Continuous Random Variables and Probability Distributions

- **4.1 Continuous Random Variables and Probability Density Functions**
- **4.2 Cumulative Distribution Functions and Expected Values**
- **4.3 The Normal Distribution**
- **4.4 The Gamma Distribution and Its Relatives**
- **4.5 Other Continuous Distributions**
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4.4 The Gamma Distribution and Its Relatives

- Gamma Function

For α>0, the gamma function Γ(α) is defined by

$$\Gamma(\alpha) = \int_0^\infty x^{a-1} e^{-x} dx$$

The most important **properties** of the gamma function are the following:

1. For any α>1, Γ(α) = (α-1) Γ(α-1) ;
2. For any positive integer n, Γ(n)=(n-1)!
3. Γ(1/2) = $\sqrt{\pi}$

4.4 The Gamma Distribution and Its Relatives

- **The Family of Gamma Distributions**

A continuous random variable X is said to have a gamma distribution if the pdf of X is

$$f(x;\alpha,\beta)=\begin{cases}\dfrac{1}{\beta^{\alpha}\Gamma(\alpha)}\,x^{\alpha\text{-}1}\mathrm{e}^{-x/\beta} & x\geq 0\\ 0 & \text{otherwise}\end{cases}$$

where the parameters α and β satisfy α >0, β > 0.

The standard gamma distribution has β = 1.

4.4 The Gamma Distribution and Its Relatives

- **Standard Gamma Distribution**

$$f(x;\alpha)=\begin{cases}\dfrac{x^{\alpha-1}e^{-x}}{\Gamma(\alpha)} & x\geq 0\\ 0 & \text{otherwise}\end{cases}$$

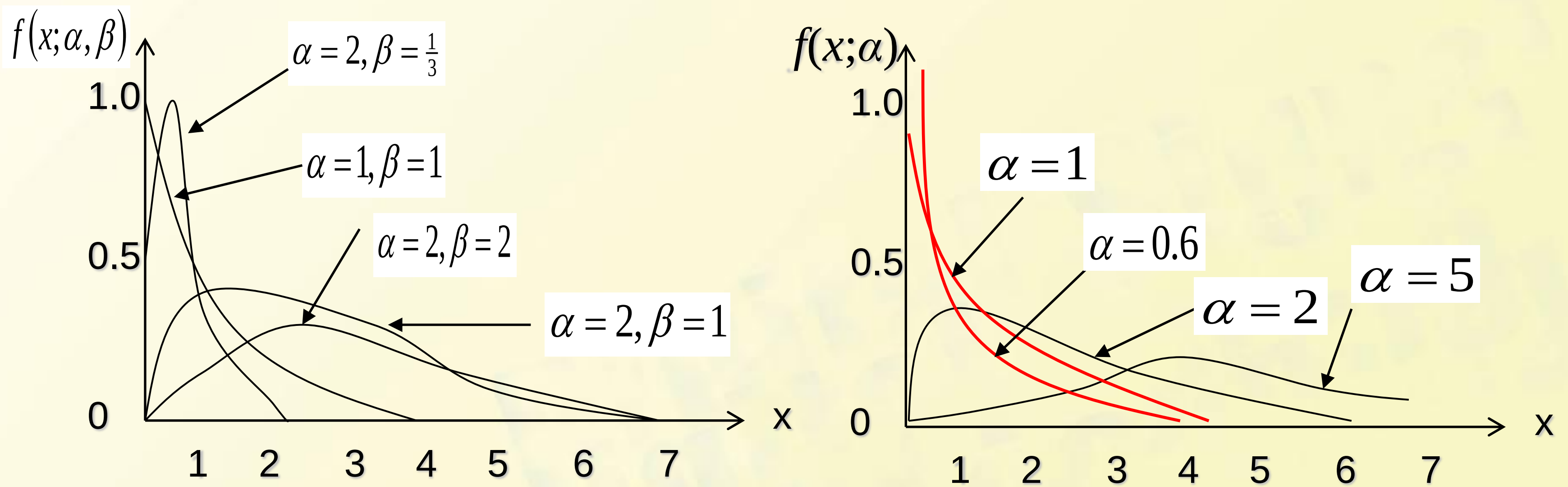
Satisfying the two Basic Properties of a pdf:

$$1: f(x;a)\geq 0$$

$$2: \int_0^{\infty} f(x;a)dx=\frac{\int_0^{\infty} x^{\alpha-1}e^{-x}dx}{\Gamma(\alpha)}=\frac{\Gamma(\alpha)}{\Gamma(\alpha)}=1$$

4.4 The Gamma Distribution and Its Relatives

- Illustrations of the Gamma pdfs



(a) Gamma density curves (b) Standard gamma density curves

4.4 The Gamma Distribution and Its Relatives

- Mean and Variance

The mean and variance of a random variable X having the gamma distribution f(x;α,β) are

$$\mathbf{E(X) = \mu = \alpha\beta}$$

$$\mathbf{V(X) = \delta^2 = \alpha\beta^{2}}$$

- The cdf of a standard gamma distribution

$$F(x;\alpha) = \int_0^x \frac{y^{\alpha-1}e^{-y}}{\Gamma(\alpha)}\,dy \qquad x > 0$$

Incomplete gamma function (or without the denominator Γ(α) sometimes)

Refer to Appendix Table A.4

4.4 The Gamma Distribution and Its Relatives

- **Example 4.22**

  Suppose the reaction time X of a randomly selected individual to a certain stimulus has a standard gamma distribution with $\alpha=2$ sec. Then

  $P(3 \le X \le 5)=?$

  $P(X>4)=?$

Solution:

$$P(3 \leq X \leq 5) = F(5;2) - F(3;2)$$
$$= 0.960 - 0.801 = 0.159$$

$$P(X>4) = 1 - P(X \leq 4) = 1 - F(4;2) = 1 - 0.908 = 0.902$$

4.4 The Gamma Distribution and Its Relatives

- Proposition

Let X have a gamma distribution with parameters α and β. Then for any $x > 0$, the **cdf** of X is given by

$$P(X \le x) = F(x;\alpha,\beta) = F\left(\frac{x}{\beta};\alpha\right)$$

where F(•; α) is the incomplete gamma function.

