Chapter 13

Two-Way Analysis of Variance

Introduction to the Practice of

STATISTICS EIGHTH EDITION

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Lecture Presentation Slides

Chapter 13
Two-Way Analysis of Variance

13.1 The Two-Way ANOVA Model

13.2 Inference for Two-Way ANOVA

13.1 The Two-Way ANOVA Model

- Advantages of two-way ANOVA
- The two-way ANOVA model
- Main effects and interactions

Two-Way Designs

In a two-way design, two factors (independent variables) are studied in conjunction with the response (dependent) variable. There are thus two ways of organizing the data, as shown in a two-way table.

Two-way table (3-by-3 design to test the attractiveness of a new website)

	Color		
Logo	Red	Green	Blue
Logo 1	14	14	14
Logo 2	14	14	14
Logo 3	14	14	14

Each number in the body of the table is the number of people who were asked to rate the attractiveness of the corresponding combination of color and logo.

When the dependent variable is quantitative, the data are analyzed with a two-way ANOVA procedure. A chi-square test is used instead if the dependent variable is categorical.

Advantages of Two-Way ANOVA

- It is **more efficient** to study two factors at once than separately.

  A two-way design requires smaller sample sizes per condition than does a series of one-way designs because the samples for all levels of factor B contribute to sampling for factor A.

- Including a second factor thought to influence the response variable helps **reduce the residual** variation in a model of the data.

  In a one-way ANOVA for factor A, any effect of factor B is assigned to the residual ("error" term). In a two-way ANOVA, both factors contribute to the "fit" part of the model.

- **Interactions** between factors can be investigated.

  The two-way ANOVA breaks down the fit part of the model between each of the main components (the two factors) and an *interaction effect*. The interaction cannot be tested with a series of one-way ANOVAs.

The Two-Way ANOVA Model

- We record a quantitative variable in a **two-way design** with I levels of factor A and J levels of factor B.
- There are $I \times J$ combinations of the two sets of factor levels. We'll use (i,j) to denote the population for which factor A is at level i and factor B is at level j.
- We assume that we have $I \times J$ **independent SRSs,** one from each population, and that each population is Normal. The population means may be different but all populations have the same σ.
- Let x_{ijk} represent the k^{th} observation from population (i,j). The statistical model is:

$$x_{ijk} = \mu_{ij} + \varepsilon_{ijk} = \mu + \alpha_i + \beta_j + (\alpha\beta)_{ij} + \varepsilon_{ijk}$$

where μ_{ij} is the mean of population (i,j) and ε_{ijk} is an error term.

μ_{ij}, the mean of combination (i,j), can be broken down into the overall mean μ, plus α_i, the effect of level i of Factor A, plus β_i, the effect of level j of Factor B, plus $(\alpha\beta)_{ij}$, the interaction effect of the two levels

Interaction

Two variables interact if a particular combination of variables leads to results that would not be anticipated on the basis of the main effects of those variables.

- Drinking alcohol increases the chance of throat cancer, as does smoking. However, people who both drink and smoke have an even higher chance of getting throat cancer. The combination of smoking and drinking is particularly dangerous: these risk factors interact.

An **interaction** implies that the effect of one variable is different at different levels of another variable.

- The effect of smoking on the probability of getting throat cancer is greater for people who drink than for people who do not drink: The effect of smoking differs depending on whether drinkers or nondrinkers are being considered.

Main Effects and Interactions

- Each factor is represented by a **main effect:** This is the impact on the response (dependent variable) of varying levels of that factor, regardless of the other factor (i.e., pooling together the levels of the other factor). There are two main effects, one for each factor.

