PROBLEM

15

Problem 1

① Gram-Schmidt $A=\begin{bmatrix}2&3\\-2&-6\\1&0\end{bmatrix}$

令 $y_1=A_1=\begin{bmatrix}2\\-2\\1\end{bmatrix}$ 则 $r_{11}=\|y_1\|_2=\sqrt{2^2+(-2)^2+1^2}=3$. 第一个单位向量是

$$q_1=\frac{y_1}{\|y_1\|_2}=\begin{bmatrix}\frac{2}{3}\\-\frac{2}{3}\\\frac{1}{3}\end{bmatrix}$$

令 $y_2=A_2-q_1q_1^TA_2=\begin{bmatrix}3\\-6\\0\end{bmatrix}-\begin{bmatrix}\frac{2}{3}\\-\frac{2}{3}\\\frac{1}{3}\end{bmatrix}\begin{bmatrix}\frac{2}{3}&-\frac{2}{3}&\frac{1}{3}\end{bmatrix}\begin{bmatrix}3\\-6\\0\end{bmatrix}=\begin{bmatrix}3\\-6\\0\end{bmatrix}-\begin{bmatrix}4\\-4\\2\end{bmatrix}=\begin{bmatrix}-1\\-2\\-2\end{bmatrix}$

$$q_2=\frac{y_2}{\|y_2\|}=\frac{y_2}{\sqrt{(-1)^2+(-2)^2+(-2)^2}}=\begin{bmatrix}-\frac{1}{3}\\-\frac{2}{3}\\-\frac{2}{3}\end{bmatrix}$$

加上第三个向量 $A_3=\begin{bmatrix}1\\0\\0\end{bmatrix}$ 得到 $y_3=A_3-q_1q_1^TA_3-q_2q_2^TA_3$

即 $y_3=\begin{bmatrix}1\\0\\0\end{bmatrix}-\begin{bmatrix}\frac{4}{9}\\-\frac{4}{9}\\\frac{2}{9}\end{bmatrix}-\begin{bmatrix}\frac{1}{9}\\\frac{2}{9}\\\frac{2}{9}\end{bmatrix}=\begin{bmatrix}\frac{4}{9}\\\frac{2}{9}\\-\frac{4}{9}\end{bmatrix}$

$$q_3=\frac{y_3}{\|y_3\|_2}=\begin{bmatrix}\frac{2}{3}\\\frac{1}{3}\\-\frac{2}{3}\end{bmatrix}$$

$$\therefore A=\begin{bmatrix}2&3\\-2&-6\\1&0\end{bmatrix}=\begin{bmatrix}\frac{2}{3}&-\frac{1}{3}&\frac{2}{3}\\-\frac{2}{3}&-\frac{2}{3}&\frac{1}{3}\\\frac{1}{3}&-\frac{2}{3}&-\frac{2}{3}\end{bmatrix}\begin{bmatrix}3&6\\0&3\\0&0\end{bmatrix}=QR.$$

② Householder.

找出豪斯霍尔德反射子将第一列向量 $x=[2\ -2\ 1]$ 移动到向量 $w=[\|x\|_2, 0, 0]$

令 $v=w-x=[3,0,0]-[2\ -2\ 1]=[1\ 2\ -1]$.

则 $P=\frac{vv^T}{v^Tv}=\frac{1}{6}\begin{bmatrix}1&2&-1\\2&4&-2\\-1&-2&1\end{bmatrix}$

$$H_1=I-2P=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}-\frac{2}{6}\begin{bmatrix}1&2&-1\\2&4&-2\\-1&-2&1\end{bmatrix}=\begin{bmatrix}\frac{2}{3}&-\frac{2}{3}&\frac{1}{3}\\-\frac{2}{3}&-\frac{1}{3}&\frac{2}{3}\\\frac{1}{3}&\frac{2}{3}&\frac{2}{3}\end{bmatrix}$$

$$H_1A=\begin{bmatrix}\frac{2}{3}&-\frac{2}{3}&\frac{1}{3}\\-\frac{2}{3}&-\frac{1}{3}&\frac{2}{3}\\\frac{1}{3}&\frac{2}{3}&\frac{2}{3}\end{bmatrix}\begin{bmatrix}2&3\\-2&-6\\1&0\end{bmatrix}=\begin{bmatrix}3&6\\0&0\\0&-3\end{bmatrix}$$

