

# Markowitz & Treynor-Black

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Version: 05 févr. 2022

In this short note, we summarize the mathematical elements of the classical portfolio theory of Markowitz and Treynor-Black.

## Arithmetic vs. Geometric mean

Let  $r_A$  and  $r_G$  be, respectively, the arithmetic and geometric means of a series of returns:

$$r_A = \frac{1}{n} \sum_{k=1}^n r_k$$
$$r_G = \prod_{k=1}^n (1 + r_k)^{1/n} - 1$$

and let  $V$  be the variance of  $r_k$ . We show that the geometric mean, which correctly represents the increase in wealth from an investment, is lower than the arithmetic mean.

The MacLaurin series for  $(1 + x)^{1/n}$  is:

$$(1 + x)^{\frac{1}{n}} = 1 + \frac{1}{n}x + \frac{1-n}{n^2} \frac{x^2}{2} + o(x^2)$$

$$r_G \approx \prod_{k=1}^n \left( 1 + \frac{1}{n}r_k + \frac{1-n}{n^2} \frac{r_k^2}{2} \right) - 1$$

Developping the product and keeping terms of order 2,

$$r_G \approx \frac{1}{n} \sum_k r_k + \frac{1}{n^2} \sum_{k \neq l} r_k r_l + \frac{1-n}{2n^2} r_k^2$$

$$r_G \approx r_A - \frac{1}{2} \left[ \frac{1}{n} \sum_k r_k^2 - \frac{1}{n^2} \left( \sum_k r_k^2 + 2 \sum_{k \neq l} r_k r_l \right) \right] \quad (1)$$

$$\approx r_A - \frac{1}{2} \left[ \frac{1}{n} \sum_k r_k^2 - \left( \frac{1}{n} \sum_k r_k \right)^2 \right] \quad (2)$$

$$\approx r_A - \frac{1}{2} V, \quad V \geq 0 \quad (3)$$

## Quadratic Programming

### QP with equality constraints

$$\begin{aligned} \min \quad & \frac{1}{2} w^T \Sigma w \\ \text{s.t.} \quad & A^T w = b \end{aligned}$$

Lagrangian:

$$L(w, \lambda) = \frac{1}{2} w^T \Sigma w - \lambda^T (A^T w - b)$$

First order conditions:

$$\begin{cases} \Sigma w - A \lambda = 0 \\ A^T w = b \end{cases}$$

or,

$$\begin{bmatrix} \Sigma & -A \\ A^T & 0 \end{bmatrix} \begin{bmatrix} w \\ \lambda \end{bmatrix} = \begin{bmatrix} 0 \\ b \end{bmatrix}$$

### Special case of Minimum Variance problem

$$A = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \quad b = \mu^*$$

Solution:

$$w = \lambda \Sigma^{-1} A$$

Normalize so that weights sum to 1:

$$w = \frac{\Sigma^{-1}\mathbf{1}}{\mathbf{1}^T \Sigma^{-1} \mathbf{1}}$$

## Mean-Variance model (Markowitz, 1952)

$$\begin{aligned} \min \quad & \frac{1}{2} w^T \Sigma w \\ \text{s.t.} \quad & \\ & \mathbf{1}^T w = 1 \\ & R^T w = R_p \end{aligned}$$

Lagrangian:

$$L(w, \lambda_1, \lambda_2) = \frac{1}{2} w^T \Sigma w - \lambda_1 (\mathbf{1}^T w - 1) - \lambda_2 (R^T w - R_p)$$

Solution of first order conditions:

$$\begin{cases} \Sigma w - \lambda_1 \mathbf{1} - \lambda_2 R = 0 \\ \mathbf{1}^T w = 1 \\ R^T w = R_p \end{cases} \quad (4)$$

Determination of  $\lambda_1$  and  $\lambda_2$ :

$$w = \Sigma^{-1}(\lambda_1 \mathbf{1} + \lambda_2 R)$$

Define:

$$\begin{aligned} a &= \mathbf{1}^T \Sigma^{-1} \mathbf{1} \\ b &= \mathbf{1}^T \Sigma^{-1} R \\ c &= R^T \Sigma^{-1} R \end{aligned}$$

Substitute in (4):

$$\begin{cases} \lambda_1 a + \lambda_2 b = 1 \\ \lambda_1 b + \lambda_2 c = R_p \end{cases}$$

