A class of functional identities associated to curves over finite fields

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Abstract

Given a geometrically irreducible smooth projective curve X over $\mathbb{F}_q$ with an $\mathbb{F}_q$-rational point ∞, a class of zeta functions was introduced by Pellarin, and related to the Anderson-Thakur function via a functional identity when the genus $g(X)$ is 0. After Anglès, Ngo Dac, and Tavares Ribeiro generalized Anderson-Thakur functions to arbitrary genus, Green and Papanikolas found a functional identity relating these so-called "special functions" to zeta functions à la Pellarin in the case $g(X) = 1$.

The aim of this paper is to prove that a generalization of those functional identities hold in arbitrary genus. Our proof exploits the topological nature of divisors on the curve X, as well as the introduction of an "adjoint shtuka function". This allows us to reinterpret Pellarin zeta functions as dual versions of special functions.

1. Introduction

Let $\mathbb{F}_q$ be the finite field with q elements, and let X be a projective, geometrically irreducible, smooth curve of genus g over $\mathbb{F}_q$, with a point $\infty \in X(\mathbb{F}_q)$. We call $A := \mathcal{O}_X(X \setminus \{\infty\})$, H the Hilbert class field of $K := \mathrm{Frac}(A)$, K_∞ the completion of K at ∞, and $\mathbb{C}_\infty$ the completion of the algebraic closure K_∞^{ac}. We fix an inclusion $H \subseteq K_\infty$ and a multiplicative sign function at ∞: $\mathrm{sgn} : \mathbb{C}_\infty(X) \to \mathbb{C}_\infty^\times$.

Throughout the paper we fix the datum $(X, \infty, \mathrm{sgn})$, and we work with a Drinfeld module ϕ - i.e. a ring homomorphism $A \to \mathbb{C}_\infty\{\tau\}$ where $\tau c = c^q \tau$ for all $c \in \mathbb{C}_\infty$ - of rank 1 and sign-normalized, also called a Drinfeld-Hayes module. Following [Tha93], we associate to ϕ a rational function f over X_H called *shtuka function*, with $\mathrm{sgn}(f) = 1$ and $\mathrm{Div}(f) = V^{(1)} - V + \Xi - \infty$ where $\Xi : \mathrm{Spec}(\mathbb{C}_\infty) \to \mathrm{Spec}(A) \hookrightarrow X$ is given by the canonical inclusion $A \hookrightarrow \mathbb{C}_\infty$, and V is the so-called *Drinfeld divisor*, an effective divisor of degree g.

A natural tool to study ϕ is the Tate algebra $\mathbb{T} := \mathbb{C}_\infty \hat{\otimes} A$, on which ϕ acts A-linearly in the second coordinate. In [ANT17], Anglès, Ngo Dac, and Tavares Ribeiro generalized the Anderson-Thakur functions introduced in [AT90]: they proved there is a class of so-called "special functions" $\mathrm{Sf}(\phi) \subseteq \mathbb{T}$ on which ϕ acts by scalar multiplication; in other words, for all $\omega \in \mathrm{Sf}(\phi)$ we have the following equalities:

$$\phi_a(\omega) = (1 \otimes a)\omega,\ \forall a \in A.$$

Another interesting class of functions are the *partial zeta functions* "à la Pellarin"; the partial

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zeta relative to an ideal $I \trianglelefteq A$ is defined as:

$$\zeta_I := \sum_{a \in I \setminus \{0\}} a^{-1} \otimes a \in \mathbb{T}.$$

They were first introduced in the case $I = A = \mathbb{F}_q[T]$ by Pellarin, who - using some log-algebraicity results of Anderson from [And94] and [And96]- proved the identity $\zeta_A \omega(1 \otimes T - T \otimes 1) = \tilde{\pi}$, where ω is the Anderson-Thakur function, and $\tilde{\pi}$ is a fundamental period of the period lattice (see [Pel11][Thm. 1]). Green and Papanikolas proved a similar result ([GP16][Thm. 7.1]) in the much wider context of $g(X) = 1$. Together with [Pel11][Thm. 1], it can be stated as follows:

Theorem (Pellarin-Green-Papanikolas). *Suppose $g(X) \leqslant 1$ and fix the unique Drinfeld-Hayes module ϕ such that the associated period lattice is $\tilde{\pi} A \subseteq \mathbb{C}_\infty$ for some $\tilde{\pi} \in \mathbb{C}_\infty^\times$. There is a function $\delta \in H \otimes A$ such that:*

$$\mathrm{Sf}(\phi) = \frac{\delta^{(1)}(\tilde{\pi} \otimes 1)}{f \zeta_A} \cdot (\mathbb{F}_q \otimes A),$$

where f is the shtuka function of ϕ.

Green and Papanikolas also described δ in terms of its divisor. Note that this result does not encompass the totality of the case $g(X) = 1$, since for each ideal class in the group $Cl(A)$ there is exactly one Drinfeld-Hayes module whose lattice is in that class.

In the present paper, we prove the following generalization for any datum $(X, \infty, \mathrm{sgn})$, where: $I \trianglelefteq A$ is any ideal, with $a_I \in I$ an element of least degree; ϕ is a Drinfeld-Hayes module with shtuka function f, dependent on the ideal class of I (see Lemma 4.25); $\tilde{\pi} \in K_\infty$ is a fundamental period of the lattice of ϕ (see Definition 7.8); V_* is the unique effective divisor of degree g with $V_* \sim 2g\infty - V$, and δ is the only function in $H \otimes A$ with $\mathrm{sgn}(\delta) = 1$ and divisor $V + V_* - 2g\infty$.

Theorem A (Theorem 6.3, complete version). *The following A-submodules of $\mathbb{T}$ coincide:*

$$\mathrm{Sf}(\phi) = \frac{\delta^{(1)}(\tilde{\pi} \otimes 1)}{f(a_I \otimes 1)\zeta_I} \cdot (\mathbb{F}_q \otimes I).$$

Remark 1.1. The elements $a_I \in I$ and $\tilde{\pi} \in K_\infty$ are uniquely determined up to a factor in $\mathbb{F}_q^\times$. Moreover, one can check that the right hand side only depends on the ideal class of I.

It's worth noting that the techniques employed by Green and Papanikolas were tailored to the case $g(X) = 1$ - for example, they used the equation of a generic elliptic curve to carry out explicit computations. In the present paper, computations are only needed in Section 7 to find the scalar factor $\tilde{\pi} \otimes 1$; the remaining content of Theorem A, expressed as a partial version thereof in Section 6, is built on the purely theoretical results of Section 3 and Section 4.

If ϕ is a Drinfeld-Hayes module and f is its shtuka function, an element $\omega \in \mathbb{T}$ is a special function if and only if $\omega^{(1)} = f\omega$. If $f \in \mathbb{T}^\times$ there is $\alpha \in K_\infty$ such that

$$(\alpha \otimes 1)^{\frac{1}{q-1}} \prod_{i \geqslant 0} \left(\frac{\alpha \otimes 1}{f} \right)^{(i)}$$

is a well-defined element of $\mathbb{T}^\times$, and it is a special function; on the other hand, as Gazda and Maurischat noticed in [GM20][Cor. 3.22], if there is an invertible special function, then $Sf(\phi) \cong A$, and it is not known if the converse is true, or in other words how restrictive is the hypothesis $f \in \mathbb{T}^\times$. We circumvent this problem and prove the following result for our datum $(X, \infty, \mathrm{sgn})$.

Theorem B (Theorem 6.7). *Fix a Drinfeld-Hayes module ϕ with shtuka function f. There is some $\alpha \in K_\infty^\times$ such that the following element of $K_\infty \hat{\otimes} K$ is well defined:*

$$\omega := (\alpha \otimes 1)^{\frac{1}{q-1}} \prod_{i \geqslant 0} \left(\frac{\alpha \otimes 1}{f} \right)^{(i)}.$$

Moreover, $\omega \in (\mathbb{F}_q \otimes K)\,\mathrm{Sf}(\phi)$, and ω does not depend on the choice of α.

In the present paper, we introduce an *adjoint shtuka function* f_* - the unique rational function over X_{K_∞} with divisor $V_* - V_*^{(1)} + \Xi - \infty$ and $\mathrm{sgn}(f_*) = 1$. In analogy with Theorem 6.7, we prove the following identity, for a certain ideal $I \trianglelefteq A$ with $a_I \in I$ an element of least degree.

Theorem C (Theorem 5.9, complete version). *The following functional identity is well posed and true in $K_\infty \hat{\otimes} K$:*

$$\zeta_I = -(a_I^{-1} \otimes a_I) \prod_{i \geqslant 0} \left((\tilde{\pi}^{1-q} \otimes 1) f_*^{(1)} \right)^{(i)}.$$

Equivalently, we prove the following identity (Proposition 7.20):

$$\frac{\left((a_I \tilde{\pi}^{-1} \otimes 1) \zeta_I \right)^{(-1)}}{(a_I \tilde{\pi}^{-1} \otimes 1) \zeta_I} = f_*.$$

Its similarity to the identity $\omega^{(1)} = f\omega$ defining a special function ω suggests this as an alternative definition of Pellarin zeta functions. If we think of the formula $\zeta_I = \sum_{a \in I \setminus \{0\}} a^{-1} \otimes a$ as a consequence of this definition, it's natural to wonder if an analogous formula exists also for special functions: this topic will be explored in a forthcoming paper by the author. As for Theorem A, the complete version of Theorem C - with the explicitation of the constant $\tilde{\pi}$ - is contained in Section 7, where we explore this duality between special functions and zeta functions.

The structure of the rest of the paper is as follows. In Section 3 we describe a functorial way of assigning a compact topology to the K_∞-points of a proper $\mathbb{F}_q$-scheme Y. We then discuss some results about divisors of curves in finite characteristic from [Mil86], and deduce a homeomorphism between certain spaces of rational functions and the spaces of their divisors; this allows us to prove statements about the convergence of the former by looking at the latter. Green and Papanikolas had already conjectured that the Jacobian variety and the divisor of V would play a role in the generalization of [GP16][Thm. 7.1], and in Section 3 and Section 4 we explain concretely how they are used.

Finally, in Section 8, we generalize another theorem of Green and Papanikolas about zeta functions "à la Anderson" ([GP16, Thm. 7.3]), which they used to prove a particular case of a log-algebraicity theorem by Anderson ([And94][Thm. 5.1.1]). In the notation of Section 8, Anderson zetas are defined as:

$$\xi_{\bar{I}} := \sum_{J \trianglelefteq A} \frac{\chi_{\bar{I}}(J)}{\chi_{\bar{I}}(J)(\Xi)} \in K_\infty \hat{\otimes} K.$$

For a certain function $h_{I,\bar{A}} \in \mathrm{Frac}(H \otimes A) \subseteq K_\infty \hat{\otimes} K$ defined as in Lemma 8.4, the result is as follows.

Theorem D (Theorem 8.7). *The function $\xi_{\bar{I}}$ is well defined, and the following identity holds:*

$$h_{I,\bar{A}} \xi_{\bar{I}} = - \left(\sum_{\sigma \in \mathrm{Gal}(H/K)} h_{I,\bar{A}}^{\sigma} \right) \zeta_A.$$

2. Notations and fundamental concepts

Recall the notations from the first paragraph of the introduction.

Throughout this paper, it's appropriate to assume $g \geqslant 1$. On one hand, all the proofs also work in the hypothesis $g = 0$ with some caveats; on the other hand, this assumption simplifies the language of the paper by avoiding a fringe case, which was already studied in [Pel11].

We introduce the following additional notations.

- The degree map $\deg : K \to \mathbb{Z}$ is defined as the opposite of the valuation at ∞, and for all A-modules $\Lambda \subseteq K$, for all integers d, we define $\Lambda(d)$ (resp. $\Lambda(\leqslant d)$) the set $\{x \in \Lambda | \deg(x) = d\}$ (resp. $\deg(x) \leqslant d$).
- For any finite field extension L/K_∞, we denote by $(\mathcal{O}_L, \mathfrak{m}_L)$ the associated local ring of integers, and by $\mathbb{F}_L$ the residue field.
- Unlabeled tensor products of modules are assumed to be over $\mathbb{F}_q$, while for unlabeled fiber products of schemes the base ring should be clear from the context.
- If Y is an R-scheme and S is an R-algebra, we denote $Y(S)$ the set of morphism of R-schemes from $\mathrm{Spec}(S)$ to Y, and with Y_S the base change $Y \times \mathrm{Spec}(S)$. If Y_S is integral, we denote $S(Y)$ the field of rational functions of Y_S.
- For all complete normed fields L, for all $\mathbb{F}_q$-vector spaces M, the module $L \otimes M$ is endowed with the sup norm induced by L; we denote its completion by $L \hat{\otimes} M$.
- In analogy to [Gos12], we use the relation symbol $a \in^* \Lambda$ to signify $a \in \Lambda \setminus \{0\}$.

Remark 2.1. Let's describe explicitly the sign function for any $h \in X(\mathbb{C}_\infty)^\times$. Since $X(\mathbb{C}_\infty)$ is the field of fractions of $\mathbb{C}_\infty \otimes A$, and sgn is a multiplicative function, we can assume $h \in \mathbb{C}_\infty \otimes A \setminus \{0\}$. We can write $h = \sum_{i=0}^k c_i \otimes a_i$, with $(a_i)_i$ in A of strictly increasing degree and $(c_i)_i$ in $\mathbb{C}_\infty^\times$, and we have:

$$\mathrm{sgn}(h) = \mathrm{sgn}\left(\sum_{i=0}^k c_i \otimes a_i\right) = \mathrm{sgn}(c_k \otimes a_k) = c_k \, \mathrm{sgn}(a_k).$$

In the rest of this section, we present some basic results that are described in great detail in [Gos12].

Let $\mathbb{C}_\infty\{\tau\}$ and $\mathbb{C}_\infty\{\tau^{-1}\}$ be the rings of non-commutative polynomials over $\mathbb{C}_\infty$, with the relations $\tau c = c^q \tau$ and $\tau^{-1} c^q = c \tau^{-1}$ for all $c \in \mathbb{C}_\infty$. There is a $\mathbb{F}_q$-linear and bijective antihomomorphism $\mathbb{C}_\infty\{\tau\} \to \mathbb{C}_\infty\{\tau^{-1}\}$ sending $\varphi := \sum_i c_i \tau^i$ to $\varphi^* := \sum \tau^{-i} c_i$.

A *Drinfeld module* of rank r is a ring homomorphism $\phi : A \to \mathbb{C}_\infty\{\tau\}$ sending a to $\phi_a := \sum_{i \geqslant 0} a_i \tau^i$ with the following properties for all $a \in^* A$:

$$\deg_x\left(\sum_{i \geqslant 0} a_i x^{q^i}\right) = q^{r \deg(a)};\ a_0 = a.$$

If moreover $r = 1$ and $a_{\deg(a)} = \mathrm{sgn}(a)$ for all $a \in^* A$, we call ϕ a *Drinfeld-Hayes module*.

Fix ϕ, ψ Drinfeld modules. An element $f = \sum_i c_i \tau^i \in \mathbb{C}_\infty\{\tau\}$ is said to be a morphism from ϕ to ψ if $f \circ \phi_a = \psi_a \circ f$ for all $a \in A$. It is known that every Drinfeld module of rank 1 is isomorphic to a Drinfeld-Hayes module.

The *exponential map* relative to a discrete A-module (called *period lattice*) $\Lambda \subseteq \mathbb{C}_\infty$ is the following analytic function from $\mathbb{C}_\infty$ to itself:

$$\exp_\Lambda(x) := x \prod_{\lambda \in^* \Lambda} \left(1 - \frac{x}{\lambda}\right) \in \mathbb{C}_\infty[[x]].$$

We can write $\exp_\Lambda(x) = \sum_{i \geqslant 0} e_i x^{q^i}$, and its (bilateral) compositional inverse exists in $\mathbb{C}_\infty[[x]]$, is denoted $\log_\Lambda = \sum_{i \geqslant 0} l_i x^{q^i}$, and is called *logarithmic map*. In particular, both functions can be seen as elements of the noncommutative power series ring $\mathbb{C}_\infty\{\{\tau\}\}$ ($\exp_\Lambda = \sum_{i \geqslant 0} e_i \tau^i$ and $\log_\Lambda = \sum_{i \geqslant 0} l_i \tau^i$), in which they are inverse to one another.

Finally, we recall that there is an equivalence between the (small) category of discrete A-submodules of $\mathbb{C}_\infty$ of rank r, with isogenies as morphisms, and the (small) category of Drinfeld modules of rank r.

3. Convergence of divisors of rational functions over X

Consider the d-th symmetric power $X^{(d)}$ for some positive integer d; for all finite field extensions L/K_∞, $X^{(d)}(L)$ is the set of L-rational divisors over X of degree d. The aim of this section is to endow $X^{(d)}(L)$ with a compact topology such that the following property holds.

Proposition 3.19. *Consider a sequence $(h_m)_m$ in $L \otimes A(\leqslant d)$.*

If the sequence $(\mathrm{Div}(h_m) + d\infty)_m$ converges to $D \in X^{(d)}(L)$, there are $(\lambda_m)_m$ in $L^\times$ such that $(\lambda_m h_m)_m$ converges to some $h \in L \otimes A(\leqslant d) \setminus \{0\}$ with $\mathrm{Div}(h) = D - d\infty$.

If the sequence $(h_m)_m$ converges to $h \in L \otimes A(\leqslant d) \setminus \{0\}$, then the sequence $(\mathrm{Div}(h_m) + d\infty)_m$ converges to $\mathrm{Div}(h) + d\infty \in X^{(d)}(L)$.

In the following sections we need several times a topology on the L-points of other projective $\mathbb{F}_q$-schemes (such as the powers $\{X^d\}_{d \geqslant 1}$ and the Jacobian variety $\mathcal{A}$ of X). To ensure their good interaction we prove that the topology that we define is functorial in Proposition 3.4.

3.1 Functorial compact topology on K_∞-rational points of $\mathbb{F}_q$-schemes

Through this subsection, L is a finite field extension of K_∞, and Y is a proper $\mathcal{O}_L$-scheme. We aim to construct a functor from proper schemes over $\mathcal{O}_L$ to compact Hausdorff topological spaces, sending Y to $Y(\mathcal{O}_L) = Y(L)$.

