Chapter 10

Inference for Regression

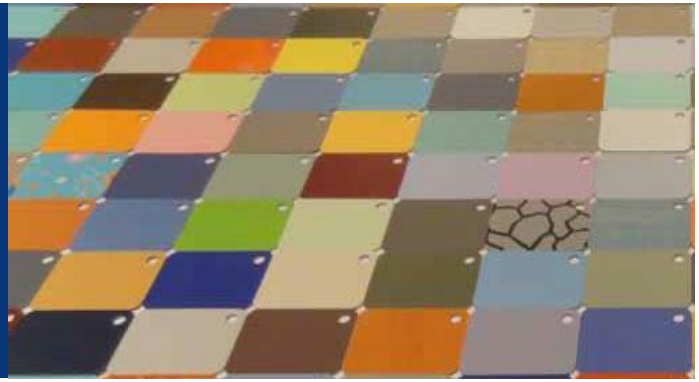
Introduction to the Practice of

STATISTICS EIGHTH EDITION

Moore / McCabe / Craig

Lecture Presentation Slides

Chapter 10
Inference for Regression



10.1 Simple Linear Regression

10.2 More Detail about Simple Linear Regression*

10.1 Simple Linear Regression

- Statistical model for linear regression
- Simple linear regression model
- Estimating the regression parameters
- Confidence intervals and significance tests
- Confidence interval for mean response
- Prediction interval

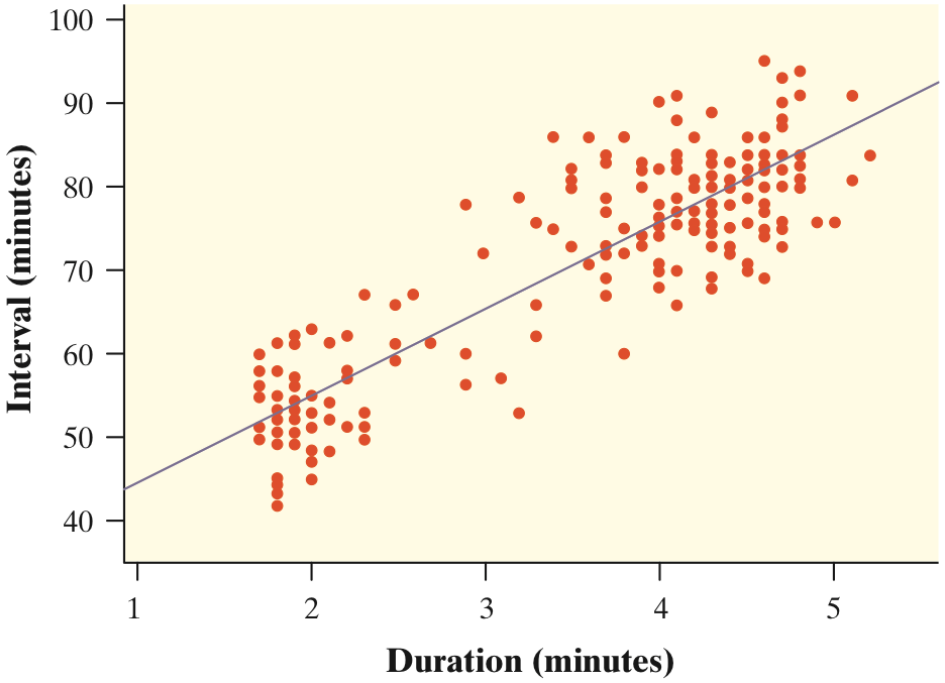
Introduction

When a scatterplot shows a linear relationship between a quantitative explanatory variable *x* and a quantitative response variable *y*, we can use the least-squares line fitted to the data to predict *y* for a given value of *x*. If the data are a random sample from a larger population, we need statistical inference to answer questions like these:

- ✓ Is there really a linear relationship between *x* and *y* in the population, or could the pattern we see in the scatterplot plausibly happen just by chance?
- ✓ What is the slope (rate of change) that relates *y* to *x* in the population, including a margin of error for our estimate of the slope?
- ✓ If we use the least-squares regression line to predict *y* for a given value of *x*, how accurate is our prediction (again, with a margin of error)?

Introduction

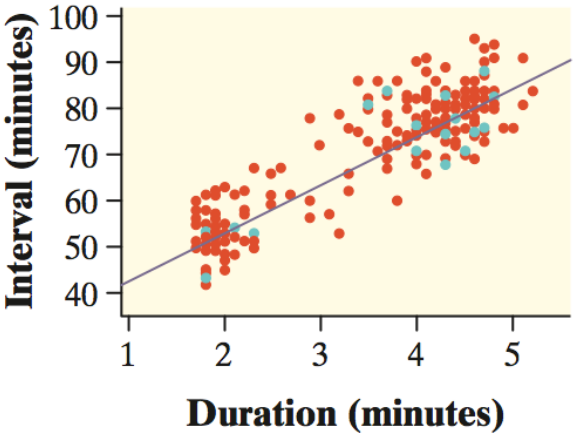
Researchers have collected data on eruptions of the Old Faithful geyser. Below is a scatterplot of the duration and interval of time until the next eruption for all 222 recorded eruptions in a single month. The least-squares regression line for this population of data has been added to the graph. It has slope 10.36 and *y*-intercept 33.97. Regarding all 222 eruptions as the population, this line is the **population regression line** (or true regression line) because it uses all the observations that month.



