LQR based Optimal Control Design of Satellite Formation Flight in Earth-centered Circular Orbit

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LQR based Optimal Control Design of Satellite Formation Flight in Earth-centered Circular Orbit

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In this paper, the formation flight control of the relative translational dynamics of satellites moving in circular orbit around the Earth has been presented. Autonomous rendezvous and proximity operations of satellites in formation are the one of the most advanced technologies in space missions. This work introduces the State space model of 2nd order approximated dynamics of satellite relative motion, which is further linearized to apply the Linear Quadratic Regulator control. The LQR optimal solution is derived with Continuous time Algebraic Riccati equation (CARE) and the system stability is proved using a suitable Lyapunov function. For the optimal solution of the quadratic cost functional, a novel approach with the help of Krotov conditions and convexity imposition has been derived. The existing cost functional for the system has been time parameterized using Extension principle and solved using a suitable Krotov function. The simulation results have been shown, for varying cases proving the satisfactory transient and steady state response of the controlled system.

I. Nomenclature

x, y, z	=	Relative position coordinates in Local Vertical Local Horizontal (LVLH) frame
$\dot{x}, \dot{y}, \dot{z}$	=	Relative velocities in LVLH frame
$\ddot{x}, \ddot{y}, \ddot{z}$	=	Relative accelerations in LVLH frame
$\dot{\theta} = \omega$	=	Angular velocity of satellite around the Earth
μ	=	Gravitational parameter for Earth
r_c	=	Radius of the Leader satellite's orbit with respect to the Earth's center
u_x, u_y, u_z	=	Control thrusts/mass along x, y and z directions
M_e	=	Mass of Earth
R_e	=	Radius of Earth
h	=	Altitude of satellite from Earth's surface
G	=	Universal Gravitational constant
t	=	Time

II. Introduction

In present-day cutting-edge technology and in the near future, the relative motion control of two or more spacecraft in orbit, as well as the automatic control of rendezvous and proximity operations are the most important missions in the space program. Many papers have been published regarding control of spacecraft motion considering the nonlinear system of both translational and attitude dynamics in optimal or in non-optimal way. In most of the

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papers, the solution is approached in an optimal way for maintaining the tradeoff between fuel consumption and trajectory generation. Thus, efficient controls are one of the most critical issues.

In spacecraft relative motion control, two things come side by side that need to be addressed. One is the translational dynamics which is related to orbital motion of both the satellites involved in Formation, and the other is Attitude dynamics, where the angular motion along the own body axis of the satellites is taken into fact. In this paper, the translational dynamics of spacecraft is being addressed, where relative motion control is considered and is desired to maintain a prescribed distance between them. The force acting in the system is the central Gravitational force (neglecting the J2 perturbations, solar radiation forces etc.). The relative motion in circular orbit is being considered, where the effect of nonlinear gravity is being approximated to 2^{nd} order dynamics and then linearized. The quaternion-based methods for spacecraft attitude determination and control are reviewed in [1]. A linearized set of equations of relative motion about a J2-perturbed elliptical reference orbit is being developed and simulated in [2]. In [3], the PID based control is done for tracking, formation and rendezvous operations in the Keplerian orbits taking nonlinear gravity into account. Both the nonlinear and linear model of translational dynamics is being modelled in State space form in [4] and controlled. The spacecraft with respect to which the dynamics is modelled is called the 'Leader' vehicle in the Formation flying concept, which is assumed in this work to move in a circular orbit around the Earth. The companion vehicle, the 'Follower' spacecraft, that is to be at a desired distance from the Leader would be controlled. The dynamics of the Leader vehicle is modelled as a circular motion in a plane with respect to the Earth frame, which is called Earth Centered Inertial (ECI) frame. Another frame of reference is described, which is though non-inertial, and is fixed with the center of mass of the Leader vehicle, called Local Vertical Local Horizontal (LVLH) frame. The coordinates of the Follower are represented in the LVLH frame, whose translational dynamics is different in the ECI frame. The second order approximated dynamics is considered in the LVLH frame in this paper, where angular motion of the Leader around the Earth is taken into calculation.

The linearized State Space model of the system is formed with respect to a particular equilibrium point of the dynamical system. The state vector consists of 3D position and velocity, and the inputs as the 3DOF actuator commands. In this model, the ideal actuators and thrusters are being considered. The optimal control (the continuous thrust control) of the system by the Linear Quadratic Regulator (LQR) controller with full state feedback is designed. The LQR is the optimal control that takes into account the performance quality of the system, as well as the energy consumption as its argument and optimizes the control output. A detailed description of LQR based control and solving the optimization problem using the Riccati equation analytically, is presented in [1]. Nonlinear control of Error dynamics of the system using PID is described in [3]. An online trained neural network for controlling the nonlinear system augmenting a LQR control of the linearized model is presented in [4], giving a robust and optimal control. A State Dependent Riccati Equation (SDRE) based nonlinear control is being approached in [5], controlling the Formation flight of satellites of both orbital and attitude dynamics. A separate type of optimal control, Neighboring Optimum feedback control law is introduced in [6] to control maneuvering the Formation flight for fuel optimality condition. A real time suboptimal control for 6 DOF motion of spacecraft maneuvering based on SDRE approach and real-time linearization of the equations of motion considering error dynamics of the system is presented in [7]. In the case of Space missions, saving the fuel is one of the key aspects, which is needed for station keeping purpose. Thus, the actuator effort is taken as the primary optimizing criteria rather than trajectory performance in many cases. As previously stated, the dynamics is linearized and modelled about stable equilibrium points to get the state matrix in the state space model. These LQR algorithms are very effective in controlling spacecraft motion and its assembly. These are proved to be globally stable in the actual nonlinear spacecraft system. In this paper, the stability is presented with the help of suitable Lyapunov functions and also the optimal control gain is derived using Krotov conditions of stability for the spacecraft motion control system, which is novel in this approach. There are many advantages using this LQR optimal control. One of the most important is that the controller structure is extremely simple, implementation is easier, reducing computational cost without hurting the performance.

The novel contribution in this approach is that to build a suitable Krotov function to apply Krotov's optimality conditions, verifying its results with the Riccati and Lyapunov method of optimal control problem and to control the 2^{nd} order approximated linear model of the translational dynamics of the relative motion of satellites.

As a common annotation, all the scalars are represented in normal, whereas, the vectors are annotated in bold. In this paper, Section III describes about the reference frames, the dynamics modelling is stated in Section IV, the State Space model and its linearization to get linear dynamics is derived in Section V, the LQR control architecture as well as the system stability using Lyapunov Function is done is Section VI, the novel Krotov condition approach for the derivation and verification of the optimal control solution and its connection with the system dynamics is presented in Section VII, and the solution to CARE, to get the control law with simulation results, the eigenvalue placement (to get closed loop poles) to get stable system and finally simulation results from MATLAB for varying the initial and final conditions are specifically described in Section VIII.

III. Frame of References

The most fundamental base with respect to which any dynamics is modeled is the Frame of Reference. It can be inertial or non-inertial determined by whether it's accelerating or not. Here two of such important reference frames on which the translational dynamics is modeled are discussed.

A. Earth-Centered Inertial (ECI):

The ECI frame is the body frame fixed to the Earth, which is inertial (though the Earth is accelerating with respect to sun, but the frame fixed to this massive object is still considered to be inertial), thus Newton's laws of motion and law of Gravity are valid in this frame. And also, most of the satellites which are studied are generally inertial pointing. To locate a body about Earth in the ECI system, Cartesian coordinates are used. The X–Y plane coincides with the Earth's equatorial plane. Sometimes Earth's orbital plane (ecliptic) about the Sun is also considered as this plane. The X-axis is fixed (not rotating) in a particular direction with respect to the celestial body. The Z-axis lies perpendicular to this plane and coincides with Earth's rotational axis (passing through the North pole). The Y-axis follows the right-hand coordinate system rule. ECI coordinate system does not rotate as Earth's rotation, it remains fixed in its position.

B. Local Vertical Local Horizontal (LVLH):

This frame of reference is fixed to the body frame of the moving satellite, particularly the Leader spacecraft in case of Formation flying. The origin of this frame moves as the satellite's COM trajectory. The principal axes of this frame are pointed towards the center of Earth, one towards its velocity vector, and other in the perpendicular frame in right-handed coordinate system notation. In this paper, the x-axis is taken in radially outward direction from the Earth, the y-axis is coinciding with the local velocity vector of the Leader satellite and the z-axis, perpendicular to them. The relative motion dynamics of the system is modelled in this frame. As the LVLH frame rotates with the Leader motion, it is not an inertial frame of reference, and similarly Newton's laws couldn't be directly applicable in this frame. The relative motion translational dynamics representing the relative orbital motion of two satellites now have to be modelled in this frame of reference.

