Physics FSE (2021-22) Homework 3

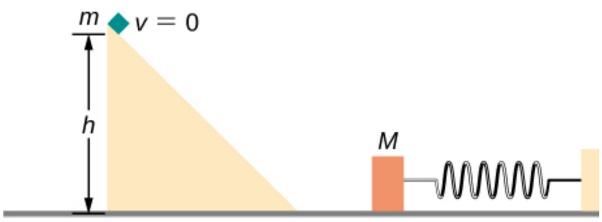
Please send the completed file to my mailbox yy.lam@qq.com by April 20th, with using the filename format:

2021xxxxxx_yourname_fse_hw3

Please answer the questions by filling on these sheets. It would be perfect if you use e-pen directly writing on the sheets. If you do not have the appropriate hardware, you may handle the questions as usual by using pieces of blank papers, then take photos and paste them onto these question sheets.

---

1. A block of mass m, after sliding down a frictionless incline, strikes another block of mass m that is attached to a spring of spring constant k. The blocks stick together upon impact and travel together. (a) Find the compression of the spring in terms of m, M, g and k when the combination comes to rest. (b) The loss of kinetic energy as a result of the bonding of the two masses upon impact is stored in the so-clalled binding energy of the two masses. Calculate the binding energy.



Solution. *(a) The potential energy converts to kinetic energy of the mass m, then after collision sticking together the kinetic energy equation is*

$$mgh = \frac{1}{2}mv_i^2 = \frac{1}{2}(M+m)v^2$$

where v_i, v are the velocities before and after the collision. Momentum must be conserved during collision, we have $mv_i = (M+m)v$. This gives

$$\begin{aligned}\frac{1}{2}(M+m)v^2 &= \frac{1}{2}\frac{m^2}{M+m}v_i^2 \\ &= \frac{1}{2}\frac{m^2}{M+m}(2gh) \\ &= \frac{m^2gh}{M+m}\end{aligned}$$

where we have used the first part of the first equation for the second step. And this should equal to the potential energy stored by the spring

$$\frac{m^2gh}{M+m} = \frac{1}{2}kx^2 \quad \Rightarrow \quad x = \sqrt{\frac{2m^2gh}{k(M+m)}}.$$

(b) The binding energy is the difference of the original potential energy and the $kx^2/2$ term, i.e.,

$$mgh - \frac{m^2gh}{M+m} = \frac{mgh(M+m) - m^2gh}{M+m} = \frac{mMgh}{M+m}.$$

2. To develop muscle tone, a woman lifts a 3.0 kg weight held in her hand. She uses her biceps muscle to flex the lower arm through an angle of 60°. (a) What is the angular acceleration if the weight is 26 cm from the elbow joint, here forearm has a moment of inertia of 0.25 kgm^2, and the net force she exerts is 760 N at an effective perpendicular lever arm of 2 cm? (b) What is the angular velocity at 60°? (c) How much work does she do?

   Solution. *(a) Since*

   $$\text{net torque} = \text{distance of the perpendicular lever arm} \times \text{net force}$$

   and also equals to $I\ddot{\theta}$ where I is the ***total*** *moment of inertia of the system, thus*

   $$(0.25 + 3 \times 0.26^2)\ddot{\theta} = 0.02 \times 760 \quad \Rightarrow \quad \ddot{\theta} = 33.6 \ rad \ s^{-2}$$

   (b) Assuming the angular acceleration $\ddot{\theta}$ being constant, the rotational kinematic equation $\dot{\theta}^2 = 2\ddot{\theta}\theta$ with the initial angular velocity zero gives

   $$\dot{\theta} = \sqrt{2 \times 33.6 \times (60 \times 2\pi/360)} = 8.4 \ rad \ s^{-1}$$

   (c) As rotational energy can be given by $\tau\theta$, we get

   $$0.02 \times 760 \times \frac{60 \times 2\pi}{360} = 15.9 \ J$$

   Note. *Using K.E.* $= \frac{1}{2}I\dot{\theta}^2$ *will give the same answer.*

3. A 200 kg rocket in deep space moves with a velocity of $(121\hat{\mathbf{i}} + 38\hat{\mathbf{j}})$ m/s. Suddenly, it explodes into three pieces, with the first (78 kg) moving at $(-321\hat{\mathbf{i}} + 228\hat{\mathbf{j}})$ m/s and the second (56 kg) moving at $(16\hat{\mathbf{i}} - 88\hat{\mathbf{j}})$ m/s. Find the velocity of the third piece.

