Lecture 2-2: Classical Cryptography

–Cryptographic Algorithms and Protocols

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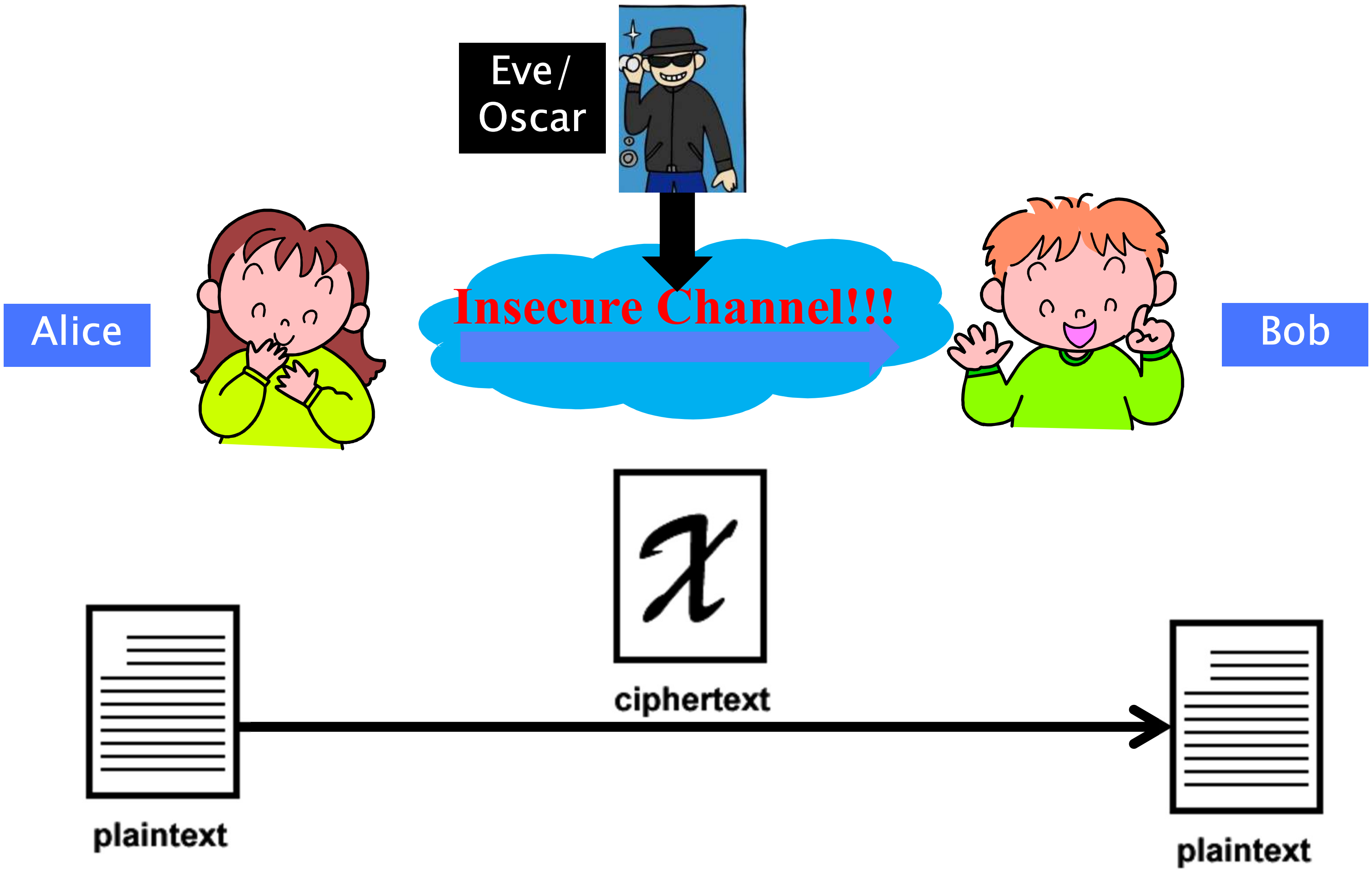
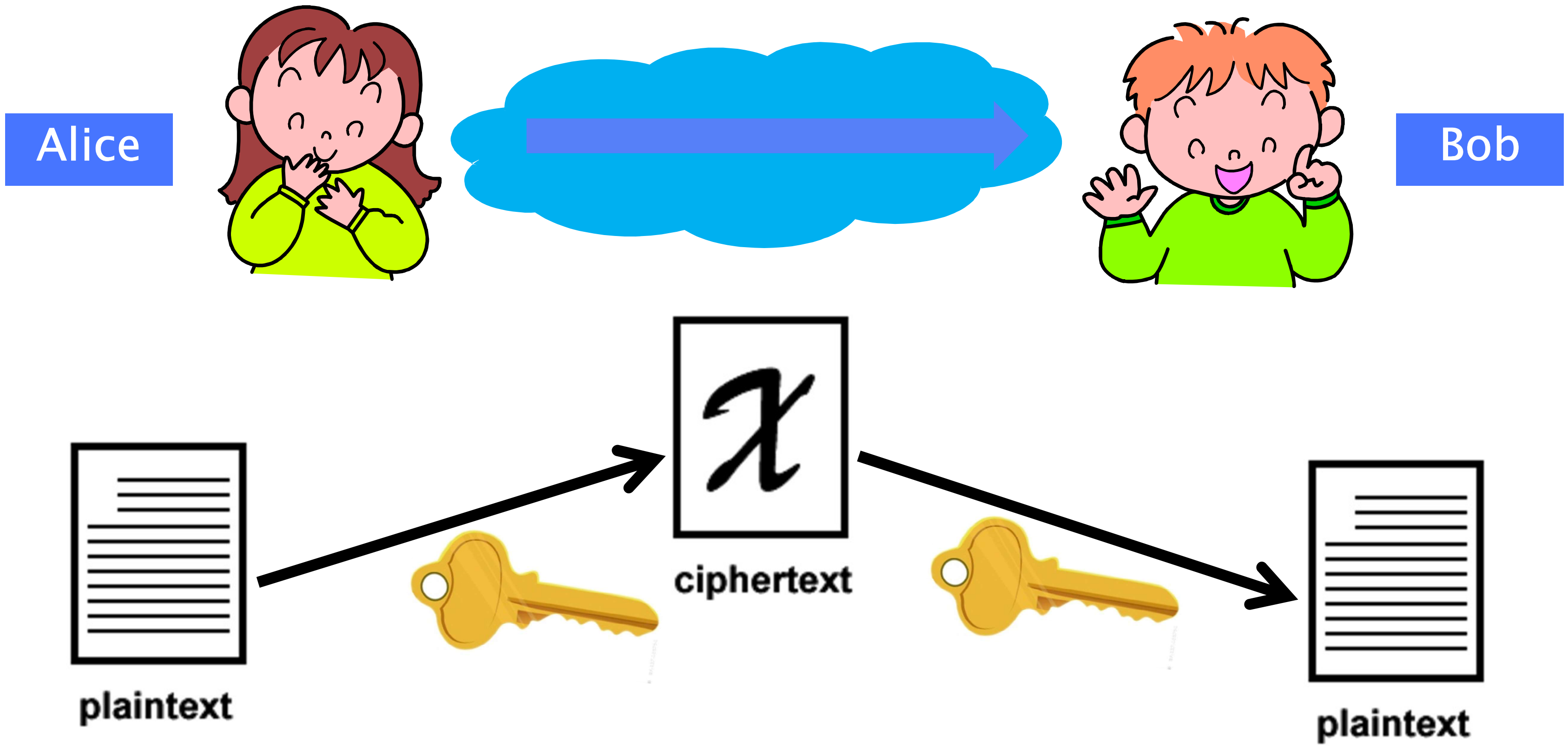
Outline

