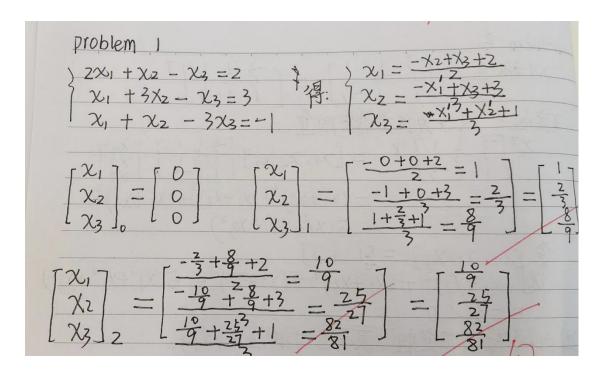
Problem 1. (20 points) Find the first two steps of the Gauss-Seidel Methods for the following linear system, using the initial vector $x_0 = [0, 0, 0]$.

$$\begin{array}{rclcrcl} 2x_1+x_2-x_3 & = & 2, \\ x_1+3x_2-x_3 & = & 3, \\ x_1+x_2-3x_3 & = & -1. \end{array}$$



		iteration w	
	[WH]	Jajuk G	[\(\frac{1}{2} \) (2-\(\frac{1}{2} \) (2-\(\frac{1}{2} \) (3-\(\fra
	Wet1 = D-1 (b-1	/Xk-0/Xk+1)=	3(3-Wet1+We)
-	[Wet]	ः ः वास्त्र	-3(-1-Uk+1-Uk+1)
-	Since 70 = [16, 70)	$W_0]^{T} = [0,0,0]$	1 - 1 - 1 - 1 - 1 - 1 - 1 - 1 - 1 - 1 -
,	[14] [\$\frac{1}{2}(2-76-	+26) [1]	[12] [\$\frac{1}{2}(2-71+711)] [\$\frac{10}{9}\$
	V1 = \$13-74+	$-W_0) = \frac{2}{3},$	1/2 = 3 (3-76+74) = 25
1 1	W3(-1-14)	-VI) [+8/9]	W2 = 3(-1-16-16) 82

Algorithm 1: Conjugate Gradient Method

```
1 x_0 be a zero vector

2 d_0 = r_0 = b - Ax_0

3 for k = 0, 1, 2, ..., n - 1 do

4 | if r_k is sufficiently small then

5 | return \ x_k

6 | \alpha_k = \frac{r_k^\top r_k}{d_k^\top A d_k}

7 | x_{k+1} = x_k + \alpha_k d_k

8 | r_{k+1} = r_k - \alpha_k A d_k

9 | \beta_k = \frac{r_{k+1}^\top r_{k+1}}{r_k^\top r_k}

10 | d_{k+1} = r_{k+1} + \beta_k d_k
```

According the above algorithm, please prove the following: Suppose that $b \neq 0$ and $r_k \neq 0$ for k < n. Then for each $1 \leq k \leq n$,

- 1. distinct residuals are pairwise orthogonal: $r_k^{\top} r_j = 0$ for j < k;
- 2. distinct vectors of a subspace span are pairwise A-conjugate: $d_k^{\top} A d_j = 0$ for j < k.

Base case (k = 1):

$$\beta_0 = \frac{r_1^\top r_1}{r_0^\top r_0} = -\frac{r_1^\top A d_0}{d_0^\top A d_0};$$

Inductive step (k > 1): Suppose that the k - 1 case hold.

The 2nd item:

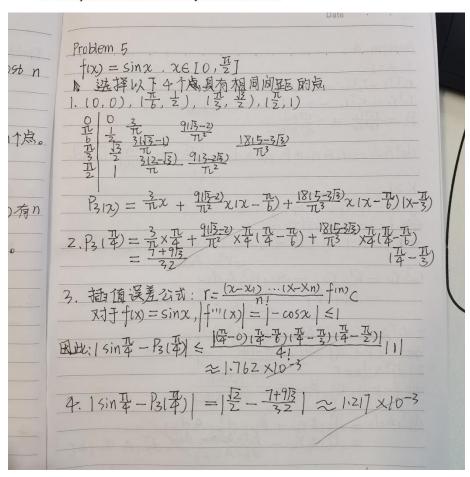
$$\textbf{0} \ \text{ If } j < k-1 \text{, then } r_j^\top r_k = 0;$$

$$\text{ If } j = k-1 \text{, then } d_{k-1}^\top A d_{k-1} = r_{k-1}^\top A d_{k-1} + \beta_{k-1} d_{k-2}^\top A d_{k-1} \\ = r_{k-1}^\top A d_{k-1};$$

$$\begin{array}{l} \bullet \ \ r_{k-1}^{\top} r_k = r_{k-1}^{\top} r_{k-1} - \alpha_{k-1} r_{k-1}^{\top} A d_{k-1} \\ = r_{k-1}^{\top} r_{k-1} - \frac{r_{k-1}^{\top} r_{k-1}}{d_{k-1}^{\top} A d_{k-1}} d_{k-1}^{\top} A d_{k-1} = 0. \end{array}$$

Problem 3. (20 points) Let $f(x) = \sin(x)$ where $x \in [0, \frac{\pi}{2}]$.