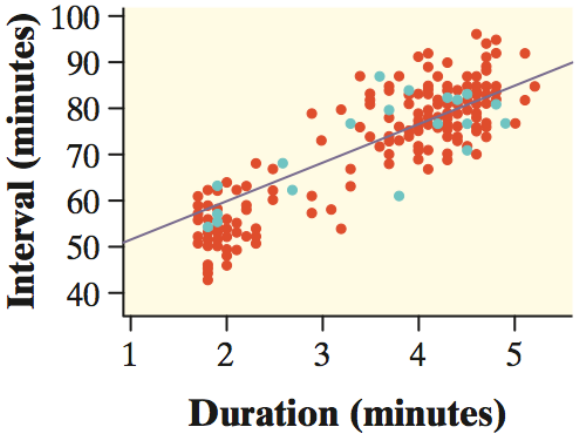
Suppose we take an SRS of 20 eruptions from the population and calculate the least-squares regression line $\hat{y} = b_0 + b_1 x$ for the sample data. How does the slope of the **sample regression line** (also called the estimated regression line, or LSRL) relate to the slope of the population regression line?

Introduction

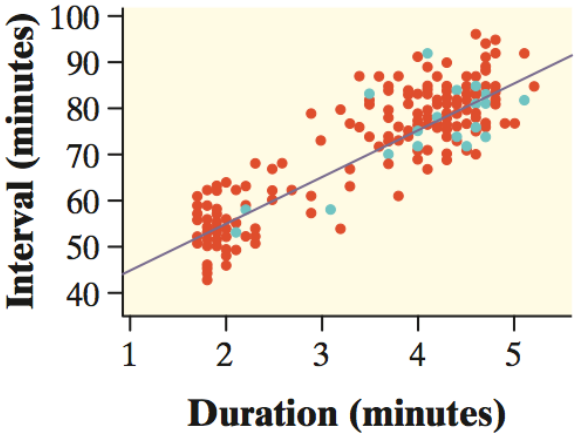
The figures below show the results of taking three different SRSs of 20 Old Faithful eruptions in this month. The green points in each graph are the selected points and the line is the LSRL for that sample of 20.



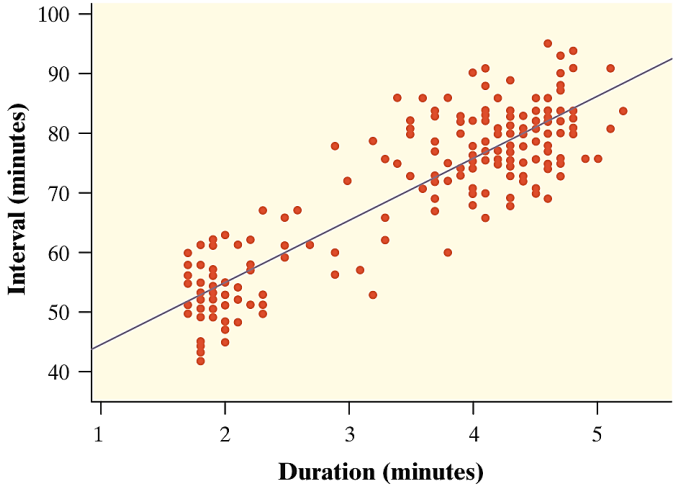
Sample 1: $\hat{y} = 32.8 + 10.2x$



Sample 2: $\hat{y} = 44.0 + 7.7x$



Sample 3: $\hat{y} = 36.0 + 9.5x$



Notice that the slopes of the sample regression lines—10.2, 7.7, and 9.5—vary quite a bit from the slope of the population regression line, 10.36.

The pattern of variation in the slope *b* is described by its sampling distribution.

Conditions for Regression Inference

The slope and intercept of the least-squares line are *statistics*. That is, we calculate them from the sample data. These statistics would take somewhat different values if we repeated the data production process. To do inference, think of b_0 and b_1 as estimates of unknown parameters β_0 and β_1 that describe the population of interest.

Conditions for Regression Inference

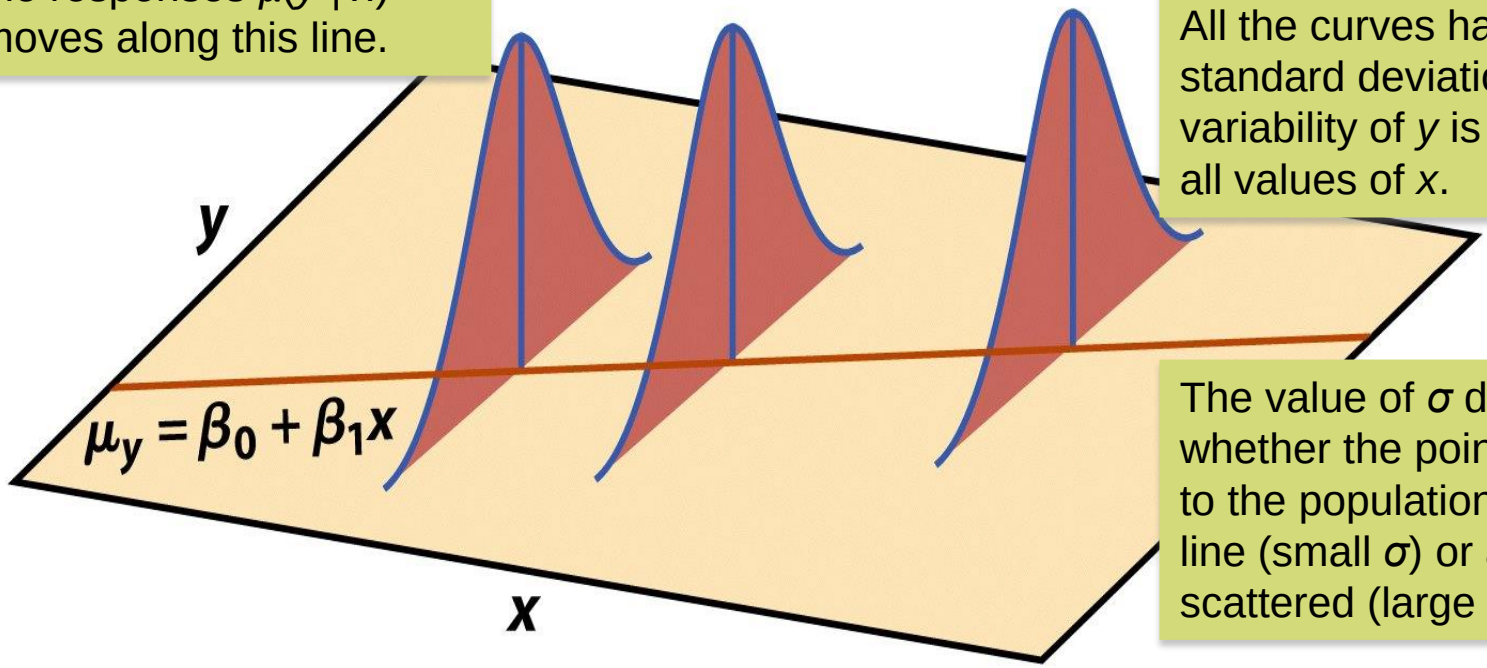
We have *n* observations on an explanatory variable *x* and a response variable *y*. Our goal is to study or predict the behavior of *y* for given values of *x*.

