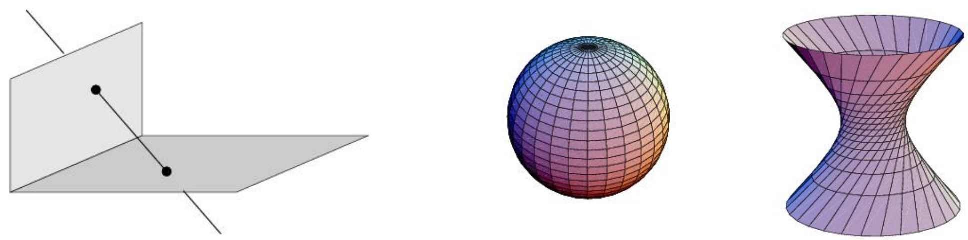
Projective 3D geometry

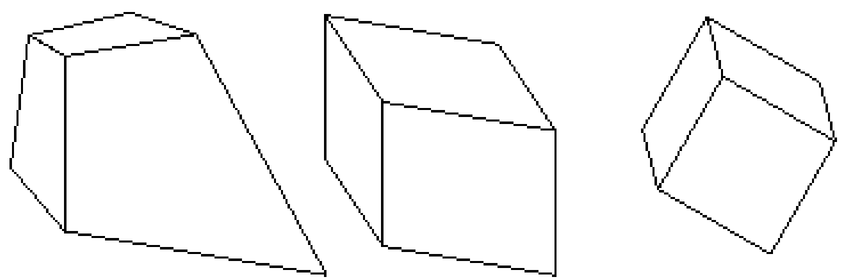
Multiple View Geometry

Projective 3D Geometry

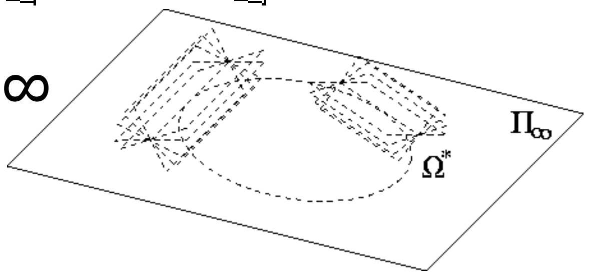
- Points, lines, planes and quadrics



- Transformations



- Π_∞, ω_∞ and Ω_∞



3D points

3D point

$(X, Y, Z)^{\mathsf{T}}$ in $\mathbf{R}^3$

$\mathrm{X} = (X_1, X_2, X_3, X_4)^{\mathsf{T}}$ in $\mathbf{P}^3$

$$\mathrm{X} = \left(\frac{X_1}{X_4}, \frac{X_2}{X_4}, \frac{X_3}{X_4}, 1 \right)^{\mathsf{T}} = (X, Y, Z, 1)^{\mathsf{T}} \qquad (X_4 \neq 0)$$

projective transformation

$$\mathbf{X}' = \mathbf{H}\mathbf{X}$$ (4x4-1=15 dof)

Planes

3D plane

$$\pi_1 X + \pi_2 Y + \pi_3 Z + \pi_4 = 0$$

$$\pi_1 X_1 + \pi_2 X_2 + \pi_3 X_3 + \pi_4 X_4 = 0$$

$$\boldsymbol{\pi}^\mathsf{T} \mathbf{X} = 0$$

Transformation

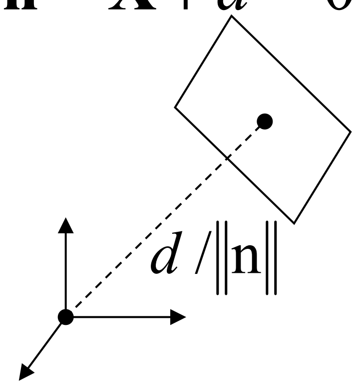
$$\mathbf{X}' = \mathbf{H}\mathbf{X}$$

$$\boldsymbol{\pi}' = \mathbf{H}^{-\mathsf{T}} \boldsymbol{\pi}$$

Euclidean representation

$$\mathbf{n}^\mathsf{T} \cdot \tilde{\mathbf{X}} + d = 0 \qquad \mathbf{n} = (\pi_1, \pi_2, \pi_3)^\mathsf{T} \qquad \tilde{\mathbf{X}} = (X, Y, Z)^\mathsf{T}$$