将 $\hat{x}=[0,-3]$ 移动到 $\hat{w}=[3,0]$. 则 $\hat{v}=\hat{w}-x=[3\ 0]-[0\ -3]=[3\ 3]$

$$\hat{P}=\frac{\hat{v}\hat{v}^T}{\hat{v}^Tv}=\begin{bmatrix}\frac{1}{2}&\frac{1}{2}\\\frac{1}{2}&\frac{1}{2}\end{bmatrix}.$$

$$\hat{H}_2=I-2P=\begin{bmatrix}1&0\\0&1\end{bmatrix}-\begin{bmatrix}1&1\\1&1\end{bmatrix}=\begin{bmatrix}0&-1\\-1&0\end{bmatrix}.$$

$$H_2H_1A=\begin{bmatrix}1&0&0\\0&0&-1\\0&-1&0\end{bmatrix}\begin{bmatrix}\frac{2}{3}&-\frac{2}{3}&\frac{1}{3}\\-\frac{2}{3}&-\frac{1}{3}&\frac{2}{3}\\\frac{1}{3}&\frac{2}{3}&\frac{2}{3}\end{bmatrix}\begin{bmatrix}2&3\\-2&-6\\1&0\end{bmatrix}=\begin{bmatrix}3&6\\0&3\\0&0\end{bmatrix}=R.$$

在两侧左乘 $H_1^{-1}H_2^{-1}=H_1H_2$ 得到QR分解：

$$A=\begin{bmatrix}2&3\\-2&-6\\1&0\end{bmatrix}=H_1H_2R=\begin{bmatrix}\frac{2}{3}&-\frac{2}{3}&\frac{1}{3}\\-\frac{2}{3}&-\frac{1}{3}&\frac{2}{3}\\\frac{1}{3}&\frac{2}{3}&\frac{2}{3}\end{bmatrix}\begin{bmatrix}1&0&0\\0&0&-1\\0&-1&0\end{bmatrix}\begin{bmatrix}3&6\\0&3\\0&0\end{bmatrix}$$

$$=\begin{bmatrix}\frac{2}{3}&-\frac{1}{3}&\frac{2}{3}\\-\frac{2}{3}&-\frac{2}{3}&\frac{1}{3}\\\frac{1}{3}&-\frac{2}{3}&-\frac{2}{3}\end{bmatrix}\begin{bmatrix}3&6\\0&3\\0&0\end{bmatrix}=QR.$$

2. 求解最小二乘问题 $\begin{bmatrix}2&3\\-2&-6\\1&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}3\\-3\\6\end{bmatrix}$.

法线方程 $A^TAx=A^Tb$ 如下： $A^TAx=A^Tb$

即 $\begin{bmatrix}2&-2&1\\3&-6&0\end{bmatrix}\begin{bmatrix}2&3\\-2&-6\\1&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}2&-2&1\\3&-6&0\end{bmatrix}\begin{bmatrix}3\\-3\\6\end{bmatrix}$

即 $\begin{bmatrix}9&18\\18&45\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}18\\27\end{bmatrix}$. 法线方程的解是 $\bar{x}_1=4$ $\bar{x}_2=-1$ $\bar{x}=[4,-1]$

余项是 $r=b-A\bar{x}=\begin{bmatrix}3\\-3\\6\end{bmatrix}-\begin{bmatrix}2&3\\-2&-6\\1&0\end{bmatrix}\begin{bmatrix}4\\-1\end{bmatrix}=\begin{bmatrix}-2\\-1\\2\end{bmatrix}$

$$\|e\|_2=\sqrt{(-2)^2+(-1)^2+2^2}=3.$$

PROBLEM2

解：①将点代入 $F_1(t)$ 中得：

$$c_1+c_2+0c_3=0$$
$$c_1+0c_2+c_3=2$$
$$c_1-c_2+0c_3=3$$
$$c_1+0c_2-c_3=1$$

即得到 $\begin{bmatrix}1&1&0\\1&0&1\\1&-1&0\\1&0&-1\end{bmatrix}\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix}=\begin{bmatrix}0\\2\\3\\1\end{bmatrix}$