- For any fixed value of *x*, the response *y* varies according to a **Normal distribution.** Repeated responses *y* are **independent** of each other.
- The mean response μ_y has a **straight line relationship** with *x* given by a population regression line $\mu_y = \beta_0 + \beta_1 x$.
- The slope and intercept are unknown parameters.
- The standard deviation of *y* (call it σ) is the same for all values of *x*. The value of σ is unknown.

Conditions for Regression Inference

The figure below shows the regression model when the conditions are met. The line in the figure is the population regression line $\mu_y = \beta_0 + \beta_1 x$.

For each possible value of the explanatory variable x, the mean of the responses $\mu(y \mid x)$ moves along this line.



The Normal curves show how y will vary when x is held fixed at different values. All the curves have the same standard deviation σ, so the variability of y is the same for all values of x.

The value of σ determines whether the points fall close to the population regression line (small σ) or are widely scattered (large σ).

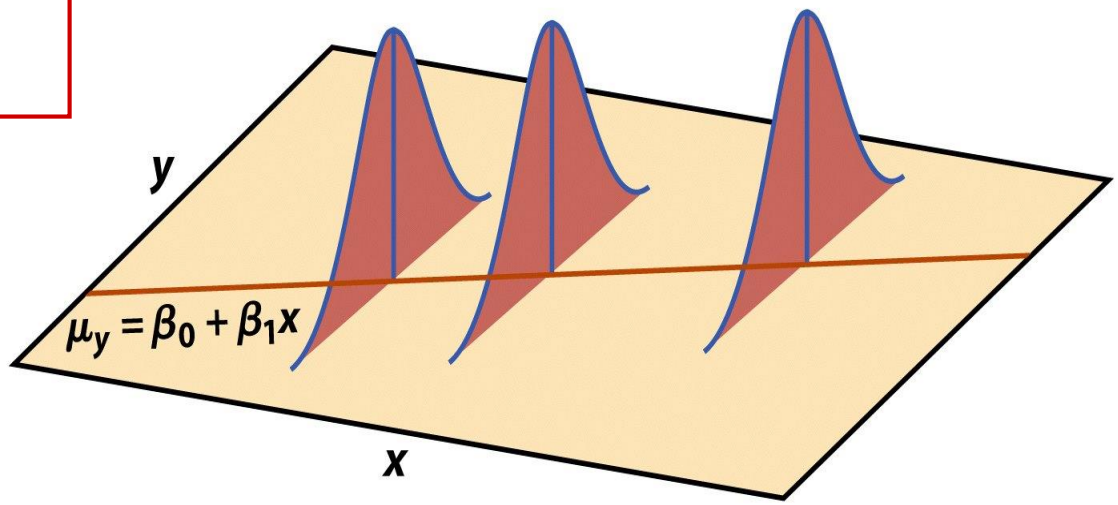
Simple Linear Regression Model

In the population, the linear regression equation is $\mu_y = \beta_0 + \beta_1 x$.

Sample data fits **simple linear regression model:**

Data =	Fit	+	Error
y_i =	$(\beta_0 + \beta_1 x_i)$	+	(ε_i)

where the ε_i are **independent** and **Normally** distributed $N(0,\sigma)$.



Linear regression assumes **equal variance of *y*** (σ is the same for all values of x).

Estimating the Parameters

$$\mu_y = \beta_0 + \beta_1 x$$

The intercept β_0, the slope β_1, and the standard deviation σ of y are the unknown parameters of the regression model. We rely on the random sample data to provide unbiased estimates of these parameters.

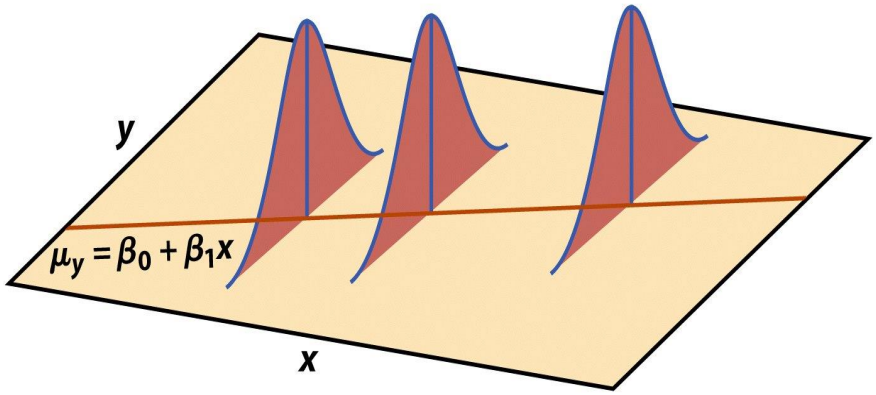
- The value of $\hat{y}$ from the least-squares regression line is really a prediction of the mean value of y (μ_y) for a given value of x.
- The least-squares regression line $(\hat{y} = b_0 + b_1 x)$ obtained from sample data is the best estimate of the true population regression line ($\mu_y = \beta_0 + \beta_1 x$).