Solution:

$$\begin{aligned}\lambda_1 &= \frac{c - bR_P}{\Delta} \\ \lambda_2 &= \frac{aR_P - b}{\Delta} \\ \Delta &= ac - b^2\end{aligned}$$

Note that:

$$\begin{aligned}\sigma_P^2 &= w^{*T} \Sigma w^* \\ &= w^{*T} \Sigma \left( \lambda_1 \Sigma^{-1} \mathbf{1} + \lambda_2 \Sigma^{-1} R \right) \\ &= \lambda_1 + \lambda_2 R_P\end{aligned}$$

Two remarkable solutions:

- Minimum variance portfolio

$$\begin{aligned}\frac{\partial \sigma_P^2}{\partial R_P} &= 0 \implies \\ \frac{2aR_P - 2b}{\Delta} &= 0 \implies \\ R_P &= \frac{b}{a} \\ \sigma_P^2 &= \frac{1}{a} \\ \lambda_1 &= \frac{1}{a} \\ \lambda_2 &= 0\end{aligned}$$

The weights of the minimum variance portfolio:

$$\begin{aligned}w_g &= \lambda_1 \Sigma^{-1} \mathbf{1} \\ &= \frac{\Sigma^{-1} \mathbf{1}}{\mathbf{1}^T \Sigma^{-1} \mathbf{1}}\end{aligned}$$

- $\lambda_1 = 0$

This second solution gives  $\lambda_2 = \frac{1}{b}$  and the optimal weights:

$$\begin{aligned}w_d &= \lambda_2 \Sigma^{-1} R \\ &= \frac{\Sigma^{-1} R}{\mathbf{1}^T \Sigma^{-1} R}\end{aligned}$$

**Theorem 1.** Any MV optimal portfolio  $w_P^*$  with expected excess return  $R_P$  can be decomposed into two MV portfolios.

$$w_P^* = Aw_g + (1 - A)w_d$$

*Proof.* Since  $w_P$  is MV optimal,

$$\begin{aligned} w_P &= \lambda_1 \Sigma^{-1} \mathbf{1} + \lambda_2 \Sigma^{-1} R \\ &= \lambda_1 a w_g + \lambda_2 b w_d \end{aligned}$$

One can verify that

$$\lambda_1 a + \lambda_2 b = 1$$

□

### MV model with riskless asset

The tangency portfolio corresponds to the point on the efficient frontier where the slope of the tangent  $\frac{R_M - r_f}{\sigma_M}$  is maximized, where:

$$\frac{R_M - r_f}{\sigma_M} = \frac{w^T (R - R_f)}{\sqrt{w^T \Sigma w}}$$

Noting that the slope is unchanged when the weights  $w$  are multiplied by a constant, the tangency portfolio is found by solving the following QP problem for an arbitrary  $R^* > R_f$ :

$$\begin{aligned} \min \quad & \frac{1}{2} w^T \Sigma w \\ \text{s.t.} \quad & \\ & \tilde{R}^T w = R^* \end{aligned}$$

with  $\tilde{R} = R - R_f$ .

Lagrangian:

$$L(w, \lambda) = \frac{1}{2} w^T \Sigma w - \lambda (\tilde{R}^T w - R^*)$$

Which yields:

$$w^* = \lambda^* \Sigma^{-1} \tilde{R} \tag{5}$$

Normalize so that the weights sum to 1:

$$w^* = \frac{\Sigma^{-1} \tilde{R}}{\mathbf{1}^T \Sigma^{-1} \tilde{R}} \tag{6}$$

The corresponding expected excess return is given by:

$$E(R_P^*) = \frac{\tilde{R}^T \Sigma^{-1} \tilde{R}}{\mathbf{1}^T \Sigma^{-1} \tilde{R}}$$

### Maximum Sharpe ratio for two risky assets

Given two assets, A and M, the allocation that maximizes the Sharpe ratio is given by:

$$w_A = \frac{R_A \sigma_M^2 - R_M \sigma_A \sigma_M \rho_{AM}}{R_A \sigma_M^2 + R_M \sigma_A^2 - (R_A + R_M) \sigma_A \sigma_M \rho_{AM}} \quad (7)$$

