

Chapter 2: Systems of Equations

Liangda Fang

Dept. of Computer Science
Jinan University



Motivation

Question: Given a nonsingular square matrix A and a vector b , how to solve a linear equation $Ax = b$?



Outline

- 1 Introduction
- 2 Preliminaries
- 3 Iterative method
 - Jacobi Method
 - Gauss-Seidel Method
 - Successive Over-Relaxation
 - Convergence of iterative methods
- 4 Methods for symmetric positive-definite matrices
 - Symmetric positive-definite matrices
 - Conjugate Gradient Method
 - Preconditioning
- 5 Conclusions



Outline

- 1 Introduction
- 2 Preliminaries
- 3 Iterative method
 - Jacobi Method
 - Gauss-Seidel Method
 - Successive Over-Relaxation
 - Convergence of iterative methods
- 4 Methods for symmetric positive-definite matrices
 - Symmetric positive-definite matrices
 - Conjugate Gradient Method
 - Preconditioning
- 5 Conclusions



Vectors

Definition (Vectors)

An n -dimensional **(column) vector** is of the following form

$$u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}.$$

Definition (Row vectors)

An n -dimensional **row vector** is of the following form

$$u = [u_1 \quad u_2 \quad \cdots \quad u_n].$$



Matrices

Definition (Matrices)

An $m \times n$ **matrix** is of the following form

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}.$$



Matrix-matrix multiplication

Definition (Matrix-matrix multiplication)

Let A be an $m \times n$ matrix, and B be $n \times p$ matrix. Then, AB , which is an $m \times p$ matrix, is defined as follows:

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1p} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{np} \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^n a_{1i}b_{i1} & \sum_{i=1}^n a_{1i}b_{i2} & \cdots & \sum_{i=1}^n a_{1i}b_{ip} \\ \sum_{i=1}^n a_{2i}b_{i1} & \sum_{i=1}^n a_{2i}b_{i2} & \cdots & \sum_{i=1}^n a_{2i}b_{ip} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{i=1}^n a_{mi}b_{i1} & \sum_{i=1}^n a_{mi}b_{i2} & \cdots & \sum_{i=1}^n a_{mi}b_{ip} \end{bmatrix}.$$



Matrix form of linear equation

Definition (Matrix form)

A system of m linear equations in n unknowns can be written in **matrix form** as

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}.$$



Some basic concepts

Definition (Identity matrix)

The $n \times n$ identity matrix I_n is the matrix with $I_{ii} = 1$ for $1 \leq i \leq n$ and $I_{ij} = 0$ for $i \neq j$.

$$\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$



Some basic concepts

Definition (Identity matrix)

The $n \times n$ identity matrix I_n is the matrix with $I_{ii} = 1$ for $1 \leq i \leq n$ and $I_{ij} = 0$ for $i \neq j$.

$$\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

Definition (Inverse)

For an $n \times n$ matrix A , the **inverse** A^{-1} of A is an $n \times n$ matrix s.t. $AA^{-1} = A^{-1}A = I_n$.



Some basic concepts

Definition (Identity matrix)

The $n \times n$ identity matrix I_n is the matrix with $I_{ii} = 1$ for $1 \leq i \leq n$ and $I_{ij} = 0$ for $i \neq j$.

$$\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

Definition (Inverse)

For an $n \times n$ matrix A , the **inverse** A^{-1} of A is an $n \times n$ matrix s.t. $AA^{-1} = A^{-1}A = I_n$.

Definition (Singular)

The $n \times n$ matrix A is **nonsingular (invertible)**, if it has a inverse A^{-1} ; otherwise, it is **singular (noninvertible)**.



Some basic concepts

Definition (Transpose)

The **transpose** of an $m \times n$ matrix A is the $n \times m$ matrix A^T whose entries are $A_{ij}^T = A_{ji}$.



Some basic concepts

Definition (Transpose)

The **transpose** of an $m \times n$ matrix A is the $n \times m$ matrix A^T whose entries are $A_{ij}^T = A_{ji}$.

Example

Let A be a 2×3 matrix as follows: $\begin{bmatrix} 1 & 1 & 3 \\ 5 & -4 & 2 \end{bmatrix}$.

Then, A^T is a 3×2 matrix as follows: $\begin{bmatrix} 1 & 5 \\ 1 & -4 \\ 3 & 2 \end{bmatrix}$.



Some basic concepts

Definition

Given a vector v ,

① p -norm: $\|v\|_p = \left(\sum_{i=1}^n |v_i|^p \right)^{\frac{1}{p}};$

② Euclidean norm (2-norm): $\|v\|_2 = \sqrt{\sum_{i=1}^n |v_i|^2};$

③ Infinity norm: $\|v\|_\infty = \max \{|v_1|, \dots, |v_n|\};$



Some basic concepts

Definition

Given a vector v ,

① p -norm: $\|v\|_p = \left(\sum_{i=1}^n |v_i|^p\right)^{\frac{1}{p}};$

② Euclidean norm (2-norm): $\|v\|_2 = \sqrt{\sum_{i=1}^n |v_i|^2};$

③ Infinity norm: $\|v\|_\infty = \max\{|v_1|, \dots, |v_n|\};$

Proposition

① *Non-negativity*: $\|v\|_p \geq 0;$

② *Absolute scalability*: $\|av\|_p = |a|\|v\|_p;$

③ *Triangle inequality*: $\|v + u\|_p \leq \|v\|_p + \|u\|_p;$

④ *Separates points*: If $\|v\|_p = 0$, then $v = 0$.

The above propositions also hold for infinity norm.



Some basic concepts

Example

Assume that $v = (2, -1)$ and $w = (1, 3)$.

- $\|v\|_2 = \sqrt{|2|^2 + |-1|^2} = \sqrt{5}$ and $\|v\|_\infty = \max\{|2|, |-1|\} = 2$;
- $\|w\|_2 = \sqrt{|1|^2 + |3|^2} = \sqrt{10}$ and $\|w\|_\infty = \max\{|1|, |3|\} = 3$;



Some basic concepts

Example

Assume that $v = (2, -1)$ and $w = (1, 3)$.

- $\|v\|_2 = \sqrt{|2|^2 + |-1|^2} = \sqrt{5}$ and $\|v\|_\infty = \max\{|2|, |-1|\} = 2$;
- $\|w\|_2 = \sqrt{|1|^2 + |3|^2} = \sqrt{10}$ and $\|w\|_\infty = \max\{|1|, |3|\} = 3$;
- $\|2v\|_2 = \sqrt{|2 \times 2|^2 + |2 \times (-1)|^2} = 2\sqrt{5}$;
- $\|3v\|_\infty = \max\{|3 \times 2|, |3 \times (-1)|\} = 6 = 3 \times 2$;



Some basic concepts

Example

Assume that $v = (2, -1)$ and $w = (1, 3)$.

- $\|v\|_2 = \sqrt{|2|^2 + |-1|^2} = \sqrt{5}$ and $\|v\|_\infty = \max\{|2|, |-1|\} = 2$;
- $\|w\|_2 = \sqrt{|1|^2 + |3|^2} = \sqrt{10}$ and $\|w\|_\infty = \max\{|1|, |3|\} = 3$;
- $\|2v\|_2 = \sqrt{|2 \times 2|^2 + |2 \times (-1)|^2} = 2\sqrt{5}$;
- $\|3v\|_\infty = \max\{|3 \times 2|, |3 \times (-1)|\} = 6 = 3 \times 2$;
- $\|v + w\|_2 = \sqrt{|2 + 1|^2 + |-1 + 3|^2} = \sqrt{13} < \sqrt{5} + \sqrt{10} = \|v\|_2 + \|w\|_2$;
- $\|v + w\|_\infty = \max\{|2 + 1|, |-1 + 3|\} = 3 < 2 + 3 = \|v\|_\infty + \|w\|_\infty$;



Some basic concepts

Example

Assume that $v = (2, -1)$ and $w = (1, 3)$.

- $\|v\|_2 = \sqrt{|2|^2 + |-1|^2} = \sqrt{5}$ and $\|v\|_\infty = \max\{|2|, |-1|\} = 2$;
- $\|w\|_2 = \sqrt{|1|^2 + |3|^2} = \sqrt{10}$ and $\|w\|_\infty = \max\{|1|, |3|\} = 3$;
- $\|2v\|_2 = \sqrt{|2 \times 2|^2 + |2 \times (-1)|^2} = 2\sqrt{5}$;
- $\|3v\|_\infty = \max\{|3 \times 2|, |3 \times (-1)|\} = 6 = 3 \times 2$;
- $\|v + w\|_2 = \sqrt{|2 + 1|^2 + |-1 + 3|^2} = \sqrt{13} < \sqrt{5} + \sqrt{10} = \|v\|_2 + \|w\|_2$;
- $\|v + w\|_\infty = \max\{|2 + 1|, |-1 + 3|\} = 3 < 2 + 3 = \|v\|_\infty + \|w\|_\infty$;
- $\|0\|_2 = 0$ and $\|0\|_\infty = 0$.



Linear Subspace

Definition

Let v_1, \dots, v_m be n -dimensional vectors.

The subspace of $V: \{v_1, \dots, v_m\}$ is $\{x \mid x = a_1 v_1 + \dots + a_m v_m\}$.

A point $y \in R^n$ is in V , if $y \in V$ (i.e., there is a vector of a_1, \dots, a_m s.t. $y = a_1 v_1 + \dots + a_m v_m$.)

Example (Orthonormal sets)

- ❶ $\{(1, 1)\}$ (Figure ??)
- ❷ $\{(1, 0, 0), (0, 1, 0)\}$; (Figures ??)



Linear Subspace

Definition

Let V and V' be two subspaces. We say V is a subspace of V' , denoted by $V \subseteq V'$, if every point of V is in V' .



Linear Subspace

Definition

Let V and V' be two subspaces. We say V is a subspace of V' , denoted by $V \subseteq V'$, if every point of V is in V' .

Lemma

Let $V: \{v_1, \dots, v_m\}$ and $V': \{v'_1, \dots, v'_n\}$ be two subspaces. Then, $V \subseteq V'$ iff $v_i \in V'$ for $1 \leq i \leq m$.



Linear Subspace

Proof.

(\Leftarrow) : Let $v \in V$.



Linear Subspace

Proof.

(\Leftarrow) : Let $v \in V$.

Then, $v = a_1 v_1 + \cdots + a_m v_m$.



Linear Subspace

Proof.

(\Leftarrow): Let $v \in V$.

Then, $v = a_1 v_1 + \cdots + a_m v_m$.

By the assumption, $v_i = b_{1i} v'_1 + \cdots + b_{ni} v'_n$ for $1 \leq i \leq m$.



Linear Subspace

Proof.

(\Leftarrow): Let $v \in V$.

Then, $v = a_1 v_1 + \cdots + a_m v_m$.

By the assumption, $v_i = b_{1i} v'_1 + \cdots + b_{ni} v'_n$ for $1 \leq i \leq m$.

$$v = a_1 v_1 + \cdots + a_m v_m$$



Linear Subspace

Proof.

(\Leftarrow): Let $v \in V$.

Then, $v = a_1 v_1 + \cdots + a_m v_m$.

By the assumption, $v_i = b_{1i} v'_1 + \cdots + b_{ni} v'_n$ for $1 \leq i \leq m$.

$$\begin{aligned} v &= a_1 v_1 + \cdots + a_m v_m \\ &= a_1 (b_{11} v'_1 + \cdots + b_{n1} v'_n) + \cdots + a_m (b_{1m} v'_1 + \cdots + b_{nm} v'_n) \end{aligned}$$



Linear Subspace

Proof.

(\Leftarrow): Let $v \in V$.

Then, $v = a_1 v_1 + \cdots + a_m v_m$.

By the assumption, $v_i = b_{1i} v'_1 + \cdots + b_{ni} v'_n$ for $1 \leq i \leq m$.

$$\begin{aligned} v &= a_1 v_1 + \cdots + a_m v_m \\ &= a_1 (b_{11} v'_1 + \cdots + b_{n1} v'_n) + \cdots + a_m (b_{1m} v'_1 + \cdots + b_{nm} v'_n) \\ &= [\sum_{i=1}^m (a_i \cdot b_{1i}) v'_1] + \cdots + [\sum_{i=1}^m (a_i \cdot b_{in}) v'_n]. \end{aligned}$$



Linear Subspace

Proof.

(\Leftarrow): Let $v \in V$.

Then, $v = a_1 v_1 + \cdots + a_m v_m$.