$\boldsymbol{\hat{y}}$ is an unbiased estimate for mean response $\boldsymbol{\mu_y}$.

$\boldsymbol{b_0}$ is an unbiased estimate for intercept $\boldsymbol{\beta_0}$.

$\boldsymbol{b_1}$ is an unbiased estimate for slope β_1.

Estimating the Parameters



The **population standard deviation** σ for *y* at any given value of *x* represents the spread of the normal distribution of the ε_i around the mean μ_y.

The **predicted values** are $\hat{y}_i = b_0 + b_1 x_i,\ i = 1,\ldots,n,$ and the **residuals** are $y_i - \hat{y}_i,\ i = 1,\ldots,n.$

The **regression standard error, s,** for *n* sample data points is calculated from the **residuals** ($y_i - \hat{y}_i$):

$$s = \sqrt{\frac{\sum residual^2}{n-2}} = \sqrt{\frac{\sum (y_i - \hat{y}_i)^2}{n-2}}$$

s is an essentially unbiased estimate of the regression standard deviation σ.

Checking the Conditions for Regression Inference

You can fit a least-squares line to any set of explanatory-response data when both variables are quantitative. If the scatterplot doesn't show a roughly linear pattern, the fitted line may be almost useless.

Before you can trust the results of inference, you must check the conditions for inference one by one.

- ✓ **The relationship is linear in the population.**
- ✓ **The response varies Normally about the population regression line.**
- ✓ **Observations are independent.**
- ✓ **The standard deviation of the responses is the same for all values of *x*.**

You can check all of the conditions for regression inference by looking at graphs of the residuals, or **residual plots.**

Confidence Interval for Regression Slope

The slope β_1 of the population regression line $\mu_y = \beta_0 + \beta_1 x$ is the rate of change of the mean response as the explanatory variable increases. We often want to estimate β_1 . The slope b_1 of the sample regression line is our point estimate for β_1.

The confidence interval for β_1 has the familiar form:

Estimate $\pm$ t^* · (standard deviation of estimate)

Because we use the statistic b as our estimate, the confidence interval is:

$$b_1 \pm t^* SE_{b1}$$

Confidence Interval for Regression Slope

A level C **confidence interval for the slope β_1** of the population regression line is:

$$b_1 \pm t^* SE_{b1}$$

Here t^* is the critical value for the t distribution with df = n – 2 having area C between $-t^*$ and t^*.

Significance Test for Regression Slope

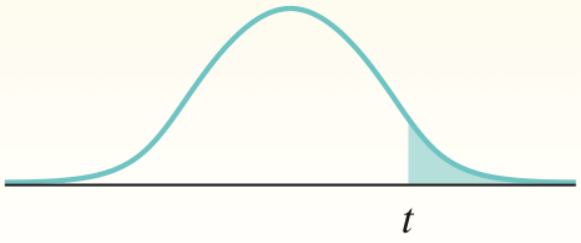
Significance Test for Regression Slope

To test the hypothesis H_0: β_1 = hypothesized value, compute the test statistic:

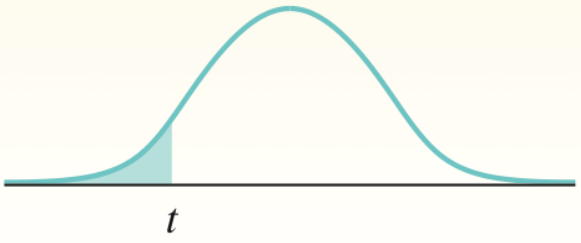
$$t = \frac{b_1 - \text{hypothesized value}}{SE_{b_1}}$$

Find the *P*-value by calculating the probability of getting a *t* statistic this large or larger in the direction specified by the alternative hypothesis H_a. Use the *t* distribution with df = $n - 2$.

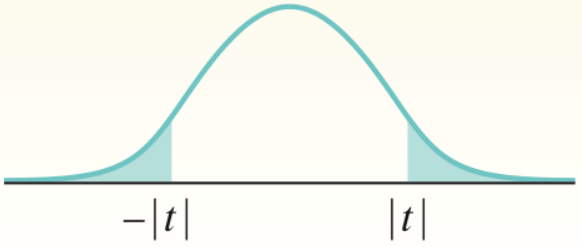
$H_a : \beta >$ hypothesized value



$H_a : \beta <$ hypothesized value



$H_a : \beta \neq$ hypothesized value



Testing the Hypothesis of No Relationship

We may look for evidence of a **significant relationship** between variables *x* and *y* in the population from which our data were drawn.

