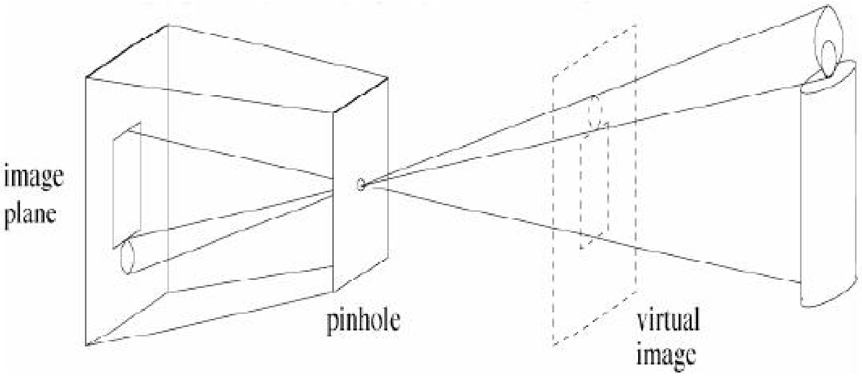
Camera Model

Multiple View Geometry

Pinhole Camera models

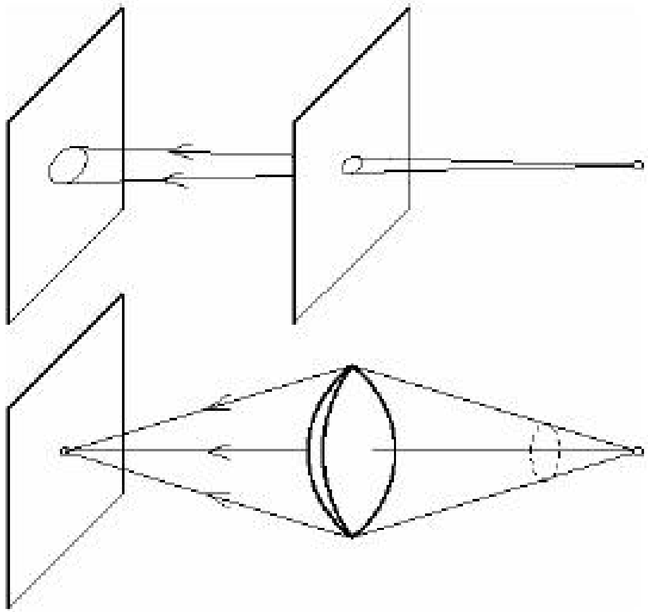
- Projection of 3D world into 2D image plane



Projection geometry for a pinhole camera

- Fundamental problem of pinhole camera

Lenses

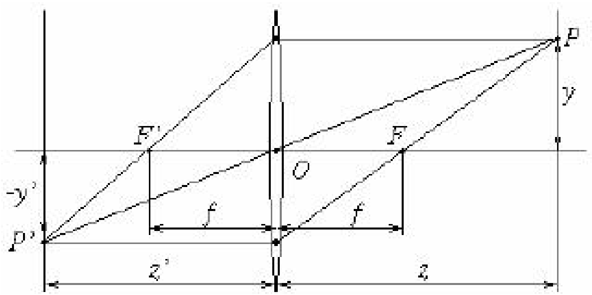


- Equip with lens

  A lens gathers a whole cone of light from every point of a visible surface, and refocuses this cone onto a single point on the sensor.

Thin lens

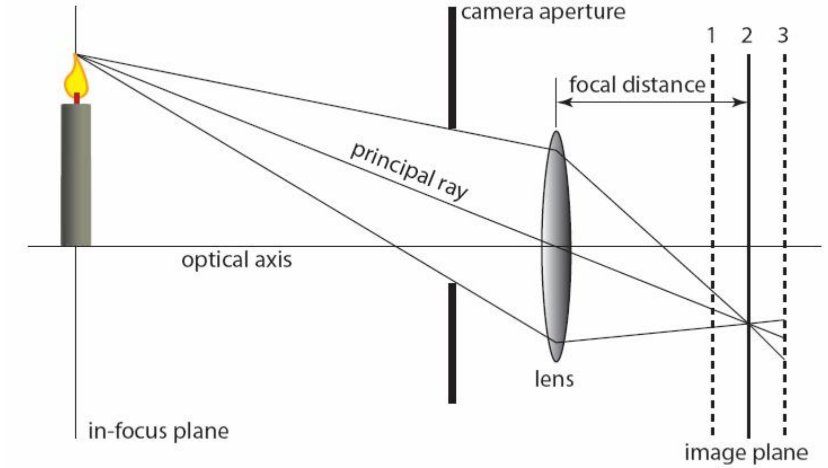
- An ideal thin lens produces the same projection with a pinhole camera, plus some finite amount of light



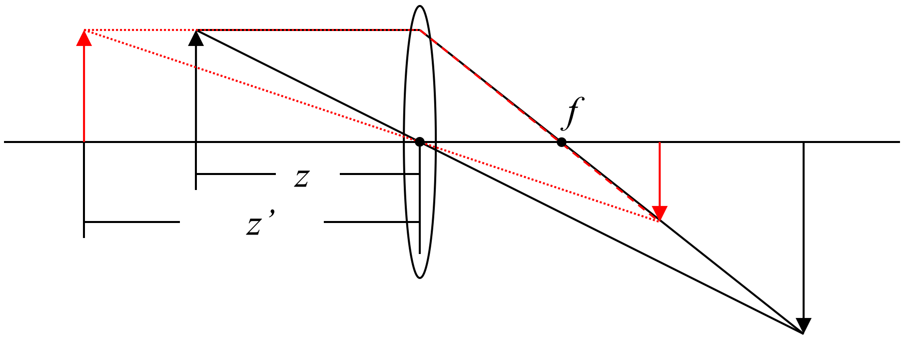
$$\frac{1}{z'} + \frac{1}{z} = \frac{1}{f}$$

