## Chapter 5 Decision Tree

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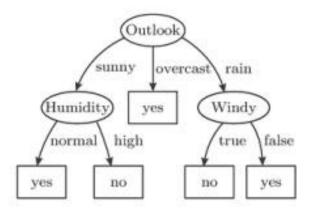
### Content

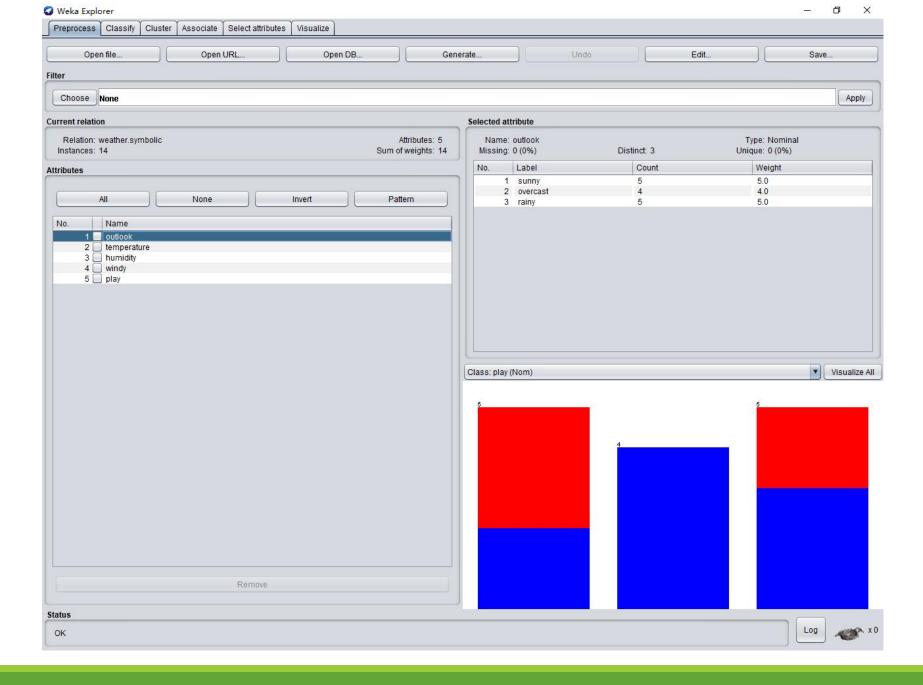
- 5.1 Basic Procedure
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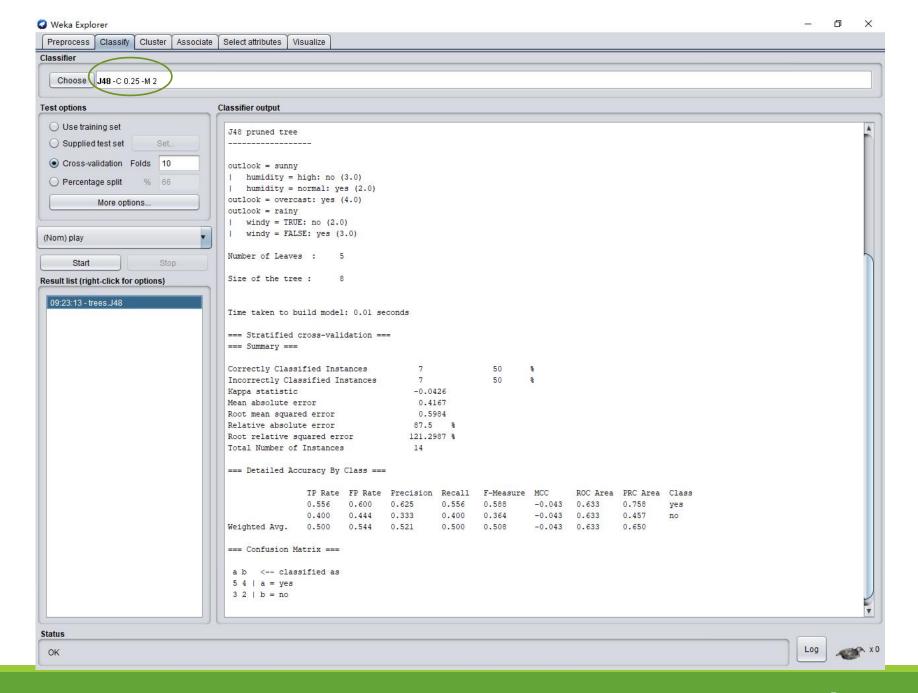
## 5.1 Basic Procedure