For that, we can test the hypothesis that the regression slope parameter β is equal to zero.

$$H_0: \beta_1 = 0 \text{ vs. } H_0: \beta_1 \neq 0$$

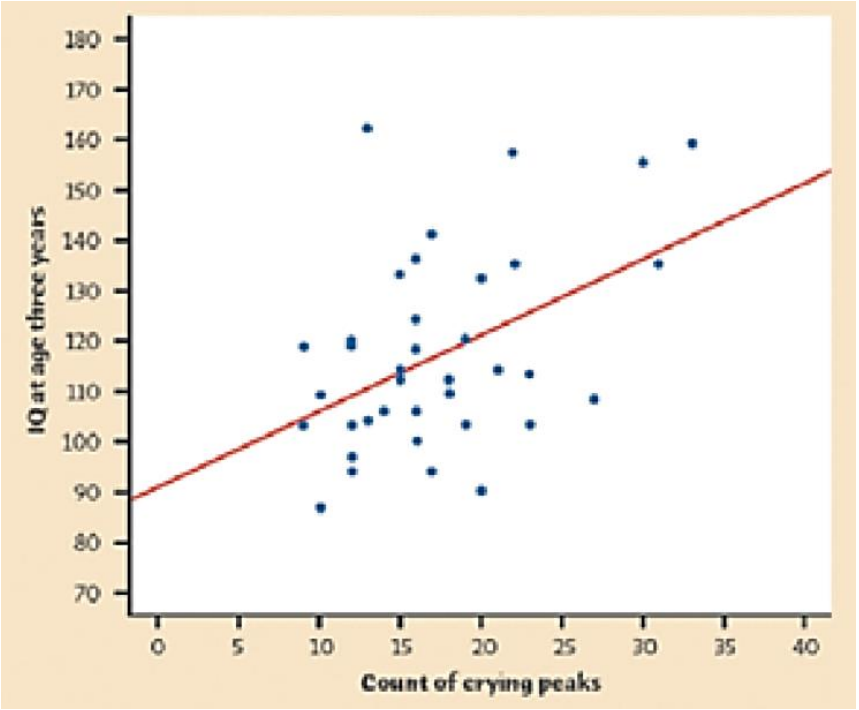
$$\text{slope } b_1 = r\frac{s_y}{s_x}$$

Testing H_0: $\beta_1 = 0$ is equivalent to testing the **hypothesis of no correlation** between *x* and *y* in the population.

Note: *A test of hypothesis for β_0 is seldom of interest, mainly because β_0 often has no practical interpretation.*

Example

Infants who cry easily may be more easily stimulated than others. This may be a sign of higher IQ. Child development researchers explored the relationship between the crying of infants 4 to 10 days old and their later IQ test scores. A snap of a rubber band on the sole of the foot caused the infants to cry. The researchers recorded the crying and measured its intensity by the number of peaks in the most active 20 seconds. They later measured the children's IQ at age three years using the Stanford-Binet IQ test. A scatterplot and Minitab output for the data from a random sample of 38 infants is below.



Regression Analysis: IQ versus Crycount

Predictor	Coef	SE Coef	T	P
Constant	91.268	8.934	10.22	0.000
Crycount	1.4929	0.4870	3.07	0.004

S = 17.50 R-Sq = 20.7% R-Sq(adj) = 18.5%

Do these data provide convincing evidence that there is a positive linear relationship between crying counts and IQ in the population of infants?

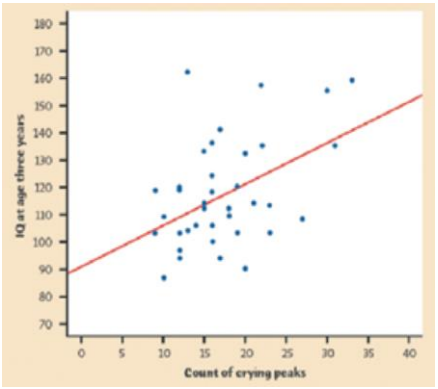
Example

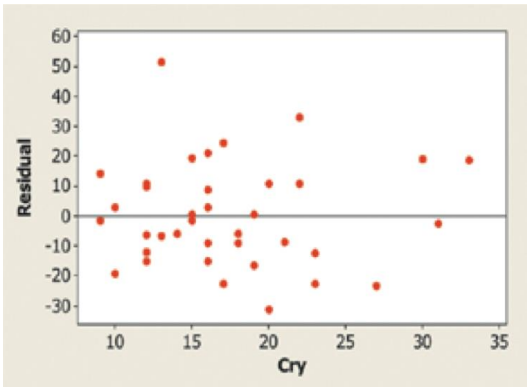
We want to perform a test of

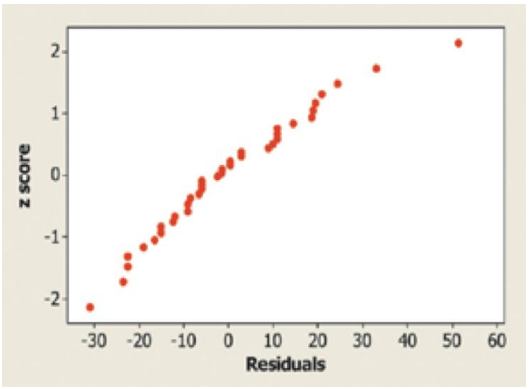
$$H_0 : \beta_1 = 0$$
$$H_a : \beta_1 > 0$$

where β_1 is the true slope of the population regression line relating crying count to IQ score.

- The scatterplot suggests a moderately positive linear relationship between crying peaks and IQ. The residual plot shows a random scatter of residuals about the line $y = 0$.







- IQ scores of individual infants should be independent.
- The Normal probability plot of the residuals shows a slight curvature, which suggests that the responses may not be Normally distributed about the line at each x-value. With such a large sample size ($n = 38$), however, the t procedures are robust against departures from Normality.
- The residual plot shows a fairly equal amount of scatter around the horizontal line at 0 for all x-values.