   Solution. *The total linear momentum of the system before and after the explosion must conserve, thus*

   $$78(-321\hat{\mathbf{i}} + 228\hat{\mathbf{j}}) + 56(16\hat{\mathbf{i}} - 88\hat{\mathbf{j}}) + (200 - 78 - 56)(v_x\hat{\mathbf{i}} + v_y\hat{\mathbf{j}}) - 200(121\hat{\mathbf{i}} + 38\hat{\mathbf{j}}) = 0$$

   where v_x, v_y are the components. It implies that the velocity of the third piece is $732\hat{\mathbf{i}} + 79.6\hat{\mathbf{j}}$.

4. Two projectiles of mass m_1 and m_2 are fired at the same speed but in opposite directions from two launch sites separated by a distance D. They both reach the same spot in their highest point and strike there. As a result of the impact they stick together and move as a single body afterwards. Show that the landing place will be $\frac{m_2}{m_1 + m_2}D$ measured from the launching position of m_1.

   Solution. *Assuming the mass m_1 is launched at 0, m_2 launched at D in a regular Cartesian coordinate system. Let $v_{1x}, v_{2x} > 0$ be the horizontal components of the launching velocities, $v_{1y}, v_{2y} > 0$ the corresponding vertical components, respectively. Since the horizontal motions are in inertia motion we have*

   $$\frac{d}{v_{1x}} = -\frac{D - d}{-v_{2x}} = t = -\frac{v_{1y}}{g}$$

where t is the instant when they hit at the d horizontal distance from 0 as well as reaching the highest position. The conservation of momentum gives

$$m_1 v_{1x} - m_2 v_{2x} = (m_1 + m_2) v_x$$

where v_x is the combined velocity in x-direction. Thus the horizontal range traveled by the combined mass from d is

$$v_x t = \frac{m_1 v_{1x} - m_2 v_{2x}}{m_1 + m_2} \left(-\frac{v_{1y}}{g} \right)$$

Write v_{1x} and v_{2x} in terms of v_{1y} by the first equation, and in fact all the v_{1y} will be canceled, then we get

$$v_x t = \frac{m_1}{m_1 + m_2} x - \frac{m_2}{m_1 + m_2} (D - d) = x - \frac{m_2}{m_1 + m_2} D$$

Therefore, the range measured from 0 is $\frac{m_2}{m_1 + m_2} D$.

5. A wind turbine is rotating counterclockwise at 0.5 rev/s and slows to a stop in 10 s. Its blades are 20 m in length. (a) What is the angular acceleration of the turbine? (b) What is the centripetal acceleration of the tip of the blades at $t = 0$ s? (c) What is the magnitude and direction of the total linear acceleration of the tip of the blades at $t = 0$ s?

   Solution. *(a) The angular acceleration is*

   $$\ddot{\theta} = \frac{0 - 0.5(2\pi)}{10} = -0.1\pi \ rad\ s^{-2}.$$

   (b) The centripetal acceleration is

   $$a_c = r\dot{\theta}^2 = 20 \times (0.5(2\pi))^2 = 20\pi^2 \ ms^{-2}.$$

   (c) The tangential acceleration is

   $$a_t = r\ddot{\theta} = 20 \times (-0.1\pi) = -2\pi \ ms^{-2}.$$

   The magnitude of the acceleration is

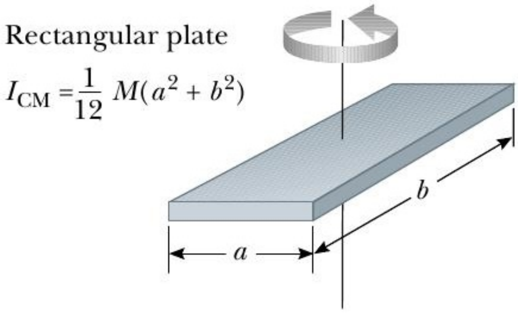
   $$a = \sqrt{a_c^2 + a_t^2} = \sqrt{(20\pi^2)^2 + (2\pi)^2} = 197.5 \ ms^{-2}.$$