- Idea: make decisions based on a tree
  - Internal node: test on some attribute(s)
  - Branch: some value of an attribute
  - Leaf: decision result

Outlook	Temp	Humidity	Windy	Golf?
rainy	hot	high	false	no
rainy	hot	high	true	no
overcast	hot	high	false	yes
sunny	mild	high	false	yes
sunny	cool	normal	false	yes
sunny	cool	normal	true	no
overcast	cool	normal	true	yes
rainy	mild	high	false	no
rainy	cool	normal	false	yes
sunny	mild	normal	false	yes
rainy	mild	normal	true	yes
overcast	mild	high	true	yes
overcast	hot	normal	false	yes
sunny	mild	high	true	no





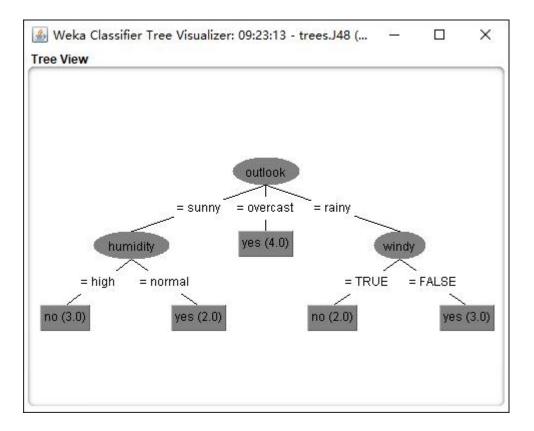


#### @relation weather.symbolic

@attribute outlook {sunny, overcast, rainy}
@attribute temperature {hot, mild, cool}
@attribute humidity {high, normal}
@attribute windy {TRUE, FALSE}
@attribute play {yes, no}

#### @data

sunny, hot, high, FALSE, no sunny, hot, high, TRUE, no overcast, hot, high, FALSE, yes rainy, mild, high, FALSE, yes rainy, cool, normal, FALSE, yes rainy, cool, normal, TRUE, no overcast, cool, normal, TRUE, yes sunny, mild, high, FALSE, no sunny, cool, normal, FALSE, yes rainy, mild, normal, FALSE, yes sunny, mild, normal, TRUE, yes overcast, mild, high, TRUE, yes overcast, hot, normal, FALSE, yes rainy, mild, high, TRUE, no



=== Stratified cross-validation ===

=== Summary ===

Correctly Classified Instances	7		50	olo
Incorrectly Classified Instances	7		50	%
Kappa statistic	-0.0426			
Mean absolute error	0.4167			
Root mean squared error	0.5984			
Relative absolute error	87.5	96		
Root relative squared error	121.2987	do		
Total Number of Instances	14			

## **Brief History**

- CLS (Concept Learning System)
  - The first decision tree learning algorithm, 1966
- ID3
  - Begin to be the mainstream technology in ML, 1979
- C4.5
  - The most commonly used algorithm, 1993
- CART (Classification And Regression Tree), 1984
- RF (Random Forest)
  - The most powerful decision tree algorithm, 2001

## Basic algorithm

(the ID3 algorithm)

- 1. If all the instances are from exactly one class, then the decision tree is an answer node containing that class name.
- 2. Otherwise,
  - (a) Define  $a_{best}$  to be an attribute with
  - (b) For each value  $v_{best,i}$  of  $a_{best}$ , grow a branch from  $a_{best}$  to a decision tree constructed recursively from all those instances with value  $v_{best,i}$  of attribute  $a_{best}$ .

