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Optimal Satellite Attitude Control: a Geometric Approach

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Abstract—Optimal nonlinear control remains one of the most challenging subjects in control theory despite a long research history. [1 2] In this paper, we present a geometric optimal control approach, which circumvents the tedious task of numerically solving online the Hamilton Jacobi Bellman (HJB) partial differential equation, which represents the dynamic programming formulation of the nonlinear global optimal control problem. Our approach makes implementation of nonlinear optimal attitude control practically feasible with low computational demand onboard a satellite. Optimal stabilizing state feedbacks are obtained from the construction of a Control Lyapunov function. Based on a phase space analysis, two natural dual optimal control objectives are considered to illustrate the application of this approach to satellite attitude control: Minimizing the norm of the control torque subject to a constraint on the convergence rate of a Lyapunov function, then maximizing the convergence rate of a Lyapunov function subject to a constraint on the control torque. Both approaches provide ease of implementation and achieve robust optimal trade-offs between attitude control rapidity and torque expenditure, without computational issues.

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1. INTRODUCTION

Two different viewpoints, both involving partial differential equations, have so far dominated nonlinear optimal control theory: the Euler-Lagrange approach from calculus of variations and the global Hamilton-Jacobi-Bellman (HJB) approach based on dynamic programming.

It has proven particularly difficult to numerically solve these PDE's when dealing with nonlinear systems. Obtaining feedback control laws adds to the inherent difficulty. There are currently no systematic numerical methods, which can be applied to solve these PDE's for nonlinear systems online. The problem is considered numerically intractable and not suitable for operational spacecraft attitude control. Inverse optimal control theory circumvents the tedious task of numerically solving a Hamilton-Jacobi-Bellman equation. The starting point of the approach is to construct a stabilizing feedback controller based on a Control Lyapunov Function. The underlying theory shows that the Lyapunov function solves the Hamilton-Jacobi-Bellman equation. The resulting controller is optimal with respect to a 'meaningful' cost-function.

Lyapunov based inverse optimal control techniques are being increasingly considered for a large range of control system applications (see references [1], [4], [5]).

We show here the application of new theory allowing for optimal attitude control of a satellite with internal torques, with a geometric Lyapunov approach based on phase space design. We exploit the fact that the convergence rate of Lyapunov functions is a natural player in optimization problems.

Based on a phase space analysis, two dual optimal control objectives are considered to illustrate the application of this geometric inverse optimal approach to satellite attitude control:

- Minimizing the norm of the control torque subject to a constraint on the convergence rate of a

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Lyapunov function.

- Maximizing the convergence rate of a Lyapunov function subject to a constraint on the control torque.

The mathematical theory of the minimum norm control strategy has first been presented in [2], where a proof of inverse optimality was provided. This minimum norm theory has been used in the field of robotics with promising results. However, the approach has surprisingly only been considered in a single paper for satellite attitude control, but the attitude parameterization chosen to simplify control design was subject to singularities (see reference [1]). The Lyapunov function was quadratic with a bilinear coupling term and the approach presented here allows for nonlinear coupling.

2. NOTATIONS

$I = [I_1, I_2, I_3]^T$: Inertia tensor of the body of the satellite about its centre of mass.

$\overline{q} = [q, q_4]^T = [q_1, q_2, q_3, q_4]^T$: Attitude quaternion of the satellite.

$\omega = [\omega_1, \omega_2, \omega_3]^T$: Vector of the angular velocity in body fixed reference frame.

$h = [h_1, h_2, h_3]^T$: Angular momentum generated by the reaction wheels in the body frame.

L: Total angular momentum in the body frame.

3. DYNAMIC MODEL

If no external disturbance torque is assumed, we have:

$$\dot{L} + \omega \times L = \mathbf{0} \tag{1}$$

The equation of the total angular momentum is:

$$L = I\omega + h \tag{2}$$

By substituting L from equation (2), into the equation (1), we obtain the general Euler's rotational equation using three orthogonal reaction wheels:

$$\begin{aligned} I_1\dot{\omega}_1 &= (I_2 - I_3)\omega_2\omega_3 + N_1 - \omega_2 h_3 + \omega_3 h_2 \\ I_2\dot{\omega}_2 &= (I_3 - I_1)\omega_3\omega_1 + N_2 - \omega_3 h_1 + \omega_1 h_3 \\ I_3\dot{\omega}_3 &= (I_1 - I_2)\omega_1\omega_2 + N_3 - \omega_1 h_2 + \omega_2 h_1 \end{aligned} \tag{3}$$

where we define: $N_i = -\dot{h}_i$, i=1,3.

Based on the quaternion parameterization of attitude kinematics, the kinematic model of a satellite is given by:

$$\begin{cases} \dot{q} = -\dfrac{1}{2}\omega^{\times} q + \dfrac{1}{2} q_4 \omega \\ \dot{q}_4 = -\dfrac{1}{2}\omega^{\mathrm{T}} q \end{cases} \tag{4}$$

Despite the fact that mathematical stability proofs are often harder to make based on this parameterization, the absence of singularities and the fact that quaternions are readily available in the onboard attitude control software of most satellites makes this parameterization more attractive than the attitude parameterization of reference [1].