IV. Orbital Motion and Translational Dynamics

As the relative translational dynamics is considered, point mass assumption is made for the Leader and Follower satellite, with central Gravitational force of attraction acting both on them. The frame of reference is the LVLH frame fitted to the Leader satellite or Hill frame. The Leader is, as explained, moving in a particular circular orbit around the Earth, therefore, having constant angular velocity $\dot{\theta}$. The translation dynamics is modelled, with respect to the LVLH frame, where x, y and z are relative position coordinates. The resulting 1st order approximate dynamics equations are known as Clohessy-Wiltshire equations in the Hill frame. But, in this case, the dynamics is modelled using the 2nd order approximate dynamics equation. Thus, the resulting 2nd order dynamics equations (as being derived in [8]) are obtained as:

$$\ddot{x} - 2\dot{y}\dot{\theta} - 3\dot{\theta}^2 x = \left(\frac{3\mu}{r_c^4}\right)\left[\frac{y^2}{2} + \frac{z^2}{2} - x^2\right] \tag{1}$$

$$\ddot{y} + 2\dot{x}\dot{\theta} = \left(\frac{3\mu}{r_c^4}\right)xy \tag{2}$$

$$\ddot{z} + \dot{\theta}^2 z = \left(\frac{3\mu}{r_c^4}\right)xz \tag{3}$$

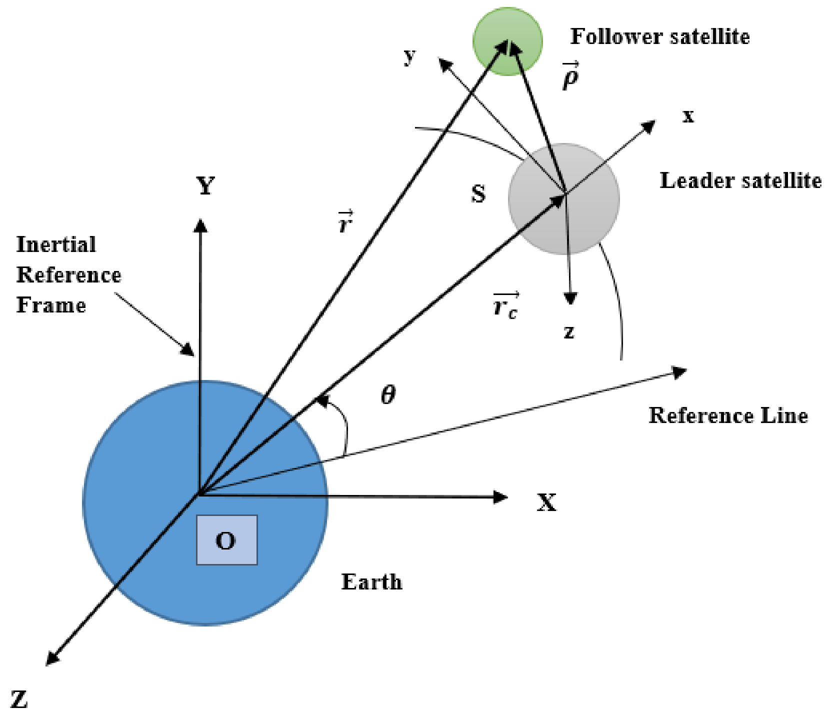

Fig 1. Formation of Leader and Follower satellite.

Equations (1)-(3) forms a set of simultaneous, nonlinear, coupled, 2nd order ordinary Differential equations, without any external control thrusts applied. Control forces are therefore applied in 3 orthogonal directions to steer the current state to the desired final state in a controlled way. These forces are acted on the body, by the thrust produced by ideal thrusters. Therefore, these equations in the continuous thrust mapping form, are represented as:

$$\ddot{x} - 2\dot{y}\dot{\theta} - 3\dot{\theta}^2 x - (\frac{3\mu}{r_c^4})[\frac{y^2}{2} + \frac{z^2}{2} - x^2] = u_x \tag{4}$$

$$\ddot{y} + 2\dot{x}\dot{\theta} - (\frac{3\mu}{r_c^4})xy = u_y \tag{5}$$

$$\ddot{z} + \dot{\theta}^2 z - (\frac{3\mu}{r_c^4})xz = u_z \tag{6}$$

where, u_x, u_y, u_z are the Force/mass in the x, y and z directions respectively, provided by the thrusters.
A configuration of the system variables that define the condition of the system is called state of the system. Any state of the body in translational dynamics is represented using positional coordinates and corresponding velocity in 3 axial directions. The state vector for the spacecraft is written as:

$$\boldsymbol{X} = [x \;\; y \;\; z \;\; \dot{x} \;\; \dot{y} \;\; \dot{z}]^T \tag{7}$$

The thrust vector can also be represented as:

$$\boldsymbol{U} = [u_x \; u_y \; u_z]^T \tag{8}$$

In compact form, the whole thrust controlled translation system dynamics can be written as:

$$\dot{\boldsymbol{X}} = \boldsymbol{G}(\boldsymbol{X}, \boldsymbol{U}) \tag{9}$$

Now, as the state of the system is defined, then the state space model will be formed after linearizing dynamics given by Eqs. (4)-(6) at some equilibrium point to apply linear control techniques.

V. Linearized Dynamics and State Space model

To apply the Linear Quadratic Regulator control to determine optimal gain to the system, the system has to be linearized to get a continuous time-invariant linear State Space model of translational dynamics given by Eqs. (4) – (9). The linearization of the system should be done about any of its equilibrium points or operating points. Thus, first of all, an equilibrium point for this system has to be determined.

The equilibrium point of the system is the state vector, where velocities in all directions are 0, thus $[\dot{x}\ \ \dot{y}\ \ \dot{z}]^T = 0$ and the same consequently gives $[\ddot{x}\ \ \ddot{y}\ \ \ddot{z}]^T = 0$. This is nothing but no net force acting on the system. Now, the translational dynamics Eqs. (4) – (6) can be solved taking all the velocity and acceleration terms = 0. The equations reduced to three conditions:

$$u_x + 3\dot{\theta}^2 x + (\frac{3\mu}{r_C^4})[\frac{y^2}{2} + \frac{z^2}{2} - x^2] = 0 \tag{10}$$

$$u_y + \left(\frac{3\mu}{r_C^4}\right) xy = 0 \tag{11}$$

$$u_z + (\frac{3\mu}{r_C^4})xz - \dot{\theta}^2 z = 0 \tag{12}$$

The equilibrium or operating point position coordinates and the acceleration due to control thrusts are related by the Eqs. (10) – (12). The operating point or equilibrium point or linearizing point for this problem is chosen as $[0,0,0]^T$ for position coordinates, which eventually gives $u_x = u_y = u_z = 0$ at operating point. For the system, as given by the Eq. (9), the equilibrium point is given by $\boldsymbol{X_{eq}} = [0, 0, 0, 0, 0, 0]^T$. Now, the system is to be linearized to get a linear form so that the linear controls can be applied.

Let a dependent variable, x is a nonlinear function of n independent variables, x_i, where $i = 1,2,3 \ldots n$. Then $\boldsymbol{x}$ is being linearized about any particular set of $\bar{x}_i$'s (operating point) according to Taylor expansion taking up to first order terms. In the similar way, the system of equations Eqs. (4) – (6) is linearized about an equilibrium point to get the following,

$$\dot{\boldsymbol{X}} - \dot{\boldsymbol{X}}_{\boldsymbol{eq}} = \boldsymbol{A}(\boldsymbol{X} - \boldsymbol{X}_{\boldsymbol{eq}}) + \boldsymbol{B}(\boldsymbol{U} - \boldsymbol{U}_{\boldsymbol{eq}}) \tag{13}$$

where, $\dot{\boldsymbol{X}} = [\dot{x}, \dot{y}, \dot{z}, \ddot{x}, \ddot{y}, \ddot{z}]^T$ and $\boldsymbol{X} = [x, y, z, \dot{x}, \dot{y}, \dot{z}]^T$. By equilibrium conditions can be given by

$$\dot{\boldsymbol{X}}_{\boldsymbol{eq}} = 0,\ \boldsymbol{X}_{\boldsymbol{eq}} = 0\ ,\boldsymbol{U}_{\boldsymbol{eq}} = 0.$$

In Eq. (13), $\boldsymbol{A}$ is called state matrix, and $\boldsymbol{B}$ is called input matrix for this Multi Input Multi Output (MIMO) system. Using 1st order Taylor expansion, the Jacobian matrices

$$\boldsymbol{A} = \left.\frac{\partial \boldsymbol{G}}{\partial \boldsymbol{X}}\right|_{\boldsymbol{X} = \boldsymbol{X}_{\boldsymbol{eq}}} ;\ \boldsymbol{B} = \left.\frac{\partial \boldsymbol{G}}{\partial \boldsymbol{U}}\right|_{\boldsymbol{U} = \boldsymbol{U}_{\boldsymbol{eq}}} \tag{14}$$

Therefore, the state and input matrices being derived, the State Space model of the linearized dynamics will be demonstrated in the following part.

State Space Model

Given the dynamical equations, the linearized system is represented in its State Space model, linearizing about $\boldsymbol{X} = 0$, and using full state estimation with no feedthrough, to obtain:

$$\dot{\boldsymbol{X}} = \boldsymbol{A}\boldsymbol{X} + \boldsymbol{B}\boldsymbol{U} \tag{15}$$

$$\boldsymbol{Y} = \boldsymbol{C}\boldsymbol{X} \tag{16}$$

where,

$$\boldsymbol{A} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 3\dot{\theta}^2 & 0 & 0 & 0 & 2\dot{\theta} & 0 \\ 0 & 0 & 0 & -2\dot{\theta} & 0 & 0 \\ 0 & 0 & -\dot{\theta}^2 & 0 & 0 & 0 \end{bmatrix}, \quad \boldsymbol{B} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \boldsymbol{C} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Linear controls are implemented in this paper to control the dynamics, or to steer the system from initial to final desired state. Linear controls are used, because it's very easy to implement, the control structure is simple and also the computation cost is much lower than nonlinear controls. In the paper, Linear Quadratic Regulator (LQR) control is being used, which needs the linearized differential equations into a state space model. According to the given input argument, the controller outputs a gain, which is optimized with respect to both of the system performance and fuel consumption. Therefore, the Linear Quadratic Regulator is needed to be designed using the derived State-Space model of the system.

