Final Lecture of Alg

INFO: Q & A class at 6.19, 18th week Mon.(Online)

Review Chapter List

Introduction 简介部分

The characteristics of an algorithm

- **Unambiguous确定性/不模棱两可**: every step is deterministic
- **Mechanical可行性**: machine can "understand"
- **Finite有穷性**: can be implemented in limited steps
- **Input/output具备输入输出**: to state the problem size and the result

Asymptotic notation: O Ω Θ

- **Caution**:
  - $\sum_{i=1}^{n} i^k = \Theta(n^{k+1})$
  - $\sum_{i=1}^{n} \frac{1}{i} = \Theta(\log n)$
  - $\log n! = \Theta(n \log n)$
  - $n^{\log_a b} = a^{\log_a n}$
  - $(\log n)^{\log n} = O(2^{(\log_2 n)^2})$
  - $(\log n)^{\log n} = \Omega(n / \log n)$
  - $n! = o(n^n)$

Common rules for asymptotic analysis

- **Master Theorem**:
  If $T(n) = aT(\lceil n/b \rceil) + O(n^d)$ for some constants $a > 0, b > 1$ and $d \geq 0$, then

$$T(n) = \begin{cases} O(n^d) & \text{if } d > \log_b a \\ O(n^d \log_b n) & \text{if } d = \log_b a \\ O(a^{\log_b n}) = O(n^{\log_b a}) & \text{if } d < \log_b a \end{cases}$$

- **Recursion Tree**
  - The sum of the values at each level of the tree is the final time complexity
- **Expression Expansion**

Divide-and-Conquer 分治算法

Hallmarks: Optimal substructure and independent sub-problem

- **Optimal substructure最优子结构**
- **Independent sub-problem独立子问题**

Master theorem and its proof

- **Proof of Master Theorem**:
  - Assume n is a power of b
  - The total work done at the kth level
    $a^k \times O(\frac{n}{b^k})^d = O(n^d) \times (\frac{a}{b^d})^k$
  - As k goes from 0 to $\log_b n$, these numbers form a geometric series with ratio a/b^d
  1. The ratio is less than 1, then the series is decreasing, and its sum is just given by the first term, $O(n^d)$
  2. The ratio is greater than 1, the series is increasing and its sum is given by its last term, $O(n^{\log_b a})$
     $n^d (\frac{a}{b^d})^{\log_b n} = n^d \left(\frac{a^{\log_b n}}{(b^{\log_b n})^d} \right) = a^{\log_b n} = a^{(\log_a n)(\log_b a)} = n^{\log_b a}$