For the sake of convenience we can represent our system as an affine control system of the form:

$$\dot{x} = f(x) + g(x)u \tag{5}$$

Where:

$x = [\overline{q}, \omega]^{\mathrm{T}}$

$u = [N_1, N_2, N_3]^{\mathrm{T}}$

$f(x) = [\dot{\overline{q}}, \dot{\omega}]_{u=0_{3\times1}}$, $g = \begin{bmatrix} \mathbf{0}_{4\times3} \\ \mathbf{1}_{3\times3} \end{bmatrix}$

4. MINIMUM NORM ATTITUDE CONTROL

For a given control Lyapunov function V, there exists by definition a control input u satisfying:

$$\frac{\partial V}{\partial x}\left(f(x) + g(x)u(x)\right) < 0 \tag{6}$$

However, u is known not to be unique in general. Inverse optimal construction is a way of determining a specific but optimal stabilizing control input.

A particularly interesting nonlinear control construction based on the existence of a Control Lyapunov function is the minimum-norm control law proposed by Freeman and Kokotovic (see reference [2]).

It is constructed from the solution to a static pointwise (for each point x) optimization problem:

$$\begin{aligned} &\text{Minimise} \quad \|u\| \\ &\text{subject to} \quad \dot{V}(x) = \frac{\partial V}{\partial x}^{\mathrm{T}}(f(x) + g(x)u) \leq -\sigma(x) \leq 0 \end{aligned} \tag{7}$$

Where the function $-\sigma(x)$ can be viewed as describing a nonlinear stability margin ([1], [2]).

This minimum-norm problem characterizes the existing link between stability margins and optimality.

A minimum-norm control problem is the pointwise optimization problem of minimizing the control effort subject to a constraint on the convergence rate of V.

This nonlinear program can be solved analytically. We can write it more compactly as a least norm problem:

$$\text{Minimise}\|u\| \quad st. \quad \langle a,u\rangle \le b \tag{8}$$

With:

$b = -L_f V(x) - \sigma(x)$

$a = L_g V(x)$

$L_f V \equiv \dfrac{\partial V^{\mathrm{T}}}{\partial x} f(x)$ and $L_g V \equiv \dfrac{\partial V^{\mathrm{T}}}{\partial x} g(x)$ respectively represent the Lie derivatives of V in the directions of the vector fields f and g.

The solution of this least norm problem is given by:

$$u = \begin{cases} \dfrac{b.a^{\mathrm{T}}}{a.a^{\mathrm{T}}} & \text{if} \quad b < 0 \\ 0 & \text{if} \quad b \ge 0 \end{cases} \tag{9}$$

The solution is a projection onto the space spanned by the vector a^{T} if the constraint is not solved by turning the controller off. Otherwise, when the constraint $b \ge 0$ is satisfied, then the minimum norm solution satisfying the constraint of equation (7) is simply $u = 0$.

The minimum-norm control law is therefore given by:

$$u_{\text{opt}} = \begin{cases} -\dfrac{(L_f V(x) + \sigma(x))(L_g V(x))^{\mathrm{T}}}{(L_g V(x))(L_g V(x))^{\mathrm{T}}} & \text{if } L_f V(x) > -\sigma \\ 0 & \text{if } L_f V(x) \le -\sigma \end{cases} \tag{10}$$

Furthermore, Freeman and Kokotovic have proven in reference [2] using differential game theory that every min-norm controller is robust and inverse optimal with respect to a meaningful cost function of the form:

$$J(x,u) = \int_0^\infty \{l(x) + r(x,u)\} dt \tag{11}$$

Where:

- $l(x)$ is a continuous positive definite function of x (with $l(0)=0$). It is lower bounded by a continuous positive definite increasing function of x.
- $r(x,u) = \gamma_x(\|u\|)$ is positive definite for some continuous increasing convex function γ_x of $\|u\|$.

The cost functional is however unspecified when stating this inverse optimal problem (see ref [2]). However, the fact that an optimization trade-off is being achieved is evident from the nonlinear program stated in equation (7), which is solved analytically. Indeed, a norm of the torque is minimized subject to a rapidity constraint, which represents the other side of the trade-off.

We consider the problem of the attitude control of a satellite with a minimum norm approach.

A natural choice of the negativity margin function $-\sigma$ is given by the time derivative of a Lyapunov function (representing a certain convergence rate), under the effect of a benchmark controller $k(x)$:

$$\sigma(x) = -\dot{V}_{k(x)} = -L_f V - L_g V k(x) \tag{12}$$

As a benchmark controller, we consider without loss of generality a standard PD law that drives q and ω to zero:

$$k(x) = u_{\text{PD}} = -k_{\text{p}} q - k_{\text{d}} \omega \tag{13}$$

By substituting σ from equation (12) into the minimum norm attitude control law of equation (10), we have:

$$u_{\text{opt}} = \begin{cases} \dfrac{[L_g V\, u_{\text{PD}}](L_g V)^{\mathrm{T}}}{(L_g V)(L_g V)^{\mathrm{T}}} & \text{if } -(L_g V) u_{\text{PD}} > 0 \\ 0 & \text{if } -(L_g V) u_{\text{PD}} \le 0 \end{cases} \tag{14}$$

Where the term between brackets is a scalar quantity and the projection is in the direction of $L_g V^{\mathrm{T}}$.

We adopt the quaternion representation of attitude kinematics in order to avoid any singularities due to attitude parameterization. For our first simulation study, we shall consider the following candidate Lyapunov function with bilinear coupling:

$$V = (k_p + \gamma k_d)((1 - q_4^2) + q^{\mathrm{T}} q) + \frac{1}{2}\omega^{\mathrm{T}} \mathbf{I} \omega + \gamma q^{\mathrm{T}} \mathbf{I} \omega \tag{15}$$

This candidate is similar to the one proposed in [8] but we consider it here as a Lyapunov function of the benchmark controller. Note that the switching function $L_g V^{\mathrm{T}}$ is a linear function of ω and q with this first choice of V. The parameter γ of the bilinear coupling term will allow for the tuning of the minimum norm controller.