Outline

Intuition on Cryptography



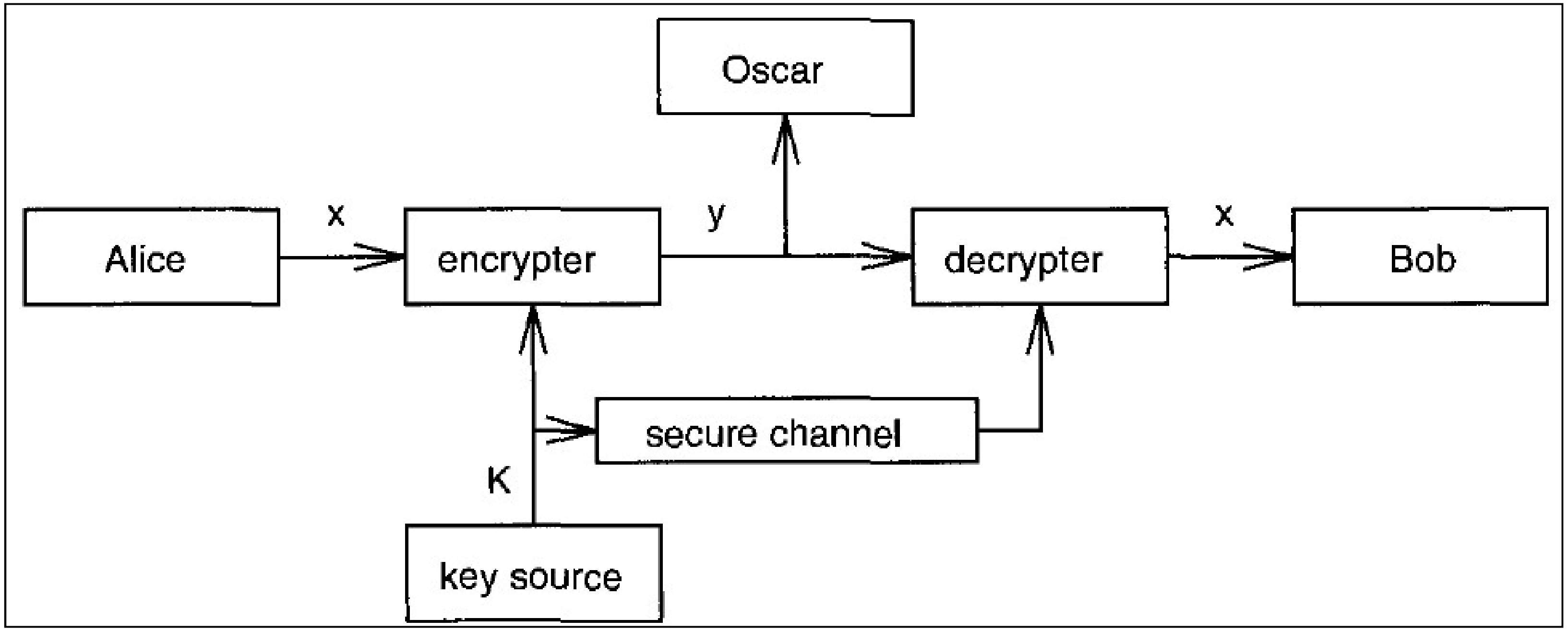
Symmetric-Key Cryptosystem (SKC)



Cryptographic Communication System

Protocol:

- A and B choose a random key K by a secure channel
- A wants to sent a string message $\mathbf{x}=x_1x_2\ldots x_n$
- A computes $y_i=e_K(x_i)$ and the resulting string $\mathbf{y}=y_1y_2\ldots y_n$ is sent over the insecure channel
- B receives $\mathbf{y}=y_1y_2\ldots y_n$ and decrypts $x_i=d_K(y_i)$ to obtain $\mathbf{x}=x_1x_2\ldots x_n$



Basics of a Cryptosystem

Objective and Solutions

- Objective: confidentially communicate over insecure channels
- SKC is one of the solutions

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A Simple Cryptosystem

A cryptosystem is a five-tuple $(\mathcal{P}, \mathcal{C}, \mathcal{K}, \mathcal{E}, \mathcal{D})$, where

1. $\mathcal{P}$ is a finite set of all possible *plaintexts*,
2. $\mathcal{C}$ is a finite set of all possible *ciphertexts*,
3. $\mathcal{K}$ is a finite set of all possible *keys*, called *keyspace*,
4. For each $K \in \mathcal{K}$, $e_K \in \mathcal{E}$, $d_K \in \mathcal{D}$
   Encryption rule $e_K : \mathcal{P} \to \mathcal{C}$; Decryption rule $d_K : \mathcal{C} \to \mathcal{P}$,
   satisfying $d_K(e_K(x)) = x$ for every $x \in \mathcal{P}$.

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   satisfying $d_K(e_K(x)) = x$ for every $x \in \mathcal{P}$.

- Encryption rule e_K is a injective function (单射), i.e., one-to-one.
  If $x_1 \neq x_2$, then $e_K(x_1) \neq e_K(x_2)$.