By the assumption, $v_i = b_{1i} v'_1 + \cdots + b_{ni} v'_n$ for $1 \leq i \leq m$.

$$\begin{aligned} v &= a_1 v_1 + \cdots + a_m v_m \\ &= a_1 (b_{11} v'_1 + \cdots + b_{n1} v'_n) + \cdots + a_m (b_{1m} v'_1 + \cdots + b_{nm} v'_n) \\ &= [\sum_{i=1}^m (a_i \cdot b_{1i}) v'_1] + \cdots + [\sum_{i=1}^m (a_i \cdot b_{ni}) v'_n]. \end{aligned}$$

(\Rightarrow): Each v_i is also a point of V . By the assumption that $V \subseteq V'$, we get that $v_i \in V'$.



Orthonormal sets

Definition

A **unit** vector is a vector whose Euclidean norm is 1, i.e., $\sum_{i=1}^n v_i^2 = 1$.



Orthonormal sets

Definition

A **unit** vector is a vector whose Euclidean norm is 1, i.e., $\sum_{i=1}^n v_i^2 = 1$.

Definition

Two vectors v and w are **orthogonal** if $v^\top w = 0$, i.e., $\sum_{i=1}^n v_i w_i = 0$.



Orthonormal sets

Definition

A **unit** vector is a vector whose Euclidean norm is 1, i.e., $\sum_{i=1}^n v_i^2 = 1$.

Definition

Two vectors v and w are **orthogonal** if $v^\top w = 0$, i.e., $\sum_{i=1}^n v_i w_i = 0$.

Definition

A set of vectors is **orthonormal** if the elements of the set are unit vectors that are pairwise orthogonal.



Orthonormal sets

Definition

A **unit** vector is a vector whose Euclidean norm is 1, i.e., $\sum_{i=1}^n v_i^2 = 1$.

Definition

Two vectors v and w are **orthogonal** if $v^\top w = 0$, i.e., $\sum_{i=1}^n v_i w_i = 0$.

Definition

A set of vectors is **orthonormal** if the elements of the set are unit vectors that are pairwise orthogonal.

Example (Orthonormal sets)

- 1 $\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\};$
- 2 $\{(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}), (\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2})\}.$



Properties of orthonormal sets

Lemma

If $\{v_1, v_2, \dots, v_n\}$ is an orthonormal set, then

$$(a_1 v_1 + \dots + a_n v_n)^\top (b_1 v_1 + \dots + b_n v_n) = \sum_{i=1}^n a_i b_i.$$



Properties of orthonormal sets

Lemma

If $\{v_1, v_2, \dots, v_n\}$ is an orthonormal set, then

$$(a_1 v_1 + \dots + a_n v_n)^\top (b_1 v_1 + \dots + b_n v_n) = \sum_{i=1}^n a_i b_i.$$

Proof.

Let $v_i = [v_{i1}, \dots, v_{im}]^\top$.



Properties of orthonormal sets

Lemma

If $\{v_1, v_2, \dots, v_n\}$ is an orthonormal set, then

$$(a_1 v_1 + \dots + a_n v_n)^\top (b_1 v_1 + \dots + b_n v_n) = \sum_{i=1}^n a_i b_i.$$

Proof.

Let $v_i = [v_{i1}, \dots, v_{im}]^\top$.

$$\begin{aligned} & (a_1 v_1 + \dots + a_n v_n)^\top (b_1 v_1 + \dots + b_n v_n) \\ &= \begin{bmatrix} \sum_{i=1}^n a_i v_{i1} & \dots & \sum_{i=1}^n a_i v_{im} \end{bmatrix} \begin{bmatrix} \sum_{j=1}^n b_j v_{j1} \\ \vdots \\ \sum_{j=1}^n b_j v_{jm} \end{bmatrix} \end{aligned}$$



Properties of orthonormal sets

Lemma

If $\{v_1, v_2, \dots, v_n\}$ is an orthonormal set, then

$$(a_1 v_1 + \dots + a_n v_n)^\top (b_1 v_1 + \dots + b_n v_n) = \sum_{i=1}^n a_i b_i.$$

Proof.

Let $v_i = [v_{i1}, \dots, v_{im}]^\top$.

$$\begin{aligned} & (a_1 v_1 + \dots + a_n v_n)^\top (b_1 v_1 + \dots + b_n v_n) \\ &= \begin{bmatrix} \sum_{i=1}^n a_i v_{i1} & \dots & \sum_{i=1}^n a_i v_{im} \end{bmatrix} \begin{bmatrix} \sum_{j=1}^n b_j v_{j1} \\ \vdots \\ \sum_{j=1}^n b_j v_{jm} \end{bmatrix} \end{aligned}$$

$$= \sum_{k=1}^m \left(\sum_{i=1}^n a_i v_{ik} \right) \cdot \left(\sum_{j=1}^n b_j v_{jk} \right)$$



Properties of orthonormal sets

Proof.

$$= \sum_{k=1}^m \sum_{i=1}^n \sum_{j=1}^n (a_i b_j v_{ik} v_{jk})$$



Properties of orthonormal sets

Proof.

$$\begin{aligned} &= \sum_{k=1}^m \sum_{i=1}^n \sum_{j=1}^n (a_i b_j v_{ik} v_{jk}) \\ &= \sum_{i=1}^n \sum_{j=1}^n [(a_i b_j) \cdot \sum_{k=1}^m (v_{ik} v_{jk})] \end{aligned}$$



Properties of orthonormal sets

Proof.

$$= \sum_{k=1}^m \sum_{i=1}^n \sum_{j=1}^n (a_i b_j v_{ik} v_{jk})$$

$$= \sum_{i=1}^n \sum_{j=1}^n [(a_i b_j) \cdot \sum_{k=1}^m (v_{ik} v_{jk})]$$

$$\text{If } i \neq j, \text{ then } \sum_{k=1}^m v_{ik} v_{jk} = v_i^\top v_j = 0.$$

$$\text{Ow, } \sum_{k=1}^m v_{ik} v_{jk} = v_i^\top v_j = 1.$$



Properties of orthonormal sets

Proof.

$$= \sum_{k=1}^m \sum_{i=1}^n \sum_{j=1}^n (a_i b_j v_{ik} v_{jk})$$

$$= \sum_{i=1}^n \sum_{j=1}^n [(a_i b_j) \cdot \sum_{k=1}^m (v_{ik} v_{jk})]$$

$$\text{If } i \neq j, \text{ then } \sum_{k=1}^m v_{ik} v_{jk} = v_i^\top v_j = 0.$$

$$\text{Ow, } \sum_{k=1}^m v_{ik} v_{jk} = v_i^\top v_j = 1.$$

$$\text{We get } \sum_{i=1}^n a_i b_i$$



Outline

- 1 Introduction
- 2 Preliminaries
- 3 Iterative method**
 - Jacobi Method
 - Gauss-Seidel Method
 - Successive Over-Relaxation
 - Convergence of iterative methods
- 4 Methods for symmetric positive-definite matrices
 - Symmetric positive-definite matrices
 - Conjugate Gradient Method
 - Preconditioning
- 5 Conclusions



Jacobi Method

Example

A linear equation is as follows:

$$3u + v = 5$$

$$u + 2v = 5.$$

We have:

$$u = \frac{5 - v}{3}$$

$$v = \frac{5 - u}{2}.$$



Jacobi Method

Example (Iteration process)

$$\textcircled{1} \begin{bmatrix} u_0 \\ v_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\textcircled{2} \begin{bmatrix} u_1 \\ v_1 \end{bmatrix} = \begin{bmatrix} \frac{5-v_0}{3} \\ \frac{5-u_0}{2} \end{bmatrix} = \begin{bmatrix} \frac{5-0}{3} \\ \frac{5-0}{2} \end{bmatrix} = \begin{bmatrix} \frac{5}{3} \\ \frac{5}{2} \end{bmatrix}$$

$$\textcircled{3} \begin{bmatrix} u_2 \\ v_2 \end{bmatrix} = \begin{bmatrix} \frac{5-v_1}{3} \\ \frac{5-u_1}{2} \end{bmatrix} = \begin{bmatrix} \frac{5-5/2}{3} \\ \frac{5-5/3}{2} \end{bmatrix} = \begin{bmatrix} \frac{5}{6} \\ \frac{5}{3} \end{bmatrix}$$

$$\textcircled{4} \begin{bmatrix} u_3 \\ v_3 \end{bmatrix} = \begin{bmatrix} \frac{5-v_2}{3} \\ \frac{5-u_2}{2} \end{bmatrix} = \begin{bmatrix} \frac{5-5/3}{3} \\ \frac{5-5/6}{2} \end{bmatrix} = \begin{bmatrix} \frac{10}{9} \\ \frac{25}{12} \end{bmatrix}$$

$\textcircled{5}$ The process converges to the solution, which is $[1, 2]$.



Jacobi Method

Example

A linear equation is as follows:

$$u + 2v = 5$$

$$3u + v = 5.$$

We have:

$$u = 5 - 2v$$

$$v = 5 - 3u.$$



Jacobi method

Example (Iteration process)

$$\textcircled{1} \begin{bmatrix} u_0 \\ v_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\textcircled{2} \begin{bmatrix} u_1 \\ v_1 \end{bmatrix} = \begin{bmatrix} 5 - 2v_0 \\ 5 - 3u_0 \end{bmatrix} = \begin{bmatrix} 5 - 2 \times 0 \\ 5 - 3 \times 0 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \end{bmatrix}$$

$$\textcircled{3} \begin{bmatrix} u_2 \\ v_2 \end{bmatrix} = \begin{bmatrix} 5 - 2v_1 \\ 5 - 3u_1 \end{bmatrix} = \begin{bmatrix} 5 - 2 \times 5 \\ 5 - 3 \times 5 \end{bmatrix} = \begin{bmatrix} -5 \\ -10 \end{bmatrix}$$

$$\textcircled{4} \begin{bmatrix} u_3 \\ v_3 \end{bmatrix} = \begin{bmatrix} 5 - 2v_2 \\ 5 - 3u_2 \end{bmatrix} = \begin{bmatrix} 5 - 2 \times (-10) \\ 5 - 3 \times (-5) \end{bmatrix} = \begin{bmatrix} 25 \\ 20 \end{bmatrix}$$

$\textcircled{5}$ The process tends to diverge.



Strictly diagonally dominance

Question: What is the condition under which Jacobi method does work?



Strictly diagonally dominance

Question: What is the condition under which Jacobi method does work?

Definition

The $n \times n$ matrix $A = (a_{ij})$ is **strictly diagonally dominant**, if for each $1 \leq i \leq n$, $|a_{ii}| > \sum_{j \neq i} |a_{ij}|$.

Example

$\begin{bmatrix} 3 & 1 & -1 \\ 2 & -5 & 2 \\ 1 & 6 & 8 \end{bmatrix}$ is strictly diagonally dominant.

① $|3| > |1| + |-1|;$

② $|-5| > |2| + |2|;$

③ $|8| > |1| + |6|.$



Strictly diagonally dominance

Example

$\begin{bmatrix} 3 & 2 & 6 \\ 1 & 8 & 1 \\ 9 & 2 & -2 \end{bmatrix}$ is not strictly diagonally dominant.

- ① $|3| < |2| + |6|;$
- ② $|-2| < |9| + |2|.$



Strictly diagonally dominance

Example

$\begin{bmatrix} 3 & 2 & 6 \\ 1 & 8 & 1 \\ 9 & 2 & -2 \end{bmatrix}$ is not strictly diagonally dominant.

- ① $|3| < |2| + |6|;$
- ② $|-2| < |9| + |2|.$

But $\begin{bmatrix} 9 & 2 & -2 \\ 1 & 8 & 1 \\ 3 & 2 & 6 \end{bmatrix}$ is strictly diagonally dominant.

- ① $|9| > |2| + |-2|;$
- ② $|8| > |1| + |1|;$
- ③ $|6| > |3| + |2|.$



Strictly diagonally dominance

Theorem

If the $n \times n$ matrix A is strictly diagonally dominant, then

- ① *A is a nonsingular matrix;*
- ② *for every vector b and every starting guess, the Jacobi Method applied to $Ax = b$ converges to the unique solution.*



The matrix form of Jacobi Method

Definition

Suppose that A is a matrix.

- D : the main diagonal of A ;
- L : the lower triangle of A ;
- U : the upper triangle of A .