## 5.2 Attribute Selection

- Information gain (信息增益)
- Gain ratio (增益率)
- Gini index (基尼指数)

## Information gain

- Information entropy
  - Measure the purity of a data set
  - The smaller the entropy, the higher the purity
  - Definition: data set D, |Y| classes, the ratio of the  $k^{\text{th}}$  class  $p_k$

$$\operatorname{Ent}(D) = -\sum_{k=1}^{|\mathcal{Y}|} p_k \log_2 p_k$$

 Information gain: to evaluate the change on the information entropy caused by a candidate partition

# Information gain

- Information gain
  - all values on a discrete attribute  $\{a^1, a^2, ..., a^v\}$
  - D<sup>v</sup>: the subset where each sample takes the value a<sup>v</sup> on a
  - The gain of the partition on a:

$$\operatorname{Gain}(D,a) = \operatorname{Ent}(D) - \sum_{v=1}^{V} \frac{|D^v|}{|D|} \operatorname{Ent}(D^v)$$

Best attribute:

$$a_* = \underset{a \in A}{\operatorname{arg max}} \operatorname{Gain}(D, a)$$
 The ID3 algorithm

The split using the feature windy results in two children nodes, one for a windy value of true and one for a windy value of false. In this data set, there are six data points with a true windy value, three of which have a play value of yes and three with a play value of no. The eight remaining data points with a windy value of false contain two no's and six yes's. The information of the windy=true node is calculated using the entropy equation above. Since there is an equal number of yes's and no's in this node, we have

$$I_E([3,3]) = -rac{3}{6}\log_2rac{3}{6} - rac{3}{6}\log_2rac{3}{6} = -rac{1}{2}\log_2rac{1}{2} - rac{1}{2}\log_2rac{1}{2} = 1$$

For the node where windy=false there were eight data points, six yes's and two no's. Thus we have

$$I_E([6,2]) = -\frac{6}{8}\log_2\frac{6}{8} - \frac{2}{8}\log_2\frac{2}{8} = -\frac{3}{4}\log_2\frac{3}{4} - \frac{1}{4}\log_2\frac{1}{4} = 0.8112781$$

To find the information of the split, we take the weighted average of these two numbers based on how many observations fell into which node.

$$I_E([3,3],[6,2]) = I_E( ext{windy or not}) = rac{6}{14} \cdot 1 + rac{8}{14} \cdot 0.8112781 = 0.8921589$$

To find the information gain of the split using windy, we must first calculate the information in the data before the split. The original data contained nine ves s and five no s.

$$I_E([9,5]) = -rac{9}{14}\log_2rac{9}{14} - rac{5}{14}\log_2rac{5}{14} = 0.940286$$

Now we can calculate the information gain achieved by splitting on the windy feature.

$$IG(\text{windy}) = I_E([9,5]) - I_E([3,3],[6,2]) = 0.940286 - 0.8921589 = 0.0481271$$

Outlook	Temp	Humidity	Windy	Golf?
rainy	hot	high	false	no
rainy	hot	high	true	no
overcast	hot	high	false	yes
sunny	mild	high	false	yes
sunny	cool	normal	false	yes
sunny	cool	normal	true	no
overcast	cool	normal	true	yes
rainy	mild	high	false	no
rainy	cool	normal	false	yes
sunny	mild	normal	false	yes
rainy	mild	normal	true	yes
overcast	mild	high	true	yes
overcast	hot	normal	false	yes
sunny	mild	high	true	no

## Gain ratio

- Information gain prefers to an attribute which has more distinct values, while gain ratio prefers to fewer values
- Definition:

$$\text{Gain\_ratio}(D,a) = \frac{\text{Gain}(D,a)}{\text{IV}(a)} \qquad \text{IV}(a) = -\sum_{v=1}^V \frac{|D^v|}{|D|} \log_2 \frac{|D^v|}{|D|}$$

The C4.5 algorithm: first Gain(D, a) > average;
 then the biggest gain ratio.