$$\pi_4 = d \qquad X_4 = 1$$



Dual: points ↔ planes, lines ↔ lines

Planes from points

Solve $\boldsymbol{\pi}$ from $\mathbf{X}_1^{\mathsf{T}}\boldsymbol{\pi}=0$, $\mathbf{X}_2^{\mathsf{T}}\boldsymbol{\pi}=0$ and $\mathbf{X}_3^{\mathsf{T}}\boldsymbol{\pi}=0$

$$\begin{bmatrix}\mathbf{X}_1^{\mathsf{T}}\\ \mathbf{X}_2^{\mathsf{T}}\\ \mathbf{X}_3^{\mathsf{T}}\end{bmatrix}\boldsymbol{\pi}=0 \qquad \text{(solve } \pi \text{ as right nullspace of } \begin{bmatrix}\mathbf{X}_1^{\mathsf{T}}\\ \mathbf{X}_2^{\mathsf{T}}\\ \mathbf{X}_3^{\mathsf{T}}\end{bmatrix}\text{)}$$

Or implicitly from coplanarity condition

$$\det\begin{bmatrix} X_1 & (X_1)_1 & (X_2)_1 & (X_3)_1 \\ X_2 & (X_1)_2 & (X_2)_2 & (X_3)_2 \\ X_3 & (X_1)_3 & (X_2)_3 & (X_3)_3 \\ X_4 & (X_1)_4 & (X_2)_4 & (X_3)_4 \end{bmatrix}=0$$

$$\det[\mathbf{X}\ \mathbf{X}_1\ \mathbf{X}_2\ \mathbf{X}_3]=0$$

$$X_1 D_{234} - X_2 D_{134} + X_3 D_{124} - X_4 D_{123} = 0$$

$$\pi = \left(D_{234}, -D_{134}, D_{124}, -D_{123}\right)^{\mathsf{T}}$$

Points from planes

Solve $\mathbf{X}$ from $\boldsymbol{\pi}_1^\mathsf{T}\mathbf{X}=0$, $\boldsymbol{\pi}_2^\mathsf{T}\mathbf{X}=0$ and $\boldsymbol{\pi}_3^\mathsf{T}\mathbf{X}=0$

$$\begin{bmatrix}\boldsymbol{\pi}_1^\mathsf{T}\\ \boldsymbol{\pi}_2^\mathsf{T}\\ \boldsymbol{\pi}_3^\mathsf{T}\end{bmatrix}\mathbf{X}=\mathbf{0}$$

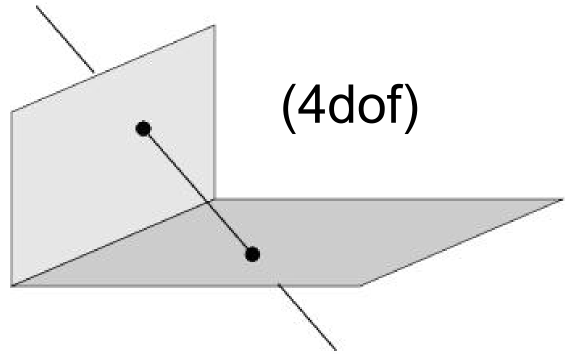
(solve X as right nullspace of $\begin{bmatrix}\pi_1^\mathsf{T}\\ \pi_2^\mathsf{T}\\ \pi_3^\mathsf{T}\end{bmatrix}$)

