Special functions and dual special functions in Drinfeld modules of arbitrary rank

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Abstract

In a previous paper ([Fer22]) the author showed in the context of Drinfeld-Hayes modules that the product between a special function and a Pellarin zeta function is rational, and that the latter can be interpreted as a dual special function. Both results are generalized in this paper.

We use a very simple functorial point of view to interpret special functions and dual special functions in Drinfeld modules of arbitrary rank, allowing us to define a universal special function ω_ϕ and a universal dual special function ζ_ϕ. In analogy to the Drinfeld-Hayes case, we prove that the latter can be expressed as an Eisenstein-like series over the period lattice, and that the scalar product $\omega_\phi \cdot \zeta_\phi$, an element of $\mathbb{C}_\infty \hat{\otimes} \Omega$, is rational.

We also describe the module of special functions in the generality of Anderson modules, as already done by Gazda and Maurischat in [GM20], and answer a question posed in that same paper about the invertibility of special functions in the context of Drinfeld-Hayes modules.

1. Introduction

Let $\mathbb{F}_q$ be the finite field with q elements, and let X be a projective, geometrically irreducible, smooth curve of genus g over $\mathbb{F}_q$, with a point $\infty \in X(\mathbb{F}_q)$. We call $A := \mathcal{O}_X(X \setminus \{\infty\})$, Ω its module of differentials, K its fraction field, K_∞ the completion of K with respect to the ∞-adic norm $\|\cdot\|$, and $\mathbb{C}_\infty$ the completion of the algebraic closure K_∞^{ac}.

A *Drinfeld module* ϕ of rank r is a graded homomorphism of rings $A \to \mathbb{C}_\infty\{\tau\}$ of degree r, where τ is the Frobenius endomorphism, such that for all $a \in A$ the constant term of ϕ_a is a. There is a unique A-module $\Lambda \subseteq \mathbb{C}_\infty$ of rank r, with an associated (surjective) exponential map $\exp_\Lambda := x \prod_{\lambda \in \Lambda \setminus \{0\}} \left(1 - \frac{x}{\lambda}\right) \in \mathbb{C}_\infty[[x]]$, such that $\phi_a \circ \exp_\Lambda(\cdot) = \exp_\Lambda(a \cdot)$.

The special case of the *Carlitz module*, with $A = \mathbb{F}_q[\theta]$ and $\phi_\theta = \theta + \tau$, is the easiest to study. The Anderson-Thakur special functions, introduced in [AT90], are defined as the series $\sum_{i \geqslant 0} c_i t^i$ in the Tate algebra $\mathbb{C}_\infty\langle t \rangle$ such that

$$\sum_i \phi_\theta(c_i) t^i = \sum_i c_i t^{i+1};$$

they form a free A-module of rank 1, generated by some ω, which holds a lot of information: for example, as shown in [AP14], ω is connected to the explicit class field theory of $\mathbb{F}_q(\theta)$, and its $\mathbb{F}_q^{ac}$-rational values interpolate Gauss-Thakur sums. In [Pel11], Pellarin proved the following

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identity relating ω to a zeta-like function:

$$\frac{t-\theta}{\omega} = \sum_{p \in \mathbb{F}_q[T]\setminus\{0\}} \frac{p(t)}{p(\theta)}.$$

The "special functions" (as defined in [ANT17]) generalize Anderson-Thakur functions to any Drinfeld module ϕ: if we denote by $\mathbb{C}_\infty \hat{\otimes} A$ the completion of $\mathbb{C}_\infty \otimes A$ with respect to the sup norm, the module of special functions $Sf_\phi(A)$ is the submodule $\{\omega \in \mathbb{C}_\infty \hat{\otimes} A | \phi_a(\omega) = (1 \otimes a)\omega \ \forall a \in A\}$, where ϕ_a acts trivially on the second component. The Pellarin zeta function has an obvious generalization as an object of $\mathbb{C}_\infty \hat{\otimes} A$: $\zeta_A := \sum_{a \in A\setminus\{0\}} a^{-1} \otimes a$; it was conjectured that the relation between Pellarin zeta functions and special functions could be generalized at least to Drinfeld modules of rank 1.

Until recently, the only result in this direction had been a paper by Green and Papanikolas ([GP16]), who studied the case $g(X) = 1$ with some assumptions on the Drinfeld divisor, exploiting the nice structure of elliptic curves. In a previous paper ([Fer22]), the author proved for all Drinfeld modules of rank 1 that the product between a special function and the Pellarin zeta function is in the fraction field of $\mathbb{C}_\infty \otimes A$, and explicitly described its divisor and the scalar proportionality constant ([Fer22][Thm. 6.3]). In the same paper, up to a slight change in the definition of ζ_A, the following identities are also proved for all $a \in A$: $\phi_a^*(\zeta_A) = (1 \otimes a)$, where $\phi^* : A \to \mathbb{C}_\infty\{\tau^{-1}\}$ is the dual Drinfeld module ([Fer22][Prop. 7.7]); this allows us to interpret ζ_A as a dual special function.

The aim of this paper is to generalize the main results of [Fer22] to a Drinfeld module ϕ of arbitrary rank, with period lattice Λ. It turns out that the algebra $\mathbb{C}_\infty \hat{\otimes} A$ is not the correct environment for (dual) special functions, and a functorial point of view is needed both to define a "canonical" (dual) special function and to formulate the correct generalization of [Fer22][Thm 6.3].

The first observation is that (dual) special functions can be easily defined as elements of $\mathbb{C}_\infty \hat{\otimes} M$ for any A-module M, and that this definition is functorial in M. In Section 2, we provide a crucial alternative interpretation for the "completed tensor product" $\hat{\otimes}$: we prove that $\mathbb{C}_\infty \hat{\otimes} M$ is canonically isomorphic to the set of continuous $\mathbb{F}_q$-linear functions from the Pontryagin dual of M, $\hat{M}$, to $\mathbb{C}_\infty$ (Lemma 2.13). The submodule of special functions is then $\mathrm{Hom}_A^{cont}(\hat{M}, \mathbb{C}_\infty{}^\phi)$, which we prove in Section 3 to be representable: it turns out that the universal object ω_ϕ, which we call *universal special function*, lies in $\mathrm{Hom}_A^{cont}\left(K_\infty\Lambda/\Lambda, \mathbb{C}_\infty{}^\phi\right)$, and is none other than the exponential function.

Similarly, in Section 5, we prove that the functor of dual special functions is represented by Λ, and that the *universal dual special function* $\zeta_\phi \in \mathbb{C}_\infty \hat{\otimes} \Lambda$ is the Eisenstein-like series $\sum_{\lambda \in \Lambda\setminus\{0\}} \lambda^{-1} \otimes \lambda$, which is the correct generalization of the Pellarin zeta function.

Finally, in Section 6, we define the canonical pairing $ev : (\mathbb{C}_\infty \hat{\otimes} \Lambda) \otimes (\mathbb{C}_\infty \hat{\otimes} \mathrm{Hom}_A(\Lambda, \Omega)) \to \mathbb{C}_\infty \hat{\otimes} \Omega$ and prove that $ev(\zeta_\phi \otimes \omega_\phi)$ is in fact a rational differential form. This is actually not a proper generalization of [Fer22][Thm. 6.3], because we do not fully describe this differential form; on the other hand, it provides an arguably more natural language for the problem, which, in the case $A = \mathbb{F}_q[\theta]$, is enough to prove that $ev(\zeta_\phi \otimes \omega_\phi) = d\theta$.

Since Gazda and Maurischat were able to describe the module of special functions in the full generality of Anderson modules in [GM20], in Section 3 we work in the same generality and come to some of the same conclusions. Moreover, in section 7, we answer affirmatively to a question they posed in their paper about the invertibility of special functions in the context of Drinfeld modules of rank 1. Both those results, while not necessary for the proof of the generalization of [Fer22][Thm. 6.3], are included to show the explainatory power of the language used in this paper.

2. Some results concerning Pontryagin duality

In this paper, compact and locally compact spaces are always assumed to be Hausdorff.

Definition 2.1 (Pontryagin duality). Call $\mathbb{S}^1 \subseteq \mathbb{C}^\times$ the unit circle. The Pontryagin duality is the contravariant functor from the category of topological abelian groups to itself, sending M to $\hat{M} := \mathrm{Hom}_{\mathbb{Z}}^{cont}(M, \mathbb{S}^1)$.