- Focal length of the lens
- Focal distance of the camera
- Center of projection (camera center)

Focusing

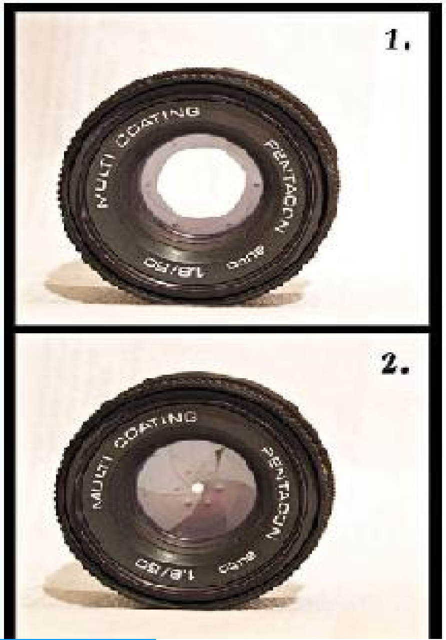


- If the image plane is at the correct focal distance (2), the lens focuses the entire cone of rays that the aperture allows through the lens onto a single point on the image plane.



- Once focus for one distance z, points on other distances will be blurred.

Focusing



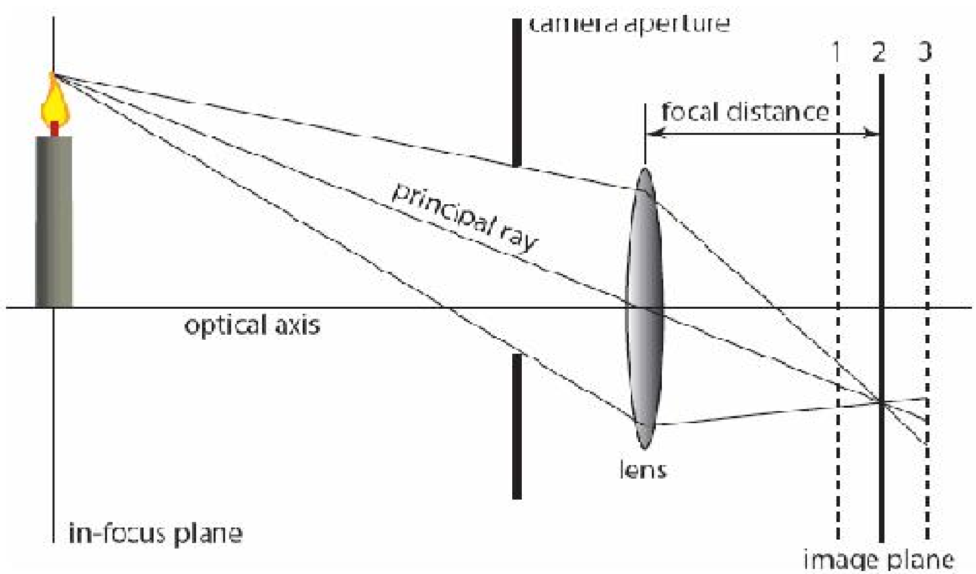
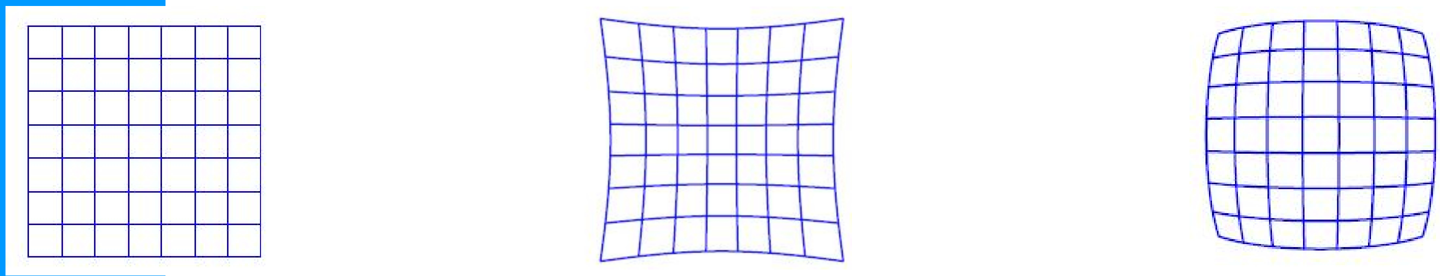


Image taken with a large aperture

Image taken with a small aperture

- Aperture: a circular diaphragm of adjustable diameter in front of the camera lens.
- Large aperture: cause a *shallow (narrow) depth of field*.
- Small aperture: increase the depth of field but longer exposure time

Radial distortion



(a) An undistorted grid (b) The grid in (a) with pincushion distortion (c) The grid in (a) with barrel distortion

- Radial distortion: moves every point in the image away from or closer to the principal point by an amount proportional to the square of their distance from it.
- A circularly symmetric function around the principal point of the image
- Equal to zero at the principal point
- Small in a sufficiently small neighborhood of the principal point