The matrix form of Jacobi Method

Example

- $A = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}$

- $D = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$

- $L = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$

- $U = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$



The matrix form of Jacobi Method

$$Ax = b$$

$$(D + L + U)x = b$$

$$Dx = b - (L + U)x$$

$$Dx_{k+1} = b - (L + U)x_k$$

$$x_{k+1} = D^{-1}(b - (L + U)x_k).$$



The matrix form of Jacobi Method

$$Ax = b$$

$$(D + L + U)x = b$$

$$Dx = b - (L + U)x$$

$$Dx_{k+1} = b - (L + U)x_k$$

$$x_{k+1} = D^{-1}(b - (L + U)x_k).$$

Jacobi Method is as follows:

$$x_0 = \text{initial vector}$$

$$x_{k+1} = D^{-1}(b - (L + U)x_k)$$

$$x_{k+1,i} = \frac{1}{a_{ii}}(b_i - \sum_{j \neq i} a_{ij}x_{k,j}) \text{ for } 1 \leq i \leq n.$$



The matrix form of Jacobi Method

Example

- $\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \end{bmatrix},$
- $$\begin{aligned} x_{k+1} &= \begin{bmatrix} u_{k+1} \\ v_{k+1} \end{bmatrix} = D^{-1}(b - (L + U)x_k) \\ &= \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \left(\begin{bmatrix} 5 \\ 5 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} u_k \\ v_k \end{bmatrix} \right) \\ &= \begin{bmatrix} \frac{5-v_k}{3} \\ \frac{5-u_k}{2} \end{bmatrix}. \end{aligned}$$



Outline

- 1 Introduction
- 2 Preliminaries
- 3 Iterative method**
 - Jacobi Method
 - **Gauss-Seidel Method**
 - Successive Over-Relaxation
 - Convergence of iterative methods
- 4 Methods for symmetric positive-definite matrices
 - Symmetric positive-definite matrices
 - Conjugate Gradient Method
 - Preconditioning
- 5 Conclusions



Gauss-Seidel Method

Question: Are there some methods converging faster than the Jacobi Method?



Gauss-Seidel Method

Question: Are there some methods converging faster than the Jacobi Method?

Answer: Yes! The Gauss-Seidel Method.



Gauss-Seidel Method

Example

- The definition of v_{k+1} uses u_{k+1} instead of u_k ;

$$\begin{bmatrix} u_{k+1} \\ v_{k+1} \end{bmatrix} = \begin{bmatrix} \frac{5-v_k}{3} \\ \frac{5-u_{k+1}}{2} \end{bmatrix}.$$

$$\textcircled{1} \quad \begin{bmatrix} u_0 \\ v_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\textcircled{2} \quad \begin{bmatrix} u_1 \\ v_1 \end{bmatrix} = \begin{bmatrix} \frac{5-v_0}{3} \\ \frac{5-u_1}{2} \end{bmatrix} = \begin{bmatrix} \frac{5-0}{3} \\ \frac{5-5/3}{2} \end{bmatrix} = \begin{bmatrix} \frac{5}{3} \\ \frac{5}{3} \end{bmatrix}$$

$$\textcircled{3} \quad \begin{bmatrix} u_2 \\ v_2 \end{bmatrix} = \begin{bmatrix} \frac{5-v_1}{3} \\ \frac{5-u_2}{2} \end{bmatrix} = \begin{bmatrix} \frac{5-5/3}{3} \\ \frac{5-10/9}{2} \end{bmatrix} = \begin{bmatrix} \frac{10}{9} \\ \frac{35}{18} \end{bmatrix}$$

$$\textcircled{4} \quad \begin{bmatrix} u_3 \\ v_3 \end{bmatrix} = \begin{bmatrix} \frac{5-v_2}{3} \\ \frac{5-u_3}{2} \end{bmatrix} = \begin{bmatrix} \frac{5-35/18}{3} \\ \frac{5-55/54}{2} \end{bmatrix} = \begin{bmatrix} \frac{55}{54} \\ \frac{215}{108} \end{bmatrix}.$$



Gauss-Seidel Method

Times	Jacobi	Gauss-Seidel
1	0.8333	0.7454
2	0.3727	0.1242
3	0.1389	0.0207

After 3 iterations, Gauss-Seidel converges faster than Jacobi.



The matrix form of Gauss-Seidel Method

$$Ax = b$$

$$(\textcolor{blue}{D} + \textcolor{red}{L} + \textcolor{brown}{U})x = b$$

$$(\textcolor{blue}{D} + \textcolor{red}{L})x = b - \textcolor{brown}{U}x$$

$$(\textcolor{blue}{D} + \textcolor{red}{L})x_{k+1} = b - \textcolor{brown}{U}x_k$$

For computation:

$$x_{k+1} = \textcolor{blue}{D}^{-1}(b - \textcolor{brown}{U}x_k - \textcolor{red}{L}x_{k+1})$$

For the proof of convergence:

$$x_{k+1} = (\textcolor{blue}{D} + \textcolor{red}{L})^{-1}(b - \textcolor{brown}{U}x_k).$$



The matrix form of Gauss-Seidel Method

$$Ax = b$$

$$(D + L + U)x = b$$

$$(D + L)x = b - Ux$$

$$(D + L)x_{k+1} = b - Ux_k$$

For computation:

$$x_{k+1} = D^{-1}(b - Ux_k - Lx_{k+1})$$

For the proof of convergence:

$$x_{k+1} = (D + L)^{-1}(b - Ux_k).$$

Gauss-Seidel Method is as follows:

$$x_0 = \text{initial vector}$$

$$x_{k+1} = D^{-1}(b - Ux_k - Lx_{k+1})$$

$$x_{k+1,i} = \frac{1}{a_{ii}}(b_i - \sum_{j>i} a_{ij}x_{k,j} - \sum_{j<i} a_{ij}x_{k+1,j}) \quad \text{for } 1 \leq i \leq n$$



The matrix form of Gauss-Seidel Method

Example

- $\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \end{bmatrix},$
- $$\begin{aligned} x_{k+1} = \begin{bmatrix} u_{k+1} \\ v_{k+1} \end{bmatrix} &= D^{-1}(b - Ux_k - Lx_{k+1}) \\ &= \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \left(\begin{bmatrix} 5 \\ 5 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_k \\ v_k \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} u_{k+1} \\ v_{k+1} \end{bmatrix} \right) \\ &= \begin{bmatrix} \frac{5-v_k}{3} \\ \frac{5-u_{k+1}}{2} \end{bmatrix}. \end{aligned}$$



Convergence of Gauss-Seidel Method

Theorem

If the $n \times n$ matrix A is strictly diagonally dominant, then

- ① *A is a nonsingular matrix;*
- ② *for every vector b and every starting guess, **the Gauss-Seidel Method** applied to $Ax = b$ converges to the unique solution.*



Outline

- 1 Introduction
- 2 Preliminaries
- 3 Iterative method**
 - Jacobi Method
 - Gauss-Seidel Method
 - **Successive Over-Relaxation**
 - Convergence of iterative methods
- 4 Methods for symmetric positive-definite matrices
 - Symmetric positive-definite matrices
 - Conjugate Gradient Method
 - Preconditioning
- 5 Conclusions



Successive Over-Relaxation

Question: Are there some methods converging faster than the Gauss-Seidel Method?



Successive Over-Relaxation

Question: Are there some methods converging faster than the Gauss-Seidel Method?

Answer: Yes! Successive Over-Relaxation, a variant of the Gauss-Seidel Method.



Relaxation parameter

Example

- Relaxation parameter ω : used to define each component of the new guess x_{k+1} as ω times, and $1 - \omega$ times the current guess x_k ;

$$\begin{bmatrix} u_{k+1} \\ v_{k+1} \end{bmatrix} = \begin{bmatrix} (1 - \omega)u_k + \omega \frac{5 - v_k}{3} \\ (1 - \omega)v_k + \omega \frac{5 - u_{k+1}}{2} \end{bmatrix}.$$

① Let $\omega = \frac{11}{10}$, so $1 - \omega = -\frac{1}{10}$;

②
$$\begin{bmatrix} u_1 \\ v_1 \end{bmatrix} = \begin{bmatrix} -\frac{u_0}{10} + \frac{11}{10} \cdot \frac{5 - v_0}{3} \\ -\frac{v_0}{10} + \frac{11}{10} \cdot \frac{5 - u_1}{2} \end{bmatrix} = \begin{bmatrix} \frac{11}{10} \cdot \frac{5 - 0}{3} \\ \frac{11}{10} \cdot \frac{5 - 11/6}{2} \end{bmatrix} \approx \begin{bmatrix} 1.8333 \\ 1.7416 \end{bmatrix}$$

③
$$\begin{bmatrix} u_2 \\ v_2 \end{bmatrix} = \begin{bmatrix} -\frac{u_1}{10} + \frac{11}{10} \cdot \frac{5 - v_1}{3} \\ -\frac{v_1}{10} + \frac{11}{10} \cdot \frac{5 - u_2}{2} \end{bmatrix} = \begin{bmatrix} -\frac{1.8333}{10} + \frac{11}{10} \cdot \frac{5 - 1.7416}{3} \\ -\frac{1.7416}{10} + \frac{11}{10} \cdot \frac{5 - 1.0114}{2} \end{bmatrix} \approx \begin{bmatrix} 1.0114 \\ 2.0196 \end{bmatrix}$$



Relaxation parameter

Example

$$\textcircled{1} \begin{bmatrix} u_3 \\ v_3 \end{bmatrix} = \begin{bmatrix} -\frac{u_2}{10} + \frac{11}{10} \cdot \frac{5-v_2}{3} \\ -\frac{v_2}{10} + \frac{11}{10} \cdot \frac{5-u_3}{2} \end{bmatrix} = \begin{bmatrix} -\frac{1.0114}{10} + \frac{11}{10} \cdot \frac{5-2.0196}{3} \\ -\frac{2.0196}{10} + \frac{11}{10} \cdot \frac{5-0.9917}{2} \end{bmatrix} \approx \begin{bmatrix} 0.9917 \\ 2.0026 \end{bmatrix}$$



Successive Over-Relaxation

Times	Jacobi	Gauss-Seidel	SOR($\omega = \frac{11}{10}$)
1	0.8333	0.7454	0.8724
2	0.3727	0.1242	0.0227
3	0.1389	0.0207	0.0087

After 3 iterations, SOR converges faster than Jacobi and Gauss-Seidel.



Relaxation parameter

- Over-relaxation: $\omega > 1$;
- The Gauss-Seidel Method: $\omega = 1$;
- Under-relaxation: $\omega < 1$.



Different relaxation parameters lead to different convergence speeds

Times	Jacobi	Gauss-Seidel	$\text{SOR}(\omega = \frac{11}{10})$	$\text{SOR}(\omega = \frac{6}{5})$
1	0.8333	0.7454	0.8724	1.0198
2	0.3727	0.1242	0.0227	0.1641
3	0.1389	0.0207	0.0087	0.0230

SOR with parameter $\frac{6}{5}$ converges faster than Jacobi, but slower than Gauss-Seidel.



The matrix form of Successive Over-Relaxation

$$Ax = b$$

$$\omega Ax = \omega b$$

$$(\omega D + \omega L + \omega U)x = \omega b$$

$$(\omega D + \omega L)x = \omega b - \omega Ux + (1 - \omega)Dx$$

$$(\omega D + \omega L)x_{k+1} = \omega b - \omega Ux_k + (1 - \omega)Dx_k$$

$$Dx_{k+1} = \omega b + (1 - \omega)Dx_k - \omega Ux_k - \omega Lx_{k+1}$$

$$x_{k+1} = (1 - \omega)x_k + D^{-1}(\omega b - \omega Ux_k - \omega Lx_{k+1})$$



The matrix form of Successive Over-Relaxation

$$Ax = b$$

$$\omega Ax = \omega b$$

$$(\omega D + \omega L + \omega U)x = \omega b$$

$$(D + \omega L)x = \omega b - \omega Ux + (1 - \omega)Dx$$

$$(D + \omega L)x_{k+1} = \omega b - \omega Ux_k + (1 - \omega)Dx_k$$

$$Dx_{k+1} = \omega b + (1 - \omega)Dx_k - \omega Ux_k - \omega Lx_{k+1}$$

$$x_{k+1} = (1 - \omega)x_k + D^{-1}(\omega b - \omega Ux_k - \omega Lx_{k+1})$$

Successive Over-Relaxation is as follows:

$$x_0 = \text{initial vector}$$

$$x_{k+1} = (1 - \omega)x_k + D^{-1}(\omega b - \omega Ux_k - \omega Lx_{k+1})$$

$$x_{k+1,i} = (1 - \omega)x_{k,i} + \frac{\omega}{a_{ii}}(b_i - \sum_{j>i} a_{ij}x_{k,j} - \sum_{j<i} a_{ij}x_{k+1,j}) \quad \text{for } 1 \leq i \leq n$$



Outline

- 1 Introduction
- 2 Preliminaries
- 3 Iterative method**
 - Jacobi Method
 - Gauss-Seidel Method
 - Successive Over-Relaxation
 - Convergence of iterative methods
- 4 Methods for symmetric positive-definite matrices
 - Symmetric positive-definite matrices
 - Conjugate Gradient Method
 - Preconditioning
- 5 Conclusions



Eigenvalues and eigenvectors

Definition (Eigenvalues and eigenvectors)

- A : an $n \times n$ matrix;
- λ : a real number;
- v : a nonzero n -dimensional real vector.