Example

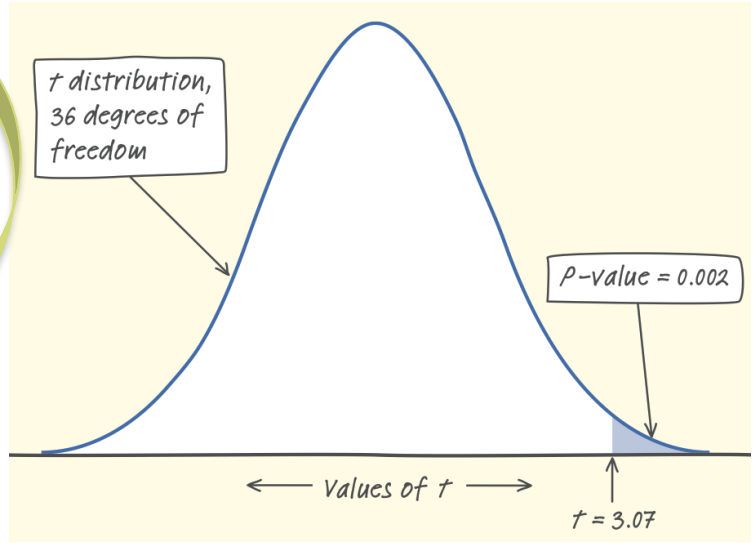
With no obvious violations of the conditions, we proceed to inference. The test statistic and *P*-value can be found in the Minitab output.

Regression Analysis: IQ versus Crycount

Predictor	Coef	SE Coef	T	P
Constant	91.268	8.934	10.22	0.000
Crycount	1.4929	0.4870	3.07	0.004

S = 17.50 R-Sq = 20.7% R-Sq(adj) = 18.5%

$$t = \frac{b_1}{SE_{b1}} = \frac{1.4929}{0.4870} = 3.07$$



The Minitab output gives $P = 0.004$ as the *P*-value for a two-sided test. The *P*-value for the one-sided test is half of this, $P = 0.002$.

The *P*-value, 0.002, is less than our $\alpha = 0.05$ significance level, so we have enough evidence to reject H_0 and conclude that there is a positive linear relationship between intensity of crying and IQ score in the population of infants.

Confidence Interval for Mean Response

We can also calculate a confidence interval for the population mean μ_y of all responses y when x takes the value x^* (within the range of data tested).

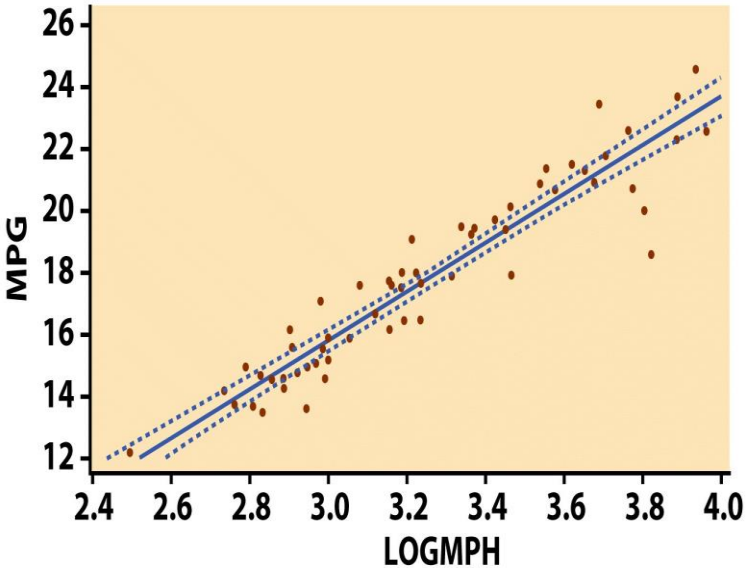
The **level *C* confidence interval for the mean response μ_y** at a given value x^* of x is:

$$\hat{\mu}_y \pm t^* SE_{\hat{\mu}}$$

where t^* is the value such that the area under the $t(n-2)$ density curve between $-t^*$ *and* t^* is C.

A separate confidence interval could be calculated for μ_y along all the values that x takes.

Graphically, the series of confidence intervals is shown as a continuous interval on either side of $\hat{y}$.

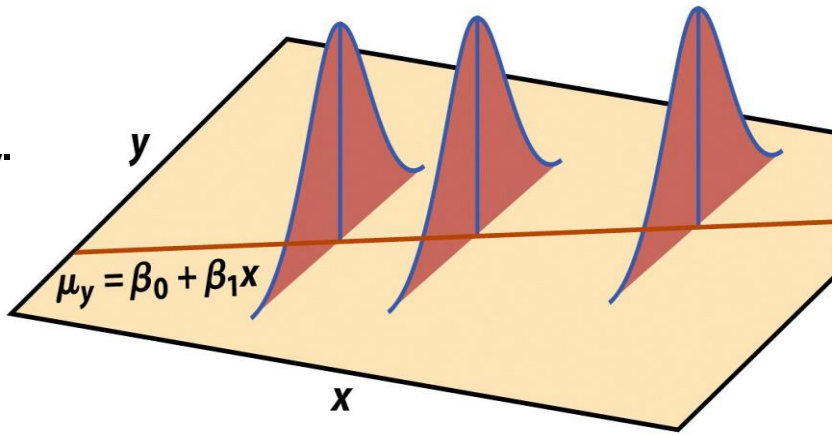


Prediction Intervals

One use of regression is for **predicting** the value of *y* at some value of *x* within the range of data tested. Reliable predictions require statistical inference.

To estimate an *individual* response *y* for a given value of *x,* we use a **prediction interval.**

If we randomly sampled many times, there would be many different values of *y* obtained for a particular *x* following $N(0, \sigma)$ around the mean response μ_y.