Remark 2.2. If M is an A-module, we can endow $\hat{M}$ with a natural structure of A-module. Moreover, since M is an $\mathbb{F}_q$-vector space, we have the following natural isomorphisms of topological A-modules:

$$\hat{M} := \mathrm{Hom}_{\mathbb{Z}}^{cont}(M, S^1) = \mathrm{Hom}_{\mathbb{Z}}^{cont}(M, \mathbb{F}_p) = \mathrm{Hom}_{\mathbb{F}_p}^{cont}(M, \mathbb{F}_p) = \mathrm{Hom}_{\mathbb{F}_q}^{cont}(M, \mathbb{F}_q).$$

The following is a well known result about Pontryagin duality, rewritten in the case of A-modules, which we do not prove:

Proposition 2.3. *For any topological A-module M, there is a natural morphism $M \to \hat{\hat{M}}$; if M is locally compact, $\hat{M}$ is locally compact, and the previous morphism is an isomorphism.*

Moreover, if M is compact (resp. discrete) $\hat{M}$ is discrete (resp. compact).

A notable result about the Pontryagin duality of A-modules is the following (see [Poo96, Theorem 8]).

Theorem 2.4 (Poonen). *The computation of the residue at ∞ induces a perfect pairing between $\Omega \otimes_A K_\infty$ and K_∞, which restricts to a perfect pairing between the discrete A-module Ω and the compact A-module K_∞/A. In other words, $\Omega \widehat{\otimes_A K_\infty} \cong K_\infty$ and $\hat{\Omega} \cong {}^{K_\infty}\!/_{A}$.*

Remark 2.5. For any discrete projective A-module Λ of finite rank r with $\Lambda^* := \mathrm{Hom}_A(\Lambda, A)$, we have the following isomorphisms of topological A-modules:

$$\widehat{\Lambda^* \otimes_A \Omega} = \mathrm{Hom}_{\mathbb{F}_q}(\Lambda^* \otimes_A \Omega, \mathbb{F}_q) \cong \mathrm{Hom}_A(\Lambda^*, \mathrm{Hom}_{\mathbb{F}_q}(\Omega, \mathbb{F}_q)) \cong \Lambda \otimes_A (K_\infty/A).$$

Retracing the isomorphisms, it's easy to check that the pairing map between $\Lambda^* \otimes_A \Omega$ and $\Lambda \otimes_A K_\infty/A$ sends $(\lambda^* \otimes \omega, \lambda \otimes b)$ to the residue of $\omega\lambda^*(\lambda)b$ at ∞.

Remark 2.6. For any discrete $\mathbb{F}_q$-vector space M, if we fix an isomorphism $\mathbb{F}_q^{\oplus I} \cong M$, i.e. an $\mathbb{F}_q$-basis $(m_i)_{i\in I}$, we induce an isomorphism of topological vector spaces between $\mathbb{F}_q^I$ and $\hat{M}$.

Definition 2.7. If M is a discrete $\mathbb{F}_q$-vector space with basis $(m_i)_{i\in I}$, we denote for all $i \in I$ m_i^* the image of $(\delta_{i,j})_{j\in I} \in \mathbb{F}_q^I$, so that for all $j \in I$ $m_i^*(m_j) = \delta_{i,j}$. We call $(m_i^*)_{i\in I}$ a *dual basis* of $\hat{M}$.

Remark 2.8. In the previous definition, a generic element $f \in \hat{M}$ corresponds to $(f(m_i))_i \in \mathbb{F}_q^I$. It's immediate to check that, for all $m \in M$, $f(m) = \sum_{i\in I} f(m_i)m_i^*(m)$, which is actually a finite sum, hence we are justified to use the following formal notation: $f = \sum_{i\in I} f(m_i)m_i^*$. The existence and uniqueness of this expression for all $f \in \hat{M}$ explains the nomenclature "dual basis" for $(m_i^*)_i$.

DEFINITION 2.9. Let M and N be topological $\mathbb{F}_q$-vector space with N locally compact. We denote $M\hat{\otimes}N := \mathrm{Hom}^{cont}_{\mathbb{F}_q}(\hat{N}, M)$.

LEMMA 2.10. *For any locally compact A-modules M, N, there is a natural isomorphism of $A\otimes A$-modules between $M\hat{\otimes}N$ and $N\hat{\otimes}M$.*

Proof. By Proposition 2.3, the Pontryagin duality induces an antiequivalence of the category of locally compact A-modules with itself, hence:

$$\mathrm{Hom}^{cont}_{\mathbb{F}_q}(\hat{N}, M) \cong \mathrm{Hom}^{cont}_{\mathbb{F}_q}(\hat{M}, \hat{\hat{N}}) \cong \mathrm{Hom}^{cont}_{\mathbb{F}_q}(\hat{M}, N);$$

$A \otimes A$-linearity is a simple check. $\square$

We now show an alternative way to think about the tensor product $\hat{\otimes}$, which makes our notation agree with the usual notation for the Tate algebra used in [GM20], [Fer22], and others.

DEFINITION 2.11. Let C be a complete topological $\mathbb{F}_q$-vector space with open subspaces $\{U_j\}_{j\in J}$ as a fundamental system of open neighborhoods of 0, and let M be a discrete $\mathbb{F}_q$-vector space. We denote $C\tilde{\otimes}M$ the completion of $C \otimes M$ with respect to $\{U_j \otimes M\}_{j\in J}$.

Remark 2.12. For all $j \in J$, since U_j is an $\mathbb{F}_q$-vector space, if $y \in C \setminus U_j$, $y + U_j \cap U_j = \emptyset$, hence U_j is also closed and C/U_j is discrete. This means that an equivalent condition is for C to be the limit of discrete $\mathbb{F}_q$-vector spaces, which is satisfied by any compact $\mathbb{F}_q$ vector space.

Moreover, being a nonarchimedean complete normed vector space over $\mathbb{F}_q$, $\mathbb{C}_\infty$ satisfies the conditions on C, as we can set $J = \mathbb{Z}$ and U_n the ball of radius $\frac{1}{q^n}$ for all integers n.

LEMMA 2.13. *There is a natural isomorphism of $\mathbb{F}_q$-vector spaces $\Phi : C\tilde{\otimes}M \to C\hat{\otimes}M$. Moreover, if C and M are A-modules, it is an isomorphism of $A \otimes A$-modules.*

Proof. Fix an $\mathbb{F}_q$-basis $(m_i)_{i\in I}$ of M; any $x \in C\tilde{\otimes}M$ can be expressed in a unique way as $\sum_{i\in I} c_i \otimes m_i$, where $c_i \in C$ for all $i \in I$, and for all $j \in J$ the set $I_j := \{i \in I | c_i \notin U_j\}$ is finite. For all $f \in \hat{M}$, we set:

$$\Phi(x)(f) := \lim_{\substack{J\subseteq I \\ \#J<\infty}} \sum_{i\in J} f(m_i)c_i,$$

which is well defined because C is complete and, for all $j \in J$, I_j is finite and U_j is an $\mathbb{F}_q$-vector space. The map $\Phi(x)$ is obviously $\mathbb{F}_q$-linear; for all $j \in J$, consider $V_j := \{f \in \hat{M} | f(m_i) = 0 \ \forall i \in I_j\} \subseteq \hat{M}$: it is a neighborhood of 0, and is contained in $\Phi(x)^{-1}(U_j)$, hence $\Phi(x)$ is also continuous. The map Φ is manifestly $\mathbb{F}_q$-linear, and, if C and M are A-modules, $A \otimes A$-linear, so we just need to prove bijectivity.

