，show the following equality：

$$x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y} = 0$$

Proof： Let $u = e^{xy}$， $v = \cos xy$， then $z = f[u, v]$

$$\frac{\partial z}{\partial x} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x} = y\frac{\partial f}{\partial u}e^{xy} - \frac{\partial f}{\partial v}y\sin xy$$

$$\frac{\partial z}{\partial y} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial y} = x\frac{\partial f}{\partial u}e^{xy} - \frac{\partial f}{\partial v}x\sin xy$$

$$\text{Left}= x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y} = xy\frac{\partial f}{\partial u}e^{xy} - x\frac{\partial f}{\partial v}y\sin xy - xy\frac{\partial f}{\partial u}e^{xy} + x\frac{\partial f}{\partial v}y\sin xy = 0$$

=Right

2. If $z = z(x, y)$ is defined by the equation $F(z + \frac{1}{x}, z - \frac{1}{y}) = 0$ implicitly,where F has continuous second order partial derivatives，and $F_u(u, v) = F_v(u, v) \neq 0$,show that

$$x^2\frac{\partial z}{\partial x} + y^2\frac{\partial z}{\partial y} = 0 \quad \text{and} \quad x^3\frac{\partial^3 z}{\partial x^2} + xy(x+y)\frac{\partial^2 z}{\partial x\partial y} + y^3\frac{\partial^2 z}{\partial y^2} = 0$$

Proof： Differentiate both sides $F(z + \frac{1}{x}, z - \frac{1}{y}) = 0$ with respect to x and y respectively, we have

$$F_u(\frac{\partial z}{\partial x}-\frac{1}{x^2})+F_v\frac{\partial z}{\partial x}=0$$

$$F_u\frac{\partial z}{\partial y}+F_v(\frac{\partial z}{\partial y}+\frac{1}{y^2})=0$$

Then

$$\frac{\partial z}{\partial x}=\frac{F_u}{x^2(F_u+F_v)},\qquad \frac{\partial z}{\partial y}=-\frac{F_v}{y^2(F_u+F_v)},$$

Therefore,

$$x^2\frac{\partial z}{\partial x}+y^2\frac{\partial z}{\partial y}=\frac{F_u-F_v}{F_u+F_v}=0$$

Differentiate both sides of the above equality with respect to x and y respectively, we have

$$2x\frac{\partial z}{\partial x}+x^2\frac{\partial^2 z}{\partial x^2}+y^2\frac{\partial^2 z}{\partial y\partial x}=0$$

$$x^2\frac{\partial^2 z}{\partial x\partial y}+2y\frac{\partial z}{\partial y}+y^2\frac{\partial^2 z}{\partial y^2}=0$$

then

$$x^3\frac{\partial^2 z}{\partial x^2}+xy^2\frac{\partial^2 z}{\partial y\partial x}=-2x^2\frac{\partial z}{\partial x}$$

$$x^2y\frac{\partial^2 z}{\partial x\partial y}+y^3\frac{\partial^2 z}{\partial y^2}=-2y^2\frac{\partial z}{\partial y}$$

Therefore,

$$\begin{aligned}x^3\frac{\partial^3 z}{\partial x^2}+xy(x+y)\frac{\partial^2 z}{\partial x\partial y}+y^3\frac{\partial^2 z}{\partial y^2}&=-2x^2\frac{\partial z}{\partial x}-2y^2\frac{\partial z}{\partial y}\\&=-2(x^2\frac{\partial z}{\partial x}+y^2\frac{\partial z}{\partial y})=0\end{aligned}$$

3. If $u(x,y,z)=\dfrac{2}{\sqrt{x^2+y^2+z^2}}$, $(x,y,z)\neq(0,0,0)$, show that $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}+\dfrac{\partial^2 u}{\partial z^2}$=0

Proof： $\dfrac{\partial u}{\partial x}=\dfrac{-2x}{(x^2+y^2+z^2)^{\frac{3}{2}}}$

$$\frac{\partial^2 u}{\partial x^2}=\frac{-2}{(x^2+y^2+z^2)^{\frac{3}{2}}}+\frac{6x^2}{(x^2+y^2+z^2)^{\frac{5}{2}}}$$

Similarly,

$$\frac{\partial^2 u}{\partial y^2}=\frac{-2}{(x^2+y^2+z^2)^{\frac{3}{2}}}+\frac{6y^2}{(x^2+y^2+z^2)^{\frac{5}{2}}}，\quad \frac{\partial^2 u}{\partial z^2}==\frac{-2}{(x^2+y^2+z^2)^{\frac{3}{2}}}+\frac{6z^2}{(x^2+y^2+z^2)^{\frac{5}{2}}}$$

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=\frac{-2}{(x^2+y^2+z^2)^{\frac{3}{2}}}+\frac{6x^2}{(x^2+y^2+z^2)^{\frac{5}{2}}}+\frac{-2}{(x^2+y^2+z^2)^{\frac{3}{2}}}+\frac{6y^2}{(x^2+y^2+z^2)^{\frac{5}{2}}}$$

$$+\frac{-2}{(x^2+y^2+z^2)^{\frac{3}{2}}}+\frac{6z^2}{(x^2+y^2+z^2)^{\frac{5}{2}}}=0$$