Outline

The Substitution Cipher

Cryptosystem 2.2: Substitution Cipher, 替换密码(on Page 21)

Let $\mathcal{P} = \mathcal{C} = \mathbb{Z}_{26}$. Let $\mathcal{K}$ consists of all possible permutations of the 26 elements in $\mathbb{Z}_{26}$. For each permutation $\pi \in \mathcal{K}$, define

- Encryption rule $e_\pi : \mathcal{P} \to \mathcal{C}$, $e_\pi(x) = \pi(x)$
- Decryption rule $d_\pi : \mathcal{C} \to \mathcal{P}$, $d_\pi(y) = \pi^{-1}(y)$

where π^{-1} is the inverse permutation to π.

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where π^{-1} is the inverse permutation to π. $\mathbb{Z}_{26} \leftrightarrow$ the English alphabet (π是英文字母表的一个置换, π^{-1}是置换π的逆置换.)

Relation of $\mathbb{Z}_{26}$ and the $26-$letter English Alphabet:

a	b	c	d	e	f	g	h	i	j	k	l	m
0	1	2	3	4	5	6	7	8	9	10	11	12

n	o	p	q	r	s	t	u	v	w	x	y	z
13	14	15	16	17	18	19	20	21	22	23	24	25

plaintext -- lower case letter, ciphertext -- upper case letter

The Substitution Cipher

An Example

Let π be the permutation over the **English alphabet** as follows:

a	b	c	d	e	f	g	h	i	j	k	l	m
X	N	Y	A	H	P	O	G	Z	Q	W	B	T

n	o	p	q	r	s	t	u	v	w	x	y	z
S	F	L	R	C	V	M	U	E	K	J	D	I

$e_\pi(\text{ihaveanapple}) = \pi(\text{ihaveanapple}) = \text{ZGXEHXSXLLBH}$

$d_\pi(\text{ZGXEHXSXLLBH}) = \pi^{-1}(\text{ZGXEHXSXLLBH}) = \text{ihaveanapple}$

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Some Issues:

- The size of Keyspace $\mathcal{K}$ is $= 26! > 4.0 \times 10^{26}$.

The Substitution Cipher

An Example

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Some Issues:

- The size of Keyspace $\mathcal{K}$ is $= 26! > 4.0 \times 10^{26}$.
- An exhaustive key search is infeasible.
  But it can easily be cryptanalyzed by some statistic methods.

An example of a ciphertext-only attack on the substitution cipher

The Shift Cipher

Cryptosystem 2.1: Shift Cipher, 移位密码(on Page 18)

Let $\mathcal{P} = \mathcal{C} = \mathcal{K} = \mathbb{Z}_{26}$. For $0 \leq K \leq 25$, define

- Encryption rule $e_K : \mathcal{P} \to \mathcal{C}$ as $e_K(x) = \underline{(x + K)} \mod 26$,
- Decryption rule $d_K : \mathcal{C} \to \mathcal{P}$ as $d_K(y) = \underline{(y - K)} \mod 26$.

where $x, y \in \mathbb{Z}_{26}$.

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where $x, y \in \mathbb{Z}_{26}$.

Some Issues:

- The Shift Cipher is a special case of the Substitution Cipher.
- When $K = 3$, it is called the *Caesar Cipher*.
- The size of Keyspace $\mathcal{K}$ is $= 26$. (only 26 permutations)
- NOT SECURE!!! (exhaustive key search)
- a necessary condition to be secure is that an exhaustive key search should be infeasible.

The Affine Cipher

Cryptosystem 2.3: Affine Cipher, 仿射密码(on Page 25)

Let $\mathcal{P} = \mathcal{C} = \mathbb{Z}_{26}$. Let the keyspace be

$$\mathcal{K} = \{(a, b) \in \mathbb{Z}_{26} \times \mathbb{Z}_{26} : \gcd(a, 26) = 1\} = \mathbb{Z}_{26}^* \times \mathbb{Z}_{26} \tag{2.1}$$

For each key $K = (a, b) \in \mathcal{K}$, define $(x, y \in \mathbb{Z}_{26})$

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Some Issues:

- Verification of $d_K(e_K(x)) = x$: $d_K(e_K(x)) = a^{-1}\left((ax + b) \bmod 26 - b\right) \mod 26 = a^{-1}ax \bmod 26 = x$.