On one hand, if $\Phi(x) \equiv 0$, we have $0 = \Phi(x)(m_i^*) = c_i$ for all $i \in I$, hence $x = 0$. On the other hand, if $g : \hat{M} \to C$ is a continuous function, and since $\hat{M}$ is compact, for all $j \in J$ the set $\{i \in I | g(m_i^*) \notin U_j\}$ is finite, hence $y := \sum_i g(m_i^*) \otimes m_i$ is an element of $C\tilde{\otimes}M$; since $\Phi(y)(m_j^*) = g(m_j^*)$ for all j, $\Phi(y) = g$. $\square$

3. Special functions

An *Anderson module* (E, ϕ) over $\mathbb{C}_\infty$ of dimension d consists of a $\mathbb{C}_\infty$-group scheme $E \cong \mathbb{G}_a^d$ and an $\mathbb{F}_q$-linear action ϕ of A over E such that $\partial(\phi_a - a) : \mathrm{Lie}(E) \to \mathrm{Lie}(E)$ is nilpotent for all $a \in A$. Fixing an isomorphism $E \cong \mathbb{G}_a^d$, we can interpret ϕ as a graded ring homomorphism $A \to \mathrm{End}_n(\mathbb{C}_\infty)\{\tau\}$ where τ is the Frobenius endomorphism.

Fix an Anderson A-module (E,ϕ); call $\exp_E : \mathrm{Lie}(E)(\mathbb{C}_\infty) \to E(\mathbb{C}_\infty)$ the associated exponential function and Λ_E its kernel, which we know is a projective A-module of finite rank. Endow $\mathrm{Lie}(E)(\mathbb{C}_\infty)$ and $E(\mathbb{C}_\infty)$ with their natural topology of finite $\mathbb{C}_\infty$-vector spaces and with the structure of A modules coming respectively from multiplication and ϕ; this makes $\exp_E$ into a morphism of topological A-modules.

Definition 3.1. The *special functions functor* $Sf_\phi : A-\mathbf{Mod} \to A-\mathbf{Mod}$ is defined by sending a discrete A-module M to $Sf_\phi(M) := \mathrm{Hom}_A^{cont}(\hat{M}, E(\mathbb{C}_\infty)) \subseteq E(\mathbb{C}_\infty)\hat{\otimes} M$.

Remark 3.2. By Lemma 2.13, our definition of $E(\mathbb{C}_\infty)\hat{\otimes}A$ coincides with the one given in [GM20]. The A-module $Sf_\phi(A)$ is the subset of $E(\mathbb{C}_\infty)\hat{\otimes}A$ of the elements on which the left and right A-actions coincide, hence it's the same as the module of special functions defined in [GM20].

The following proposition examines the functor Sf_ϕ.

Proposition 3.3. *Suppose that E is uniformizable. The functor Sf_ϕ is naturally isomorphic to $\mathrm{Hom}_A(\Lambda_E^* \otimes_A \Omega, \_)$, and the universal object in $\mathrm{Hom}_A^{cont}\left({K_\infty\Lambda_E}/{\Lambda_E}, E(\mathbb{C}_\infty)\right)$ sends $c \in K_\infty\Lambda_E$ to $\exp_E(c)$.*

Proof. Since E is uniformizable, $E(\mathbb{C}_\infty)$ is isomorphic to ${\mathrm{Lie}(E)(\mathbb{C}_\infty)}/{\Lambda_E}$ as a topological A-module. Since $K_\infty\Lambda_E \subseteq \mathrm{Lie}(E)(\mathbb{C}_\infty)$ is a finite K_∞-vector space, it admits a topological complement C, which induces an isomorphism $E(\mathbb{C}_\infty) \cong {K_\infty\Lambda_E}/{\Lambda_E} \bigoplus C$. For any discrete A-module M, for any $\omega \in Sf_\phi(M)$, its projection $\overline{\omega}$ onto $C\hat{\otimes}M$ is in $\mathrm{Hom}_A^{cont}(\hat{M}, C)$; since $\hat{M}$ is compact, the image of $\overline{\omega}$ must be a compact sub-A-module of C, but since $C \subseteq \mathrm{Lie}(E)(\mathbb{C}_\infty)$, for any $c \in C \setminus \{0\}$ the set $A \cdot c$ is unbounded, hence the only compact sub-A-module is the trivial one. We deduce the following natural isomorphisms:

$$Sf_\phi(M) = \mathrm{Hom}_A^{cont}(\hat{M}, E(\mathbb{C}_\infty)) \cong \mathrm{Hom}_A^{cont}\left(\hat{M}, {K_\infty\Lambda_E}/{\Lambda_E}\right) \cong \mathrm{Hom}_A(\Lambda_E^* \otimes_A \Omega, M),$$

where we used Lemma 2.10 for the second isomorphism.

Setting $M := \Lambda_E^* \otimes_A \Omega$, and following the identity along the chain of isomorphisms, we deduce that the universal object in $\mathrm{Hom}_A^{cont}\left({K_\infty\Lambda_E}/{\Lambda_E}, E(\mathbb{C}_\infty)\right)$ is the continuous A-linear map sending $c \in K_\infty\Lambda_E$ to $\exp_E(c)$. $\square$

For the sake of completeness, let's prove a statement which does not assume uniformizability.

Proposition 3.4. *If we restrict the functor Sf_ϕ to the subcategory of torsionless A-modules, it is naturally isomorphic to $\mathrm{Hom}_A(\Lambda_E^* \otimes_A \Omega, \_)$.*

Proof. The map $\exp_E$ is open because its Jacobian at all points is the identity; call C its image. Since C is an $\mathbb{F}_q$-vector space, for all $y \in E(\mathbb{C}_\infty) \setminus C$, the open set $y + C$ does not intersect C, hence C is also closed. In particular, the quotient ${E(\mathbb{C}_\infty)}/{C}$ is a discrete A-module.

A discrete A-module M is torsionless if and only if it has no nontrivial compact submodules; in this case, $\hat{M}$ is a compact A-module with no nontrivial discrete quotients. In particular, any function $f \in Sf_\phi(M) = \mathrm{Hom}_A^{cont}(\hat{M}, E(\mathbb{C}_\infty))$, projected onto ${E(\mathbb{C}_\infty)}/{C}$, is trivial, hence its image must be contained in C. The rest of the proof is the same as Proposition 3.3 up to substituting $E(\mathbb{C}_\infty)$ with C. $\square$

Corollary 3.5. *The following isomorphism of A-modules holds:*

$$Sf_\phi(A) = \{\omega \in E(\mathbb{C}_\infty)\hat{\otimes}A | \phi_a(\omega) = (1 \otimes a)\omega \ \forall a \in A\} \cong \Omega^* \otimes_A \Lambda_E.$$

Remark 3.6. If we fix an $\mathbb{F}_q$-basis $\{\mu_i\}_i$ of $\mathrm{Hom}_A(\Lambda_E, \Omega)$, with $\{\mu_i^*\}$ dual basis of ${}^{K_\infty \Lambda_E}\!/\!_{\Lambda_E}$, by Remark 2.12 and Lemma 2.13, we can express the universal object in the following alternative way as an element of $E(\mathbb{C}_\infty)\hat{\otimes}\,\mathrm{Hom}_A(\Lambda_E, \Omega)$:

$$\exp_E\left(\sum_i \mu_i^* \otimes \mu_i\right) = \sum_i \exp_E(\mu_i^*) \otimes \mu_i.$$

4. Two notable infinite series in $\mathbb{C}_\infty$

Fix a finitely generated projective A-module $\Lambda \subseteq \mathbb{C}_\infty$ of rank r.

Remark 4.1. Since A is a Dedekind domain, every finitely generated and torsion-free A-module is projective, so the latter condition is superfluous.

We denote $\exp_\Lambda \in \mathbb{C}_\infty\{\{\tau\}\}$ the exponential function relative to Λ, and $\exp_\Lambda^* \in \mathbb{C}_\infty\{\{\tau^{-1}\}\}$ the dual exponential function.

We now follow a construction due to Poonen, who proved a duality result of central importance to this section ([Poo96][Theorem 10]).

Definition 4.2. For all $\beta \in \ker(\exp_\Lambda^*) \setminus \{0\}$, call $g_\beta \in \mathbb{C}_\infty\{\{\tau\}\}$ the unique function such that $\exp_\Lambda^* \circ \beta = g_\beta^* \circ (1 - \tau^{-1})$.

Remark 4.3. This definition is well posed for all $\beta \in \ker(\exp_\Lambda^*) \setminus \{0\}$, because $\mathbb{F}_q \subseteq \ker(\exp_\Lambda^* \circ \beta)$. Moreover, since $(1 - \tau) \circ g_\beta = \beta \exp_\Lambda$, $g_\beta|_\Lambda$ has image in $\mathbb{F}_q$.

