Reviewing Exercises

Chapter One

1. The domain of the function $f(x)=\sqrt{x^2-1}+\ln(4-x^2)$ is $\underline{\{x\,|\,1\le x^2<4\}}$ or $\underline{(-2,\ -1]\bigcup[1,\ 2)}$.

2. The domain of $y=\dfrac{x+2}{4-\sqrt{x^2-9}}$ is $\underline{\{x\,|\,x^2\ge 9 \text{ and } x^2\ne 25\}}$.

3. The domain of the function $y=\dfrac{\log(2-x)}{\sqrt{|x|-1}}$ is

$\underline{(-\infty,-1)\cup(1,2) \quad \text{or} \quad x<-1 \; or \; 1<x<2}$.

4. The domain of the function $y=\dfrac{\log(3-x)}{\sqrt{|x|-2}}$ is $\underline{(-\infty,-2)\cup(2,3)}$.

5. The domain of the function $y=\ln(1-x)+\arccos(|x|-1)$ is $\underline{\quad [-2,\ 1) \quad}$.

6. The domain of the function $y=\log(2-x)+\sqrt{x^2-1}$ is $\underline{\quad (-\infty,-1]\bigcup[1,2) \quad}$.

Chapter Two

7. $\lim\limits_{x\to -1}(-x^3+2x-5)=$ $\underline{-6}$

8. $\lim\limits_{x\to\infty} x\sin\dfrac{1}{x}=$ $\underline{\quad 1 \quad}$

9. $\lim\limits_{x\to+\infty}(x\sin\dfrac{1}{x}+\dfrac{1}{x}\sin x)=$ $\underline{\quad 1 \quad}$

10. $\lim\limits_{x\to+\infty}(\sqrt{x+2}-\sqrt{x})=$ $\underline{\quad 0 \quad}$

11. $\lim\limits_{x\to+\infty}(\sqrt{x+2}-\sqrt{x+1})=$ $\underline{\quad 0 \quad}$

12. $\lim\limits_{n\to\infty}\left(\sqrt{n+4}-\sqrt{n-4}\right)=$ $\underline{\quad 0 \quad}$

13. $\lim\limits_{n\to\infty}(\sqrt{n+2}-\sqrt{n+1})=$ $\underline{\quad 0 \quad}$

14. $\lim\limits_{x\to\infty}\left(\dfrac{1+x}{x}\right)^{3x}=$ $\underline{e^3}$

15. $\lim\limits_{x\to\infty}\left(\frac{1+x}{x}\right)^{2x} = \underline{e^2}$

16. $\lim\limits_{x\to 0}(1+2x)^{\frac{1}{x}} = \underline{e^2}$

17. $\lim\limits_{x\to 0}\sqrt[x]{1-2x} = \underline{e^{-2}}$.

18. $\lim\limits_{n\to\infty}\left(1-\frac{2}{n}\right)^{n} = \underline{e^{-2}}$

19. $\lim\limits_{n\to\infty} 3^n \sin\frac{\pi}{3^n} = \underline{\pi}$

20. $\lim\limits_{n\to\infty} 2^n \sin\frac{\pi}{2^n} = \underline{\quad\pi\quad}$

21. If $\lim\limits_{x\to 2}\frac{x^2+ax+b}{x^2-x-2} = 2$, then (B)

A. $a=-8,\ b=2$; B. $a=2,\ b=-8$;

C. $a=2$, b is arbitrary; D. a, b are arbitrary.

22. $\lim\limits_{n\to\infty}\left(1-\frac{1}{n}\right)^{n} =$ (B)

A. e; B. e^{-1}; C. 1; D. ∞.

23. For the following limits, (B) exists

A. $\lim\limits_{x\to 0}\frac{1}{e^x-1}$; B. $\lim\limits_{x\to\infty}\frac{x^2}{1-x^2}$; C. $\lim\limits_{x\to\infty}\sin x$; D. $\lim\limits_{x\to 0} e^{\frac{1}{x}}$.

24. If for any x, $h(x)\le f(x)\le g(x)$, $\lim\limits_{x\to\infty}[g(x)-h(x)]=0$, then $\lim\limits_{x\to\infty} f(x)$ (D)

A. exists and the limit is 0; B. exists but the limit is not 0; C. doesn't exist; D. may exist.

25. Find the following limits:

(1) $\lim\limits_{x\to 2}(-x^2+5x-2)$

Solution: $\lim\limits_{x\to 2}(-x^2+5x-2) = -4+10-2 = 4$

(2) $\lim\limits_{t\to 1}\frac{t^2+t-2}{t^2-1}$

Solution: $\lim_{t\to1}\frac{t^2+t-2}{t^2-1}=\lim_{t\to1}\frac{(t+2)(t-1)}{(t+1)(t-1)}=\lim_{t\to1}\frac{t+2}{t+1}=\frac{3}{2}$

(3) $\lim_{x\to-2}\frac{-2x-4}{x^3+2x^2}$

Solution: $\lim_{x\to-2}\frac{-2x-4}{x^3+2x^2}=\lim_{x\to-2}\frac{-2(x+2)}{x^2(x+2)}=\lim_{x\to-2}\frac{-2}{x^2}=\frac{-2}{4}=-\frac{1}{2}$

(4) $\lim_{x\to9}\frac{\sqrt{x}-3}{x-9}$

Solution:

$$\lim_{x\to9}\frac{\sqrt{x}-3}{x-9}=\lim_{x\to9}\frac{(\sqrt{x}-3)(\sqrt{x}+3)}{(x-9)(\sqrt{x}+3)}=\lim_{x\to9}\frac{x-9}{(x-9)(\sqrt{x}+3)}=\lim_{x\to9}\frac{1}{\sqrt{x}+3}=\frac{1}{\sqrt{9}+3}=\frac{1}{6}$$

(5) $\lim_{x\to1}\frac{x-1}{\sqrt{x+3}-2}$

Solution:

$$\lim_{x\to1}\frac{x-1}{\sqrt{x+3}-2}=\lim_{x\to1}\frac{(x-1)(\sqrt{x+3}+2)}{(\sqrt{x+3}-2)(\sqrt{x+3}+2)}=\lim_{x\to1}\frac{(x-1)(\sqrt{x+3}+2)}{x-1}=\lim_{x\to1}(\sqrt{x+3}+2)=4$$

(6) $\lim_{x\to0}\frac{\sqrt{2+x}-\sqrt{2}}{x}$

Solution:

$$\lim_{x\to0}\frac{\sqrt{2+x}-\sqrt{2}}{x}=\lim_{x\to0}\frac{(\sqrt{2+x}-\sqrt{2})(\sqrt{2+x}+\sqrt{2})}{x(\sqrt{2+x}+\sqrt{2})}$$

$$=\lim_{x\to0}\frac{x}{x(\sqrt{2+x}+\sqrt{2})}=\lim_{x\to0}\frac{1}{\sqrt{2+x}+\sqrt{2}}$$

$$=\frac{1}{\sqrt{2}+\sqrt{2}}=\frac{1}{2\sqrt{2}}$$

26. Find the following limits:

(1) $\lim_{y\to0}\frac{\sin 3y}{4y}=\lim_{y\to0}\frac{\sin 3y}{3y}\frac{3}{4}=1\cdot\frac{3}{4}=\frac{3}{4}$