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- The size of keyspace is $\phi(26) \times 26 = 12 \times 26 = 312$. (312 permutations)

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- The Affine Cipher is a special case of the Substitution Cipher.
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- The Affine Cipher is a special case of the Substitution Cipher.
- For the general case of $\mathcal{P} = \mathcal{C} = \mathbb{Z}_m$, $\mathcal{K} = \mathbb{Z}_m^{*} \times \mathbb{Z}_m$?
  The size of keyspace is $\phi(m) \times m$.

Cryptanalysis of the Affine Cipher

A ciphertext-only attack on the Affine cipher

Oscar knows a string of ciphertext, $\mathbf{y}$, and that is obtained using the Affine cipher.

The Affine cipher can be broken by using the Statistical properties of the English language.

The Vigenere Cipher

Cryptosystem 2.4: Vigenere Cipher (on Page 26)

Let $n \in \mathbb{Z}^+$. Let $\mathcal{P} = \mathcal{K} = \mathcal{C} = (\mathbb{Z}_{26})^n$. For a key $K = (k_1, k_2, \cdots, k_n)$,

- Encryption: $e_K(x_1, x_2, \cdots, x_n) = (x_1 + k_1, x_2 + k_2, \cdots, x_n + k_n)$;
- Decryption: $d_K(y_1, y_2, \cdots, y_n) = (y_1 - k_1, y_2 - k_2, \cdots, y_n - k_n)$.

where all operations are performed in $\mathbb{Z}_{26}$.

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where all operations are performed in $\mathbb{Z}_{26}$.

An example

Assume the keyword is "C I P H E R" which $\leftrightarrow K = (2, 8, 15, 7, 4, 17)$.

Plaintext: thiscrytosystemisnotsecure

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$\leftrightarrow 19 \;\; 7 \;\; 8 \;\; 18 \;\; 2 \;\; 17 \;\; 24 \;\; 15 \;\; 19 \;\; 14 \;\; 18 \;\; 24 \;\; 18 \;\; 19 \;\; 4 \;\; 12 \;\; 8 \;\; 18 \cdots$

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$$
\begin{array}{cccccc|cccccc|cccccc|c}
\leftrightarrow 19 & 7 & 8 & 18 & 2 & 17 & 24 & 15 & 19 & 14 & 18 & 24 & 18 & 19 & 4 & 12 & 8 & 18 & \cdots \\
\oplus \quad 2 & 8 & 15 & 7 & 4 & 17 & 2 & 8 & 15 & 7 & 4 & 17 & 2 & 8 & 15 & 7 & 4 & 17 & \cdots
\end{array}
$$

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Cryptosystem 2.4: Vigenere Cipher (on Page 26)

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\oplus & 2 & 8 & 15 & 7 & 4 & 17 & 2 & 8 & 15 & 7 & 4 & 17 & 2 & 8 & 15 & 7 & 4 & 17 & \cdots \\
=: & 21 & 15 & 23 & 25 & 6 & 8 & 0 & 23 & 8 & 21 & 22 & 15 & 20 & 1 & 19 & 19 & 12 & 9 & \cdots
\end{array}
$$

Ciphertext: VPXZGIAXICWPUBTTMJPWIZITWZT

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Plaintext: thiscrytosystemisnotsecure

$\leftrightarrow$	19	7	8	18	2	17	24	15	19	14	18	24	18	19	4	12	8	18	$\cdots$
$\oplus$	2	8	15	7	4	17	2	8	15	7	4	17	2	8	15	7	4	17	$\cdots$
$=:$	21	15	23	25	6	8	0	23	8	21	22	15	20	1	19	19	12	9	$\cdots$

Ciphertext: VPXZGIAXICWPUBTTMJPWIZITWZT

A plaintext letter may be decrypted as different cipher letters.

$\boxed{\text{polyalphabetic}}$

The Vigenere Cipher

monoalphabetic (单表) v.s. polyalphabetic (多表)

- the monoalphabetic (单表) cryptosystem: each alphabetic character is mapped to a unique alphabetic character.
- the polyalphabetic (多表) cryptosystem: an alphabetic character can be mapped to one of several possible alphabetic characters.

Some Issues:

- The Vigenere cipher is polyalphabetic:
  1. the key is a string of length n;
  2. n characters are encrypted at a time;
  3. an alphabetic character can be mapped to one of n possible alphabetic characters if the keyword contains n distinct characters.
- It is named after Blaise de Vigenere (in the sixteenth century).
- The size of Keyspace is 26^n; $26^5 \approx 1.19 \times 10^7$, $26^{10} \approx 1.4 \times 10^{14}$.
- It can be analyzed by some statistical methods.