Representing a plane by its span

$$\mathrm{X}=\mathbf{M}\,\mathrm{x} \qquad \mathbf{M}=\left[\mathbf{X}_1\mathbf{X}_2\mathbf{X}_3\right] \qquad \pi=(a,b,c,d)^\mathsf{T}$$

$$\boldsymbol{\pi}^\mathsf{T}\mathbf{M}=\mathbf{0} \qquad \mathbf{M}=\begin{bmatrix}\mathbf{p}^\mathrm{T}\\ \mathrm{I}\end{bmatrix} \qquad \mathbf{p}=\left(-\frac{b}{a},-\frac{c}{a},-\frac{d}{a}\right)^\mathrm{T}$$

Lines



$$\mathbf{W} = \begin{bmatrix} \mathbf{A}^\mathsf{T} \\ \mathbf{B}^\mathsf{T} \end{bmatrix} \qquad \lambda\mathbf{A} + \mu\mathbf{B}$$

$$\mathbf{W}^* = \begin{bmatrix} \mathbf{P}^\mathsf{T} \\ \mathbf{Q}^\mathsf{T} \end{bmatrix} \qquad \lambda\mathbf{P} + \mu\mathbf{Q}$$

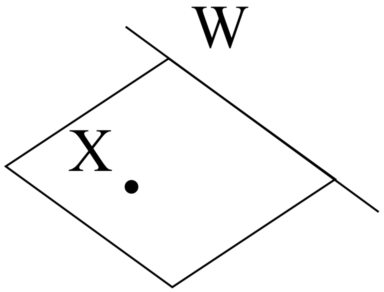
$$\mathrm{W}^*\mathrm{W}^\mathsf{T} = \mathrm{W}\mathrm{W}^{*\mathsf{T}} = 0_{2\times 2}$$

Example: X-axis

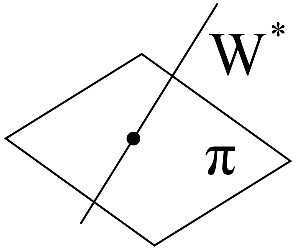
$$\mathrm{W} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \qquad \mathrm{W}^* = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Points, lines and planes

$$\mathbf{M} = \begin{bmatrix} \mathbf{W} \\ \mathbf{X}^{\mathsf{T}} \end{bmatrix} \qquad \mathbf{M}\boldsymbol{\pi} = 0$$



$$\mathbf{M} = \begin{bmatrix} \mathbf{W}^{*} \\ \boldsymbol{\pi}^{\mathsf{T}} \end{bmatrix} \qquad \mathbf{M}\mathbf{X} = 0$$



Quadrics and dual quadrics

$X^T Q X = 0$ (Q : 4x4 symmetric matrix)

$$Q = \begin{bmatrix} \bullet & \bullet & \bullet & \bullet \\ \circ & \bullet & \bullet & \bullet \\ \circ & \circ & \bullet & \bullet \\ \circ & \circ & \circ & \bullet \end{bmatrix}$$

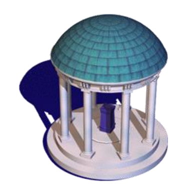
1. 9 d.o.f.
2. in general 9 points define quadric
3. det Q=0 ↔ degenerate quadric
4. pole – polar $\pi = QX$
5. (plane ∩ quadric)=conic $C = M^T Q M$ $\pi : X = Mx$
6. transformation $Q' = H^{-T} Q H^{-1}$

$\pi^T Q^* \pi = 0$

1. relation to quadric $Q^* = Q^{-1}$ (non-degenerate)
2. transformation $Q'^* = H Q^* H^T$