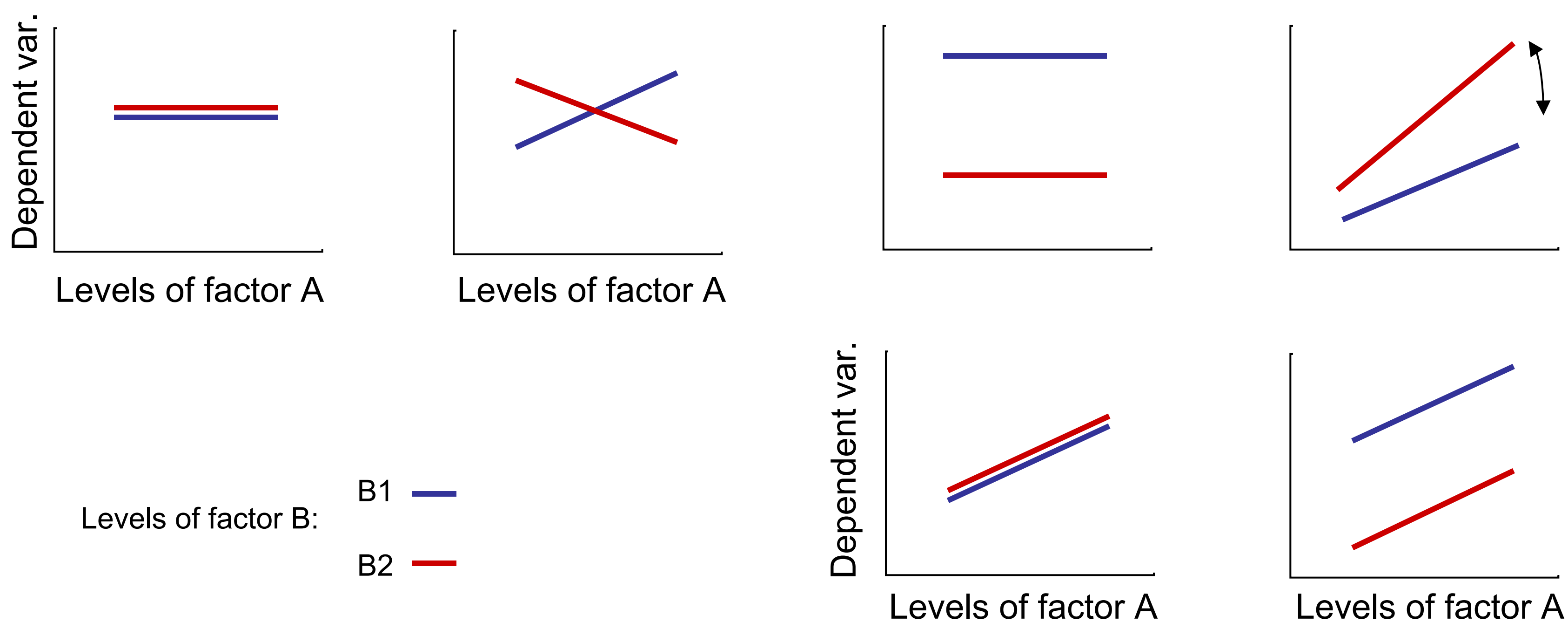
- The interaction of both factors is also studied and is described by the **interaction effect.**

- When there is no clear interaction, the main effects are enough to describe the data. In the presence of interaction, the main effects could mask what is really going on with the data.