VI. Linear Quadratic Regulator design and System Stability

The goal of optimal control is to control the dynamic system, but at minimum cost. This cost function is described in case of LQR control as a quadratic function of state vector and input thrusts. It takes two input parameters, $\boldsymbol{Q}$ and $\boldsymbol{R}$ matrices. These matrices make the trade-off between the performance of the system and energy consumption or fuel consumption or actuator effort to provide an optimal state feedback gain to the system. Correspondingly the LQR design is to find an optimal feedback gain ($\boldsymbol{K}$) from the following control law:

$$\boldsymbol{U} = -\boldsymbol{KX} \tag{17}$$

where, $\boldsymbol{K}$ is the optimal gain matrix provided by LQR controller to minimize the scalar quadratic cost function, L:

$$L = \frac{1}{2}\int_0^\infty (\boldsymbol{X}^T\boldsymbol{QX} + \boldsymbol{U}^T\boldsymbol{RU})dt = \int_0^\infty I\, dt \tag{18}$$

where $\boldsymbol{Q}$ and $\boldsymbol{R}$ are positive definite matrices. The term $\boldsymbol{X}^T\boldsymbol{QX}$ determines the cost of the performance of the system, which penalizes the system when the system is off the reference or desired state vector. The term $\boldsymbol{U}^T\boldsymbol{RU}$ determines the cost of energy lost or fuel consumed, which penalize the system when the actuator effort is more.

The optimization problem can be being solved by the method of control Hamiltonian formulation, where Eq. (18) is to be minimized given the state/input constraints Eq. (15) and Eq. (16). The formulation can be given by:

$$H = I + \boldsymbol{\lambda^T}\dot{\boldsymbol{X}} = \frac{1}{2}(\boldsymbol{X}^T\boldsymbol{QX} + \boldsymbol{U}^T\boldsymbol{RU}) + \boldsymbol{\lambda^T}(\boldsymbol{AX} + \boldsymbol{BU}) \tag{19}$$

where, H that combines the objective and state functions (in case of Dynamic optimization problem like this) needs to be minimized. The variables $\boldsymbol{\lambda}$ are known as costate variables analogous to Lagrange multipliers in Lagrange Method of Multipliers formulation (as in case of Static optimization). The costate equation is given by:

$$\dot{\boldsymbol{\lambda}} = -\left(\frac{\partial H}{\partial \boldsymbol{X}}\right) = -\left(\boldsymbol{QX} + \boldsymbol{A^T\lambda}\right) \tag{20}$$

This is an infinite-horizon, continuous time LQR problem, where the optimal solution for the LQR design is given by:

$$\frac{dH}{d\boldsymbol{U}} = 0 \Rightarrow \boldsymbol{U}(t) = -\boldsymbol{R}^{-1}\boldsymbol{B}^T\boldsymbol{PX}(t) \tag{21}$$

where P is a constant positive definite matrix, which is the solution of continuous time algebraic Riccati equation (CARE):

$$-\boldsymbol{PA} - \boldsymbol{A}^T\boldsymbol{P} + \boldsymbol{PBR}^{-1}\boldsymbol{B}^T\boldsymbol{P} - \boldsymbol{Q} = 0 \tag{22}$$

Now, given the $\boldsymbol{Q}$ and $\boldsymbol{R}$ matrices, the $\boldsymbol{P}$ matrix can be analytically determined for this specific dynamical system described by $\boldsymbol{A}$ and $\boldsymbol{B}$ matrices. The matrix $\boldsymbol{Q}$ determines the penalty on the state performance of the system, whereas $\boldsymbol{R}$ determines the penalty on fuel consumption. Both are positive definite and diagonal matrices. There is no hard and fast rule to determine a perfect $\boldsymbol{Q}$ and $\boldsymbol{R}$ matrix. These matrices are chosen according to how the system is desired to behave. If the value of $\boldsymbol{R}$ is greater than $\boldsymbol{Q}$, then more care is given about its fuel consumption (i.e., more penalty the cost function will provide if the fuel consumption or actuator effort is high) than state performance and vice versa.

This Algebraic Riccati equation (CARE) is a type of nonlinear matrix equation that arises in the context of solving continuous time infinite horizon optimal problems. The optimal problem stated here, i.e., control of formation flight is also nothing but a continuous time infinite horizon problem. As the control forces are modelled as continuous time thrusts, thus it’s continuous time problem. And, due to the time-independent nature of cost matrices $\boldsymbol{Q}$ and $\boldsymbol{R}$, it’s an infinite horizon problem. Therefore, it is nevertheless to state that the solution of Eq. (22) gives the desired feedback, $\boldsymbol{U}$. In the optimal gain formulation, the Riccati equation Eq. (22) needs to be solved to get the gain matrix using Eq. (21). The solution of CARE takes both the system demonstrated by $\boldsymbol{A}$ and $\boldsymbol{B}$ with the cost arguments $\boldsymbol{Q}$ and $\boldsymbol{R}$ into calculation. The Eq. (22) can be solved numerically and analytically. Now, the analytic solution for the CARE is described.

The LQR control ensures the optimal feedback gain by optimizing the cumulative control system error and fuel consumption. Solving the algebraic Riccati equation, the optimal gain can be determined methodically. Let the matrix is defined as:

$$\boldsymbol{P} = \begin{bmatrix} P_{11} & P_{12} & P_{13} & P_{14} & P_{15} & P_{16} \\ P_{21} & P_{22} & P_{23} & P_{24} & P_{25} & P_{26} \\ P_{31} & P_{32} & P_{33} & P_{34} & P_{35} & P_{36} \\ P_{41} & P_{42} & P_{43} & P_{44} & P_{45} & P_{46} \\ P_{51} & P_{52} & P_{53} & P_{54} & P_{55} & P_{56} \\ P_{61} & P_{62} & P_{63} & P_{64} & P_{65} & P_{66} \end{bmatrix}$$

Now, as P is a positive definite matrix (all positive eigenvalues), it clearly tells that $\boldsymbol{P}$ is a symmetric matrix; therefore, $P_{ij} = P_{ji}$. Thus, 36 unknowns of $\boldsymbol{P}$ reduces down to 21.

Thus, these matrices are applied in the Eq. (22), to get,

$$-\boldsymbol{PA} - \boldsymbol{A}^T\boldsymbol{P} + \boldsymbol{PBR}^{-1}\boldsymbol{B}^T\boldsymbol{P} - \boldsymbol{Q} = 0$$

Thus, the expanded form, by putting matrix $\boldsymbol{P}$, as defined, the following is obtained:

$$\Rightarrow \begin{bmatrix} P_{14}^2 + P_{15}^2 + P_{16}^2 & P_{14}P_{24} + P_{15}P_{25} + P_{16}P_{26} & P_{14}P_{34} + P_{15}P_{35} + P_{16}P_{36} & P_{14}P_{44} + P_{15}P_{45} + P_{16}P_{46} & P_{14}P_{45} + P_{15}P_{55} + P_{16}P_{56} & P_{14}P_{46} + P_{15}P_{56} + P_{16}P_{66} \\ . & P_{24}^2 + P_{25}^2 + P_{26}^2 & P_{24}P_{34} + P_{25}P_{35} + P_{26}P_{36} & P_{24}P_{44} + P_{25}P_{45} + P_{26}P_{46} & P_{24}P_{45} + P_{25}P_{55} + P_{26}P_{56} & P_{24}P_{46} + P_{25}P_{56} + P_{26}P_{66} \\ . & . & P_{34}^2 + P_{35}^2 + P_{36}^2 & P_{34}P_{44} + P_{35}P_{45} + P_{36}P_{46} & P_{34}P_{45} + P_{35}P_{55} + P_{36}P_{56} & P_{34}P_{46} + P_{35}P_{56} + P_{36}P_{66} \\ . & . & . & P_{44}^2 + P_{45}^2 + P_{46}^2 & P_{44}P_{45} + P_{45}P_{55} + P_{46}P_{56} & P_{44}P_{46} + P_{45}P_{56} + P_{46}P_{66} \\ . & . & . & . & P_{45}^2 + P_{55}^2 + P_{56}^2 & P_{45}P_{46} + P_{55}P_{56} + P_{56}P_{66} \\ . & . & . & . & . & P_{46}^2 + P_{56}^2 + P_{66}^2 \end{bmatrix}$$

$$= \begin{bmatrix} Q_{11} + 6\dot{\theta}^2 P_{14} & 3\dot{\theta}^2 P_{24} & 3\dot{\theta}^2 P_{34} - \dot{\theta}^2 P_{16} & 3\dot{\theta}^2 P_{44} + P_{11} - 2\dot{\theta} P_{15} & 3\dot{\theta}^2 P_{45} + P_{12} + 2\dot{\theta} P_{14} & 3\dot{\theta}^2 P_{46} + P_{13} \\ . & Q_{22} & -\dot{\theta}^2 P_{26} & P_{12} - 2\dot{\theta} P_{25} & P_{22} + 2\dot{\theta} P_{24} & P_{23} \\ . & . & Q_{33} - 2\dot{\theta}^2 P_{36} & P_{13} - 2\dot{\theta} P_{35} - \dot{\theta}^2 P_{46} & P_{23} + 2\dot{\theta} P_{34} - \dot{\theta}^2 P_{56} & P_{33} - \dot{\theta}^2 P_{66} \\ . & . & . & Q_{44} + 2P_{14} - 4\dot{\theta} P_{45} & P_{24} + 2\dot{\theta} P_{44} + P_{15} - 2\dot{\theta} P_{55} & P_{34} + P_{16} - 2\dot{\theta} P_{56} \\ . & . & . & . & Q_{55} + 2P_{25} + 4\dot{\theta} P_{45} & P_{35} + P_{26} + 2\dot{\theta} P_{46} \\ . & . & . & . & . & Q_{66} + 2P_{36} \end{bmatrix} \quad (23)$$