With the Lyapunov function of equation (15), the control torque given in equation (14) is switched off between two straight lines in phase space domain: $L_g V=0$ and $u_{\text{PD}}=0$. The first switching curve is a straight line because $L_g V = \omega + \gamma q$.

New generalized Lyapunov function

Given that the Lyapunov function V of the benchmark controller explicitly affects the control law with the minimum norm approach (see equation 14), we seek a generalization of it's expression into a class of functions with nonlinear (not bilinear) coupling, thence not restricting the switching curve $LgV=0$ to be a straight line.

A motivation for considering a nonlinear coupling is that, with a linear switching function $L_g V$ (for rest to rest manoeuvres with a PD feedback as a benchmark), the time during which the control is switched off between two switching lines remains the same (because of a constant angular velocity). It is desirable to seek stable nonlinear switching functions allowing for the control torque to be switched off for variable time durations depending on the amplitude of the maneuver. We shall also demonstrate that nonlinear coupling can enhance control accuracy.

We propose the generalized candidate Lyapunov function:

$$V = 2(k_p + \gamma k_d)(1 - q_4) + \frac{1}{2}\omega^{\mathrm{T}}\mathbf{I}\omega + \gamma F^{\mathrm{T}}(q)\mathbf{I}\omega \qquad (16)$$

$F(q)$ is a 3 dimensional vector: $F(q) = [F_{\mathrm{i}}(q_{\mathrm{i}}), \mathrm{i} = 1{,}3]$. Note that the corresponding switching function is given by $L_g V = \omega + \gamma F(q) = 0$.

Let $\sigma_{\min}$ denote the minimum eigenvalue of the matrix **I**.
Let $\sigma_{\max}$ denote the maximum eigenvalue of the matrix **I**.

Positivity of V— The following inequalities hold for any choice of the symmetric positive definite matrix **I**:

$$\omega^T \mathbf{I}\omega \geq \omega^T \sigma_{\min}\omega \qquad (17)$$

$$F^T \mathbf{I}\omega \geq -b\|\omega\|\|q\|\sigma_{\max} \qquad (18)$$

Where $\| \ \|$ denotes a standard Euclidian norm.

We note that the condition $\|F(q)\| < b\|q\|$ was assumed, which is satisfied with F continuous on the finite interval where q is defined. We now investigate the positivity condition of the Lyapunov function V.

The inequality $1 - q_4 \geq \|q\| = \sqrt{q_1^2 + q_2^2 + q_3^2}$ holds at all times (It is not an equality because q_4 can be positive or negative). The positivity of V stems from the inequality:

$$V \geq [\|q\| \ \|\omega\|]\mathbf{K}[\|q\| \ \|\omega\|]^{\mathrm{T}} \qquad (19)$$

$$\mathbf{K} = \begin{bmatrix} k_p + \gamma k_{\mathrm{d}} & -\frac{\gamma\sigma_{\max}}{2}b \\ -\frac{\gamma\sigma_{\max}}{2}b & \sigma_{\min} \end{bmatrix} \qquad (20)$$

The parameter γ can be chosen small enough to make the matrix K positive definite.

Negativity of $\dot{V}$ — The time derivative of V is given by:

$$\dot{V} = (k_p + \gamma k_d)q^{\mathrm{T}}\omega + \omega^{\mathrm{T}}\mathbf{I}\dot{\omega} + \gamma F'\dot{q}^{\mathrm{T}}\mathbf{I}\omega + \gamma F^T\mathbf{I}\dot{\omega} \qquad (21)$$

Where $F' = \dfrac{\partial F(q)}{\partial q}$ and we denote $F(q)$ *as F.*

By expanding the derivative terms of equation (21), we have:

$$\dot{V} = (k_p + \gamma k_d)q^{\mathrm{T}}\omega + \omega^{\mathrm{T}}(-\omega\times(\mathbf{I}\omega + h) + N)$$
$$+ \gamma F'\left(\frac{1}{2}q_4\omega - \frac{1}{2}\omega^{\times}q\right)\mathbf{I}\omega + \gamma F^{\mathrm{T}}(-\omega\times(\mathbf{I}\omega + h) + N) \qquad (22)$$

We consider a Proportional plus Derivative control torque:

$$\begin{aligned}\dot{V} &= (k_p + \gamma k_d)q^T\omega - k_p\omega^T q - \gamma k_d F^T\omega \\ &+ \omega^{\mathrm{T}}(-\omega\times(\mathbf{I}\omega + h)) \\ &- k_{\mathrm{d}}\|\omega\|^2 - \gamma k_p F^T q \\ &+ \gamma F'\left(\frac{1}{2}q_4\omega - \frac{1}{2}\omega^{\times}q\right)\mathbf{I}\omega + \gamma F^T(-\omega\times(\mathbf{I}\omega + h))\end{aligned} \qquad (23)$$

The fourth term is zero because it represents a scalar product of two orthogonal vectors. Equation (23) reduces to:

$$\begin{aligned}\dot{V} &= \gamma k_d(q^{\mathrm{T}} - F^T)\omega \\ &- k_{\mathrm{d}}\|\omega\|^2 - \gamma k_p F^T q \\ &+ \gamma F'\left(\frac{1}{2}q_4\omega - \frac{1}{2}\omega^{\times}q\right)\mathbf{I}\omega + \gamma F^T(-\omega\times(\mathbf{I}\omega + h))\end{aligned} \qquad (24)$$

The function $F(q)$, defined on the finite intervals where quaternions are defined, has to be continuous ($F_{\mathrm{i}}(q_{\mathrm{i}})$ continuous, i=1,3), to satisfy the condition $c\|q\| \leq F(q) \leq d\|q\|$. In other words, $F(q)$ is a Lipschitz function. This condition is required to impose a bound to the first and last term of equation (24).