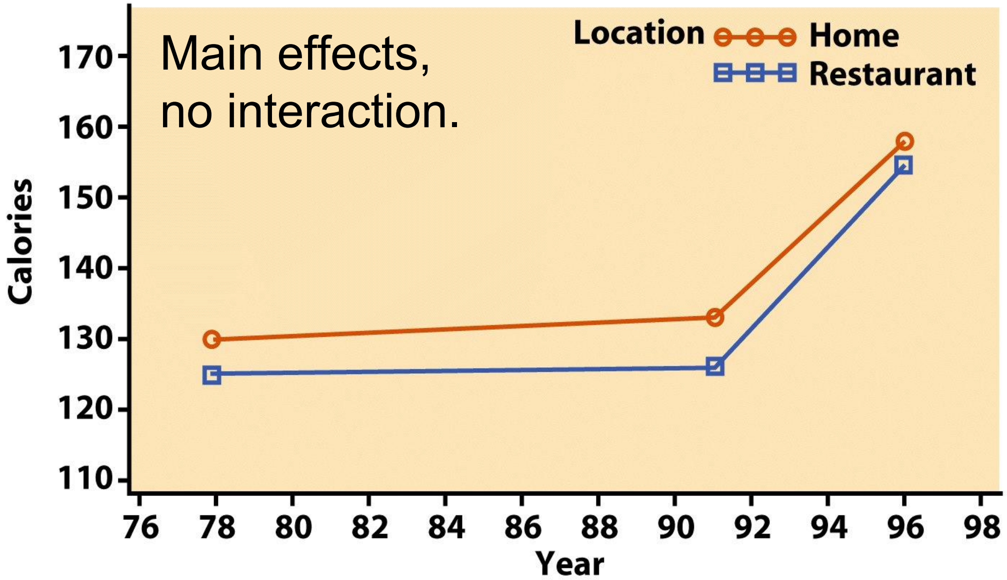
Main Effects and Interactions

In a two-way design, statistical significance can be found for each factor, for the interaction effect, or for any combination of these.

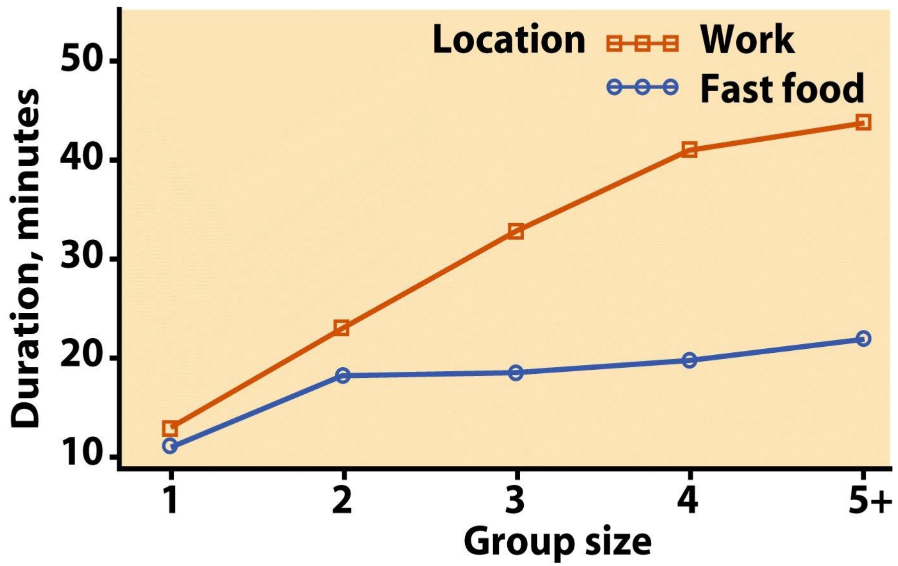
Neither factor is significant	Neither factor is significant	Only one factor is significant	Both factors are significant
No interaction	Interaction effect is significant	No interaction	With or without significant interaction