## Gini index

- Measure the probability that two randomly chosen samples belong to different classes
- The smaller the Gini, the higher the purity
- Definition:

$$\begin{aligned} \operatorname{Gini}(D) &= \sum_{k=1}^{|\mathcal{Y}|} \sum_{k' \neq k} p_k p_{k'} \\ &= 1 - \sum_{k=1}^{|\mathcal{Y}|} p_k^2 \ . \end{aligned}$$

$$Gini\_index(D, a) = \sum_{v=1}^{V} \frac{|D^v|}{|D|} Gini(D^v)$$
 $a_* = \underset{a \in A}{a \in A} Gini\_index(D, a).$ 

The CART algorithm

	sur	nny
	Y	2
class	N	3

Gini (sunny) = 1 - 
$$(\frac{2}{5})^2$$
 -  $(\frac{3}{5})^2$  = 0.48

	ra	in
	Y	3
class	N	2

Gini (rain) = 1 - 
$$(\frac{3}{5})^2$$
 -  $(\frac{2}{5})^2$  = 0.48

	over	cast
,	Y	4
class	N	0

Gini (overcast) = 1 - 
$$(\frac{4}{4})^2$$
 -  $(\frac{0}{4})^2$  = 0

Gini<sub>split</sub> (Outlook) = 
$$\frac{5}{14}$$
 0.48 +  $\frac{5}{14}$  0.48 +  $\frac{4}{14}$  0 = 0.343

$$\operatorname{Gini\_index}(D,a) = \sum_{v=1}^{V} \frac{|D^v|}{|D|} \operatorname{Gini}(D^v)$$

Outlook	Temp	Humidity	Windy	Golf?
rainy	hot	high	false	no
rainy	hot	high	true	no
overcast	hot	high	false	yes
sunny	mild	high	false	yes
sunny	cool	normal	false	yes
sunny	cool	normal	true	no
overcast	cool	normal	true	yes
rainy	mild	high	false	no
rainy	cool	normal	false	yes
sunny	mild	normal	false	yes
rainy	mild	normal	true	yes
overcast	mild	high	true	yes
overcast	hot	normal	false	yes
sunny	mild	high	true	no

Attribute	Feature	Y	N	GIN I (i)	GIN I <sub>split</sub>	
	sunny	2	3	0.48		
Outlook	rain	3	2	0.48	0.343	
	overcasi	4	0	u		
	hot	2	-2	0.5	0.44	
Temperature	cool	3	1	0.375		
	mild	4	2	0.444		
	high	3	4	0.490	0.368	
Humidity	normal	6	1	0. 245		
	false	6	2	0.375	0 420	
Windy	true	3	3	0.5	0.429	

# 5.4 Continuous Value and Missing Value

- Bi-partition (二分法) for continuous value
  - Sort *n* distinct values on a continuous attribute  $\{a^1, a^2, ..., a^n\}$

$$D_{\scriptscriptstyle t}^{\scriptscriptstyle +}$$
 and  $D_{\scriptscriptstyle t}^{\scriptscriptstyle -}$ 

Split D into

w.r.t. a splitting point *t* 

Candidate

$$T_a = \left\{ \frac{a^i + a^{i+1}}{2} \mid 1 \leqslant i \leqslant n - 1 \right\}$$

$$Gain(D, a) = \max_{t \in T_a} Gain(D, a, t)$$

Choose the best i

$$= \max_{t \in T_a} \operatorname{Ent}(D) - \sum_{\lambda \in \{-,+\}} \frac{|D_t^{\lambda}|}{|D|} \operatorname{Ent}(D_t^{\lambda})$$

1. (10 points) Consider the following data set. Each training example has the form of (x, y). The input x has three attributes: refund (categorical), marital status (categorical), taxable income (continuous). The output y = cheat belongs to the set  $\{yes, no\}$ .