A positive constant then exists such that $\|q - F(q)\| \leq a\|q\|$.

Similarly, F' must be continuous with $q_{\mathrm{i}} \in$[-1,1], i=1,3 to satisfy a positive bound on it's norm $\|F'\| \leq \beta$.

The expression of the time derivative of V finally satisfies:

$$\begin{aligned}\dot{V} &\leq -k_{\mathrm{d}}\|\omega\|^2 - \gamma k_p F^{\mathrm{T}}q + \gamma k_d a\|q\|\|\omega\| \\ &+ \gamma F'\left(\frac{1}{2}q_4\omega - \frac{1}{2}\omega^{\times}q\right)\mathbf{I}\omega + \gamma F^T(-\omega\times(\mathbf{I}\omega + h))\end{aligned} \qquad (25)$$

The signs of $F(q)$ and q must agree to make the second term smaller than a negative definite function (a quadratic term with negative sign).

The Lyapunov stability conditions are that $q^{\mathrm{T}}F(q) \geq 0$ and $F(q)$ continuous, which implies that $F(0)=0$.

The Lipschitz property of a continuous function F ($\|F\| > c\|q\|$) can serve as a bound for the second term.

We can now conclude that the time derivative of V satisfies the following inequality:

$$\dot{V} \le -\left[\|q\|\ \|\omega\|\right]\mathbf{M}\left[\|q\|\ \|\omega\|\right]^{\mathrm{T}} \tag{26}$$

A reasonable additional assumption (finiteness) is required to establish the stability proof: $\alpha = \sup_{t\ge 0}\|\omega\|$.

The inequality (27) is now satisfied with:

$$\mathbf{M} = \begin{bmatrix} \gamma k_{\mathrm{p}} c & -\left(\frac{1}{2}+\frac{\beta}{4}\right)\alpha\gamma\sigma_{\max} - \frac{\gamma k_{\mathrm{d}}}{2}a \\ -\left(\frac{1}{2}+\frac{\beta}{4}\right)\alpha\gamma\sigma_{\max} - \frac{\gamma k_{\mathrm{d}}}{2}a & k_{\mathrm{d}} - \gamma\beta\sigma_{\max} \end{bmatrix} \tag{28}$$

Therefore, by choosing γ to make the matrix **M** positive definite, the function V given by equation (16) is a Lyapunov function for the system of equations (3) and (4).

Sufficient stability condition—To summarize the stability analysis, sufficient stability conditions are:

- $F(q)$ has to be a class C^1 function.
- The function F is contained in the first and third quadrant: $q^T F(q) \ge 0$.

These two conditions also imply that $F(0)=0$.

The last mathematical proof of stability shows that the switching curve can be chosen more freely in a minimum norm approach than previously attempted (The switching function L_gV in reference [1] can only be a straight line).

The switching curve where the torque is first switched off is given by $L_gV(x) = \omega + \gamma F(q) = 0$. The function F can be any nonlinear function satisfying the above stability conditions. Generalizing the choice of L_gV to nonlinear functions is also suitable because L_gV is the direction where the torque is projected in the regions of the phase space lying outside of the switching curves $L_gV(x)=0$ and $u_{PD}=0$ where the torque is respectively switched off and back on.

Among the advantages associated with the flexibility of the choice of F, it should be noted that between two straight lines as in reference [1] ($F(q)=q$), the torque is turned off for the same time duration, for any rest to rest maneuver after fixing the value of γ. In practice, larger maneuvers can require the control torque to be switched off for longer time durations and that can only be achieved if the function $F(x)$ is nonlinear and satisfies the stability conditions.

Generalised (Modified) minimum-norm approach

A practically convenient implementation of minimum norm control that demonstrates trade-off improvement over a benchmark controller has recently been proposed in references [4] and [5] for the control of a pendulum. The control objective is to impose the torque expenditure of a benchmark controller by minimizing the functional:

$$u_{\mathrm{opt}} = \underset{u\in \mathrm{Kv(x)}}{\mathrm{Argmin}}\left\{\|u - k(x)\|\right\} \tag{27}$$

where k(x) is a benchmark stabilising controller for a given Lyapunov function V.

For our satellite attitude control simulation of this controller, we consider the Lyapunov function given in equation (15).

A rapidity constraint is added to this minimization problem. The control input belongs to the rapidity constrained set:

$$\mathrm{Kv(x)} = \left\{u : L_f V + L_g V\, u < -\sigma(x), u \in \mathrm{U}\right\} \tag{28}$$

The generalised minimum norm control law satisfying the above conditions is given by:

$$u_{\mathrm{opt}} = \begin{cases} -\dfrac{\psi(x)\left(L_g V\right)^{\mathrm{T}}}{\left(L_g V\right)\left(L_g V\right)^{\mathrm{T}}} + k(x) & \text{if } \psi > 0 \\ k(x) & \text{if } \psi \le 0 \end{cases} \tag{29}$$

$$\psi = L_f V + L_g V\, k(x) + \sigma(x)$$

We choose the negativity margin function as:

$$\sigma(x) = \eta\left(q^{\mathrm{T}} q + \omega^{\mathrm{T}}\omega\right) \tag{30}$$

Where η is a positive constant parameter.