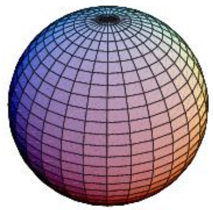
Quadric classification

Rank	Sign.	Diagonal	Equation	Realization
4	4	(1,1,1,1)	$X^2+Y^2+Z^2+1=0$	No real points
	2	(1,1,1,-1)	$X^2+Y^2+Z^2=1$	Sphere
	0	(1,1,-1,-1)	$X^2+Y^2=Z^2+1$	Hyperboloid (1S)
3	3	(1,1,1,0)	$X^2+Y^2+Z^2=0$	Single point
	1	(1,1,-1,0)	$X^2+Y^2=Z^2$	Cone
2	2	(1,1,0,0)	$X^2+Y^2=0$	Single line
	0	(1,-1,0,0)	$X^2=Y^2$	Two planes
1	1	(1,0,0,0)	$X^2=0$	Single plane

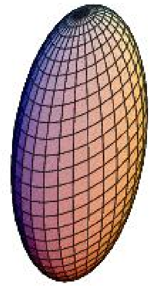


Quadric classification

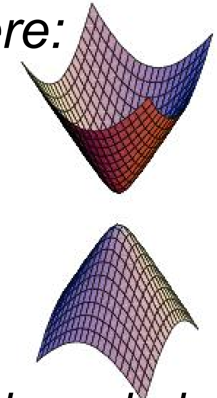
Projectively equivalent to *sphere:*



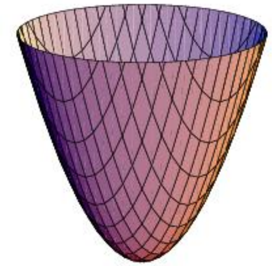
sphere



ellipsoid

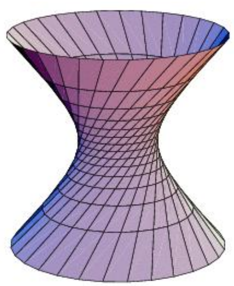


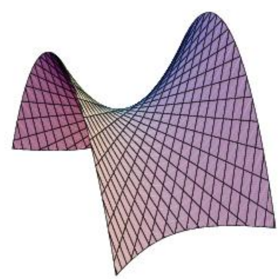
hyperboloid of two sheets



paraboloid

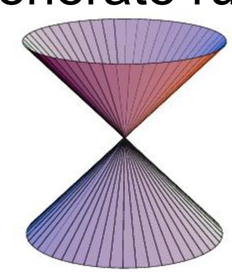
Ruled quadrics:



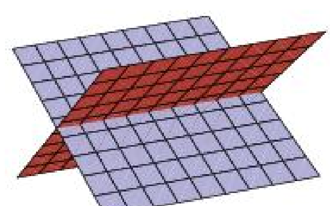


hyperboloids of one sheet

Degenerate ruled quadrics:



cone



two planes

The plane at infinity

$$\boldsymbol{\pi}'_\infty = \mathbf{H}_A^{-\mathsf{T}} \boldsymbol{\pi}_\infty = \begin{bmatrix} \mathbf{A}^{-\mathsf{T}} & \mathbf{0} \\ -\mathbf{A}\mathbf{t} & 1 \end{bmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \boldsymbol{\pi}_\infty$$

The plane at infinity π_∞ is a fixed plane under a projective transformation H iff H is an affinity

1. canical position $\pi_\infty = (0,0,0,1)^{\mathsf{T}}$
2. contains directions $\mathrm{D} = (X_1, X_2, X_3, 0)^{\mathsf{T}}$
3. two planes are parallel $\Leftrightarrow$ line of intersection in π_∞
4. line // line (or plane) $\Leftrightarrow$ point of intersection in π_∞

The absolute conic

The absolute conic Ω_∞ is a (point) conic on π_∞.