(2) $\lim_{x\to0}\frac{\sin 5x}{\sin 4x}=\lim_{x\to0}\frac{\frac{\sin 5x}{5x}}{\frac{\sin 4x}{4x}}\cdot\frac{5}{4}=\frac{1}{1}\cdot\frac{5}{4}=\frac{5}{4}$

(3) $\lim_{x\to0}\frac{x+x\cos x}{\sin x\cos x}=\lim_{x\to0}\frac{x(1+\cos x)}{\sin x\cos x}=\lim_{x\to0}\frac{1+\cos x}{\frac{\sin x}{x}\cos x}=\frac{1+1}{1\cdot1}=2$

27. The discontinuous points of the function $f(x)=\frac{x+1}{x^2-4x+3}$ are $\underline{x=1 \text{ and } x=3}$

28. The function $f(x)=\frac{x}{x^2+4x-5}$ is discontinuous at ___-5 and 1_______.

29. The discontinuous points of the function $y=\frac{x-1}{(x-1)(x-2)}$ are _*x*=1 and *x*=2

30. If $f(x)=\begin{cases} x\sin\frac{1}{x}, & x>0 \\ a+x^2, & x\le 0 \end{cases}$ is continuous on $(-\infty,+\infty)$, then $a=$___0______

31. If $f(x)=\begin{cases} 1+x\sin\frac{1}{x}, & x>0 \\ a+x^2, & x\le 0 \end{cases}$ is continuous on $(-\infty,+\infty)$, then $a=$___1______

32. If $f(x)=\begin{cases} 4x+1, & x\ge1 \\ x^2+k, & x<1 \end{cases}$ is continuous at $x=1$, then $k=$___4___ .

33. If $f(x)=\begin{cases} x^2+a, & x\ne0 \\ 4, & x=0 \end{cases}$ is continuous at $x=0$, then $a=(D)$.

A. 1 B. 2 C. 3 D. 4

34. If $f(x)=\begin{cases} x+3, & x<0 \\ x^2-2x+k, & x\ge0 \end{cases}$ is continuous on R, then k=(A)

A. 3 B. 2 C. 1 D. 0

35. If $f(x)=\begin{cases} x\sin\frac{1}{x}, & x>0 \\ \ln(a+x^2), & x\le0 \end{cases}$ is continuous on $(-\infty,+\infty)$, then $a=$ (D)

A. $\frac{1}{e}$; B. e; C. 2; D. 1

36. If $f(x)=\frac{e^{\frac{1}{x}}+e}{e^{\frac{1}{x}}-e}$, then (A)

A. $x=0$ is the jump discontinuity; B. $x=0$ is the removable discontinuity;
C. $x=1$ is the jump discontinuity; D. $x=1$ is the removable discontinuity.

37. If $f(x)=\begin{cases} x^2\sin\dfrac{1}{x}, & x\neq 0 \\ 0, & x=0 \end{cases}$, then at $x=0$, $f(x)$ is (C)

A. discontinuous; B. continuous but non-differentiable;

C. continuous and differentiable; D. discontinuous but differentiable.

38. For what value of *a* is

$$f(x)=\begin{cases} x^2-1, & x<3 \\ 2ax, & x\geq 3 \end{cases}$$

continuous at every *x*?

Solution: Since $f(x)$ is continuous at $x=3$, then

$$\lim_{x\to 3^+} f(x)=\lim_{x\to 3^-} f(x)=f(3)=6a$$

But $\lim_{x\to 3^-} f(x)=\lim_{x\to 3^-}(x^2-1)=8$, then

$$6a=8.$$

Therefore, $a=\dfrac{4}{3}$.

39. For what value of *b* is

$$g(x)=\begin{cases} x, & x<-2 \\ bx^2, & x\geq -2 \end{cases}$$

continuous at every *x*?

Solution: Since $g(x)$ is continuous at $x=-2$, then

$$\lim_{x\to -2^+} g(x)=\lim_{x\to -2^-} g(x)=g(-2)=4b$$

But $\lim_{x\to -2^-} g(x)=\lim_{x\to -2^-} x=-2$, then

$$4b=-2.$$

Therefore, $a=-\dfrac{1}{2}$.

40. For the function $f(x)=\dfrac{x^2+2x-3}{x^2-3x+2}$, the horizontal asymptote of its graph is $\underline{y=1}$, the vertical asymptote is $\underline{x=1}$.

41. The horizontal asymptote of the curve $y=\dfrac{x^2}{x^2-1}$ is $\underline{y=1}$

42. Given $f(x)=\dfrac{x^2-9}{x^2+3x}$, the vertical asymptote is $\underline{x=0}$, the horizontal asymptote is $\underline{y=1}$_

43. The horizontal asymptote of the curve $y=\dfrac{x^2+2x-3}{2x^2-x+1}$ is $\underline{y=\dfrac{1}{2}}$

44. If $f(x)=\dfrac{\sqrt{x^2+1}}{3x-5}$, find all the horizontal asymptotes and vertical asymptotes of its graph.

Solution: Since

$$\lim_{x\to\frac{5}{3}^-}\frac{\sqrt{x^2+1}}{3x-5}=-\infty,\ \lim_{x\to\frac{5}{3}^+}\frac{\sqrt{x^2+1}}{3x-5}=+\infty$$

then the line $x=\dfrac{5}{3}$ is a vertical asymptote of the graph of $f(x)$.

Since

$$\lim_{x\to+\infty}\frac{\sqrt{x^2+1}}{3x-5}=\frac{1}{3},\ \lim_{x\to-\infty}\frac{\sqrt{x^2+1}}{3x-5}=-\frac{1}{3}$$

then the lines $y=\dfrac{1}{3}$ and $y=-\dfrac{1}{3}$ are horizontal asymptotes of the graph of $f(x)$.

45. Find the following limits:

(1) $$\lim_{x\to\infty}\frac{2x^3+7}{x^3-x^2+x+7}=\lim_{x\to\infty}\frac{2+\frac{7}{x^3}}{1-\frac{1}{x}+\frac{1}{x^2}+\frac{7}{x^3}}=\frac{2+0}{1-0+0+0}=2$$

(2) $$\lim_{x\to\infty}\frac{x+1}{x^2+3}=\lim_{x\to\infty}\frac{\frac{1}{x}+\frac{1}{x^2}}{1+\frac{3}{x^2}}=\frac{0+0}{1+0}=0$$

(3) $$\lim_{x\to+\infty}(\sqrt{x+9}-\sqrt{x+4})$$

$$=\lim_{x\to+\infty}\frac{(\sqrt{x+9}-\sqrt{x+4})(\sqrt{x+9}+\sqrt{x+4})}{\sqrt{x+9}+\sqrt{x+4}}=\lim_{x\to+\infty}\frac{5}{\sqrt{x+9}+\sqrt{x+4}}$$

$$=\lim_{x\to+\infty}\frac{\frac{5}{\sqrt{x}}}{\sqrt{1+\frac{9}{x}}+\sqrt{1+\frac{4}{x}}}=\frac{0}{1+1}=0$$

46. Find the limits of the following functions.

(1) $\lim\limits_{x\to -1}\dfrac{x^2+2x+1}{x^2-2x-3}$

Solution: $\lim\limits_{x\to -1}\dfrac{x^2+2x+1}{x^2-2x-3}=\lim\limits_{x\to -1}\dfrac{(x+1)^2}{(x+1)(x-3)}=\lim\limits_{x\to -1}\dfrac{x+1}{x-3}=\dfrac{-1+1}{-1-3}=0$