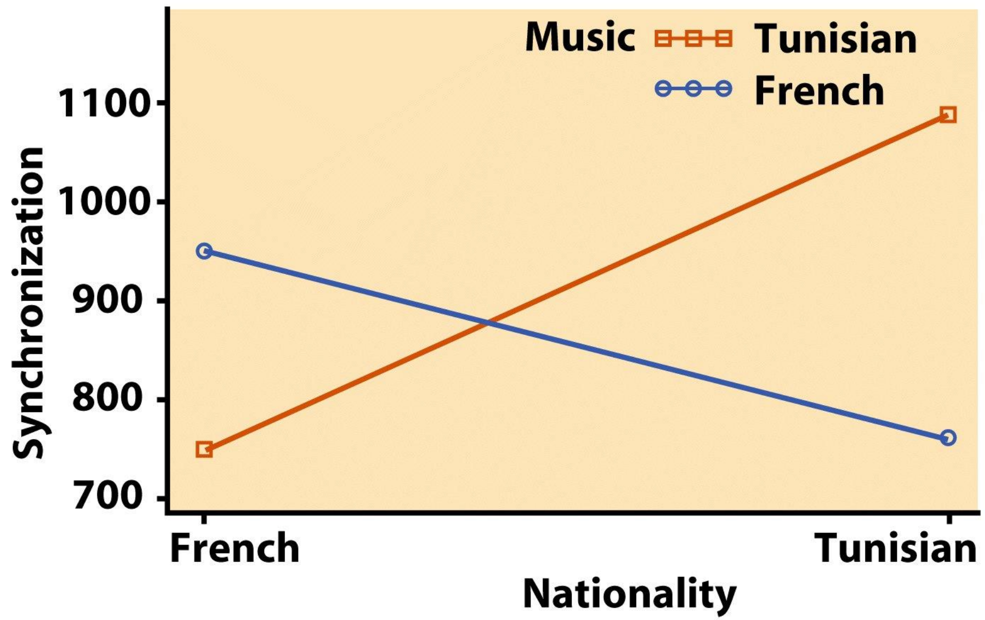


Main Effects and Interactions



Interaction effect: The main effects don’t tell the whole story.





Important interaction effect: The main effects are misleading.

13.2 Inference for Two-Way Analysis of Variance

- The ANOVA table for two-way ANOVA

Inference for Two-Way ANOVA

- A one-way ANOVA tests the following model of your data:

  Data (“total”) = fit (“groups”) + residual (“error”)

  so that the sums of squares and degrees of freedom are:

  SST = SSG + SSE

  DFT = DFG + DFE

- When the **sample sizes are equal**, a two-way design breaks down the “fit” part of the model into more specific subcomponents, so that:

  SST = SSA + SSB + SSAB + SSE

  DFT = DFA + DFB + DFAB + DFE

  where A and B are the two **main effects** from each of the two factors, and AB represents the **interaction** of factors A and B.

The Two-Way ANOVA Table

Source of variation	DF	Sum of squares SS	Mean square MS	F	*P*-value
Factor A	DFA = I -1	SSA	MSA= SSA/DFA	F_A= MSA/MSE	**for F_A**
Factor B	DFB = J - I	SSB	MSB= SSB/DFB	F_B= MSB/MSE	**for F_B**
Interaction	DFAB = $(I-1)(J-1)$	SSAB	MSAB= SSAB/DFAB	F_{AB}= MSAB/MSE	**for F_{AB}**
Error	DFE = N - IJ	SSE	MSE= SSE/DFE		
Total	DFT = N – 1 =DFA+DFB+DFAB+DFE	SST =SSA+SSB+SSAB+SSE			

Main effects: *P*-value for factor A, *P*-value for factor B.

Interaction: *P*-value for the interaction effect of A and B.

Error: Represents variability in the measurements within the groups.

MSE is an unbiased estimate of the population variance σ^2.

Example

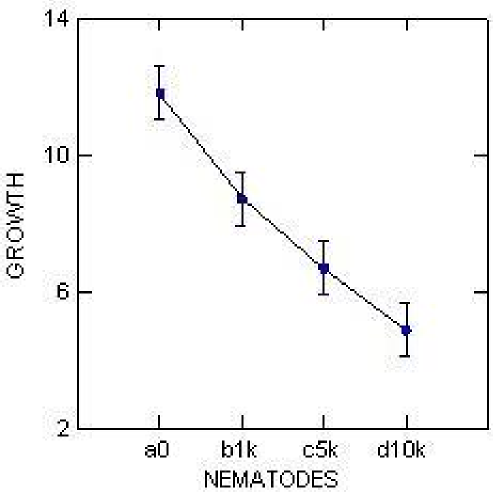
Do nematodes affect plant growth? A botanist prepares 16 identical planting pots and adds different numbers of nematodes into the pots. Seedling growth (in mm) is recorded two weeks later. We analyzed these data with a one-way ANOVA in the previous chapter.