Theorem 4.4 (Poonen). *The function* $\ker(\exp_\Lambda^*) \to \widehat{\ker(\exp_\Lambda)}$ *sending* β *to* g_β *is an isomorphism of topological* A*-modules, where* A *acts via* ϕ^* *on the LHS and via multiplication on the RHS.*

We now state the main propositions which will be proven in this section.

Proposition 4.5. *For all* $\beta \in \ker(\exp_\Lambda^*) \setminus \{0\}$*, the following identity holds in* $\mathbb{C}_\infty$*:*

$$\beta = -\sum_{\lambda \in \Lambda \setminus \{0\}} \frac{g_\beta(\lambda)}{\lambda}.$$

The previous proposition can be viewed as an explicit formula for the inverse of Poonen's isomorphism in Theorem 4.4.

Proposition 4.6. *For all* $c \in K_\infty \setminus \{0\}$ *with* $\|c\| < 1$*, the following identity holds in* $\mathbb{C}_\infty$*:*

$$c = -\sum_{\lambda \in \Lambda \setminus \{0\}} \frac{\exp_\Lambda(c\lambda)}{\lambda}.$$

4.1 Locally finite subspaces

Definition 4.7. A *locally finite subspace* of $V \subseteq \mathbb{C}_\infty$ is an $\mathbb{F}_q$-vector space such that for any positive real number r there are finitely many elements of V of norm at most r.

An *ordered basis* of V is a sequence $(v_i)_{i \geqslant 1}$ with the following property: for all $m \geqslant 1$, v_m is an element of $V \setminus \mathrm{Span}(\{v_i\}_{i<m})$ of least norm.

We call the sequence of real numbers $(\|v_i\|)_{i \geqslant 1}$ the *norm sequence* of V.

The next two results aim to justify the nomenclature "ordered basis" and the well-posedness of the norm sequence.

Lemma 4.8. *If $(v_i)_{i\geqslant 1}$ is an ordered basis of a locally finite subspace $V\subseteq\mathbb{C}_\infty$, it is a basis of V as an $\mathbb{F}_q$-vector space.*

Proof. For all $m\geqslant 1$ $v_m\notin\mathrm{Span}(\{v_i\}_{i\leqslant m-1})$, hence the v_i's are $\mathbb{F}_q$-linearly independent. Since for all $r\in\mathbb{R}$ there is a finite number of elements of V with norm at most r, the norm sequence $\{\|v_i\|\}_{i\geqslant 1}$ tends to infinity; in particular, for all $v\in V$ there is an integer m such that $\|v_m\|>\|v\|$, so $v\in V_{m-1}$ by definition of v_m. □

Proposition 4.9. *If $(v_i)_{i\geqslant 1}$ is an ordered basis of a locally finite subspace $V\subseteq\mathbb{C}_\infty$, and $(v'_i)_{i\geqslant 1}$ is a sequence of elements in V that are $\mathbb{F}_q$-linearly independent and with increasing norm, $\|v'_i\|\geqslant\|v_i\|$ for all i. In particular, the norm sequence of V does not depend on the ordered basis of V.*

Proof. If for some m $\|v'_m\|<\|v_m\|$, for all $i\leqslant m$ $\|v'_i\|\leqslant\|v'_m\|<\|v_m\|$, so $v'_i\in V_{m-1}$; since V_m has dimension $m-1$ as an $\mathbb{F}_q$-vector space, we reach a contradiction. If we take $(v'_i)_i$ to be another ordered basis, by this reasoning we get both $\|v'_m\|\geqslant\|v_m\|$ and $\|v_m\|\geqslant\|v'_m\|$, hence the norm sequence is independent from the choice of the ordered basis. □

Finally, we show that the norm sequence is reasonably well behaved with regard to subspaces.

Lemma 4.10. *Take two locally finite subspaces $W\subseteq V\subseteq\mathbb{C}_\infty$ with $\dim_{\mathbb{F}_q}\left(V/W\right)=1$, with norm sequences respectively $(s_i)_{i\geqslant 1}$ and $(r_i)_{i\geqslant 1}$. Then, there is a positive integer m such that for all $i<m$ $s_i=r_i$ and for all $i\geqslant m$ $s_i=r_{i+1}$.*

Proof. Let's fix an ordered basis $\{w_i\}_{i\geqslant 1}$ of W and an element $u\in V\setminus W$ of least norm. Let m be the least positive integer such that $\|w_m\|>\|u\|$, and define the following vectors v_i for all positive integers i:

$$v_i=\begin{cases} w_i \text{ if } i<m\\ u \text{ if } i=m\\ w_{i-1} \text{ if } i>m\end{cases}.$$

We want to prove that the sequence $(v_i)_{i\geqslant 1}$ is an ordered basis of V, i.e. that, for all k, v_k is an element of least norm not contained in $\mathrm{Span}(\{v_i\}_{i<k})$. For $k\leqslant m$ it's obvious by the definition of u and $(w_i)_{i\geqslant 0}$. For $k>m$, $v_{k+1}=w_k$ is an element of W of least norm not contained in $\mathrm{Span}(\{w_i\}_{i<k})$, and is also not contained in $\mathrm{Span}(\{v_i\}_{i<k+1})=\mathrm{Span}(\{w_i\}_{i<k}\cup\{u\})$. We need to prove that any element of $V\setminus W$ not contained in $\mathrm{Span}(\{v_i\}_{i<k+1})$ has norm at least $\|w_k\|$; since $\dim_{\mathbb{F}_q}\left(V/W\right)=1$, it can be written as $u+w$ with $w\in W$ and $w\notin\mathrm{Span}(\{v_i\}_{i<k+1})$, so $\|w\|\geqslant\|w_k\|>\|u\|$ and $\|u+w\|=\|w\|\geqslant\|w_k\|$.

As a consequence, for all $i<m$ $s_i=\|w_i\|=\|v_i\|=r_i$ and for all $i\geqslant m$ $s_i=\|w_i\|=\|v_{i+1}\|=r_{i+1}$. □

4.2 Estimation of the coefficients of g_β and $\exp_\Lambda$

The following result is similar to a well known lemma (see [Gos12][Lemma 8.8.1]).

Lemma 4.11. *Call $S_{n,d}(x_1,\dots,x_n)\in\mathbb{F}_q[x_1,\dots,x_n]$ the sum of the d-th powers of all the homogeneous linear polynomials. Suppose that the coefficient of monomial $x_1^{d_1}\cdots x_n^{d_n}$ in the expansion of $S_{n,d}(x_1,\dots,x_n)$ is nonzero: then, for all $1\leqslant j\leqslant n$, $\sum_{i=1}^j d_i\geqslant q^j-1$. In particular, if $d<q^n-1$, $S_{n,d}=0$.*

Proof. The coefficient $c_{d_1,\dots,d_n}$ of the monomial $x_1^{d_1}\cdots x_n^{d_n}$ is:

$$\frac{d!}{d_1!\cdots d_n!}\sum_{a_1,\dots,a_n\in\mathbb{F}_q} a_1^{d_1}\cdots a_n^{d_n} = \frac{d!}{d_1!\cdots d_n!}\prod_{i=1}^n\left(\sum_{a_i\in\mathbb{F}_q} a_i^{d_i}\right),$$

where by convention we set $0^0=1$. On one hand, the multinomial coefficient $\frac{d!}{d_1!\cdots d_n!}$ is nonzero in $\mathbb{F}_q$ if and only if $C(d)=C(d_1)+\cdots+C(d_n)$, where we denote by $C(m)$ the sum of the digits in base q of the nonnegative integer m; in particular, for $1\leqslant j\leqslant n$ this implies $C(d_1+\cdots+d_j)=C(d_1)+\cdots+C(d_j)$. On the other hand, $\sum_{a_i\in\mathbb{F}_q} a_i^{d_i}\neq 0$ if and only if $d_i>0$ and $q-1|d_i$; in particular, this implies $C(d_i)\geqslant q-1$ for all i.

If $c_{d_1,\dots,d_n}\neq 0$, for $1\leqslant j\leqslant n$ we have:

$$C\left(\sum_{i=1}^j d_i\right)=\sum_{i=1}^j C(d_i)\geqslant (q-1)j,$$

hence $\sum_{i=1}^j d_i\geqslant q^j-1$. Applying this to $j=n$ we get the condition $d\geqslant q^n-1$, therefore $S_{n,d}=0$ for all $d<q^n-1$. □

DEFINITION 4.12. For a locally finite subspace $V\subseteq\mathbb{C}_\infty$, for all integers $i\geqslant 0$ we define:

$$e_{V,i} := \sum_{\substack{I\subseteq V\setminus 0\\ |I|=q^i-1}}\prod_{v\in I} v^{-1}$$

(by convention, $e_{V,0}=1$).