(2) $\lim\limits_{x\to 0}\dfrac{x^2}{1-\sqrt{1+x^2}}$

Solution:

$$\lim_{x\to 0}\frac{x^2}{1-\sqrt{1+x^2}}=\lim_{x\to 0}\frac{x^2(1+\sqrt{1+x^2})}{(1-\sqrt{1+x^2})(1+\sqrt{1+x^2})}=\lim_{x\to 0}\frac{x^2(1+\sqrt{1+x^2})}{-x^2}$$

$$=\lim_{x\to 0}-(1+\sqrt{1+x^2})$$

$$=-2$$

(3) $\lim\limits_{x\to 0}\dfrac{\tan x-\sin x}{x^3}$

Solution:

$$\lim_{x\to 0}\frac{\tan x-\sin x}{x^3}=\lim_{x\to 0}\frac{\sin x}{x}\frac{1}{\cos x}\frac{1-\cos x}{x^2}$$

$$=\lim_{x\to 0}\frac{1-\cos x}{x^2}$$

$$=\lim_{x\to 0}\frac{1-\cos x}{x^2}$$

$$=\lim_{x\to 0}\frac{2\sin^2\frac{x}{2}}{x^2}=\frac{1}{2}$$

(4) Find $\lim\limits_{n\to\infty}\left(\dfrac{1}{n^2+\pi}+\dfrac{2}{n^2+2\pi}+\cdots+\dfrac{n}{n^2+n\pi}\right)$

Solution: Since

$$\frac{n(n+1)}{2(n^2+n\pi)}=\frac{1+2+\cdots+n}{n^2+n\pi}\le\frac{1}{n^2+\pi}+\frac{2}{n^2+2\pi}+\cdots+\frac{n}{n^2+n\pi}\le\frac{1+2+\cdots+n}{n^2+\pi}=\frac{n(n+1)}{2(n^2+\pi)}$$

and

$$\lim_{n\to\infty}\frac{n(n+1)}{2(n^2+\pi)}=\frac{1}{2},\qquad \lim_{n\to\infty}\frac{n(n+1)}{2(n^2+n\pi)}=\frac{1}{2}$$

then by Squeeze theorem

$$\lim_{n\to\infty}\left(\frac{1}{n^2+\pi}+\frac{2}{n^2+2\pi}+\cdots+\frac{n}{n^2+n\pi}\right)=\frac{1}{2}$$

(5) $\lim\limits_{n\to\infty}\left(\dfrac{1}{\sqrt{n^2+\pi}}+\dfrac{1}{\sqrt{n^2+2\pi}}+\cdots+\dfrac{1}{\sqrt{n^2+n\pi}}\right)$

Solution: Since

$$\frac{n}{\sqrt{n^2+n\pi}}\le\frac{1}{\sqrt{n^2+\pi}}+\frac{1}{\sqrt{n^2+2\pi}}+\cdots+\frac{1}{\sqrt{n^2+n\pi}}\le\frac{n}{\sqrt{n^2+\pi}}$$

and

$$\lim_{n\to\infty}\frac{n}{\sqrt{n^2+n\pi}}=1,\qquad \lim_{n\to\infty}\frac{n}{\sqrt{n^2+\pi}}=1$$

then by Squeeze theorem

$$\lim_{n\to\infty}n\left(\frac{1}{n^2+\pi}+\frac{1}{n^2+2\pi}+\cdots+\frac{1}{n^2+n\pi}\right)=1$$

(6) Find $\lim\limits_{n\to\infty}\left(\dfrac{1}{n^2+\frac{1}{n}}+\dfrac{2}{n^2+\frac{2}{n}}+\dfrac{3}{n^2+\frac{3}{n}}+\cdots+\dfrac{n}{n^2+1}\right)$

Solution: Since

$$\frac{n(n+1)}{2(n^2+1)}=\sum_{k=1}^{n}\frac{k}{n^2+1}\le\sum_{k=1}^{n}\frac{k}{n^2+\frac{k}{n}}\le\sum_{k=1}^{n}\frac{k}{n^2}=\frac{n(n+1)}{2n^2}=\frac{n+1}{2n},$$

$$\lim_{n\to\infty}\frac{n(n+1)}{2(n^2+1)}=\frac{1}{2},\qquad \lim_{n\to\infty}\frac{n+1}{2n}=\frac{1}{2},$$

then by squeeze theorem

$$\lim_{n\to\infty}\left(\frac{1}{n^2+\frac{1}{n}}+\frac{2}{n^2+\frac{2}{n}}+\frac{3}{n^2+\frac{3}{n}}+\cdots+\frac{n}{n^2+1}\right)=\frac{1}{2}$$

47. If $f(x)\in C([0,\ 2])$, and $f(0)=f(2)=1$, show that there exists a $\xi\in[0,\ 2]$, such that $f(\xi)=\xi$.

Proof: Let $g(x)=f(x)-x$, then $g(x)\in C([0,\ 2])$. Since

$$g(0)=f(0)-0=1>0$$

$$g(2)=f(2)-2=-1<0$$

then by intermediate value theorem, there exists a $\xi \in [0,\ 2]$, such that

$$g(\xi)=0$$

that is $f(\xi)=\xi$.

Chapter Three

48. If $f(x)=-x^2+\dfrac{2}{\sqrt{x}}-\pi$, then $f'(x)=\underline{-2x-x^{-\frac{3}{2}}}$

49. If $f'(x_0)=5$, then $\lim\limits_{\varepsilon\to 0}\dfrac{f(x_0-2\varepsilon)-f(x_0)}{5\varepsilon}=\underline{-2}$.

50. If $f'(x_0)=5$, then $\lim\limits_{\varepsilon\to 0}\dfrac{f(x_0-3\varepsilon)-f(x_0)}{5\varepsilon}=\underline{-3}$.

51. If $f'(x_0)=1$, then $\lim\limits_{\varepsilon\to 0}\dfrac{f(x_0-5\varepsilon)-f(x_0)}{3\varepsilon}=\underline{-\dfrac{5}{3}}$.

52. If $f'(x_0)=1$, then $\lim\limits_{\varepsilon\to 0}\dfrac{f(x_0-3\varepsilon)-f(x_0)}{5\varepsilon}=\underline{-\dfrac{3}{5}}$.

53. If $f'(x_0)=2$, then $\lim\limits_{\varepsilon\to 0}\dfrac{f(x_0-2\varepsilon)-f(x_0)}{5\varepsilon}=\underline{-\dfrac{4}{5}}$.

54. If $f(x)=x(x+1)(x+2)\cdots(x+n),\ (n\ge 2)$, then $f'(0)=\underline{\quad n!\quad}$

55. If $f(x)=x(x+1)(x+2)\cdots(x+100)$, then $f'(0)=\underline{100!}$

56. If $y=2\ln\sqrt{x}$, then $y'=\underline{\dfrac{1}{x}}$

57. The derivative of $y=x^{\frac{1}{x}}\ (x>0)$ is $\underline{x^{\frac{1}{x}-2}(1-\ln x)}$

58. The derivative of $y=x^x\ (x>0)$ is $\underline{x^x(1+\ln x)}$

59. If $f(x)$ has second derivative at any x, $y=f(\ln x)$, then $y''=$

$\underline{\dfrac{1}{x^2}[f''(\ln x)-f'(\ln x)]}$

60. The slope of the tangent line to the curve $y=2^x+x$ at the point (0,1) is $\underline{\quad \ln 2+1\quad}$

61. The equation for the line that is tangent to the curve $y = x^3 - x$ at the point $(-1,\ 0)$ is

$\underline{y = 2x + 2}$.