In a metric frame:

$$\left.\begin{array}{c} X_1^2 + X_2^2 + X_3^2 \\ X_4 \end{array}\right\} = 0$$

or conic for directions: $(X_1, X_2, X_3)\mathrm{I}(X_1, X_2, X_3)^\mathsf{T}$

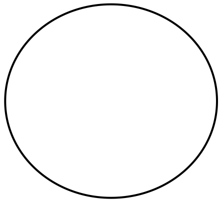
(with no real points)

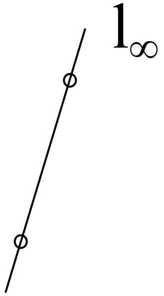
> The absolute conic Ω_∞ is a fixed conic under the projective transformation $\mathbf{H}$ iff $\mathbf{H}$ is a similarity

1. Ω_∞ is only fixed as a set
2. Circle intersect Ω_∞ in two points
3. Spheres intersect π_∞ in Ω_∞

The circular points

"circular points"





$$x_1^2 + x_2^2 + dx_1x_3 + ex_2x_3 + fx_3^2 = 0$$
$$x_3 = 0$$

$$x_1^2 + x_2^2 = 0$$

$$\mathrm{I} = (1, i, 0)^{\mathsf{T}}$$

$$\mathrm{J} = (1, -i, 0)^{\mathsf{T}}$$

Algebraically, encodes orthogonal directions

$$\mathrm{I} = (1,0,0)^{\mathsf{T}} + i(0,1,0)^{\mathsf{T}}$$

The circular points

$$\mathrm{I} = \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} \qquad \mathrm{J} = \begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix}$$

$$\mathrm{I}' = \mathbf{H}_S \, \mathrm{I} = \begin{bmatrix} s\cos\theta & -s\sin\theta & t_x \\ s\sin\theta & s\cos\theta & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} = se^{i\theta} \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} = \mathrm{I}$$

The circular points I, J are fixed points under the projective transformation $\mathbf{H}$ iff $\mathbf{H}$ is a similarity

Conic dual to the circular points

$$\mathbf{C}^*_\infty = \mathrm{IJ}^\mathsf{T} + \mathrm{JI}^\mathsf{T} \qquad \mathbf{C}^*_\infty = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{C}^*_\infty = \mathbf{H}_S \mathbf{C}^*_\infty \mathbf{H}_S^\mathsf{T}$$

The dual conic $\mathbf{C}^*_\infty$ is **fixed** conic under the projective transformation $\mathbf{H}$ iff $\mathbf{H}$ is a similarity

Note: $\mathbf{C}^*_\infty$ has 4DOF

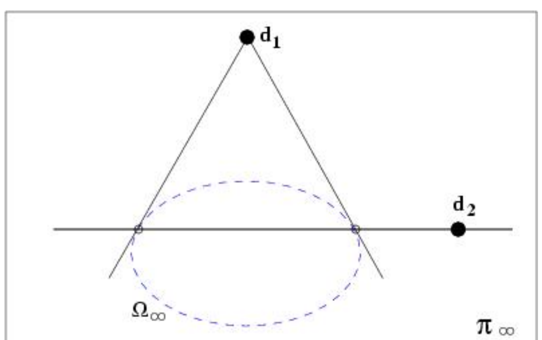
l_∞ is the nullvector

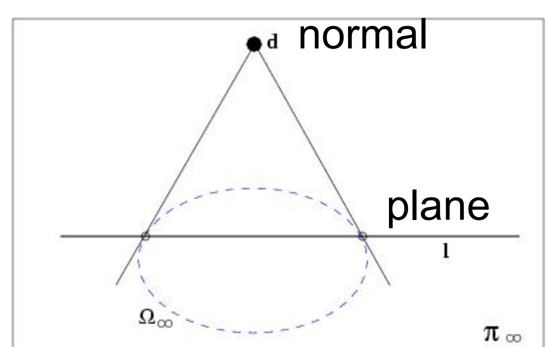
The absolute conic

Euclidean: $$\cos\theta = \frac{\left(\mathbf{d}_1^{\mathsf{T}}\mathbf{d}_2\right)}{\sqrt{\left(\mathbf{d}_1^{\mathsf{T}}\mathbf{d}_1\right)\left(\mathbf{d}_2^{\mathsf{T}}\mathbf{d}_2\right)}}$$