Prediction Intervals

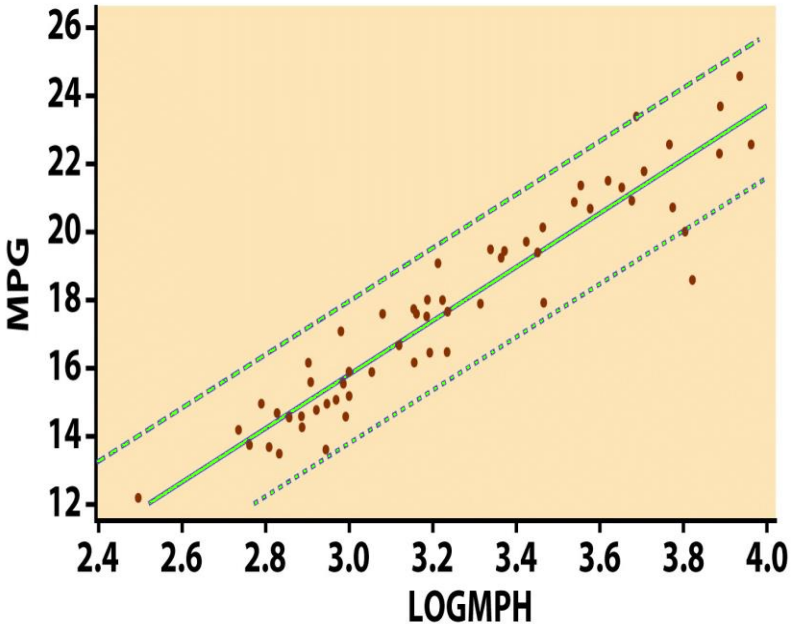
The **level *C* prediction interval for a single observation** on *y* when *x* takes the value *x** is:

$$\hat{y} \pm t^* SE_{\hat{y}}$$

*t** is the critical value for the ***t* (*n* – 2)** distribution with area C between –*t** and +*t**.

The prediction interval accounts for error in estimating β_0 and β_1 as well as uncertainty about the value of *y* being predicted.

Graphically, the series of prediction intervals is shown as a continuous interval on either side of $\hat{y}$. These intervals are wider than the corresponding confidence intervals for μ_y.



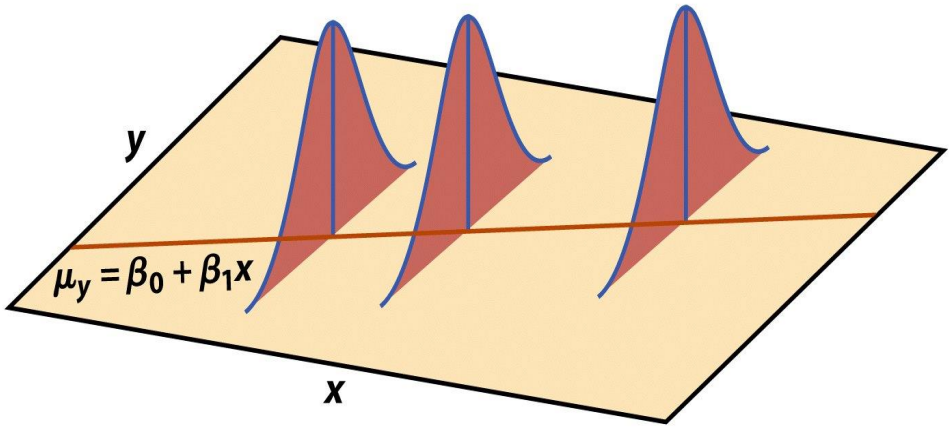
10.2 More Detail About Simple Linear Regression*

- Analysis of variance for regression
- The ANOVA *F* test
- Calculations for regression inference
- Inference for correlation

Analysis of Variance for Regression

The regression model is:

Data =	fit	+	error
y_i =	$(\beta_0 + \beta_1 x_i)$	+	(ε_i)



where the ε_i are **independent** and **Normally** distributed $N(0,\sigma)$, and σ is the same for all values of x.

It resembles an ANOVA, which also assumes equal variance, where

SST =	SS model	+	SS error	and
DFT =	DF model	+	DF error	

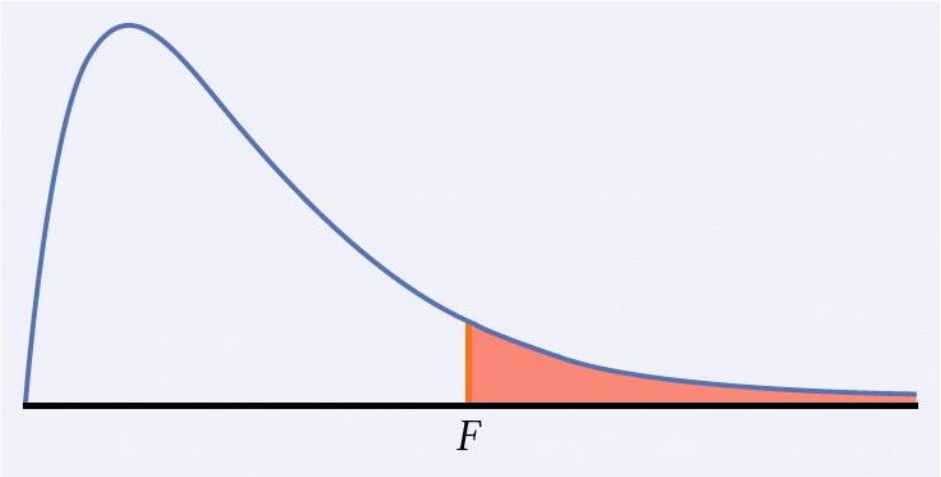
The ANOVA *F* Test

For a simple linear relationship, the ANOVA tests the hypotheses

$$H_0: \beta_1 = 0 \text{ versus } H_a: \beta_1 \neq 0$$

by comparing MSM (model) to MSE (error): F = MSM/MSE

When H_0 is true, F follows the $F(1, n - 2)$ distribution. The P-value is $P(F \geq f)$.