Tid	Refund	Marita I Status	Taxable Income	Cheat
1	Yes	Single	125K	No
2	No	Married	100K	No
3	No	Single	70K	No
4	Yes	Married	120K	No
5	No	Divorced	95K	Yes
6	No	Married	60K	No
7	Yes	Divorced	220K	No
8	No	Single	85K	Yes
9	No	Married	75K	No
10	No	Single	90K	Yes

- a) (4 points) According to the information gain, which attribute is the first attribute chosen to construct the best decision tree? Why?
- b) (4 points) Draw the whole decision tree learned from the data set.
- c) (2 points) Given the following testing example, what is the value of *cheat* according to the learned decision tree?

$$\operatorname{Gain}(D,a) = \operatorname{Ent}(D) - \sum_{v=1}^V \frac{|D^v|}{|D|} \operatorname{Ent}(D^v) \qquad \qquad \operatorname{Ent}(D) = -\sum_{k=1}^{|\mathcal{Y}|} p_k \log_2 p_k$$
, where

**Here, Ent(D)**= 
$$-\left(\frac{3}{10}\log\frac{3}{10} + \frac{7}{10}\log\frac{7}{10}\right) = 0.8813$$

For the taxable income attribute, since it is continuous, we should determine the best splitting value.

First, sort all values on taxable income:

{60, 70, 75, 85, 90, 95, 100, 120, 125, 220}**₽** 

Second, according to the bi-partition method, candidates of splitting points are:

$$T = \{65, 72.5, 80, 87.5, 92.5, 107.5, 110, 122.5, 127.5\}$$

The formula is:+

$$\begin{aligned} \operatorname{Gain}(D, a) &= \max_{t \in T_a} \ \operatorname{Gain}(D, a, t) \\ &= \max_{t \in T_a} \ \operatorname{Ent}(D) - \sum_{\lambda \in \{-, +\}} \frac{|D_t^{\lambda}|}{|D|} \operatorname{Ent}(D_t^{\lambda}) \end{aligned}$$

Gain(D, "Taxable Income", t=72.5)=0.1178+

Gain(D, "Taxable Income", t=87.5)=0.0058

Gain(D, "Taxable Income", t=92.5)=0.0348

Gain(D, "Taxable Income", t=97.5)=0.2813

Gain(D, "Taxable Income", t=110)=0.1916

Gain(D, "Taxable Income", t=122.5)=0.1178

Gain(D, "Taxable Income", t=127.5)=0.0548

The best splitting point is t = 97.5,

Therefore, the first partition attribute is marital status or Taxable Income < 97.5.

Gain(D, "Taxable Income", t=65)=0.0548

Gain(D, "Taxable Income", t=80)=0.1916

0.88-[0.9\*(-3/9log3/9-6/9log6/9)+0.1\*0]=0.0548 0.88-[0.8\*(-5/8loq5/8-3/8loq3/8)+0.2\*0]=0.1178 Taxable

Income

125K

100K

70K

120K

95K

60K

220K

85K

75K

90K

Cheat

No

No

No

No

Yes

No

No

Yes

No

Yes

Refund

Yes

No

No

Yes

No

No

Yes

No

No

No

6

8

9

10

Marital

Status

Single

Married

Single

Married

Divorced

Divorced

Single

Married

Single

Married

#### @relation weather

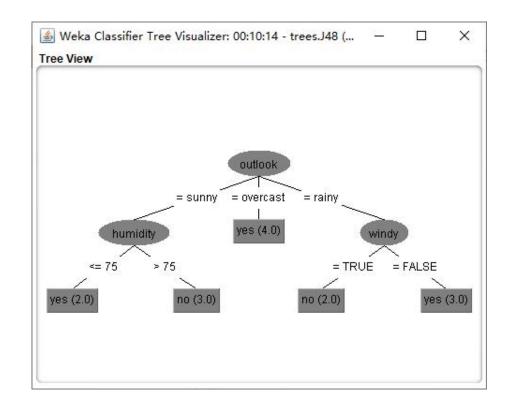
@attribute outlook {sunny, overcast, rainy}