Projective: $$\cos\theta = \frac{\left(\mathbf{d}_1^{\mathsf{T}}\mathbf{\Omega}_\infty\mathbf{d}_2\right)}{\sqrt{\left(\mathbf{d}_1^{\mathsf{T}}\mathbf{\Omega}_\infty\mathbf{d}_1\right)\left(\mathbf{d}_2^{\mathsf{T}}\mathbf{\Omega}_\infty\mathbf{d}_2\right)}}$$

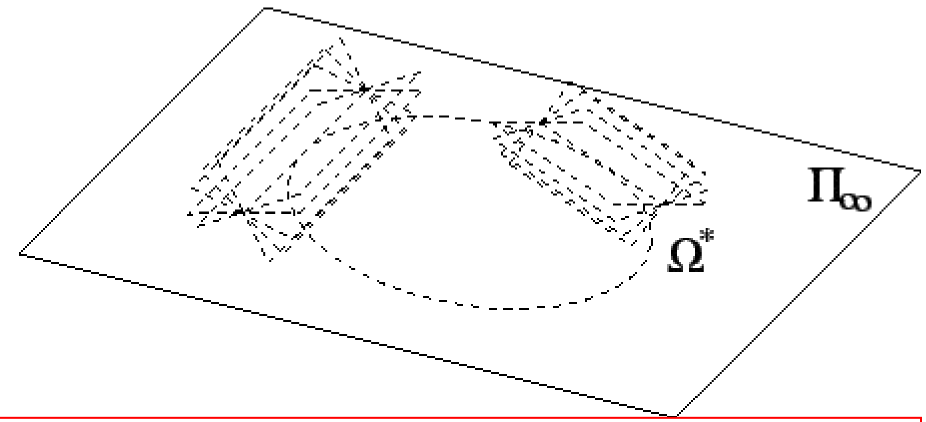
$\mathbf{d}_1^{\mathsf{T}}\mathbf{\Omega}_\infty\mathbf{d}_2 = 0$ (orthogonality=conjugacy)





The absolute dual quadric

$$\boldsymbol{\Omega}^*_\infty = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0}^\mathsf{T} & 0 \end{bmatrix}$$

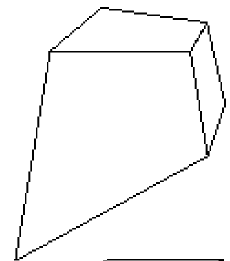
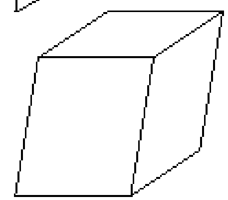
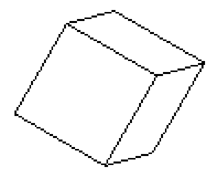
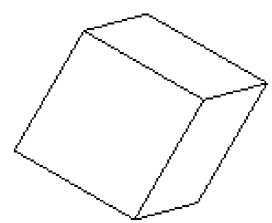


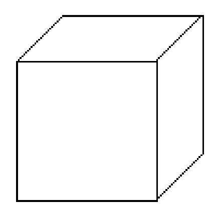
The absolute conic $\boldsymbol{\Omega}^*_\infty$ is a fixed conic under the projective transformation $\mathbf{H}$ iff $\mathbf{H}$ is a similarity

1. 8 dof
2. plane at infinity π_∞ is the nullvector of Ω_∞
3. Angles:

$$\cos\theta = \frac{\pi_1^\mathsf{T}\Omega^*_\infty\pi_2}{\sqrt{\left(\pi_1^\mathsf{T}\Omega^*_\infty\pi_1\right)\left(\pi_2^\mathsf{T}\Omega^*_\infty\pi_2\right)}}$$