The matrix equation Eq. (23) forms a set of 21 simultaneous non-linear coupled algebraic equations, spanning 21 unknowns of the matrix $\boldsymbol{P}$. It needs to be solved analytically, to get a full description of $\boldsymbol{P}$. From there, calculating eigenvalues, it can be clearly verified that $\boldsymbol{P}$ is a constant positive definite matrix. It gives the solution to the algebraic Riccati equation and in turn provides the optimal control solution for this system. In the following, the stability of the closed loop system given by Eqs. (15) – (17).

System Stability

The LQR controller does the action of placing the eigenvalues or the closed loop poles of the system dynamics, for which the system will be bounded, minimizing the quadratic cost function, which optimizes the system performance and fuel consumption according to the chosen input $\boldsymbol{Q}$ and $\boldsymbol{R}$ matrices. This adds an optimal feedback gain to the system, $\boldsymbol{U} = -\boldsymbol{KX}$ to control it. The LQR control thus correctly places the eigenvalues. The designed Linear Quadratic Regulator controller is stable under given system dynamics. In the following, the stability of the overall system is shown.

By choosing a proper Lyapunov function for this optimal control problem of the translational dynamics, the global stability of the system will be derived. Let the Lyapunov function be defined as:

$$V = \frac{1}{2}\boldsymbol{X}^T\boldsymbol{PX} \tag{24}$$

where, $\boldsymbol{P}$ is the solution of algebraic Riccati equation, Eq. (22). As $\boldsymbol{P}$ is a time independent positive definite matrix, the Lyapunov function V is always positive (except origin, where it's 0) for $\forall\, t \in [0, \infty)$.

Now,

$$\begin{aligned}\dot{\boldsymbol{V}} &= \frac{1}{2}\dot{\boldsymbol{X}}^T\boldsymbol{PX} + \frac{1}{2}\boldsymbol{X}^T\boldsymbol{P}\dot{\boldsymbol{X}} \\ &= \frac{1}{2}(\boldsymbol{X}^T\boldsymbol{A}^T + \boldsymbol{U}^T\boldsymbol{B}^T)\boldsymbol{PX} + \frac{1}{2}\boldsymbol{X}^T\boldsymbol{P}(\boldsymbol{AX} + \boldsymbol{BU}) \\ &= \frac{1}{2}\boldsymbol{X}^{\boldsymbol{T}}(\boldsymbol{A}^{\boldsymbol{T}}\boldsymbol{P} + \boldsymbol{PA})\boldsymbol{X} - \boldsymbol{X}^T\boldsymbol{PBR}^{-1}\boldsymbol{B}^T\boldsymbol{PX} \\ &= -\frac{1}{2}\boldsymbol{X}^{\boldsymbol{T}}(\boldsymbol{Q} + \boldsymbol{PBR}^{-\boldsymbol{1}}\boldsymbol{B}^T\boldsymbol{P})\mathrm{X}\end{aligned} \tag{25}$$

In Eq. (25), as $\boldsymbol{Q}$ and $\boldsymbol{R}$ are positive diagonal matrices, $\boldsymbol{P}$ is a positive definite matrix, $\boldsymbol{B}$ is also positive. Therefore, as $\boldsymbol{Q} > \boldsymbol{0}$, then $\boldsymbol{Q} + \boldsymbol{PBR}^{-1}\boldsymbol{B}^T\boldsymbol{P}$ is always positive definite. Henceforth, $\dot{\boldsymbol{V}} < \boldsymbol{0}\ \forall\, \boldsymbol{X} \neq \boldsymbol{0}$ and $\dot{\boldsymbol{V}} = \boldsymbol{0}$ for $\boldsymbol{X} = \boldsymbol{0}$ only, which defines $\boldsymbol{V}$ as a positive definite function**.** This led to the globally asymptotic stability of the closed loop system according to the Lyapunov theory, where the optimal control is given by, $\overline{\boldsymbol{U}} = -\boldsymbol{R}^{-1}\boldsymbol{B}^T\boldsymbol{P}\overline{\boldsymbol{X}}$. Thus, the closed loop system formed by $\boldsymbol{A}$ and $\boldsymbol{B}$, with feedback $\boldsymbol{K}$, as determined for given $\boldsymbol{Q}$ and $\boldsymbol{R}$ matrices, is asymptotically stable in the whole space $\boldsymbol{X}$.

Feedback with setpoint (reference) input:

The LQR control regulates the system state to 0, when no reference input is given. The setpoint could not be directly fed into the input, unless the Step response performance is satisfactory, that is with no steady state error. If not, then the input is fed with a gain. Though the optimization problem doesn't change with the addition of reference input, as far as the system has excellent step performance with the suitable combination of $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{Q}, \boldsymbol{R}$ and $\boldsymbol{K}$**.** If the input reference is $\boldsymbol{R_{ef}}(t)$, then the input of the modified model is,

$$\boldsymbol{U}(t) = -\boldsymbol{KX} + \boldsymbol{R_{ef}}(t) \tag{26}$$

Thus, the modified State Space equation with reference input is given by,

$$\dot{\boldsymbol{X}} = (\boldsymbol{A} - \boldsymbol{BK})\boldsymbol{X} + \boldsymbol{R_{ef}} \tag{27}$$

Therefore, the continuous time infinite horizon problem is being solved using the Riccati equation, Eq. (22). The solution method has taken into account the fact that $\boldsymbol{P}$ is a positive definite matrix. But due to the imposition of constraint, and the process of obtaining the closed form solution, given by Eq. (21), it exploited the relation between the states of the system (i.e., here $\boldsymbol{X}$) and the costate variables $\boldsymbol{\lambda}$ used in the cost optimization problems, where the multipliers are being multiplied with the state/input constraint equations in the minimization formulation in Eq. (19). But the optimal theory proposed by Krotov, is freed from all these constraints, where the problem is solved using an equivalent extension of the cost functional specifically designed for the system. The following section is novel in the approach, where Krotov condition is implemented in a naïve way for the first time in a problem like the spacecraft relative motion control, and followed by the verification of this method with the good old optimal solution using Riccati's method.

VII.Verification the Optimal solution based on Krotov conditions

In case of Generic Optimal Control Problem, the control law is computed to minimize an overall cost function J,

$$J(\boldsymbol{X}(t), \boldsymbol{U}(t), t) \;=\; \int_{\tau}^{T} c(\boldsymbol{X}(t), \boldsymbol{U}(t), t)dt \;+\; F(\boldsymbol{X}(T)) \tag{28}$$

on the set of states ($\boldsymbol{X}$), optimizing the performance and fuel consumption of the system. The integral term characterizes the quality of the system in the control interval (τ, T). And $F(\boldsymbol{X}(T))$ is the quality of control at final time. But in this problem, as the Final state contains just a single point thus the final addendum reduces to 0. The modified cost function becomes,

$$J(\boldsymbol{X}(t), \boldsymbol{U}(t), t) \;=\; \int_{0}^{\infty} c(\boldsymbol{X}(t), \boldsymbol{U}(t), t)dt \tag{29}$$

This is a continuous time problem where the final time is not finite, thus the integration in the time domain is so justified in Eq. (29). The goal of the problem is to find an optimal element from the set $\boldsymbol{U}$, satisfying the input and state constraints. But in many optimization problems, it might happen that the optimal solution is not found in the solution set. In such a case, as proposed by Krotov in [10], the same cost functional is being equivalently extended to find a solution in its higher solution set, and its total decomposition with respect to time, to be done.

A. Equivalent extension:

The original minimization principle is replaced or extended using the introduction of parameters, which is to be optimized, such that the extended problem is simpler than the original. This new model of the minimization problem is called 'equivalent extension' of the original problem.

As previously stated, if in case of optimization problem, the optimal solution is not found, the original cost functional (here represented by Eq. (29)) is being extended to a newly defined, simpler and another cost functional, where the solution set is also extended to a higher set, say $\boldsymbol{U}'$, where $\boldsymbol{U} \subset \boldsymbol{U}'$**.** In doing so, in case of discrete time problems, as well as continuous time problems, the cost functional Eq. (29) is being parameterized in time (or decomposed in time in Discrete time problems). This simplifies the problem, by breaking it into several miniature optimization problems in time to be solved in case of discrete time, and a complete integral problem in the control interval time domain, for which the satellite system has been controlled. The solution gives rise to the Krotov optimal condition, to get an admissible optimal process.