Kasiski test, index of coincidence --> n

The Hill Cipher

Cryptosystem 2.5: Hill Cipher (on Page 32)

Let $n \in \mathbb{Z}^+$ and $n \geq 2$. Let $\mathcal{P} = \mathcal{K} = \mathcal{C} = (\mathbb{Z}_{26})^n$. Let

$$\mathcal{K} = \{n \times n \text{ invertible matrices over } \mathbb{Z}_{26}\}. \tag{2.2}$$

For a key $K \in \mathcal{K}$, define

- Encryption: $e_K(x) = xK$;
- Decryption: $d_K(y) = yK^{-1}$.

where all operations are performed in $\mathbb{Z}_{26}$.

Theorem 2.3 on Page 30

$K^{-1}=(\det K)^{-1}K^*$

K^* is the adjoint matrix of K

$k_{ij}^*=(-1)_{i+j} \det K_{ji}$

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- Encryption: $e_K(x) = \underline{x}K$;
- Decryption: $d_K(y) = \underline{y}K^{-1}$.

where all operations are performed in $\mathbb{Z}_{26}$.

Some Issues:

- The Hill cipher is another polyalphabetic cryptosystem, invented in 1929 by Lester S. Hill.
- The Hill cipher can be broken using the known plaintext attack, which assumes that Oscar knows at least n distinct plaintext-ciphertext pairs.

Outline

The Permutation Cipher

Cryptosystem 2.6: Permutation Cipher, 置换密码(on Page 33)

Let $m \in \mathbb{Z}^+$. Let $\mathcal{P} = \mathcal{C} = (\mathbb{Z}_{26})^m$. Let $\mathcal{K}$ consist of all permutations of $\mathcal{M} = \{1, 2, \cdots, m\}$. For a key $\pi \in \mathcal{K}$, define

- Encryption: $e_\pi(x_1, x_2, \cdots, x_m) = (x_{\pi(1)}, x_{\pi(2)}, \cdots, x_{\pi(m)})$;
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此处，π是位置集合$\mathcal{M} = \{1, 2, \cdots, m\}$上的一个置换, π^{-1}是π的逆置换.

The Permutation Cipher

Cryptosystem 2.6: Permutation Cipher, 置换密码(on Page 33)

Let $m \in \mathbb{Z}^+$. Let $\mathcal{P} = \mathcal{C} = (\mathbb{Z}_{26})^m$. Let $\mathcal{K}$ consist of all permutations of $\mathcal{M} = \{1, 2, \cdots, m\}$. For a key $\pi \in \mathcal{K}$, define

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Some Issues:

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  keeps the plaintext characters unchanged, but alters their positions.
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Examples

See Pages 32-33, Example 2.7.

The Permutation Cipher is a special case of the Hill Cipher.

Outline

Stream Ciphers

Block cipher v.s. Stream Cipher

- Block cipher: successive plaintext elements are encrypted by the same key:

$$\mathbf{y} = y_1 y_2 \cdots = e_K(x_1) e_K(x_2) \cdots$$

- Stream cipher: to generate a Keystream $\mathbf{z} = z_1 z_2 \cdots$, and use it to encrypt a plaintext string:

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A block cipher can be seen as a special case of a stream cipher.

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A block cipher can be seen as a special case of a stream cipher.

Synchronous v.s. Non-Synchronous

- Is the keystream constructed from the key which is independent of the plaintext and ciphertext?

The Synchronous Stream Cipher

Definition 2.6: Synchronous Stream Cipher, SSC (on Page 35)

An SSC is a six-tuple $(\mathcal{P}, \mathcal{C}, \mathcal{K}, \mathcal{L}, \mathcal{E}, \mathcal{D})$ with a function g:

1. $\mathcal{P}$ is a finite set of all possible *plaintexts*,
2. $\mathcal{C}$ is a finite set of all possible *ciphertexts*,
3. $\mathcal{K}$ is a finite set of all possible *keys*, called *keyspace*,
4. $\mathcal{L}$ is a finite set called the *keystream alphabet*,
5. g is the *keystream generator*. g takes a key $K \in \mathcal{K}$ as input, and generates an infinite string $z_1 z_2 \cdots$ called the *keystream*, $z_i \in \mathcal{L}$,
6. For each $z \in \mathcal{L}$, there is $e_z \in \mathcal{E}$ & $d_z \in \mathcal{D}$
   Encryption rule $e_z : \mathcal{P} \rightarrow \mathcal{C}$ & Decryption rule $d_z : \mathcal{C} \rightarrow \mathcal{P}$, satisfying $d_z(e_z(x)) = x$ for every $x \in \mathcal{P}$.

