Problem 1. (20 points) Find the first two steps of the Gauss-Seidel Methods for the following linear system, using the initial vector $x_0 = [0, 0, 0]$.

$$\begin{aligned} 2x_1 + x_2 - x_3 &= 2, \\ x_1 + 3x_2 - x_3 &= 3, \\ x_1 + x_2 - 3x_3 &= -1. \end{aligned}$$

problem 1

$$\begin{cases} 2x_1 + x_2 - x_3 = 2 \\ x_1 + 3x_2 - x_3 = 3 \\ x_1 + x_2 - 3x_3 = -1 \end{cases} \quad 得: \quad \begin{cases} x_1 = \dfrac{-x_2+x_3+2}{2} \\ x_2 = \dfrac{-x_1+x_3+3}{3} \\ x_3 = \dfrac{x_1+x_2+1}{3} \end{cases}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}_1 = \begin{bmatrix} \frac{-0+0+2}{2} = 1 \\ \frac{-1+0+3}{3} = \frac{2}{3} \\ \frac{1+\frac{2}{3}+1}{3} = \frac{8}{9} \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{2}{3} \\ \frac{8}{9} \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}_2 = \begin{bmatrix} \frac{-\frac{2}{3}+\frac{8}{9}+2}{2} = \frac{10}{9} \\ \frac{-\frac{10}{9}+\frac{8}{9}+3}{3} = \frac{25}{27} \\ \frac{\frac{10}{9}+\frac{25}{27}+1}{3} = \frac{82}{81} \end{bmatrix} = \begin{bmatrix} \frac{10}{9} \\ \frac{25}{27} \\ \frac{82}{81} \end{bmatrix}$$

1. the fixed-point iteration with $x_k = \begin{bmatrix} u_k \\ v_k \\ w_k \end{bmatrix}$ is

$$\begin{bmatrix} u_{k+1} \\ v_{k+1} \\ w_{k+1} \end{bmatrix} = D^{-1}(b - Ux_k - Lx_{k+1}) = \begin{bmatrix} \frac{1}{2}(2 - v_k + w_k) \\ \frac{1}{3}(3 - u_{k+1} + w_k) \\ -\frac{1}{3}(-1 - u_{k+1} - v_{k+1}) \end{bmatrix}$$

Since $x_0 = [u_0, v_0, w_0]^T = [0, 0, 0]^T$,

$$\begin{bmatrix} u_1 \\ v_1 \\ w_1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}(2 - v_0 + w_0) \\ \frac{1}{3}(3 - u_1 + w_0) \\ -\frac{1}{3}(-1 - u_1 - v_1) \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{2}{3} \\ +\frac{8}{9} \end{bmatrix}, \quad \begin{bmatrix} u_2 \\ v_2 \\ w_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}(2 - v_1 + w_1) \\ \frac{1}{3}(3 - u_2 + w_1) \\ -\frac{1}{3}(-1 - u_2 - v_2) \end{bmatrix} = \begin{bmatrix} \frac{10}{9} \\ \frac{25}{27} \\ \frac{82}{81} \end{bmatrix}$$

Problem 2. (20 points)

Algorithm 1: Conjugate Gradient Method

x_0 be a zero vector
$d_0 = r_0 = b - Ax_0$
for $k = 0, 1, 2, \ldots, n-1$ **do**
  if r_k *is sufficiently small* **then**
    return x_k
  $\alpha_k = \frac{r_k^\top r_k}{d_k^\top A d_k}$
  $x_{k+1} = x_k + \alpha_k d_k$
  $r_{k+1} = r_k - \alpha_k A d_k$
  $\beta_k = \frac{r_{k+1}^\top r_{k+1}}{r_k^\top r_k}$
  $d_{k+1} = r_{k+1} + \beta_k d_k$

According the above algorithm, please prove the following:
Suppose that $b \neq 0$ and $r_k \neq 0$ for $k < n$. Then for each $1 \leq k \leq n$,

1. distinct residuals are pairwise orthogonal: $r_k^\top r_j = 0$ for $j < k$;
2. distinct vectors of a subspace span are pairwise A-conjugate: $d_k^\top A d_j = 0$ for $j < k$.