@attribute temperature numeric @attribute humidity numeric

@attribute windy {TRUE, FALSE}
@attribute play {yes, no}

#### @data

sunny, 85, 85, FALSE, no sunny, 80, 90, TRUE, no overcast, 83, 86, FALSE, yes rainy, 70, 96, FALSE, yes rainy, 68, 80, FALSE, yes rainy, 65, 70, TRUE, no overcast, 64, 65, TRUE, yes sunny, 72, 95, FALSE, no sunny, 69, 70, FALSE, yes rainy, 75, 80, FALSE, yes rainy, 75, 70, TRUE, yes overcast, 72, 90, TRUE, yes overcast, 81, 75, FALSE, yes rainy, 71, 91, TRUE, no



=== Stratified cross-validation ===

=== Summary ===

Correctly Classified Instances	9	64.2857 %
Incorrectly Classified Instances	5	35.7143 %
Kappa statistic	0.186	
Mean absolute error	0.2857	
Root mean squared error	0.4818	
Relative absolute error	60 %	
Root relative squared error	97.6586 %	
Total Number of Instances	14	

### 5.3 Continuous Value and Missing Value

Weight for missing value

•  $D \subset D$  : set where each sample has a value on a

 $\begin{array}{ll} {}^\bullet \tilde{D}^V \subseteq \tilde{D} & \text{: set where each sample takes value } a^V \text{ on } a \\ {}^\bullet \tilde{D}_k \subseteq \tilde{D} & \text{: set where each sample belongs to the } k^{\text{th}} \end{array}$ 

$$\tilde{D} = \bigcup_{k=1}^{|\mathcal{Y}|} \tilde{D}_k, \qquad \tilde{D} = \bigcup_{v=1}^{V} \tilde{D}^v.$$

$$\sum_{k=1}^{|\mathcal{Y}|} \tilde{p}_k = 1, \sum_{v=1}^{V} \tilde{r}_v = 1.$$

$$\rho = \frac{\sum_{\boldsymbol{x} \in \tilde{D}} w_{\boldsymbol{x}}}{\sum_{\boldsymbol{x} \in D} w_{\boldsymbol{x}}},$$

$$\tilde{p}_k = \frac{\sum_{\boldsymbol{x} \in \tilde{D}_k} w_{\boldsymbol{x}}}{\sum_{\boldsymbol{x} \in \tilde{D}} w_{\boldsymbol{x}}} \quad (1 \leqslant k \leqslant |\mathcal{Y}|),$$

$$\tilde{r}_v = \frac{\sum_{\boldsymbol{x} \in \tilde{D}^v} w_{\boldsymbol{x}}}{\sum_{\boldsymbol{x} \in \tilde{D}} w_{\boldsymbol{x}}} \quad (1 \leqslant v \leqslant V).$$

# 5.3 Continuous Value and Missing Value

Information gain:

$$\begin{aligned} \operatorname{Gain}(D,a) &= \rho \times \operatorname{Gain}(\tilde{D},a) & \operatorname{Ent}(\tilde{D}) &= -\sum_{k=1}^{|\mathcal{Y}|} \tilde{p}_k \log_2 \tilde{p}_k \\ &= \rho \times \left( \operatorname{Ent}\left(\tilde{D}\right) - \sum_{v=1}^{V} \tilde{r}_v \operatorname{Ent}\left(\tilde{D}^v\right) \right) \end{aligned}$$