3. The ratio is exactly 1, in this case $O(\log_b n)$ terms of the series are equal to $O(n^d)$

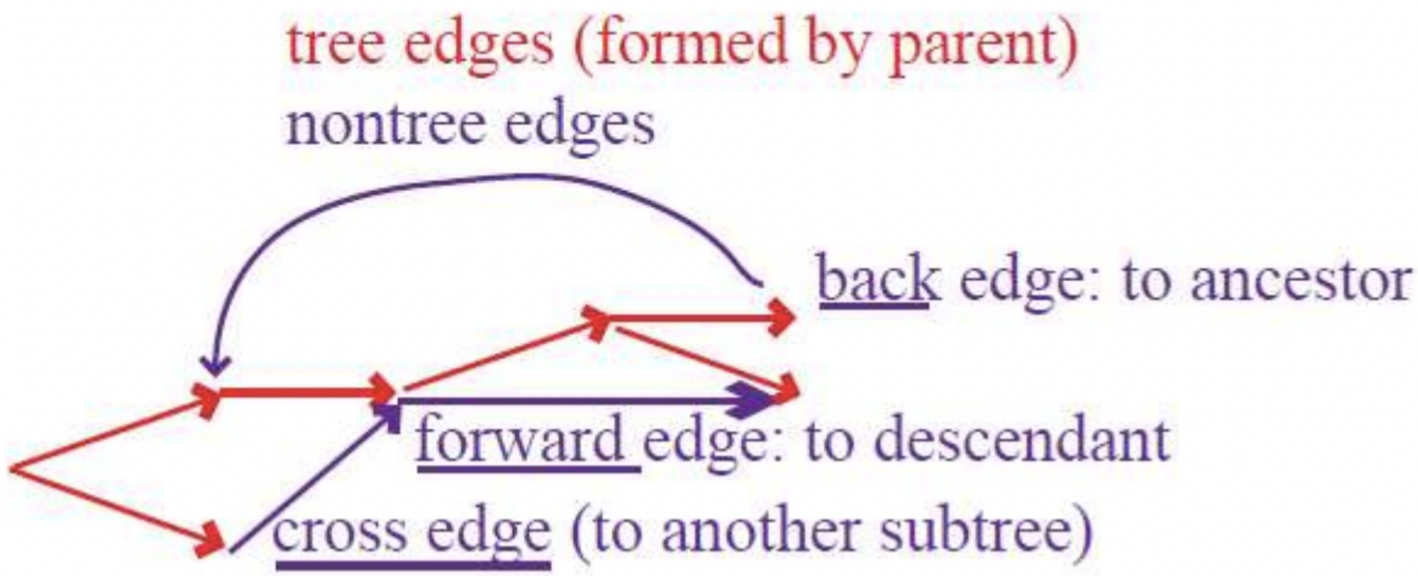
Merge sort $(O(n \log n))$

- $T(n) = 2T(n/2) + (n-1)$

Selection $(O(n))$

Graph Algorithms 图算法

Explore $(O(E))$ and DFS $(O(|V|+|E|))$



DAG: topological ordering, Shortest paths $(O(|V|+|E|))$

BFS $(O(|V|+|E|))$

Dijkstra's algorithm $(O(|V|+|E|) \log |V|)$

- Assume non-negative weight edges

Priority queue implementations: array, binary heap and d-ary heap

binary heap:

- heapify $O(\log n)$
  - Correct a single violation
- build heap $O(n \log n)$
  - Produce a heap from an unordered array
- heap sort $O(n \log n)$