Lemma 3.1. *The natural maps $\big(\mathrm{red}_{L,k} : Y(\mathcal{O}_L) \to Y(\mathcal{O}_L/\mathfrak{m}_L^k)\big)_{k \geqslant 1}$ induce a bijection $Y(\mathcal{O}_L) \cong \varprojlim_k Y(\mathcal{O}_L/\mathfrak{m}_L^k)$.*

Proof. Since $\mathrm{Spec}(\mathcal{O}_L) \cong \varinjlim_k \mathrm{Spec}(\mathcal{O}_L/\mathfrak{m}_L^k)$, we have:

$$Y(\mathcal{O}_L) \cong \mathrm{Hom}_{\mathcal{O}_L}\left(\varinjlim_k \mathrm{Spec}(\mathcal{O}_L/\mathfrak{m}_L^k), Y\right) \cong \varprojlim_k \mathrm{Hom}_{\mathcal{O}_L}\left(\mathrm{Spec}(\mathcal{O}_L/\mathfrak{m}_L^k), Y\right) \cong \varprojlim_k Y(\mathcal{O}_L/\mathfrak{m}_L^k).$$

□

Remark 3.2. The limit topology induced on $\varprojlim_k Y(\mathcal{O}_L/\mathfrak{m}_L^k)$ - where the indexed spaces are endowed with the discrete topology - is Hausdorff.

Since Y is finite-type over $\mathcal{O}_L$ and $\mathcal{O}_L/\mathfrak{m}_L^k$ is finite for all k, $Y(\mathcal{O}_L/\mathfrak{m}_L^k)$ is finite for all k, so $Y(L)$ is compact. Moreover, it can be endowed with an ultrametric distance $\bar{d}$ as follows:

$$\bar{d}(P, Q) := \min_{k \in \mathbb{Z}} \left\{ \frac{1}{q^k} \,\middle|\, \mathrm{red}_{L,k}(P) \neq \mathrm{red}_{L,k}(Q) \right\}.$$

DEFINITION 3.3. Fix an inclusion $\mathbb{F}_L \hookrightarrow \mathcal{O}_L$: it induces a section of $\mathrm{red}_{L,1} : Y(\mathcal{O}_L) \to Y(\mathbb{F}_L)$. We denote $\mathrm{red}_L : Y(\mathcal{O}_L) \to Y(\mathcal{O}_L)$ the composition of the two maps.

From this point onwards, unless otherwise stated, we interpret the set $Y(\mathcal{O}_L) = Y(L)$ as endowed with this topology, which we call *compact topology*. Similarly, if Y' is a proper $\mathbb{F}_q$-scheme, the set $Y'(L) = Y'_{\mathcal{O}_L}(L)$ is always endowed with the compact topology.

PROPOSITION 3.4. *The map associating to a proper $\mathcal{O}_L$-scheme Y the topological space $Y(\mathcal{O}_L)$ can be extended to a functor F_L.*

Proof. For every morphism $\varphi : Z \to Y$ of proper $\mathcal{O}_L$-schemes, the map $\varphi(\mathcal{O}_L) : Z(\mathcal{O}_L) \to Y(\mathcal{O}_L)$ is continuous because it induces a system of maps $(\varphi(\mathcal{O}_L/\mathfrak{m}_L^k) : Z(\mathcal{O}_L/\mathfrak{m}_L^k) \to Y(\mathcal{O}_L/\mathfrak{m}_L^k))_k$ which commute with the transition maps of the projective systems $(Z(\mathcal{O}_L/\mathfrak{m}_L^k))_k$ and $(Y(\mathcal{O}_L/\mathfrak{m}_L^k))_k$.

If we set $F_L(\varphi) := \varphi(\mathcal{O}_L)$ for all morphisms, it's easy to check that F_L sends the identity map to the identity map and preserves composition, hence it is a functor. □

Remark 3.5. We also obtain a functor from proper $\mathbb{F}_q$-schemes to topological spaces, sending a scheme Y to $Y(\mathcal{O}_L) = Y(L)$, by precomposing F_L with the base change $Y \mapsto Y_{\mathcal{O}_L}$.

LEMMA 3.6. *Let $f : Z \to Y$ be a morphism of proper $\mathcal{O}_L$-schemes. Fix a subset $V \subseteq Y(L)$ with preimage $U \subseteq Z(L)$, such that $F_L(f)|_U : U \to V$ is bijective. Then $F_L(f)|_U$ is a homeomorphism.*

Proof. The map $F_L(f) : Z(L) \to Y(L)$ is closed, being a continuous map between compact Hausdorff spaces. Any closed set of U can be written as $C \cap U$, with $C \subseteq Z(L)$ closed. We have:

$$F_L(f)(C \cap U) = F_L(f)\left(C \cap F_L(f)^{-1}(V)\right) = F_L(f)(C) \cap V,$$

which is closed in V because $F_L(f)(C)$ is closed in $Y(L)$. This means that $F_L(f)|_U$ is closed, and since it induces a bijection between U and V, it is a homeomorphism. □

Remark 3.7. In the case of the projective space $\mathbb{P}^n$ of dimension n over $\mathbb{F}_q$, the set $\mathbb{P}^n(L)$ is in bijection with $L^{n+1} \setminus \{0\}/L^\times$; since the latter has a natural topology induced by L, the former also does, and it's easy to check that it's the same as the compact topology we defined.

The following statements show that the functor F_L sends group schemes to topological groups.

LEMMA 3.8. *The topological spaces $F_L(Y \times_{\mathcal{O}_L} Y)$ and $F_L(Y) \times F_L(Y)$ are naturally isomorphic.*

Proof. The projections $\pi_1, \pi_2 : Y \times Y \to Y$ induce a natural continuous map $F_L(Y \times_{\mathcal{O}_L} Y) \to F_L(Y) \times F_L(Y)$. Since both spaces are compact and Hausdorff, the map is closed; since the underlying function is the natural bijection $(Y \times_{\mathcal{O}_L} Y)(L) \cong Y(L) \times Y(L)$, the map is a homeomorphism. □

PROPOSITION 3.9. *If Y is a (commutative) group scheme over $\mathcal{O}_L$, the metric on $Y(L)$ is translation invariant, and makes it into a (commutative) topological group.*

Proof. By Lemma 3.8, we identify $F_L(Y \times_{\mathcal{O}_L} Y) \cong F_L(Y) \times F_L(Y)$ via a natural homeomorphism.

Call e the identity, i the inverse, and m the multiplication of Y. Then $F_L(Y)$ has a natural structure of topological group, with identity $F_L(e)$, inverse $F_L(i)$ and multiplication $F_L(m)$, because all the necessary diagrams commute by functoriality. For the same reason, if Y is commutative, $Y(L)$ is also commutative.

To prove the invariance of the metric, we need to show that every translation is an isometry. Fix a morphism of $\mathcal{O}_L$-schemes $P : \mathrm{Spec}(\mathcal{O}_L) \to Y$ (i.e. $P \in Y(\mathcal{O}_L)$), and consider the following:

$$l_P : Y \cong \mathrm{Spec}(\mathcal{O}_L) \times_{\mathcal{O}_L} Y \xrightarrow{P \times id_Y} Y \times_{\mathcal{O}_L} Y \xrightarrow{m} Y,$$

so that $F_L(l_P) : Y(L) \to Y(L)$ is the left translation by P. It's immediate to check that, if we call $-P$ the inverse of P in $Y(\mathcal{O}_L)$, l_{-P} is the two-sided inverse of l_P, therefore they are isomorphisms. In particular l_P induces a family of bijections $\{Y(\mathcal{O}_L/\mathfrak{m}_L^k) \to Y(\mathcal{O}_L/\mathfrak{m}_L^k)\}_{k\geqslant 1}$, whose limit is precisely $F_L(l_P)$, hence $F_L(l_P)$ is an isometry. The proof for right translations is essentially the same. □

COROLLARY 3.10. *Suppose that Y is commutative. Denote with addition the group law on $Y(L)$ and with 0 its identity element. If $(P_i)_{i\in\mathbb{N}}$ is a sequence in $Y(L)$ converging to 0, then the series $\sum_i P_i$ is a well defined element of $Y(L)$ (i.e. the sequence of partial sums converge).*

Proof. Call $\bar{d}$ the distance on $Y(L)$. Since d is ultrametric, we just need $\lim_k \bar{d}(S_k, S_{k-1}) = 0$, with $S_k := \sum_{i=0}^k P_i$. Since the metric is translation invariant, $\lim_k \bar{d}(S_k, S_{k-1}) = \lim_k \bar{d}(P_k, 0)$, which is zero by hypothesis. □

3.2 Topology of the space of divisors

In this subsection we state some propositions about the symmetric powers of a curve and its Jacobian. Most results are stated and proven in [Mil86].

Recall the definition of X; S_d is the permutation group of d elements. We have the following (see [Mil86][Prop. 3.1, Prop. 3.2]).

PROPOSITION 3.11. *Fix a positive integer d. Consider the natural right action of S_d on X^d and call its quotient $X^{(d)}$. Then $X^{(d)}$ is a proper smooth $\mathbb{F}_q$-scheme.*

The following result (see [Mil86][Thm. 3.13]) gives us the functorial interpretation of the symmetric power $X^{(d)}$.

THEOREM 3.12. *Consider the functor Div_X^d which sends an $\mathbb{F}_q$-algebra R to the set of relative effective Cartier divisors of degree d on X_R over R (i.e. effective Cartier divisors on X_R which are finite and flat of rank d over R). This functor is represented by $X^{(d)}$.*

COROLLARY 3.13. *For every field $E/\mathbb{F}_q$, $X^{(d)}(E)$ is in bijection with the E-subschemes of X_E of degree d.*

Let's continue with the fundamental property of the Jacobian variety (see [Mil86][Thm. 1.1]).

THEOREM 3.14. *The functor from $\mathbb{F}_q$-algebras to abelian groups*

$$R \mapsto \{\mathcal{L} \in \mathrm{Pic}(X_R) | \deg(\mathcal{L}_t) = 0\ \forall t \in \mathrm{Spec}(R)\}/\pi_R^*(\mathrm{Pic}(R))$$

is represented by an abelian variety $\mathcal{A}$ over $\mathbb{F}_q$, called the Jacobian variety of X.

The following result clarifies the relation between the symmetric powers of X and $\mathcal{A}$ (see [Mil86][Thm. 5.2]).

THEOREM 3.15. *For all $d \geqslant 1$, the point $\infty \in X(\mathbb{F}_q)$ induces a natural morphism of $\mathbb{F}_q$-schemes $J^d : X^{(d)} \to \mathcal{A}$. Moreover, the morphism $J^g : X^{(g)} \to \mathcal{A}$ is birational and surjective.*

Remark 3.16. For every field $E/\mathbb{F}_q$, at the level of E-points the morphism J^d sends an effective divisor D of degree d to the class of $D - d\infty$.

Finally, a result on the fibers of the map J^d (see [Mil86][Rmk. 5.6.(c)] and [Har77][Prop. II.7.12]).

Proposition 3.17. *Fix a field extension* $E/\mathbb{F}_q$ *and a point* $D \in X^{(d)}(E)$, *corresponding to a sheaf of* $\mathcal{O}_{X_E}$*-ideals* $\mathcal{I}_D$, *with* $P := J^d \circ D \in \mathcal{A}(E)$. *Call* V *the* E*-vector space of the global sections of the* $\mathcal{O}_{X_E}$*-sheaf* $\mathcal{I}_D^{-1}$. *The fiber* $(J^d)^*P$ *is naturally isomorphic as an* E*-scheme to* $\mathbb{P}(V)$.

For any field extension E'/E, *for all* $f \in E' \otimes_E V$, *the isomorphism sends the line* $E' \cdot f \in \mathbb{P}(V)(E')$ *to* $\mathrm{Div}(f) + D \in X^{(d)}(E')$.

Corollary 3.18. *Let* $D \in X^{(d)}(E^{ac})$ *with* $h^0(D) = 1$. *If* $J^d \circ D \in \mathcal{A}(E^{ac})$ *factors through some* $P \in \mathcal{A}(E)$, D *factors through some* $D' \in X^{(d)}(E)$.

Proof. Since D factors through some finite extension $\Phi : \mathrm{Spec}(E') \to \mathrm{Spec}(E)$, we can assume $D \in X^{(d)}(E')$ without loss of generality. By Proposition 3.17, the pullback of $P \circ \Phi \in \mathcal{A}(E')$ along J^d is a morphism $\mathrm{Spec}(E') \to X^{(d)}$, hence it is exactly D. If $Z \to X^{(d)}$ is the pullback of P along J^d, $Z \times_{\mathrm{Spec}(E)} \mathrm{Spec}(E')$ is isomorphic to $\mathrm{Spec}(E')$; we deduce that $Z \cong \mathrm{Spec}(E)$, and D factors through $Z \to X^{(d)}$. $\square$

With the following proposition we can finally switch between convergence of functions and convergence of divisors, which is essential to prove the functional identity of Theorem 5.9.

Proposition 3.19. *Fix a finite field extension* L/K_∞ *and a sequence* $(h_m)_m$ *in* $L \otimes A(\leqslant d)$.

If the sequence $(\mathrm{Div}(h_m) + d\infty)_m$ *converges to* $D \in X^{(d)}(L)$, *there are* $(\lambda_m)_m$ *in* $L^\times$ *such that* $(\lambda_m h_m)_m$ *converges to some* $h \in L \otimes A(\leqslant d) \setminus \{0\}$ *with* $\mathrm{Div}(h) = D - d\infty$.

If the sequence $(h_m)_m$ *converges to* $h \in L \otimes A(\leqslant d) \setminus \{0\}$, *the sequence* $(\mathrm{Div}(h_m) + d\infty)_m$ *converges to* $\mathrm{Div}(h) + d\infty \in X^{(d)}(L)$.

Proof. Call $V := \Gamma(\mathcal{I}_{d\infty}^{-1}, X)$ and call Z_d the pullback of the closed subscheme $0 \in \mathcal{A}$ along $J^d : X^d \to \mathcal{A}$, so that $\mathrm{Div}(h_m) + d\infty \in Z_d(L)$ for all m. As we noted in Remark 3.7, $\mathbb{P}(V)(L)$ is homeomorphic to $(L \otimes A(\leqslant d) \setminus \{0\})/L^\times$ endowed with the quotient topology. On the other hand, by Proposition 3.17 (setting $E = \mathbb{F}_q$ and $D = d\infty$), the $\mathbb{F}_q$-schemes $\mathbb{P}(V)$ and Z_d are isomorphic; in particular, the induced map $\mathbb{P}(V)(L) \to Z_d(L)$, which sends a line $L \cdot f \in L \otimes A(\leqslant d)$ to $\mathrm{Div}(f) + d\infty$, is a homeomorphism in the compact topology, by Remark 3.5.

If the sequence $(\mathrm{Div}(h_m) + d\infty)_m$ converges to $D \in Z_d(L)$, this proves that the equivalence classes $([h_m])_m$ in $L \otimes A(\leqslant d)/L^\times$ do converge to an equivalence class $[h]$ whose divisor is $D - d\infty$. Since the projection is open, we can lift this convergence to $L \otimes A(\leqslant d)$ up to scalar multiplication.

The map $L \otimes A(\leqslant d) \setminus \{0\} \to Z(L)$ sending a function f to the effective divisor $\mathrm{Div}(f) + d\infty$ is continuous. In particular, if the sequence $(h_m)_m$ converges to $h \in L \otimes A(\leqslant d) \setminus \{0\}$, the sequence $(\mathrm{Div}(h_m) + d\infty)_m$ converges to $\mathrm{Div}(h) + d\infty \in Z_d(L)$. $\square$

4. Frobenius and divisors

Fix an ideal $I \trianglelefteq A$, with ideal class $\bar{I} \in Cl(A)$; with slight abuse of notation, call I also the corresponding effective divisor of X. Call $\Xi \in X(K)$ the morphism $\mathrm{Spec}(K) \to X \setminus \infty$ corresponding to the canonical inclusion $A \hookrightarrow K$.

In the first subsection, we recall the notion of Frobenius twist $P^{(1)}$ for a point $P \in X^{(d)}(K_\infty)$, and study its behavior with respect to the compact topology. The main result is Proposition 4.6, where we prove that the sequence $(P^{(m)})_m$ converges to $\mathrm{red}_{K_\infty} P$.

In the second subsection, we study the divisor of a rational function h with respect to its expansion $\sum_{i \geqslant k} c_i u^i$ as an element of $K_\infty \hat{\otimes} K \cong K((u))$. Among several useful results, the most significant is Proposition 4.16, where we state the identity $\mathrm{Div}(c_k) = \mathrm{red}_{K_\infty}(\mathrm{Div}(h))$.

Finally, in the third subsection, we construct the divisors $\{V_{I,*,m}\}_{m>0}$ and $V_{I,*}$ in $X^{(g)}(K_\infty)$ (see Lemma 4.25), uniquely defined by the following properties for $m \gg 0$:

$$\begin{cases} V_{I,*,m} - V_{I,*,m}^{(1)} \sim \Xi^{(m)} - \Xi^{(1)} \\ \mathrm{red}_{K_\infty}(V_{I,*,m}) \sim (\deg(I)+g)\infty - I \end{cases} ; \qquad \begin{cases} V_{I,*} - V_{I,*}^{(1)} \sim \infty - \Xi \\ \mathrm{red}_{K_\infty}(V_{I,*}) \sim (\deg(I)+g)\infty - I \end{cases} .$$

The main result is the convergence of the sequence $(V_{I,*,m})_{m\gg 0}$ to $V_{I,*}^{(1)}$ in $X^{(g)}(K_\infty)$ (Proposition 4.28).

4.1 Frobenius twist

In this subsection we define the Frobenius twist for a (proper) $\mathbb{F}_q$-scheme Y and study its behavior with respect to the topology of $Y(K_\infty)$. The fundamental results are Proposition 4.6 and Lemma 4.10.

Definition 4.1. Let Y be an $\mathbb{F}_q$-scheme, and R an $\mathbb{F}_q$-algebra. Consider the morphism $\mathrm{Frob}_R : \mathrm{Spec}(R) \to \mathrm{Spec}(R)$ of $\mathbb{F}_q$-schemes induced by raising to the q^{th} power. The morphism Frob_R and the identity on Y induce an endomorphism of Y_R over $\mathrm{Spec}(\mathbb{F}_q)$: we denote it $F_R^Y : Y_R \to Y_R$.

Call $\pi_Y : Y_R \to Y$ and $\pi_R : Y_R \to \mathrm{Spec}(R)$ the natural projections. For all $P \in Y(R)$, denote $\overline{P}$ the unique element of $\mathrm{Hom}_R(\mathrm{Spec}(R), Y_R)$ such that $P = \pi_Y \circ \overline{P}$. We call Frobenius twist of P, and denote $P^{(1)} \in Y(R)$ the only element such that $\overline{P^{(1)}}$ is the pullback of $\overline{P}$ along F_R^Y. The n-th iteration of the twist is denoted $P^{(n)}$ for all $n \in \mathbb{N}$.