B. Krotov sufficient optimality conditions:

According to Krotov sufficient conditions, global optimality of the control processes can be achieved by decomposing the original control problem with respect to time or time parameterizing using Extension principle as stated above. It provides an optimal solution, $[\overline{\boldsymbol{X}}(t), \overline{\boldsymbol{U}}(t)]$ that satisfies the constraints or the state and input equation, the solution thus can be concluded to be an admissible control process, ensuring global optimality by the Krotov conditions.

Optimal control using Extension principle:

The goal is to minimize the cost, in the equivalent extended form of the original global optimal control problem to get an admissible optimal process. Now in the problem, as it is an infinite horizon continuous time LQR problem (any final time terms are set to zero), the equivalent representation in the time interval $t \in [0, \infty]$ is extended.

Theorem 1:

Let $\phi(\boldsymbol{X}(t), t)$ be a continuously differentiable function in time and space, then the cost function of this Global Optimal Control Problem can be extended using the Extension principle as follows:

$$J_{eq}(\boldsymbol{X}(t), \boldsymbol{U}(t), t) = \; \phi(\boldsymbol{X}(t_0), t_0) \; + \int_{0}^{\infty} S(\boldsymbol{X}(t), \boldsymbol{U}(t), t)dt \tag{30}$$

where (t_0 being initial time),

$$S(\boldsymbol{X}(t), \boldsymbol{U}(t), t) \;\triangleq\; \frac{\partial\phi}{\partial t} \;+\; \frac{\partial\phi}{\partial \boldsymbol{X}} G(\boldsymbol{X}(t), \boldsymbol{U}(t), t) \;+\; c(\boldsymbol{X}(t), \boldsymbol{U}(t), t) \tag{31}$$

If any $[\boldsymbol{X}(t), \boldsymbol{U}(t)]$ pair satisfies the dynamical equation of the system, $\dot{\boldsymbol{X}} = G(\boldsymbol{X}(t), \boldsymbol{U}(t), t)$ along with the state and input constraints, then it is an admissible process for the system. And the extended equivalent cost functional, J_{eq} gives the global optimality of the admissible process. For this problem of LQR control, the integrable cost penalty $c(\boldsymbol{X}(t), \boldsymbol{U}(t), t)$ or more formally, the quality of the system in the control interval, where the penalty parameters for state and the actuator are $\boldsymbol{Q}$ and $\boldsymbol{R}$ matrices respectively, the function $S(\boldsymbol{X}(t), \boldsymbol{U}(t), t)$, given the system and cost by Eq. (9) and Eq. (18) respectively, can be written as:

$$S(\boldsymbol{X}(t), \boldsymbol{U}(t), t) \triangleq \frac{\partial \phi}{\partial t} + \frac{\partial \phi}{\partial \boldsymbol{X}}[\boldsymbol{AX} + \boldsymbol{BU}] + \boldsymbol{X}^T\boldsymbol{QX} + \boldsymbol{U}^T\boldsymbol{RU} \tag{32}$$

Theorem 2: If any admissible process $[\overline{\boldsymbol{X}}(t), \overline{\boldsymbol{U}}(t)]$ satisfies:

$$S(\overline{\boldsymbol{X}}(t), \overline{\boldsymbol{U}}(t), t) = \min_{X \in \boldsymbol{X}, U \in \boldsymbol{U}} S(\boldsymbol{X}(t), \boldsymbol{U}(t), t), \quad \forall\, t \in [0, \infty) \tag{33}$$

then $[\overline{\boldsymbol{X}}(t), \overline{\boldsymbol{U}}(t)]$ is the optimal solution of the problem, and $\overline{\boldsymbol{U}}(t)$ is the optimal feedback for the system. $\boldsymbol{X}$ and $\boldsymbol{U}$ are whole set of admissible state $\boldsymbol{X}(t)$ and input $\boldsymbol{U}(t)$ vectors respectively which satisfies the system dynamics $\dot{\boldsymbol{X}}(t) = G(\boldsymbol{X}(t), \boldsymbol{U}(t), t)$. It's clearly understood that minimizing $S(\boldsymbol{X}(t), \boldsymbol{U}(t), t)$ in an iterative way as in Eq. (33), or minimizing it over the whole control interval as in continuous time problem, it eventually gives the minimal or optimal solution of $J_{eq}(\boldsymbol{X}(t), \boldsymbol{U}(t), t)$, which is desired.

The differentiable function $\phi(\boldsymbol{X}(t), t)$ is called Krotov function, and is totally problem specific. It changes the cost function, J accordingly and so the optimization problem. It's also called a solving function and needs to be selected efficiently. Now, in the following part, a problem specific Krotov function is being proposed, which will lead to the feedback, ensuring optimal control for this LQR problem.

Proposition:
For this equivalent Optimal Control Problem for satellite formation flight control, the suitable Krotov function is chosen as:

$$\phi(\boldsymbol{X}(t), t) = \frac{1}{2}\boldsymbol{X}^T\boldsymbol{PX} \tag{34}$$

where, matrix $\boldsymbol{P}$ or every element of matrix $\boldsymbol{P}$ is differentiable in the time domain $\forall\, t \in [0, \infty)$. Further, another constraint is also imposed on $\boldsymbol{P}$ that it's symmetric, which is novel in this approach and can be concluded as:

a) $S(\boldsymbol{X}, \boldsymbol{U}, t)$ is a convex function on $[\boldsymbol{X}, \boldsymbol{U}]$.

b) The matrix $\boldsymbol{P}$ satisfies the following matrix inequality

$$\frac{1}{2}\dot{\boldsymbol{P}} + \boldsymbol{PA} + \boldsymbol{A}^T\boldsymbol{P} + \boldsymbol{Q} - \boldsymbol{PBR}^{-1}\boldsymbol{B}^T\boldsymbol{P} \geq 0 \quad \forall\, t \in [0, \infty). \tag{35}$$

Though, the imposition of the symmetric constraint on $\boldsymbol{P}$, makes it still more generalised than the Riccati approach.

Proof: The selected $\phi(\boldsymbol{X}(t), t)$ is applied to the function, $S(\boldsymbol{X}(t), \boldsymbol{U}(t), t)$ as framed in Eq. (32), to get

$$\begin{aligned} S &= \frac{1}{2}\boldsymbol{X}^T\dot{\boldsymbol{P}}\boldsymbol{X} + \boldsymbol{X}^T\boldsymbol{P}\dot{\boldsymbol{X}} + \boldsymbol{X}^T\boldsymbol{P}[\boldsymbol{AX} + \boldsymbol{BU}] + \boldsymbol{X}^T\boldsymbol{QX} + \boldsymbol{U}^T\boldsymbol{RU} \\ &= \frac{1}{2}\boldsymbol{X}^T\dot{\boldsymbol{P}}\boldsymbol{X} + 2(\boldsymbol{X}^T\boldsymbol{PAX} + \boldsymbol{X}^T\boldsymbol{PBU}) + \boldsymbol{X}^T\boldsymbol{QX} + \boldsymbol{U}^T\boldsymbol{R} \end{aligned} \tag{36}$$

As $\boldsymbol{X}^T\boldsymbol{PAX}$ is a scalar then $\boldsymbol{X}^T\boldsymbol{PAX} = \boldsymbol{XA}^T\boldsymbol{PX}^T$ and provided $\boldsymbol{P}^T = \boldsymbol{P}$. Therefore, from Eq. (36),

$$S = \frac{1}{2}\boldsymbol{X}^T\dot{\boldsymbol{P}}\boldsymbol{X} + \boldsymbol{X}^T\boldsymbol{PAX} + \boldsymbol{XA}^T\boldsymbol{PX}^T + 2\boldsymbol{X}^T\boldsymbol{PBU} + \boldsymbol{X}^T\boldsymbol{QX} + \boldsymbol{U}^T\boldsymbol{RU} \tag{37}$$

Adding and subtracting the term $\boldsymbol{X}^T\boldsymbol{PBR}^{-1}\boldsymbol{B}^T\boldsymbol{PX}$ to Eq. (37),

$$S = \boldsymbol{X}^T(\frac{1}{2}\dot{\boldsymbol{P}} + \boldsymbol{PA} + \boldsymbol{A}^T\boldsymbol{P} + \boldsymbol{Q} - \boldsymbol{PBR}^{-1}\boldsymbol{B}^T\boldsymbol{P})\boldsymbol{X} + 2\boldsymbol{X}^T\boldsymbol{PBU} + \boldsymbol{U}^T\boldsymbol{RU} + \boldsymbol{X}^T\boldsymbol{PBR}^{-1}\boldsymbol{B}^T\boldsymbol{PX} \tag{38}$$