If $Av = \lambda v$, then λ is called an **eigenvalue** of A , and v is its corresponding **eigenvector**.



Eigenvalues and eigenvectors

Definition (Eigenvalues and eigenvectors)

- A : an $n \times n$ matrix;
- λ : a real number;
- v : a nonzero n -dimensional real vector.

If $Av = \lambda v$, then λ is called an **eigenvalue** of A , and v is its corresponding **eigenvector**.

Proposition

*Eigenvalues are the roots of the **characteristic polynomial***
 $\det(A - \lambda I) = 0$.



Eigenvalues and eigenvectors

Definition (Eigenvalues and eigenvectors)

- A : an $n \times n$ matrix;
- λ : a real number;
- v : a nonzero n -dimensional real vector.

If $Av = \lambda v$, then λ is called an **eigenvalue** of A , and v is its corresponding **eigenvector**.

Proposition

*Eigenvalues are the roots of the **characteristic polynomial***
 $\det(A - \lambda I) = 0$.

Proposition

Suppose that A is an $n \times n$ matrix, λ is an eigenvalue of A , v is the eigenvector of A w.r.t. λ .

Then, $A^m v = \lambda^m v$.



Eigenvalues and eigenvectors

Example

- $A = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix};$



Eigenvalues and eigenvectors

Example

- $A = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix};$
- $\det(A - \lambda I) = \begin{vmatrix} 1 - \lambda & 3 \\ 2 & 2 - \lambda \end{vmatrix} = (1 - \lambda)(2 - \lambda) - 6 = (\lambda - 4)(\lambda + 1);$
- The eigenvalues: 4 and -1 ;



Eigenvalues and eigenvectors

Example

- $A = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix};$
- $\det(A - \lambda I) = \begin{vmatrix} 1 - \lambda & 3 \\ 2 & 2 - \lambda \end{vmatrix} = (1 - \lambda)(2 - \lambda) - 6 = (\lambda - 4)(\lambda + 1);$
- The eigenvalues: 4 and -1 ;
- $(A - 4I)x = 0 \Rightarrow \begin{bmatrix} -3 & 3 \\ 2 & -2 \end{bmatrix} x = 0 \Rightarrow -3x_1 + 3x_2 = 0 \Rightarrow x_1 = x_2;$
- The eigenvectors wrt $\lambda = 4$: all nonzero multiples of $\begin{bmatrix} 1 & 1 \end{bmatrix}^T$;



Eigenvalues and eigenvectors

Example

- $A = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix};$
- $\det(A - \lambda I) = \begin{vmatrix} 1 - \lambda & 3 \\ 2 & 2 - \lambda \end{vmatrix} = (1 - \lambda)(2 - \lambda) - 6 = (\lambda - 4)(\lambda + 1);$
- The eigenvalues: 4 and -1 ;
- $(A - 4I)x = 0 \Rightarrow \begin{bmatrix} -3 & 3 \\ 2 & -2 \end{bmatrix} x = 0 \Rightarrow -3x_1 + 3x_2 = 0 \Rightarrow x_1 = x_2;$
- The eigenvectors wrt $\lambda = 4$: all nonzero multiples of $\begin{bmatrix} 1 & 1 \end{bmatrix}^T$;
- $(A + I)x = 0 \Rightarrow \begin{bmatrix} 2 & 3 \\ 2 & 3 \end{bmatrix} x = 0 \Rightarrow 2x_1 + 3x_2 = 0 \Rightarrow x_1 = -\frac{3}{2}x_2;$
- The eigenvectors wrt $\lambda = -1$: all nonzero multiples of $\begin{bmatrix} 3 & -2 \end{bmatrix}^T$;



Diagonalizable

Definition (Similarity)

Two $n \times n$ matrices A_1 and A_2 are **similar**, denoted by $A_1 \sim A_2$, if there exists a nonsingular $n \times n$ matrix S s.t. $A_1 = SA_2S^{-1}$.

Definition (Diagonal matrix)

The $n \times n$ diagonal matrix D_n is the matrix with $I_{ii} \neq 0$ for $1 \leq i \leq n$.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

Definition (Diagonalizable matrix)

An $n \times n$ matrix A is **diagonalizable**, if it is similar to a diagonal matrix B .



Diagonalizable

Proposition

- A : an $n \times n$ matrix;
- $\lambda_1, \lambda_2, \dots, \lambda_n$: the eigenvalues of A ;
- v_i : the eigenvector of A wrt λ_i ;
- $S = [v_1 \ v_2 \ \dots \ v_n]$;
- $B = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{bmatrix}$.

Then, $A = SBS^{-1}$.



Diagonalizable

Example

- $A = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix};$
- The eigenvalues: 4 and -1 ;



Diagonalizable

Example

- $A = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix};$
- The eigenvalues: 4 and -1 ;
- $B = \begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix};$



Diagonalizable

Example

- $A = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix};$
- The eigenvalues: 4 and -1 ;
- $B = \begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix};$
- An eigenvector wrt $\lambda_1 = 4$: $\begin{bmatrix} 1 \\ 1 \end{bmatrix};$
- An eigenvector wrt $\lambda_2 = -1$: $\begin{bmatrix} 3 \\ -2 \end{bmatrix};$



Diagonalizable

Example

- $A = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix};$
- The eigenvalues: 4 and -1 ;
- $B = \begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix};$
- An eigenvector wrt $\lambda_1 = 4$: $\begin{bmatrix} 1 \\ 1 \end{bmatrix};$
- An eigenvector wrt $\lambda_2 = -1$: $\begin{bmatrix} 3 \\ -2 \end{bmatrix};$
- $S = \begin{bmatrix} 1 & 3 \\ 1 & -2 \end{bmatrix};$



Diagonalizable

Example

- $A = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix};$
- The eigenvalues: 4 and -1 ;
- $B = \begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix};$
- An eigenvector wrt $\lambda_1 = 4$: $\begin{bmatrix} 1 \\ 1 \end{bmatrix};$
- An eigenvector wrt $\lambda_2 = -1$: $\begin{bmatrix} 3 \\ -2 \end{bmatrix};$
- $S = \begin{bmatrix} 1 & 3 \\ 1 & -2 \end{bmatrix};$
- $\begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 1 & -2 \end{bmatrix}^{-1}.$



Spectral radius

Definition

Let A be an $n \times n$ matrix and $\lambda_1, \dots, \lambda_n$ be the eigenvalues of A . The **spectral radius** $\rho(A)$ is defined as $\max\{|\lambda_1|, \dots, |\lambda_n|\}$.

The spectral radius: the upper bound of $\frac{\|Ax\|_2}{\|x\|_2}$.



Spectral radius

Definition

Let A be an $n \times n$ matrix and $\lambda_1, \dots, \lambda_n$ be the eigenvalues of A . The **spectral radius** $\rho(A)$ is defined as $\max\{|\lambda_1|, \dots, |\lambda_n|\}$.

The spectral radius: the upper bound of $\frac{\|Ax\|_2}{\|x\|_2}$.

Example

- $A = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}$;
- The eigenvalues: 4 and -1 ;
- $\rho(A) = \max\{|4|, |-1|\} = 4$.



Convergence

Theorem

- A : an $n \times n$ matrix with spectral radius $\rho(A) < 1$;
- b : an vector.

For any initial vector x_0 , the iteration $x_{k+1} = Ax_k + b$ converges, i.e., there exists a unique x_* s.t. $\lim_{k \rightarrow \infty} x_k = x_*$ and $x_* = Ax_* + b$.



Proof of the convergence of Jacobi Method

Proof.

- Matrix form: $x_{k+1} = -D^{-1}(L + U)x_k + D^{-1}b$;
- Reduce the proof of convergence to that of $\rho(D^{-1}(L + U)) < 1$;
- Let λ be any eigenvalue of $D^{-1}(L + U)$ with corresponding eigenvector v ;
- m : the index such that $|v_m| \geq |v_i|$ for $1 \leq i \neq m \leq n$;
- $D^{-1}(L + U)v = \lambda v \Rightarrow (L + U)v = \lambda Dv$.



Proof of the convergence of Jacobi Method

Proof.

- Take absolute values of the m th component of this vector equation:

$$\begin{aligned} |\lambda| |v_m| |a_{mm}| &= |\lambda a_{mm} v_m| \\ &= \left| \sum_{i \neq m} a_{mi} v_i \right| \\ &\leq |v_m| \sum_{i \neq m} |a_{mi}| \\ &< |v_m| |a_{mm}|. \end{aligned}$$

- Hence, $|\lambda| < 1$.
- Since λ is an arbitrary eigenvalue, we have that $\rho(D^{-1}(L + U)) < 1$.



Proof of the convergence of Gauss-Seidel Method

Proof.

- Matrix form: $x_{k+1} = -(L + D)^{-1}Ux_k + (L + D)^{-1}b$
- Reduce the proof of convergence to that of $\rho((L + D)^{-1}U) < 1$;
- Let λ be any eigenvalue of $(L + D)^{-1}U$ with corresponding eigenvector v ;
- m : the index such that $|v_m| \geq |v_i|$ for $1 \leq i \neq m \leq n$;
- $(L + D)^{-1}Uv = \lambda v \Rightarrow Uv = \lambda(D + L)v$.



Proof of the convergence of Gauss-Seidel Method

Proof.



$$\begin{aligned}
 |\lambda| |v_m| \cdot \sum_{i>m} |a_{mi}| &< |\lambda| |v_m| \cdot (|a_{mm}| - \sum_{i<m} |a_{mi}|) \\
 &\leq |\lambda| \cdot (|a_{mm} v_m| - \sum_{i<m} |a_{mi} v_i|) \\
 &\leq |\lambda| \cdot |a_{mm} v_m + \sum_{i<m} a_{mi} v_i| \\
 &= |\sum_{i>m} a_{mi} v_i| \\
 &\leq |v_m| \sum_{i>m} |a_{mi}|
 \end{aligned}$$

- Hence, $|\lambda| < 1$.
- Since λ is arbitrary, we have that $\rho((L + D)^{-1}U) < 1$.



Outline

- 1 Introduction
- 2 Preliminaries
- 3 Iterative method
 - Jacobi Method
 - Gauss-Seidel Method
 - Successive Over-Relaxation
 - Convergence of iterative methods
- 4 Methods for symmetric positive-definite matrices
 - Symmetric positive-definite matrices
 - Conjugate Gradient Method
 - Preconditioning
- 5 Conclusions



Methods for symmetric positive-definite matrices

Sometimes, we handle some special matrices, e.g., symmetric and positive-definite.

Question: For this type of matrices, are there some methods converging faster?



Methods for symmetric positive-definite matrices

Sometimes, we handle some special matrices, e.g., symmetric and positive-definite.

Question: For this type of matrices, are there some methods converging faster?

Answer: Yes! Conjugate Gradient Method.



Symmetric positive-definite matrices

Definition

The $n \times n$ matrix A is

- **symmetric:** $A^\top = A$;
- **positive-definite:** $x^\top Ax > 0$ for all vectors $x \neq 0$.

Example

$A = \begin{bmatrix} 2 & 2 \\ 2 & 5 \end{bmatrix}$ is symmetric positive-definite.

- symmetric: $A^\top = \begin{bmatrix} 2 & 2 \\ 2 & 5 \end{bmatrix} = A$;
- positive-definite:
$$\begin{aligned} x^\top Ax &= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 2 & 2 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ &= 2x_1^2 + 4x_1x_2 + 5x_2^2 \\ &= 2(x_1 + x_2)^2 + 3x_2^2 \\ &> 0 \text{ if } x_1, x_2 > 0 \end{aligned}$$



Symmetric positive-definite matrices

Example

$A = \begin{bmatrix} 2 & 4 \\ 4 & 5 \end{bmatrix}$ is not positive-definite.

$$\begin{aligned} x^{\top} A x &= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ &= 2x_1^2 + 8x_1x_2 + 5x_2^2 \\ &= 2(x_1 + 2x_2)^2 - 3x_2^2 \\ &= 2(-2 + 2 \cdot 1)^2 - 3 \cdot 1^2 \quad (x_1 = -2, x_2 = 1) \\ &= -3 \\ &< 0 \end{aligned}$$



Properties of positive-definite matrices

Lemma

A matrix is nonsingular iff $Ax = 0$ implies that $x = 0$.