   The angle of direction is

   $$\theta = \tan^{-1} \frac{-2\pi}{20\pi^2} = -1.82^\circ$$

6. Drive the formula for I_{CM} as shown in the figure.

   Solution. *The formula of moment of inertia is given by $\int r^2 \mathrm{d}m$. Written in terms of Cartesian coordinate system that $r^2 = x^2 + y^2$ and $\mathrm{d}m = \rho(\mathrm{d}x\mathrm{d}y)$ where ρ is the mass per*

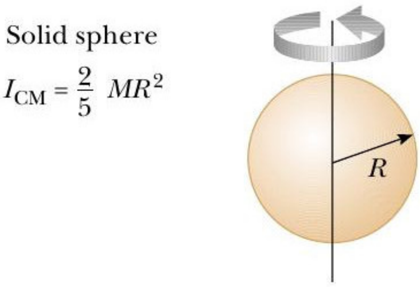


unit area. Thus,

$$
\begin{aligned}
I &= \int_{-b/2}^{b/2}\int_{-a/2}^{a/2} \rho(x^2+y^2)\mathrm{d}x\mathrm{d}y \\
&= \int_{-b/2}^{b/2}\int_{-a/2}^{a/2} \rho x^2\mathrm{d}x\mathrm{d}y + \int_{-b/2}^{b/2}\int_{-a/2}^{a/2} \rho y^2\mathrm{d}x\mathrm{d}y \\
&= \int_{-b/2}^{b/2} \rho\frac{x^3}{3}\Big|_{-a/2}^{a/2}\mathrm{d}y + \int_{-b/2}^{b/2} \rho x y^2\Big|_{-a/2}^{a/2}\mathrm{d}y \\
&= \frac{\rho}{12}a^3 y\Big|_{-b/2}^{b/2} + \rho a\frac{y^3}{3}\Big|_{-b/2}^{b/2} \\
&= \frac{\rho}{12}a^3 b + \frac{\rho}{12}ab^3
\end{aligned}
$$

Since $M = \rho ab$, the moment of inertia is $I_{CM} = \frac{1}{12}M(a^2+b^2)$. □

7. Drive the formula for I_{CM} as shown in the figure. (Hint: Imagine a solid sphere being the infinite sum of a stack of infinitesimal varying size of thin solid cylinders.)



Solution. *Rewrite the moment of inertia of a solid cylinder formula rotating about the symmetric axis $I_{cylinder} = \frac{1}{2}MR^2$ as an infinitesimal thin piece*

$$
\mathrm{d}I = \frac{1}{2}(dm)y^2 = \frac{1}{2}(\rho\pi y^2\mathrm{d}x)y^2 = \frac{1}{2}\rho\pi y^4\mathrm{d}x
$$

*where for convenience the cylinder element rotates about the x-axis with y as the **varying radius** (i.e., turn the diagram in 90 degrees), ρ being the density per **unit volume** and $\mathrm{d}x$ the infinitesimal thickness. Integrating $\mathrm{d}I$ over the interval $[-R, R]$ with the circle* *equation* $R^2 = x^2 + y^2$ *gives*

$$\begin{aligned}
I &= \frac{1}{2}\rho\pi \int_{-R}^{R} y^4 \mathrm{d}x = \frac{1}{2}\rho\pi \int_{-R}^{R} (R^2 - x^2)^2 \mathrm{d}x \\
&= \frac{1}{2}\rho\pi \left[R^4 x - \frac{2}{3}R^2 x^3 + \frac{1}{5}x^5 \right]_{-R}^{R} \\
&= \frac{1}{2}\rho\pi \left(2R^5 - \frac{4}{3}R^5 + \frac{2}{5}R^5 \right) \\
&= \frac{8}{15}\rho\pi R^5 \\
&= \frac{4}{3}\rho\pi R^3 \cdot \frac{3}{4} \cdot \frac{8}{15} R^2 \\
&= \frac{2}{5} M R^2
\end{aligned}$$

$\square$