Answers to Exercises

Section 5.1 9, 12, 18, 19

Section 5.2 24，26，33，35

Section 5-1

9.

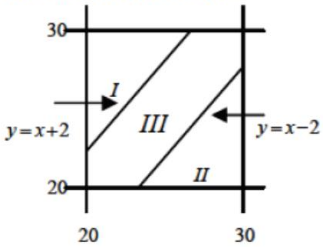
a. $1=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(x,y)dxdy=\int_{20}^{30}\int_{20}^{30}K(x^2+y^2)dxdy=K\int_{20}^{30}\int_{20}^{30}x^2dydx+K\int_{20}^{30}\int_{20}^{30}y^2dxdy$

$=10K\int_{20}^{30}x^2dx+10K\int_{20}^{30}y^2dy=20K\cdot\left(\frac{19,000}{3}\right)\Rightarrow K=\frac{3}{380,000}.$

b. $P(X<26 \text{ and } Y<26)=\int_{20}^{26}\int_{20}^{26}K(x^2+y^2)dxdy=K\int_{20}^{26}\left[x^2y+\frac{y^3}{3}\right]_{20}^{26}dx=K\int_{20}^{26}(6x^2+3192)dx=$

$K(38,304)=.3024.$

c. The region of integration is labeled *III* below.



$P(|X-Y|\le 2)=\iint_{III}f(x,y)dxdy=1-\iint_{I}f(x,y)dxdy-\iint_{II}f(x,y)dxdy=$

$1-\int_{20}^{28}\int_{x+2}^{30}f(x,y)dydx-\int_{22}^{30}\int_{20}^{x-2}f(x,y)dydx=.3593$(after much algebra).

d. $f_X(x)=\int_{-\infty}^{\infty}f(x,y)dy=\int_{20}^{30}K(x^2+y^2)dy=10Kx^2+K\left.\frac{y^3}{3}\right|_{20}^{30}=10Kx^2+.05$, for $20\le x\le 30$.

e. $f_Y(y)$ can be obtained by substituting y for x in (d); clearly $f(x,y)\neq f_X(x)\cdot f_Y(y)$, so X and Y are not independent.

12.

a. $P(X > 3) = \int_3^\infty \int_0^\infty xe^{-x(1+y)}dydx = \int_3^\infty e^{-x}dx = .050.$

b. The marginal pdf of X is $f_X(x) = \int_0^\infty xe^{-x(1+y)}dy = e^{-x}$ for $x \geq 0$. The marginal pdf of Y is $f_Y(y) = \int_3^\infty xe^{-x(1+y)}dx = \frac{1}{(1+y)^2}$ for $y \geq 0$. It is now clear that $f(x,y)$ is not the product of the marginal pdfs, so the two rvs are not independent.

c. P(at least one exceeds 3) $= P(X > 3 \text{ or } Y > 3) = 1 - P(X \leq 3 \text{ and } Y \leq 3)$

$$= 1 - \int_0^3 \int_0^3 xe^{-x(1+y)}dydx = 1 - \int_0^3 \int_0^3 xe^{-x}e^{-xy}dy$$

$$= 1 - \int_0^3 e^{-x}(1 - e^{-3x})dx = e^{-3} + .25 - .25e^{-12} = .300.$$

18.

a. $P_{y|X}(y|1)$ results from dividing each entry in x = 1 row of the joint probability table by $p_x(1) = .34$:

$$P_{y|x}(0\,|\,1) = \frac{.08}{.34} = .2353$$

$$P_{y|x}(1\,|\,1) = \frac{.20}{.34} = .5882$$

$$P_{y|x}(2\,|\,1) = \frac{.06}{.34} = .1765$$

b. $P_{y|X}(x|2)$ is requested; to obtain this divide each entry in the y = 2 row by $p_x(2) = .50$:

y	0	1	2
$P_{y\|X}(y\|2)$	.12	.28	.60

c. $P(Y \leq 1 \mid x = 2) = P_{y|X}(0|2) + P_{y|X}(1|2) = .12 + .28 = .40$

d. $P_{X|Y}(x|2)$ results from dividing each entry in the y = 2 column by $p_y(2) = .38$:

x	0	1	2
$P_{x\|y}(x\|2)$	.0526	.1579	.7895

19.