Base case ($k = 1$):

1. $r_0^\top r_1 = r_0^\top r_0 - \alpha_0 r_0^\top A d_0 = r_0^\top r_0 - \frac{r_0^\top r_0}{d_0^\top A d_0} d_0^\top A d_0 = 0;$
2. $r_1^\top r_1 = r_1^\top r_0 - \alpha_0 r_1^\top A d_0 = -\frac{r_0^\top r_0}{d_0^\top A d_0} r_1^\top A d_0;$
3. $\beta_0 = \frac{r_1^\top r_1}{r_0^\top r_0} = -\frac{r_1^\top A d_0}{d_0^\top A d_0};$
4. $d_0^\top A d_1 = d_0^\top A r_1 + \beta_0 d_0^\top A d_0 = d_0^\top A r_1 - \frac{r_1^\top A d_0}{d_0^\top A d_0} d_0^\top A d_0 = 0.$

Inductive step ($k > 1$): Suppose that the $k-1$ case hold.
The 2nd item:

1. $r_j^\top r_k = r_j^\top r_{k-1} - \alpha_{k-1} r_j^\top A d_{k-1};$
2. If $j < k-1$, then $r_j^\top r_k = 0;$
3. If $j = k-1$, then $$d_{k-1}^\top A d_{k-1} = r_{k-1}^\top A d_{k-1} + \beta_{k-1} d_{k-2}^\top A d_{k-1} = r_{k-1}^\top A d_{k-1};$$
4. $$r_{k-1}^\top r_k = r_{k-1}^\top r_{k-1} - \alpha_{k-1} r_{k-1}^\top A d_{k-1} = r_{k-1}^\top r_{k-1} - \frac{r_{k-1}^\top r_{k-1}}{d_{k-1}^\top A d_{k-1}} d_{k-1}^\top A d_{k-1} = 0.$$

Problem 3. (20 points) Let $f(x) = \sin(x)$ where $x \in [0, \frac{\pi}{2}]$.

1. By using Newton's divided differences method, write down a degree 3 polynomial $P_3(x)$ with evenly spaced points;
2. Calculate $P_3(\frac{\pi}{4})$;
3. Give an error bound for the approximation in (2);
4. Compare the actual error to your error bound.

Date

Problem 5

$f(x) = \sin x,\ x \in [0, \frac{\pi}{2}]$

选择以下4个~~点~~具有相同间距的点

1. $(0,0), (\frac{\pi}{6}, \frac{1}{2}), (\frac{\pi}{3}, \frac{\sqrt{3}}{2}), (\frac{\pi}{2}, 1)$

0	0			
		$\frac{3}{\pi}$		
$\frac{\pi}{6}$	$\frac{1}{2}$		$\frac{9(\sqrt{3}-2)}{\pi^2}$	
		$\frac{3(\sqrt{3}-1)}{\pi}$		$\frac{18(5-3\sqrt{3})}{\pi^3}$
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$		$\frac{9(3-2\sqrt{3})}{\pi^2}$	
		$\frac{3(2-\sqrt{3})}{\pi}$		
$\frac{\pi}{2}$	1			

$$P_3(x) = \frac{3}{\pi}x + \frac{9(\sqrt{3}-2)}{\pi^2}x(x-\frac{\pi}{6}) + \frac{18(5-3\sqrt{3})}{\pi^3}x(x-\frac{\pi}{6})(x-\frac{\pi}{3})$$

2. $P_3(\frac{\pi}{4}) = \frac{3}{\pi} \times \frac{\pi}{4} + \frac{9(\sqrt{3}-2)}{\pi^2} \times \frac{\pi}{4}(\frac{\pi}{4}-\frac{\pi}{6}) + \frac{18(5-3\sqrt{3})}{\pi^3} \times \frac{\pi}{4}(\frac{\pi}{4}-\frac{\pi}{6})(\frac{\pi}{4}-\frac{\pi}{3})$

$= \frac{7+9\sqrt{3}}{32}$

3. 插值误差公式：$r = \frac{(x-x_1)\cdots(x-x_n)}{n!} f^{(n)}(c)$

对于 $f(x) = \sin x$，$|f'''(x)| = |-\cos x| \le 1$

因此：$|\sin\frac{\pi}{4} - P_3(\frac{\pi}{4})| \le \frac{|(\frac{\pi}{4}-0)(\frac{\pi}{4}-\frac{\pi}{6})(\frac{\pi}{4}-\frac{\pi}{3})(\frac{\pi}{4}-\frac{\pi}{2})|}{4!}|1|$

$\approx 1.762 \times 10^{-3}$

4. $|\sin\frac{\pi}{4} - P_3(\frac{\pi}{4})| = |\frac{\sqrt{2}}{2} - \frac{7+9\sqrt{3}}{32}| \approx 1.217 \times 10^{-3}$

Problem 4. (20 points) Let $T_n(x) = \begin{cases} 1, & n = 0; \\ x, & n = 1; \\ 2xT_{n-1}(x) - T_{n-2}(x), & n > 1. \end{cases}$

Please prove the following:

1. $\deg(T_n) = n$;
2. The leading coefficient of T_n is 2^{n-1} for $n \geq 1$;
3. $T_n(1) = 1$ and $T_n(-1) = (-1)^n$;
4. $T_n(x) = \cos(n \arccos(x))$ for $-1 \leq x \leq 1$;
5. $|T_n(x)| \leq 1$ for $-1 \leq x \leq 1$;
6. All roots of $T_n(x)$ are located between -1 and 1.