Distortion is a function of the distance:

$$r = \sqrt{x^2 + y^2}$$

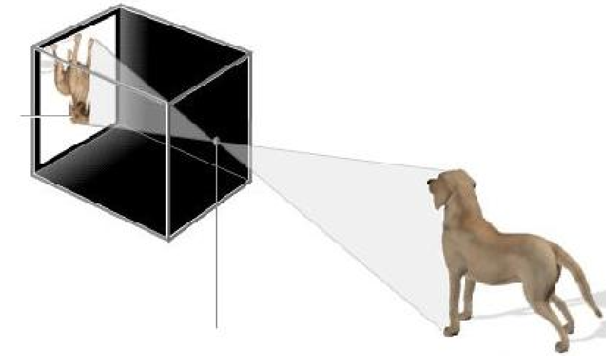
Distorted position of the point is:

$$x_d = x d(r) \quad \text{and} \quad y_d = y d(r)$$

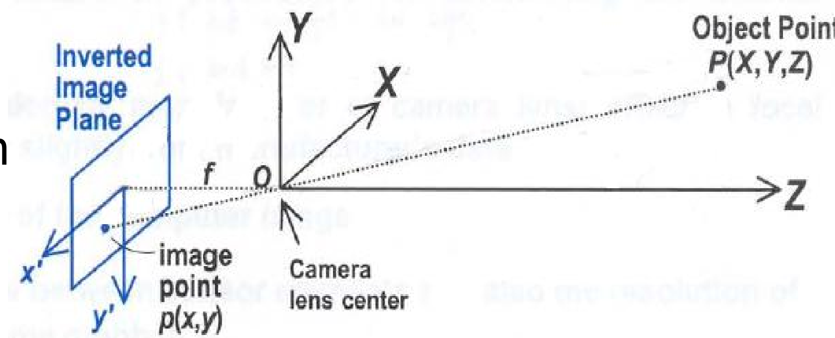
$d(r)$ is defined as the distortion function:

$$d(r) = 1 + k_2 r^2 + k_4 r^4$$

Pinhole camera model

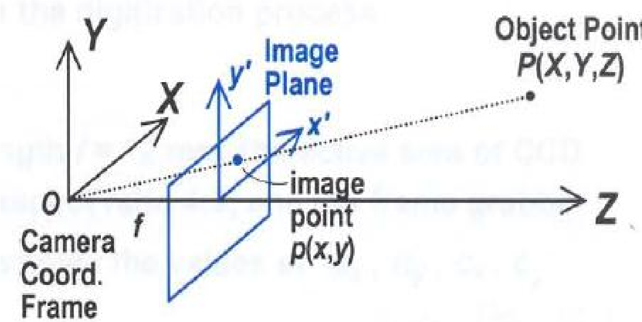


- For convenience, put image plane at focal length f in front of camera lens center to avoid inverting image
- Coordinate systems:
  - camera coordinate system
  - image coordiante system

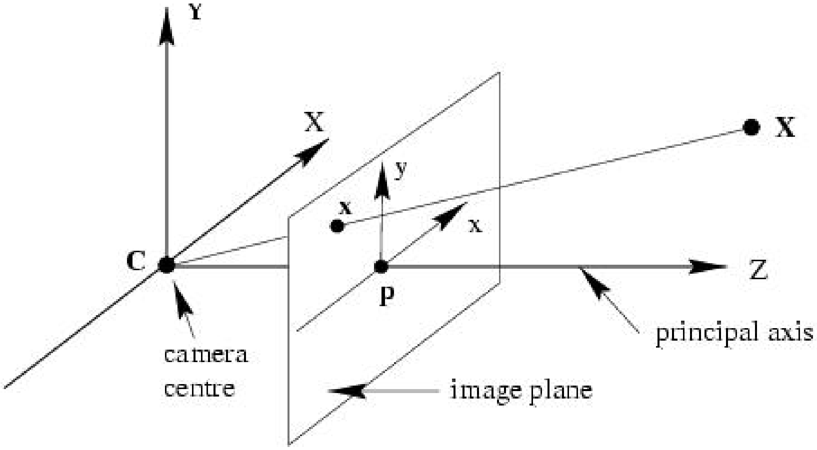


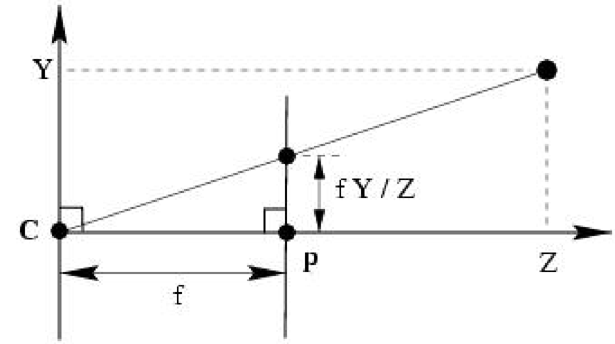
- Object point P (X,Y,Z) is projected onto the image plane:

$$\frac{X}{x} = \frac{Y}{y} = \frac{Z}{f}$$



Pinhole camera model

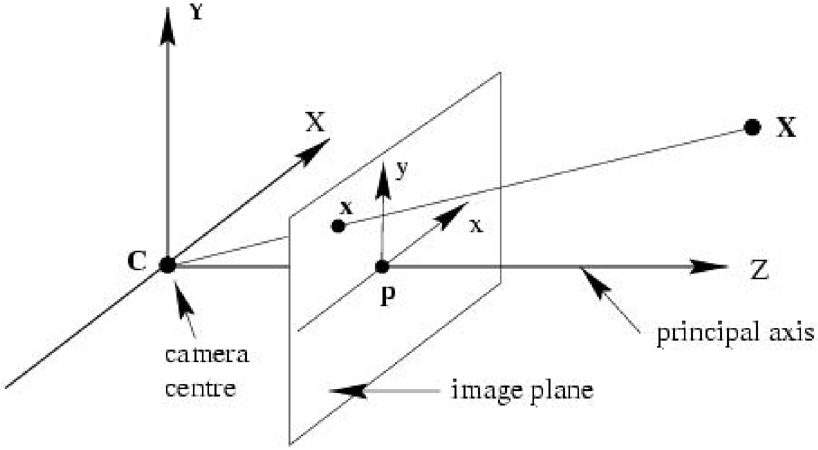


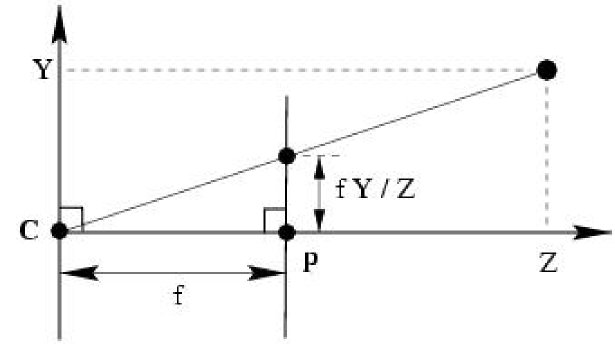


Object point $\boldsymbol{P}\ (X,Y,Z)^{\mathrm{T}}$ is projected onto the image point $(x,y)^{\mathrm{T}}$:

$$\frac{X}{x} = \frac{Y}{y} = \frac{Z}{f}$$

Pinhole camera model





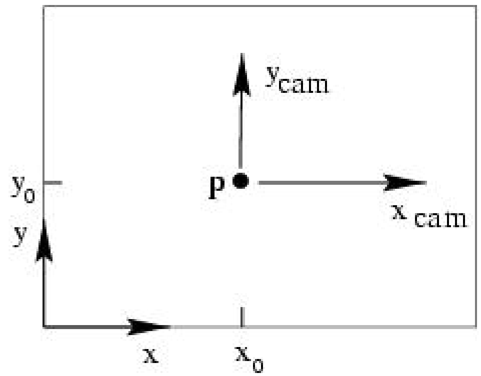
$$\begin{pmatrix} fX \\ fY \\ Z \end{pmatrix} = \begin{bmatrix} f & & & 0 \\ & f & & 0 \\ & & 1 & 0 \end{bmatrix}\begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix}$$

$$\mathrm{x} = \mathrm{P}\mathrm{X}$$

$$\begin{pmatrix} fX \\ fY \\ Z \end{pmatrix} = \begin{bmatrix} f & & \\ & f & \\ & & 1 \end{bmatrix}\begin{bmatrix} 1 & & & 0 \\ & 1 & & 0 \\ & & 1 & 0 \end{bmatrix}\begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix}$$

$$\mathrm{P} = \mathrm{diag}(f, f, 1)\left[\mathrm{I} \mid 0\right]$$

Principal point offset

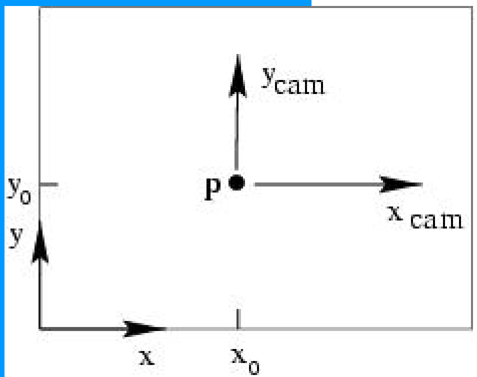


$$(X,Y,Z)^T \mapsto (fX/Z + p_x, fY/Z + p_y)^T$$

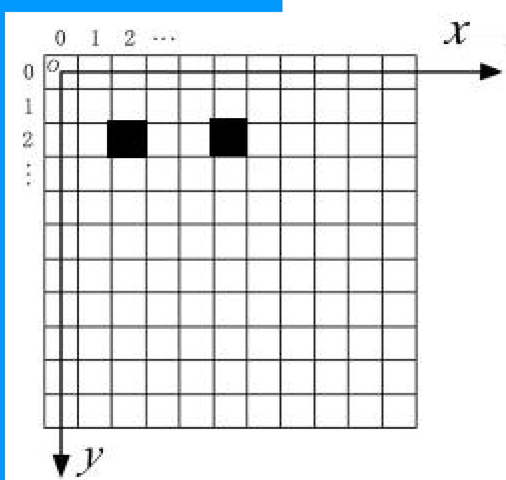
$(p_x, p_y)^T$ principal point

$$\begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} fX + Zp_x \\ fY + Zp_y \\ Z \end{pmatrix} = \begin{bmatrix} f & & p_x & 0 \\ & f & p_y & 0 \\ & & 1 & 0 \end{bmatrix} \begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix}$$