a. $f_{Y|X}(y \mid x) = \dfrac{f(x,y)}{f_X(x)} = \dfrac{k(x^2+y^2)}{10kx^2+.05}$ $20 \le y \le 30$

$f_{X|Y}(x \mid y) = \dfrac{k(x^2+y^2)}{10ky^2+.05}$ $20 \le x \le 30$ $\left(k = \dfrac{3}{380{,}000}\right)$

b. $P(Y \ge 25 \mid X = 22) = \int_{25}^{30} f_{Y|X}(y \mid 22)\,dy$

$= \int_{25}^{30} \dfrac{k((22)^2+y^2)}{10k(22)^2+.05}\,dy = \cancel{.783}$ 0. 556

$P(Y \ge 25) = \int_{25}^{30} f_Y(y)\,dy = \int_{25}^{30} (10ky^2+.05)\,dy = \cancel{.75}$ 0. 549

c. $E(Y \mid X=22) = \int_{-\infty}^{\infty} y \cdot f_{Y|X}(y \mid 22)\,dy = \int_{20}^{30} y \cdot \dfrac{k((22)^2+y^2)}{10k(22)^2+.05}\,dy$

$= 25.372912$

$E(Y^2 \mid X=22) = \int_{20}^{30} y^2 \cdot \dfrac{k((22)^2+y^2)}{10k(22)^2+.05}\,dy = 652.028640$

$V(Y \mid X = 22) = E(Y^2 \mid X=22) - [E(Y \mid X=22)]^2 = 8.243976$

$\sigma = \sqrt{V(Y \mid X = 22)}$ =2.87

Section 5-2

24. Let $h(X, Y)$ = the number of individuals who handle the message. A table of the possible values of (X, Y) and of $h(X, Y)$ are displayed in the accompanying table.

	$h(x, y)$	y = 1	2	3	4	5	6
	1	-	2	3	4	3	2
	2	2	-	2	3	4	3
x	3	3	2	-	2	3	4
	4	4	3	2	-	2	3
	5	3	4	3	2	-	2
	6	2	3	4	3	2	-

Since $p(x,y) = \frac{1}{30}$ for each possible (x, y),

$E[h(X,Y)] = \sum_x \sum_y h(x,y) \cdot p(x,y) = \sum_x \sum_y h(x,y) \cdot \frac{1}{30} = \cdots = \frac{84}{30} = 2.80$.

26. Revenue = $3X + 10Y$, so $E(\text{revenue}) = E(3X + 10Y)$

$$= \sum_{x=0}^{5}\sum_{y=0}^{2}(3x+10y)\cdot p(x,y) = 0\cdot p(0,0)+\ldots+35\cdot p(5,2) = 15.4 = \$15.40.$$

33. Since $E(XY) = E(X)\cdot E(Y)$, $\text{Cov}(X, Y) = E(XY) - E(X)\cdot E(Y) = E(X)\cdot E(Y) - E(X)\cdot E(Y) = 0$, and since $\text{Corr}(X, Y) = \dfrac{\text{Cov}(X,Y)}{\sigma_X\sigma_Y}$, then $\text{Corr}(X, Y) = 0$.

35.

a. $\text{Cov}(aX + b, cY + d) = E[(aX + b)(cY + d)] - E(aX + b)\cdot E(cY + d)$
$= E[acXY + adX + bcY + bd] - (aE(X) + b)(cE(Y) + d)$
$= acE(XY) + adE(X) + bcE(Y) + bd - [acE(X)E(Y) + adE(X) + bcE(Y) + bd]$
$= acE(XY) - acE(X)E(Y) = ac[E(XY) - E(X)E(Y)] = ac\text{Cov}(X, Y)$.

b. $\text{Corr}(aX + b, cY + d) = \dfrac{\text{Cov}(aX+b, cY+d)}{SD(aX+b)SD(cY+d)} = \dfrac{ac\text{Cov}(X,Y)}{|a|\cdot|c|\,SD(X)SD(Y)} = \dfrac{ac}{|ac|}\text{Corr}(X,Y)$. When a and c have the same signs, $ac = |ac|$, and we have
$\text{Corr}(aX + b, cY + d) = \text{Corr}(X, Y)$

c. When a and c differ in sign, $|ac| = -ac$, and we have $\text{Corr}(aX + b, cY + d) = -\text{Corr}(X, Y)$.