所以 $A=\begin{bmatrix}1&1&0\\1&0&1\\1&-1&0\\1&0&-1\end{bmatrix}$ $b=\begin{bmatrix}0\\2\\3\\1\end{bmatrix}$

则 $A^TA=\begin{bmatrix}4&0&0\\0&2&0\\0&0&2\end{bmatrix}$ $A^Tb=\begin{bmatrix}6\\-3\\1\end{bmatrix}$

由 $A^TAc=A^Tb$ 得：

$c_1=\frac{3}{2}$, $c_2=-\frac{3}{2}$, $c_3=\frac{1}{2}$

则 $F_1(t)=\frac{3}{2}-\frac{3}{2}\cos 2\pi t+\frac{1}{2}\sin 2\pi t$

所以 $e_1=b-Ac=\begin{bmatrix}0\\2\\3\\1\end{bmatrix}-\begin{bmatrix}0\\2\\3\\1\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}$

即 $\|e_1\|_2=0$

②将点代入 $F_2(t)$ 中得：

$$c_1+c_2+0c_3+c_4=0$$
$$c_1+0c_2+c_3-c_4=2$$
$$c_1-c_2+0c_3+c_4=3$$
$$c_1+0c_2-c_3-c_4=1$$

即得 $\begin{bmatrix}1&1&0&1\\1&0&1&-1\\1&-1&0&1\\1&0&-1&-1\end{bmatrix}\begin{bmatrix}c_1\\c_2\\c_3\\c_4\end{bmatrix}=\begin{bmatrix}0\\2\\3\\1\end{bmatrix}$

所以 $A=\begin{bmatrix}1&1&0&1\\1&0&1&-1\\1&-1&0&1\\1&0&-1&-1\end{bmatrix}$ $b=\begin{bmatrix}0\\2\\3\\1\end{bmatrix}$

得：$A^TA=\begin{bmatrix}4&0&0&0\\0&2&0&0\\0&0&2&0\\0&0&0&4\end{bmatrix}$ $A^Tb=\begin{bmatrix}6\\-3\\1\\0\end{bmatrix}$

由 $A^TAc=A^Tb$ 得：

$c_1=\frac{3}{2}$ $c_2=-\frac{3}{2}$ $c_3=\frac{1}{2}$ $c_4=0$

则 $F_2(t)=\frac{3}{2}-\frac{3}{2}\cos 2\pi t+\frac{1}{2}\sin 2\pi t+0\cos 4\pi t$
$=\frac{3}{2}-\frac{3}{2}\cos 2\pi t+\frac{1}{2}\sin 2\pi t$

与 $F_1(t)$ 相同

所以 $\|e_2\|_2=\|e_1\|_2=0$

PROBLEM3

1: (1) let λ be eigenvalues and v be eigenvectors of A.

then we have : $AV = \lambda V$

$$\Rightarrow |A-\lambda E| = \left|\begin{bmatrix} 3-\lambda & -1 \\ -1 & 3-\lambda \end{bmatrix}\right|$$

$$= (3-\lambda)^2 - (-1)^2$$

$$= 0$$

$\therefore \lambda_1 = 2$, $\lambda_2 = 4$

① for $\lambda_1 = 2$, $V_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

② for $\lambda_2 = 4$, $V_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

(2) initial vector : $x_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

then : $x_1 = A \cdot x_0 = \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$

$$x_2 = A \cdot x_1 = \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 10 \\ -6 \end{bmatrix}$$

$$x_3 = A \cdot x_2 = \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 10 \\ 6 \end{bmatrix} = \begin{bmatrix} 36 \\ -28 \end{bmatrix}$$

(3) ①逆向幂迭代(平移 $s=1$)，平移后特征值：1，3，倒数最大为($\frac{1}{1}$)，故定位 2 附近的特征值；

②逆向幂迭代(平移 $s=5$)，平移后特征值 -3，-1，倒数最大为($\frac{1}{-1}$)，故定位4附近的特征值。

PROBLEM4

: we have $h = 0.1$, $x = 0$

then $f'(0) = \frac{f(0+0.1) - f(0-0.1)}{2 \times 0.1}$

$= \frac{f(0.1) - f(-0.1)}{0.2}$

$= \frac{e^{0.1} - e^{-0.1}}{0.2} \approx 1.0017$

And $f'(x) = e^x$, $f'(0) = e^0 = 1$

$\therefore$ error $= |1.0017 - 1| = 0.0017$

PROBLEM5

$err = |1.00166?5 - 1| = \ldots$

Problem 5.