Proposition

*A positive-definite matrix is **nonsingular**.*

Proof.

We want to prove that $x = 0$ when A is positive-definite and $Ax = 0$.

Since $y^T Ay > 0$ for every nonzero vector y , we have $Ay \neq 0$.

Hence, x must be zero vector.

So A is nonsingular.



Properties of symmetric matrices

Theorem (The finite-dimensional spectral theorem)

Let A be a symmetric $n \times n$ matrix. Then the set of unit eigenvectors of A is an orthonormal set $\{v_1, \dots, v_n\}$ forming a basis of \mathbb{R}^n .



Properties of symmetric matrices

Theorem (The finite-dimensional spectral theorem)

Let A be a symmetric $n \times n$ matrix. Then the set of unit eigenvectors of A is an orthonormal set $\{v_1, \dots, v_n\}$ forming a basis of \mathbb{R}^n .

Example

- $A = \begin{bmatrix} 2 & 2 \\ 2 & 5 \end{bmatrix};$



Properties of symmetric matrices

Theorem (The finite-dimensional spectral theorem)

Let A be a symmetric $n \times n$ matrix. Then the set of unit eigenvectors of A is an orthonormal set $\{v_1, \dots, v_n\}$ forming a basis of \mathbb{R}^n .

Example

- $A = \begin{bmatrix} 2 & 2 \\ 2 & 5 \end{bmatrix};$
- $\det(A - \lambda I) = \begin{vmatrix} 2 - \lambda & 2 \\ 2 & 5 - \lambda \end{vmatrix} = (2 - \lambda)(5 - \lambda) - 4 = (\lambda - 6)(\lambda - 1);$
- The eigenvalues: 6 and 1.



Properties of symmetric matrices

Example

- $(A - 6I)x = 0 \Rightarrow \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix} x = 0 \Rightarrow 2x_1 - x_2 = 0 \Rightarrow x_1 = \frac{1}{2}x_2;$
- An eigenvector wrt $\lambda = 6$: $\begin{bmatrix} 1 & 2 \end{bmatrix}^\top;$
- The unit eigenvector wrt $\lambda = 6$ via normalization:
$$\begin{bmatrix} \frac{v_1}{\|v\|_2} & \frac{v_2}{\|v\|_2} \end{bmatrix}^\top = \begin{bmatrix} \frac{1}{\sqrt{2^2+1^2}} & \frac{2}{\sqrt{2^2+1^2}} \end{bmatrix}^\top = \begin{bmatrix} \frac{\sqrt{5}}{5} & \frac{2\sqrt{5}}{5} \end{bmatrix}^\top;$$



Properties of symmetric matrices

Example

- $(A - 6I)x = 0 \Rightarrow \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix} x = 0 \Rightarrow 2x_1 - x_2 = 0 \Rightarrow x_1 = \frac{1}{2}x_2;$

- An eigenvector wrt $\lambda = 6$: $\begin{bmatrix} 1 & 2 \end{bmatrix}^\top;$

- The unit eigenvector wrt $\lambda = 6$ via normalization:

$$\begin{bmatrix} \frac{v_1}{\|v\|_2} & \frac{v_2}{\|v\|_2} \end{bmatrix}^\top = \begin{bmatrix} \frac{1}{\sqrt{2^2+1^2}} & \frac{2}{\sqrt{2^2+1^2}} \end{bmatrix}^\top = \begin{bmatrix} \frac{\sqrt{5}}{5} & \frac{2\sqrt{5}}{5} \end{bmatrix}^\top;$$

- $(A - I)x = 0 \Rightarrow \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} x = 0 \Rightarrow x_1 + 2x_2 = 0 \Rightarrow x_1 = -2x_2;$

- An eigenvector wrt $\lambda = 1$: $\begin{bmatrix} 2 & -1 \end{bmatrix}^\top;$

- The unit eigenvector wrt $\lambda = 1$ via normalization:

$$\begin{bmatrix} \frac{v_1}{\|v\|_2} & \frac{v_2}{\|v\|_2} \end{bmatrix}^\top = \begin{bmatrix} \frac{2}{\sqrt{2^2+(-1)^2}} & -\frac{1}{\sqrt{2^2+(-1)^2}} \end{bmatrix}^\top = \begin{bmatrix} \frac{2\sqrt{5}}{5} & -\frac{\sqrt{5}}{5} \end{bmatrix}^\top;$$



Properties of symmetric matrices

Example

- $(A - 6I)x = 0 \Rightarrow \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix} x = 0 \Rightarrow 2x_1 - x_2 = 0 \Rightarrow x_1 = \frac{1}{2}x_2$;
- An eigenvector wrt $\lambda = 6$: $\begin{bmatrix} 1 & 2 \end{bmatrix}^\top$;
- The unit eigenvector wrt $\lambda = 6$ via normalization:

$$\begin{bmatrix} \frac{v_1}{\|v\|_2} & \frac{v_2}{\|v\|_2} \end{bmatrix}^\top = \begin{bmatrix} \frac{1}{\sqrt{2^2+1^2}} & \frac{2}{\sqrt{2^2+1^2}} \end{bmatrix}^\top = \begin{bmatrix} \frac{\sqrt{5}}{5} & \frac{2\sqrt{5}}{5} \end{bmatrix}^\top$$
;
- $(A - I)x = 0 \Rightarrow \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} x = 0 \Rightarrow x_1 + 2x_2 = 0 \Rightarrow x_1 = -2x_2$;
- An eigenvector wrt $\lambda = 1$: $\begin{bmatrix} 2 & -1 \end{bmatrix}^\top$;
- The unit eigenvector wrt $\lambda = 1$ via normalization:

$$\begin{bmatrix} \frac{v_1}{\|v\|_2} & \frac{v_2}{\|v\|_2} \end{bmatrix}^\top = \begin{bmatrix} \frac{2}{\sqrt{2^2+(-1)^2}} & -\frac{1}{\sqrt{2^2+(-1)^2}} \end{bmatrix}^\top = \begin{bmatrix} \frac{2\sqrt{5}}{5} & -\frac{\sqrt{5}}{5} \end{bmatrix}^\top$$
;
- $\left\{ \begin{bmatrix} \frac{\sqrt{5}}{5} & \frac{2\sqrt{5}}{5} \end{bmatrix}^\top, \begin{bmatrix} \frac{2\sqrt{5}}{5} & -\frac{\sqrt{5}}{5} \end{bmatrix}^\top \right\}$ are an orthonormal set of R^2 .



Properties of symmetric positive-definite matrices

Proposition

Suppose that the $n \times n$ matrix A is symmetric.

Then, A is positive-definite iff all of its eigenvalues are positive.

Proof.

(\implies) A is positive-definite and $Av = \lambda v$ for any nonzero vector v .

$$0 < v^T Av = v^T \lambda v = \lambda \|v\|_2^2.$$

Since $\|v\|_2^2 > 0$, $\lambda > 0$.



Properties of symmetric positive-definite matrices

$$\lambda > 0$$

Proof.

(\Leftarrow) By the finite-dimensional spectral theorem, any nonzero vector x can be represented by

$$x = c_1 v_1 + \dots + c_n v_n$$

where v_1, \dots, v_n are the eigenvectors of A and not all c_i are zero.

$$\begin{aligned} x^\top A x &= (c_1 v_1 + \dots + c_n v_n)^\top A (c_1 v_1 + \dots + c_n v_n) \\ &= (c_1 v_1 + \dots + c_n v_n)^\top (\lambda_1 c_1 v_1 + \dots + \lambda_n c_n v_n) \\ &= \lambda_1 c_1^2 + \dots + \lambda_n c_n^2 \\ &> 0 \end{aligned}$$



Properties of symmetric positive-definite matrices

Definition

A principal submatrix of a square matrix A is a square submatrix whose diagonal entries are diagonal entries of A .

Proposition

Any principal submatrix of a symmetric positive-definite matrix is symmetric positive-definite.

Example

If
$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$
 is symmetric positive-definite,
then so is
$$\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}.$$



Outline

- 1 Introduction
- 2 Preliminaries
- 3 Iterative method
 - Jacobi Method
 - Gauss-Seidel Method
 - Successive Over-Relaxation
 - Convergence of iterative methods
- 4 Methods for symmetric positive-definite matrices
 - Symmetric positive-definite matrices
 - Conjugate Gradient Method
 - Preconditioning
- 5 Conclusions



A-Conjugate

Assume we have a symmetric positive-definite $n \times n$ matrix A .

Definition

For two n -vectors v and w , define the **A -inner product** as

$$(v, w)_A = v^\top A w.$$

The vectors v and w are **A -conjugate** if $(v, w)_A = 0$.

$$v^\top A w = 0$$



A-Conjugate

Assume we have a symmetric positive-definite $n \times n$ matrix A .

Definition

For two n -vectors v and w , define the **A -inner product** as

$$(v, w)_A = v^\top A w.$$

The vectors v and w are **A -conjugate** if $(v, w)_A = 0$.

Proposition

- *Symmetry*: $(v, w)_A = (w, v)_A$;
- *Linearity*: $(\alpha v, w)_A = \alpha(v, w)_A$ **and** $(v, \alpha w)_A = \alpha(v, w)_A$;
- *Positive-definiteness*: $(v, v)_A > 0$ if $v \neq 0$;
- *Classical inner-product*: $(v, w) = (v, w)_I$.



Conjugate Gradient Method

Lemma

Let $D = \{d_1, \dots, d_n\}$ a set of n mutually conjugate vectors wrt to A . Then D forms a basis for R^n .



Conjugate Gradient Method

Lemma

Let $D = \{d_1, \dots, d_n\}$ a set of n mutually conjugate vectors wrt to A . Then D forms a basis for R^n .

Proposition

The solution x^ of $Ax = b$ can be represented by*

$$x^* = \sum_{k=1}^n \alpha_k d_k.$$



Question: How to compute the mutually conjugate vectors D and their corresponding coefficients α_k 's?



Some notations of Conjugate Gradient Method

- ❶ d_k : the k -th mutually conjugate vector;
- ❷ α_k : the coefficient of d_k for x^* ensuring $(d_k, r_{k+1}) = 0$;
- ❸ x_k : the approximate solution at step k ;
 - the projection of x^* onto $\{d_1, \dots, d_{k-1}\}$, i.e., $\sum_{i=1}^{k-1} \alpha_i d_i$;



Some notations of Conjugate Gradient Method

- ❶ d_k : the k -th mutually conjugate vector;
- ❷ α_k : the coefficient of d_k for x^* ensuring $(d_k, r_{k+1}) = 0$;
- ❸ x_k : the approximate solution at step k ;
 - the projection of x^* onto $\{d_1, \dots, d_{k-1}\}$, i.e., $\sum_{i=1}^{k-1} \alpha_i d_i$;
- ❹ r_k : the residual of x_k at step k , i.e., $b - Ax_k$;
 - $(r_i, r_k) = 0$ for $0 \leq i < k$;
- ❺ β_k : the coefficient ensuring $(d_k, d_{k+1})_A = 0$.



An iterative framework for Conjugate Gradient Method

Algorithm 1: Conjugate Gradient Method

```
1  $x_0$  = initial guess
2  $d_0 = r_0 = b - Ax_0$ 
3 for  $k = 0, 1, 2, \dots, n - 1$  do
4   if  $r_k$  is sufficiently small then
5     return  $x_k$ 
6   Compute the parameters  $\alpha_k, x_{k+1}, r_{k+1}, \beta_k$  and  $d_{k+1}$ .
```



An iterative framework for Conjugate Gradient Method

Algorithm 2: Conjugate Gradient Method

```
1  $x_0$  = initial guess
2  $d_0 = r_0 = b - Ax_0$ 
3 for  $k = 0, 1, 2, \dots, n - 1$  do
4   if  $r_k$  is sufficiently small then
5     return  $x_k$ 
6   Compute the parameters  $\alpha_k, x_{k+1}, r_{k+1}, \beta_k$  and  $d_{k+1}$ .
```

- $(r_i, r_k) = 0$ for $0 \leq i < k$ implies the method ends in n iterations.
- To achieve this, we need $(r_{k+1}, d_k) = 0$ for each iteration.
- In addition, $(d_k, d_{k+1})_A = 0$ is guaranteed for each iteration.