Remark 4.13. For all $c\in\mathbb{C}_\infty$, since $V\subseteq\mathbb{C}_\infty$ is locally finite, the infinite product $c\prod_{v\in V}\left(1-\frac{c}{v}\right)$ converges in $\mathbb{C}_\infty$, and is equal to $\sum_{n\geqslant 0} e_{V,n}c^{q^n}$. In particular, $\sum_{n\geqslant 0} e_{V,n}x^{q^n}\in\mathbb{C}_\infty[[x]]$ is the only analytic function which converges everywhere, whose zeroes are simple and coincide with V, and with leading coefficient equal to 1.

LEMMA 4.14. *Fix a locally finite subspace $V\subseteq\mathbb{C}_\infty$, and an ordered basis $(v_i)_{i\geqslant 1}$. Call $V_m :=$ $\mathrm{Span}(\{v_i\}_{i\leqslant m})$ for all $m\geqslant 0$, and define $r_i=\|v_i\|$ for all $i\geqslant 1$. We have:*

- *for all $k\geqslant 0$:*

$$\|e_{V,k}\|\leqslant\prod_{i=1}^k r_i^{q^{i-1}-q^i};$$

- *for all $m>0$, for all $k>0$:*

$$\left\|\sum_{v\in V_m} v^{q^k-1}\right\|\begin{cases}=0 \text{ if } k<m\\ \leqslant r_m^{q^k-q^m}\prod_{i=1}^m r_i^{q^i-q^{i-1}} \text{ if } k\geqslant m\end{cases}.$$

Proof. For the first part, if $k=0$ $e_{V,k}=1$, so there is nothing to prove. If $k>0$, we have:

$$\|e_{V,k}\|=\left\|\sum_{\substack{I\subseteq V\setminus\{0\}\\ |I|=q^k-1}}\prod_{v\in I}v^{-1}\right\|\leqslant\max_{\substack{I\subseteq V\setminus\{0\}\\ |I|=q^k-1}}\left\|\prod_{v\in I}v^{-1}\right\|=\left\|\prod_{v\in V_k}v^{-1}\right\|=\prod_{i=1}^k r_i^{q^{i-1}-q^i}.$$

For the second part, note that the element whose norm we are trying to estimate is equal to $S_{m,q^k-1}(v_1,\dots,v_m)$, in the notation of Lemma 4.11. By that lemma, if $k<m$, the element is

zero, otherwise we have the following inequality:

$$\|S_{m,q^k-1}(v_1,\ldots,v_m)\| \leqslant \max_{\substack{d_1,\ldots,d_m\\ d_1+\cdots+d_m=q^k-1\\ \forall j\ d_1+\cdots+d_j\geqslant q^j-1}} \left\|v_1^{d_1}\cdots v_m^{d_m}\right\|.$$

It's easy to see that the maximum norm of the product $v_1^{d_1}\cdots v_m^{d_m}$ under the specified conditions is obtained when we set $d_i = q^i - q^{i-1}$ for $i < m$ and $d_m = q^k - q^{m-1}$, therefore we get the desired inequality. □

Remark 4.15. Since $\Lambda \subseteq K_\infty\Lambda$ is discrete and $K_\infty\Lambda \cong {K_\infty}^r$ is locally compact, Λ is a locally finite subspace of $\mathbb{C}_\infty$. Moreover, $e_{\Lambda,n}$ is exactly the coefficient in degree q^n of the exponential function $\exp_\Lambda$.

Lemma 4.16. *For all $\beta \in \ker(\exp_\Lambda)\setminus\{0\}$, $\ker(g_\beta)$ is an $\mathbb{F}_q$-vector subspace of Λ of codimension 1. In particular, $g_\beta = \beta\sum_{n\geqslant 0} e_{\ker(g_\beta),n}\tau^n$.*

Proof. Let's denote $V_\beta := \ker(g_\beta)$. If $c \in V_\beta$ then $\exp_\Lambda(c) = \beta^{-1}(1-\tau)(g_\beta(c)) = 0$, hence $c \in \Lambda$. Moreover, $g_\beta|_\Lambda$ is an $\mathbb{F}_q$-linear function with image in $\mathbb{F}_q$, hence its kernel V_β has codimension at most 1 in Λ. It is exactly 1 because $g_\beta|_\Lambda$ is not identically zero by Proposition 4.4.

From the identity $(1-\tau)\circ g_\beta = \beta\exp_\Lambda$, since the zeroes of $\exp_\Lambda$ are simple, we deduce the same for the zeroes of g_β, therefore $g_\beta = c_\beta\sum_{n\geqslant 0} e_{V_\beta,n}\tau^n$ for some constant $c_\beta \in \mathbb{C}_\infty$ by Remark 4.13. Finally, from the same identity we deduce that the coefficient of τ in g_β is β, hence $c_\beta = \beta$. □

4.3 Proof of the identities

We can finally prove the main propositions of this section.

Proposition 4.5. *For all $\beta \in \ker(\exp_\Lambda^*)\setminus\{0\}$, the following identity holds in $\mathbb{C}_\infty$:*

$$\beta = -\sum_{\lambda\in\Lambda\setminus\{0\}} \frac{g_\beta(\lambda)}{\lambda}.$$

Proof. First of all, the series converges because the denominators belong to the locally finite subspace Λ and the numerators to $\mathbb{F}_q$. Fix an ordered basis $(\lambda_i)_{i\geqslant 1}$ of Λ and define $\Lambda_m := \mathrm{Span}(\{\lambda_i\}_{i\leqslant m})$ for all $m \geqslant 0$. Call $V_\beta := \ker(g_\beta)$; by Lemma 4.16, $V_\beta \subseteq \Lambda$ has codimension 1, hence by Lemma 4.10, if we denote $(r_i)_{i\geqslant 1}$ and $(s_i)_{i\geqslant 1}$ the norm sequences respectively of Λ and V_β, there is a positive integer N such that for all $i < N$ $s_i = r_i$, and for all $i \geqslant N$ $s_i = r_{i+1}$. For all $m \geqslant N$, we define:

$$S_m := \beta + \sum_{\lambda\in\Lambda_m\setminus\{0\}} \frac{g_\beta(\lambda)}{\lambda} = \beta\sum_{k\geqslant 1} e_{V_\beta,k}\sum_{\lambda\in\Lambda_m}\lambda^{q^k-1}.$$

By Lemma 4.14, we have:

$$\begin{aligned}
\|\beta^{-1}S_m\| &= \left\| \sum_{k\geqslant 1} e_{V_\beta,k} \sum_{\lambda\in\Lambda_m} \lambda^{q^k-1} \right\| \leqslant \max_{k\geqslant m}\left\{ \|e_{V_\beta,k}\| \left\| \sum_{\lambda\in\Lambda_m} \lambda^{q^k-1} \right\| \right\} \\
&\leqslant \max_{k\geqslant m}\left\{ \left(\prod_{i=1}^{k} s_i^{q^{i-1}-q^i} \right) \left(r_m^{q^k-q^m} \prod_{i=1}^{m} r_i^{q^i-q^{i-1}} \right) \right\} \\
&= \max_{k\geqslant m}\left\{ \left(\prod_{i=N}^{k} r_{i+1}^{q^{i-1}-q^i} \right) \left(r_m^{q^k-q^m} \prod_{i=N}^{m} r_i^{q^i-q^{i-1}} \right) \right\} \\
&= \max_{k\geqslant m}\left\{ \left(\prod_{i=N}^{m} \left(\frac{r_i}{r_{i+1}}\right)^{q^i-q^{i-1}} \right) \left(\prod_{i=m+1}^{k} \left(\frac{r_m}{r_i}\right)^{q^i-q^{i-1}} \right) \right\} \\
&= \prod_{i=N}^{m} \left(\frac{r_i}{r_{i+1}}\right)^{q^i-q^{i-1}} = \left(\frac{r_N}{r_{m+1}}\right)^{q^N-q^{N-1}} \prod_{i=N+1}^{m} \left(\frac{r_i}{r_{m+1}}\right)^{q^i-2q^{i-1}+q^{i-2}} \leqslant \left(\frac{r_N}{r_{m+1}}\right)^{q^N-q^{N-1}}.
\end{aligned}$$