- 1. By using Newton's divided differences method, write down a degree 3 polynomial $P_3(x)$ with evenly spaced points;
- 2. Calculate $P_3(\frac{\pi}{4})$;
- 3. Give an error bound for the approximation in (2);
- 4. Compare the actual error to your error bound.



Problem 4. (20 points) Let
$$T_n(x)=\left\{\begin{array}{ll} 1, & n=0;\\ x, & n=1;\\ 2xT_{n-1}(x)-T_{n-2}(x), & n>1. \end{array}\right.$$
 Please prove the following:

1.
$$deg(T_n) = n;$$

2. The leading coefficient of T_n is 2^{n-1} for $n \ge 1$;

3.
$$T_n(1) = 1$$
 and $T_n(-1) = (-1)^n$;

4.
$$T_n(x) = \cos(n \arccos(x))$$
 for $-1 \le x \le 1$;

5.
$$|T_n(x)| \le 1$$
 for $-1 \le x \le 1$;

6. All roots of $T_n(x)$ are located between -1 and 1.

```
T_{n(x)} = \begin{cases} x \\ 2xT_{n-1}(x) - T_{n-2}(x) \end{cases} 
 Problem 7
 1. 对"当n=0与n=1时, deg(To(x))=0, deg(Ti(x))=1

\pm n_{7}
, 
\deg(T_n) = \deg(Z_{X_1} T_{n-1} T_{X_1}) - T_{n-2} T_{x_2})

= \deg(T_{n-1} T_{x_1}) + 1

      由此可得当的, deg (Th) 为其前一项的复数+1,
      因此追纳旅谈可钨 deg (Tn+1X)) = n-1
=> deg (Tn(X)) = n-1 +1 = n
Z. 当n=1与n=文时, lc(T1(x))=1c(x)=1, lc(Tz(x))=1c(zx2)
    RJ + n71, 1c(Tn) = ((2xTn-1x)-Tn-2(x))
                       = Z· (C(Tn+(X))
    由归纳假设可得: 1c(Tn+10x1)=2n-2
                 因此: (ctTn)=2·2n-2=2n-1
3. 当 n=1, n=2, Tn(1)=1, Tn(-1)=(-1)n 成成之,
  - 般地 Tn+1(1) = 2x1 Tn(1) - Tn-1(1)= 2-1=1
           T_{n+1}(-1) = 2 \times (-1) T_{n}(-1) - T_{n-1}(-1)
= -2 \times (-1)^n - (-1)^{n-1}
= (-1)^{n-1} \times (-1)
                      =(-1)^{n+1}
   2 y = arccosx, cozy=x (=(-1 ≤x≤1)
4. T_{n+1}(x) = 2x T_n(x) - T_{n-1}(x)
             = 2x cos my - cos (my-y)
             = zxcosmy - cosmy cosy - Sin ny Siny
             = 2 cosy cosmy - cusmy cosy - sin my siny
             = cosny cosy - sinny siny
              = & COS (m+1) y .
                                        (Thix) = cosin arccosx)
              = cos (n+1) arc (cosx)
```

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9.
   (1) Base case :
                           To (-1) = 1 = (-1)
  To(1) = 1 (n=0)
                                T, (-1) = -1 = (-1)
  Ti(1) = 1 (n=1)
  Inductive steep (n >1)
   T_{n}(t) = 2 T_{n-1}(t) - T_{n-2}(t) = 2-1 = 1
          T_{n(-1)} = -2 T_{n-1}(-1) - T_{n-2}(-1) = -2 \times (-1)^{n-1} - (-1)^{n-2}
                                         = 2x(-1)" - (-1)"
                                         = (-1)"
  thence, Ta(1)=1, Ta(-1) = (-1)"
  (2) let y = arccour.
  Base case:
    To (x) = 1 = 005 (0 = arccos(x))
      Tila) = 2 = Os (Ix arcces(x))
 Inductive steep (n>1):
        T_n(x) = 2 x T_{n-1}(x) - T_{n-2}(x)
               = 2 x 000 (n-1)y - 005 (n-2)y
              = 22 casin-1) y - cos[in-1)y-y]
               = 2 x cos(n-1)y - cos(n-1)y cosy - sin(n-1)y siny
               = 2x005(n-1)y - x005(n-1)y - sin(n-1)y sing
               = Cos(n-1) ysogy - sin(n-1)ysiny
               = Cos[(n-1)y+y]
               = cosny = cos(narcosx)
(3) According to (2), |T_n(x)| = |\cos(n\alpha\cos x)| \le 1, for -1 \le x \le 1.
   ( ALL cosx in the range of [-1.13)
(4) T_n(x) = \cos(n \operatorname{arccos} x) = 0
  \Rightarrow narccosx = (k + \frac{1}{2})\pi, kee
   \Rightarrow arcosx = \frac{2kH}{2n}\pi
  \Rightarrow \qquad \chi = \cos \frac{2k+1}{2n} \pi
Hexe, -1 = x = 000 3K+1 x =1
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