The Lie derivatives of the Lyapunov function in the directions of the vector fields f and g are given by:

$$L_g V(x) = \left[\omega_1 + \gamma q_1, \omega_2 + \gamma q_2, \omega_3 + \gamma q_3\right]$$

$$L_f V(x) = \frac{\partial V^{\mathrm{T}}}{\partial q}\dot{q} + \left[\frac{\partial V^{\mathrm{T}}}{\partial \omega}\dot{\omega}\right]_{u=0_{3\times 1}}$$

5. MAXIMUM RATE ATTITUDE CONTROL

We consider the following maximum-rate control problem:

$$\text{Maximise } -\dot{V}(x) \quad \text{st.} \quad \|u\| \le u_{\max} \tag{31}$$

Where:

$$\dot{V}(x) = \frac{\partial V^{\mathrm{T}}(x)}{\partial x} f(x) + \frac{\partial V^{\mathrm{T}}(x)}{\partial x} g(x) u \tag{32}$$

The optimisation objective is to maximize the convergence rate of a Lyapunov function of the system, subject to a maximum absolute value of the norm of the torque.

To illustrate the maximum rate control concept, a convenient system to consider is the double integrator. The double integrator is generally considered as an appropriate mathematical model for single axis spacecraft rest to rest maneuvers, when the gyroscopic torque is small.

The equation of a double integrator is:

$$\ddot{\theta} = u \Leftrightarrow \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = u \end{cases} \qquad (33)$$

Where $x = [x_1, x_2] = [\theta, \dot{\theta}]$, $f(x) = [\dot{\theta}, 0]^T$, $g(x) = [0,1]^T$

Since the torque u is scalar for this single input system, the norm being considered is simply the absolute value of the torque. We shall see that the double integrator is a particularly convenient example to compare the maximum rate control law with a minimum-time controller.

Provided that $\dot{V}$ can be made negative with a sufficiently high gain, the maximum rate (or minimum time sliding mode) control law is given by:

$$u_i = -u_{\mathrm{i}\max}.sign\left(\frac{\partial V^{\mathrm{T}}}{\partial x} g(x)\right)_{\mathrm{i}} \qquad (34)$$

For the double integrator, the maximum-rate control problem is solved with the Lyapunov function:

$$V = \frac{1}{2}\left(\dot{\theta} + \gamma\theta\right)^2 \qquad (35)$$

The maximum rate control law is:

$$u = -u_{\max}.sign\left(\dot{\theta} + \gamma\theta\right) \qquad (36)$$

The time derivative of V, with the control law given by equation (36) is given by:

$$\dot{V}(x) = -\left(\dot{\theta} + \gamma\theta\right)u_{\max} sign(\dot{\theta} + \gamma\theta) + \gamma\dot{\theta}\left(\theta + \gamma\dot{\theta}\right) \qquad (37)$$

$\dot{V}$ is therefore negative under the condition that:

$$u_{\max} \geq \gamma\left|\dot{\theta}\right|_{\max} \qquad (38)$$

where $\dot{\theta}$ solves:

$$\frac{\dot{\theta}^2}{2} - \frac{\dot{\theta}^2(0)}{2} = -u_{\max}\left(\theta - \theta(0)\right) \qquad (39)$$

Minimizing time—We now investigate the possibility of achieving near minimum-time manoeuvres with a maximum rate controller.
The switching-time of the bang-bang minimum-time solution is well-known to be:

$$t_{\mathrm{s}} = \frac{t_{\mathrm{f}}}{2} \qquad (40)$$

A minimum time solution can in theory be achieved when the linear switching function intersects the extremal trajectory '$u= -u_{\max}$' at $\theta = \frac{\theta(0) - \theta_{\mathrm{f}}}{2}$. We show an example of this in the numerical simulations section.

By integrating both sides of the double integrator, the state trajectories are given by:

$$\frac{x_2^{\ 2}}{2} - \frac{x_{20}^{\ 2}}{2} = (x_1 - x_{10})\left(\pm u_{\max}\right) \qquad (41)$$

Removing chattering— The sign function will in practice cause a high frequency oscillation phenomenon, known as chattering (see figure (5a)). Indeed, the switching frequency cannot be infinite. As an alternative to that, we propose to replace the sign function by a smooth function with a boundary layer of the form:

$$\zeta(x) = \frac{x}{|x| + \varepsilon.e^{-(x^2/2\sigma^2)}} \qquad (42)$$

This smooth approximation of the sign function has the advantage of approximating the sign function more closely when the tracking error is large and smoothly when tracking errors are smaller. Chattering can therefore be removed without needlessly degrading control rapidity.

6. NUMERICAL SIMULATIONS AND ANALYSIS

The system parameters used for the simulations are:
$I_1 = 10\, kg\, m^2, I_2 = 14\, kg\, m^2, I_3 = 12\, kg\, m^2$

which corresponds to a slightly asymmetric micro-satellite.

The initial conditions are:

$\omega i(0)$=0, i=1, 2, 3.

$q_1(0) = 0.3062, q_2(0) = 0.1768, q_3(0) = 0.1768$

$q_4(0) = 0.9186$

This corresponds to initial attitude errors of 30 degrees on all three axes for a 1-2-3 Euler rotation sequence.

The benchmark controller gains are: kp= 0.02, kd= 0.5.