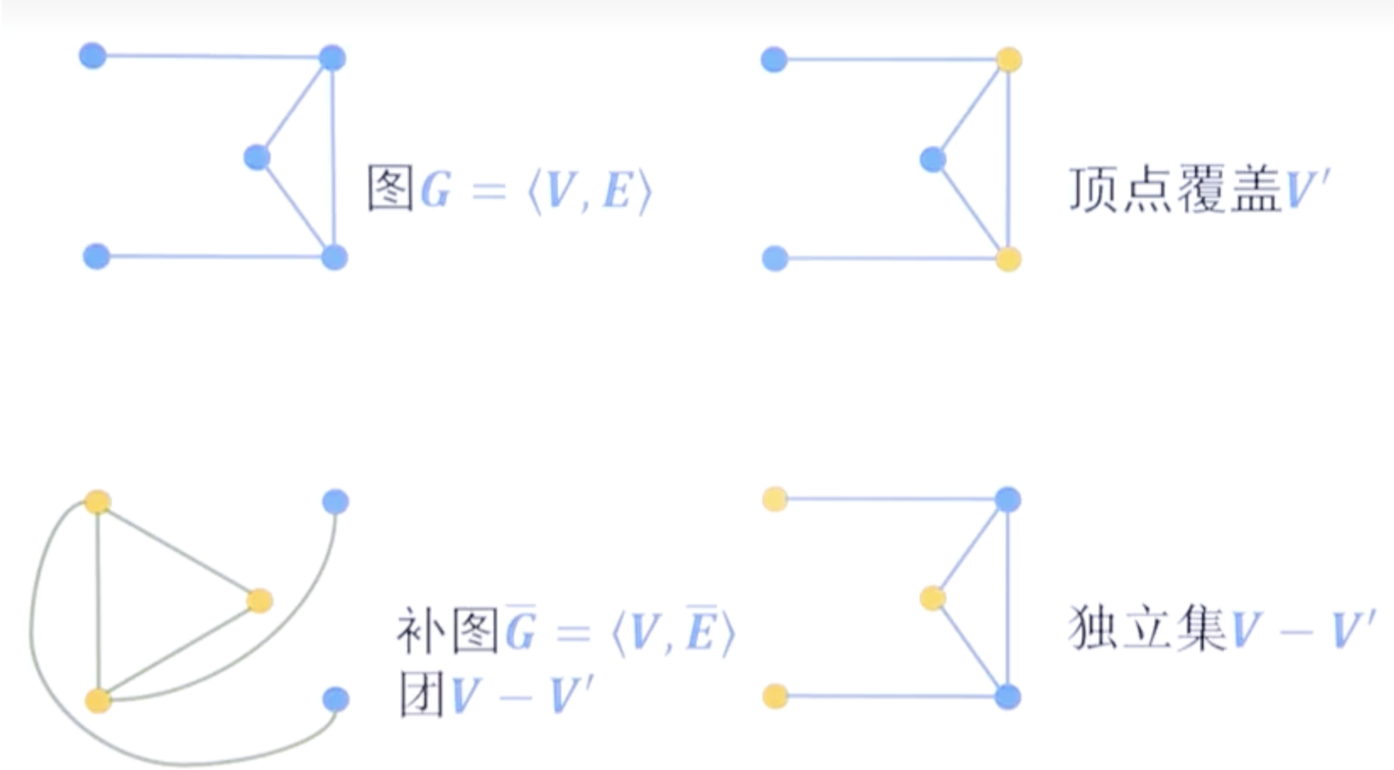
Implementation	Delete Min	Insert/Decrease Key	Djikstra's Algorithm Time Cost($\|V\| \times \text{Delete Min} + (\|V\|+\|E\|) \times \text{Insert/Decrease Key}$)
Array	$O(\|V\|)$	$O(1)$	$O(\|V\|^2)$
Binary heap	$O(\log \|V\|)$	$O(\log \|V\|))$	$O((\|E\|+\|V\|) \log \|V\|)$
d-ary heap	$O(\frac{d \log \|V\|}{\log d})$	$O(\frac{\log \|V\|}{\log d})$	$O((\|V\| \cdot d + \|E\|)\ \frac{\log \|V\|}{\log d})$
Fibonacci heap	$O(\log \|V\|)$	$O(1)$(amortized)	$O(\|V\| \log \|V\| + \|E\|)$

Bellman-Ford algorithm $(O(|V||E|))$

- 每次遍历所有顶点V，根据顶点的边更新下一次抵达的顶点的最短距离

Conversion of graph concept

- **Vertex Cover**顶点覆盖，顶点能够连接图中所有的边
- **Set Cover**集合覆盖，其实相当于顶点覆盖，一个顶点连接的所有边构成一个集合
- **Independent Set**独立集，各顶点之间不存在联系
- **Clique**团，顶点之间两两连接
- **Complement Graph**补图，原来有边的地方没有边，原来没边的地方有边
- **Minimum Spanning Tree**最小生成树，边覆盖，边连接所有顶点，且总权值最小



Greedy Algorithms 贪心算法

Tree's propeties

- A tree on n nodes has $n-1$ edges.
- Any connected, undirected graph $G=(V,E)$ with $|E|=|V|-1$ is a tree.
- An undirected graph is a tree if and only if there is a unique path between any pair of nodes

MST: Kruskal's algorithm $(O(|V|+|E|)\log|V|)$, Prim's algorithm $(O(|V|+|E|)\log|V|)$

	Kruskal's Algorithm	Prim's Algorithm
Sort all edges?	Yes	No
Minimum	The lightest edge in the remaining edges	The lightest edge among the cross edges
Data Structure	Disjoint Set(Connect Component)	Binary Heap(Priorty Queue)
Time Complexity	$O((\|V\|+\|E\|)\log\|V\|)$	$O((\|V\|+\|E\|)\log\|V\|)$

Disjoint Set

- **Properties**:
  - For any $x \neq \pi(x)$, rank(x) < rank$(\pi(x))$
  - Any root node of rank k has at least 2^k nodes in its tree
  - If there are n elements overall, there can be at most $n/2^k$ nodes of rank k (All the trees have height ≤ $\log n$)
- Proof of Property 2 (at least):

- merge two trees with height k.
- Proof of Property 3 (at most):
  - $k = 0$: forest of n singleton trees with height 0.
  - $k = 1$: $n/2$ single-child trees with height 1.
  - assume when rank = k, the property holds
  - how to produce the most nodes at rank $k+1$? Merge equal-height trees as many as possible so that the number of nodes of rank $k+1$ is at most $(n/2^k)/2$.