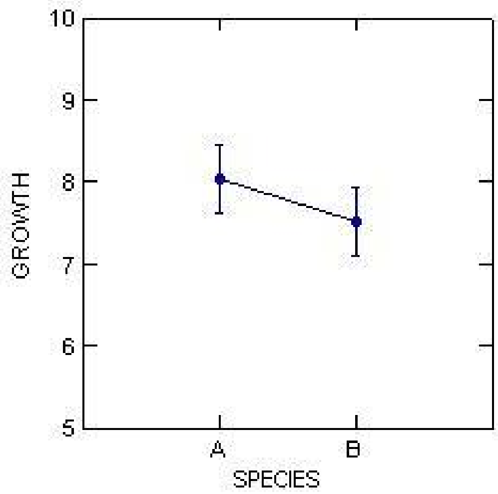
We also have data for another plant species. We can study the effect of nematode amounts (four levels) on seedling growth for both plant species (two levels, species A and B).

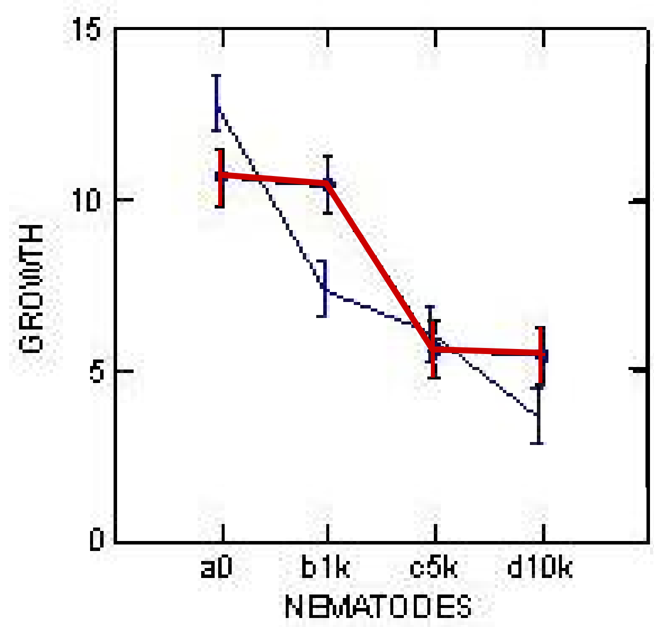
Plant species A was also grown with pesticide. In a separate ANOVA involving only species A, we can analyze seedling growth for combinations of nematodes and pesticide conditions.

Example

Source	SS	df	MS	F-ratio	P
NEMATODES	254.645	3	84.882	31.002	0.000
SPECIES	2.101	1	2.101	0.767	0.390
NEMATODES*SPECIES	34.124	3	11.375	4.154	0.017
Error	65.710	24	2.738		







Plant A is in red.

Example

The results in the previous ANOVA table show that **tests for main effects can be misleading in light of a significant interaction.**

The insignificance of the Species *F*-test (*P* = 0.39) does not mean that there is *never* a difference in average seedling growth for species A and B. It just means that there is no significant difference between the **average of all data for species A** and the **average of all data for species B**.