Minimum norm controller with F(q)=q

Simulations of the minimum norm controller are first carried out with the function $F(q) = q$, corresponding to a Lyapunov function with bilinear coupling (given in equation (15)). The phase portrait of the system described by equations (3) and (4), with the control law (10) is presented in figure (1).We observe that the control torque is turned off between two straight lines in the phase space domain, determined by the Lyapunov function and the benchmark controller.

By decreasing the values of the parameter γ, the system proceeds more directly toward the origin in phase space, the torque expenditure is reduced, while rapidity is slightly degraded but the overall trade-off is improved (see figures (1), (2a), (2b)).

Compared to the PD benchmark, a significant amount of energy is saved, without significantly deteriorating rapidity. For instance, with γ=0.02, the average torque expenditure is divided by 2 while the 2% settling time increases by less than 50 seconds for a 300 seconds manoeuvre. We shall however demonstrate more rigorously that a better trade-off is achieved by minimum-norm techniques.

In reference [1], both rapidity and torque were improved, only because the gains k_p and k_d were inappropriately selected as the system was excessively under-damped. We have chosen the gains of the PD controller to achieve a 6% single overshoot (slightly under-damped) to undertake a fairer and more realistic comparison. Minimum norm controllers would achieve an even higher trade-off improvement if the PD gains happened to be excessively under-damped or over-damped.

Minimum norm controller with a nonlinear function F(q)

As an example, we consider the Tangent hyperbolic function:

$$F(q) = \tanh(\lambda q), 0 < \lambda < 1 \tag{43}$$

For λ=5, we observe on figures (4a) and (4B) that this choice of F (Lyapunov function with nonlinear coupling) achieves higher accuracy at steady state than the function $F(q) = q$. Note that for comparison purposes, we imposed a similar torque expenditure $\int_0^{t_F} u^T(t)u(t)dt = 4.10^{-3}(Nm)^2$ by setting $\gamma = 0.025$ for $F(q)=q$ and $\gamma = 0.0075$ for F nonlinear.

Trade-off improvement by generalized minimum norm control

Even though the cost functional is unspecified with our inverse optimal minimum norm approach, we can show more rigorously that the energy-rapidity trade-off of a benchmark controller (PD in our study) can be outperformed by minimum norm laws. One way of achieving this is to compare rapidity after reducing the gains of the PD benchmark until the torque expenditure is similar to the one achieved with a minimum norm controller for a given value of the parameter γ, for the same manoeuvre. We prefer however to compare rapidity, not only for a similar torque and a similar manoeuvre but also for identical gains.

This is done by the generalised minimum norm approach. The control law is given in equation (29). k(x) is chosen to be a PD law. We consider the same PD gains as in previous simulations and we set γ =0.02, η=0.01.

We observe on figures (3a), (3b) and (3c) that attitude manoeuvres can be achieved more rapidly for a similar torque expenditure with a generalised minimum norm approach. The 1% settling time passes from 228s to 124s on the X axis, from 242s to 214s on the Y axis, and from 204s to 164s on the Z axis. Even the overall average torque is slightly reduced from 6.10^{-4}Nm with the PD benchmark to 5.83 10^{-4}Nm with the generalized minimum norm law.

Maximum rate controller

Minimum time case—We take as an example the initial condition: $\theta(0) = 1, \theta_f = 0$, with u_{max}= 0.5. For the double integrator, a minimum time response is obtained when the state variables at the switching point are given by:

$$x_1(t_s) = \frac{x_{10} - x_{1f}}{2} = 0.5$$

$$x_2(t_s) = -\sqrt{2|x_{1s} - x_{10}|u_{max}} = -\sqrt{0.5}$$

The maximum rate controller performs as a minimum time controller (see figure 5b) with the following choice of the parameter γ (slope of the switching curve):

$$\gamma = \frac{-x_2(t_s)}{x_1(t_s)} = \sqrt{2} = 1.414$$

The robustness advantage of the maximum rate approach to time minimization is that the control is in a closed loop.

Trade-offs obtained by varying γ—When different values of the parameter γ are chosen, rapidity is traded-off against torque expenditure in a similar way to what was observed with a minimum norm law (see figure (5a)).

Chattering removal—The boundary layer proposed in equation (42) removes chattering as noticed in figures (6a) (6b), with γ=1.2, ε=0.02, σ=0.5.

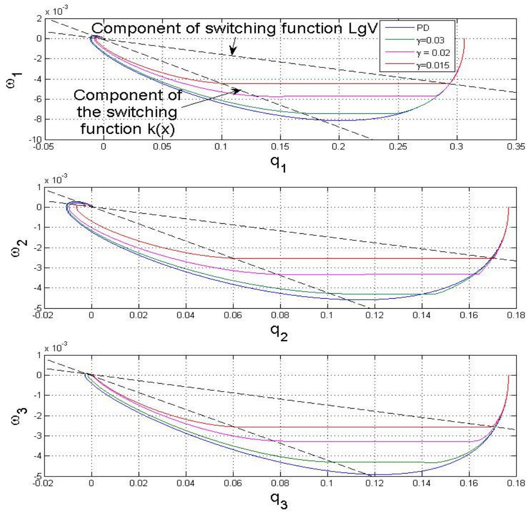

Figure 1: Phase portraits of the minimum norm controller with different values of γ