Hierarchy of transformations

Projective 15dof	$\begin{bmatrix} \mathrm{A} & \mathrm{t} \\ \mathrm{v}^{\mathsf{T}} & v \end{bmatrix}$		Intersection and tangency
Affine 12dof	$\begin{bmatrix} \mathrm{A} & \mathrm{t} \\ 0^{\mathsf{T}} & 1 \end{bmatrix}$		Parallellism of planes, Volume ratios, centroids, **The plane at infinity π_∞**
Similarity 7dof	$\begin{bmatrix} s\,\mathrm{R} & \mathrm{t} \\ 0^{\mathsf{T}} & 1 \end{bmatrix}$		**The absolute conic Ω_∞**
Euclidean 6dof	$\begin{bmatrix} \mathrm{R} & \mathrm{t} \\ 0^{\mathsf{T}} & 1 \end{bmatrix}$		Volume



Singular Value Decomposition

$$A_{m\times n} = U_{m\times m}\Sigma_{m\times n}V^{\mathsf{T}}_{n\times n} \qquad m \geq n$$

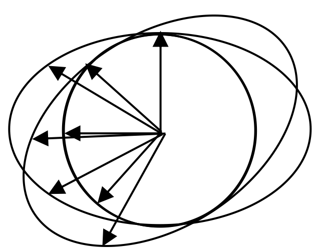
$$\Sigma = \begin{bmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_n \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

$$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n \geq 0$$

$$U^{\mathsf{T}}U = I$$

$$V^{\mathsf{T}}V = I$$

$$A = U_1\,\sigma_1\,V_1^{\mathsf{T}} + U_2\,\sigma_2\,V_2^{\mathsf{T}} + \cdots + U_n\,\sigma_n\,V_n^{\mathsf{T}}$$



$$U\Sigma V^{\mathsf{T}} X$$

Singular Value Decomposition

$$A = U\Sigma V^{\mathsf{T}}$$

- Homogeneous least-squares

$$\min \|AX\| \text{ subject to } \|X\| = 1 \qquad \text{solution } X = V_n$$

- Span and null-space

$$\mathbf{S}_L = [U_1 U_2]; \mathbf{N}_L = [U_3 U_4]$$
$$\mathbf{S}_R = [V_1 V_2]; \mathbf{N}_R = [V_3 V_4]$$

$$\Sigma = \begin{bmatrix} \sigma_1 & 0 & 0 & 0 \\ 0 & \sigma_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

- Closest rank r approximation

$$\tilde{A} = U\tilde{\Sigma} V^{\mathsf{T}} \qquad \tilde{\Sigma} = diag(\sigma_1, \sigma_2, \cdots, \sigma_r, 0, \ldots, 0)$$

- Pseudo inverse

$$A^{+} = V\Sigma^{+} U^{\mathsf{T}} \quad \Sigma^{+} = diag(\sigma_1^{-1}, \sigma_2^{-1}, \cdots, \sigma_r^{-1}, 0, \cdots, 0)$$

Projective geometry of 1D

$$(x_1, x_2)^{\mathsf{T}} \qquad x_2 = 0$$

$$\bar{\mathrm{x}}' = \mathbf{H}_{2\times 2}\,\bar{\mathrm{x}} \qquad \text{3DOF (2x2-1)}$$

The cross ratio

$$Cross(\bar{\mathrm{x}}_1, \bar{\mathrm{x}}_2, \bar{\mathrm{x}}_3, \bar{\mathrm{x}}_4) = \frac{|\bar{\mathrm{x}}_1, \bar{\mathrm{x}}_2||\bar{\mathrm{x}}_3, \bar{\mathrm{x}}_4|}{|\bar{\mathrm{x}}_1, \bar{\mathrm{x}}_3||\bar{\mathrm{x}}_2, \bar{\mathrm{x}}_4|} \qquad |\bar{\mathrm{x}}_i, \bar{\mathrm{x}}_j| = \det \begin{bmatrix} x_{i1} & x_{j1} \\ x_{i2} & x_{j2} \end{bmatrix}$$

Invariant under projective transformations