4.4 The Gamma Distribution and Its Relatives

- **Example 4.24**

Suppose the survival time X in weeks of a randomly selected male mouse exposed to 240 rads of gamma radiation has a gamma distribution with $\alpha=8$ and $\beta=15$, then

(1) The probability that a mouse survives between 60 and 120 weeks is ?

(2) The probability that a mouse survives at least 30 weeks is

Solution:

(1) $P(X \le 10) = F(10; 0.2) = 1 - e^{-(0.2)(10)} = 0.865$

(2) $P(5 \le X \le 10) = F(10; 0.2) - F(5; 0.2) = 0.233$

4.4 The Gamma Distribution and Its Relatives

- **The Exponential Distribution**

X is said to have an exponential distribution with parameter λ (λ>0) if the pdf of X is

$$f(x;\lambda) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Just a special case of the general gamma pdf

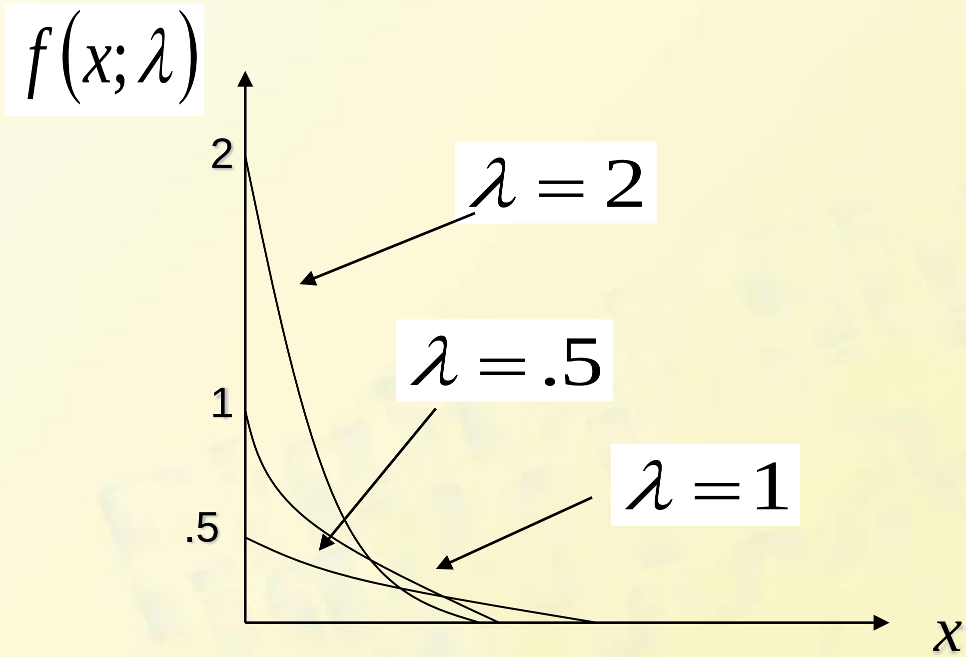
$\alpha=1$ and $\beta = 1/\lambda$

therefore, we have

$E(X) = \alpha\beta = 1/\lambda$; $V(X) = \alpha\beta^2 = 1/\lambda^2$

4.4 The Gamma Distribution and Its Relatives

- Illustrations of the Exponential pdfs



4.4 The Gamma Distribution and Its Relatives

- **The cdf of Exponential Distribution**

Unlike the general gamma pdf, the exponential pdf can be easily integrated.

$$F(x;\lambda)=\begin{cases} 0 & x<0 \\ 1-e^{-\lambda x} & x\geq 0 \end{cases}$$

4.4 The Gamma Distribution and Its Relatives

- **Example**

Suppose the response time X at a certain on-line computer terminal (the elapsed time between the end of a user's inquiry and the beginning of the system's response to inquiry) has an exponential distribution with expected response time equal to 5 sec. then E(X) =1/ λ =5, so λ=0.2. the probability that the response tine is at most 10 sec is

$$P(X \le 10) = F(10;0.2) = 1 - e^{-(0.2)(10)} = 0.865$$

The probability that response time is between 5 and 10 sec is

$$P(5 \le X \le 10) = F(10;0.2) - F(5;0.2) = 0.233$$

The Chi-Squared Distribution

Let v be a positive integer. Then a random variable X is said to have a chi-squared distribution with parameter v if the **pdf of X** is the gamma density with $\alpha = v/2$ and $\beta = 2$. The pdf of a chi-squared rv is thus

$$f(x, v) = \begin{cases} \dfrac{1}{2^{v/2}\Gamma(v/2)} x^{(v/2)-1} e^{-x/2} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

The parameter v is called the number of degrees of freedom of X. The symbol χ^2 is often used in place of "chi-squared."