Since this number tends to zero as m tends to infinity, we have the following identities in $\mathbb{C}_\infty$:

$$0 = \lim_m S_m = \lim_m \left(\beta + \sum_{\lambda\in\Lambda_m\setminus\{0\}} \frac{g_\beta(\lambda)}{\lambda} \right) = \beta + \sum_{\lambda\in\Lambda\setminus\{0\}} \frac{g_\beta(\lambda)}{\lambda}.$$

□

Proposition 4.6. *For all $c \in K_\infty \setminus \{0\}$ with $\|c\| < 1$, the following identity holds in $\mathbb{C}_\infty$:*

$$c = -\sum_{\lambda\in\Lambda\setminus\{0\}} \frac{\exp_\Lambda(c\lambda)}{\lambda}.$$

Proof. First of all, the series converges because the denominators belong to the locally finite subspace Λ and the numerators to the compact subspace $\exp_\Lambda(K_\infty\Lambda) \cong {K_\infty\Lambda}/{\Lambda}$. Fix an ordered basis $(\lambda_i)_{i\geqslant 1}$ of Λ and define $\Lambda_m := \mathrm{Span}(\{\lambda_i\}_{i\leqslant m})$ for all $m \geqslant 0$; denote $(r_i)_{i\geqslant 1}$ the norm sequence of Λ. For all $m \geqslant 0$, define:

$$S_m := c + \sum_{\lambda\in\Lambda_m\setminus\{0\}} \frac{\exp_\Lambda(c\lambda)}{\lambda} = \sum_{k\geqslant 1} e_{\Lambda,k} c^{q^k} \sum_{\lambda\in\Lambda_m} \lambda^{q^k-1}.$$

By Lemma 4.14, if $\|c\| < 1$ we have:

$$\begin{aligned}
\|S_m\| &= \left\| \sum_{k\geqslant 1} e_{\Lambda,k} c^{q^k} \sum_{\lambda\in\Lambda_m} \lambda^{q^k-1} \right\| \leqslant \max_{k\geqslant m}\left\{ \left\|e_{\Lambda,k} c^{q^k}\right\| \left\| \sum_{\lambda\in\Lambda_m} \lambda^{q^k-1} \right\| \right\} \\
&\leqslant \max_{k\geqslant m}\left\{ \|c\|^{q^k} \left(\prod_{i=1}^{k} r_i^{q^{i-1}-q^i} \right) \left(r_m^{q^k-q^m} \prod_{i=1}^{m} r_i^{q^i-q^{i-1}} \right) \right\} \\
&= \max_{k\geqslant m}\left\{ \|c\|^{q^k} \left(\prod_{i=m+1}^{k} \left(\frac{r_m}{r_i}\right)^{q^i-q^{i-1}} \right) \right\} \leqslant \|c\|^{q^m}.
\end{aligned}$$

This number tends to zero as m tends to infinity, hence we have the following identities in $\mathbb{C}_\infty$:

$$0 = \lim_m S_m = \lim_m \left(c + \sum_{\lambda \in \Lambda_m \setminus \{0\}} \frac{\exp_\Lambda(c\lambda)}{\lambda} \right) = c + \sum_{\lambda \in \Lambda \setminus \{0\}} \frac{\exp_\Lambda(c\lambda)}{\lambda}.$$

□

Remark 4.17. The previous result is optimal, in the sense that it cannot be meaningfully extended to elements in K_∞ of higher norm. For $c = 1$ the RHS is 0, and the LHS 1; therefore, for all $x \in K_\infty$ with $\|x\| > 1$, setting $c = x$ and $c = x + 1$ yields the same RHS and different LHS.

5. Dual special functions

Fix a Drinfeld module ϕ with lattice $\Lambda \subseteq \mathbb{C}_\infty$. If ϕ is a Drinfeld-Hayes module, we have the following result from [Fer22][Prop. 7.7,Prop. 7.17]):

Proposition 5.1. *Let* $\zeta_\phi := -\sum_{\lambda \in \Lambda \setminus \{0\}} \lambda^{-1} \otimes \lambda \in \mathbb{C}_\infty \hat{\otimes} \Lambda$. *For all* $a \in A \setminus \{0\}$*:*

$$\phi_a^*(\zeta_\phi) = (1 \otimes a)\zeta_\phi.$$

In this section, we prove a generalization to Drinfeld modules of arbitrary rank.

Definition 5.2. Denote ${\mathbb{C}_\infty}^{\phi^*}$ the topological $\mathbb{F}_q$-vector space $\mathbb{C}_\infty$ endowed with the A-module structure induced by ϕ^*.

The *dual special function functor* $Sf_{\phi^*} : A-\mathbf{Mod} \to A-\mathbf{Mod}$ is defined by sending a discrete A-module M to $Sf_{\phi^*}(M) := \mathrm{Hom}_A^{cont}(\hat{M}, {\mathbb{C}_\infty}^{\phi^*}) \subseteq {\mathbb{C}_\infty}^{\phi^*} \hat{\otimes} M$.

Proposition 5.3. *The functor* Sf_{ϕ^*} *is naturally isomorphic to* $\mathrm{Hom}_A(\Lambda, \_)$, *and the universal object in* ${\mathbb{C}_\infty}^{\phi^*} \hat{\otimes} \Lambda$ *is* $-\sum_{\lambda \in \Lambda \setminus \{0\}} \lambda^{-1} \otimes \lambda$.

Proof. The map $\exp_\Lambda^* : {\mathbb{C}_\infty}^{\phi^*} \to \mathbb{C}_\infty$ is a continuous A-linear morphism; for any A-module M, it induces a morphism $Sf_{\phi^*}(M) \to \mathrm{Hom}_A^{cont}(\hat{M}, \mathbb{C}_\infty)$. Fix some $\zeta \in Sf_{\phi^*}(M)$, with image $\overline{\zeta}$: since $\hat{M}$ is compact, the image of $\hat{\zeta}$ must be a compact sub-A-module of $\mathbb{C}_\infty$, but for any $c \in \mathbb{C}_\infty \setminus \{0\}$ the set $A \cdot c$ is unbounded, hence $\overline{\zeta} \equiv 0$. We deduce that the image of $\zeta : \hat{M} \to {\mathbb{C}_\infty}^{\phi^*}$ must be contained in $\ker \exp_\Lambda^*$, which by Proposition 4.4 is isomorphic as a topological A-module to $\hat{\Lambda}$; we have the following natural isomorphisms:

$$Sf_{\phi^*}(M) = \mathrm{Hom}_A^{cont}(\hat{M}, \ker \exp_\Lambda^*) \cong \mathrm{Hom}_A^{cont}\left(\widehat{\ker \exp_\Lambda^*}, M\right) \cong \mathrm{Hom}_A(\Lambda, M),$$

where we used Lemma 2.10 for the second isomorphism.

The universal object $\zeta_\phi \in {\mathbb{C}_\infty}^{\phi^*} \hat{\otimes} \Lambda$ is given by the natural morphism of Proposition 4.4 $\psi : \hat{\Lambda} \cong \ker \exp_\Lambda^* \subseteq {\mathbb{C}_\infty}^{\phi^*}$, which by Proposition 4.5 sends $g \in \hat{\Lambda}$ to $-\sum_{\lambda \in \Lambda \setminus \{0\}} \frac{g(\lambda)}{\lambda}$.