Principal point offset



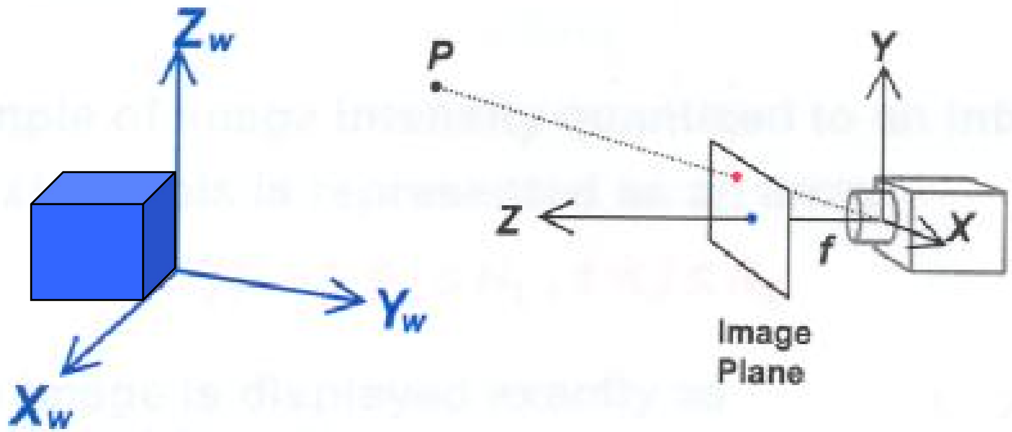
$$\begin{pmatrix} fX + Zp_x \\ fY + Zp_x \\ Z \end{pmatrix} = \begin{bmatrix} f & & p_x & 0 \\ & f & p_y & 0 \\ & & 1 & 0 \end{bmatrix} \begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix}$$



$$\lambda \mathbf{x} = \mathbf{K}[\mathbf{I}|\mathbf{0}]\mathbf{X}_{\mathbf{cam}}$$

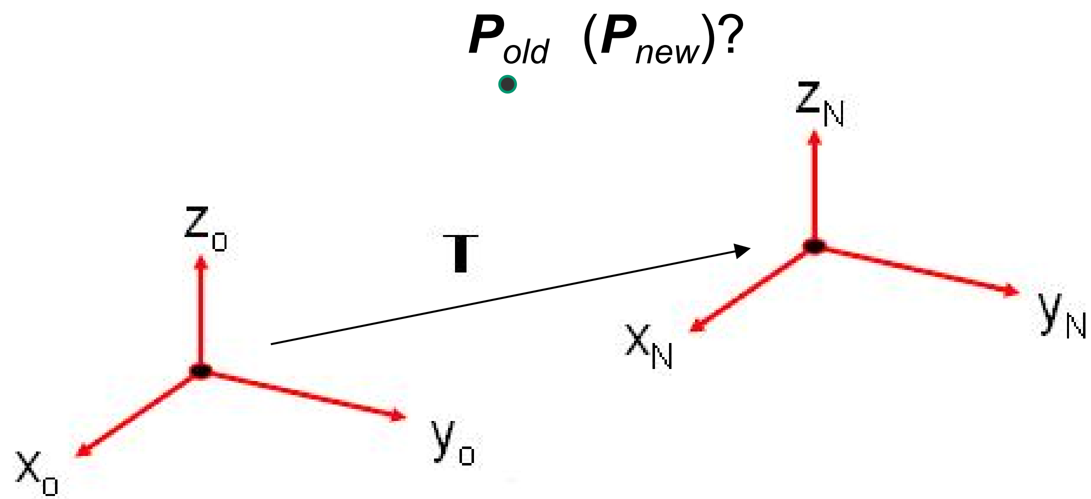
$$K = \begin{bmatrix} f & & p_x \\ & f & p_y \\ & & 1 \end{bmatrix}$$ calibration matrix

Camera rotation and translation



$$\mathbf{P}_w \xrightarrow{\ ?\ } \mathbf{P}_{cam}$$

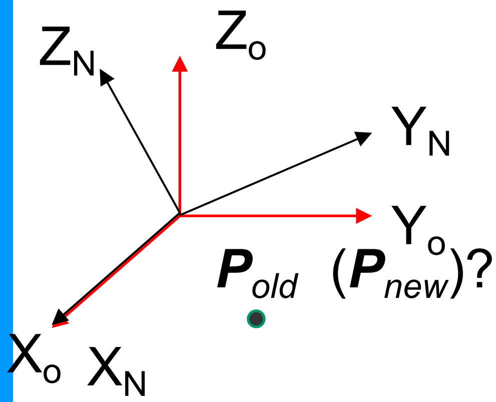
3D coordinate system translation



$$P_{old} = \begin{bmatrix} X_o \\ Y_o \\ Z_o \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X_N \\ Y_N \\ Z_N \\ 1 \end{bmatrix} = \begin{bmatrix} X_N + t_x \\ Y_N + t_y \\ Z_N + t_z \\ 1 \end{bmatrix} = TP_{new}$$

3D coordinate system rotation

- Counterclockwise rotation about X axis



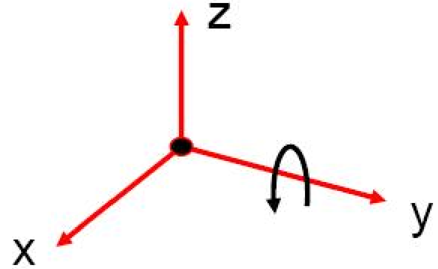
$$\mathbf{P}_{old} = \mathbf{R}\mathbf{P}_{new}$$

$$\mathbf{R} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

3D coordinate system rotation

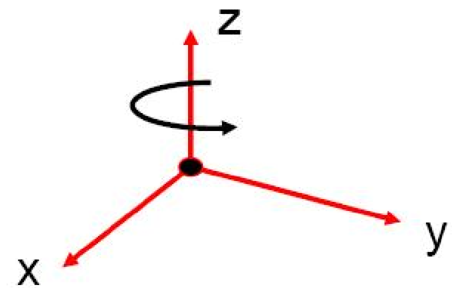
- Counterclockwise rotation about Y axis

$$\mathbf{P}_{old} = \mathbf{R}\mathbf{P}_{new} \qquad \mathbf{R} = \begin{bmatrix} \cos\theta & 0 & -\sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ \sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