62. At the point where the curve of $y = \ln x$ intersects the line $x = e$, the tangent line of curve $y = \ln x$ is (A)

A. $x - ey = 0$; B. $x - ey - 2 = 0$; C. $ex - y = 0$; D. $ex - y - 2 = 0$

63. If $f(x)$ is differentiable at $x = a$, then $f'(a) =$ (D)

A. $\lim_{h \to 0} \frac{f(a) - f(a+h)}{h}$; B. $\lim_{h \to 0} \frac{f(a-h) - f(a)}{h}$;

C. $\lim_{h \to 0} \frac{f(a+2h) - f(a)}{h}$; D. $\lim_{h \to 0} \frac{f(a+2h) - f(a+h)}{h}$.

64. Suppose that $f(x)$ has second derivative, $y = f(\ln x)$, then $y'' =$ (D)

A. $f''(\ln x)$; B. $\frac{1}{x^2} f''(\ln x)$;

C. $\frac{1}{x^2}[f''(\ln x) + f'(\ln x)]$; D. $\frac{1}{x^2}[f''(\ln x) - f'(\ln x)]$

65. If $\lim_{x \to 0} \frac{f(x)}{1 - \cos x} = -1$, and $f(x)$ is continuous at $x = 0$, then at $x = 0$ (D)

A. $f(x)$ is not differentiable; B. $f(x)$ is differentiable, but $f'(0) \neq 0$;

C. $f(x)$ attains its relative minimum; D. $f(x)$ attains its relative maximum.

66. If $f(x)$ is continuous at $x = 0$ and $\lim_{x \to 0} \frac{f(x)}{x^2} = -1$, then at $x = 0$, $f(x)$ (C)

A. is non-differentiable; B. is differentiable and $f'(0) \neq 0$;

C.takes local maximum; D. takes local minimum.

67. If $f(x) = \begin{cases} x^2 + a, & x \neq 0 \\ 4, & x = 0 \end{cases}$ is continuous at $x = 0$, then $a = (D)$.

A. 1 B. 2 C. 3 D. 4

68. If $f(x) = xe^x$, then $f'(0) =$ (A)

A. 1 B . 2 C. 3 D. 4

69. Find $\frac{d}{dx}\ln(x^2+9)$.

Solution:

$$\frac{d}{dx}\ln(x^2+9)=\frac{1}{x^2+9}\frac{d}{dx}(x^2+9)=\frac{1}{x^2+9}2x=\frac{2x}{x^2+9}.$$

70. Find the derivative of $f(x)=\begin{cases} x^2\sin\frac{1}{x}, & x\neq 0 \\ 0, & x=0 \end{cases}$

Solution:

$$f'(x)=\begin{cases} 2x\sin\frac{1}{x}-\cos\frac{1}{x}, & x\neq 0 \\ 0, & x=0 \end{cases}$$

The derivative at x=0 must be obtained by definition:

$$f'(0)=\lim_{h\to 0}\frac{f(0+h)-f(0)}{h}=\lim_{h\to 0}\frac{h^2\sin\frac{1}{h}}{h}=\lim_{h\to 0}h\sin\frac{1}{h}=0$$

71. Find the derivative of the following functions:

(1) $f(x)=\sin x+\sqrt{x}+\arctan x-e^{-x}+\ln x$

Solution: $f'(x)=\cos x+\frac{1}{2\sqrt{x}}+\frac{1}{1+x^2}+e^{-x}+\frac{1}{x}$

(2) $f(x)=\frac{1}{x}+\cos x-\arcsin x+\sin 1+2^x$

Solution: $f'(x)=-\frac{1}{x^2}-\sin x-\frac{1}{\sqrt{1-x^2}}+2^x\ln 2$

(3) $y=(x^2+1)(x+5+\frac{1}{x})$

Solution: $y'=2x(x+5+\frac{1}{x})+(x^2+1)(1-\frac{1}{x^2})=3x^2+10x+3-\frac{1}{x^2}$

(4) $y=\frac{2x+5}{3x-2}$

Solution: $y'=\frac{2(3x-2)-3(2x+5)}{(3x-2)^2}=\frac{-19}{(3x-2)^2}$

(5) $y=e^{-x}\cos 2x$

Solution:

$$\frac{dy}{dx}=(\cos 2x)'e^{-x}+\cos 2x(e^{-x})'$$

$$= -2\sin 2x e^{-x} - \cos 2x e^{-x}$$

(6) $y = \sqrt{3x^2 - 4x + 6}$

Solution: $y' = \dfrac{3x-2}{\sqrt{3x^2-4x+6}}$

(7) $y = \dfrac{\ln t}{t}$

Solution: $y' = \dfrac{\frac{1}{t}t - \ln t}{t^2} = \dfrac{1-\ln t}{t^2}$

(8) $y = \sqrt{x(x+1)}$

Solution: $\ln y = \dfrac{1}{2}(\ln|x| + \ln|x+1|)$

$$\frac{y'}{y} = \frac{1}{2}\left(\frac{1}{x} + \frac{1}{1+x}\right)$$

$$y' = \frac{1}{2}\left(\frac{1}{x} + \frac{1}{1+x}\right)y = \frac{1}{2}\left(\frac{1}{x} + \frac{1}{1+x}\right)\sqrt{x(x+1)}$$

(9) $y = x^x$

Solution: $\ln y = x\ln x$

$$\frac{y'}{y} = \ln x + 1$$

$$y' = (\ln x + 1)y = (\ln x + 1)x^x$$

(10) If $f(x) = -x^2 + \dfrac{2}{\sqrt{x}} - \pi$, find $f'(x)$.

Solution: $f'(x) = -2x - x^{-\frac{3}{2}}$

(11) If $y = (2x+1)(\sqrt[3]{x} - x^3)$, find y'.

Solution:

$$f'(x) = 2(\sqrt[3]{x} - x^3) + (2x+1)(\frac{1}{3}x^{-\frac{2}{3}} - 3x^2)$$

$$= -8x^3 - 3x^2 + \frac{8}{3}\sqrt[3]{x} + \frac{1}{3}x^{-\frac{2}{3}}$$

(12) If $f(t)=\dfrac{t^2+2t}{\sqrt{t}-1}$, find $\dfrac{df}{dt}$.

Solution: $$\frac{df}{dt}=\frac{(2t+2)(\sqrt{t}-1)-(t^2+2t)\dfrac{1}{2\sqrt{t}}}{(\sqrt{t}-1)^2}=\frac{1.5t\sqrt{t}-2t+\sqrt{t}-2}{(\sqrt{t}-1)^2}$$

(13) If $\rho=\theta\sin\theta+\dfrac{1}{2}\cos\theta$, find $\left.\dfrac{d\rho}{d\theta}\right|_{\theta=\frac{\pi}{4}}$.