Computation of x_{k+1} , r_{k+1} and d_{k+1}

By definition:

$$\begin{aligned}x_{k+1} &= x_k + \alpha_k d_k \\ b - Ax_{k+1} &= b - Ax_k - \alpha_k A d_k \\ r_{k+1} &= r_k - \alpha_k A d_k\end{aligned}$$

Update d_{k+1} by r_{k+1} and d_k :

$$d_{k+1} = r_{k+1} + \beta_k d_k.$$



Computation of α_k

Choose α_k s.t. $r_{k+1}^\top d_k = 0$:



Computation of α_k

Choose α_k s.t. $r_{k+1}^\top d_k = 0$:

$$\begin{aligned} r_{k+1} &= r_k - \alpha_k A d_k \\ 0 &= d_k^\top r_{k+1} = d_k^\top r_k - \alpha_k d_k^\top A d_k \\ \alpha_k &= \frac{d_k^\top r_k}{d_k^\top A d_k} \end{aligned}$$



Computation of α_k

Choose α_k s.t. $r_{k+1}^\top d_k = 0$:

$$\begin{aligned} r_{k+1} &= r_k - \alpha_k A d_k \\ 0 &= d_k^\top r_{k+1} = d_k^\top r_k - \alpha_k d_k^\top A d_k \\ \alpha_k &= \frac{d_k^\top r_k}{d_k^\top A d_k} \end{aligned}$$

α_k can be transformed as $\frac{r_k^\top r_k}{d_k^\top A d_k}$ because the following

$$\begin{aligned} d_k - r_k &= \beta_{k-1} d_{k-1} \\ r_k^\top d_k - r_k^\top r_k &= 0 & (r_k^\top d_{k-1} = 0) \end{aligned}$$



Computation of β_k

Choose β_k s.t. $d_k^\top A d_{k+1} = 0$:



Computation of β_k

Choose β_k s.t. $d_k^\top A d_{k+1} = 0$:

$$\begin{aligned}d_{k+1} &= r_{k+1} + \beta_k d_k \\ 0 = d_k^\top A d_{k+1} &= d_k^\top A r_{k+1} + \beta_k d_k^\top A d_k \\ \beta_k &= -\frac{d_k^\top A r_{k+1}}{d_k^\top A d_k}.\end{aligned}$$



Computation of β_k

Choose β_k s.t. $d_k^\top A d_{k+1} = 0$:

$$\begin{aligned} d_{k+1} &= r_{k+1} + \beta_k d_k \\ 0 = d_k^\top A d_{k+1} &= d_k^\top A r_{k+1} + \beta_k d_k^\top A d_k \\ \beta_k &= -\frac{d_k^\top A r_{k+1}}{d_k^\top A d_k}. \end{aligned}$$

β_k can be simplified as $\frac{r_{k+1}^\top r_{k+1}}{r_k^\top r_k}$ because the following

- $d_k^\top A r_{k+1} = \frac{1}{\alpha_k} (r_k - r_{k+1})^\top r_{k+1} = -\frac{1}{\alpha_k} r_{k+1}^\top r_{k+1};$
- $d_k^\top A d_k = (r_k + \beta_{k-1} d_{k-1})^\top A d_k = \frac{1}{\alpha_k} r_k^\top (r_k - r_{k+1}) = \frac{1}{\alpha_k} r_k^\top r_k.$



Conjugate Gradient Method

Algorithm 3: Conjugate Gradient Method

```
1  $x_0 =$  initial guess
2  $d_0 = r_0 = b - Ax_0$ 
3 for  $k = 0, 1, 2, \dots, n - 1$  do
4   if  $r_k$  is sufficiently small then
5     return  $x_k$ 
6    $\alpha_k = \frac{r_k^\top r_k}{d_k^\top A d_k}$ 
7    $x_{k+1} = x_k + \alpha_k d_k$ 
8    $r_{k+1} = r_k - \alpha_k A d_k$ 
9    $\beta_k = \frac{r_{k+1}^\top r_{k+1}}{r_k^\top r_k}$ 
10   $d_{k+1} = r_{k+1} + \beta_k d_k$ 
```



Example

Example (Initialize x_0 , d_0 and r_0)

Solve $\begin{bmatrix} 2 & 2 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \end{bmatrix}$ using the Conjugate Gradient Method. .

- $x_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix};$

- $r_0 = d_0 = \begin{bmatrix} 6 \\ 3 \end{bmatrix}.$



Example

Example (1st step)

$$\bullet \alpha_0 = \frac{\begin{bmatrix} 6 & 3 \end{bmatrix} \begin{bmatrix} 6 \\ 3 \end{bmatrix}}{\begin{bmatrix} 6 & 3 \end{bmatrix} \begin{bmatrix} 2 & 2 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} 6 \\ 3 \end{bmatrix}} = \frac{45}{6 \cdot 18 + 3 \cdot 27} = \frac{5}{21};$$

$$\bullet x_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \frac{5}{21} \begin{bmatrix} 6 \\ 3 \end{bmatrix} = \begin{bmatrix} \frac{10}{7} \\ \frac{5}{7} \end{bmatrix};$$

$$\bullet r_1 = \begin{bmatrix} 6 \\ 3 \end{bmatrix} - \frac{5}{21} \begin{bmatrix} 18 \\ 27 \end{bmatrix} = 12 \begin{bmatrix} \frac{1}{7} \\ -\frac{2}{7} \end{bmatrix};$$

$$\bullet \beta_0 = \frac{144 \cdot 5 / 49}{36 + 9} = \frac{16}{49};$$

$$\bullet d_1 = 12 \begin{bmatrix} \frac{1}{7} \\ -\frac{2}{7} \end{bmatrix} + \frac{16}{49} \begin{bmatrix} 6 \\ 3 \end{bmatrix} = \begin{bmatrix} \frac{180}{49} \\ -\frac{120}{49} \end{bmatrix}.$$



Example

Example (2nd step)

- $$\alpha_1 = \frac{\begin{bmatrix} \frac{12}{7} & -\frac{24}{7} \end{bmatrix} \begin{bmatrix} \frac{12}{7} \\ -\frac{24}{7} \end{bmatrix}}{\begin{bmatrix} \frac{180}{49} & -\frac{120}{49} \end{bmatrix} \begin{bmatrix} 2 & 2 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} \frac{180}{49} \\ -\frac{120}{49} \end{bmatrix}} = \frac{7}{10};$$
- $$x_2 = \begin{bmatrix} \frac{10}{7} \\ \frac{5}{7} \end{bmatrix} + \frac{7}{10} \begin{bmatrix} \frac{180}{49} \\ -\frac{120}{49} \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \end{bmatrix};$$
- $$r_2 = 12 \begin{bmatrix} \frac{1}{7} \\ -\frac{2}{7} \end{bmatrix} - \frac{7}{10} \begin{bmatrix} 2 & 2 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} \frac{180}{49} \\ -\frac{120}{49} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix};$$
- The solution is $x_2 = \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$ since $r_2 = 0$.



Main theorem

Theorem (Main theorem)

Let $b \neq 0$, $x_0 = 0$, and $r_k \neq 0$ for $k < n$. Then for each $1 \leq k \leq n$,

- 1 the following three subspaces of R^n are equal:

$$(x_1, \dots, x_k) = (r_0, \dots, r_{k-1}) = (d_0, \dots, d_{k-1});$$

- 2 distinct residuals are pairwise orthogonal:

$$r_k^\top r_j = 0 \text{ for } j < k;$$

- 3 distinct vectors of a subspace span are pairwise A -conjugate:

$$d_k^\top A d_j = 0 \text{ for } j < k.$$



Proof of 1st item

Proof.

- Base case ($k = 1$): $(x_1) = (r_0) = (d_0)$ since $x_1 = x_0 + \alpha d_0 = \alpha_0 d_0 = \alpha_0 r_0$.



Proof of 1st item

Proof.

- Base case ($k = 1$): $(x_1) = (r_0) = (d_0)$ since $x_1 = x_0 + \alpha d_0 = \alpha_0 d_0 = \alpha_0 r_0$.
- Inductive step ($k > 1$): Suppose that the $k - 1$ case hold.



Proof of 1st item

Proof.

- Base case ($k = 1$): $(x_1) = (r_0) = (d_0)$ since $x_1 = x_0 + \alpha d_0 = \alpha_0 d_0 = \alpha_0 r_0$.
- Inductive step ($k > 1$): Suppose that the $k - 1$ case hold.

$$\textcircled{1} \quad (x_1, \dots, x_k) \subseteq (d_0, \dots, d_{k-1}): \quad x_k = \sum_{i=0}^{k-1} \alpha_i d_i;$$



Proof of 1st item

Proof.

- Base case ($k = 1$): $(x_1) = (r_0) = (d_0)$ since $x_1 = x_0 + \alpha d_0 = \alpha_0 d_0 = \alpha_0 r_0$.
- Inductive step ($k > 1$): Suppose that the $k - 1$ case hold.

① $(x_1, \dots, x_k) \subseteq (d_0, \dots, d_{k-1})$: $x_k = \sum_{i=0}^{k-1} \alpha_i d_i$;

② $(x_1, \dots, x_k) \supseteq (d_0, \dots, d_{k-1})$:
$$x_k = x_{k-1} + \alpha_{k-1} d_{k-1} \Rightarrow d_{k-1} = \frac{1}{\alpha_{k-1}} x_k - \frac{1}{\alpha_{k-1}} x_{k-1}.$$



Proof of 1st item

Proof.

- Base case ($k = 1$): $(x_1) = (r_0) = (d_0)$ since $x_1 = x_0 + \alpha d_0 = \alpha_0 d_0 = \alpha_0 r_0$.
- Inductive step ($k > 1$): Suppose that the $k - 1$ case hold.

$$\textcircled{1} \quad (x_1, \dots, x_k) \subseteq (d_0, \dots, d_{k-1}): \quad x_k = \sum_{i=0}^{k-1} \alpha_i d_i;$$

$$\textcircled{2} \quad (x_1, \dots, x_k) \supseteq (d_0, \dots, d_{k-1}):$$
$$x_k = x_{k-1} + \alpha_{k-1} d_{k-1} \Rightarrow d_{k-1} = \frac{1}{\alpha_{k-1}} x_k - \frac{1}{\alpha_{k-1}} x_{k-1}.$$

$$\textcircled{1} \quad (r_0, \dots, r_{k-1}) \subseteq (d_0, \dots, d_{k-1}):$$
$$d_{k-1} = r_{k-1} + \beta_{k-2} d_{k-2} \Rightarrow r_{k-1} = d_{k-1} - \beta_{k-2} d_{k-2};$$



Proof of 1st item

Proof.

- Base case ($k = 1$): $(x_1) = (r_0) = (d_0)$ since $x_1 = x_0 + \alpha d_0 = \alpha_0 d_0 = \alpha_0 r_0$.
- Inductive step ($k > 1$): Suppose that the $k - 1$ case hold.

$$\textcircled{1} \quad (x_1, \dots, x_k) \subseteq (d_0, \dots, d_{k-1}): \quad x_k = \sum_{i=0}^{k-1} \alpha_i d_i;$$

$$\textcircled{2} \quad (x_1, \dots, x_k) \supseteq (d_0, \dots, d_{k-1}):$$

$$x_k = x_{k-1} + \alpha_{k-1} d_{k-1} \Rightarrow d_{k-1} = \frac{1}{\alpha_{k-1}} x_k - \frac{1}{\alpha_{k-1}} x_{k-1}.$$

$$\textcircled{1} \quad (r_0, \dots, r_{k-1}) \subseteq (d_0, \dots, d_{k-1}):$$

$$d_{k-1} = r_{k-1} + \beta_{k-2} d_{k-2} \Rightarrow r_{k-1} = d_{k-1} - \beta_{k-2} d_{k-2};$$

$$\textcircled{2} \quad (r_0, \dots, r_{k-1}) \supseteq (d_0, \dots, d_{k-1}):$$

$$d_{k-2} = \sum_{i=0}^{k-2} \gamma_i r_i \Rightarrow$$

$$d_{k-1} = r_{k-1} + \beta_{k-2} d_{k-2} = r_{k-1} + \sum_{i=0}^{k-2} (\beta_{k-2} \gamma_i) r_i.$$



Lemmas of the 2nd and 3rd items

Lemma

$(r_j, d_k)_A = 0$ for $0 \leq j < k$ or $0 \leq k < j + 1$.



Lemmas of the 2nd and 3rd items

Lemma

$(r_j, d_k)_A = 0$ for $0 \leq j < k$ or $0 \leq k < j + 1$.

Proof.

Here we only prove the case that $0 \leq j < k$.

If $j = 0$, then $d_k^\top A r_0 = d_k^\top A d_0 = 0$;



Lemmas of the 2nd and 3rd items

Lemma

$(r_j, d_k)_A = 0$ for $0 \leq j < k$ or $0 \leq k < j + 1$.