If we fix an $\mathbb{F}_q$-basis $(\lambda_i)_i$ of Λ, with $(\lambda_i^*)_i$ dual basis of $\hat{\Lambda}$, by Remark 2.12 and Lemma 2.13 we can write:

$$\zeta_\phi = \sum_i \psi(\lambda_i^*) \otimes \lambda_i = \sum_i \left(-\sum_{\lambda \in \Lambda \setminus \{0\}} \frac{\lambda_i^*(\lambda)}{\lambda} \right) \otimes \lambda_i = -\sum_{\lambda \in \Lambda \setminus \{0\}, i} \lambda^{-1} \otimes \lambda_i^*(\lambda)\lambda_i = -\sum_{\lambda \in \Lambda \setminus \{0\}} \lambda^{-1} \otimes \lambda.$$

□

COROLLARY 5.4. *For all discrete A-modules M, $Sf_{\phi^*}(M)$ is isomorphic to* $\mathrm{Hom}_A(\Lambda, M)$ *as an $A\otimes A$-module. In particular, for a fixed we have the following equality between subsets of $\mathbb{C}_\infty\hat{\otimes}M$:*

$$Sf_{\phi^*}(M) = \left\{ \sum_{\lambda\in\Lambda} \lambda^{-1} \otimes l(\lambda) \middle| l \in \mathrm{Hom}_A(\Lambda, M) \right\}.$$

6. Pairing between special functions and dual special functions

DEFINITION 6.1. For any Drinfeld module ϕ with lattice $\Lambda \subseteq \mathbb{C}_\infty$, we define the *universal special function* $\omega_\phi \in \mathbb{C}_\infty\hat{\otimes}\,\mathrm{Hom}_A(\Lambda,\Omega)$ and the *universal zeta function* $\zeta_\phi \in \mathbb{C}_\infty\hat{\otimes}\Lambda$ as the universal objects of the functors Sf_ϕ and Sf_{ϕ^*}, respectively.

If ϕ is a Drinfeld-Hayes module, we have the following rationality result linking zeta functions and special functions (see [Fer22][Thm. 6.3]).

THEOREM 6.2 (Ferraro). *Let ϕ be a Drinfeld-Hayes module. The product of an element in $Sf_{\phi^*}(A)$ and an element in $Sf_\phi(A)$ is rational.*

To generalize this result, we need a preliminary lemma.

LEMMA 6.3. *The following morphism is well defined:*

$$\begin{array}{ccccc}
\mathbb{C}_\infty\hat{\otimes}\Lambda & \otimes & \mathbb{C}_\infty\hat{\otimes}(\Lambda^*\otimes_A\Omega) & \xrightarrow{\quad ev \quad} & \mathbb{C}_\infty\hat{\otimes}\Omega \\
 & & & & \\
\sum_i c_i\otimes\lambda_i & \otimes & \sum_j d_j\otimes(\lambda_j^*\otimes\omega_j) & \mapsto & \sum_{i,j}(c_id_j)\otimes(\lambda_j^*(\lambda_i)\omega_j) \\
 & & \overset{..}{\|} & & \overset{..}{\|} \\
 & & f & & g
\end{array}$$

Moreover, considering g and f as continuous functions respectively from ${}^{K_\infty}\!/_{A}$ and $\Lambda\otimes_A {}^{K_\infty}\!/_{A}$ to $\mathbb{C}_\infty$, for all $b \in {}^{K_\infty}\!/_{A}$ we have:

$$g(b) = \sum_i c_i f(\lambda_i\otimes b).$$

Proof. The morphism is well defined because for all $\varepsilon > 0$ there are finitely many pairs of indices (i,j) such that $\|c_id_j\| > \varepsilon$. Call $\mathrm{res} : \Omega\otimes_{\mathbb{F}_q} K_\infty/A \to \mathbb{F}_q$ and $\mathrm{res}_\Lambda : (\Lambda^*\otimes_A\Omega)\otimes_{\mathbb{F}_q}\left(\Lambda\otimes_A {}^{K_\infty}\!/_{A}\right) \to \mathbb{F}_q$ the two perfect pairings. By Remark 2.5 we have:

$$g(b) = \sum_{i,j} c_id_j\,\mathrm{res}(\lambda_j^*(\lambda_i)\omega_j, b) = \sum_i c_i \sum_j d_j\,\mathrm{res}_\Lambda(\lambda_j^*\otimes\omega_j, \lambda_i\otimes b) = \sum_i c_i f(\lambda_i\otimes b).$$

□

THEOREM 6.4. *Fix a Drinfeld module ϕ, and let the morphism ev be defined like in the previous lemma. The following property holds for all $b \in K_\infty$ with $\|b\| < 1$: as an element of* $\mathrm{Hom}^{cont}_{\mathbb{F}_q}\left({}^{K_\infty}\!/_{A}, \mathbb{C}_\infty\right)$, *$ev(\zeta_\phi\otimes\omega_\phi))$ sends b to itself. In particular there is some $h \in \mathbb{C}_\infty\otimes A$ such that $h\cdot ev(\zeta_\phi\otimes\omega_\phi) \in \mathbb{C}_\infty\otimes\Omega$.*

Proof. As an element of $\mathrm{Hom}^{cont}_{\mathbb{F}_q}\left({}^{K_\infty\Lambda}\!/_{\Lambda}, \mathbb{C}_\infty\right)$, ω_ϕ sends $c \in K_\infty\Lambda$ to $\exp_\Lambda(c)$. By Lemma 6.3, since $\zeta_\phi = -\sum_{\lambda\in\Lambda\setminus\{0\}} \lambda^{-1}\otimes\lambda$, for all $b\in K_\infty$ we have:

$$ev(\zeta_\phi\otimes\omega_\phi)(b) = -\sum_{\lambda\in\Lambda\setminus\{0\}} \frac{\exp_\Lambda(b\lambda)}{\lambda}.$$

By Proposition 4.6, if $\|b\|<1$, $ev(\zeta_\phi\otimes\omega_\phi)(b)=b$. Fix some $a\in A\setminus\{0\}$; for all $b\in K_\infty$ with $\|ab\|<1$, we have:

$$((a\otimes 1-1\otimes a)ev(\zeta_\phi\otimes\omega_\phi))(b) = a\cdot ev(\zeta_\phi\otimes\omega_\phi)(b) - ev(\zeta_\phi\otimes\omega_\phi)(ab) = ab-ab=0.$$

Fix an ordered basis $\{\mu_i\}_{i\geqslant 1}$ of Ω, with $\{\mu_i^*\}_{i\geqslant 1}$ dual basis of ${K_\infty}/{A}$, and write:

$$(a\otimes 1-1\otimes a)ev(\zeta_\phi\otimes\omega_\phi) = \sum_i c_i\otimes\mu_i$$

for some c_i's in $\mathbb{C}_\infty$. Since $\lim_i \mu_i^*=0$ in ${K_\infty}/{A}$, there is some k such that, for all $j>k$, there is a lifting $b_j\in K_\infty$ of μ_j^* with $\|b_j\|<\|a\|^{-1}$; in particular, for all $j>k$:

$$0=(a\otimes 1-1\otimes a)ev(\zeta_\phi\otimes\omega_\phi)(b_j) = (a\otimes 1-1\otimes a)ev(\zeta_\phi\otimes\omega_\phi)(\mu_j^*) = \left(\sum_i c_i\otimes\mu_i\right)(\mu_j^*) = c_j,$$

hence $(a\otimes 1-1\otimes a)ev(\zeta_\phi\otimes\omega_\phi) = \sum_{i\leqslant k} c_i\otimes\mu_i \in\mathbb{C}_\infty\otimes\Omega$. □

Remark 6.5. Let's consider the case $A=\mathbb{F}_q[t]$, with $K_\infty=\mathbb{F}_q[[t^{-1}]]$ and $\Omega=\mathbb{F}_q[t]dt$, where $dt: K_\infty\to\mathbb{F}_q$ sends t^n to $\delta_{-1,n}$.

The pairing $ev(\zeta_\phi\otimes\omega_\phi)(t^n)$ is equal to 0 if $n\geqslant 0$, since $t^n\in A$, while it's equal to t^n if $n<0$ by Theorem 6.4, since $\|t^n\|<1$. We have the following:

$$(t\otimes 1-1\otimes t)ev(\zeta_\phi\otimes\omega_\phi)(t^n) = t\cdot ev(\zeta_\phi\otimes\omega_\phi)(t^n) - ev(\zeta_\phi\otimes\omega_\phi)(t^{n+1}) = \begin{cases}1 \text{ if } n=-1\\ 0 \text{ if } n\neq -1\end{cases},$$

hence $(t\otimes 1-1\otimes t)ev(\zeta_\phi\otimes\omega_\phi)=dt$.