Solution: $$\frac{d\rho}{d\theta}=\sin\theta+\theta\cos\theta-\frac{1}{2}\sin\theta=\frac{1}{2}\sin\theta+\theta\cos\theta$$

$$\left.\frac{d\rho}{d\theta}\right|_{\theta=\frac{\pi}{4}}=\frac{\sqrt{2}}{4}+\frac{\sqrt{2}\pi}{8}$$

(14) $f(x)=\dfrac{1}{3^{x^2+2x}}+\log_2(1-2x+x^3)$

Solution:

$$f'(x)=\frac{1}{3^{x^2+2x}}\ln\frac{1}{3}(2x+2)+\frac{1}{(1-2x+x^3)\ln 2}(-2+3x^2)$$
$$=\frac{-2(x+1)\ln 3}{3^{x^2+2x}}+\frac{-2+3x^2}{(1-2x+x^3)\ln 2}$$

(15) $g(\mathrm{x})=3x\sqrt{2x^2+3}$

Solution:

$$g'(\mathrm{x})=3\sqrt{2x^2+3}+3x\times\frac{1}{2\sqrt{2x^2+3}}\times 4x=3\sqrt{2x^2+3}+\frac{6x^2}{\sqrt{2x^2+3}}$$
$$=\frac{12x^2+9}{\sqrt{2x^2+3}}$$

(16) $g(x)=x^{\cos x}+\arctan e^x$

Solution:

$$g'(x)=x^{\cos x}(-\sin x\ln x+\frac{\cos x}{x})+\frac{e^x}{1+e^{2x}}$$

(17) $h(\mathrm{x})=\left(\dfrac{1}{x^2}-5\right)^{-2}$

Solution: $$h'(\mathrm{x})=-2\left(\frac{1}{x^2}-5\right)^{-3}\times(-2)x^{-3}=4\left(\frac{1}{x^2}-5\right)^{-3}x^{-3}=4\left(\frac{1}{x}-5x\right)^{-3}$$

72. If $2y = 1 + xe^{xy}$, then $\left.\frac{dy}{dx}\right|_{x=0} = \underline{\frac{1}{2}}$

73. Given $y - xy^2 + x^2 + 1 = 0$. Find y'.

Solution:

We differentiate $F(x, y)$ implicitly, treating y as a function of x:

$$\frac{d}{dx}[y - xy^2 + x^2 + 1] = \frac{d}{dx}0$$

$$\frac{d}{dx}y - \frac{d}{dx}(xy^2) + \frac{d}{dx}x^2 + \frac{d}{dx}1 = 0$$

$$y' - (x \cdot 2yy' + y^2) + 2x = 0$$

$$y' - 2xyy' - y^2 + 2x = 0$$

$$(1 - 2xy)y' = y^2 - 2x$$

$$y' = \frac{y^2 - 2x}{1 - 2xy}$$

74. If r is a function of θ which is defined by the equation $\cos r + \cot\theta = e^{r\theta}$, find $\frac{dr}{d\theta}$.

Solution: Differentiate both sides w.r.t. θ, then

$$-\sin r\frac{dr}{d\theta} - \csc^2\theta = e^{r\theta}(\theta\frac{dr}{d\theta} + r)$$

$$-(\sin r + \theta e^{r\theta})\frac{dr}{d\theta} = re^{r\theta} + \csc^2\theta$$

$$\frac{dr}{d\theta} = -\frac{re^{r\theta} + \csc^2\theta}{\sin r + \theta e^{r\theta}}$$

75. If the equation $y - xe^y = 1$ define the function *y* of *x* implicitly, find $\left.y'\right|_{x=0}$.

Solution: Differentiate both sides of the equation with respect to *x*, we get

$$y' - e^y - xe^y y' = 0$$

then $y' = \frac{e^y}{1 - xe^y}$.

Substitute *x*=0 into the original equation, we have

$$y - 0e^y = 1$$

then $y=1$,

Substitute $x=0$ and $y|_{x=0}=1$ into the derivative, then

$$y'|_{x=0}=\left.\frac{e^y}{1-xe^y}\right|_{x=0}=e$$

Chapter Four

76. For $f(x)=\dfrac{x-1}{x^2-x+2}$, the y intercept of its graph is (C)

A. (1, 0) B. $(-0.5,\ 1)$ C. $(0,\ -0.5)$ D. (0, 1)

77. Let $g(x)=e^x-x-3$, the intervals where the graph of g is increasing is (A).

A. $(-\infty,0]$ B. $(-\infty,-3]$ C. $[0,+\infty)$ D. $(-\infty,+\infty)$

78. If (1, 3) is the inflection point of the curve $f(x)=ax^3+bx^2$, then (A)

A. $a=-\dfrac{3}{2},\ b=\dfrac{9}{2}$; B. $a=\dfrac{3}{2},\ b=-\dfrac{9}{2}$; C. $a=-\dfrac{3}{2},\ b=-\dfrac{9}{2}$; D. $a=\dfrac{3}{2},\ b=\dfrac{9}{2}$

79. Finding limits:

(1) $\lim\limits_{x\to 0}\dfrac{(1+x)^{\frac{1}{x}}-e}{x}$

Solution:

$$\lim_{x\to 0}\frac{(1+x)^{\frac{1}{x}}-e}{x}=\lim_{x\to 0}\frac{(1+x)^{\frac{1}{x}}\dfrac{\frac{1}{1+x}x-\ln(x+1)}{x^2}}{1}=\lim_{x\to 0}(1+x)^{\frac{1}{x}}\frac{\frac{1}{1+x}x-\ln(x+1)}{x^2}$$

$$=\lim_{x\to 0}(1+x)^{\frac{1}{x}}\lim_{x\to 0}\frac{x-(1+x)\ln(x+1)}{x^2(1+x)}=e\lim_{x\to 0}\frac{1-\ln(x+1)-1}{2x+3x^2}$$

$$=e\lim_{x\to 0}\frac{-\frac{1}{1+x}}{2+6x}=-\frac{1}{2}e$$

(2) $\lim\limits_{x\to 0}\dfrac{e^x-e^{-x}-2x}{x-\sin x}$

Solution:

$$\lim_{x\to 0}\frac{e^x-e^{-x}-2x}{x-\sin x}=\lim_{x\to 0}\frac{e^x+e^{-x}-2}{1-\cos x}=\lim_{x\to 0}\frac{e^x-e^{-x}}{\sin x}=\lim_{x\to 0}\frac{e^x+e^{-x}}{\cos x}=\frac{2}{1}=2$$

(3) $\lim_{x\to 1}\left(\frac{x}{x-1}-\frac{1}{\ln x}\right)$

Solution :

$$\lim_{x\to 1}\left(\frac{x}{x-1}-\frac{1}{\ln x}\right)=\lim_{x\to 1}\frac{x\ln x-x+1}{(x-1)\ln x}=\lim_{x\to 1}\frac{\ln x+1-1}{\ln x+\frac{x-1}{x}}=\lim_{x\to 1}\frac{x\ln x}{x\ln x+x-1}=\lim_{x\to 1}\frac{\ln x+1}{\ln x+2}=\frac{1}{2}$$

(4) Find $\lim_{x\to 0}\left(\frac{1}{e^x-1}-\frac{1}{x}\right)$

Solution:

$$\lim_{x\to 0}\left(\frac{1}{e^x-1}-\frac{1}{x}\right)=\lim_{x\to 0}\frac{x+1-e^x}{x(e^x-1)}=\lim_{x\to 0}\frac{1-e^x}{xe^x+e^x-1}$$

$$=\lim_{x\to 0}\frac{-e^x}{xe^x+2e^x}=-\frac{1}{2}$$

(5) $\lim_{x\to 0}\left(\frac{a_1^x+a_2^x}{2}\right)^{\frac{1}{x}}$ $\qquad (a_1>0, a_2>0)$

Solution:

$$\lim_{x\to 0}\left(\frac{a_1^x+a_2^x}{2}\right)^{\frac{1}{x}}=\lim_{x\to 0}e^{\frac{\ln\left(\frac{a_1^x+a_2^x}{2}\right)}{x}}$$

$$=e^{\lim_{x\to 0}\frac{\ln\left(\frac{a_1^x+a_2^x}{2}\right)}{x}}$$

$$=e^{\lim_{x\to 0}\frac{\frac{1}{\frac{a_1^x+a_2^x}{2}}\cdot\frac{a_1^x\ln a_1+a_2^x\ln a_2}{2}}{1}}$$

$$=e^{\frac{\ln a_1+\ln a_2}{2}}=e^{\ln\sqrt{a_1a_2}}=\sqrt{a_1a_2}$$

80. (1) Determine the intervals where $y=x^{\frac{1}{x}}$, $(x>0)$ is increasing and where is decreasing.