Problem 7 $\quad T_n(x) = \begin{cases} 1 & n=0 \\ x & n=1 \\ 2xT_{n-1}(x) - T_{n-2}(x) & n>1 \end{cases}$

1. 当 $n=0$ 与 $n=1$ 时，$\deg(T_0(x)) = 0$，$\deg(T_1(x)) = 1$

当 $n>1$，$\deg(T_n) = \deg(2xT_{n-1}(x) - T_{n-2}(x))$

$= \deg(T_{n-1}(x)) + 1$

由此可得当 $n>1$，$\deg(T_n)$ 为其前一项的度数 $+1$，

因此归纳假设可得 $\deg(T_{n-1}(x)) = n-1$

$\Rightarrow \deg(T_n(x)) = n-1+1 = n$

2. 当 $n=1$ 与 $n=2$ 时，$lc(T_1(x)) = lc(x) = 1$，$lc(T_2(x)) = lc(2x^2-1) = 2$

对于 $n>1$，$lc(T_n) = lc(2xT_{n-1}(x) - T_{n-2}(x))$

$= 2 \cdot lc(T_{n-1}(x))$

由归纳假设可得：$lc(T_{n-1}(x)) = 2^{n-2}$

因此：$lc(T_n) = 2 \cdot 2^{n-2} = 2^{n-1}$

3. 当 $n=1, n=2$，$T_n(1) = 1$，$T_n(-1) = (-1)^n$ 成立，

一般地 $T_{n+1}(1) = 2 \times 1\, T_n(1) - T_{n-1}(1) = 2-1 = 1$

$T_{n+1}(-1) = 2 \times (-1)\, T_n(-1) - T_{n-1}(-1)$

$= -2 \times (-1)^n - (-1)^{n-1}$

$= (-1)^{n-1} \times (-1)$

$= (-1)^{n+1}$

令 $y = \arccos x$，$\cos y = x \Leftarrow (-1 \leq x \leq 1)$

4. $T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$

$= 2x\cos ny - \cos(ny - y)$

$= 2x\cos ny - \cos ny \cos y - \sin ny \sin y$

$= 2\cos y \cos ny - \cos ny \cos y - \sin ny \sin y$

$= \cos ny \cos y - \sin ny \sin y$

$= \cos(n+1)y \quad \because T_n(x) = \cos(n \arccos x)$

$= \cos(n+1) \operatorname{arc}(\cos x)$

9.

(1) Base case:

$T_0(1) = 1 \quad (n=0) \qquad T_0(-1) = 1 = (-1)^0$

$T_1(1) = 1 \quad (n=1) \qquad T_1(-1) = -1 = (-1)^1$

Inductive steep $(n>1)$:

$T_n(1) = 2T_{n-1}(1) - T_{n-2}(1) = 2-1 = 1$

$T_n(-1) = -2T_{n-1}(-1) - T_{n-2}(-1) = -2\times(-1)^{n-1} - (-1)^{n-2}$

$= 2\times(-1)^n - (-1)^n$

$= (-1)^n$

Hence, $T_n(1) = 1$, $T_n(-1) = (-1)^n$

(2) let $y = \arccos x$,

Base case:

$T_0(x) = 1 = \cos(0 \times \arccos(x))$

$T_1(x) = x = \cos(1 \times \arccos(x))$

Inductive steep $(n>1)$:

$T_n(x) = 2x\,T_{n-1}(x) - T_{n-2}(x)$

$= 2x\cos(n-1)y - \cos(n-2)y$

$= 2x\cos(n-1)y - \cos[(n-1)y - y]$

$= 2x\cos(n-1)y - \cos(n-1)y\cos y - \sin(n-1)y\sin y$

$= 2x\cos(n-1)y - x\cos(n-1)y - \sin(n-1)y\sin y$

$= \cos(n-1)y\cos y - \sin(n-1)y\sin y$

$= \cos[(n-1)y + y]$

$= \cos ny \quad = \cos(n\arccos x)$

(3) According to (2), $|T_n(x)| = |\cos(n\arccos x)| \le 1$, for $-1 \le x \le 1$.

(ALL $\cos x$ in the range of $[-1, 1]$)

(4) $T_n(x) = \cos(n\arccos x) = 0$

$\Rightarrow n\arccos x = (k+\frac{1}{2})\pi \quad , \quad k \in \mathbb{Z}$

$\Rightarrow \quad \arccos x = \frac{2k+1}{2n}\pi$

$\Rightarrow \qquad x = \cos\frac{2k+1}{2n}\pi$

Hexe, $-1 \le x = \cos\frac{2k+1}{2n}\pi \le 1$.