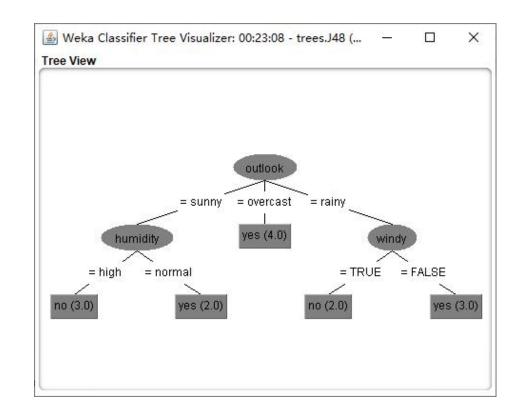
- Reset the weight  $w_x$  of sample x if need:
  - If x has some value on a, just keep W<sub>x</sub>
  - Otherwise, first join x into each node corresponding to  $a^v$  and then set the weight of x to

#### brelation weather.symbolic

@attribute outlook {sunny, overcast, rainy}
@attribute temperature {hot, mild, cool}
@attribute humidity {high, normal}
@attribute windy {TRUE, FALSE}
@attribute play {yes, no}

#### @data

sunny, ?, high, FALSE, no
sunny, hot, high, TRUE, no
overcast, ?, high, FALSE, yes
rainy, mild, high, FALSE, yes
rainy, cool, normal, FALSE, yes
rainy, cool, normal, TRUE, no
overcast, cool, normal, TRUE, yes
sunny, mild, high, FALSE, no
sunny, cool, normal, FALSE, yes
rainy, ?, normal, FALSE, yes
sunny, mild, normal, TRUE, yes
overcast, mild, high, TRUE, yes
overcast, hot, normal, FALSE, yes
rainy, mild, high, TRUE, no



=== Stratified cross-validation === === Summarv ===

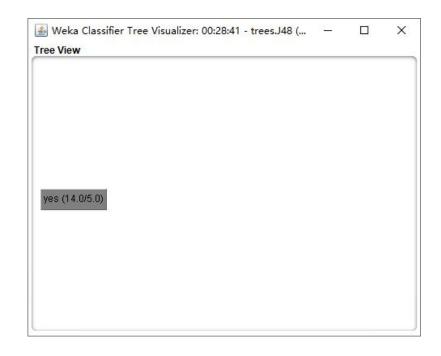
Correctly Classified Instances	8	57.1429 %
Incorrectly Classified Instances	6	42.8571 %
Kappa statistic	0.1429	
Mean absolute error	0.369	
Root mean squared error	0.5405	
Relative absolute error	77.5 %	
Root relative squared error	109.5471 %	
Total Number of Instances	14	

#### @relation weather.symbolic

@attribute outlook {sunny, overcast, rainy}
@attribute temperature {hot, mild, cool}
@attribute humidity {high, normal}
@attribute windy {TRUE, FALSE}
@attribute play {yes, no}

#### @data

?, ?, high, FALSE, no
sunny, hot, high, TRUE, no
overcast, ?, high, FALSE, yes
?, mild, high, FALSE, yes
?, cool, normal, FALSE, yes
rainy, cool, normal, TRUE, no
overcast, cool, normal, TRUE, yes
?, mild, high, FALSE, no
sunny, cool, normal, FALSE, yes
rainy, ?, normal, FALSE, yes
?, mild, normal, TRUE, yes
?, mild, high, TRUE, yes
overcast, hot, normal, FALSE, yes
rainy, mild, high, TRUE, no



=== Summary ===			
Correctly Classified Instances	7	50	90
Incorrectly Classified Instances	7	50	of
Kappa statistic	-0.1395		
Mean absolute error	0.5403		
Root mean squared error	0.5727		
Relative absolute error	113.4615 %		
Root relative squared error	116.0707 %		
Total Number of Instances	14		

## 5.4 Random Forest

Algorithm 15.1 Random Forest for Regression or Classification.