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Some Issues:

- A stream cipher is *periodic* with period d if $z_{i+d} = z_i$ for all integers $i \geq 1$. The Vigenere Cipher can be defined as a periodic SSC.

The SSC over Binary Alphabets

Let $\mathcal{P} = \mathcal{C} = \mathcal{L} = \mathbb{Z}_2$, $\mathcal{K} = \{k_1, k_2, \cdots, k_m; c_0, c_1 \cdots, c_{m-1}\} = \mathbb{Z}_2^{2m}$.

1. The keystream is generated by a linear recurrence g of degree m:
$$\boxed{z_{i+m} = \sum_{j=0}^{m-1} c_j z_{i+j} \mod 2,} \tag{4.1}$$
for all $i \geq 1$, where $c_0, \cdots, c_{m-1} \in \mathbb{Z}_2$ are constants, and $z_i = k_i, 1 \leq i \leq m$.
2. For each $z \in \mathcal{L}$, define
Encryption rule $e_z(x) = (x + z) \mod 2$;
Decryption rule $d_z(y) = (y + z) \mod 2$.

- g: degree m (if $c_0 = 1$), linear, period $2^m - 1$ (for binary case) if $c_0, \cdots, c_{m-1}$ are proper;

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- keystream can be produced in hardware by a Linear Feedback Shift Register

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- keystream can be produced in hardware by a Linear Feedback Shift Register
- See Pages 36-37, Example 2.8.

Cryptanalysis on the SSC over Binary Alphabets

A known plaintext attack on the LFSR Stream cipher

(See Section 2.2.5)

A Non-Synchronous Stream Cipher

A non-synchronous stream cipher: each keystream element z_i depends on previous plaintext or ciphertext elements as well as the key K

Cryotosystem 2.7: Autokey Cipher (Non-Synchronous) (on Page 38)

Let $\mathcal{P} = \mathcal{C} = \mathcal{K} = \mathcal{L} = \mathbb{Z}_{26}$. Let $z_1 = K$, and define $z_i = x_{i-1}$ for all $i \geq 2$. For $0 \leq z \leq 25$, define
Encryption rule $e_z(x) = (x + z) (\bmod\ 26)$;
Decryption rule $d_z(y) = (y - z) (\bmod\ 26)$.

Examples

See Pages 37-38, Example 2.9.

Outline

Cryptanalysis

A general assumption

- Kerckhoff's Principle: The opponent, Oscar, knows the cryptosystem being used.

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The most Common Types of Attack Models:

The attack model specifies the information available to the adversary.

1. ciphertext-only attack: a string of ciphertext $\mathbf{y}$
2. known plaintext attack: plaintext $\mathbf{x}$ and the corresponding $\mathbf{y}$
3. chosen plaintext attack: access Enc, choose $\mathbf{x}$ and construct the corresponding $\mathbf{y}$
4. chosen ciphertext attack: access Dec, choose $\mathbf{y}$ and construct the corresponding $\mathbf{x}$

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Objective of the attack

- To determine the key K.

Cryptanalysis

Some Issues:

- The weakest type of attack is the ciphertext–only attack.
- The statistical properties of the English language (see Page 40) are usually used in cryptanalysis.

Cryptanalysis: To determine the key K

1. A Ciphertext-only attack on the Affine cipher:
   By using Statistical properties of the English language, See pages 40-42.
2. A Ciphertext-only attack on the Substitution cipher:
   By using Statistical properties of the English language, See pages 42-44.
3. A Ciphertext-only attack on the Vigenere cipher:
   To determine the keyword length: Kasiski test or the index of coincidence;
   To determine the keyword: Statistical methods, See pages 45-48.
4. A known plaintext attack on the Hill cipher: (See pages 48-49)
5. A known plaintext attack on the LFSR Stream cipher: (See pages 49-51)

Outline

Conclusions

Homework 1

Problem Set 1

Exercises: 2.1, 2.7, 2.8, 2.9, 2.10, 2.15(a), 2.16, 2.18, 2.23, 2.30(with executable codes)

Thanks for your attention!

Questions?

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