The cut property

- **Cut割**: Any partition of the vertices into two groups, S and $V - S$.
- **Cut property**: Suppose edges X are part of a minimum spanning tree of $G = (V, E)$. Pick any subset of nodes S for which X does not cross between S and $V - S$, and let e be the lightest edge across this partition. Then $X \cup \{e\}$ is a part of some MST.
- 这段话描述的是图论中MST（最小生成树）的一个重要性质，即在一个由边组成的集合X是G的最小生成树的情况下，任取G的一个点集S，选取S和$V - S$之间的最轻边e，一定有$X \cup \{e\}$ 是G的某个最小生成树的一部分。
- Proof of the cut property
  - Assume X is part of some MST T, and e is not in T
  - Construct a different MST T' containing $X \cup \{e\}$
    - Adding e to T will create a cycle
    - $T' = T \cup \{e\} - \{e'\}$
    - T' has the same number of edges as T, so T' is a tree
    - $\text{weight}(T') = \text{weight}(T) + w(e) - w(e')$, since $w(e) \leq w(e')$, $\text{weight}(T') \leq \text{weight}(T)$. T' is also a MST

Huffman encoding $(O(n \log n))$

- 使用优先队列（二叉树实现）的复杂度为$O(n \log n)$

Greedy algorithm for set cover (approximation ratio $\ln n$)

- 这个问题的想法是，每次挑选出连接边数最多一个点，将其加入顶点覆盖的解集中，再将与这个顶点相关的所有边全部删除，以此类推获得顶点覆盖的近似解
- The approximation ratio **depends on** n, not a vaild approximation algorithm

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Dynamic Programming 动态规划

Hallmarks: optimal substructure and overlapping sub-problem

- **Optimal substructure最优子结构**
- **Overlapping sub-problem覆盖子问题**

Longest Increasing Subsequence(LIS) $(O(n^2))$

- 题意是要求寻找一个序列中的最长递增子**序列**，记长度为n
- 利用动态规划法，采用在线处理思想，维护一个记录与序列元素下标相同的数组，记录以当前位置为结尾的序列所具有的最长递增子序列的长度
- 每次读取到一个新的元素k时，将其与前$k-1$个元素比较，若比其中的某个元素x大，可以将当前元素k的最长递增子序列记录数组记录为x元素的记录$+1$
- 因此，总共需要读取n个元素，对于每个元素，需要读取比它更小的$k-1$个元素，总体的时间复杂度为$O(n^2)$

Edit distance $(O(mn))$

- 题意是求两个字符串间，通过删除、插入或修改这三种操作，将其中一个字符串变换为另一个字符串所需的最小操作次数
- 时间复杂度$O(mn)$，其中m指的是第一个字符串的长度，n指的是第二个字符串的长度
- 基本思路为：
  - 选定第一个字符串或第二个字符串为参照，这里取第一个字符串为参照
  - 将第二个字符串与第一个字符串进行逐字符比对，记第一个字符串的比较字符为c_i，第二个字符串的比较字符为c_j，其存在以下几种结果：
    - 字符与原字符串的字符相等（$c_i = c_j$），那么无需进行操作，直接继承双方各个的上一次字符的操作结果
    - 字符与原字符串的字符不相等（$c_i != c_j$），那么观察继承双方上一个字符是删除、插入或修改操作中哪一个所需的操作次数最小，便采用之，最后加1代表此字符进行了操作