Lemma 4.2. *In the notation of Definition 4.1, we have $P^{(1)} = P \circ \mathrm{Frob}_R$.*

Proof. We have the following cartesian diagram:

$$\begin{tikzcd} \mathrm{Spec}(R) \arrow[r, "\overline{P^{(1)}}"] \arrow[d, "\mathrm{Frob}_R"'] \arrow[dr, phantom, "\square"] & Y_R \arrow[r, "\pi_R"] \arrow[d, "F_R^Y"] \arrow[dr, phantom, "\square"] & \mathrm{Spec}(R) \arrow[d, "\mathrm{Frob}_R"] \\ \mathrm{Spec}(R) \arrow[r, "\overline{P}"'] & Y_R \arrow[r, "\pi_R"'] & \mathrm{Spec}(R). \end{tikzcd}$$

Since $\pi_Y \circ F_R^Y = \pi_Y$, we get: $P^{(1)} = \pi_Y \circ \overline{P^{(1)}} = \pi_Y \circ F_R^Y \circ \overline{P^{(1)}} = \pi_Y \circ \overline{P} \circ \mathrm{Frob}_R = P \circ \mathrm{Frob}_R$. $\square$

Remark 4.3. In light of Lemma 4.2, if Frob_R is an isomorphism, for all $P \in Y(R)$ we can redefine $P^{(k)} \in Y(R)$ as $P \circ (\mathrm{Frob}_R)^k$ for all $k \in \mathbb{Z}$.

Lemma 4.4. *Fix a positive integer d, an $\mathbb{F}_q$-scheme Y, and an $\mathbb{F}_q$-algebra R, and consider a point $(P_1, \dots, P_d) \in Y^d(R)$. Its Frobenius twist is $(P_1^{(1)}, \dots, P_d^{(1)})$.*

Proof. The i-th projection $\pi_i : Y^d \to Y$ is such that $\pi_i \circ (P_1, \dots, P_d) = P_i$. By Remark 4.2:

$$\pi_i \circ \left((P_1, \dots, P_d)^{(1)}\right) = \pi_i \circ (P_1, \dots, P_d) \circ \mathrm{Frob}_R = P_i \circ \mathrm{Frob}_R = P_i^{(1)}. \qquad \square$$

Remark 4.5. The analogous statement, with the same proof, is true for any product of $\mathbb{F}_q$-schemes.

Let L/K_∞ be a finite field extension and Y a proper $\mathcal{O}_L$-scheme. Recall the notation red_L of Definition 3.3.

Proposition 4.6. *Fix a point $P \in Y(L)$, and set k_L such that $\#\mathbb{F}_L = q^{k_L}$. The sequence $(P^{(mk_L+r)})_m$ converges to $\mathrm{red}_L(P)^{(r)}$ in $Y(L)$.*

Proof. Since $\mathrm{Spec}(\mathcal{O}_L)$ only has one closed point, we can choose an open affine subscheme $U \subseteq Y$ with $B := \mathcal{O}_Y(U)$ such that $P \in U(\mathcal{O}_L)$: P corresponds to a map of $\mathcal{O}_L$-algebras $\chi_P : B \to \mathcal{O}_L$; its reduction modulo $\mathfrak{m}_L$, composed with the immersion $\mathbb{F}_L \hookrightarrow \mathcal{O}_L$, is equal to the morphism $\chi_{\mathrm{red}_L(P)} : B \to \mathcal{O}_L$ corresponding to $\mathrm{red}_L(P)$ by Definition 3.3. For all i, $P^{(i)}$ corresponds to the map $(\cdot)^{q^i} \circ \chi_P$, which modulo $\mathfrak{m}_L^{q^i}$ is the same as $\chi_{\mathrm{red}_L(P)^{(i)}}$, hence the projections of $P^{(i)}$ and $\mathrm{red}_L(P)^{(i)}$ onto $Y(\mathcal{O}_L/\mathfrak{m}_L^{q^i})$ coincide. Since $\mathrm{red}_L(P)^{(mk_L+i)} = \mathrm{red}_L(P)^{(i)}$ for all $m \geqslant 0$, this proves the convergence. □

Remark 4.7. For any effective divisor D of the curve X_L over $\mathrm{Spec}(L)$, we can define its twist $D^{(1)}$ as the pullback along F_L^X. Obviously, if $D = \sum P_i$ with $P_i \in X_L(L_i)$, $D^{(1)} = \sum P_i^{(1)}$.

Definition 4.8. Let h be a rational function on X_L, i.e. a morphism of L-schemes $X_L \to \mathbb{P}^1_L$. We define the Frobenius twist $h^{(1)} := h \circ F_L^X$.

Remark 4.9. The field rational functions of X_L is $\mathrm{Frac}(L \otimes A)$, and if $h = \sum_i l_i \otimes a_i \in L \otimes A$, $h^{(1)} = \sum_i l_i^q \otimes a_i$.

We show that the Frobenius twists of divisors and rational functions are compatible.

Lemma 4.10. *Let $h \in L(X)$, and call* $\mathrm{Div}(h)$ *its divisor. Then,* $\mathrm{Div}(h^{(1)}) = (\mathrm{Div}(h))^{(1)}$.

Proof. For any closed point $P \in \mathbb{P}^1_L$, $(h^{(1)})^*(P) = (F_L^X)^* \circ h^*(P)$; setting $P = 0$ and $P = \infty$, since the Frobenius twist on the divisors is induced by the pullback via F_L^X, we get our thesis. □

4.2 Rational functions on X_{K_∞} as Laurent series

Fix a finite field extension L/K_∞ and a uniformizer $u \in \mathcal{O}_L$. Call K' the fraction field of $X_{\mathbb{F}_L}$, i.e. $\mathbb{F}_L K$, and $A' := \mathbb{F}_L A$.

Remark 4.11. Since $L(X)$ is the fraction field of $\mathcal{O}_L \otimes_{\mathbb{F}_L} K'$, which has $\mathfrak{m}_L K' = (u \otimes 1)\mathcal{O}_L \otimes_{\mathbb{F}_L} K'$ as a maximal ideal, we can endow $L(X)$ with the $\mathfrak{m}_L K'$-adic metric.

Lemma 4.12. *There is a natural immersion of fields $L(X) \hookrightarrow K'((u))$, which is a completion with respect to the $\mathfrak{m}_L K'$-adic metric. Moreover, it induces an isomorphism between the completion $L\hat{\otimes}A$ of $L \otimes A \subseteq L(X)$ and $A[[u]][u^{-1}]$.*

Proof. The natural isomorphisms $(\mathcal{O}_L \otimes_{\mathbb{F}_L} K'/\mathfrak{m}_L^n K' \cong K'[u]/u^n)_{n \geqslant 1}$ pass to the limit and to fraction fields, giving a natural isometry between the completion of $L(X)$ and $K'((u))$.

The inclusion $L \otimes A \subseteq A[[u]][u^{-1}]$ is obvious, and by the previous reasoning it is an isometry with respect to the natural metric of $L \otimes A$; on the other hand each element in $A[[u]][u^{-1}]$ is the limit of its truncated expansions, which are in $L \otimes A$. □

Lemma 4.13. *For all positive integers d, the induced inclusion of $L \otimes_{\mathbb{F}_L} A'(\leqslant d) \cong L \otimes A(\leqslant d)$, endowed with its natural metric of finite L-vector space, into $K'((u))$ is a closed immersion.*

Proof. The restriction of the $\mathfrak{m}_L K'$-adic metric of $L(X)$ to $L \otimes_{\mathbb{F}_L} A'(\leqslant d)$ is the natural metric of a finite L-vector space. By Lemma 4.12, the inclusion into $K'((u))$ is an isometry, hence a closed immersion. □

Remark 4.14. For all $h \in L(X) \subseteq K'((u))$, if we write $h = \sum_{j \geqslant m} c_j u^j$, then $h^{(1)} = \sum_{j \geqslant m} c_j^{(1)} u^{qj}$.

To better understand the usefulness of $K'((u))$, let's state a couple of propositions. First, we prove a very natural result, analogous to Lemma 4.10 but with the reduction instead of the twist.

Definition 4.15. For all nonzero $h \in L(X) \subseteq K'((u))$, write $h = \sum_{j \geqslant m} c_j u^j$ with $c_m \neq 0$, and set $\mathrm{red}_u(h) := c_m$.

Proposition 4.16. *For all nonzero $h \in L(X)$, $\mathrm{Div}(\mathrm{red}_u(h)) = \mathrm{red}_L(\mathrm{Div}(h))$, where both are $\mathbb{F}_L$-rational divisors of $X_{\mathbb{F}_L}$.*

Proof. Since for any nonzero $h \in L(X)$ there is a positive integer d and $h_+, h_- \in \mathcal{O}_L \otimes_{\mathbb{F}_L} A'(\leqslant d)$ such that $h = \frac{h_+}{h_-}$, we can assume $h \in \mathcal{O}_L \otimes_{\mathbb{F}_L} A'(\leqslant d)$. Up to a factor in $L^\times$, we can also assume $h = \sum_{i \geqslant 0} c_i u^i \in K'[[u]]$ with $c_0 \in A'(\leqslant d) \setminus \{0\}$. By Remark 4.14, the sequence $(h^{(mk_L)})_m$ is equal to $(\sum_{j \geqslant 0} c_j u^{jmk_L})_m$, hence it converges to c_0 in $K'[[u]]$; by Lemma 4.13 this convergence lifts to $L \otimes_{\mathbb{F}_L} A'(\leqslant d)$. The sequence of divisors $(\mathrm{Div}(h^{(mf_L)}) + d\infty)_m$, by Proposition 3.19, converges to $\mathrm{Div}(c_0) + d\infty$ in $X^{(d)}(L)$; on the other hand, by Proposition 4.6, it converges to $\mathrm{red}_L(\mathrm{Div}(h)) + d\infty$, hence we have the desired equality. □

We prove now that the immersion $L(X) \hookrightarrow K'((u))$ behaves reasonably well with evaluations.

Proposition 4.17. *Fix $h \in L \otimes A(d)$, and expand $h = \sum_i h_{(i)} u^i$ as an element of $K'((u))$; fix $P \in X_{\mathbb{F}_L}(L^{ac}) \setminus \{\infty\}$, corresponding to a $\mathbb{F}_L$-linear homomorphism $\chi_P : A' \to L^{ac}$. Then, $h_{(i)} \in A'$ for all i and $h(P) = \sum_i \chi_P(h_{(i)}) u^i$.*

Proof. We can write $h = \sum_j \gamma_j a_j$, with $\gamma_j \in L = \mathbb{F}_L((u))$ and $a_j \in A'(\leqslant d)$, hence $h_{(i)} \in A'(\leqslant d)$ for all i. For all integers m define $\gamma_{j,m}$ as the truncation of $\gamma_j \in \mathbb{F}_L((u))$ at the degree m, and define $h_m := \sum_j \gamma_{j,m} a_j \in K'[u^{\pm 1}]$, so that $h_m = \sum_{i \leqslant m} h_{(i)} u^i$. We have the equalities:

$$h_m(P) = \chi_P\left(\sum_j \gamma_{j,m} a_j\right) = \sum_j \gamma_{j,m} \chi_P(a_j); \quad h_m(P) = \chi_P\left(\sum_{i \leqslant m} h_{(i)} u^i\right) = \sum_{i \leqslant m} \chi_P(h_{(i)}) u^i;$$

where we used that both summations are finite. Since the sequence $(\gamma_{j,m})_m$ converges to γ_j in $\mathbb{F}_L((u))$ for all j, the first equation tells us that the sequence $(h_m(P))_m$ converges to $h(P)$. On the other hand, from the second equation, it also converges to $\sum_i \chi_P(h_{(i)}) u^i$. □

Proposition 4.18. *Let $h = \sum_i h_{(i)} u^i \in A'[[u]][u^{-1}]$ be a rational function over X_L; for any finite field extension E/L, fix $P \in X_{\mathbb{F}_L}(L)$ such that $\mathrm{red}_L(P) \neq \infty$, corresponding to a function $\chi_P : A' \to \mathcal{O}_L$. Then P is not a pole of h, and $h(P) = \sum_i \chi_P(h_{(i)}) u^i$.*

Proof. For $N \gg 0$, we have a strict inclusion between the following global sections of sheaves over X_L: $\mathcal{L}(N\infty - \mathrm{Div}_-(h) - P) \subsetneq \mathcal{L}(N\infty - \mathrm{Div}_-(h))$; we can fix h_- in their difference, and set $h_+ := hh_-$; by definition, $h_+, h_- \in L \otimes A$.

If we write $h_+ = \sum_i h_{+,(i)} u^i$ and $h_- = \sum_i h_{-,(i)} u^i$, we have for all integers k the equation $h_{+,(k)} = \sum_{i+j=k} h_{(i)} h_{-,(j)}$, which commutes with evaluation, being a finite sum. Since χ_P has image in $\mathcal{O}_L$, the series $\sum_i \chi_P(h_{(i)}) u^i$ converges, hence by Proposition 4.17 we get the following equation in $\mathcal{O}_L$:

$$\begin{aligned} h_+(P) &= \sum_k \chi_P(h_{+,(k)}) u^k = \sum_k \sum_{i+j=k} \chi_P(h_{(i)}) \chi_P(h_{-,(j)}) u^k \\ &= \left(\sum_i \chi_P(h_{(i)}) u^i\right) \left(\sum_j \chi_P(h_{-,(j)}) u^j\right) = \left(\sum_i \chi_P(h_{(i)}) u^i\right) h_-(P). \end{aligned}$$

Since $h_- \in \mathcal{L}(N\infty - \mathrm{Div}_-(h)) \setminus \mathcal{L}(N\infty - \mathrm{Div}_-(h) - P)$, if P is a pole of h, then P is a zero of h_- of the same order, and $h_+(P) \neq 0$, hence we reach a contradiction by the previous equation. Since P is not a pole of h, then $h_-(P) \neq 0$, and since $h_+(P) = h(P)h_-(P)$ we get:

$$h(P) = \frac{h_+(P)}{h_-(P)} = \sum_i \chi_P(h_{(i)})u^i. \qquad \square$$

Corollary 4.19. *Let $h = \sum_i h_{(i)}u^i \in K'[[u]]$ be a rational function over X_L. Then, h is in $A'[[u]][u^{-1}]$ if and only if all its poles reduce to ∞.*

Proof. Suppose $h \in A'[[u]][u^{-1}]$ and take a pole $P \in X_L(E)$ of h. Define $A'' := A \otimes \mathbb{F}_E$ and $K'' := K \otimes \mathbb{F}_E$, where $\mathbb{F}_E$ is the residue field of E, and fix a uniformizer v of $\mathcal{O}_E$. The natural immersion $K'((u)) \subseteq K''((v))$ sends h into $A''[[v]][v^{-1}]$; applying Proposition 4.18 to the field E, $\mathrm{red}_E(P) = \infty$.

If vice versa $h \notin A'[[u]][u^{-1}]$, call m the least integer such that $h_{(m)} \notin A'$ and set $h' := \sum_{i<m} h_{(i)}u^i$. By Lemma 4.16:

$$\mathrm{Div}(h_{(m)}) = \mathrm{Div}(\mathrm{red}_u(h - h')) = \mathrm{red}_L(\mathrm{Div}(h - h'))$$

and, since $h_{(m)} \notin A'$, $h - h'$ has a pole at a point P which does not reduce to ∞; on the other hand, since $h' \in A'[[u]][u^{-1}]$, it does not have a pole at P, hence P is a pole of $h = (h - h') + h'$. $\square$

Corollary 4.20. *Let $h = \sum_i h_{(i)}u^i \in A'[[u]][u^{-1}]$ be a nonzero rational function over X_L, and suppose that the coefficients $(h_{(i)})_i$ are all contained some prime ideal $P \trianglelefteq A'$. Then P, as a closed point of X_L, is a zero of h.*

Proof. If we define $\mathbb{F}_P := A'/P$, we can take a point $Q \in X_L(\mathbb{F}_P L)$ with support at P. By Proposition 4.18, we get $h(Q) = \sum_i h_{(i)}(Q)u^i = 0$, hence P is a zero of h. $\square$

4.3 Notable divisors and convergence results

As foreshadowed by Lemma 4.10, in this subsection we explore the relation between Frobenius twists, divisors, and the compact topology.

Lemma 4.21 (Drinfeld's vanishing lemma). *Let E/K be a field extension, $W \in X^{(d)}(E)$ for some $d \leqslant g$, $P, Q \in X(E)$. Suppose that $[W - W^{(m)}] = [P - Q]$, where $P \neq Q^{(sm)}$ for $0 \leqslant s + d \leqslant 2g$; then $d = g$ and $h^0(W) = 1$.*

Proof. Call $W_0 := W$ and set $W_{i+1} = W_i + Q^{(im)}$ for all $i \in \mathbb{Z}$. Note that, since $\deg(W_k) = d + k$, $h^0(W_{-d-1}) = 0$ and $h^0(W_{2g-d-1}) = g$. For all i, $h^0(W_i) \leqslant h^0(W_{i+1}) \leqslant h^0(W_i) + 1$, so there is a least integer $k \in [-d, 0]$ such that $h^0(W_k) = 1$.

Let's prove that for all $i \in [-d, 2g - d - 1[$, if $h^0(W_i) \geqslant 1$, then $h^0(W_{i+1}) = h^0(W_i) + 1$. We have two relations:

$$W_{i+1} = (W + Q + \cdots + Q^{((i-1)m)}) + Q^{(im)} = W_i + Q^{(im)},$$
$$W_{i+1} = (W + Q^{(m)} + \cdots + Q^{(im)}) + Q = (W_i^{(m)} - W^{(m)} + W) + Q \sim W_i^{(m)} + P;$$

they induce two inclusions of vector spaces - $\mathcal{L}(W_i^{(m)}) \subseteq \mathcal{L}(W_{i+1})$ and $\mathcal{L}(W_i) \subseteq \mathcal{L}(W_{i+1})$. To prove that those inclusions are strict, we need that $\mathcal{L}(W_i) \neq \mathcal{L}(W_i^{(m)})$ as subspaces of $\mathcal{L}(W_{i+1})$; they have the same dimension (because the Frobenius twist induces an isomorphism between the two vector spaces), so we just need $W_i \not\sim W_i^{(m)}$, but:

$$W_i \not\sim W_i^{(m)} \Leftrightarrow W - W^{(m)} \not\sim Q^{(im)} - Q \Leftrightarrow P \not\sim Q^{(im)} \Leftrightarrow P \neq Q^{(im)},$$

which is implied by our hypothesis. In particular, since W_{2g-d-1} has degree $2g-1$, we get that

$$g = h^0(W_{2g-d-1}) = h^0(W_k) + 2g - d - 1 - k = 2g - d - k,$$

therefore $g = d + k$; but $k \leqslant 0$ and $d \leqslant g$ implies $d = g$ and $k = 0$, therefore $h^0(W) = 1$. □

The previous lemma ensures that if such a divisor W exists, it has no other effective divisors in its same equivalence class. On the other hand, the existence of such W is ensured by the following propositions.