(2) Find the largest term of the sequence $\left\{\sqrt[n]{n}\right\}$

(3) Find $\lim_{x\to+\infty}x^{\frac{1}{x}}$

Solution: (1) Since

$$y' = x^{\frac{1}{x}} \frac{1-\ln x}{x^2},$$

then $y' > 0$, $x \in (0,e)$, therefore, $y = x^{\frac{1}{x}}$ is increasing on $(0,e)$;

$y' < 0$, $x \in (e,+\infty)$, therefore, $y = x^{\frac{1}{x}}$ is decreasing on $(e,+\infty)$.

(2) $\lim\limits_{x\to\infty} x^{\frac{1}{x}} = e^{\lim\limits_{x\to+\infty} \frac{\ln x}{x}} = e^0 = 1$;

(3) For the sequence $\left\{\sqrt[n]{n}\right\}$, it can attain its maximum value as n is close to e.

Since $2 < e < 3$, we have to compare $\sqrt{2}$, $\sqrt[3]{3}$

Since $8 = \left(\sqrt{2}\right)^6 < \left(\sqrt[3]{3}\right)^6 = 9$, then

$$\sqrt{2} < \sqrt[3]{3},$$

So the largest term of the sequence $\left\{\sqrt[n]{n}\right\}$ is $\sqrt[3]{3}$;

81. If the function $f(x) = \begin{cases} \dfrac{\ln(1+kx)}{2x}, & x \neq 0 \\ -1, & x = 0 \end{cases}$ is differentiable at $x = 0$, find k and $f'(0)$

Solution: Since $f(x)$ is differentiable at $x = 0$, then $f(x)$ is continuous at $x = 0$, therefore

$$\lim_{x\to 0} f(x) = f(0) = -1,$$

That is $\lim\limits_{x\to 0} f(x) = \lim\limits_{x\to 0} \dfrac{\ln(1+kx)}{2x} = -1,$

Since

$$\lim_{x\to 0} \frac{\ln(1+kx)}{2x} = \lim_{x\to 0} \frac{k}{2(1+kx)} = \frac{k}{2}$$

then $k = -2.$

$$f'(0) = \lim_{x\to 0} \frac{f(x) - f(0)}{x} = \lim_{x\to 0} \frac{\ln(1-2x) + 2x}{2x^2}$$

$$= \lim_{x\to 0} \frac{\dfrac{2}{2x-1} + 2}{4x}$$

$$= \lim_{x\to 0} \frac{-\dfrac{4}{(2x-1)^2}}{4} = -1$$

82. For $f(x)=-2x^3+6x^2-3$

(1) Find all critical points.
(2) Find the intervals on which the function is increasing and decreasing.
(3) Identify the function's local extreme values, if any, saying where they occur.
(4) Find where the graph of f is concave up and where it is concave down.
(5) Find all inflection points.

Solution: Since $f'(x)=-6x^2+12x=-6x(x-2)$

$$f''(x)=-12x+12=-12(x-1)$$

then let $f'(x)=-6x(x-2)=0$, we get $x_1=0,\ x_2=2$. They are **critical points.**

Let $f''(x)=-12(x-1)=0$, we get $x_3=1$.

x	$(-\infty, 0)$	0	(0, 1)	1	(1, 2)	2	$(2, +\infty)$
Sign of $f'(x)$	—	0	+		+	0	—
Sign of $f''(x)$	+		+	0	—		—
$f(x)$		—3		1		5	

The sign table indicate that $f(x)$ is increasing on [0, 2], decreasing on $(-\infty, 0]$ and $[2, +\infty)$. The local minimum is —3 at x=0, the local maximum is 5 at x=2.

From the sign table, we know the graph of $f(x)$ is concave up on $(-\infty, 1]$, and concave down on $[1, +\infty)$. The inflection point is (1, 1).

83. For $f(x)=\dfrac{x^2-4}{x^2-2}$

(1) Find all critical points.
(2) Find the open intervals on which the function is increasing and decreasing.
(3) Identify the function's local extreme values, if any, saying where they occur.
(4) Find where the graph of f is concave up and where it is concave down.
(5) Find all inflection points.
(6) Find all asymptotes.

Solution: The domain is $D=\{x\mid x\neq\pm\sqrt{2}\}$, since

$$f'(x)=\frac{2x(x^2-2)-2x(x^2-4)}{(x^2-2)^2}=\frac{4x}{(x^2-2)^2},$$

$$f''(x)=\frac{4(x^2-2)^2-4x\cdot4x(x^2-2)}{(x^2-2)^4}=\frac{-8-12x^2}{(x^2-2)^3}$$

then let $f'(x)=\dfrac{4x}{(x^2-2)^2}=0$, we get $x_1=0$. It is the only critical point.

Let $f''(x)=\dfrac{-8-12x^2}{(x^2-2)^3}=0$, there is no solution.

x	$(-\infty,-\sqrt{2})$	$(-\sqrt{2},\ 0)$	0	$(0,\sqrt{2})$	$(\sqrt{2},+\infty)$
Sign of $f'(x)$	—	—	0	+	+
Sign of $f''(x)$	+	—		—	+
$f(x)$			2		

The sign table indicate that $f(x)$ is increasing on $[0,\sqrt{2})$ and $(\sqrt{2},+\infty)$, decreasing on $(-\infty,-\sqrt{2})$ and $(-\sqrt{2},\ 0]$. The local minimum is 2 at x=0. No local maximum.

From the sign table, we know the graph of $f(x)$ is concave up on $(-\infty,-\sqrt{2})$ and $(\sqrt{2},+\infty)$, and concave down on $(-\sqrt{2},\ \sqrt{2})$. No inflection point.

Since

$$\lim_{x\to\infty} f(x)=\lim_{x\to\infty}\frac{x^2-4}{x^2-2}=1$$

then $y=1$ is the horizontal asymptote.