Knapsack $(O(nW))$

- 背包问题的意思是：给定容量W的背包，给定n件物品，每件物品有自己对应的价值，也有自己对应的体积，要求取当前条件下背包能放入物品价值的最大值
- 时间复杂度为$O(nW)$，n是物品数量，W是背包的容量
- 在动态规划算法中，我们需要做的是：
  - 维护一个以背包容量和物品种类数为轴的二维表
  - 每次从一个固定的背包容量开始，开始遍历每个物品种类
    - 若为01背包，对单个物品有取或不取两种选择
    - 若为普通背包，对单个物品可取多件或不取
  - 在取这件物品时，以当前循环下背包的最大容量减去该物品所占的容量，若可以取，再比较取了该物品的价值是否会比当前已记录的最大价值更大

Chain matrix multiplication $(O(n^3))$

- 从最小规模的2个矩阵之间相乘开始计算，eg: 2x3与3x4的矩阵相乘需要进行2x3x4次乘法，记录下来
- 接下来从3个矩阵之间相乘开始计算，根据已知的2个矩阵相乘的结果，可以看作找1个矩阵与另一个矩阵（上一次已经求出结果的2个矩阵的乘积）进行相乘，以此类推
- 因此，以整个矩阵相乘链为序，从第i个矩阵到第j个矩阵的最少相乘次数为**min(从第i个矩阵到第k个矩阵的相乘次数 + 从第k+1个矩阵到第j个矩阵的相乘次数 + 这两部分相乘所需的次数)**
- 每次循环都需要从第一个矩阵开始，每次都需要遍历当前规模（2，3，4……直到n），因此实现计算范围的确定（第i个矩阵及第j个矩阵）已经需要$O(n^2)$的复杂度，当我们进入规模内部，需要遍历当前规模下的所有可能两部分矩阵的划分方案，因此也需要$O(m)$的复杂度，m指代当前规模的大小，$m \leq n$因此最终的复杂度为$O(n^3)$

NP-complete NP完全问题

P, NP, Reduction, NP-completeness(NPC)

- NP Problem:
  - SAT
  - 3-SAT
  - Independent Set
  - 3D Matching
  - Vertex Cover
  - Clique
  - ZOE
  - Subset Sum
  - ILP
  - Rudrata Cycle
  - TSP

Example for reduction

- SAT $\rightarrow$ 3-SAT
- 3-SAT $\rightarrow$ Independent Set
  - 画图，将3-SAT中的每个子句转化为一个三角形的图，然后将诸如x的顶点与每个$\bar{x}$的顶点相连，保证x与$\bar{x}$不能被同时选取
- Independent Set $\rightarrow$ Vertex Cover
  - 独立集与顶点覆盖的关系是：在一个图G中，所有的顶点V，记一个顶点覆盖的顶点集合为S，那么$V-S$就是一个独立集；反之亦然
- Independent Set $\rightarrow$ Clique
  - 独立集与团的关系是：在一个图G中，取其补图G'，即有边的地方没有边，没有边的地方有边，那么原来的独立集在补图中就是一个团
- 3D Matching $\rightarrow$ SAT
- Rudrata Cycle $\rightarrow$ SAT

Approximated algorithm for vertex cover(approximation ratio 2)

- 这个问题的想法是，每次挑选出一条边，边两端的顶点加入顶点覆盖的解集中，再将与这两个顶点相关的所有边全部删除，以此类推获得顶点覆盖的近似解
- The approximation ratio is **constant**
- A vertex cover of a graph $G=(V,E)$ is a subset of vertices $S \subseteq V$ that includes at least one endpoint of every edge in E. Give a 2-approximation algorithm for the following task.

- Let $U \subseteq E$ be the set of all the edges that are picked by **Approximation_Vertex_Cover**. The optimal vertex cover must include at least one endpoint of each edge in U(and other edges). Furthermore, no two edges in U share an endpoint. Therefore, $|U|$ is a lower bound for C_{opt}. Namely, $C_{\text{opt}} \geq |U|$. The number of vertices in V' returned by **Approximation_Vertex_Cover** is $2 \cdot |U|$. Therefore, $C = |V'| = 2 \cdot |U| \leq 2C_{\text{opt}}$. Hence, $C \leq 2 \cdot C_{\text{opt}}$.