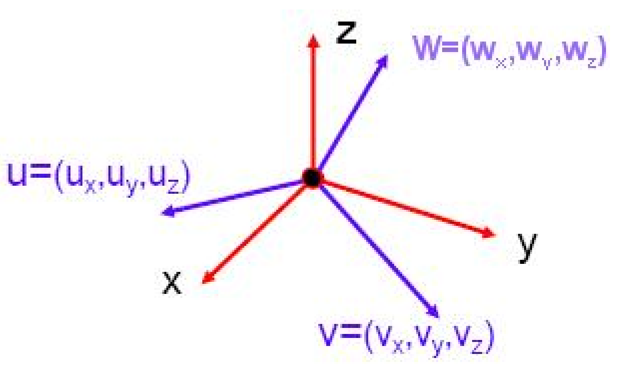
- Counterclockwise rotation about Z axis

$$\mathbf{P}_{old} = \mathbf{R}\mathbf{P}_{new} \qquad \mathbf{R} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



Composed rotation

- R_x, R_y and R_z can perform any rotation about an axis passing through the origin
- Problem: Given the XYZ orthonormal coordinate system, find a transformation M that maps XYZ into the orthogonal system UVW, with the same origin



Change of coordinate

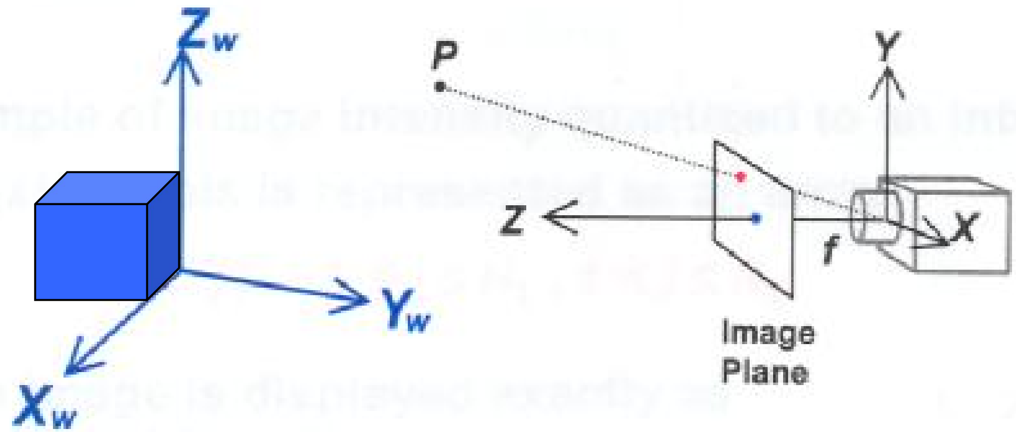
- Solution: **M** is the rotation matrix whose rows are **U**,**V** and **W** respectively

$$\mathbf{M} = \begin{bmatrix} u_x & v_x & w_x & 0 \\ u_y & v_y & w_y & 0 \\ u_z & v_z & w_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Note: the inverse transform is the transpose

$$\mathbf{M}^{-1} = \mathbf{M}^T = \begin{bmatrix} u_x & u_y & u_z & 0 \\ v_x & v_y & v_z & 0 \\ w_x & w_y & w_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Camera rotation and translation



Step1: translation $\mathbf{P}_w = \mathbf{T}\mathbf{P}_N$

Step2: Rotation $\mathbf{P}_N = \mathbf{R}\mathbf{P}_{cam}$

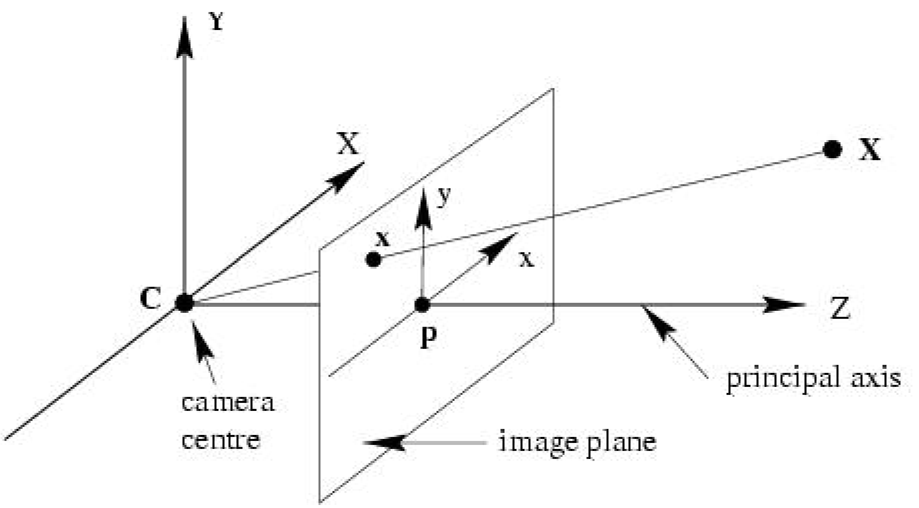
$$\mathbf{P}_w = \mathbf{T}\mathbf{R}\mathbf{P}_{cam} \longrightarrow \mathbf{P}_{cam} = \mathbf{R}^T\mathbf{T}\mathbf{P}_w = \begin{bmatrix} \mathbf{R}^T & -\mathbf{R}^T\tilde{\mathbf{C}} \\ \mathbf{0} & 1 \end{bmatrix}\mathbf{P}_w$$