*Proof.* Use equation (6) with

$$\Sigma = \begin{bmatrix} \sigma_A^2 & \rho \sigma_A \sigma_M \\ \rho \sigma_A \sigma_M & \sigma_M^2 \end{bmatrix} \quad (8)$$

$$\Sigma^{-1} = \frac{1}{(1 - \rho^2) \sigma_A^2 \sigma_M^2} \begin{bmatrix} \sigma_M^2 & -\rho \sigma_A \sigma_M \\ -\rho \sigma_A \sigma_M & \sigma_A^2 \end{bmatrix}$$

□

### Treynor-Black Model (Treynor & Black, 1973)

Assets excess return is modeled by a single factor model:

$$R_i = \alpha_i + \beta_i R_M + e_i$$

where  $\alpha_i$  is the idiosyncratic excess return of asset  $i$ , and  $e_i \sim N(0, \sigma_i^2)$  is the specific risk.

### Calculation of the active portfolio

The active portfolio is determined by the idiosyncratic excess return and the specific risk of each asset.

The specific risks are assumed to be independent:

$$\Sigma_A = \begin{bmatrix} \sigma^2(e_1) & & \\ & \ddots & \\ & & \sigma^2(e_n) \end{bmatrix}$$

Using equation (6), we get:

$$w_{Ai} = \frac{\alpha_i / \sigma_i^2}{\sum \alpha_i / \sigma_i^2}$$

So that the active portfolio has an excess return and variance given by:

$$R_A = \alpha_A + \beta_A R_M$$

$$\sigma_A^2 = \beta_A^2 \sigma_M^2 + \sigma^2(e_A)$$

with

$$\alpha_A = \sum w_{Ai} \alpha_i$$

$$\beta_A = \sum w_{Ai} \beta_i$$

$$\sigma^2(e_A) = \sum w_{Ai}^2 \sigma^2(e_i)$$

### Allocation of wealth between the active portfolio and the market portfolio

A fraction  $w_A$  of wealth is allocated to the active portfolio, and the balance  $(1 - w_A)$  to the market portfolio so as to maximize the Sharpe ratio of the global portfolio  $x_A + (1 - x)M$ .

Using equation (7) we get after some algebra:

$$w_A = \frac{\alpha_A \sigma_M^2}{\alpha_A \sigma_M^2 (1 - \beta_A) + R_M \sigma^2(e_A)}$$

### Separability of the Sharpe ratio in the active portfolio

The first order condition for the optimal active portfolio is:

$$w_A = \lambda_A \Sigma^{-1} \alpha \tag{9}$$

Substitute in the expression

$$\alpha_A = w_A^T \alpha$$

to get:

$$\frac{\alpha_A}{\lambda_A} = \alpha^T \Sigma^{-1} \alpha \tag{10}$$

We next get an expression for  $\lambda_A$  in terms of known quantities:

$$\begin{aligned} \sigma^2(e_A) &= w_A^T \Sigma w_A \\ &= \lambda_A^2 \alpha^T \Sigma^{-1} \Sigma \Sigma^{-1} \alpha \\ &= \lambda_A^2 \alpha^T \Sigma^{-1} \alpha \end{aligned}$$

Therefore,

$$\begin{aligned}\frac{\sigma^2(e_A)}{\lambda_A^2} &= \alpha^T \Sigma^{-1} \alpha \\ &= \frac{\alpha_A}{\lambda_A}\end{aligned}$$

Which yields:

$$\lambda_A = \frac{\sigma^2(e_A)}{\alpha_A}$$

Use this result in equation (10) to get:

$$\begin{aligned}\frac{\alpha_A^2}{\sigma^2(e_A)} &= \alpha^T \Sigma^{-1} \alpha \\ &= \sum_i \frac{\alpha_i^2}{\sigma^2(e_i)}\end{aligned}$$

which shows that the square of the Sharpe ratio of the active portfolio is the sum of the squares of the Sharpe ratios of the assets forming that portfolio.

### **The Treynor-Black model in the notation of the 1973 paper and separability of the Sharpe ratio between the active and market portfolios**

The investment universe is composed of  $n$  assets with asset-specific excess return:

$$r_i = \alpha_i + \beta_i r_M + e_i \quad i = 1, \dots, n \quad (11)$$

$$E(r_i) = \alpha_i + \beta_i E(r_M) = \mu_i \quad (12)$$

and of the market asset itself. Let  $w_i, i = 1, \dots, n$  be the investment in the assets with asset-specific excess returns, and  $w_M$  the investment in the market asset.