As usual, let L be a finite field extension of K_∞, with residue field $\mathbb{F}_L$ and $q^{k_L} := \#\mathbb{F}_L$

PROPOSITION 4.22. *Call $\mathcal{A}_0(L)$ the kernel of $\mathrm{red}_L : \mathcal{A}(L) \to \mathcal{A}(\mathbb{F}_L)$ (which is a continuous homomorphism). The map $\mathcal{A}(L) \to \mathcal{A}_0(L) \times \mathcal{A}(\mathbb{F}_L)$ sending a point D to the couple $(D - D^{(k_L)}, \mathrm{red}_L(D))$ is an isomorphism of topological groups.*

Proof. The map is obviously a continuous group homomorphism. Since domain and codomain are both compact and Hausdorff, it's sufficient to prove bijectivity.

On one hand, to prove injectivity, if we suppose $D - D^{(k_L)} = 0$ we have $D \in \mathcal{A}(\mathbb{F}_L)$, so if $\mathrm{red}_L(D) = 0$ we can deduce that $D = 0$.

On the other hand, to prove surjectivity, we fix $(D_0, \tilde{D}) \in \mathcal{A}_0(L) \times \mathcal{A}(\mathbb{F}_L)$ and show that they are the image of some $D \in \mathcal{A}(L)$. By Proposition 4.6 we have that the sequence $(D_0^{(ik_L)})_i$ converges to $\mathrm{red}_L(D_0) = 0$, hence by Corollary 3.10 the series $\tilde{D} + \sum_{i\geqslant 0} D_0^{(ik_L)}$ converges to some point $D \in \mathcal{A}(L)$. Since the Frobenius twist and the reduction red_L are continuous endomorphisms of $\mathcal{A}(L)$, we get the following equations:

$$D - D^{(k_L)} = \tilde{D} + \sum_{i\geqslant 0} D_0^{(ik_L)} - \tilde{D}^{(k_L)} - \sum_{i\geqslant 1} D_0^{(ik_L)} = D_0; \qquad \mathrm{red}_L(D) = \mathrm{red}_L(\tilde{D}) = \tilde{D},$$

hence the image of D is $(D_0, \tilde{D})$. □

From now on, given an effective divisor $W \in X^{(d)}(L)$, we denote with $J(W)$ its image via the morphism $J^d : X^{(d)} \to \mathcal{A}$, i.e. the equivalence class of $W - d\infty$ in the Jacobian.

COROLLARY 4.23. *Fix a point $D \in \mathcal{A}(\mathbb{F}_L)$, and let $P, Q \in X(L)$ such that $\mathrm{red}_L(P) = \mathrm{red}_L(Q)$, with $P \neq Q^{(s)}$ for $|s| < 2g$.*

Then, there is a unique effective divisor W such that: $W - W^{(k_L)} \sim P - Q$, $\mathrm{red}_L(J(W)) = D$, and $\deg(W) \leqslant g$. Moreover, a fortiori, $W \in X^{(g)}(L)$ and, if R is a point in the support of W, $R \notin X(\mathbb{F}_q^{ac})$.

Proof. By Proposition 4.22 there is an element $D' \in \mathcal{A}(L)$ such that $D' - D'^{(k_L)} = [P - Q]$ and $\mathrm{red}_L(D') = D$. Since the morphism J^g is surjective, there is a divisor $W \in X^{(g)}(L^{ac})$ such that $J(W) = D'$. By Drinfeld's vanishing lemma, there is only one divisor of degree $\leqslant g$ with the requested properties, hence $h^0(W) = 1$; by Corollary 3.18, W is L-rational.

Now, call $W' \leqslant W$ a maximal $\mathbb{F}_q^{ac}$-rational effective divisor ($W' \in X^{(d)}(\mathbb{F}_q^{ac})$), and call G the group of L-linear field automorphisms of L^{ac}, which acts naturally on $X(L^{ac})$. Since $W \in X^{(g)}(L)$, it is fixed by the induced action of G; moreover, this action sends $X(\mathbb{F}_q^{ac})$ to itself, hence $W' \leqslant W$ is also fixed by G: since W' is both L-rational and $\mathbb{F}_q^{ac}$-rational, $W' \in X^{(d)}(\mathbb{F}_L)$. We have:

$$(W - W') - (W - W')^{(k_L)} = (W - W^{(k_L)}) + (W' - W'^{(k_L)}) = W - W^{(k_L)} \sim P - Q,$$

but $\deg(W - W') = g$ from Drinfeld's vanishing lemma, hence $d = deg(W') = 0$. □

Recall the notations of $I, \bar{I}, \Xi$ from the start of this section.

Lemma 4.24. *We have the identity* $\mathrm{red}_{K_\infty}(\Xi) = \infty$ *in* $X(K_\infty)$.

Proof. Since the image of the canonical inclusion $A \hookrightarrow K_\infty$ is not contained in $\mathcal{O}_{K_\infty}$, the morphism $\Xi : \mathrm{Spec}(K_\infty) \to X \setminus \infty$ does not factor through $\mathrm{Spec}(\mathcal{O}_{K_\infty})$, which means that $\mathrm{red}_{K_\infty}(\Xi) \notin X(\mathbb{F}_q) \setminus \infty$, so $\mathrm{red}_{K_\infty}(\Xi) = \infty$. $\square$

Next, we construct some notable divisors.

Lemma 4.25. *The following effective divisors of* X_{K_∞} *exist and are unique:*

- *a divisor* $V_{\bar{I}}$ *of degree at most* g*, such that* $V_{\bar{I}} - V_{\bar{I}}^{(1)} \sim \Xi - \infty$ *and* $\mathrm{red}_{K_\infty}(J(V_{\bar{I}})) = J(I)$*;*
- *for* $m \geqslant 1$*, a divisor* $V_{\bar{I},m}$ *of degree at most* g*, such that* $V_{\bar{I},m} - V_{\bar{I},m}^{(1)} \sim \Xi^{(1)} - \Xi^{(m+1)}$ *and* $\mathrm{red}_{K_\infty}(J(V_{\bar{I}})) = J(I)$*;*
- *a divisor* $V_{\bar{I},*}$ *of degree at most* g *such that* $J(V_{\bar{I},*}) + J(V_{\bar{I}}) = 0$*;*
- *for* $m \gg 0$*, a divisor* $V_{\bar{I},*,m}$ *of degree at most* g *such that* $J(V_{\bar{I},*,m}) + J(V_{\bar{I},m}) = 0$.

Moreover, they all are in $X^{(g)}(K_\infty)$.

Proof. Let's first note that the divisors, if they exist, are well defined: since for all $a, b \in^* A$ $J(aI) = J(bI)$, the properties of the divisors we want to construct only depend on the ideal class $\bar{I} \in Cl(A)$ of I.

Since $\mathrm{red}_{K_\infty}(\Xi) = \infty$ by Lemma 4.24, we can apply Corollary 4.23 to $P = \Xi$ and $Q = \infty$ (resp. $P = \Xi^{(1)}$ and $Q = \Xi^{(m+1)}$ for $m \geqslant 1$), so the divisor $V_{\bar{I}}$ (resp. $V_{\bar{I},m}$) exists, is unique, and is contained in $X^{(g)}(K_\infty)$.

Since $J^g(K_\infty^{ac}) : X^{(g)}(K_\infty^{ac}) \to \mathcal{A}(K_\infty^{ac})$ is surjective, there is at least one effective divisor $V_{\bar{I},*}$ of degree at most g such that $J(V_{\bar{I},*}) = -J(V_{\bar{I}})$. It has the following properties:

$$[V_{\bar{I},*} - V_{\bar{I},*}^{(1)}] = [V_{\bar{I}}^{(1)} - V_{\bar{I}}] = [\infty - \Xi]; \qquad \mathrm{red}_{K_\infty}(J(V_{\bar{I},*})) = -\,\mathrm{red}_{K_\infty}(J(V_{\bar{I}})) = -J(I).$$

By Corollary 4.23 applied to $P = \infty$ and $Q = \Xi$, $V_{\bar{I},*}$ is unique, K_∞-rational, and of degree g.

In the same way we can prove the existence and uniqueness of $V_{\bar{I},*,m}$ for $m \gg 0$, since it has the following properties: $V_{\bar{I},*,m} - V_{\bar{I},*,m}^{(1)} \sim \Xi^{(m+1)} - \Xi^{(1)}$, and $\mathrm{red}_{K_\infty}(J(V_{\bar{I},*,m})) = -J(I)$. $\square$

Remark 4.26. The classical construction of the *Drinfeld divisors* $\{V_{\bar{I}}\}_{\bar{I}}$ (see [Tha93]) is simpler and gives arguably more information than the one presented in this paper. On the other hand, a topological point of view is more consistent with the rest of the work.

Remark 4.27. Since we fixed an inclusion $H \subseteq K_\infty$, the divisors $\{V_{\bar{I}}\}_{\bar{I}}$ are actually H-rational, and the natural action of $\mathrm{Gal}(H/K)$ on this set is free and transitive (see [Hay79][Prop. 3.2, Thm. 8.5]). Call $\bar{I}^\sigma \in Cl(A)$ the element such that $V_{\bar{I}^\sigma} = V_{\bar{I}}^\sigma$. Since this action commutes with morphisms of schemes, for all $\sigma \in \mathrm{Gal}(H/K)$, for all $\bar{I} \in Cl(A)$, we have that

$$[V_{\bar{I},*}^\sigma - g\infty] = [V_{\bar{I},*} - g\infty]^\sigma = [g\infty - V_{\bar{I}}]^\sigma = [g\infty - V_{\bar{I}}^\sigma] = [g\infty - V_{\bar{I}^\sigma}] = [V_{\bar{I}^\sigma,*} - g\infty];$$

hence $V_{\bar{I},*}^\sigma = V_{\bar{I}^\sigma,*}$ by Lemma 4.25 because of uniqueness.

Finally, we state the main result of this subsection, which is central to the proof of Theorem 5.9.

Proposition 4.28. *The sequences* $(V_{\bar{I},m})_m$ *and* $(V_{\bar{I},*,m})_m$ *converge respectively to the divisors* $V_{\bar{I}}^{(1)}$ *and* $V_{\bar{I},*}^{(1)}$ *in* $X^{(g)}(K_\infty)$.

Proof. Define $U := \{D \in X^{(g)}(K_\infty) | h^0(D) = 1\}$, so that the restriction $J^g(K_\infty)|_U$ induces a bijection of U with its image in $\mathcal{A}(K_\infty)$, which by Lemma 3.6 is a homeomorphism. By Lemma 4.25 - for $m \gg 0$ - $h^0(V_{\bar{I},m}) = h^0(V_{\bar{I}}^{(1)}) = 1$, so $V_{\bar{I},m}, V_{\bar{I}}^{(1)} \in U$, and it suffices to prove the convergence of their images in $\mathcal{A}(K_\infty)$.

If we identify $\mathcal{A}(K_\infty)$ and $\mathcal{A}(\mathbb{F}_q) \times \mathcal{A}_0(K_\infty)$ by Proposition 4.22, we have:

$$\begin{aligned}\lim_m V_{\bar{I},m} &= \lim_m \left(\mathrm{red}_{K_\infty}(J^g(V_{\bar{I},m})), [V_{\bar{I},m} - V_{\bar{I},m}^{(1)}]\right) = \lim_m \left(J(I), [\Xi^{(1)} - \Xi^{(m+1)}]\right) \\ &= \left(J(I), [\Xi^{(1)} - \infty]\right) = \left(\mathrm{red}_{K_\infty}(J^g(V_{\bar{I}}^{(1)})), [V_{\bar{I}}^{(1)} - V_{\bar{I}}^{(2)}]\right) = V_{\bar{I}}^{(1)}.\end{aligned}$$

Similarly, for the other statement, it suffices to prove that the sequence $(J(V_{\bar{I},m,*}))_m$ converges to $J(V_{\bar{I},*}^{(1)})$, which is obvious because $J(V_{\bar{I},m,*}) = -J(V_{\bar{I},m})$ for all $m \gg 0$ and $J(V_{\bar{I},*}^{(1)}) = -J(V_{\bar{I}}^{(1)})$. □

5. Pellarin zeta functions

Throughout this and all the next sections we fix a uniformizer $u \in K_\infty$ and an ideal $I \trianglelefteq A$. As in the previous section, we also call I the corresponding closed subscheme of $X \setminus \infty$, and d its degree.

The following definition is a generalization of the zeta functions à la Pellarin introduced in [Pel11].

Definition 5.1. The (partial) *Pellarin zeta function* relative to I is defined as the series:

$$\zeta_I := \sum_{a \in^* I} a^{-1} \otimes a \in K_\infty \hat{\otimes} A.$$

In this section, we first define the rational approximations $\{\zeta_{I,m}\}$ of ζ_I and compute their divisors, in analogy to what already done by Chung, Ngo Dac and Pellarin in the case $I = A$ (see [CDP21][Lemma 2.1]). Afterwards, we use Proposition 3.19 to prove a functional identity regarding ζ_I in the shape of an infinite product, i.e. the partial version of Theorem 5.9.

The complete version of the theorem, stated in the introduction, is proven at the end of Section 7.

5.1 The approximations of ζ_I and their divisors

For $m \in \mathbb{N}$, call j_m the least integer such that $\dim_{\mathbb{F}_q}(I(\leqslant j_m)) = h^0(j_m\infty - I) = m+1$. We call $a_I \in I$ the nonzero element with least degree (i.e. $a_I \in I(j_0)$) and sign 1.

Remark 5.2. Since $\deg(j_m\infty - I) + 1 - g \leqslant h^0(j_m\infty - I) \leqslant \deg(j_m\infty - I) + 1$, we get the inequality:

$$m + d \leqslant j_m \leqslant m + g + d.$$

Moreover, for $m \gg 0$, the rightmost inequality becomes an equality.

Definition 5.3. We set for all $m \geqslant 0$:

$$\zeta_{I,m} := \sum_{a \in^* I(\leqslant j_m)} a^{-1} \otimes a \in K_\infty \otimes A.$$

Remark 5.4. The sequence $\zeta_{I,m}$ converges to ζ_I in $K_\infty\hat{\otimes}A \cong A[[u]]$.

Proposition 5.5. *The divisor of $\zeta_{I,m}$ is $\Xi^{(1)}+\dots+\Xi^{(m)}+I+W_m-j_m\infty$ for some effective divisor W_m with $h^0(W_m)=1$. Moreover, for $m\gg 0$, $j_m=m+g+d$ and $W_m=V_{\bar{I},*,m}$.*

We use the following technical lemma (see [Gos12][Lemma 8.8.1]).

Lemma 5.6. *Let $U\subseteq \mathbb{C}_\infty$ be a finite $\mathbb{F}_q$-vector space of dimension α, let a be a nonnegative integer, and denote a_i - with $0\leqslant a_i<q$ - the i-th digit of the expansion of a in base q. Then, if $\sum_i a_i<(q-1)\alpha$, the polynomial $\sum_{w\in U}(x+w)^a\in\mathbb{C}_\infty[x]$ is identically zero. In particular, this happens for $0\leqslant a<q^\alpha-1$.*

Proof of Proposition 5.5. Since $\zeta_{I,m}$ is sum of elements whose divisor contains I, it's obvious that $\mathrm{Div}^+(\zeta_{I,m})\geqslant I$. For any positive integer k we have:

$$\zeta_{I,m}(\Xi^{(k)})=\sum_{a\in I(\leqslant j_m)} a^{q^k-1},$$

which by Lemma 5.6 is zero when $k\leqslant \dim(I(\leqslant j_m))-1=m$. Since the only poles are at ∞, and have multiplicity at most j_m, $\mathrm{Div}(\zeta_{I,m})=\Xi^{(1)}+\dots+\Xi^{(m)}+I+W_m-j_m\infty$ for some effective divisor W_m. To study $h^0(W_m)$, call $D_n:=j_m\infty-I-\sum_{i=1}^n\Xi^{(i)}$ for all nonnegative integers n.

Note that, since $(j_m\infty-I)^{(1)}=j_m\infty-I$, we deduce for all $n\geqslant 0$:

$$\mathcal{L}(D_{n+1})\subseteq\mathcal{L}(D_n),\qquad \mathcal{L}(D_{n+1})\subseteq\mathcal{L}(D_n^{(1)}),\qquad \mathcal{L}(D_n)\cap\mathcal{L}(D_n^{(1)})=\mathcal{L}(D_{n+1}).$$

Let's prove that, if $h^0(D_n)\geqslant 1$, then $h^0(D_{n+1})=h^0(D_n)-1$. If this were not the case, since for $k\gg 0$ $h^0(D_k)<\deg(D_k)<0$, we could fix the maximum n such that $h^0(D_{n+1})=h^0(D_n)>0$, and we would get the following implications:

$$\begin{aligned}\left(\mathcal{L}(D_{n+1}^{(1)})+\mathcal{L}(D_{n+1})\subseteq\mathcal{L}(D_n^{(1)})\right)&\Rightarrow\left(\mathcal{L}(D_n)=\mathcal{L}(D_{n+1})=\mathcal{L}(D_{n+1}^{(1)})\right)\\ \Rightarrow\left(\mathcal{L}(D_{n+1})=\mathcal{L}(D_{n+1})\cap\mathcal{L}(D_{n+1}^{(1)})=\mathcal{L}(D_{n+2})\right)&\Rightarrow\left(h^0(D_{n+2})=h^0(D_{n+1})\right),\end{aligned}$$

contradicting the maximality hypothesis on n. In particular, since $h^0(W_m)\geqslant 1$, we have:

$$h^0(W_m)=h^0(D_m)=h^0(D_0)-m=h^0(j_m\infty-I)-m=1.$$

On one hand, $\deg(W_m)=\deg(j_m\infty-\Xi^{(1)}-\dots-\Xi^{(m)}-I)=j_m-m-d$, which is $\leqslant g$ by Remark 5.2. On the other hand, by Lemma 4.10 and Proposition 4.16 we have:

$$\begin{aligned}&0\sim\mathrm{Div}(\zeta_{I,m})-\mathrm{Div}(\zeta_{I,m})^{(1)}=\Xi^{(1)}-\Xi^{(m+1)}+W_m-W_m^{(1)},\\ &0\sim\mathrm{Div}(\mathrm{red}_u(\zeta_{I,m}))\sim\mathrm{red}_{K_\infty}(\mathrm{Div}(\zeta_{I,m}))=I+\mathrm{red}_{K_\infty}(W_m)-(d+g)\infty;\end{aligned}$$

so $W_m-W_m^{(1)}\sim\Xi^{(m+1)}-\Xi^{(1)}$, and $\mathrm{red}_{K_\infty}(W_m-g\infty)\sim d\infty-I$. Therefore, for $m\gg 0$, $W_m=V_{\bar{I},*,m}$ by Lemma 4.25. □

5.2 The function ζ_I as an infinite product

Proposition 5.7. *In $\mathcal{O}_{K_\infty}\otimes K\subseteq K[[u]]$ there are functions $f'_{\bar{I},*},f'_{\bar{I}}\in 1+uK[[u]]$, with divisors $V_{\bar{I},*}-V_{\bar{I},*}^{(1)}+\Xi-\infty$ and $V_{\bar{I}}^{(1)}-V_{\bar{I}}+\Xi-\infty$, respectively. Moreover, there is a rational function $\delta'_{\bar{I}}\in\mathcal{O}_{K_\infty}\otimes K$, with divisor $V_{\bar{I}}+V_{\bar{I},*}-2g\infty$, such that $\frac{\delta_{\bar{I}}'^{(1)}}{\delta'_{\bar{I}}}=\frac{f'_{\bar{I}}}{f'_{\bar{I},*}}$.*

Proof. From the definition of $V_{\bar{I},*}$, the divisor $V_{\bar{I},*} - V_{\bar{I},*}^{(1)} + \Xi - \infty$ is principal, hence it comes from some $f'_{\bar{I},*} \in K_\infty(X)$. Moreover, by Lemma 4.16:

$$\mathrm{Div}(\mathrm{red}_u(f'_{\bar{I},*})) = \mathrm{red}_{K_\infty}(\mathrm{Div}(f'_{\bar{I},*})) = \mathrm{red}_{K_\infty}(V_{\bar{I},*}) - \mathrm{red}_{K_\infty}(V_{\bar{I},*})^{(1)} = 0,$$

hence $\mathrm{red}_u(f'_{\bar{I},*}) \in \mathbb{F}_q$; up to scalar multiplication, we can assume $f'_{\bar{I},*} = 1 + O(u)$. The existence of $f'_{\bar{I}}$ can be proven in the same way.