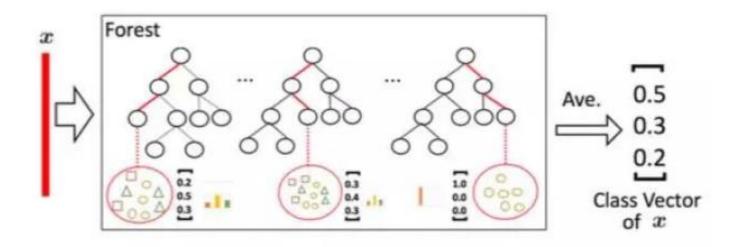
- 1. For b = 1 to B:
  - (a) Draw a bootstrap sample  $\mathbb{Z}^*$  of size N from the training data.
  - (b) Grow a random-forest tree  $T_b$  to the bootstrapped data, by recursively repeating the following steps for each terminal node of the tree, until the minimum node size  $n_{min}$  is reached.
    - i. Select m variables at random from the p variables.
    - ii. Pick the best variable/split-point among the m.
    - iii. Split the node into two daughter nodes.
- 2. Output the ensemble of trees  $\{T_b\}_1^B$ .

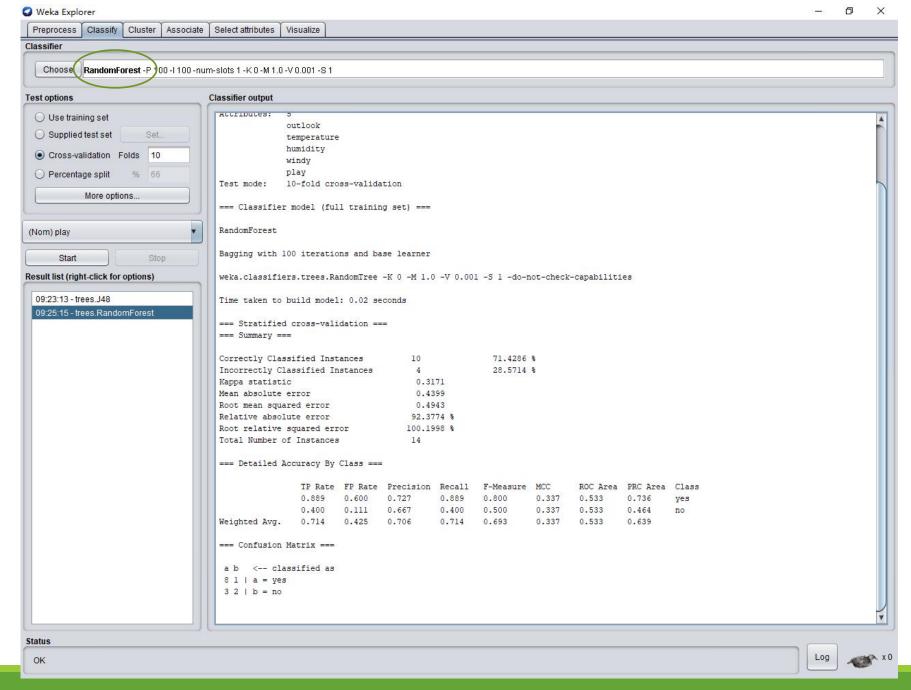
To make a prediction at a new point x:

Regression: 
$$\hat{f}_{rf}^B(x) = \frac{1}{B} \sum_{b=1}^B T_b(x)$$
.

Classification: Let  $\hat{C}_b(x)$  be the class prediction of the bth random-forest tree. Then  $\hat{C}_{rf}^B(x) = majority\ vote\ \{\hat{C}_b(x)\}_1^B$ .

# Example





## 4.4 Random Forests

RF vs. other ensembles