From the LQR control logic, $\boldsymbol{R}$ matrix is the cost penalty matrix for fuel consumption; hence, $\boldsymbol{R}$ is always a diagonal matrix, having all elements positive; therefore, $\boldsymbol{R} > 0$ and represented as a square of another unique positive definite matrix, $\widetilde{\boldsymbol{R}}$, as $\widetilde{\boldsymbol{R}}^2 = \boldsymbol{R}$ and vice versa for inverses. Now, Eq. (38) can be rewritten as:

$$S = \boldsymbol{X}^T\left(\frac{1}{2}\dot{\boldsymbol{P}} + \boldsymbol{PA} + \boldsymbol{A}^T\boldsymbol{P} + \boldsymbol{Q} - \boldsymbol{PBR}^{-1}\boldsymbol{B}^T\boldsymbol{P}\right)\boldsymbol{X} + \left[\widetilde{\boldsymbol{R}}\boldsymbol{U} + \widetilde{\boldsymbol{R}}^{-1}\boldsymbol{B}^T\boldsymbol{PX}\right]^T\left[\widetilde{\boldsymbol{R}}\boldsymbol{U} + \widetilde{\boldsymbol{R}}^{-1}\boldsymbol{B}^T\boldsymbol{PX}\right] \qquad (39)$$

The second term in Eq. (39) is positive definite and strictly convex. Therefore, the condition for which the function, S can become convex is:

$$\left(\frac{1}{2}\dot{\boldsymbol{P}} + \boldsymbol{PA} + \boldsymbol{A}^T\boldsymbol{P} + \boldsymbol{Q} - \boldsymbol{PBR}^{-1}\boldsymbol{B}^T\boldsymbol{P}\right) \geq 0 \qquad \forall\, t \in [0, \infty)$$

In the above, the linear matrix inequality (LMI) relation has been derived by applying the convexity of S. In case of optimization problems, a convex optimization function is always preferred, due to the presence of global extremum in this situation. Now, if final time boundary condition and the time independence of $\boldsymbol{P}$ matrix, $\dot{\boldsymbol{P}} = 0$ is imposed in this LMI, for this infinite horizon problem of Satellite relative motion control, then equating Eq. (35) to 0, it will reduce down to the continuous time algebraic Riccati equation, Eq. (22) in non-iterative method. The main goal of the Krotov's method is to formulate a more generalized and simpler way to solve the optimization problem, given the cost matrices and satisfying the constraints to apply for this practical problem like Rendezvous and Proximity operations of satellites. The method, as presented in the Section VI, provides the optimal solution for this, by directly solving the Riccati equation which is obtained using the optimization method of Lagrange multipliers as stated in Eq. (19). But it is being derived in the preceding in a more general way, that provides the relation Eq. (35), which is encompassed just using the symmetric $\mathbf{P}$ constraint. Connecting with the system, the Riccati equation in Section VI and this relation Eq. (35), represents the same system, when proper boundary conditions for this particular system are being imposed, which is completely being verified as the relation above reduces to Eq. (22).

Corollary:
For the chosen Krotov function, $\phi(\boldsymbol{X}(t), t)$, these can be concluded as:

a) The function, $\phi(\boldsymbol{X}(t), t)$ is the solution function for this Optimal Control Problem.

b) The global optimal control law, or the optimal feedback for the system is given by:

$$\overline{\boldsymbol{U}} = -\boldsymbol{R}^{-1}\boldsymbol{B}^T\boldsymbol{P}\overline{\boldsymbol{X}} \qquad (40)$$

where, $\boldsymbol{P}$ is the solution of the matrix differential equation (specific for this system):

$$\boldsymbol{PA} + \boldsymbol{A}^T\boldsymbol{P} + \boldsymbol{Q} - \boldsymbol{PBR}^{-1}\boldsymbol{B}^T\boldsymbol{P} = 0$$

Proof: An admissible process is therefore found, $\overline{\boldsymbol{X}}$, which is the optimal solution of the extended equivalent form of cost function. As the convexity condition of S is independent of $\boldsymbol{X}$,and satisfies the matrix $\boldsymbol{P}$, thus, ϕ is the solving function for this Optimal Control. This Krotov function is specific for this optimization problem, whereas changing it will be suitable for optimization problems, for some other system dynamics, but related to the optimal solution admissible process still using the Krotov's conditions.

The corresponding optimal feedback for this problem can be also found, as the minimization of Eq. (39) is satisfied by the 2nd part of the equation, going to 0. Therefore,

$$\widetilde{\boldsymbol{R}}\boldsymbol{U} + \widetilde{\boldsymbol{R}}^{-1}\boldsymbol{B}^T\boldsymbol{P} = 0. \qquad (41)$$

From Eq. (41), $\overline{\boldsymbol{U}}$ is solved as

$$\overline{\boldsymbol{U}} = -\boldsymbol{R}^{-1}\boldsymbol{B}^T\boldsymbol{P}\overline{\boldsymbol{X}}$$

This is the desired optimal solution to the problem. Therefore, according to the Krotov sufficient conditions regarding the minimization of the cost functional for this system [from Eq. (29)],

$$J = \frac{1}{2}\int_0^\infty (\boldsymbol{X}^T\boldsymbol{QX} + \boldsymbol{U}^T\boldsymbol{RU})dt,$$

optimizing the performance index and fuel consumption satisfying the input/ state constraints in a non-iterative way using optimal feedback given by,

$$\overline{\boldsymbol{U}} = -\boldsymbol{R}^{-1}\boldsymbol{B}^{T}\boldsymbol{P}\overline{\boldsymbol{X}}.$$

Therefore, the result of the optimization problem in this Krotov's method, given by Eq. (40) is exactly matching with what is being obtained for the cost minimization problem by using Riccatis's equation, given by Eq. (21). In this continuous time infinite horizon optimal control problem of satellite relative motion control, the method used in Section VI is the solving of optimization problem using the Hamiltonian's method, given by Eq. (19). But in this Krotov's method, the idea of cost functional over the set of states, is being extended to get an equivalent extension, given by Eq. (30), which is solved by taking a suitable, problem specific Krotov function in a more generic way to obtain a Linear Matrix Inequality relation, Eq. (35). This just reduced down to the Riccati Equation form in Eq. (22) using the system's nature and boundary conditions. Thus, by both methods the same optimal control problem gives the same optimal feedback, $\boldsymbol{U}$, though the solution methodology is different. So, the control problem by Krotov's condition, stated here is perfectly verified to give stable optimal feedback and being used for such an essential purpose of optimal control of relative translational motion of satellites, revolving around Earth.

VIII. Results and Discussions

We demonstrated the satellite formation flight control, where Leader is in the circular orbit, that is it's in a particular orbit of fixed radius around the Earth, in a dedicated orbital plane, and the Follower satellite is at a desired and fixed distance from Leader. The control approach used is Linear Quadratic Regular, with the system translation linearized at one of the equilibrium points, which gives an optimal gain optimizing the system state performance and fuel consumption.

The parameter $\dot{\theta}$ appearing in the equations, is the angular rate of rotation of the Leader satellite, around the Earth in a circular orbit, which solely depends on the altitude of the satellite above Earth's surface. For our problem, we took the altitude of the Leader satellite as $h = 340\ km$ from the Earth's surface. So, we get the angular velocity of satellite about Earth as,

$$\dot{\theta} = \omega = \sqrt{\frac{GM_e}{(R_e+h)^3}} \tag{42}$$

where, $M_e = 5.972e + 24\ kg$ is the mass of Earth, $G = 6.67408e - 11\ m^3kg^{-1}s^{-2}$ is Universal Gravitational constant, $R_e = 6370\ km$ is radius of Earth and $h = 340\ km$ is altitude of the Leader from Earth surface, to get $\dot{\theta} = 0.0011486\ rad/s$. Therefore, this parameter is being used in the equations, as the entries in A matrix. The state space model is thus represented by equations:

$$\begin{aligned} \dot{\boldsymbol{X}} &= \boldsymbol{AX} + \boldsymbol{BU} \\ \boldsymbol{Y} &= \boldsymbol{CX} \end{aligned}$$

where,

$$\text{matrix,}\quad \boldsymbol{A} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 3.9578e-6 & 0 & 0 & 0 & 2.2972e-3 & 0 \\ 0 & 0 & 0 & -2.2972e-3 & 0 & 0 \\ 0 & 0 & -1.3193e-6 & 0 & 0 & 0 \end{bmatrix},$$

with $\boldsymbol{B}$ and $\boldsymbol{C}$ are as defined earlier. Firstly, the open loop stability of the system has been checked, by the method of eigenvectors as previously explained. The eigenvalues of $\boldsymbol{A}$ are thus calculated in MATLAB. The eigenvalues are found to lie on the imaginary axis only and repeated eigenvalues are present on the imaginary axis. From which we can conclude that the system is totally unbounded. Thus, we have to apply a feedback control to the system to stabilize it. Hence, LQR control comes into play. As previously stated, we are using full state observation of the states. This is validated by checking rank of the observability matrix of $\boldsymbol{A}$ and $\boldsymbol{C}$ in MATLAB and found to be full rank, i.e., 6.

LQR control through the State Space model has one of its important aspects, which is controllability. Which is by suitable pole placing if we can steer a system from unstable to stable or not done by the feedback gain. This is validated by checking rank of the controllability matrix of $\boldsymbol{A}$ and $\boldsymbol{B}$ in MATLAB and found to be full rank, i.e., 6.