Since $V_{\bar{I}} + V_{\bar{I},*} - 2g\infty \sim 0$, we can pick $\tilde{\delta}'_{\bar{I}} \in K_\infty \otimes A(\leqslant 2g)$ with that divisor, and up to scalar multiplication we can assume $\tilde{\delta}'_{\bar{I}} = c_0 + O(u)$ for some $c_0 \in K$. We get:

$$\mathrm{Div}(\tilde{\delta}'_{\bar{I}})^{(1)} - \mathrm{Div}(\tilde{\delta}'_{\bar{I}}) = \mathrm{Div}(f'_{\bar{I}}) - \mathrm{Div}(f'_{\bar{I},*}) \Longrightarrow \frac{\tilde{\delta}'^{(1)}_{\bar{I}}}{\tilde{\delta}'_{\bar{I}}} = \lambda \frac{f'_{\bar{I}}}{f'_{\bar{I},*}}$$

for some $\lambda \in K_\infty$; moreover, by considering the expansion in $K((u))$, $\lambda = 1 + O(u)$, hence it admits a $q-1$-th root $\mu \in \mathcal{O}_{K_\infty}$. If we set $\delta'_{\bar{I}} := \mu \tilde{\delta}'_{\bar{I}}$ we obtain the desired equation. □

Remark 5.8. The choices of $f'_{\bar{I}}, f'_{\bar{I},*}, \delta'_{\bar{I}}$ are not unique.

THEOREM 5.9 (Partial version). *The infinite product* $(a_I^{-1} \otimes a_I) \prod_{i \geqslant 1} f'^{(i)}_{\bar{I},*}$ *exists in* $\mathcal{O}_{K_\infty} \hat{\otimes} K$ *and is equal, up to a factor* $\lambda \otimes 1 \in^* \mathcal{O}_{K_\infty} \otimes \mathbb{F}_q$, *to* ζ_I. *We can also write:*

$$\zeta_I = -(a_I^{-1} \otimes a_I) \prod_{i \geqslant 0} \left((\lambda \otimes 1)^{1-q} f'^{(1)}_{\bar{I},*} \right)^{(i)}.$$

Proof. Let's immerse $\mathcal{O}_{K_\infty} \hat{\otimes} K$ into $K[[u]]$. By Proposition 5.7, $f'_{\bar{I},*} = 1 + O(u)$, hence $f'^{(i)}_{\bar{I},*} = 1 + O(u^{q^i})$ for all $i \geqslant 0$, and the convergence of the infinite product is obvious. For all $m \geqslant 0$:

$$\mathrm{red}_u(\zeta_{I,m}) = \sum_{\mu \in \mathbb{F}_q^\times} (\mu a_I)^{-1} \otimes (\mu a_I) = -a_I^{-1} \otimes a_I.$$

In particular, by Proposition 4.16, for $m \gg 0$ we have:

$$\mathrm{Div}(1 \otimes a_I) = \mathrm{Div}(\mathrm{red}_u(\zeta_{I,m})) = \mathrm{red}_{K_\infty}(\mathrm{Div}(\zeta_{I,m})) = I + \mathrm{red}_{K_\infty}(V_{\bar{I},*,m}) - (g+d)\infty;$$

since $\mathrm{red}_{K_\infty} : X^{(g)}(K_\infty) \to X^{(g)}(K_\infty)$ is a continuous map, and the sequence $(V_{\bar{I},*,m})_m$ converges to $V_{\bar{I},*}$ in $X^{(g)}(K_\infty)$ by Lemma 4.28, the equality passes to the limit:

$$\mathrm{red}_{K_\infty}(V_{\bar{I},*}) = \mathrm{Div}(1 \otimes a_I) + (g+d)\infty - I.$$

Define the rational function $\alpha_m := \delta'^{(1)}_{\bar{I}} \frac{\zeta_{I,m}}{f'^{(1)}_{\bar{I},*} \cdots f'^{(m)}_{\bar{I},*}}$ for $m \gg 0$ and look at its divisor:

$$\mathrm{Div}(\alpha_m) = I + V_{\bar{I},*}^{(m+1)} + V_{\bar{I},*,m} + V_{\bar{I}}^{(1)} - (3g+d)\infty \Longrightarrow \alpha_m \in K_\infty \otimes A(\leqslant 3g+d).$$

By Lemma 4.28, the sequence $(\mathrm{Div}(\alpha_m) + (3g+d)\infty)_m$ converges to

$$I + \mathrm{red}_{K_\infty}(V_{\bar{I},*}) + V_{\bar{I},*}^{(1)} + V_{\bar{I}}^{(1)} = (\mathrm{Div}(1 \otimes a_I) + (g+d)\infty) + (\mathrm{Div}(\delta'^{(1)}_{\bar{I}}) + 2g\infty)$$

in $X^{(3g+d)}(K_\infty)$. Moreover, since $(\alpha_m)_m$ converges in $K((u))$ to $\delta'^{(1)}_{\bar{I}} \zeta_I \left(\prod_{i \geqslant 1} f'^{(i)}_{\bar{I},*} \right)^{-1}$, by Lemma 4.13 the latter is an element of $K_\infty \otimes A(\leqslant 3g+d)$. By Proposition 3.19, we have:

$$\mathrm{Div}\left(\delta'^{(1)}_{\bar{I}} \frac{\zeta_I}{\prod_{i \geqslant 1} f'^{(i)}_{\bar{I},*}} \right) = \mathrm{Div}(\lim_m \alpha_m) = \lim_m \mathrm{Div}(\alpha_m) = \mathrm{Div}(1 \otimes a_I) + \mathrm{Div}(\delta'^{(1)}_{\bar{I}}).$$

In particular, there is some $\lambda \in K_\infty$ (a fortiori in $\mathcal{O}_{K_\infty}$) such that:

$$\zeta_I = (\lambda \otimes 1)(a_I^{-1} \otimes a_I) \prod_{i \geqslant 1} {f'_{\bar{I},*}}^{(i)}.$$

As elements of $K((u))$, $\zeta_I(a_I \otimes a_I^{-1}) = -1 + O(u)$, and ${f'_{\bar{I},*}}^{(i)} = 1 + O(u)$ for all $i \geqslant 0$, hence $\lambda \otimes 1 = -1 + u\mathbb{F}_q[[u]] \subseteq \mathbb{F}_q((u))$. In particular, the infinite product $\prod_{i \geqslant 0} (\lambda^{1-q} \otimes 1)^{q^i}$ converges in $\mathbb{F}_q[[u]]$ to $-\lambda \otimes 1$, so we deduce the following rearrangement:

$$\zeta_I = -(a_I^{-1} \otimes a_I) \prod_{i \geqslant 0} \left((\lambda^{1-q} \otimes 1) {f'_{\bar{I},*}}^{(1)} \right)^{(i)}. \qquad \square$$

Definition 5.10. Define the functions $f_{\bar{I}}, f_{\bar{I},*}, \delta_{\bar{I}}$ respectively as the unique scalar multiples of the functions $f'_{\bar{I}}.f'_{\bar{I},*}, \delta'_{\bar{I}}$ such that $\mathrm{sgn}(f_{\bar{I}}) = \mathrm{sgn}(f_{\bar{I},*}) = \mathrm{sgn}(\delta_{\bar{I}}) = 1$.

We call $\{f_{\bar{I}}\}_{\bar{I} \in Cl(A)}$ the *shtuka functions* and $\{f_{\bar{I},*}\}_{\bar{I} \in Cl(A)}$ the *adjoint shtuka functions.*

Remark 5.11. We have the equality $\frac{\delta_{\bar{I}}^{(1)}}{\delta_{\bar{I}}} = \frac{f_{\bar{I}}}{f_{\bar{I},*}}$, since both sides have the same divisor and the same sign.

Remark 5.12. The functions $\{f_{\bar{I}}\}_{\bar{I} \in Cl(A)}, \{f_{\bar{I},*}\}_{\bar{I} \in Cl(A)}, \{\delta_{\bar{I}}\}_{\bar{I} \in Cl(A)}$ all have sign equal to 1, and their divisors are all H-rational by Remark 4.27, so all these functions are in $\mathrm{Frac}(H \otimes A)$. From Remark 4.27 we also know that, for all $\bar{I} \in Cl(A), \sigma \in G(H/K) \cong Cl(A)$:

$$\mathrm{Div}(f_{\bar{I}}^\sigma) = \mathrm{Div}(f_{\bar{I}})^\sigma = \left(V_{\bar{I}}^{(1)}\right)^\sigma - V_{\bar{I}}^\sigma + \Xi - \infty = V_{\bar{I}^\sigma}^{(1)} - V_{\bar{I}^\sigma} + \Xi - \infty = \mathrm{Div}(f_{\bar{I}^\sigma}),$$

and since both function have sign equal to 1 we get $f_{\bar{I}}^\sigma = f_{\bar{I}^\sigma}$. Similarly, $f_{\bar{I},*}^\sigma = f_{\bar{I}^\sigma,*}$ and $\delta_{\bar{I}}^\sigma = \delta_{\bar{I}^\sigma}$.

Corollary 5.13. *There is $\gamma_I \in \mathbb{C}_\infty$, unique up to a factor in $\mathbb{F}_q^\times$, such that $\frac{((\gamma_I \otimes 1)\zeta_I)^{(-1)}}{(\gamma_I \otimes 1)\zeta_I} = f_{\bar{I},*}$.*

6. The module of special functions

Fix an ideal $I \trianglelefteq A$. By the *shtuka correspondence* (see [Gos12][Section 6.2]), we can associate a Drinfeld-Hayes module ϕ to the shtuka function $f_{\bar{I}}$.

Let's quickly review the notion of special function, introduced in [ANT17].

Definition 6.1 (Special functions). Set $\mathbb{T} := \mathbb{C}_\infty \hat{\otimes} A$. The set of special functions relative to a shtuka function $f_{\bar{I}}$ is defined as $\mathrm{Sf}_{\bar{I}} := \{\omega \in \mathbb{T} | \omega^{(1)} = f_{\bar{I}}\omega\}$.

Remark 6.2. The following fundamental property holds for all $\omega \in \mathbb{T}$ (see [ANT17][Lemma 3.6] and [ANT17][Rmk. 3.10]):

$$\omega \in \mathrm{Sf}_{\bar{I}} \iff \forall a \in A \ \phi_a(\omega) = (a \otimes 1)\omega.$$

In fact, the property on the right hand side constitutes the original definition of Anglès, Ngo Dag, and Tavares Ribeiro.

In this section, we use Theorem 5.9 (in its partial version) to describe somewhat explicitly the module of special functions relative to ϕ.

Set $\zeta := (\gamma_I \otimes 1)\zeta_I$, with γ_I defined as in Corollary 5.13, so that $\zeta^{(-1)} = f_*\zeta$.

Theorem 6.3 (Partial version). *The A-module $\mathrm{Sf}_{\bar{I}}$ coincides with $(\mathbb{F}_q \otimes I)\frac{\delta_{\bar{I}}}{\zeta^{(-1)}}$.*

Remark 6.4. A nice consequence of this result is that $Sf_{\bar I}$ and I are isomorphic as A-modules. Except for the Carlitz modules, and the Drinfeld modules studied by Green and Papanikolas in [GP16], the description of the isomorphism class of the module of special functions was an open problem until Gazda and Maurischat solved it in the broad context of Anderson modules (see [GM20][Thm. 3.11]).

Before the proof of Theorem 6.3, let's state some preliminary results.

Remark 6.5. By Lemma 4.12, we know that $K_\infty\hat\otimes A \cong A[[u]][u^{-1}]$. A rational function over X_{K_∞} is in $\mathbb{T}$ if and only if it's contained in $A[[u]][u^{-1}]$, which by Corollary 4.19 happens if and only if its poles all reduce to ∞.

LEMMA 6.6. *The subset of $\mathbb{C}_\infty\hat\otimes K$ fixed by the Frobenius twist is $\mathbb{F}_q\otimes K$.*

Proof. Fix an $\mathbb{F}_q$-basis $\{b_i\}_i$ of K: any element $c\in\mathbb{C}_\infty\hat\otimes K$ can be written in a unique way as a possibly infinite sum $\sum_i a_i\otimes b_i$, with $a_i\in\mathbb{C}_\infty$ for all i. If $c=c^{(1)}$, we need to have for all i the equality $a_i^q=a_i$, hence $a_i\in\mathbb{F}_q$ for all i. □

Proof of Theorem 6.3. First, let's show that $(\mathbb{F}_q\otimes K)\,\mathrm{Sf}_{\bar I}=(\mathbb{F}_q\otimes K)\frac{\delta_{\bar I}}{\zeta^{(-1)}}$. Pick any $\omega\in\mathrm{Sf}_{\bar I}$; since $\omega^{(1)}=f_{\bar I}\omega$, $\delta_{\bar I}^{(1)}=\frac{f_{\bar I}}{f_{\bar I,*}}\delta_{\bar I}$, and $\zeta=\frac{1}{f_{\bar I,*}}\zeta^{(-1)}$, we have:

$$\left(\frac{\omega\zeta^{(-1)}}{\delta_{\bar I}}\right)^{(1)}=\frac{\omega^{(1)}\zeta}{\delta_{\bar I}^{(1)}}=\frac{(f_{\bar I}\omega)(f_{\bar I,*}^{-1}\zeta^{(-1)})}{f_{\bar I}f_{\bar I,*}^{-1}\delta_{\bar I}}=\frac{\omega\zeta^{(-1)}}{\delta_{\bar I}},$$

hence $\frac{\omega\zeta^{(-1)}}{\delta_{\bar I}}\in\mathbb{F}_q\otimes K$ by Lemma 6.6, or equivalently $(\mathbb{F}_q\otimes K)\omega=(\mathbb{F}_q\otimes K)\frac{\delta_{\bar I}}{\zeta^{(-1)}}$.

We can twist everything and multiply by $\gamma_I\otimes 1$ without loss of generality: the thesis is now that $(1\otimes\lambda)\frac{\delta_{\bar I}^{(1)}}{\zeta_I}\in A[[u]][u^{-1}]$ if and only if $\lambda\in I$. Suppose $\lambda\in I$, and consider the sequence $\left((1\otimes\lambda)\frac{\delta_{\bar I,m}^{(1)}}{\zeta_{I,m}}\right)_m$ in $K((u))$, whose limit is $(1\otimes\lambda)\frac{\delta_{\bar I}^{(1)}}{\zeta_I}$. The divisor of the m-th element of the sequence (for $m\gg 0$) is

$$V_{\bar I,m}^{(1)}-(\Xi^{(1)}+\cdots+\Xi^{(m)})-I+(m+d-g)\infty+\mathrm{Div}(1\otimes\lambda);$$

since $\lambda\in I$, the only poles of the function reduce to ∞, hence $(1\otimes\lambda)\frac{\delta_{\bar I,m}^{(1)}}{\zeta_{I,m}}\in A[[u]][u^{-1}]$ by Corollary 4.19, and so does the limit.

Vice versa, suppose $(1\otimes\lambda)\frac{\delta_{\bar I}^{(1)}}{\zeta_I}\in A[[u]][u^{-1}]$. Since the coefficients of $(1\otimes\lambda^{-1})\zeta_I\in K((u))$ are all contained in $\lambda^{-1}I$, $\delta_{\bar I}^{(1)}=\left((1\otimes\lambda)\frac{\delta_{\bar I}^{(1)}}{\zeta_I}\right)\left((1\otimes\lambda^{-1})\zeta_I\right)$ has all coefficients in $\lambda^{-1}I$, so the same is true for $\delta_{\bar I}$. If by contradiction $\lambda\notin I$, there is a prime ideal $P\trianglelefteq A$ which divides the fractional ideal $\lambda^{-1}I$, hence all the coefficients of $\delta_{\bar I}$ are in $A\cap\lambda^{-1}I\subseteq P$, which by Corollary 4.20 means that P is a zero of $\delta_{\bar I}$. Since $P\in X(\mathbb{F}_q^{ac})$ and $\mathrm{Div}(\delta_{\bar I})=V_{\bar I}+V_{\bar I,*}-2g\infty$, this is a contradiction because, by Corollary 4.23, neither $V_{\bar I}$ or $V_{\bar I,*}$ have $\mathbb{F}_q^{ac}$-rational points in their support. □

To end this section, let's include an analogous result to Theorem 5.9 for special functions.

THEOREM 6.7. *There is some $\alpha\in K_\infty^\times$ such that the following element of $K_\infty\hat\otimes K$ is well defined:*

$$\omega:=(\alpha\otimes 1)^{\frac{1}{q-1}}\prod_{i\geqslant 0}\left(\frac{\alpha\otimes 1}{f_{\bar I}}\right)^{(i)}.$$

Moreover, $\omega \in (\mathbb{F}_q \otimes K)\,\mathrm{Sf}_{\bar{I}}$, and ω does not depend on the choice of α.

