Homework

Assignment 1:

Section 2.5: 1(b) (Using Jacobi Method and Gauss-Seidel Method to calculate the first two iterations)

## 2.5 Exercises

**Solutions** for Exercises numbered in **blue** can be found at goo.gl/p12UpD

**1.** Compute the first two steps of the Jacobi and the Gauss–Seidel Methods with starting vector $[0, \dots, 0]$.

(a) $\begin{bmatrix} 3 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$

(c) $\begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix}\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \\ 5 \end{bmatrix}$

Assignment 2:

Section 2.6: 1(b) (Using the SOR Method to calculate the first two iterations, $\omega = 1.2, 1.5$)

## 2.6 Exercises

**Solutions** for Exercises numbered in **blue** can be found at goo.gl/RdWwnA

**1.** Show that the following matrices are symmetric positive-definite by expressing $x^T Ax$ as a sum of squares.

$(a)\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}(b)\begin{bmatrix} 1 & 3 \\ 3 & 10 \end{bmatrix}(c)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

Assignment 3:

Section 2.6: 1(b), 2(b), 13(a)

## 2.6 Exercises

**Solutions** for Exercises numbered in **blue** can be found at goo.gl/RdWwnA

**1.** Show that the following matrices are symmetric positive-definite by expressing $x^T Ax$ as a sum of squares.

$(a)\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}(b)\begin{bmatrix} 1 & 3 \\ 3 & 10 \end{bmatrix}(c)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

2. Show that the following symmetric matrices are not positive-definite by finding a vector $x \neq 0$ such that $x^T Ax < 0$.

$(a)\begin{bmatrix} 1 & 0 \\ 0 & -3 \end{bmatrix}(b)\begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix}(c)\begin{bmatrix} 1 & -1 \\ -1 & 0 \end{bmatrix}(d)\begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

**13.** Solve the problems by carrying out the Conjugate Gradient Method by hand.

$(a)\begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}(b)\begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

Assignment 4:

Reasoning the three items of THEOREM 2.16

**THEOREM 2.16** Let $A$ be a symmetric positive-definite $n \times n$ matrix and let $b \neq 0$ be a vector. In the Conjugate Gradient Method, assume that $r_k \neq 0$ for $k < n$ (if $r_k = 0$ the equation is solved). Then for each $1 \leq k \leq n$,

(a) The following three subspaces of $R^n$ are equal:

$$\langle x_1, \ldots, x_k \rangle = \langle r_0, \ldots, r_{k-1} \rangle = \langle d_0, \ldots, d_{k-1} \rangle,$$

(b) the residuals $r_k$ are pairwise orthogonal: $r_k^T r_j = 0$ for $j < k$,

(c) the directions $d_k$ are pairwise $A$-conjugate: $d_k^T A d_j = 0$ for $j < k$. ■

Assignment 5:

Section 3.1: 1(a) (Using the Lagrange interpolation and Newton's divided difference formula to solve the problem)

## 3.1 Exercises

**Solutions** for Exercises numbered in **blue** can be found at

1. Use Lagrange interpolation to find a polynomial that passes through the points.

(a) $(0, 1), (2, 3), (3, 0)$
(b) $(-1, 0), (2, 1), (3, 1), (5, 2)$
(c) $(0, -2), (2, 1), (4, 4)$

Assignment 6:

Section 3.2: 1, 2

## 3.2 Exercises

**Solutions** for Exercises numbered in **blue** can be found at goo.gl/B0TSfm

1. (a) Find the degree 2 interpolating polynomial $P_2(x)$ through the points $(0, 0)$, $(\pi/2, 1)$, and $(\pi, 0)$. (b) Calculate $P_2(\pi/4)$, an approximation for $\sin(\pi/4)$. (c) Use Theorem 3.3 to give an error bound for the approximation in part (b). (d) Using a calculator or MATLAB, compare the actual error to your error bound.

2. (a) Given the data points $(1, 0), (2, \ln 2), (4, \ln 4)$, find the degree 2 interpolating polynomial. (b) Use the result of (a) to approximate ln 3. (c) Use Theorem 3.3 to give an error bound for the approximation in part (b). (d) Compare the actual error to your error bound.