Therefore, our model is controllable by LQR, and with full state observation. But the LQR takes two input parameters $\boldsymbol{Q}$ and $\boldsymbol{R}$ matrix for optimization performance and energy by minimizing the quadratic cost function. For our control problem, we choose $\boldsymbol{Q} = diag(1,1,1,1,1,1)$ and $\boldsymbol{R} = diag(1,1,1)$, i.e., we are equally penalizing the state performance and fuel consumption. And also take care of each state and each actuator by the same amount, therefore all entries of both $\boldsymbol{Q}$ and $\boldsymbol{R}$ are set to identity. Thus, to get the optimal gain, we solved the algebraic Riccati equation as described, given the $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{Q}$ and $\boldsymbol{R}$ matrices. The solution of $\boldsymbol{P}$ is given by,

$$\boldsymbol{P} = \begin{bmatrix} 1.7320541 & 0 & 0 & 1.0000031 & 0.0013263 & 0 \\ 0 & 1.7320518 & 0 & -0.0013263 & 0.9999991 & 0 \\ 0 & 0 & 1.7320501 & 0 & 0 & 0.9999987 \\ 1.0000031 & -0.0013263 & 0 & 1.7320526 & 0 & 0 \\ 0.0013263 & 0.9999991 & 0 & 0 & 1.7320503 & 0 \\ 0 & 0 & 0.9999987 & 0 & 0 & 1.7320501 \end{bmatrix}$$

We can see that $\boldsymbol{P}$ is a positive definite matrix (symmetric and having positive eigenvalues). Therefore, by the optimal solution of the LQR problem, the optimal feedback is given by,

$$\overline{\boldsymbol{U}} = -\boldsymbol{R}^{-1}\boldsymbol{B}^T\boldsymbol{P}\overline{\boldsymbol{X}} = -\boldsymbol{K}\boldsymbol{X}$$

The optimal gain matrix $\boldsymbol{K}$ is thus,

$$\boldsymbol{K} = \begin{bmatrix} 1 & -0.0013 & 0 & 1.7321 & 0 & 0 \\ 0.0013 & 1 & 0 & 0 & 1.7321 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1.7321 \end{bmatrix}$$

Hence, our closed loop system becomes,

$$\begin{aligned} \dot{\boldsymbol{X}} &= (\boldsymbol{A} - \boldsymbol{B}\boldsymbol{K})\boldsymbol{X} = \overline{\boldsymbol{A}}\boldsymbol{X} \\ \boldsymbol{Y} &= \boldsymbol{C}\boldsymbol{X} \end{aligned} \tag{43}$$

where, $$\overline{\boldsymbol{A}} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0.0013 & 0 & -1.7321 & 0.0023 & 0 \\ -0.0013 & -1 & 0 & -0.0023 & -1.7321 & 0 \\ 0 & 0 & -1 & 0 & 0 & -1.7321 \end{bmatrix}$$

So, to check the closed loop stability of the system, by the method of eigenvalues, we check the eigenvalues of $\overline{\boldsymbol{A}}$ matrix in MATLAB.

All eigenvalues are found to be in the left half of the plane, thus having a negative real part. Henceforth we can conclude that our feedback system is totally controlled and bounded for any time $t > 0$. We also checked the Step response of the closed loop system,

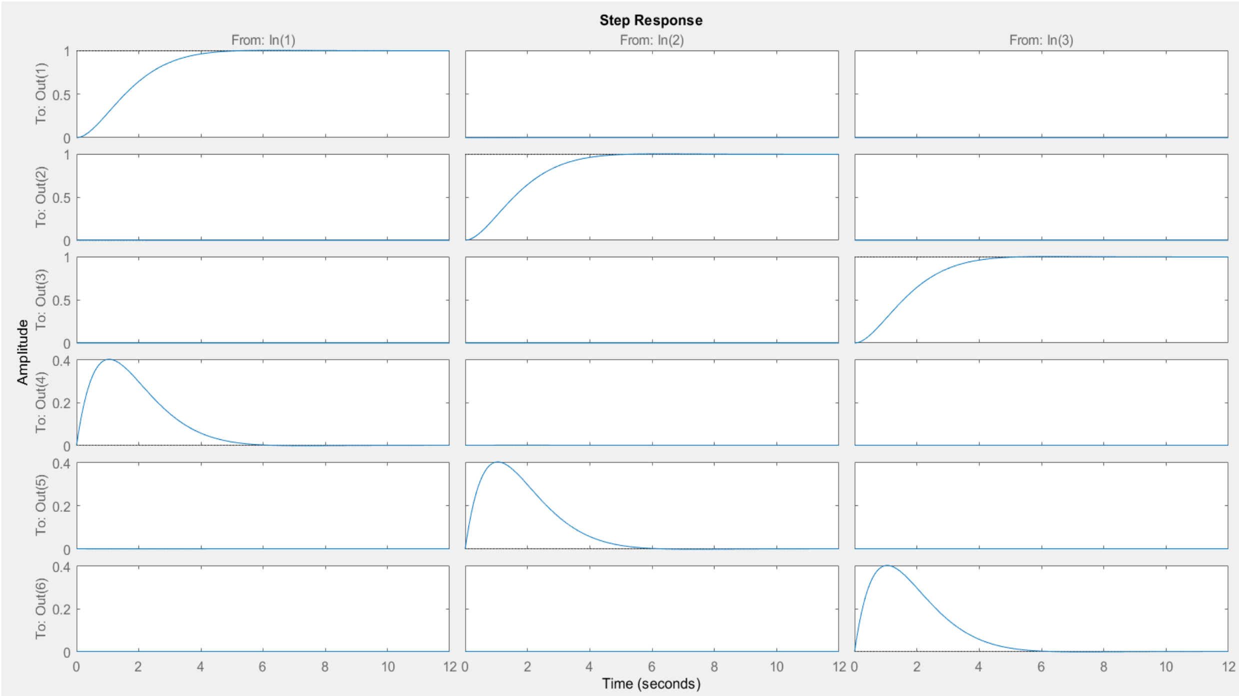

Fig.2: The step response of the closed loop feedback system.

It's clearly seen from the step response in Fig. 2, that for each of the corresponding Input actuator thrusts represented by In, the corresponding positional coordinate correctly responds to the step response, giving no steady state error. Similarly, the velocity coordinates are also behaving satisfactorily. Thus, as previously stated in Section VI, for this closed loop, the input reference positional coordinates can be fed, without any further gain. The whole simulation is done in SIMULINK, and necessary codes are run in MATLAB. The simulation results are as follows. For all simulations, we have chosen the same Q and R matrix.

The first simulation is carried out with the initial state as $\boldsymbol{X}(0) = [1000,1000,1000,0,0,0]^T$ and the desired final state is set to $\boldsymbol{X_{des}} = [200,-50,100,0,0,0]^T$. The simulation results are shown in Fig. (3). The Fig. 3(A) is showing the variation of the relative position coordinates of the Follower satellite, starting initially from $[1000,1000,1000]^T$. It clearly seemed to have converged to the desired final state position $[200,-50,100]^T$, within 7 seconds from the start of its observed motion under this controlled trajectory. Therefore, satisfactory transient and steady state performance is observed for the model. The Fig. 3(B), is the variation of x, y and z relative velocities of the Follower that converged to 0 within 7 seconds. The plot in Fig. 3(C) is the control thrusts/mass provided by the thrusters in 3 directions. The thrusts correctly reached convergence to 0 within 7 seconds which is also expected, to reach equilibrium.

The second simulation is carried out with the initial state as $\boldsymbol{X}(0) = [750,1000,-1000,0,50,0]^T$ and the desired final state is set to $\boldsymbol{X_{des}} = [0,400,100,0,0,0]^T$. The simulation results are shown in Fig. (4). The plot in Fig. 4(A) is showing the variation of the relative position coordinates of the Follower satellite, starting initially from $[750,1000,-1000]^T$. It clearly seemed to have converged to the desired final state position $[0,400,100]^T$, within 8 seconds from the start of its observed motion under this controlled trajectory. The Fig. 4(B) plot, is the variation of x, y and z relative velocities of the Follower that have reached convergence within 8 seconds. The plot in Fig. 4(C) is the control thrusts/mass provided by the thrusters in 3 directions. The thrusts correctly converged to 0 within 7 seconds to reach the equilibrium.

The third simulation is carried out with the initial state as $\boldsymbol{X}(0) = [0,600,-100,100,0,-10]^T$ and the desired final state is set to $\boldsymbol{X_{des}} = [100, step, 100,0,0,0]^T$. Step input is given is y direction as 500 for first 10 seconds and 100 for next 10 seconds. The simulation results are shown in Fig. (5). The plot in Fig. 5(A) is showing the variation of the relative position coordinates of the Follower satellite, starting initially from $[0,600,-100]^T$. Step input was applied which seems to perfectly converge with the step values. The Fig. 5(B) plot, is the variation of x, y and z

relative velocities of the Follower. The initial velocity states were chosen $[100,0,-10]^T$. It varies as expected with the step response given and reached convergence. The plot in Fig. 5(C) is the control thrusts/mass provided by the thrusters in 3 directions. The thrusts correctly converged to reach the equilibrium.
Therefore, in each of the simulations, as our LQR control is designed, all the results are stable and convergent, maintaining a specific position in formation flight.