Proof. Take $\alpha, \beta \in K_\infty^\times$ such that the following elements of $K_\infty \hat{\otimes} K \cong K((u))$ are well defined:

$$\omega(\alpha) := (\alpha \otimes 1)^{\frac{1}{q-1}} \prod_{i \geqslant 0} \left(\frac{\alpha \otimes 1}{f_{\bar{I}}} \right)^{(i)}, \qquad \omega(\beta) := (\beta \otimes 1)^{\frac{1}{q-1}} \prod_{i \geqslant 0} \left(\frac{\beta \otimes 1}{f_{\bar{I}}} \right)^{(i)}.$$

The infinite products converge only if $\frac{\alpha\otimes 1}{f_{\bar{I}}}, \frac{\beta\otimes 1}{f_{\bar{I}}} = 1 + O(u)$ in $K((u))$; in particular, we have $\gamma := \alpha\beta^{-1} = 1 + O(u)$ in $\mathbb{F}_q[[u]]$, therefore:

$$\omega(\alpha) := (\gamma\beta \otimes 1)^{\frac{1}{q-1}} \prod_{i \geqslant 0} \left(\frac{\gamma\beta \otimes 1}{f_{\bar{I}}} \right)^{(i)} = \omega(\beta)(\gamma \otimes 1)^{\frac{1}{q-1}} \prod_{i \geqslant 0} (\gamma^{q^i} \otimes 1) = \omega(\beta),$$

and this proves that, if $\omega(\alpha)$ is well defined, it does not depend on α. On the other hand, by Proposition 5.7, we can choose $f'_{\bar{I}} \in \mathcal{O}_{K_\infty} \hat{\otimes} K$ and $\alpha \in K_\infty \otimes \mathbb{F}_q$ such that $f'_{\bar{I}} = 1 + O(u)$ and $f_{\bar{I}} = (\alpha \otimes 1) f'_{\bar{I}}$, hence $\omega := \omega(\alpha)$ is well defined. Finally, we have:

$$\frac{\omega^{(1)}}{\omega} = \left((\alpha \otimes 1)^{\frac{1}{q-1}} \right)^q \prod_{i \geqslant 0} \left(\frac{\alpha \otimes 1}{f_{\bar{I}}} \right)^{(i+1)} \left((\alpha \otimes 1)^{\frac{1}{q-1}} \prod_{i \geqslant 0} \left(\frac{\alpha \otimes 1}{f_{\bar{I}}} \right)^{(i)} \right)^{-1} = (\alpha \otimes 1) \frac{f_{\bar{I}}}{\alpha \otimes 1} = f_{\bar{I}},$$

so $\omega \in (\mathbb{F}_q \otimes K)\,\mathrm{Sf}_{\bar{I}}$ by the same considerations expressed in the proof of Theorem 6.3. □

7. Relation between Pellarin zeta functions and period lattices

The aim of this section is to compute more explicitly the constant γ_I defined in Corollary 5.13. To do so, we first study more in depth the zeta function ζ_I and its coefficients as a series in $K[[u]]$; afterwards, we draw a correspondence between the adjoint shtuka function $f_{I,*}$ and a certain Drinfeld-Hayes module ϕ, obtaining the following result.

Proposition 7.19. *The period lattice of ϕ is $\gamma_I^{-1} I \subseteq \mathbb{C}_\infty$.*

Finally, we state a complete version of Theorem 6.3 - which generalizes [GP16][Thm. 7.1] - and Theorem 5.9.

7.1 Evaluations of the Pellarin zeta functions

The aim of this subsection, expressed in the following proposition, is to show that there is a well behaved notion of evaluation for the Pellarin zeta function ζ_I at any point $P \in X(\mathbb{C}_\infty) \setminus \{0\}$.

From now on, for any series $s \in K[[u^{\frac{1}{q^n}}]][u^{-1}]$ for some n, we denote by $s_{(i)}$ the coefficient of u^i. We can extend the valuation v: we denote $v(s)$ the least element in $\frac{1}{q^n}\mathbb{Z}$ such that $s_{(v(s))} \neq 0$.

Proposition 7.1. *For all points $P \in X(\mathbb{C}_\infty) \setminus \{\infty\}$, corresponding to maps $\chi_P : A \to \mathbb{C}_\infty$, the sequence $(\zeta_{I,m}(P))_m$ and the series $\sum_{i \geqslant 0} \chi_P\left((\zeta_I)_{(i)}\right) u^i$ converge to the same element of $\mathbb{C}_\infty$.*

To prove the proposition, we first need some results on the coefficients $\left((\zeta_I)_{(i)}\right)_i$.

Lemma 7.2. *For all integers $i \geqslant 0$, we have $\deg((\zeta_{I,m})_{(i)}) \leqslant \log_q(i+1) + g + \deg(I) + 1$ for $m \geqslant 0$.*

Proof. Recall the definition of j_m, and that $m + \deg(I) + 1 \leqslant j_m \leqslant m + g + \deg(I)$, from Remark 5.2. The coefficients of $\zeta_{I,0}$ have degree $j_0 \leqslant g + \deg(I)$, so the lemma holds for $m = 0$. Since $v(\zeta_{I,m}) = j_0$ for all $m \geqslant 0$, the coefficient $(\zeta_{I,m})_{(0)}$, is nonzero if and only if $I = A$; in that case, it's equal to $\sum_{a\in\mathbb{F}_q^\times} a^{-1}\otimes a = -1$, and its valuation is 0, so the lemma also holds for $i=0$. Let's prove the lemma for $i\geqslant 1$, $m\geqslant 1$.

We claim that it suffices to prove the following inequality, for all $m\geqslant 1$:

$$v\left(\sum_{a\in I(j_m)} a^{-1}\otimes a\right) = v(\zeta_{I,m}-\zeta_{I,m-1}) \geqslant q^{m-1}.$$

If the inequality is true for $m\geqslant 1$, fix $i>0$, and set $n := \lfloor \log_q(i)\rfloor + 1$, so that $q^{n-1}\leqslant i < q^n$; then, for $m\geqslant n$:

$$\deg((\zeta_{I,m})_{(i)}) = \deg\left(\left(\sum_{k=0}^{m}\zeta_{I,k}-\zeta_{I,k-1}\right)_{(i)}\right) = \deg\left(\left(\sum_{k=0}^{n}\zeta_{I,k}-\zeta_{I,k-1}\right)_{(i)}\right) \leqslant j_n,$$

which is at most $n+g+\deg(I) = \lfloor\log_q(i)\rfloor + g + \deg(I) + 1 \leqslant \log_q(i+1) + g + \deg(I) + 1$.

For $m\geqslant 1$, $\zeta_{I,m}(\Xi)-\zeta_{I,m-1}(\Xi) = 1-1 = 0$. By Proposition 5.5, on one hand, $\zeta_{I,m}-\zeta_{I,m-1}$ has only one pole, of degree at most j_m, at ∞, and has I and $\Xi^{(1)},\dots,\Xi^{(m-1)}$ among its zeroes; on the other hand,

$$h^0(W_m) = h^0(j_m\infty - I - \Xi - \cdots - \Xi^{(m-1)}) = h^0(j_m\infty - I - \Xi^{(1)} - \cdots - \Xi^{(m)}) = 1,$$

hence the remaining set of zeroes is $W_m^{(-1)}$, and $\zeta_{I,m}-\zeta_{I,m-1}$ is a scalar multiple of $\zeta_{I,m}^{(-1)}$.

If we fix $b\in I(j_m)$ with $\operatorname{sgn}(b)=1$, we get the following:

$$\begin{aligned}\left((\zeta_{I,m}-\zeta_{I,m-1})(\Xi^{(-1)})\right)^q &= \sum_{a\in I(j_m)} a^{1-q} = -\sum_{\substack{a\in I(j_m)\\ \operatorname{sgn}(a)=1}} a^{1-q} = -\sum_{c\in I(<j_m)} (b+c)^{1-q}\\ &= -b^{1-q}\sum_{c\in I(<j_m)}\sum_{i\geqslant 0}\binom{1-q}{i}\frac{c^i}{b^i} = -b^{1-q}\sum_{i\geqslant 0} b^{-i}\binom{1-q}{i}\sum_{c\in I(<j_m)} c^i.\end{aligned}$$

On the other hand, by Lemma 5.6, we have:

$$\sum_{c\in I(<j_m)} c^i = 0,\quad \forall i < q^{\dim(I(<j_m))}-1 = q^m-1.$$

As elements of $\mathcal{O}_{K_\infty}\cong\mathbb{F}_q[[u]]\subseteq K[[u]]$, $v\left(\frac{c}{b}\right)\geqslant 1$ for all $c\in I(<j_m)$, and $v(b^{-1}) = j_m$, so we get:

$$\begin{aligned}q\cdot v\left((\zeta_{I,m}-\zeta_{I,m-1})(\Xi^{(-1)})\right) &= (1-q)v(b) + v\left(\sum_{i\geqslant q^m-1}\binom{1-q}{i}\sum_{c\in I(<j_m)}\left(\frac{c}{b}\right)^i\right)\\ &\geqslant (1-q)v(b) + \min_{\substack{c\in I(<j_m)\\ i\geqslant q^m-1}}\left\{i\cdot v\left(\frac{c}{b}\right)\right\} \geqslant j_m(q-1)+q^m-1.\end{aligned}$$

Since $\left(\zeta_{I,m}^{(-1)}\right)(\Xi^{(-1)}) = (\zeta_{I,m}(\Xi))^{\frac{1}{q}} = -1$, we get:

$$\sum_{a\in I(j_m)} a^{-1}\otimes a = \zeta_{I,m}-\zeta_{I,m-1} = -\zeta_{I,m}^{(-1)}\cdot(\zeta_{I,m}-\zeta_{I,m-1})(\Xi^{(-1)}).$$

Its valuation, since $\zeta_{I,m}^{(-1)}\in K[[u^{\frac{1}{q}}]]$, is at least $j_m\frac{q-1}{q} + q^{m-1} - \frac{1}{q} \geqslant q^{m-1}$, for $m\geqslant 1$. $\square$

COROLLARY 7.3. *For all $k \geqslant 0$, for all $i \in \frac{1}{q^k}\mathbb{N}$, $\deg((\zeta_I^{(-k)})_{(i)}) \leqslant \log_q(i+1)+k+g+\deg(I)+1$. For all points $P \in X(\mathbb{C}_\infty) \setminus \{\infty\}$, corresponding to maps $\chi_P : A \to \mathbb{C}_\infty$, for all $k \geqslant 0$, the following series converges:*

$$\sum_{i\geqslant 0} \chi_P\left((\zeta_I^{(-k)})_{(i)}\right) u^i.$$

First, a simple lemma.

LEMMA 7.4. *Fix a point $P \in X(\mathbb{C}_\infty) \setminus \{\infty\}$, corresponding to map $\chi_P : A \to \mathbb{C}_\infty$. There is a positive real constant k_P such that $v(\chi_P(a)) \geqslant -k_P \deg(a)$ for all $a \in A$.*

Proof. We can pick a finite set $\{a_1, \dots, a_n\}$ such that for all $a \in A \setminus \mathbb{F}_q$ there is a product a' of a_i's with $\deg(a') = \deg(a)$. Let's define $k_P := \max\left\{\frac{-v(\chi_P(a_i))}{\deg(a_i)}\right\}_i$, so that $v(\chi_P(a_i)) \geqslant -k_P \deg(a_i)$ for all i. We prove the lemma by induction on $\deg(a)$.

If $\deg(a) = 0$ the claim is trivially true. If $\deg(a) > 0$ there is a product $a' := \lambda \prod_i a_i^{e_i}$, with $\lambda \in \mathbb{F}_q$, of the same degree and sign, hence $\deg(a - a') < \deg(a)$. We have:

$$v(\chi_P(a-a')) \geqslant -k_P \deg(a-a') > -k_P \deg(a), \text{ by inductive hypothesis;}$$
$$v(\chi_P(a')) = \sum_i e_i \cdot v(\chi_P(a_i)) \geqslant -\sum_i k_P e_i \cdot \deg(a_i) = -k_P \deg(a') = -k_P \deg(a).$$

Hence, $v(\chi_P(a)) \geqslant \min\{v(\chi_P(a')), v(\chi_P(a-a'))\} \geqslant -k_P \deg(a)$. □

Proof of Corollary 7.3. The first part of the statement for $k = 0$ follows from the inequality of Lemma 7.2, using the fact that for all i the sequence $((\zeta_{I,m})_{(i)})_m$ is eventually equal to $(\zeta_I)_{(i)}$. For $k > 0$ and $i \in \frac{1}{q^k}\mathbb{N}$, we get:

$$\deg\left((\zeta_I^{(-k)})_{(i)}\right) = \deg\left((\zeta_I)_{(iq^k)}\right) \leqslant \log_q(iq^k+1)+g+\deg(I)+1 \leqslant \log_q(i+1)+k+g+\deg(I)+1.$$

Let's define k_P as in Lemma 7.4. Then, for all $i > 0$ we have:

$$v\left(\chi_P\left((\zeta_I^{(-k)})_{(i)}\right) u^i\right) \geqslant -k_P \deg\left((\zeta_I^{(-k)})_{(i)}\right) + i \geqslant i - k_P \log_q(i+1) - k_P(k+g+\deg(I)+1),$$

which tends to infinity as i tends to infinity, proving the convergence of $\sum_{i\geqslant 0} \chi_P\left((\zeta_I^{(-k)})_{(i)}\right) u^i$. □

Finally we can prove Proposition 7.1.

Proof of Proposition 7.1. Define k_P as in Lemma 7.4. For $m \geqslant 0$, by Lemma 7.2 we have:

$$v(\zeta_I - \zeta_{I,m}) = v\left(\sum_{m'\geqslant m} \zeta_{I,m'+1} - \zeta_{I,m'}\right) \geqslant \min_{m'\geqslant m} v\left(\zeta_{I,m'+1} - \zeta_{I,m'}\right) \geqslant q^m.$$

For all $i \geqslant q^m$, by Corollary 7.3, we have:

$$\begin{aligned}\deg\left((\zeta_I - \zeta_{I,m})_{(i)}\right) &\leqslant \max\left\{\deg\left((\zeta_I)_{(i)}\right), \deg\left((\zeta_{I,m})_{(i)}\right)\right\}\\ &\leqslant \max\{\log_q(i+1)+g+\deg(I)+1, j_m\} = \log_q(i+1)+g+\deg(I)+1,\end{aligned}$$

since $j_m \leqslant m+g+\deg(I)+1$ and $m \leqslant \log_q(i+1)$. In particular:

$$\begin{aligned} v\left(\sum_i \chi_P\left((\zeta_I-\zeta_{I,m})_{(i)}\right)u^i\right) &= v\left(\sum_{i\geqslant q^m} \chi_P\left((\zeta_I-\zeta_{I,m})_{(i)}\right)u^i\right) \\ &\geqslant \min_{i\geqslant q^m}\left\{i-k_P\cdot\deg\left((\zeta_I-\zeta_{I,m})_{(i)}\right)\right\} \\ &\geqslant \min_{i\geqslant q^m}\left\{i-k_P(\log_q(i+1)+g+\deg(I)+1)\right\}, \end{aligned}$$

which tends to infinity as m tends to infinity. By Proposition 4.17, $\zeta_{I,m}(P)=\sum_i \chi_P\left((\zeta_{I,m})_{(i)}\right)u^i$, hence we get that

$$\lim_m \zeta_{I,m}(P) - \sum_{i\geqslant 0}\chi_P\left((\zeta_I)_{(i)}\right)u^i = \lim_m\left(\sum_i \chi_P\left((\zeta_{I,m}-\zeta)_{(i)}\right)u^i\right) = 0. \qquad \square$$

Definition 7.5. We define the evaluation of ζ_I at P as $\zeta_I(P) := \sum_i (\zeta_I)_{(i)}(P)u^i$.

Corollary 7.6. *For all $i \geqslant 1$, we have $\zeta_I(\Xi^{(i)})=0$. Similarly, for all $k\geqslant 0$, for all $i\geqslant 1$, $\sum_j \chi_{\Xi^{(i-k)}}((\zeta_I^{(-k)})_{(j)})u^j = 0$ (where j varies among $\frac{1}{q^k}\mathbb{N}$).*

Proof. For the first identity we use that, for all $i\geqslant 1$, $\zeta_{I,m}(\Xi^{(i)})=0$ for $m \gg 0$. For the second identity, note that

$$\left(\sum_{j\in\frac{1}{q^k}\mathbb{N}} \chi_{\Xi^{(i-k)}}((\zeta_I^{(-k)})_{(j)})u^j\right)^{q^k} = \sum_{j\in\mathbb{N}} \chi_{\Xi}^{(i)}((\zeta_I)_{(j)})u^j = 0. \qquad \square$$

7.2 Adjoint Drinfeld modules and adjoint shtuka functions

From now on, in this section we use the following notations: $V_* := V_{\bar{I},*}$, $f_* := f_{\bar{I},*}$, $\zeta := (\gamma_I\otimes 1)\zeta_I$, with γ_I defined as in Corollary 5.13, so that $\zeta^{(-1)} = f_*\zeta$.

Proposition 7.7 shows a connection between adjoint Drinfeld modules, adjoint shtuka functions, and zeta functions, which is meant to mirror the correspondence between Drinfeld modules, shtuka functions, and special functions (see for example [Tha93][Eq.(∗∗)] and [And94][Eq.(46)]).

Afterwards, we present some basic definitions and results concerning the coefficients of exponential and logarithmic functions (see for example [Gos12]) to prove the interesting Proposition 7.18. On the surface the proposition resembles a log-algebraicity result, and could be linked to this rich branch of research (see for example [And94], [And96], [ADT16]); on the other hand, it encourages a greater focus on the adjoint exponential function, whose kernel was already studied in works such as [Poo96].

Proposition 7.7. *Set $e_d := \prod_{i=0}^{d-1} f_*^{(-i)}$ for all nonnegative integers d. The collection $\{e_d\}_{d\geqslant 0}$ is a basis of the $\mathbb{C}_\infty\otimes 1$-vector space $\mathcal{O}(V_*^{(1)}) := \bigcup_{d\geqslant 0}\mathcal{L}(V_*^{(1)}+d\infty)$.*

For all $a\in A$ we have the equality $1\otimes a = \sum_{i=0}^{\deg(a)}(a_i\otimes 1)e_i$ with $a_i^{q^i}\in K_\infty$, and the function $\phi^* : A \to \mathbb{C}_\infty\{\tau^{-1}\}$ sending a to $\sum_i a_i\tau^{-i}$ is the adjoint of a Drinfeld-Hayes module ϕ.