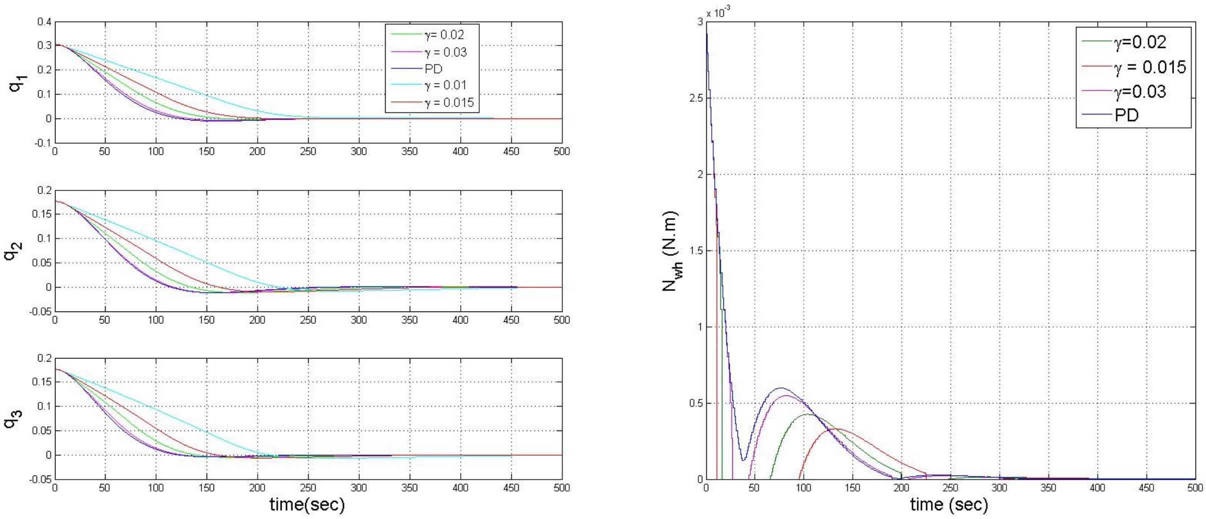

Figure 2: a- Attitude control response of the minimum norm controller, b- Norm of the control torque of the wheels

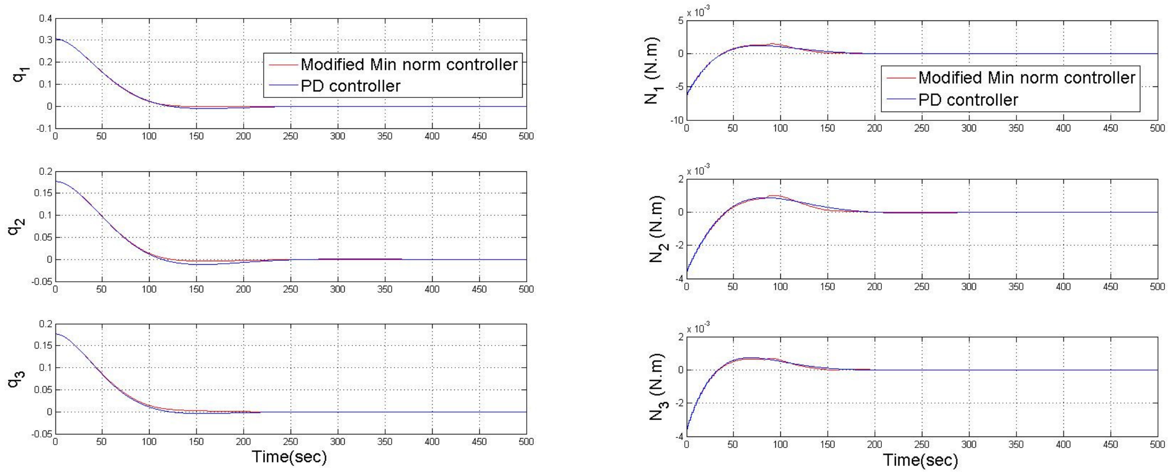

Figure 3: a-Comparative attitude control response, b- Comparative control torque of the wheels
Red: Generalised (modified) minimum norm controller, blue: PD benchmark controller

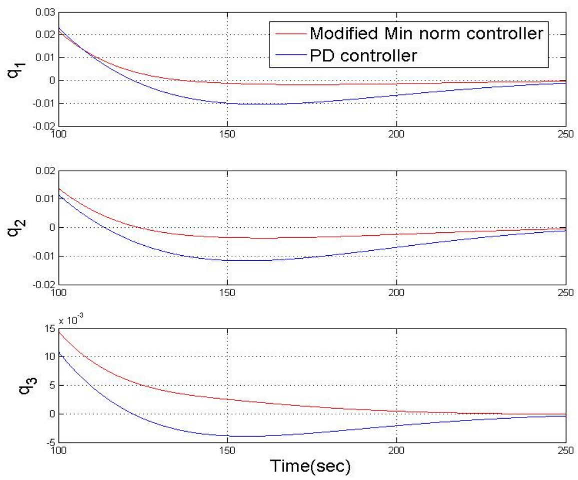

Figure 3-c: Comparative attitude control response of the PD and min-norm controller closer to the origin (from t=100 sec)