Proof.

Here we only prove the case that $0 \leq j < k$.

If $j = 0$, then $d_k^\top A r_0 = d_k^\top A d_0 = 0$;

Otherwise,

$$d_k^\top A r_j = d_k^\top A (d_j - \beta_{j-1} d_{j-1}) = d_k^\top A d_j - \beta_{j-1} d_k^\top A d_{j-1} = 0.$$



Lemmas of the 2nd and 3rd items

Lemma

$(r_j, d_k)_A = 0$ for $0 \leq j < k$ or $0 \leq k < j + 1$.

Proof.

Here we only prove the case that $0 \leq j < k$.

If $j = 0$, then $d_k^\top A r_0 = d_k^\top A d_0 = 0$;

Otherwise,

$$d_k^\top A r_j = d_k^\top A (d_j - \beta_{j-1} d_{j-1}) = d_k^\top A d_j - \beta_{j-1} d_k^\top A d_{j-1} = 0.$$

Lemma

$$d_k^\top A d_k = r_k^\top A d_k.$$



Lemmas of the 2nd and 3rd items

Lemma

$(r_j, d_k)_A = 0$ for $0 \leq j < k$ or $0 \leq k < j + 1$.

Proof.

Here we only prove the case that $0 \leq j < k$.

If $j = 0$, then $d_k^\top A r_0 = d_k^\top A d_0 = 0$;

Otherwise,

$$d_k^\top A r_j = d_k^\top A (d_j - \beta_{j-1} d_{j-1}) = d_k^\top A d_j - \beta_{j-1} d_k^\top A d_{j-1} = 0.$$

Lemma

$$d_k^\top A d_k = r_k^\top A d_k.$$

Proof.

$$d_k^\top A d_k = r_k^\top A d_k + \beta_{k-1} d_{k-1}^\top A d_k = r_k^\top A d_k.$$



Proof of the 2nd and 3rd items

Proof.

Base case ($k = 1$):

$$\textcircled{1} \quad r_0^\top r_1 = r_0^\top r_0 - \alpha_0 r_0^\top A d_0 = r_0^\top r_0 - \frac{r_0^\top r_0}{d_0^\top A d_0} d_0^\top A d_0 = 0;$$



Proof of the 2nd and 3rd items

Proof.

Base case ($k = 1$):

$$\textcircled{1} \quad r_0^\top r_1 = r_0^\top r_0 - \alpha_0 r_0^\top A d_0 = r_0^\top r_0 - \frac{r_0^\top r_0}{d_0^\top A d_0} d_0^\top A d_0 = 0;$$

$$\textcircled{2} \quad \beta_0 = -\frac{r_1^\top A d_0}{d_0^\top A d_0};$$



Proof of the 2nd and 3rd items

Proof.

Base case ($k = 1$):

$$\textcircled{1} \quad r_0^\top r_1 = r_0^\top r_0 - \alpha_0 r_0^\top A d_0 = r_0^\top r_0 - \frac{r_0^\top r_0}{d_0^\top A d_0} d_0^\top A d_0 = 0;$$

$$\textcircled{2} \quad \beta_0 = -\frac{r_1^\top A d_0}{d_0^\top A d_0};$$

$$\textcircled{3} \quad d_0^\top A d_1 = d_0^\top A r_1 + \beta_0 d_0^\top A d_0 = d_0^\top A r_1 - \frac{r_1^\top A d_0}{d_0^\top A d_0} d_0^\top A d_0 = 0.$$



Proof of the 2nd and 3rd items

Proof.

Inductive step ($k > 1$): Suppose that the $k - 1$ case hold.

The 2nd item:

$$\textcircled{1} \quad r_j^\top r_k = r_j^\top r_{k-1} - \alpha_{k-1} r_j^\top A d_{k-1};$$



Proof of the 2nd and 3rd items

Proof.

Inductive step ($k > 1$): Suppose that the $k - 1$ case hold.

The 2nd item:

- ① $r_j^\top r_k = r_j^\top r_{k-1} - \alpha_{k-1} r_j^\top A d_{k-1};$
- ② If $j < k - 1$, then $r_j^\top r_k = 0;$



Proof of the 2nd and 3rd items

Proof.

Inductive step ($k > 1$): Suppose that the $k - 1$ case hold.

The 2nd item:

- ① $r_j^\top r_k = r_j^\top r_{k-1} - \alpha_{k-1} r_j^\top A d_{k-1}$;
- ② If $j < k - 1$, then $r_j^\top r_k = 0$;
- ③ If $j = k - 1$, then $\alpha_{k-1} = \frac{r_{k-1}^\top r_{k-1}}{d_{k-1}^\top A d_{k-1}}$;



Proof of the 2nd and 3rd items

Proof.

Inductive step ($k > 1$): Suppose that the $k - 1$ case hold.

The 2nd item:

$$\textcircled{1} \quad r_j^\top r_k = r_j^\top r_{k-1} - \alpha_{k-1} r_j^\top A d_{k-1};$$

$$\textcircled{2} \quad \text{If } j < k - 1, \text{ then } r_j^\top r_k = 0;$$

$$\textcircled{3} \quad \text{If } j = k - 1, \text{ then } \alpha_{k-1} = \frac{r_{k-1}^\top r_{k-1}}{d_{k-1}^\top A d_{k-1}};$$

$$\begin{aligned} \textcircled{4} \quad r_{k-1}^\top r_k &= r_{k-1}^\top r_{k-1} - \alpha_{k-1} r_{k-1}^\top A d_{k-1} \\ &= r_{k-1}^\top r_{k-1} - \frac{r_{k-1}^\top r_{k-1}}{d_{k-1}^\top A d_{k-1}} d_{k-1}^\top A d_{k-1} = 0. \end{aligned}$$



Proof of the 2nd and 3rd items

Proof.

Inductive step ($k > 1$): Suppose that the $k - 1$ case hold.

The 3rd item:

$$\textcircled{1} \quad d_j^\top A d_k = d_j^\top A r_k + \beta_{k-1} d_j^\top A d_{k-1};$$



Proof of the 2nd and 3rd items

Proof.

Inductive step ($k > 1$): Suppose that the $k - 1$ case hold.

The 3rd item:

- ① $d_j^\top A d_k = d_j^\top A r_k + \beta_{k-1} d_j^\top A d_{k-1};$
- ② If $j < k - 1$, then $A d_j = \frac{r_j - r_{j+1}}{\alpha_j}$ is orthogonal to r_k , i.e.,
 $d_j^\top A r_k = 0;$



Proof of the 2nd and 3rd items

Proof.

Inductive step ($k > 1$): Suppose that the $k - 1$ case hold.

The 3rd item:

- ① $d_j^\top A d_k = d_j^\top A r_k + \beta_{k-1} d_j^\top A d_{k-1}$;
- ② If $j < k - 1$, then $A d_j = \frac{r_j - r_{j+1}}{\alpha_j}$ is orthogonal to r_k , i.e.,
 $d_j^\top A r_k = 0$;
- ③ So $d_j^\top A d_k = 0$.



Proof of the 2nd and 3rd items

Proof.

Inductive step ($k > 1$): Suppose that the $k - 1$ case hold.

The 3rd item:

- ① $d_j^\top A d_k = d_j^\top A r_k + \beta_{k-1} d_j^\top A d_{k-1}$;
- ② If $j < k - 1$, then $A d_j = \frac{r_j - r_{j+1}}{\alpha_j}$ is orthogonal to r_k , i.e.,
 $d_j^\top A r_k = 0$;
- ③ So $d_j^\top A d_k = 0$.
- ④ If $j = k - 1$, then $\beta_{k-1} = -\frac{r_k^\top A d_{k-1}}{d_{k-1}^\top A d_{k-1}}$.



Proof of the 2nd and 3rd items

Proof.

Inductive step ($k > 1$): Suppose that the $k - 1$ case hold.

The 3rd item:

- ① $d_j^\top A d_k = d_j^\top A r_k + \beta_{k-1} d_j^\top A d_{k-1}$;
- ② If $j < k - 1$, then $A d_j = \frac{r_j - r_{j+1}}{\alpha_j}$ is orthogonal to r_k , i.e.,
 $d_j^\top A r_k = 0$;
- ③ So $d_j^\top A d_k = 0$.
- ④ If $j = k - 1$, then $\beta_{k-1} = -\frac{r_k^\top A d_{k-1}}{d_{k-1}^\top A d_{k-1}}$.
- ⑤ So $d_{k-1}^\top A d_k = d_{k-1}^\top A r_k + \beta_{k-1} d_{k-1}^\top A d_{k-1}$
 $= d_{k-1}^\top A r_k - \frac{r_k^\top A d_{k-1}}{d_{k-1}^\top A d_{k-1}} d_{k-1}^\top A d_{k-1} = 0$.



Outline

- 1 Introduction
- 2 Preliminaries
- 3 Iterative method
 - Jacobi Method
 - Gauss-Seidel Method
 - Successive Over-Relaxation
 - Convergence of iterative methods
- 4 **Methods for symmetric positive-definite matrices**
 - Symmetric positive-definite matrices
 - Conjugate Gradient Method
 - **Preconditioning**
- 5 Conclusions



Motivation of Preconditioning

- Suppose that $Ax = b$ is a linear system.
- Condition number of A : a bound on how inaccurate the solution x will be after approximation.
- A is ill-conditioned: the condition number of A is very large.
- A is ill-conditioned \Rightarrow Conjugate Gradient Method fails.



Motivation of Preconditioning

- Suppose that $Ax = b$ is a linear system.
- Condition number of A : a bound on how inaccurate the solution x will be after approximation.
- A is ill-conditioned: the condition number of A is very large.
- A is ill-conditioned \Rightarrow **Conjugate Gradient Method fails.**
- Questions: Is it possible to handle the ill-conditioned matrix?
- Answer: Precondition.



Motivation of Preconditioning

Definition

Suppose $M = M_1 M_2$ is nonsingular and $Ax = b$ is a linear system. Let $\tilde{A}\tilde{x} = \tilde{b}$ be the linear system where

- $\tilde{A} = M_1^{-1} A M_2^{-1}$;
- $\tilde{x} = M_2 x$;
- $\tilde{b} = M_1^{-1} b$.

The matrix M is called a **preconditioner**.



Motivation of Preconditioning

Definition

Suppose $M = M_1 M_2$ is nonsingular and $Ax = b$ is a linear system. Let $\tilde{A}\tilde{x} = \tilde{b}$ be the linear system where

- $\tilde{A} = M_1^{-1} A M_2^{-1}$;
- $\tilde{x} = M_2 x$;
- $\tilde{b} = M_1^{-1} b$.

The matrix M is called a **preconditioner**.

- An effective preconditioner reduces the condition number, i.e., $\text{cond}(M_1^{-1} A M_2^{-1})$ is small.
- Two criterion of choosing M :
 - 1 M as close to A ;
 - 2 M is simple to invert.



Three preconditioners

Let $A = L + D + L^\top$.

- 1 **Jacobi preconditioner:** $M = D$;
- 2 **Gauss-Seidel preconditioner:** $M = (D + L)D^{-1}(D + L)^\top$;
- 3 **SSOR preconditioner:** $M = (D + \omega L)D^{-1}(D + \omega L)^\top$ where $0 \leq \omega \leq 2$.



Preconditioner

Lemma

Let M be symmetric positive-definite matrix. Then, there exists a unique symmetric positive-definite matrix C s.t. $M = C^2$.



Preconditioner

Lemma

Let M be symmetric positive-definite matrix. Then, there exists a unique symmetric positive-definite matrix C s.t. $M = C^2$.

Since A is symmetric positive-definite, we choose a symmetric positive-definite preconditioner M .

Definition

Suppose $M = C^2$ is **symmetric positive-definite** and $Ax = b$ is a linear system. Let $\tilde{A}\tilde{x} = \tilde{b}$ be the linear system where

- $\tilde{A} = C^{-1}AC^{-1}$;
- $\tilde{x} = Cx$;
- $\tilde{b} = C^{-1}b$.

The matrix M is called a **preconditioner**.



Some notations of Conjugate Gradient Method to preconditioned linear system

We use the Conjugate Gradient method to solve

$$\tilde{A}\tilde{x} = \tilde{b}.$$

- ① \tilde{d}_k : the k -th mutually conjugate vector wrt $C^{-1}AC^{-1}$;
- ② $\tilde{\alpha}_k$: the coefficient of \tilde{d}_k for \tilde{x}^* ;
- ③ \tilde{x}_k : the approximate solution to \tilde{x}^* at step k ;
- ④ \tilde{r}_k : the residual of \tilde{x}_k of preconditioned system at step k , i.e.,
 $\tilde{b} - \tilde{A}\tilde{x}_k = C^{-1}(b - Ax_k) = C^{-1}r_k$;
- ⑤ $\tilde{\beta}_k$: the coefficient ensuring $(\tilde{d}_k, \tilde{d}_{k+1})_{\tilde{A}} = 0$.