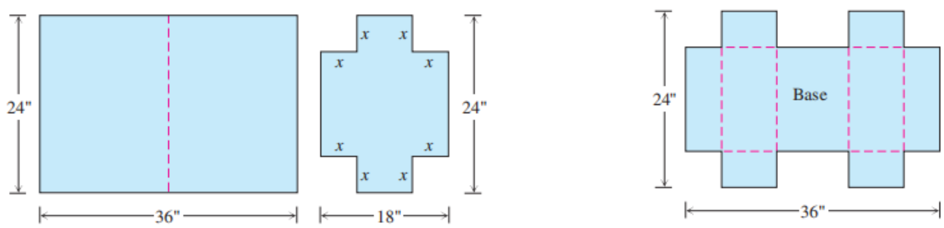
Since

$$\lim_{x\to\sqrt{2}^+}\frac{x^2-4}{x^2-2}=-\infty,\quad \lim_{x\to\sqrt{2}^-}\frac{x^2-4}{x^2-2}=+\infty$$

$$\lim_{x\to-\sqrt{2}^+}\frac{x^2-4}{x^2-2}=+\infty,\quad \lim_{x\to-\sqrt{2}^-}\frac{x^2-4}{x^2-2}=-\infty$$

then the lines $x=\sqrt{2}$ and $x=-\sqrt{2}$ are vertical asymptotes.

84. **Designing a suitcase** A 24-in.-by-36-in. sheet of cardboard is folded in half to form a 24-in.-by-18-in. rectangle as shown in the accompanying figure. Then four congruent squares of side length *x* are cut from the corners of the folded rectangle. The sheet is unfolded, and the six tabs are folded up to form a box with sides and a lid. Find the value of x such that the box holds as much as possible.



Solution: The area of the base is

$$A(x)=2x(24-2x)$$

The height of the box is $h=18-2x$, then the volume of the box is

$$V(x) = A(x)h = 2x(24-2x)(18-2x) = 8(x^3 - 21x^2 + 108x), \quad 0 < x < 9$$

Since

$$V'(x) = 8(3x^2 - 42x + 108) = 24(x^2 - 14x + 36) = 24(x - 7 + \sqrt{13})(x - 7 - \sqrt{13})$$

Then let $V'(x) = 24(x - 7 + \sqrt{13})(x - 7 - \sqrt{13}) = 0$, we get

$$x_1 = 7 - \sqrt{13}, \quad x_2 = 7 + \sqrt{13} \quad (reject)$$

When $0 < x < 7 - \sqrt{13}, \quad V'(x) > 0;$ when $7 - \sqrt{13} < x < 9, \quad V'(x) < 0;$ Then $V(x)$ attains its absolute maximum at $x = 7 - \sqrt{13}$.

Therefore, when $x = 7 - \sqrt{13}$, the volume of the box is maximum.

85. A rectangular storage container with an open top is to have a volume of 10 cubic meters. The length of its base is twice the width. Material for the base costs $10 per square meter. Material for the sides costs $6 per square meter. Find the cost of materials for the cheapest such container.

Solution: Let x be the width of the base, h be the height of the container, then the volume is

$$10 = x \cdot 2x \cdot h$$

The total cost of materials is

$$C = x \cdot 2x \cdot 10 + (2x + 2x + x + x) \cdot h \cdot 6 = 20x^2 + 36xh = 20x^2 + \frac{180}{x}, \quad x > 0$$

Since $C' = 40x - \frac{180}{x^2}$, then let $C' = 40x - \frac{180}{x^2} = 0$, then the critical value is

$$x = \sqrt[3]{\frac{9}{2}}$$

Since $C'' = 40 + \frac{360}{x^3} > 0, x > 0$, then C attains its absolute minimum value at $x = \sqrt[3]{\frac{9}{2}}$, and the absolute minimum value is

$$C(\sqrt[3]{\frac{9}{2}}) = \left(20x^2 + \frac{180}{x}\right)\Bigg|_{x=\sqrt[3]{\frac{9}{2}}} = \left(\frac{20x^3 + 180}{x}\right)\Bigg|_{x=\sqrt[3]{\frac{9}{2}}} = 270\sqrt[3]{\frac{2}{9}} = 90\sqrt[3]{6}.$$

86. If $f(x)$ is continuous on $[0,1]$, and differentiable on $(0,1)$,

$$f(0) = 0, \ f(1) = \frac{1}{2}, \ f\left(\frac{1}{2}\right) = 1,$$

then show that

(1) There exists at least a $\xi_1 \in (\frac{1}{2},\ 1)$ such that $f(\xi_1) = \xi_1$.

(2) There exists at least a $\xi \in (0,1)$ such that $f'(\xi) = 1$.

Proof: (1) Let $\varphi(x) = f(x) - x$, since $f(x)$ is continuous on $[0,1]$, then $\varphi(x)$ is continuous on $[\frac{1}{2},\ 1]$, and

$$\varphi(\frac{1}{2}) = f(\frac{1}{2}) - \frac{1}{2} = \frac{1}{2} > 0$$

$$\varphi(1) = f(1) - 1 = -\frac{1}{2} < 0$$

By the intermediate value theorem, there exists at least a $\xi_1 \in (\frac{1}{2},\ 1)$ such that

$$\varphi(\xi_1) = 0$$

That is, $f(\xi_1) = \xi_1$.

(2)Since $\varphi(x)$ is continuous on $[0,\ \xi_1]$, and differentiable on $[0,\ \xi_1]$, and

$$\varphi(0) = 0, \quad \varphi(\xi_1) = 0$$

By Rolle's theorem, there exists at least a $\xi \in (0,1)$ such that

$$\varphi'(\xi) = 0$$

That is, $f'(\xi) = 1$

87. If $f(x)$ is continuous on [0, 1], and differentiable on (0, 1), $f(0) = 0$, show that there exists at least a $\xi \in (0,1)$ such that

$$f'(\xi) = \frac{3f(\xi)}{1-\xi}.$$

Proof: Let $\varphi(x) = (1-x)^3 f(x)$, then $\varphi(x)$ is continuous on [0, 1], differentiable on (0, 1), and

$$\varphi(0) = \varphi(1) = 0.$$

By Rolle's Theorem, there exists at least a $\xi \in (0,1)$ such that

$$\varphi'(\xi) = -3(1-\xi)^2 f(\xi) + (1-\xi)^3 f'(\xi) = 0.$$

It is equivalent to

$$f'(\xi) = \frac{3f(\xi)}{1-\xi}.$$

Chapter Five

88. $\int (2x - \sqrt{x})dx = \underline{x^2 - \frac{2}{3}x^{\frac{3}{2}} + C}$

89. $\int (\frac{1}{x} - 3\sqrt{x} + \frac{1}{\sqrt{1-x^2}})dx = \underline{\ln|x| - 2x^{\frac{3}{2}} + \arcsin x + C}$

90. $\int (\sqrt{x} - \sin x + \frac{1}{1+x^2})dx = \underline{\frac{2}{3}x^{\frac{3}{2}} + \cos x + \arctan x + C}$

91. If $\int f(x)dx = 2xe^x + C$, then $f(x) =$ （ D ）

A. $2xe^x$ **;** **B.** $x\ e^x$ **;**

C. $x + e^x$ **D.** $2e^x(1+x)$

92. If $\boldsymbol{f(x)}$ is continuous, and $\int \boldsymbol{f(x)dx = F(x)+C}$, then (C)

A. $\int \boldsymbol{f(2x)dx = F(2x)+C}$; B. $\int \boldsymbol{f(x^2)x\,dx = F(x^2)+C}$;

C. $\int \boldsymbol{f(e^x)e^x\,dx = F(e^x)+C}$; D. $\int \boldsymbol{f(\cos x)\sin x\,dx = F(\cos x)+C}$

93. If $f'(e^x) = x$, then $f(e^x) =$ （ D ）

A. $\frac{1}{2}x^2$ B. $\frac{1}{2}x^2 + C$ C. $xe^x + e^x + C$ D. $xe^x - e^x + C$

94. If $\frac{\sin x}{x}$ is one anti-derivative of $f(x)$, then $\int xf'(x)dx =$ (C)

A. $\frac{\sin x}{x} + C$; B. $\frac{1+\sin x}{x^2} + C$; C. $\cos x - \frac{2\sin x}{x} + C$; D. $\cos x + \frac{2\sin x}{x} + C$