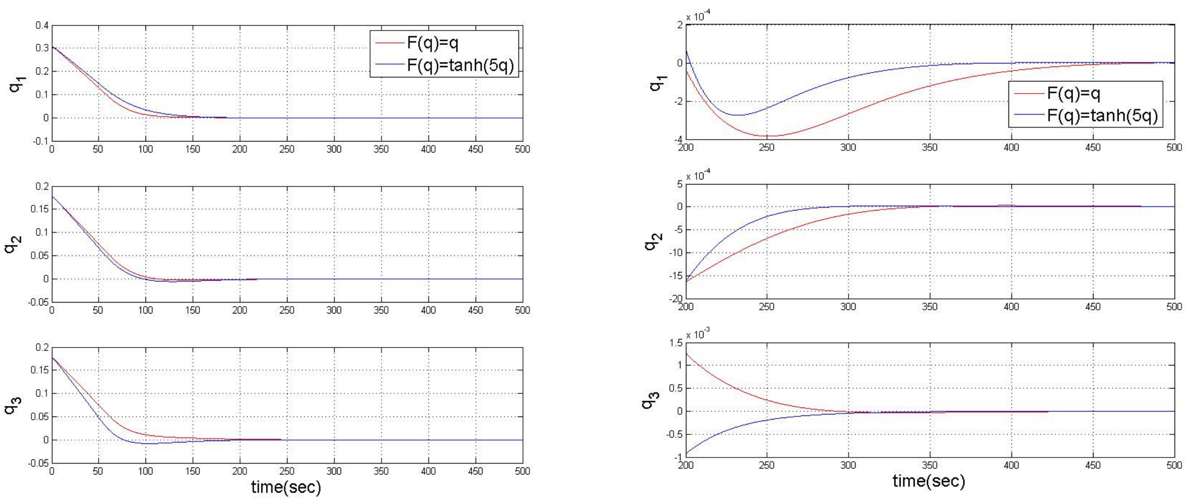

Figure 4:
a- Comparative attitude error with $F(q)$ linear (red) and $F(q)$ nonlinear(blue)
b- Comparative attitude error with $F(q)$ linear and $F(q)$ nonlinear at steady state (after t=200 seconds)

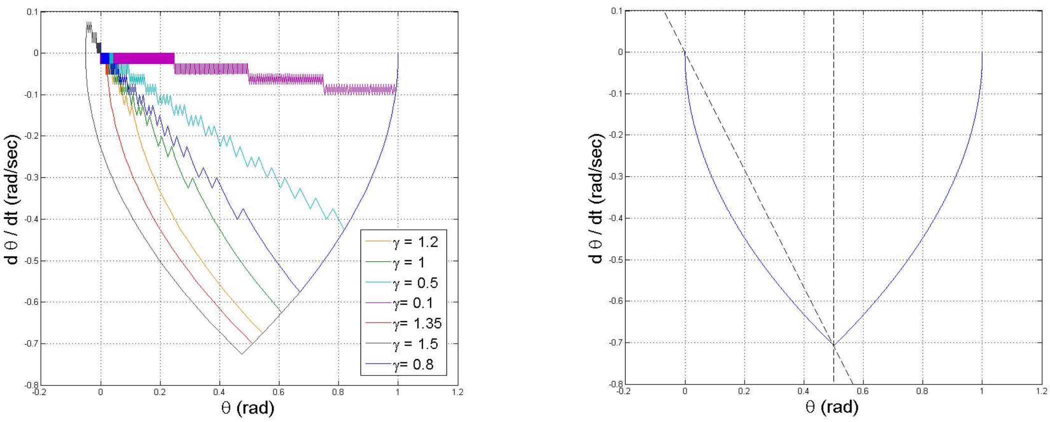

Figure 5: a- Phase portrait of the maximum rate control with γ varying.
b- Phase portrait of the maximum rate controller that achieves a minimum time response

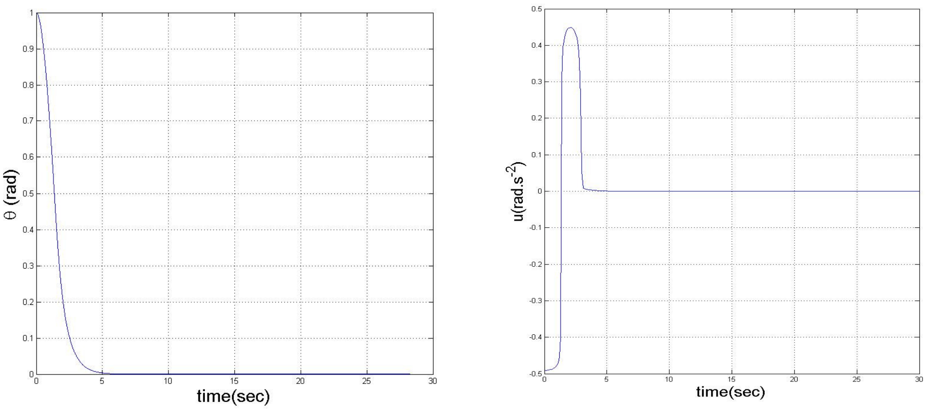

Figure 6: a- Rotation angle with a maximum rate controller and a smooth boundary layer
b- Control effort of the maximum rate controller with a smooth boundary layer

7. Conclusion

Optimal trade-offs between attitude control rapidity and torque expenditure, have been achieved with a Lyapunov approach. A new generalized Lyapunov function has been proposed for consideration with minimum norm controllers, with the advantage of control design flexibility. We have shown that generalized minimum norm laws decisively enhance the overall attitude control performance of standard benchmark controllers such as PD laws. Maximizing the convergence rate of Lyapunov function has also been shown to be a simple but viable practical solution for closed loop rest to rest near minimum time maneuvers.

These optimization approaches have the practical advantages of guaranteed global stability and of being free from computational issues, which is highly desirable given the limited resources onboard satellites.

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BIOGRAPHY

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