Finally, for all $a,b\in A$, $\phi^*_{ab}(\zeta) = (\phi^*_a\circ\phi^*_b)(\zeta)$.

Proof. For the first part, we just need to prove that, for all $d\geqslant 0$, $e_d\in\mathcal{L}(V_*^{(1)})$ and it has a pole of multiplicity exactly d at ∞; using that $\mathrm{Div}(f_*^{(-i)}) = V_*^{(-i)} - V_*^{(1-i)} + \Xi^{(-i)} - \infty$, we get:

$$\mathrm{Div}(e_d) = \mathrm{Div}\left(\prod_{i=0}^{d-1} f_*^{(-i)}\right) = V_*^{(1-d)} - V_*^{(1)} + \sum_{i=0}^{d-1} \Xi^{(-i)} - d\infty.$$

If we fix $a \in A$ of degree d, $1 \otimes a \in \mathcal{L}(V_*^{(1)} + d\infty)$, hence it can be written as $\sum_{i=0}^{d}(a_i \otimes 1)e_i$. Moreover, if we twist k times and evaluate at Ξ for all $0 \leqslant k \leqslant d$ we get the following triangular system of equations in the variables $(a_i)_i$:

$$\left\{ a = \sum_{i=0}^{k}\left(a_i^{q^k} \prod_{j=k-i}^{k} f_*^{(j)}(\Xi)\right)\right\}_k \Longrightarrow \left\{ a_k^{q^k} = \left(\prod_{j=0}^{k} f_*^{(j)}(\Xi)\right)^{-1}\left(a - \sum_{i=0}^{k-1} a_i^{q^k} \prod_{j=k-i}^{k} f_*^{(j)}(\Xi)\right)\right\}_k.$$

From this system we can deduce that $a_0 = a$ and, since $f_*^{(j)}(\Xi) \in K_\infty$ for all $j \geqslant 0$, that $a_k^{q^k} \in K_\infty$ for all k. Finally, since $\deg(a) = \deg(e_d)$, and for all $i \geqslant 0$ $\mathrm{sgn}(f_*^{(i)}) = \mathrm{sgn}(f_*) = 1$, we have:

$$\mathrm{sgn}(a) = \mathrm{sgn}\left(\sum_{i=0}^{d}(a_i^{q^d} \otimes 1)e_i^{(d)}\right) = \mathrm{sgn}((a_d^{q^d} \otimes 1)e_d^{(d)}) = a_d^{q^d}\,\mathrm{sgn}(e_d^{(d)}) = a_d^{q^d}\,\mathrm{sgn}\left(\prod_{i=1}^{d} f_*^{(i)}\right) = a_d^{q^d},$$

so $a_d = \mathrm{sgn}(a)$. For all $a \in A$, write $\phi_a^* := \sum_i a_i\tau^{-i}$. Since for all $k \geqslant 0$ and for all $a \in A$ we have $\zeta e_k = \zeta^{(-k)}$ and $1 \otimes a = (1 \otimes a)^{(-k)} = \sum_i (a_i^{\frac{1}{q^k}} \otimes 1)e_i^{(-k)}$, we get the following equations for all $k \geqslant 0$ and $a, b \in A$:

$$\begin{aligned}
(1 \otimes a)\zeta^{(-k)} &= \sum_i (a_i^{\frac{1}{q^k}} \otimes 1)(e_i\zeta)^{(-k)} = \sum_i (a_i^{\frac{1}{q^k}} \otimes 1)\zeta^{(-k-i)} = \tau^{-k} \circ \phi_a^*(\zeta);\\
\phi_{ab}^*(\zeta) &= (1 \otimes ab)\zeta = (1 \otimes a)\left((1 \otimes b)\zeta\right) = \sum_i (1 \otimes a)\left((b_i \otimes 1)\zeta^{(-i)}\right)\\
&= \sum_i (b_i \otimes 1)\left((1 \otimes a)\zeta^{(-i)}\right) = \sum_i (b_i \otimes 1)\left(\tau^{-i} \circ \phi_a^*(\zeta)\right) = (\phi_b^* \circ \phi_a^*)(\zeta).
\end{aligned}$$

Since the elements $(\zeta^{(-i)})_{i\geqslant 0} = (\zeta e_i)_{i\geqslant 0}$ are all $\mathbb{C}_\infty \otimes 1$-linearly independent, we have the equality $\phi_{ab}^* = \phi_b^* \circ \phi_a^*$. Together with the fact that $\deg(\phi_a^*) = \deg(a)$ and $a_{\deg(a)} = \mathrm{sgn}(a)$, this means that the function $\phi := (\phi^*)^* : A \to K_\infty\{\tau\}$ is a Drinfeld-Hayes module. $\square$

From this point onwards, ϕ and ϕ^* are defined as in Proposition 7.7.

Definition 7.8. We call $\Lambda' \subseteq \mathbb{C}_\infty$ the unique rank 1 projective A-module such that, for all $a \in A$, we have the following identity in $\mathbb{C}_\infty\{\{\tau\}\}$:

$$\exp_{\Lambda'} \circ (a\tau^0) = \phi_a \circ \exp_{\Lambda'}.$$

We define $\exp := \exp_{\Lambda'}$ and $\log := \log_{\Lambda'}$. We call $\Lambda \subseteq K$ the unique fractional ideal isogenous to Λ' such that $\Lambda(\leqslant 0) = \mathbb{F}_q$. We choose a nonzero element of least degree in Λ' and call it $\tilde{\pi}_{\Lambda'}$ - for simplicity we denote it $\tilde{\pi}$ for the rest of the section.

Remark 7.9. Since $\mathrm{rk}(\Lambda') = 1$, a lattice isogenous to Λ' is uniquely determined by its nonzero elements of greatest norm, hence the hypothesis that for Λ they are $\mathbb{F}_q$ already implies $\Lambda \subseteq K$. Our choice of $\tilde{\pi}$ is up to a factor in $\mathbb{F}_q^\times$, and we have $\tilde{\pi}\Lambda = \Lambda'$.

Definition 7.10. Define $\exp^* := \sum_i e_i^{\frac{1}{q^i}}\tau^{-i} \in \mathbb{C}_\infty\{\{\tau^{-1}\}\}$, where for all i e_i is the i-th coefficient of $\exp \in \mathbb{C}_\infty\{\{\tau\}\}$. We call $\exp^*$ the *adjoint exponential function.*

Remark 7.11. Since $\exp\circ(a\tau^0) = \phi_a \circ \exp$ for all $a \in A$, we easily deduce the following identity in $\mathbb{C}_\infty\{\{\tau^{-1}\}\}$ for all $a \in A$:

$$a \exp^* = \exp^* \circ \phi_a^*.$$

Remark 7.12. If we write $\exp := \sum_i e_i\tau^i$ and $\phi_a := \sum_j a_j\tau^j$ for some $a \in A$ (with $a_0 = a$), the equation $\exp\circ(a\tau^0) = \phi_a \circ \exp$ becomes:

$$\sum_k (e_k a^{q^k})\tau^k = \sum_k \left(\sum_{i+j=k} a_j e_i^{q^j}\right)\tau^k \Rightarrow e_k(a^{q^k} - a) = \sum_{i=0}^{k-1} a_{k-i} e_i^{q^{k-i}}.$$

Since $e_0 = 1$, we get that $e_k \in K_\infty$ for all $k \geqslant 0$ by induction.

Definition 7.13. For any rank 1 projective A-module $L \subseteq \mathbb{C}_\infty$, we define, for all $k \geqslant 1$:

$$S_k(L) := \sum_{\substack{\lambda_1,\dots,\lambda_k \in^* L \\ i\neq j \Rightarrow \lambda_i \neq \lambda_j}} (\lambda_1\cdots\lambda_k)^{-1}; \qquad P_k(L) := \sum_{\lambda\in^* L} \lambda^{-k}.$$

We also set $S_0(L) := 1$ and $P_0(L) := -1$.

Remark 7.14. By definition $\exp_L(x) := x\prod_{\lambda\in^* L}\left(1 - \frac{x}{\lambda}\right) \in \mathbb{C}_\infty[[x]]$, and by absolute convergence we can expand the product and rearrange the terms of the series, so we get $\exp_L(x) = \sum_{i\geqslant 0} S_i(L)x^i$; in particular, if $i+1$ is not a power of q, $S_i(L) = 0$.

Remark 7.15. Note that in the summation that defines $S_{q^i-1}(\tilde{\pi}\Lambda)$ there is a unique summand of greatest norm, given by the product of the $q^i - 1$ nonzero elements of lower degree of $\tilde{\pi}\Lambda$. Since $\mathbb{F}_q^\times$ are the nonzero elements of lowest degree of Λ, this product has valuation at least:

$$\sum_{j=0}^{i-1}(q^{j+1} - q^j)(j + v(\tilde{\pi})) = iq^i - \left(\sum_{j=0}^{i-1} q^j\right) + q^i v(\tilde{\pi}) \geqslant (i - 1 + v(\tilde{\pi}))q^i.$$

In particular, since $\lim_i \frac{1}{q^i} v\left(S_{q^i-1}(\tilde{\pi}\Lambda)\right) = \infty$, $\exp : \mathbb{C}_\infty \to \mathbb{C}_\infty$ is an entire function with an infinite radius of convergence, and $\exp^* : \mathbb{C}_\infty \to \mathbb{C}_\infty$, while not being a power series, is continuous, converges everywhere, and sends 0 to 0; moreover, $\lim_{\|z\|\to 0} \exp^*(z)z^{-1} = 1$.

Remark 7.16. Since they are continuous $\mathbb{F}_q$-linear endomorphisms of $\mathbb{C}_\infty$, we can extend uniquely both $\exp$ and $\exp^*$ to continuous $\mathbb{F}_q \otimes K$-linear endomorphisms of $\mathbb{C}_\infty\hat{\otimes}K$.

Lemma 7.17. *For all rank 1 lattices $L \subseteq \mathbb{C}_\infty$, $\log_L = -\sum_i P_{q^i-1}(L)\tau^i$.*

Proof. Newton's identities for an infinite number of variables tell us that, for all $i \geqslant 1$, $iS_i(L) = \sum_{j=0}^{i-1} S_j(L)P_{i-j}(L)$. Setting $i = q^k - 1$ with $k \geqslant 1$, since $S_j(L) = 0$ if $j+1$ is not a power of q, we get:

$$-S_{q^k-1}(L) = \sum_{j=0}^{k-1} S_{q^j-1}(L)P_{q^k-q^j}(L) = \sum_{j=0}^{k-1} S_{q^j-1}(L)(P_{q^{k-j}-1}(L))^{q^j}.$$

In particular:

$$\exp_L \circ \left(-\sum_{i\geqslant 0} P_{q^i-1}(L)\tau^i\right) = \sum_{k\geqslant 0}\left(-\sum_{j=0}^{k} S_{q^j-1}(L)(P_{q^{k-j}-1}(L))^{q^j}\right)\tau^k = -S_0(L)P_0(L) = 1.$$

The uniqueness of right inverses proves the thesis. $\square$

Proposition 7.18. *We have the following functional identity:*

$$\exp^*(\zeta) = 0.$$

Proof. By Remark 7.11, for all $a \in A$ we have $\exp^* \circ \phi_a^* = (a \otimes 1)\exp^*$ as endomorphisms of $\mathbb{C}_\infty \hat{\otimes} K$; by Proposition 7.7, $\phi_a^*(\zeta) = (1 \otimes a)\zeta$. Hence, for all $a \in A$:

$$0 = \exp^*(0) = \exp^*(\phi_a^*(\zeta) - (1 \otimes a)\zeta) = (a \otimes 1 - 1 \otimes a)\exp^*(\zeta).$$

For $a \notin \mathbb{F}_q$, $a \otimes 1 - 1 \otimes a$ is invertible in $\mathbb{C}_\infty \hat{\otimes} K$ - with inverse $\sum_{i \geqslant 0} a^{-i-1} \otimes a^i$ - so we get the thesis. $\square$

7.3 The fundamental period $\tilde{\pi}$

Finally, in this subsection we are able to link the zeta function ζ_I and the fundamental period $\tilde{\pi}$.

Proposition 7.19. *Fix an element $a_I \in^* I$ of least degree. We have $a_I^{-1} I = \Lambda$.*

Proof. Since the nonzero elements of least degree of both $a_I^{-1}I$ and Λ are $\mathbb{F}_q^\times$, it suffices to show that I and Λ are isogenous. Let's first give an intuitive rundown of the proof.

For all $n \geqslant 0$, for all $k \geqslant 0$, if $n < k$ then $\zeta^{(-k)}(\Xi^{(-n)}) = (\zeta(\Xi^{(k-n)}))^{\frac{1}{q^k}} = 0$ by Corollary 7.6, while if $n \geqslant k$ then:

$$\zeta^{(-k)}(\Xi^{(-n)}) = \gamma_I^{\frac{1}{q^k}} \sum_{a \in^* I} a^{\frac{1}{q^n} - \frac{1}{q^k}} = \left(\gamma_I \sum_{a \in^* I} \left(\frac{a}{\gamma_I} \right)^{1-q^{n-k}} \right)^{\frac{1}{q^n}} = \left(\gamma_I P_{q^{n-k}-1}(\gamma_I^{-1} I)\right)^{\frac{1}{q^n}}.$$

Since $\exp^*(\zeta) = 0$ by Proposition 7.18, evaluating $\exp_*(\zeta)$ at $\Xi^{(-n)}$ we should get:

$$0 = \exp_*(\zeta)(\Xi^{(-n)}) = \sum_{k \geqslant 0} S_{q^k-1}(\Lambda')^{\frac{1}{q^k}} \zeta^{(-k)}(\Xi^{(-n)}) = \sum_{0 \leqslant k \leqslant n} S_{q^k-1}(\Lambda')^{\frac{1}{q^k}} \left(\gamma_I P_{q^{n-k}-1}(\gamma_I^{-1} I)\right)^{\frac{1}{q^n}} =$$

$$= \left(\gamma_I \sum_{0 \leqslant k \leqslant n} P_{q^{n-k}-1}(\gamma_I^{-1} I) S_{q^k-1}(\Lambda')^{q^{n-k}} \right)^{\frac{1}{q^n}},$$

which by Lemma 7.17 implies that $\log_{\gamma_I^{-1}I} \circ \exp = 1$. In particular, $\exp = \exp_{\gamma_I^{-1}I}$, therefore their zero loci are the same, which means that $\gamma_I^{-1} I = \Lambda' = \tilde{\pi}\Lambda$.

The previous reasoning is not rigorous only when it assumes that evaluation at Ξ^{-n} commutes with the expansion of $\exp_*(\zeta)$, therefore to prove the theorem it suffices to show that $\sum_{0 \leqslant k \leqslant n} S_{q^k-1}(\Lambda')^{\frac{1}{q^k}} \zeta^{(-k)}(\Xi^{(-n)}) = 0$. For all $k \in \mathbb{N}$, define $c_k := S_{q^k-1}(\tilde{\pi}\Lambda)^{\frac{1}{q^k}}$; by Remark 7.12, $c_k \in \mathbb{F}_q((u^{\frac{1}{q^k}}))$. By Corollary 7.3 and Remark 7.15 respectively, we have the following inequalities for all $i \in \frac{1}{q^k}\mathbb{N}$, for all $k \in \mathbb{N}$:

$$\deg\left((\zeta^{(-k)})_{(i)}\right) \leqslant \log_q(i+1) + k + g + \deg(I) + 1, \qquad v(c_k) \geqslant k - 1 + v(\tilde{\pi}) =: k'.$$

Fix a positive integer n. Since $\exp^*(\zeta) = 0$, for any arbitrarily large N we can choose a positive integer $m \geqslant n$ such that $v\left(\sum_{k=0}^m c_k \zeta^{(-k)}\right) \geqslant N$. For all $k \leqslant m$ we can write the following, where the index i varies among $\frac{1}{q^m}\mathbb{Z}$:

$$c_k = \sum_{i \geqslant k'} \lambda_{k,i} u^i \in \mathbb{F}_q\left(\left(u^{\frac{1}{q^m}}\right)\right) \text{ with } \lambda_{k,i} \in \mathbb{F}_q.$$

Let's rearrange $\sum_{k=0}^m c_k\zeta^{(-k)}$, with the indexes i and j varying among $\frac{1}{q^m}\mathbb{Z}$:

$$\sum_{k=0}^m\sum_{j\geqslant k'}\lambda_{k,j}\zeta^{(-k)}u^j=\sum_{k=0}^m\sum_{j\geqslant k'}\lambda_{k,j}\sum_{i\geqslant 0}\left(\zeta^{(-k)}\right)_{(i)}u^{i+j}=\sum_{i\geqslant 0}\left(\sum_{k=0}^m\sum_{j=k'}^{i}\lambda_{k,j}\left(\zeta^{(-k)}\right)_{(i-j)}\right)u^i.$$

Since $v\left(\sum_{k=0}^m c_k\zeta^{(-k)}\right)\geqslant N$, we get that, for $i\in\frac{1}{q^m}\mathbb{Z}$ and $i<N$:

$$\sum_{k=0}^m\sum_{j=k'}^{i}\lambda_{k,j}\left(\zeta^{(-k)}\right)_{(i-j)}=0.$$

Using this result and Corollary 7.3, the evaluation $\sum_{k=0}^m c_k\zeta^{(-k)}(\Xi^{(-n)})$ can be rearranged as follows:

$$\sum_{k=0}^m\sum_{j\geqslant k'}\lambda_{k,j}\zeta^{(-k)}(\Xi^{(-n)})u^j=\sum_{k=0}^m\sum_{j\geqslant k'}\lambda_{k,j}\sum_{i\geqslant 0}\left(\zeta^{(-k)}\right)_{(i)}(\Xi^{(-n)})u^{i+j}$$
$$=\sum_{i\geqslant 0}\left(\sum_{k=0}^m\sum_{j=k'}^{i}\lambda_{k,j}\left(\zeta^{(-k)}\right)_{(i-j)}\right)(\Xi^{(-n)})u^i=\sum_{i\geqslant N}\left(\sum_{k=0}^m\sum_{j=k'}^{i}\lambda_{k,j}\left(\zeta^{(-k)}\right)_{(i-j)}\right)(\Xi^{(-n)})u^i.$$

For $i-j,k\geqslant 0$, since $j\geqslant k'\geqslant v(\tilde{\pi})-1$, and since $\log_q(x)\leqslant x$ for all $x>0$, we have:

$$\deg\left((\zeta^{(-k)})_{(i-j)}\right)\leqslant\log_q(i-j+1)+k+g+\deg(I)+1\leqslant i+k+g+\deg(I)+3-v(\tilde{\pi})=:i+C,$$

so each summand has valuation at least $i-\frac{i+C}{q^n}\geqslant N-\frac{N+C}{q^n}$, which tends to infinity as N tends to infinity. Since $m=m(N)$ depends on N and tends to infinity as N does, we have:

$$0=\lim_{N\to\infty}\sum_{k=0}^{m(N)}c_k\zeta^{(-k)}(\Xi^{(-n)})=\lim_{m\to\infty}\sum_{k=0}^{m}c_k\zeta^{(-k)}(\Xi^{(-n)})=\sum_{k=0}^{n}c_k\zeta^{(-k)}(\Xi^{(-n)}),$$

where we used that $\zeta^{(-k)}(\Xi^{(-n)})=0$ for $k>n$ by Corollary 7.6. This concludes the proof. □

PROPOSITION 7.20. *The following identity holds in $\mathbb{C}_\infty\hat{\otimes}K$:*

$$\frac{\left((a_I\tilde{\pi}^{-1}\otimes 1)\zeta_I\right)^{(-1)}}{(a_I\tilde{\pi}^{-1}\otimes 1)\zeta_I}=f_*,$$

Proof. From the definition of γ_I we have $\frac{\zeta_I}{\zeta_I^{(1)}}=(\gamma_I\otimes 1)^{q-1}f_*^{(1)}$. Since $\Lambda=a_I^{-1}I$ and $\tilde{\pi}\Lambda=\gamma_I^{-1}I$, we deduce $\gamma_I=\frac{a_I}{\tilde{\pi}}$ up to a factor in $\mathbb{F}_q^\times$. □

We can finally state and prove more precise versions of Theorems 5.9 and 6.3.