15

① Trapezoid Rule: $\int_0^1 x^2 dx \approx \frac{h}{2}(y_0 + y_1) = \frac{1-0}{2} \times (0+1) = \frac{1}{2}$

② Simpson's Rule: $\int_0^1 x^2 dx \approx \frac{h}{3}(y_0 + 4y_1 + y_2) = \frac{\frac{1}{2}}{3} \times [0 + 1 + 4 \times (\frac{1}{2})^2] = \frac{1}{3}$

③ Midpoint Rule: $\int_0^1 x^2 dx \approx 1 \times (\frac{1}{2})^2 = \frac{1}{4}$

exact value $\int_0^1 x^2 dx = \frac{1}{3} x^3 \Big|_0^1 = \frac{1}{3}$

Trapezoid Rule $err = |\frac{1}{2} - \frac{1}{3}| = \frac{1}{6}$

Simpson's Rule $err = |\frac{1}{3} - \frac{1}{3}| = 0$

Midpoint Rule $err = |\frac{1}{4} - \frac{1}{3}| = \frac{1}{12}$

Problem 6: ... first: $(w-x)^T(w+x) = w^T w - x^T w + w^T x - x^T x = \|w\|_2 - \|x\|_2 =$

PROBLEM6

① $Hx = (I - 2vv^T) \cdot x$

$= x - 2vv^T \cdot x$

$= (w - u) - 2\frac{u}{\|u\|_2} \cdot \frac{u^T}{\|u\|_2} \cdot x$

$= w - \frac{uu^T \cdot u}{\|u\|_2^2} - \frac{uu^T x}{\|u\|_2^2} - \frac{uu^T(w-u)}{\|u\|_2^2}$

$= w - \frac{uu^T(w+x)}{\|u\|_2^2}$

$= w - \frac{u(w-x)^T(w+x)}{\|u\|_2^2}$

$\because \|x\|_2 = \|w\|_2$, we have $(w-x)^T \cdot (w+x) = 0$

$\therefore Hx = w$

② Since $Hx = w$, we have $H^{-1} \cdot Hx = H^{-1}w$

$x = H^{-1}w$

$\because H$ 正交

$\therefore H^{-1} = H^T \Rightarrow x = H^T w$

And $H^T = (I - 2vv^T)^T = I - 2vv^T = H$

$\therefore \cancel{4} x = H^T w = Hw$

$\Rightarrow Hw = x$

PROBLEM7

由定理：设 A 是一个 $m\times n$ 矩阵，A^TA 的特征值非负的，

对于一个 $m\times n$ 矩阵 A，$n\times n$ 矩阵 A^TA 对称，因而它的特征向量正交，特征值为非负实数，可表示为 $s_1^2\geq\cdots\geq s_n^2\geq 0$，其对应的正交特征向量集为 $\{v_1,\cdots v_n\}$。

对于 u_i，如果 $s_i\neq 0$，使用方程 $s_iu_i=Av_i$ 定义 u_i，

如果 $s_i=0$，选择任一单位向量 u_i，与 $u_1\cdots u_{i-1}$ 正交，$(1\leq i\leq m)$

因此 $u_1,\cdots u_m$ 会是两两正交的单位向量，为 R^m 的另外一个正交基，构成 AA^T 的正交特征向量。

综上，题目的定理得以证明。

PROBLEM8

The insights behind PageRank

- A webpage with good score have inlinks from those with good scores;
- A webpage with good score have outlinks to those with good scores;
- Inlinks from good webpages should carry more weight than inlinks from marginal webpages.

The insights behind PageRank

1. Webpages vote for the importance of other webpages by linking to them;
   - The more inlinks a page has, the more important it is.
2. One webpage has only one vote;
   - If a webpage has more than one outlinks, its vote must be split.
3. A link to webpage i from an important page increases webpage i's importance more than a link from an unimportant one.
   - It matters who your supporters are.

PROBLEM9

要简单描述，要给出公式，要有对比