Remark 6.6. For general A, if we call $K_{<1} := \{b\in K \,|\, \|b\|<1\}$, by Riemann-Roch we have $\dim_{\mathbb{F}_q}\left({K\setminus A}/{K_{<1}}\right)=g$. In particular, Theorem 6.4 determines $ev(\zeta_\phi\otimes\omega_\phi)$ up to a $\mathbb{C}_\infty$-linear subspace of $\mathbb{C}_\infty\hat{\otimes}\Omega$ of dimension g.

7. Invertible special functions in Drinfeld modules of rank 1

We use the considerations of section 3, applied to the context of a Drinfeld-Hayes module ϕ with lattice Λ, to answer a question posed by Gazda and Maurischat in [GM20].

We know that there is a some $f\in\mathrm{Frac}(\mathbb{C}_\infty\otimes A)$, called *shtuka function*, such that, for all $\omega\in\mathbb{C}_\infty\hat{\otimes}A$, $\omega\in Sf_\phi(A)$ if and only if $\omega^{(1)}=f\omega$. In particular, if there is some $\omega\in Sf_\phi(A)$ which is an invertible element of the ring $\mathbb{C}_\infty\hat{\otimes}A$, for all $\omega'\in Sf_\phi(A)$ we have $\left(\frac{\omega'}{\omega}\right)^{(1)}=\frac{\omega'}{\omega}$, i.e. $\frac{\omega'}{\omega}\in\mathbb{F}_q\otimes A$, hence $Sf_\phi(A)=A\cdot\omega$. The conjecture of Gazda and Maurischat in [GM20] was that the converse is also true.

First, we prove two results to show that duality is well-behaved with respect to norms. For starters, we endow the space $\widehat{K_\infty}\cong\Omega\otimes_A K_\infty$ with a norm $|\cdot|$ such that it is a normed vector space over $(K_\infty,\|\cdot\|)$, and we use the same notation for the induced norm on the quotient $\hat{A}$; note that $|\cdot|$ is unique up to a scalar factor in $\mathbb{R}^+$.

Proposition 7.1. *For all $f \in \widehat{K} \setminus \{0\}$ we have*

$$|f|^{-1} = \min\{\|\lambda\| \text{ s.t. } \lambda \in K_\infty, f(\lambda) \neq 0\}.$$

Proof. We can identify K_∞ with $\mathbb{F}_q((t))$, and $\widehat{K_\infty}$ with $\mathbb{F}_q((t))dt$; for all $p(t) = \sum_{i\in\mathbb{Z}} \lambda_i t^i \in \mathbb{F}_q((t))$ with leading term $\lambda_k t^k$, we set $\|p(t)\| = q^{-k}$, $|dt| = q^{-1}$, and $dt(p) = \lambda_{-1}$.

Take $\mu \in \mathbb{F}_q((t))dt$ with leading term $b_k t^k dt$, so that $|\mu| = q^{-k-1}$: if $p \in \mathbb{F}_q((t))$ has $\|p\| < q^{k+1}$, its leading term has degree at least $-k$, hence $\mu(p) = 0$; on the other hand $\|t^{-k-1}\| = q^{k+1}$ and $\mu(t^{-k-1}) = b_k \neq 0$. In particular:

$$|\mu|^{-1} = \min\{\|p\| \text{ s.t. } p \in \mathbb{F}_q((t)), \mu(p) \neq 0\}.$$

□

Proposition 7.2. *Fix an $\mathbb{F}_q$-basis $(a_i)_{i\in I}$ of A strictly ordered by degree, with $(a_i^*)_{i\in I}$ dual basis of $\hat{A} \cong {}^{\Omega\otimes_A K_\infty}/_{\Omega}$. The sequence $(|a_i^*|)_{i\in I}$ is strictly decreasing.*

Proof. We can assume $I \subseteq \mathbb{Z}$ to be the set of degrees of elements in A, and that a_i has degree i for all $i \in I$. For all $i \in I$ set $b_i := a_i$, while for all $i \in \mathbb{Z} \setminus I$ choose some $b_i \in K_\infty$ with valuation $-i$: every $c \in K_\infty$ can be expressed in a unique way as $\sum_{i\in\mathbb{Z}} \lambda_i b_i$ where $\lambda_i \in \mathbb{F}_q$ for all $i \in \mathbb{Z}$ and $\lambda_i = 0$ for $i \gg 0$. Denote $(b_i^*)_{i\in\mathbb{Z}}$ the sequence in $\widehat{K_\infty}$ determined by the property $b_i^*(b_j) = \delta_{i,j}$ for all $i, j \in \mathbb{Z}$: every $c \in \widehat{K_\infty}$ can be expressed in a unique way as $\sum_{i\in\mathbb{Z}} \lambda_i b_i^*$ where $\lambda_i \in \mathbb{F}_q$ for all $i \in \mathbb{Z}$ and $\lambda_i = 0$ for $i \gg 0$. By Proposition 7.1, up to rescaling $|\cdot|$, we have for all $i \in \mathbb{Z}$:

$$|b_i^*|^{-1} = \min\{\|c\| \text{ s.t. } c \in K_\infty \text{ and } b_i^*(c) \neq 0\} = \min\left\{\left\|\sum_{j\in\mathbb{Z}} \lambda_j b_j\right\| \text{ s.t. } \lambda_i \neq 0\right\} = \|b_i\|.$$

For any $c \in \widehat{K_\infty}$, call $\bar{c}$ its projection onto $\hat{A}$. Since $(b_i)_{i\in I} = (a_i)_{i\in I}$ is an $\mathbb{F}_q$-basis of A, $\overline{b_i^*}$ is a_i^* if $i \in I$, and 0 otherwise. For all $i \in I$, we have:

$$|a_i^*| = \min\{|c| \text{ s.t. } c \in \widehat{K}, \bar{c} = a_i^*\} = \min\left\{\left|\sum_{j\in\mathbb{Z}} \lambda_j b_j^*\right| \text{ s.t. } \lambda_j = \delta_{i,j} \forall j \in I\right\} = |b_i^*| = \|a_i\|^{-1}.$$

□

Proposition 7.3. *Suppose $Sf_\phi(A) \cong A$. Then, there is a special function in $Sf_\phi(A)$ which is invertible as an element of $\mathbb{C}_\infty \hat{\otimes} A$.*

Proof. Fix an $\mathbb{F}_q$-basis $(a_i)_{i\geqslant 0}$ of A strictly ordered by degree, with $a_0 = 1$, and let $(a_i^*)_{i\in I}$ be the dual basis of $\hat{A} \cong {}^{\Omega\otimes_A K_\infty}/_{\Omega} \cong {}^{K_\infty\Lambda}/_{\Lambda}$.

By Remark 3.6, the universal special function $\omega_\phi \in \mathbb{C}_\infty \hat{\otimes} A$ can be written as $\sum_i \exp_\Lambda(a_i^*) \otimes a_i$. To prove it is invertible, it suffices to show that for all $i > 0$ $\|\exp_\Lambda(a_0^*)\| > \|\exp_\Lambda(a_i^*)\|$: in this case, if we call $\omega := (\exp_\Lambda(a_0^*)^{-1} \otimes 1)\omega_\phi$, the series $\sum_n (1-\omega)^n$ converges in $\mathbb{C}_\infty \hat{\otimes} A$, and is an inverse to ω.

For all indices i, choose a lifting $c_i \in K_\infty\Lambda \subseteq \mathbb{C}_\infty$ of $a_i^* \in {}^{K_\infty\Lambda}/_{\Lambda}$ with the least norm; in particular, there are no $\lambda \in \Lambda$ such that $\|\lambda\| = \|c_i\|$, so we have:

$$\|\exp_\Lambda(a_i^*)\| = \|c_i\| \prod_{\lambda\in\Lambda\setminus\{0\}} \left\|1 - \frac{c_i}{\lambda}\right\| = \|c_i\| \prod_{\substack{\lambda\in\Lambda\setminus\{0\} \\ \|\lambda\|\leqslant\|c_i\|}} \left\|1 - \frac{c_i}{\lambda}\right\| = \|c_i\| \prod_{\substack{\lambda\in\Lambda\setminus\{0\} \\ \|\lambda\|<\|c_i\|}} \left\|\frac{c_i}{\lambda}\right\|.$$

By Proposition 7.2, the sequence $(\|c_i\|)_i$ is strictly decreasing, hence the sequence $(\|\exp_\Lambda(a_i^*)\|)_i$ is strictly decreasing. □

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