THEOREM 5.9 (Complete version). *The following functional identity is well posed and true in $K_\infty\hat{\otimes}K$:*

$$\zeta_I=-(a_I^{-1}\otimes a_I)\prod_{i\geqslant 0}\left((\tilde{\pi}^{1-q}\otimes 1)f_*^{(1)}\right)^{(i)}.$$

Proof. From the partial version of this theorem, we have the following identity in $\mathcal{O}_{K_\infty}\hat{\otimes}K$:

$$\zeta_I=-(a_I^{-1}\otimes a_I)\prod_{i\geqslant 0}\left((\lambda^{1-q}\otimes 1){f'_{\bar{I},*}}^{(1)}\right)^{(i)},$$

where $f'_{\bar{I},*}$ is a scalar multiple of $f_{\bar{I},*}$, and $\lambda \in \mathcal{O}_{K_\infty}$ is some constant. We deduce:

$$\frac{\zeta_I}{\zeta_I^{(1)}} = (a_I^{q-1} \otimes 1)(\lambda^{1-q} \otimes 1){f'_{\bar{I},*}}^{(1)}.$$

On the other hand, by Corollary 7.20, we know that

$$\frac{\zeta_I}{\zeta_I^{(1)}} = \left(\frac{a_I}{\tilde{\pi}} \otimes 1\right)^{q-1} f_*^{(1)},$$

hence $(\lambda^{1-q} \otimes 1){f'_{I,*}}^{(1)} = (\tilde{\pi}^{1-q} \otimes 1)f_*^{(1)}$ and we get the desired identity. □

Recall the notations as in Definition 5.10. We restate Theorem 6.3 to make it a proper generalization of [GP16][Thm. 7.1].

Theorem 6.3 (Complete version). *The following A-submodules of $\mathbb{C}_\infty \hat{\otimes} A$ coincide:*

$$\mathrm{Sf}_{\bar{I}} = (\mathbb{F}_q \otimes I)\frac{\delta_{\bar{I}}^{(1)}(\tilde{\pi} \otimes 1)}{f_{\bar{I}}(a_I \otimes 1)\zeta_I}.$$

Proof. From the partial version of this theorem, we have the A-module $\mathrm{Sf}_I \subseteq \mathbb{C}_\infty \hat{\otimes} A$ coincides with $(\mathbb{F}_q \otimes I)\frac{\delta_I}{\zeta}$, where $\zeta := (\gamma_I \otimes 1)\zeta_I$. Since $\gamma_I = \frac{a_I}{\tilde{\pi}}$ up to a factor in $\mathbb{F}_q^\times$ by Corollary 7.20, and $\delta_{\bar{I}}^{(1)} f_{\bar{I},*} = \delta_{\bar{I}} f_{\bar{I}}$ by Remark 5.11, we deduce the thesis. □

8. Relation with zeta functions à la Anderson

In this section, we prove the generalization of [GP16, Thm. 7.3] in the form of Theorem 8.7. We define a zeta function "à la Anderson" $\xi_{\bar{I}}$ relative to a class $\bar{I} \in Cl(A)$, and then relate it to the Pellarin zeta function ζ_A.

Let $\phi : A \to K_\infty\{\tau\}$ be the Drinfeld-Hayes module corresponding to the Drinfeld divisor $V_{\bar{I}}$. We can extend ϕ to ideals, sending $J = (a, b) \trianglelefteq A$ to the generator ϕ_J of the left ideal $(\phi_a, \phi_b) < K_\infty\{\tau\}$, following a construction of Hayes (see [Hay79]).

Definition 8.1. Fix $\omega \in \mathrm{Sf}_{\bar{I}}$, and for $J \trianglelefteq A$ define $\chi_{\bar{I}}(J) := \frac{\phi_J(\omega)}{\omega}$.

Remark 8.2. The previous definition does not depend on the choice of ω. For $a \in A$, $J \trianglelefteq A$, since $\phi_{aJ} = \phi_J \circ \phi_a$ we have that

$$\chi_{\bar{I}}(aJ) = \frac{\phi_{aJ}(\omega)}{\omega} = \frac{\phi_J \circ \phi_a(\omega)}{\omega} = \frac{\phi_J((1 \otimes a)\omega)}{\omega} = (1 \otimes a)\frac{\phi_J(\omega)}{\omega} = \chi_{\bar{I}}(a)\chi_{\bar{I}}(J).$$

It's easy to check that we can extend χ to all fractional ideals in a unique way such that for all $a \in K$ and for all fractional ideals J we have $\chi_{\bar{I}}(a)\chi_{\bar{I}}(J) = \chi_{\bar{I}}(aJ)$.

Proposition 8.3. *For all $J \trianglelefteq A$, $\chi_{\bar{I}}(J) \in \mathrm{Frac}(K_\infty \otimes A)$, and $\mathrm{Div}(\chi_{\bar{I}}(J)) = V_{\bar{I}-\bar{J}} + J - V_{\bar{I}} - \deg(J)\infty$.*

Proof. Consider $\mathcal{O}(V_{\bar{I}}) := \bigcup_{n \geqslant 0} \mathcal{L}(V_{\bar{I}} + n\infty)$, which admits as a flag base $\{f_{\bar{I}} \cdots f_{\bar{I}}^{(k)}\}_{k \geqslant -1}$. By definition, for $a \in A$, if $\phi_a = \sum_i a_i \tau^i$ we have $1 \otimes a = \sum_{i \geqslant 0}(a_i \otimes 1)f_{\bar{I}} \cdots f_{\bar{I}}^{(i)}$. If we multiply everything by some $\omega \in \mathrm{Sf}_{\bar{I}}$ we get that $(1 \otimes a)\omega = \sum_i (a_i \otimes 1)\omega^{(i)} = \phi_a(\omega)$, hence $\chi_{\bar{I}}(a) = 1 \otimes a$. For a fixed non principal ideal $J = (a, b) \trianglelefteq A$, if we write $\phi_J = \sum_{i=0}^{\deg(J)}(c_i \otimes 1)\tau^i$, we get:

$$\chi_{\bar{I}}(J) = \frac{\phi_J(\omega)}{\omega} = \sum_{i=0}^{\deg(J)} c_i f_{\bar{I}} \cdots f_{\bar{I}}^{(i-1)} \in \mathcal{L}(V_{\bar{I}} + \deg(J)\infty).$$

Moreover, if we write $\phi_J = \psi_1 \circ \phi_a + \psi_2 \circ \phi_b$ for some $\psi_1, \psi_2 \in K_\infty\{\tau\}$, we get:

$$\chi_{\bar{I}}(J) = \frac{\phi_J(\omega)}{\omega} = \frac{\psi_1 \circ \phi_a(\omega) + \psi_2 \circ \phi_b(\omega)}{\omega} = (1 \otimes a)\frac{\psi_1(\omega)}{\omega} + (1 \otimes b)\frac{\psi_2(\omega)}{\omega}.$$

Since $1 \otimes a, 1 \otimes b \in \mathcal{O}(-J)$, $\frac{\psi_1(\omega)}{\omega}, \frac{\psi_2(\omega)}{\omega} \in \mathcal{O}(V_{\bar{I}})$, and the degree of $\chi_{\bar{I}}(J)$ is $\deg(J)$, we get $\chi_{\bar{I}}(J) \in \mathcal{L}(V_{\bar{I}} - J + \deg(J)\infty)$. The divisor $D := V_{\bar{I}} + \deg(J)\infty - J$ has degree g and is such that:

$$D - D^{(1)} \sim V_{\bar{I}} - V_{\bar{I}}^{(1)} \sim \Xi - \infty \qquad \operatorname{red}(D - g\infty) \sim I - J + (\deg(J) - \deg(I))\infty.$$

By Lemma 4.25, $D \sim V_{\bar{I}-\bar{J}}$, and $h^0(V_{\bar{I}-\bar{J}}) = 1$, hence $\operatorname{Div}(\chi_{\bar{I}}(J)) = V_{\bar{I}-\bar{J}} + J - V_{\bar{I}} - \deg(J)\infty$. Since $\chi_{\bar{I}}(J)(\Xi) = c_0 \in K_\infty$ and $\operatorname{Div}^+(\chi_{\bar{I}}(J)), \operatorname{Div}^-(\chi_{\bar{I}}(J))$ are K_∞-rational, $\chi_{\bar{I}}(J) \in \operatorname{Frac}(K_\infty \otimes A)$. □

Let's prove a lemma before the last proposition.

LEMMA 8.4. *Fix an ideal $I \trianglelefteq A$, with degree d_I. Then, for all ideal classes $\bar{J} \in Cl(A)$, there is some H-rational function $h_{I,\bar{J}}$ with divisor:*

$$\operatorname{Div}(h_{I,\bar{J}}) = V_{\bar{J},*}^{(1)} + V_{\bar{I}+\bar{J}} - I - \Xi - (2g - d_I - 1)\infty.$$

Moreover, we can choose $\{h_{I,\bar{J}}\}_{\bar{J} \in Cl(A)}$ such that for all $\bar{J} \in Cl(A)$, $\frac{h_{I,\bar{A}}}{h_{I,\bar{J}}}(\Xi) = 1$.

Proof. Fix some ideal $J \trianglelefteq A$, call d_J its degree, and define $D := I + \Xi + (2g - d_I - 1)\infty$. Consider the divisor $D - V_{\bar{J},*}^{(1)}$: we want to prove that it is equivalent to $V_{\bar{I}+\bar{J}}$. First of all, its degree is g, hence it is equivalent to some effective divisor. Moreover, we have the following equivalences:

$$\begin{aligned}
\operatorname{red}_{K_\infty}(D - V_{\bar{J},*}^{(1)}) &\sim \operatorname{red}_{K_\infty}(D) - \operatorname{red}_{K_\infty}(V_{\bar{J},*}) \\
&\sim (I + (2g - d_I)\infty) - ((d_J + g)\infty - J) \\
&\sim (I + J) + (g - d_J - d_I)\infty \sim \operatorname{red}_{K_\infty}(V_{\bar{I}+\bar{J}}); \\
(D - V_{\bar{J},*}^{(1)}) - (D - V_{\bar{J},*}^{(1)})^{(1)} &\sim (D - D^{(1)}) - (V_{\bar{J},*} - V_{\bar{J},*}^{(1)})^{(1)} \\
&\sim (\Xi - \Xi^{(1)}) - (\infty - \Xi)^{(1)} \sim \Xi - \infty \sim V_{\bar{I}+\bar{J}} - V_{\bar{I}+\bar{J}}^{(1)}.
\end{aligned}$$

By Lemma 4.25, the two conditions imply that $D - V_{\bar{J},*}^{(1)} \sim V_{\bar{I}+\bar{J}}$.

By Remark 4.27, the divisors $\{V_{\bar{J},*}^{(1)} + V_{\bar{I}+\bar{J}} - D\}_{\bar{J} \in Cl(A)}$ are H-rational. Moreover, by the same reasoning as Remarks 4.27 and 5.12, they are all conjugated by the action of $\operatorname{Gal}(H/K)$. We can therefore fix a function $h'_{I,\bar{A}} \in \operatorname{Frac}(H \otimes A)$ with divisor $V_{\bar{A},*}^{(1)} + V_{\bar{I}} - D$, and set $h'_{I,\bar{J}} := {h'_{I,\bar{A}}}^{\sigma_{\bar{J}}}$ for all $\bar{J} \in Cl(A)$, where $\sigma_{\bar{J}}$ is the appropriate element of $\operatorname{Gal}(H/K)$.

Now, for all $\sigma \in \operatorname{Gal}(H/K)$, we set $c_\sigma := \frac{{h'_{\bar{A}}}^\sigma}{h'_{\bar{A}}}(\Xi) \in H^\times$. For all $\sigma, \tau \in \operatorname{Gal}(H/K)$ we have $c_{\sigma\tau} = c_\sigma c_\tau^\sigma$, so by Hilbert 90 there is some $b \in H$ such that, for all $\sigma \in \operatorname{Gal}(H/K)$, $c_\sigma = \frac{b}{b^\sigma}$. Finally, we can set $h_{I,\bar{J}} := h'_{I,\bar{J}} \cdot b^{\sigma_{\bar{J}}}$, so that both the conditions we were looking for are satisfied. □

Remark 8.5. The quotients $\frac{h_{I,\bar{A}}}{h_{I,\bar{J}}}$ only depends on the class $\bar{I}$.

DEFINITION 8.6. The *Anderson zeta function* relative to the Drinfeld module ϕ is defined as:

$$\xi_{\bar{I}} := \sum_{J \trianglelefteq A} \frac{\chi_{\bar{I}}(J)}{\chi_{\bar{I}}(J)(\Xi)} \in K_\infty \hat{\otimes} K \cong K((u)).$$

In the notations of Lemma 8.4, we have the following result.

Theorem 8.7. *The function $\xi_{\bar{I}}$ is well defined, and the following identity holds:*

$$h_{I,\bar{A}}\xi_{\bar{I}} = -\left(\sum_{\sigma\in\mathrm{Gal}(H/K)} h^{\sigma}_{I,\bar{A}}\right)\zeta_A.$$

Proof. Let's fix representatives $J_i \trianglelefteq A$ for each ideal class $\bar{J}_i \in Cl(A)$. To prove convergence we rearrange the terms:

$$\sum_{J\trianglelefteq A}\frac{\chi_{\bar{I}}(J)}{\chi_{\bar{I}}(J)(\Xi)} = \sum_i \sum_{\substack{J\trianglelefteq A\\ \bar{J}=\bar{J}_i}} \frac{\chi_{\bar{I}}(J)}{\chi_{\bar{I}}(J)(\Xi)} = \sum_i \sum_{\substack{a\in^* J_i^{-1}\\ \mathrm{sgn}(a)=1}} \frac{\chi_{\bar{I}}(aJ_i)}{\chi_{\bar{I}}(aJ_i)(\Xi)} = -\sum_i \sum_{a\in^* J_i^{-1}} \frac{\chi_{\bar{I}}(aJ_i)}{\chi_{\bar{I}}(aJ_i)(\Xi)}$$

$$= -\sum_i\left(\frac{\chi_{\bar{I}}(J_i)}{\chi_{\bar{I}}(J_i)(\Xi)}\sum_{a\in^* J_i^{-1}}\frac{\chi_{\bar{I}}(a)}{\chi_{\bar{I}}(a)(\Xi)}\right) = -\sum_i \frac{\chi_{\bar{I}}(J_i)}{\chi_{\bar{I}}(J_i)(\Xi)}\zeta_{J_i^{-1}}.$$

We now express explicitly the quotient $\frac{\zeta_{J_i^{-1}}}{\zeta_A}$. In $K((u))$, it is the limit of the sequence $\left(\frac{\zeta_{J_i^{-1},m}}{\zeta_{A,m}}\right)_m$, whose divisor for $m \gg 0$ is $V_{-\bar{J}_i,*,m} - V_{\bar{A},*,m} - J_i + \deg(J_i)\infty$, which we can rearrange as:

$$(V_{-\bar{J}_i,*,m} + V_{-\bar{J}_i,m} - 2g\infty) - (V_{\bar{A},*,m} + V_{-\bar{J}_i,m} + J_i - (2g+\deg(J_i))\infty).$$

Both divisors in the parentheses are principal, and their positive components converge; by Proposition 3.19, the function $\frac{\zeta_{J_i^{-1}}}{\zeta_A}$ is rational, with divisor:

$$\lim_m \left(V_{-\bar{J}_i,*,m} - V_{\bar{A},*,m} - J_i + \deg(J_i)\infty\right) = V^{(1)}_{-\bar{J}_i,*} - V^{(1)}_{\bar{A},*} - J_i + \deg(J_i)\infty.$$

If we multiply $\frac{\zeta_{J_i^{-1}}}{\zeta_A}$ by $\chi_{\bar{I}}(J_i)$, the resulting divisor is $V^{(1)}_{-\bar{J}_i,*} + V_{\bar{I}-\bar{J}_i} - V^{(1)}_{\bar{A},*} - V_{\bar{I}}$, which is the divisor of $\frac{h_{I,-\bar{J}_i}}{h_{I,\bar{A}}}$ by Lemma 8.4. Since $-\zeta_{J_i^{-1}}(\Xi) = -\zeta_A(\Xi) = \frac{h_{I,-\bar{J}_i}}{h_{I,\bar{A}}}(\Xi) = 1$, we have the equality $\frac{\zeta_{J_i^{-1}}}{\zeta_A}\chi_{\bar{I}}(J) = \chi_{\bar{I}}(J)(\Xi)\frac{h_{I,-\bar{J}_i}}{h_{I,\bar{A}}}$, so we can rewrite the Anderson zeta as:

$$\xi_{\bar{I}} = -\zeta_A \sum_i \frac{h_{I,-\bar{J}_i}}{h_{I,\bar{A}}} = -\zeta_A \sum_{\bar{J}\in Cl(A)} \frac{h_{I,\bar{J}}}{h_{I,\bar{A}}} = -\zeta_A \sum_{\sigma\in\mathrm{Gal}(H/K)} \frac{h^{\sigma}_{I,\bar{A}}}{h_{I,\bar{A}}}.$$ □

Remark 8.8. Evaluating at Ξ, we get - modulo the characteristic of $\mathbb{F}_q$ - $\xi_{\bar{I}}(\Xi) = \#Cl(A)$.

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