Computation of the forementioned notations

$$\textcircled{1} \quad \tilde{\alpha}_k = \frac{\tilde{r}_k^\top \tilde{r}_k}{\tilde{d}_k^\top \tilde{A} \tilde{d}_k};$$

$$\textcircled{2} \quad \tilde{x}_{k+1} = \tilde{x}_k + \tilde{\alpha}_k \tilde{d}_k;$$

$$\textcircled{3} \quad \tilde{r}_{k+1} = \tilde{r}_k - \tilde{\alpha}_k \tilde{A} \tilde{d}_k;$$

$$\textcircled{4} \quad \tilde{\beta}_k = \frac{\tilde{r}_{k+1}^\top \tilde{r}_{k+1}}{\tilde{r}_k^\top \tilde{r}_k};$$

$$\textcircled{5} \quad \tilde{d}_{k+1} = \tilde{r}_{k+1} + \tilde{\beta}_k \tilde{d}_k.$$



The Extra Computation of Direct Usage of CGM

- 1 Decompose M as $C \cdot C$;
- 2 Compute C^{-1} ;
- 3 Compute $\tilde{A} = C^{-1}AC^{-1}$;
- 4 Compute $\tilde{b} = C^{-1}b$;
- 5 Compute $x_k = C^{-1}\tilde{x}_k$.



The Extra Computation of Direct Usage of CGM

- 1 Decompose M as $C \cdot C$;
- 2 Compute C^{-1} ;
- 3 Compute $\tilde{A} = C^{-1}AC^{-1}$;
- 4 Compute $\tilde{b} = C^{-1}b$;
- 5 Compute $x_k = C^{-1}\tilde{x}_k$.

To reduce the extra computations, we incorporate the above computations into CGM.



Simplified Computation

- ❶ Let $z_k = M^{-1}r_k$;
- ❷ $\tilde{r}_k^\top \tilde{r}_k = r_k^\top (C^{-1})^\top C^{-1} r_k = r_k^\top M^{-1} r_k = r_k^\top z_k$;
- ❸ Let $d_k = C^{-1} \tilde{d}_k$;
- ❹ $\tilde{d}_k^\top \tilde{A} \tilde{d}_k = d_k^\top C^\top C^{-1} A C^{-1} C d_k = d_k^\top A d_k$.



Simplified Computation

$$\textcircled{1} \quad \tilde{\alpha}_k = \frac{\tilde{r}_k^\top \tilde{r}_k}{\tilde{d}_k^\top \tilde{A} \tilde{d}_k} = \frac{r_k^\top z_k}{d_k^\top A d_k};$$



Simplified Computation

$$\textcircled{1} \quad \tilde{\alpha}_k = \frac{\tilde{r}_k^\top \tilde{r}_k}{\tilde{d}_k^\top \tilde{A} \tilde{d}_k} = \frac{r_k^\top z_k}{d_k^\top A d_k};$$

$$\textcircled{2} \quad \tilde{x}_{k+1} = \tilde{x}_k + \tilde{\alpha}_k \tilde{d}_k \Rightarrow Cx_{k+1} = Cx_k + \tilde{\alpha}_k Cd_k \Rightarrow \\ Cx_{k+1} = C(x_k + \tilde{\alpha}_k d_k) \Rightarrow x_{k+1} = x_k + \tilde{\alpha}_k d_k;$$



Simplified Computation

$$\textcircled{1} \quad \tilde{\alpha}_k = \frac{\tilde{r}_k^\top \tilde{r}_k}{\tilde{d}_k^\top \tilde{A} \tilde{d}_k} = \frac{r_k^\top z_k}{d_k^\top A d_k};$$

$$\textcircled{2} \quad \tilde{x}_{k+1} = \tilde{x}_k + \tilde{\alpha}_k \tilde{d}_k \Rightarrow Cx_{k+1} = Cx_k + \tilde{\alpha}_k Cd_k \Rightarrow \\ Cx_{k+1} = C(x_k + \tilde{\alpha}_k d_k) \Rightarrow x_{k+1} = x_k + \tilde{\alpha}_k d_k;$$

$$\textcircled{3} \quad \tilde{r}_{k+1} = \tilde{r}_k - \tilde{\alpha}_k \tilde{A} \tilde{d}_k \Rightarrow \\ C^{-1}r_{k+1} = C^{-1}r_k - \tilde{\alpha}_k C^{-1}A C^{-1}Cd_k \Rightarrow \\ C^{-1}r_{k+1} = C^{-1}(r_k - \tilde{\alpha}_k A d_k) \Rightarrow \\ r_{k+1} = r_k + \tilde{\alpha}_k A d_k;$$



Simplified Computation

$$\textcircled{1} \quad \tilde{\alpha}_k = \frac{\tilde{r}_k^\top \tilde{r}_k}{\tilde{d}_k^\top \tilde{A} \tilde{d}_k} = \frac{r_k^\top z_k}{d_k^\top A d_k};$$

$$\textcircled{2} \quad \tilde{x}_{k+1} = \tilde{x}_k + \tilde{\alpha}_k \tilde{d}_k \Rightarrow Cx_{k+1} = Cx_k + \tilde{\alpha}_k Cd_k \Rightarrow \\ Cx_{k+1} = C(x_k + \tilde{\alpha}_k d_k) \Rightarrow x_{k+1} = x_k + \tilde{\alpha}_k d_k;$$

$$\textcircled{3} \quad \tilde{r}_{k+1} = \tilde{r}_k - \tilde{\alpha}_k \tilde{A} \tilde{d}_k \Rightarrow \\ C^{-1}r_{k+1} = C^{-1}r_k - \tilde{\alpha}_k C^{-1}A C^{-1}Cd_k \Rightarrow \\ C^{-1}r_{k+1} = C^{-1}(r_k - \tilde{\alpha}_k A d_k) \Rightarrow \\ r_{k+1} = r_k + \tilde{\alpha}_k A d_k;$$

$$\textcircled{4} \quad z_{k+1} = M^{-1}r_{k+1};$$



Simplified Computation

$$\textcircled{1} \quad \tilde{\alpha}_k = \frac{\tilde{r}_k^\top \tilde{r}_k}{\tilde{d}_k^\top \tilde{A} \tilde{d}_k} = \frac{r_k^\top z_k}{d_k^\top A d_k};$$

$$\textcircled{2} \quad \tilde{x}_{k+1} = \tilde{x}_k + \tilde{\alpha}_k \tilde{d}_k \Rightarrow Cx_{k+1} = Cx_k + \tilde{\alpha}_k Cd_k \Rightarrow \\ Cx_{k+1} = C(x_k + \tilde{\alpha}_k d_k) \Rightarrow x_{k+1} = x_k + \tilde{\alpha}_k d_k;$$

$$\textcircled{3} \quad \tilde{r}_{k+1} = \tilde{r}_k - \tilde{\alpha}_k \tilde{A} \tilde{d}_k \Rightarrow \\ C^{-1}r_{k+1} = C^{-1}r_k - \tilde{\alpha}_k C^{-1}A C^{-1}Cd_k \Rightarrow \\ C^{-1}r_{k+1} = C^{-1}(r_k - \tilde{\alpha}_k A d_k) \Rightarrow \\ r_{k+1} = r_k + \tilde{\alpha}_k A d_k;$$

$$\textcircled{4} \quad z_{k+1} = M^{-1}r_{k+1};$$

$$\textcircled{5} \quad \tilde{\beta}_k = \frac{\tilde{r}_{k+1}^\top \tilde{r}_{k+1}}{\tilde{r}_k^\top \tilde{r}_k} = \frac{r_{k+1}^\top z_{k+1}}{r_k^\top z_k};$$



Simplified Computation

$$\textcircled{1} \quad \tilde{\alpha}_k = \frac{\tilde{r}_k^\top \tilde{r}_k}{\tilde{d}_k^\top \tilde{A} \tilde{d}_k} = \frac{r_k^\top z_k}{d_k^\top A d_k};$$

$$\textcircled{2} \quad \tilde{x}_{k+1} = \tilde{x}_k + \tilde{\alpha}_k \tilde{d}_k \Rightarrow Cx_{k+1} = Cx_k + \tilde{\alpha}_k Cd_k \Rightarrow \\ Cx_{k+1} = C(x_k + \tilde{\alpha}_k d_k) \Rightarrow x_{k+1} = x_k + \tilde{\alpha}_k d_k;$$

$$\textcircled{3} \quad \tilde{r}_{k+1} = \tilde{r}_k - \tilde{\alpha}_k \tilde{A} \tilde{d}_k \Rightarrow \\ C^{-1}r_{k+1} = C^{-1}r_k - \tilde{\alpha}_k C^{-1}A C^{-1}Cd_k \Rightarrow \\ C^{-1}r_{k+1} = C^{-1}(r_k - \tilde{\alpha}_k A d_k) \Rightarrow \\ r_{k+1} = r_k + \tilde{\alpha}_k A d_k;$$

$$\textcircled{4} \quad z_{k+1} = M^{-1}r_{k+1};$$

$$\textcircled{5} \quad \tilde{\beta}_k = \frac{\tilde{r}_{k+1}^\top \tilde{r}_{k+1}}{\tilde{r}_k^\top \tilde{r}_k} = \frac{r_{k+1}^\top z_{k+1}}{r_k^\top z_k};$$

$$\textcircled{6} \quad \tilde{d}_{k+1} = \tilde{r}_{k+1} + \tilde{\beta}_k \tilde{d}_k \Rightarrow Cd_{k+1} = C^{-1}r_{k+1} + \tilde{\beta}_k Cd_k \Rightarrow \\ d_{k+1} = M^{-1}r_{k+1} + \tilde{\beta}_k d_k \Rightarrow d_{k+1} = z_{k+1} + \tilde{\beta}_k d_k.$$



Some notations of preconditioned Conjugate Gradient Method

- ❶ d_k : the k -th mutually conjugate vector wrt A ;
- ❷ z_k : the auxiliary vector instead of \tilde{r}_k ;
- ❸ $\tilde{\alpha}_k$: the coefficient of \tilde{d}_k for \tilde{x}^* ;
- ❹ x_k : the approximate solution to x^* at step k ;
- ❺ r_k : the residual of x_k of original system at step k ;
- ❻ $\tilde{\beta}_k$: the coefficient ensuring $(\tilde{d}_k, \tilde{d}_{k+1})_{\tilde{A}} = 0$.



Preconditioned Conjugate Gradient Method

Algorithm 4: Preconditioned Conjugate Gradient Method

```

1  $x_0 =$  initial guess
2  $r_0 = b - Ax_0$ 
3  $d_0 = z_0 = M^{-1}r_0$ 
4 for  $k = 0, 1, 2, \dots, n - 1$  do
5   if  $r_k$  is sufficiently small then
6     return  $x_k$ 
7    $\tilde{\alpha}_k = \frac{r_k^\top z_k}{d_k^\top A d_k}$ 
8    $x_{k+1} = x_k + \tilde{\alpha}_k d_k$ 
9    $r_{k+1} = r_k - \tilde{\alpha}_k A d_k$ 
10   $z_{k+1} = M^{-1}r_{k+1}$ 
11   $\tilde{\beta}_k = \frac{r_{k+1}^\top z_{k+1}}{r_k^\top z_k}$ 
12   $d_{k+1} = z_{k+1} + \tilde{\beta}_k d_k$ 

```



Outline

- 1 Introduction
- 2 Preliminaries
- 3 Iterative method
 - Jacobi Method
 - Gauss-Seidel Method
 - Successive Over-Relaxation
 - Convergence of iterative methods
- 4 Methods for symmetric positive-definite matrices
 - Symmetric positive-definite matrices
 - Conjugate Gradient Method
 - Preconditioning
- 5 Conclusions



Conclusions

- Three iterative methods for strictly diagonally dominant matrix
 - ① Jacobi Method
 - ② Gauss-Seidel Method
 - ③ Successive Over-Relaxation



Conclusions

- Three iterative methods for strictly diagonally dominant matrix
 - 1 Jacobi Method
 - 2 Gauss-Seidel Method
 - 3 Successive Over-Relaxation
- Two iterative methods for symmetric positive-definite matrix
 - 1 Conjugate Gradient Method
 - 2 Preconditioned Conjugate Gradient Method



Thank you!