The ANOVA test and the two-sided t*-test for* H_0*:* $\beta_1 = 0$ *yield the same P-value.*

Software output for regression may provide t, F, *or both, along with the P-value.*

The ANOVA Table

Source	Sum of squares SS	DF	Mean square MS	*F*	*P*-value
Model	$SSM = \sum_{i=1}^{n}(\hat{y}_i - \bar{y})^2$	1	*MSM=SSM/DFM*	*MSM/MSE*	Tail area above F
Error	$SSE = \sum_{i=1}^{n}(y_i - \hat{y}_i)^2$	*n* − *2*	*MSE=SSE/DFE*		
Total	$SST = \sum_{i=1}^{n}(y_i - \bar{y})^2$	*n* − 1			

SST = SSM + SSE *DFT = DFM + DFE* *F=MSM/MSE*

The standard deviation, *s,* of the *n* residuals $e_i = y_i - \hat{y}_i$, *I = 1,…,n*, is calculated from the following quantity:

s is an approximately unbiased estimate of the regression standard deviation ***σ*.**

Calculations for Regression Inference

To assess variation in the estimates of β_0 and β_1, we calculate the standard errors for the estimated regression coefficients.

The standard error of the slope estimate b_1 is:

$$SE_{b1} = \frac{s}{\sqrt{\sum (x_i - \overline{x}_i)^2}}$$

The standard error of the intercept estimate b_0 is:

$$SE_{b0} = s\sqrt{\frac{1}{n} + \frac{\overline{x}^2}{\sum (x_i - \overline{x}_i)^2}}$$

Calculations for Regression Inference

To estimate mean responses or predict future responses, we calculate the following standard errors:

The standard error of the estimate of the mean response μ_y is:

$$SE_{\hat{\mu}} = s\sqrt{\frac{1}{n} + \frac{(x^* - \overline{x})^2}{\sum(x - \overline{x})^2}}$$

The standard error for predicting an individual response *y* is:

$$SE_{\hat{y}} = s\sqrt{1 + \frac{1}{n} + \frac{(x^* - \overline{x})^2}{\sum(x - \overline{x})^2}}$$

Inference for Correlation

To test the null hypothesis of no linear association, we have the choice of also using the **correlation parameter ρ.**

$$b_1 = r\frac{s_y}{s_x}$$

- When x is clearly the explanatory variable, this test is equivalent to testing the hypothesis H_0: $\beta = 0$.
- When there is no clear explanatory variable (e.g., arm length vs. leg length), a regression of x on y is no less legitimate than one of y on x. In that case, the correlation test of significance should be used.

Inference for Correlation

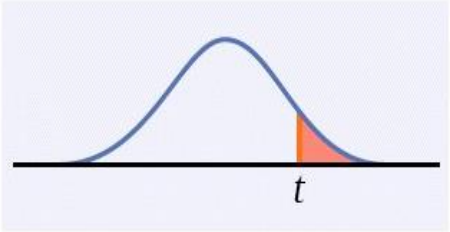
The test of significance for ρ uses the one-sample t-test for: H_0: $\rho = 0$.

We compute the t statistic for sample size n and correlation coefficient r.

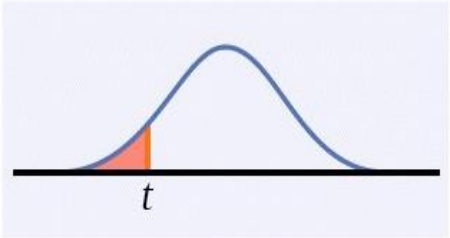
$$t = \frac{r\sqrt{n-2}}{\sqrt{1-r^2}}$$

The P-value is the area under $t(n-2)$ for values of T as or more extreme than t in the direction of H_a.

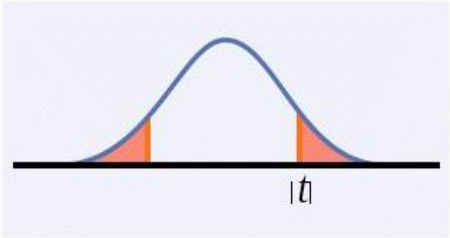
$H_a: \rho > 0$ is $P(T \geq t)$



$H_a: \rho < 0$ is $P(T \leq t)$



$H_a: \rho \neq 0$ is $2P(T \geq |t|)$



Inference for Correlation

When the hypothesis H_0: $\rho = 0$ is rejected, it is safe to assume that there is some sort of relationship between the variables x and y.

Recall, though, from Chapter 2 that correlation measures *linear* relationships. As such it is not always a reliable indicator of *nonlinear* relationships.

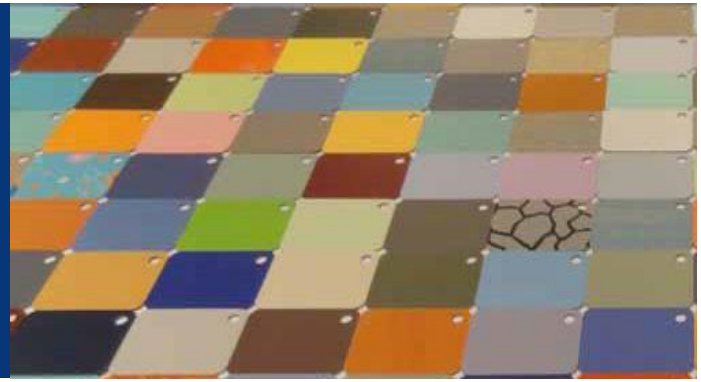
When the hypothesis H_0: $\rho = 0$ is *not* rejected, do not assume that the variables are unrelated.

✓First of all, it is possible that a Type II error may have occurred!

✓Secondly, it is possible that x and y are related in a nonlinear way that the correlation coefficient r has no chance of detecting.

✓A good way to investigate the second possibility is to examine a scatterplot.

Chapter 10
Inference for Regression



10.1 Simple Linear Regression

10.2 More Detail about Simple Linear Regression*