$x \sim K[I|0]X_{cam}$

$x \sim KR^T[I|-\tilde{C}]X$ $\quad \mathrm{P} = \mathrm{K}[\mathrm{R} \mid \mathrm{t}] \quad \mathrm{t} = -\mathrm{R}\tilde{\mathrm{C}}$

Camera anatomy

Camera center
Column points
Principal plane
Axis plane
Principal point
Principal ray



Camera center

null-space camera projection matrix

$$\mathrm{PC} = 0$$

$$\mathrm{X} = \mathrm{A} + \alpha \mathrm{C}$$

$$\mathrm{x} \sim \mathrm{PX} = \mathrm{PA} + \alpha \mathrm{PC}$$

For any A, all points on AC project on image of A,

therefore C is camera center

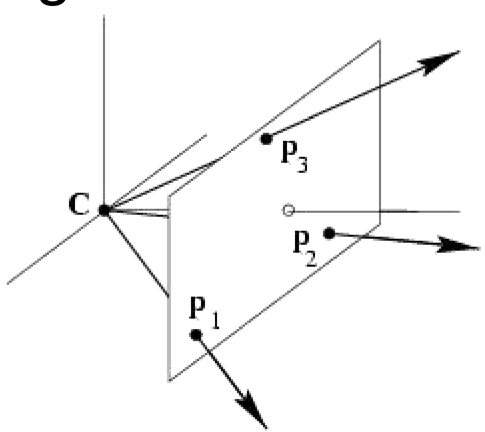
Image of camera center is $(0,0,0)^{\mathsf{T}}$, i.e. undefined

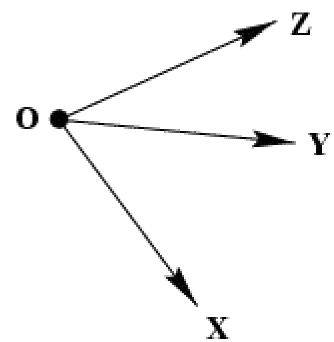
Finite cameras: $C = \begin{pmatrix} -\mathrm{M}^{-1}\mathrm{p}_4 \\ 1 \end{pmatrix}$

Column vectors

$$[\mathrm{p}_2] = [\mathrm{p}_1 \mathrm{p}_2 \mathrm{p}_3 \mathrm{p}_4]\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

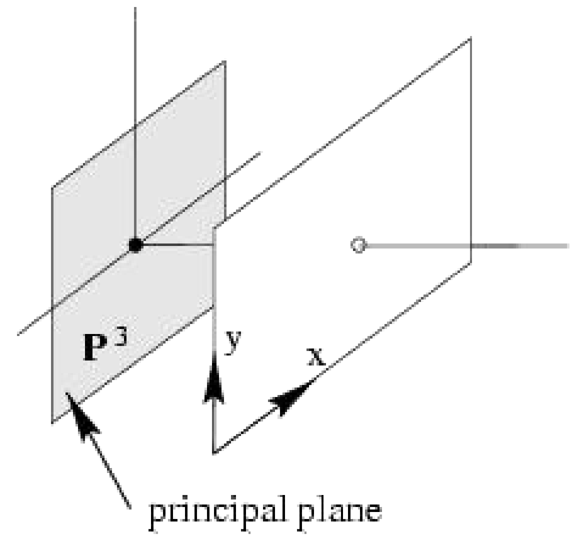
Image points corresponding to X,Y,Z directions and origin



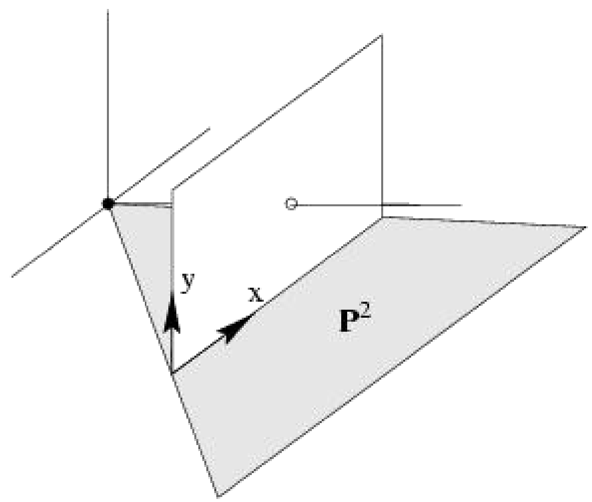


Row vectors

$$\begin{bmatrix} \mathbf{p}^{1\mathsf{T}} \\ \mathbf{p}^{2\mathsf{T}} \\ \mathbf{p}^{3\mathsf{T}} \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$$