Assignment 7:

Section 3.3: 1(a) (Additionally compute the upper bound of $|\prod_{i=1}^{n}(x-x_i)|$);

## 3.3 Exercises

**Solutions**

1. List the Chebyshev interpolation nodes $x_1, \ldots, x_n$ in the given interval. (a) $[-1, 1], n = 6$ (b) $[-2, 2], n = 4$ (c) $[4, 12], n = 6$ (d) $[-0.3, 0.7], n = 5$

Assignment 8:

Section 4.1: 1(a)

Section 4.2: 1(a), 3(a)

## 4.1 Exercises

**Solutions** for Exercises numbered in **blue** can be found at goo.gl/TP3ocv

1. Solve the normal equations to find the least squares solution and 2-norm error for the following inconsistent systems:

$$\text{(a)}\ \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} \quad \text{(b)}\ \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \quad \text{(c)}\ \begin{bmatrix} 1 & 2 \\ 1 & 1 \\ 2 & 1 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 3 \\ 2 \end{bmatrix}$$

## 4.2 Exercises

**Solutions** for Exercises numbered in **blue** can be found at goo.gl/QJUrKq

1. Fit data to the periodic model $y = F_3(t) = c_1 + c_2 \cos 2\pi t + c_3 \sin 2\pi t$. Find the 2-norm error and the RMSE.

(a)

| $t$ | $y$ |
|---|---|
| 0 | 1 |
| 1/4 | 3 |
| 1/2 | 2 |
| 3/4 | 0 |

(b)

| $t$ | $y$ |
|---|---|
| 0 | 1 |
| 1/4 | 3 |
| 1/2 | 2 |
| 3/4 | 1 |

(c)

| $t$ | $y$ |
|---|---|
| 0 | 3 |
| 1/2 | 1 |
| 1 | 3 |
| 3/2 | 2 |

3. Fit data to the exponential model by using linearization. Find the 2-norm of the difference between the data points $y_i$ and the best model $c_1 e^{c_2 t_i}$.

(a)

| $t$ | $y$ |
|---|---|
| −2 | 1 |
| 0 | 2 |
| 1 | 2 |
| 2 | 5 |

(b)

| $t$ | $y$ |
|---|---|
| 0 | 1 |
| 1 | 1 |
| 1 | 2 |
| 2 | 4 |

Assignment 9:

Section 4.3: 1(c)(Apply both classical Gram-Schmidt orthogonalization and Householder reflectors to find the full QR factorization of the matrices), 9

## 4.3 Exercises

**Solutions** for Exercises numbered in **blue** can be found at goo.gl/tlBDfJ

1. Apply classical Gram–Schmidt orthogonalization to find the full QR factorization of the following matrices:

$$\text{(a)}\ \begin{bmatrix} 4 & 0 \\ 3 & 1 \end{bmatrix} \quad \text{(b)}\ \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \quad \text{(c)}\ \begin{bmatrix} 2 & 1 \\ 1 & -1 \\ 2 & 1 \end{bmatrix} \quad \text{(d)}\ \begin{bmatrix} 4 & 8 & 1 \\ 0 & 2 & -2 \\ 3 & 6 & 7 \end{bmatrix}$$

9. Prove that a square matrix is orthogonal if and only if its columns are pairwise orthogonal unit vectors.

Assignment 10:

Section 12.2: 1(c)

Section 12.3: 1(c), 2(b)

## 12.2 Exercises

Solutions for Exercises numbered in **blue** can be found at goo.gl/HVdJaD

1. Put the following matrices in upper Hessenberg form:

(a) $\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ (c) $\begin{bmatrix} 2 & 1 & 0 \\ 4 & 1 & 1 \\ 3 & 0 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 1 & 0 \\ 2 & 3 & 1 \\ 2 & 1 & 0 \end{bmatrix}$

## 12.3 Exercises

Solutions for Exercises numbered in **blue** can be found at goo.gl/D7d9vx

1. Find the SVD of the following symmetric matrices by hand calculation, and describe geometrically the action of the matrix on the unit circle:

(a) $\begin{bmatrix} -3 & 0 \\ 0 & 2 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & 0 \\ 0 & 3 \end{bmatrix}$ (c) $\begin{bmatrix} \frac{3}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{3}{2} \end{bmatrix}$

(d) $\begin{bmatrix} -\frac{3}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{3}{2} \end{bmatrix}$ (e) $\begin{bmatrix} 0.75 & 1.25 \\ 1.25 & 0.75 \end{bmatrix}$

2. Find the SVD of the following matrices by hand calculation:

(a) $\begin{bmatrix} 3 & 0 \\ 4 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 6 & -2 \\ 8 & \frac{3}{2} \end{bmatrix}$ (c) $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

(d) $\begin{bmatrix} -4 & -12 \\ 12 & 11 \end{bmatrix}$ (e) $\begin{bmatrix} 0 & -2 \\ -1 & 0 \end{bmatrix}$