Test

December 7, 2023

Problem 1. (15 points)

1. Apply classical Gram-Schimdt orthogonalization and Householder reflectors to find the full QR factorization of the matrix $\begin{bmatrix} 2 & 3 \\ -2 & -6 \\ 1 & 0 \end{bmatrix}$.
2. Please find the least squares solution and 2-norm error $\|e\|_2$ for the following inconsistent systems: $\begin{bmatrix} 2 & 3 \\ -2 & -6 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \\ 6 \end{bmatrix}$.

Problem 2. (10 points) Fit the data to periodic models $F_1(t) = c_1 + c_2 \cos 2\pi t + c_3 \sin 2\pi t$ and $F_2(t) = c_1 + c_2 \cos 2\pi t + c_3 \sin 2\pi t + c_4 \cos 4\pi t$. Find and compare the 2-norm errors $\|e\|_2$ of F_1 and F_2. The points are as follows: $(0, 0), (\frac{1}{4}, 2), (\frac{1}{2}, 3), (\frac{3}{4}, 1)$.

Problem 3. (10 points) Given a square matrix $A = \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}$, please answer the following questions:

1. Find all eigenvalues and eigenvectors of A;
2. Apply three steps of Power Iteration with initial vector $x_0 = (1, 0)$;
3. Predict the result of applying Inverse Power Iteration with shift $s = 1$ and $s = 5$ respectively, and explain the reason.

Problem 4. (10 points) Use the three-point centered-difference formula to approximate $f'(0)$, where $f(x) = e^x$, and find the approximation error for $h = 0.1$.

Problem 5. (15 points) Apply the Trapezoid Rule, Simpson's Rule and Midpoint Rule to approximate the integral $\int_0^1 x^2 dx$. Compute the error by comparing with the exact value from calculus.

Problem 6. (10 points) Given the following:

- x and w: two vectors with $\|x\|_2 = \|w\|_2$;
- $u = w - x$ and $v = \frac{u}{\|u\|_2}$;
- $H = I - 2vv^{-1}$.

Please prove that $Hx = w$ and $Hw = x$.

Problem 7. (10 points) Please prove the following:
Let A be an $m \times n$ matrix where $m \leq n$. There are two orthonormal bases $\{v_1, \cdots, v_n\}$ of R^n and $\{u_1, \cdots, u_m\}$ of R^m, and real numbers $s_1 \geq s_2 \geq \cdots \geq 0$ s.t. $Av_i = s_i u_i$ for $1 \leq i \leq m$.

Problem 8. (10 points) Please briefly introduce the main idea of Google Page Rank.

Problem 9. (10 points) Please briefly introduce the idea of Trapezoid Rule, Simpson's Rule, Composite Newton-Cotes formulas and Midpoint Rule and compare them.