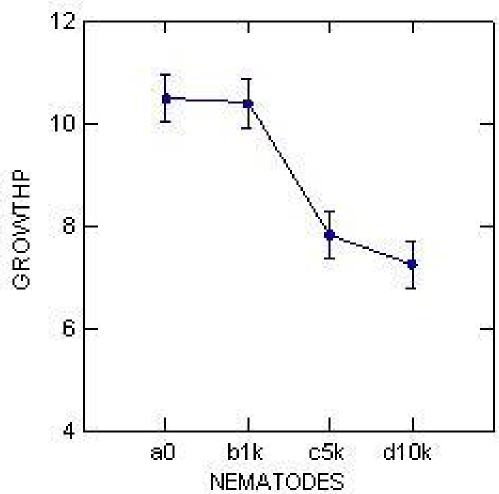
Note that the test for interaction is significant (*P* = 0.017). *To fully appreciate the effects of nematode level and species, one must understand the nature of the interaction.*

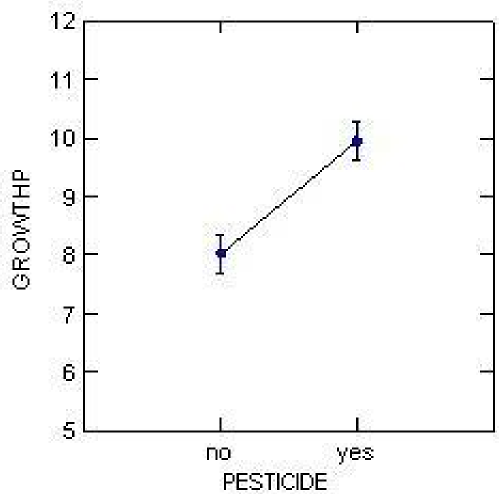
The right most plot on the previous slide helps us do so. First of all, it is clear that whether we consider plant A or plant B, average seedling growth decreases with an increase in nematodes. This explains the significance of the nematode *F*-test (*P* = 0.000).

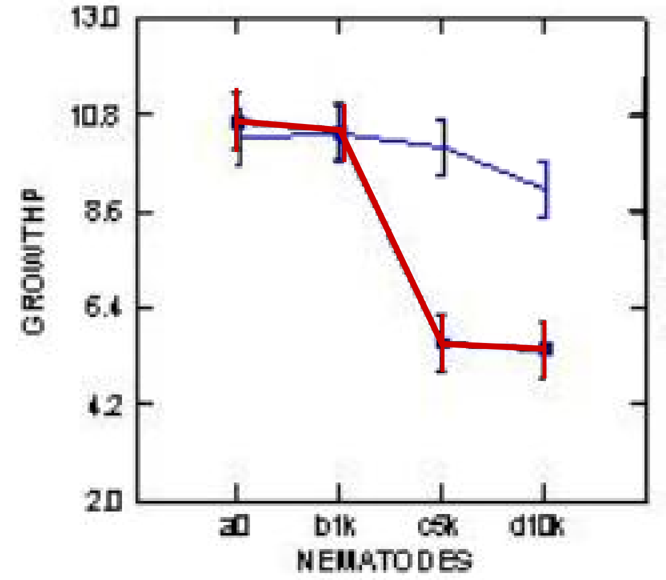
The plot suggests the following interpretation of the interaction: nematodes have a less profound effect on seedling growth of plant A than they do on seedling growth of plant B.

Example

Source	SS	df	MS	F-ratio	P
NEMATODES	68.343	3	22.781	13.195	0.000
PESTICIDE	29.070	1	29.070	16.837	0.000
NEMATODES*PESTICIDE	36.711	3	12.237	7.087	0.001
Error	41.438	24	1.727		







Both main effects are very significant.

The interaction is significant (*P*-value = 0.001): We can see from the third plot that the detrimental effect of nematodes is much stronger in pesticide-free pots (in red).

Cautions for Two-Way ANOVA

1. Always construct a plot of factor level means.

   Examine for possible interaction.

2. Examine the test for interaction first.

   Presence of a strong interaction may influence the interpretation of the main effects.

3. When an interaction is present, carefully examine the means to properly interpret the data.

   The marginal means do not tell the whole story. An examination of treatment means may provide a better interpretation than the ANOVA F tests for main effects. (A treatment mean is the mean response to a particular combination of factor levels.)

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