Treynor and Black restate this portfolio as an investment in  $n + 1$  assets, asset 1 to  $n$  being only exposed to the specific risk, and the  $n + 1$  asset being only exposed to the market risk:

$$w_{n+1} = w_M + \sum_{i=1}^n \beta_i w_i$$

Note that these  $n + 1$  assets are independent. The mean and variance of the portfolio are:



$$E(r_P) = \sum_{i=1}^{n+1} w_i E(r_i) = \mu_P \quad (13)$$

$$\sigma_P^2 = \sum_{i=1}^{n+1} w_i^2 \sigma_i^2 \quad (14)$$

As usual, maximize the Sharpe ratio by solving:

$$\begin{aligned} \min \quad & \frac{1}{2} w^T \Sigma w \\ \text{s.t.} \quad & \\ & \mu^T w = \mu_P \end{aligned}$$

Keeping in mind that the assets are independent, the Lagrangian is:

$$L(w, \lambda) = \sum_{i=1}^{n+1} w_i^2 \sigma_i^2 - 2\lambda \left( \sum_{i=1}^{n+1} w_i \mu_i - \mu_P \right)$$

First order conditions for optimality yield:

$$2w_i \sigma_i^2 - 2\lambda \mu_i = 0 \quad i = 1, \dots, n+1$$

or,

$$w_i = \lambda \frac{\mu_i}{\sigma_i^2} \quad (15)$$

Substitute in (12) to get:

$$\mu_P = \lambda \sum_{i=1}^{n+1} \mu_i^2 / \sigma_i^2 \quad (16)$$

$$\sigma_P^2 = \lambda^2 \sum_{i=1}^{n+1} \mu_i^2 \sigma_i^2 \quad (17)$$

so that,

$$\lambda = \frac{\sigma_P^2}{\mu_P}$$

To summarize, the weights of the assets in the active portfolio are:

$$w_i = \frac{\mu_i}{\mu_P} \frac{\sigma_P^2}{\sigma_i^2} \quad i = 1, \dots, n$$

To determine the investment in the market asset,  $w_M$ , recall that,

$$\mu_{n+1} = E(r_M) = \mu_M \quad (18)$$

$$\sigma_{n+1}^2 = \sigma_M^2 \quad (19)$$

Thus,

$$w_{n+1} = \sum_{i=1}^n w_i \beta_i + w_M \quad (20)$$

$$= \lambda \frac{\mu_M}{\sigma_M^2} \quad (21)$$

From equation (15), we have:

$$\sum_{i=1}^n w_i \beta_i = \lambda \sum_{i=1}^n \frac{\beta_i \mu_i}{\sigma_i^2}$$

So that the investment in the market asset can be written as

$$w_M = \lambda \left[ \frac{\mu_M}{\sigma_M^2} - \sum_{i=1}^n \frac{\beta_i \mu_i}{\sigma_i^2} \right]$$

To establish the separability of the Sharpe ratio between the active and the market portfolios, combine equations (16) and (17) to get:

$$\frac{\mu_P^2}{\sigma_P^2} = \sum_{i=1}^{n+1} \frac{\mu_i^2}{\sigma_i^2}$$

Denoting  $S_A, S_M, S_P$  the Sharpe ratios of, respectively, the active, market and overall portfolios, we can restate the previous equation as:

$$S_P^2 = \sum_{i=1}^n \frac{\mu_i^2}{\sigma_i^2} + S_M^2 \quad (22)$$

$$= \frac{\alpha_A^2}{\sigma_A^2 + S_M^2} \quad (23)$$

$$S_A^2 + S_M^2 \quad (24)$$

Treynor and Black call  $\alpha_A = \sum_{i=1}^n w_i \alpha_i$  the “appraisal premium” and  $\sigma_A^2 = \sum_{i=1}^n w_i^2 \sigma_i^2$  the “appraisal risk”.

## **Bibliography**

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Treynor, J. L., & Black, F. (1973). How to Use Security Analysis to Improve Portfolio Selection. *The Journal of Business*, 46(1), 66–86. <http://www.jstor.org/stable/2351280>