95. Find the following integrals.

(1) $\int (\sqrt{x} - \sin x + 2^x - \frac{1}{1+x^2} + 1)dx$

Solution: $\int(\sqrt{x}-\sin x+2^{x}-\frac{1}{1+x^{2}}+1)dx=\frac{2}{3}x^{\frac{3}{2}}+\cos x+\frac{2^{x}}{\ln 2}-\arctan x+x+C$

(2) $\int(\frac{1}{x}-\cos x+x-\frac{1}{\sqrt{1-x^{2}}}+2)dx$

Solution: $\int(\frac{1}{x}-\cos x+x-\frac{1}{\sqrt{1-x^{2}}}+2)dx=\ln|x|-\sin x+\frac{x^{2}}{2}-\arcsin x+2x+C$

(3) $\int(\frac{2}{x}+\frac{1}{\sqrt{x}}-\sin x+2-\frac{1}{1+x^{2}})\,dx$

Solution:

$\int(\frac{2}{x}+\frac{1}{\sqrt{x}}-\sin x+2-\frac{1}{1+x^{2}})\,dx = 2\ln|x|+2\sqrt{x}+\cos x+2x-\arctan x+c$

(4) $\int(2\cos 2x-3\sin 3x)dx$

Solution: $\int(2\cos 2x-3\sin 3x)dx=\sin 2x+\cos 3x+C$

96. Finding the following indefinite integrals:

(1) $\int\frac{12+5t-3t^{2}}{t^{3}}dt$

Solution: $\int\frac{12+5t-3t^{2}}{t^{3}}dt=-\frac{6}{t^{2}}-\frac{5}{t}-3\ln|\mathrm{t}|+\mathrm{C}$

(2) $\int\frac{2x+2}{x^{2}+2x}dx$

Solution: $\int\frac{2x+2}{x^{2}+2x}dx=\ln|x^{2}+2x|+\mathrm{C}$

(3) $\int 2t^{2}(t^{3}+4)^{-2}dt$

Solution: $\int 2t^{2}(t^{3}+4)^{-2}dt=-\frac{2}{3(t^{3}+4)}+C$

(4) $\int x\sqrt{x-1}dx$

Solution: Let $t=\sqrt{x-1}$, then $x=t^{2}+1$, $dx=2tdt$,

$$\int x\sqrt{x-1}dx = \int (t^2+1)t \times 2tdt$$
$$= \int (2t^4+2t^2)dt$$
$$= \frac{2}{5}t^5 + \frac{2}{3}t^3 + C = \frac{2}{5}(x-1)^{\frac{5}{2}} + \frac{2}{3}(x-1)^{\frac{3}{2}} + C$$

(5) $\int \frac{x^2}{\sqrt{4x^3-1}}dx$

Solution:

$$\int \frac{x^2}{\sqrt{4x^3-1}}dx = \frac{1}{12}\int \frac{12x^2}{\sqrt{4x^3-1}}dx = \frac{1}{12}2\sqrt{4x^3-1}+C$$
$$= \frac{1}{6}\sqrt{4x^3-1}+C$$

97. Finding the following indefinite integrals:

(1) $\int \frac{e^x}{e^x+1}dx$

Solution: $\int \frac{e^x}{e^x+1}dx = \ln(e^x+1)+C$

(2) $\int \frac{1}{1+e^x}\,dx$

Solution:

$$\int \frac{1}{1+e^x}\,dx = \int \frac{1+e^x-e^x}{1+e^x}\,dx = \int (1-\frac{e^x}{1+e^x})dx = x-\ln(1+e^x)+C$$

(3) Find $\int x\arctan x dx$

Solution:

$$\int x\arctan x dx = \frac{1}{2}x^2\arctan x - \int \frac{1}{2}x^2\frac{1}{1+x^2}\,dx$$
$$= \frac{1}{2}x^2\arctan x - \frac{1}{2}\int (1-\frac{1}{1+x^2})dx$$
$$= \frac{1}{2}x^2\arctan x - \frac{1}{2}(x-\arctan x)+C$$

(4) $\int \frac{x+2}{x^2-2x+5}dx$

Solution:

$$\int \frac{x+2}{x^2-2x+5}dx = \int \frac{\frac{1}{2}(2x-2)+3}{x^2-2x+5}dx$$

$$= \frac{1}{2}\int \frac{2x-2}{x^2-2x+5}dx + 3\int \frac{1}{x^2-2x+5}dx$$

$$= \frac{1}{2}\int \frac{1}{x^2-2x+5}d(x^2-2x+5) + 3\int \frac{1}{(x-1)^2+4}d(x-1)$$

$$= \frac{1}{2}\ln(x^2-2x+5) + \frac{3}{2}\arctan\frac{x-1}{2} + C$$

(5) $\int \frac{x+\sqrt{\arctan x}}{1+x^2}dx$

Solution:

$$\int \frac{x+\sqrt{\arctan x}}{1+x^2}dx = \int \frac{x}{1+x^2}dx + \int \frac{\sqrt{\arctan x}}{1+x^2}dx$$

$$= \frac{1}{2}\int \frac{1}{1+x^2}d(1+x^2) + \int \sqrt{\arctan x}d\arctan x$$

$$= \frac{1}{2}\ln(1+x^2) + \frac{2}{3}(\arctan x)^{\frac{3}{2}} + C$$

(6) Find $\int x\arctan xdx$

Solution:

$$\int x\arctan xdx = \frac{1}{2}x^2\arctan x - \int \frac{1}{2}x^2\frac{1}{1+x^2}dx \qquad \text{(3 marks)}$$

$$= \frac{1}{2}x^2\arctan x - \frac{1}{2}\int (1-\frac{1}{1+x^2})dx \qquad \text{(4 marks)}$$

$$= \frac{1}{2}x^2\arctan x - \frac{1}{2}(x-\arctan x) + C \qquad \text{(6 marks)}$$

(7) $\int \frac{\arctan e^x}{e^x}\mathrm{d}x$

Solution:

$$\int \frac{\arctan e^x}{e^x}\mathrm{d}x = -e^{-x}\arctan e^x + \int e^{-x}\frac{e^x}{1+e^{2x}}dx$$

$$= -e^{-x}\arctan e^x + \int (1-\frac{e^{2x}}{1+e^{2x}})dx$$

$$= -e^{-x}\arctan e^{x} + x - \frac{1}{2}\ln(1+e^{2x}) + C$$

(8) $\frac{\sin x}{x}$ is one anti-derivative of $f(x)$, find $\int xf'(x)dx$

Solution:

$$\int xf'(x)dx = xf(x) - \int f(x)dx = x\left(\frac{\sin x}{x}\right)' - \frac{\sin x}{x} + C$$

$$= \frac{x\cos x - \sin x}{x} - \frac{\sin x}{x} + C = \frac{x\cos x - 2\sin x}{x} + C$$