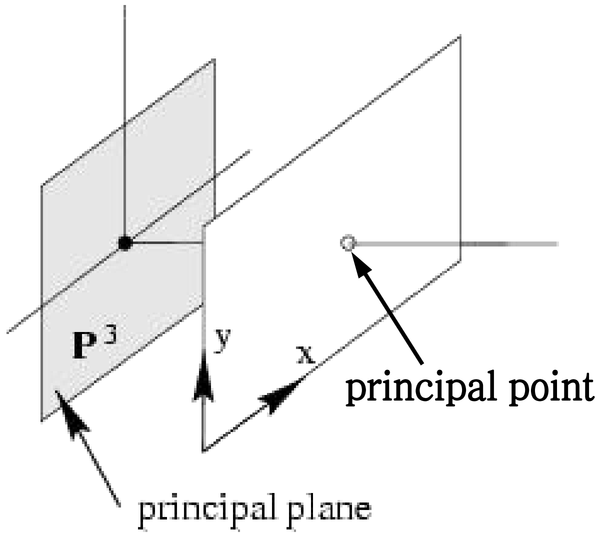


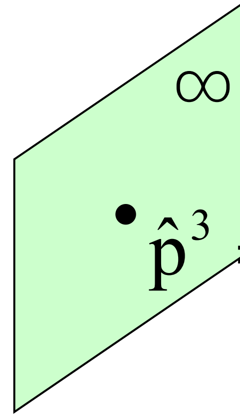
$$\begin{bmatrix} \mathbf{p}^{1\mathsf{T}} \\ \mathbf{p}^{2\mathsf{T}} \\ \mathbf{p}^{3\mathsf{T}} \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ 0 \\ w \end{bmatrix}$$



note: p^1,p^2 dependent on image reparametrization

The principal point





$$\hat{p}^3 = (p_{31}, p_{32}, p_{33}, 0)$$

$$x_0 = P\hat{p}^3 = Mm^3$$

Action of projective camera on point

Forward projection

$$x = PX$$

$$x = PD = \left[M \mid p_4\right]D = Md$$

Back-projection

$$PC = 0$$

$$X = P^+x \qquad P^+ = P^\mathsf{T}\left(PP^\mathsf{T}\right)^{-1} \qquad PP^+ = I$$

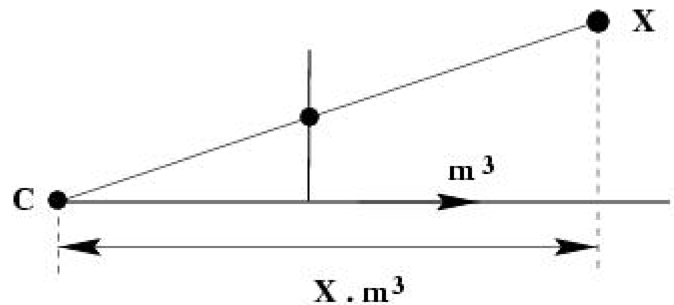
(pseudo-inverse)

$$X(\lambda) = P^+x + \lambda C$$

$$d = M^{-1}x$$

$$X(\lambda) = \mu\underbrace{\begin{pmatrix} M^{-1}x \\ 0 \end{pmatrix}}_{D} + \underbrace{\begin{pmatrix} -M^{-1}p_4 \\ 1 \end{pmatrix}}_{C} = \begin{pmatrix} M^{-1}(\mu x - p_4) \\ 1 \end{pmatrix}$$

Depth of points



$$w = P^{3^T} X = P^{3^T}(X - C) = m^{3^T}(\tilde{X} - \tilde{C})$$

(PC=0) (dot product)

If $\det M > 0; \|m^3\| = 1$,
then m^3 unit vector in positive direction

$$\boxed{\mathrm{depth}(X; P) = \frac{\mathrm{sign}(\det M)\, w}{T \|m^3\|}}$$

$$X = (X, Y, Z, T)^T$$

Camera matrix decomposition

Finding the camera center

$$\mathrm{PC} = 0$$ (use SVD to find null-space)

$$X = \det([\mathrm{p}_2, \mathrm{p}_3, \mathrm{p}_4]) \quad Y = -\det([\mathrm{p}_1, \mathrm{p}_3, \mathrm{p}_4])$$

$$Z = \det([\mathrm{p}_1, \mathrm{p}_2, \mathrm{p}_4]) \quad T = -\det([\mathrm{p}_1, \mathrm{p}_2, \mathrm{p}_3])$$

Finding the camera orientation and internal parameters

$$\mathrm{M} = \mathrm{KR}$$ (use RQ decomposition ~QR)

(if only QR, invert)

$$\boxed{} = (\boxed{Q}\ R)^{-1} = R^{-1}\ \boxed{Q}^{-1}$$

Summary of the properties

1. Camera centre: the 1-dimentional right null-space $\mathbf{C}$ of $\mathbf{P}$

$$\mathbf{C} = \begin{pmatrix} -\mathbf{M}^{-1}\mathbf{p}_4 \\ 1 \end{pmatrix}$$

2. Column points:
   $\boldsymbol{p}_1$ vanishing point of X axis
   $\boldsymbol{p}_2$ vanishing point of Y axis
   $\boldsymbol{p}_3$ vanishing point of Z axis
   $\boldsymbol{p}_4$ image of the coordinate origin
3. Principal plane: the last row of $\mathbf{P}$, i.e. $\mathbf{P}^3$
4. Axis planes:
   $\mathbf{P}^1$ plane in space through the camera center and the image line $x=0$
   $\mathbf{P}^2$ plane in space through the camera center and the image line $y=0$
5. Principal point: the image point $\mathbf{x}_0=\mathbf{M}\mathbf{m}^3$
6. Principal axis vector: $\mathbf{v}=\det(\mathbf{M})\mathbf{m}^3$