数值计算基础（20级信科院）

Sorted out by $\mathrm{H3Art}$

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1. (10 Points) Apply classic Gram-Schimdt orthogonalization to find the full QR factorization of the matrix $\begin{bmatrix} 2 & 3 \\ -2 & -6 \\ 1 & 0 \end{bmatrix}$.

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2. (10 Points) Given a square matrix $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$, please answer the following questions:

(1) Find all eigenvalues and eigenvectors of $\mathbf{A}$;

(2) Apply three steps of Power Iteration with initial vector $\mathbf{x_0} = (1, 0)$.

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3. **【果园这边没教】**(20 Points) Apply the Trapezoid Rule, Simpson's Rule and Mid-point Rule to approximate the integral $\int_0^1 x^2 \mathrm{d}x$. Compute the error by comparing with the exact value from calculus.

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4. (15 Points) If the $n \times n$ matrix $\mathbf{A}$ is strictly diagonally dominant, then for every vector $\mathbf{b}$ and every starting guess, the Gauss-Seidel Method applied to $\mathbf{Ax} = \mathbf{b}$ converges to the unique solution.

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5. (15 Points) Please prove the following: Let $\mathbf{A}$ be an $n \times n$ matrix with real eigenvalues $\lambda_1, \ldots, \lambda_n$, satisfying $|\lambda_1| > |\lambda_2| \geq |\lambda_3| \geq \cdots \geq |\lambda_n|$. Assume that the eigenvalues of $\mathbf{A}$ span R^n. For almost every initial vector, the Power Iteration method converges to an eigenvector associated to λ_1.

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6. (15 Points) Given the following:

- $\mathbf{x}$ and $\mathbf{w}$: two vectors with $||\mathbf{x}||_2 = ||\mathbf{w}||_2$;
- $\mathbf{u} = \mathbf{w} - \mathbf{x}$ and $\mathbf{v} = \dfrac{\mathbf{u}}{||\mathbf{u}||_2}$
- $\mathbf{H} = \mathbf{I} - \mathbf{2vv}^\top$

Please prove that $\mathbf{Hx} = \mathbf{w}$ and $\mathbf{Hw} = \mathbf{x}$.

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7. (15 Points) Please compare the following methods: Jacobi Method, Gauss-Seidel Method, SOR, Conjugate Gradient Method and Preconditioned Conjugate Gradient Method.