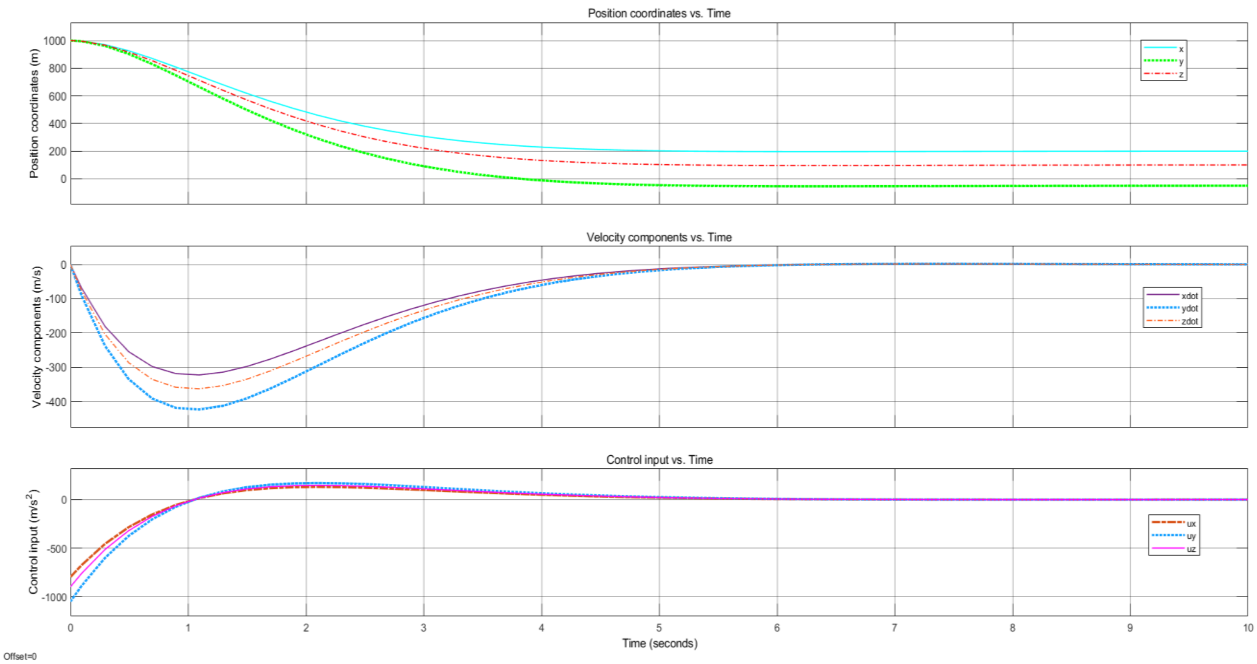

Fig.3: (A) Position states with time; (B) Velocity states with time; (C) Control thrusts per unit mass with time.

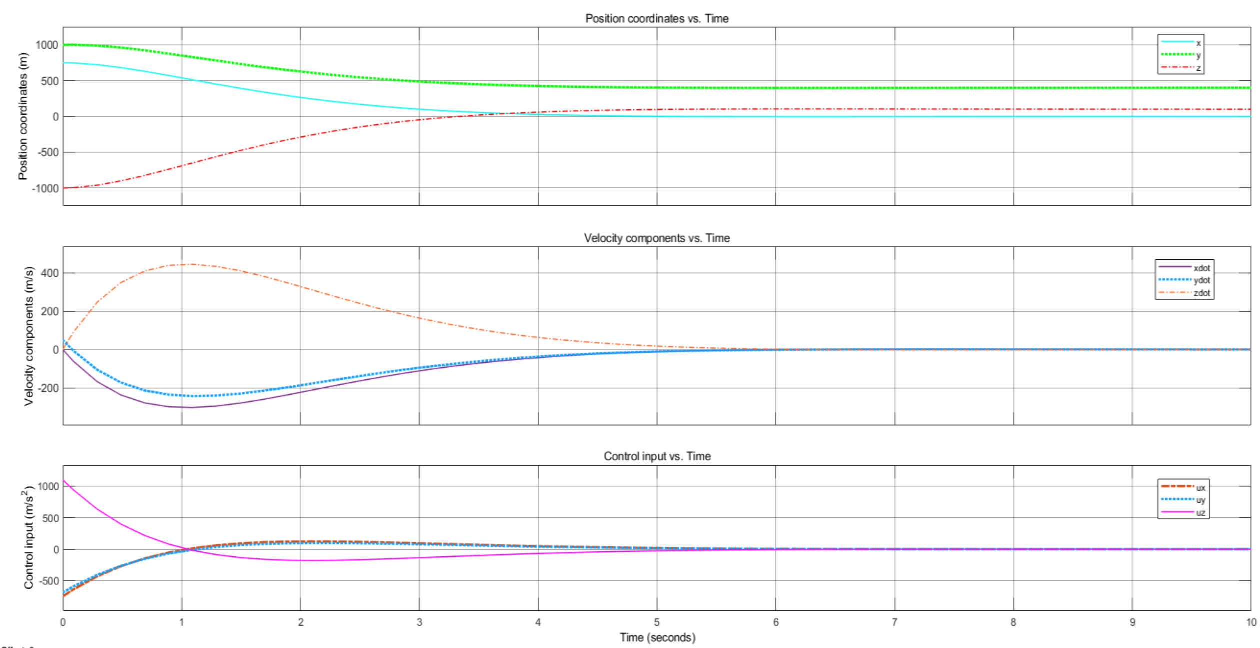

Fig.4: (A) Position states with time; (B) Velocity states with time; (C) Control thrusts per unit mass with time.

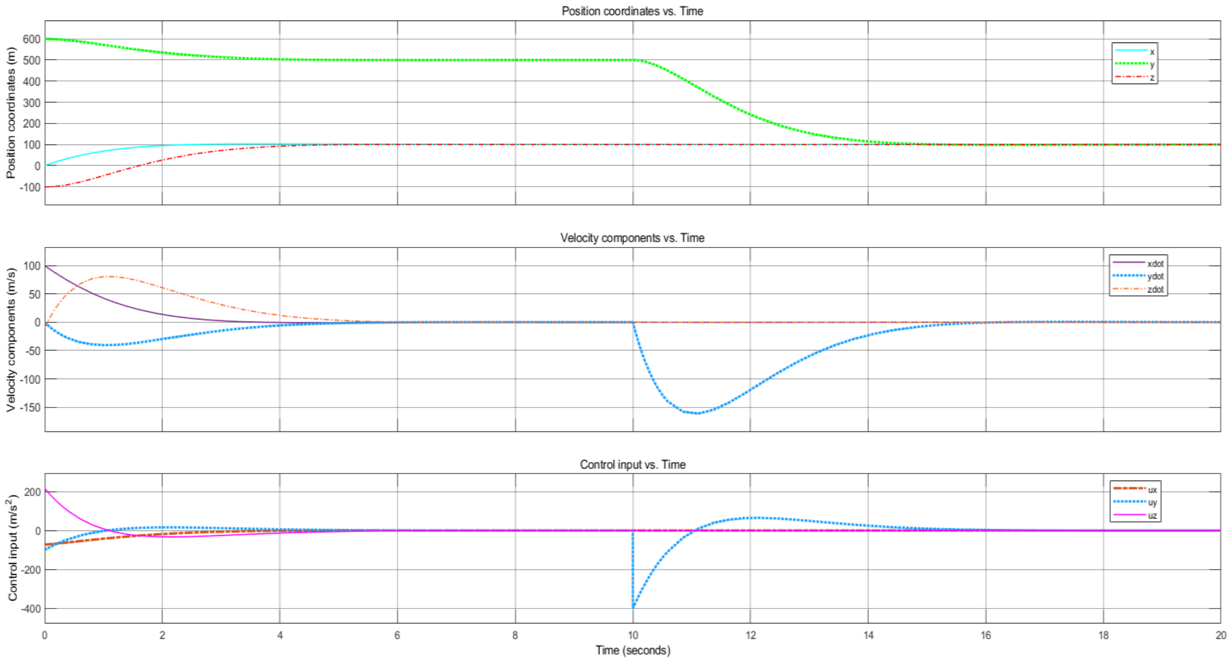

Fig.5: (A) Position states with time; (B) Velocity states with time; (C) Control thrusts per unit mass with time.

[Fig (A), (B) and (C) are arranged in top to bottom fashion for each subplot]

IX.Conclusion

In our paper, we presented the Linear Quadratic Regulator control of the linearized 2nd order translational dynamics of the relative motion between Leader and Follower satellite. The Leader is constrained to move in a circular orbit around Earth and the Follower is desired to keep a fixed location with respect to the former spacecraft. The LQR design is demonstrated with the explanation of every detail which is being implemented, and our model is performing very satisfactorily under non-perturbed circumstances and its global stability is also explained with the novel implementation of stability of this LQR system under Krotov conditions. The necessary and sufficiency conditions were presented for the optimality of this problem and the solution methodology has been stated in an iterative way using Equivalent extension of the optimal problem and in a non-iterative way using convex imposition on the same. The transient and steady state responses are also up to the mark. In the discussions, simulation results are presented with varying initial, final states and $\boldsymbol{Q}$ and $\boldsymbol{R}$ matrices. Thus, it is concluded that this LQR design can be successfully implemented.

The future research includes the translation motion control in elliptic or Keplerian orbits. Further along with translational motion, the attitude dynamics can also be designed to optimally control using LQR, which gives the full 6DOF control of the relative satellite motion.

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