Black-Litterman

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```
library(xts)
library(hornpa)
library(lubridate)
library(xtable)
library(PerformanceAnalytics)
library(TTR)
library(lubridate)
library(roll)
library(Hmisc)
library(nFactors)
library(kableExtra)
#library(broom)
library(quadprog)
```

Principle

Bayesian approach:

- The expected returns are random variables
- CAPM equilibrium distribution as prior
- additional probabilistic views combined with prior to get posterior distribution of expected return.

Distribution of asset returns:

$$r \sim \mathcal{N}(\mu, \Sigma)$$

Assume quadratic utility function:

$$U(w) = w^T \Pi - \frac{\delta}{2} w^T \Sigma w$$

Solve first order conditions for optimality to get

$$\Pi = \delta \Sigma w_{eq}$$

The expected return μ is also a random variable. The bayesian prior is such that

$$\mu = \Pi + \epsilon^{(e)}$$

with

$$\epsilon^{(e)} \sim \mathcal{N}(0, \tau \Sigma)$$

where τ is a scalar.

Views are expressed as portfolios whose returns are independent random normal variables.

$$P\mu = Q + \epsilon^{(v)}$$

with

$$\epsilon^{(v)} \sim \mathcal{N}(0, \Omega)$$

Posterior distribution

GLS linear model

Consider the linear model

$$Y = X\beta + E$$

with $\mathrm{Cov}(E|X) = \Omega = \sigma^2 V$

$$\hat{\beta} = (X^T \Omega^{-1} X)^{-1} X^T \Omega^{-1} Y$$

Proof:

Set $\Omega = K^T K$ and define $Z = K^{-1}Y, B = K^{-1}X, g = K^{-1}\epsilon$, the linear model becomes:

$$Z = B\beta + g$$

with $E(g) = 0$ and $V(g) = \sigma^2 I$. Applying OLS to this model yields the desired result.

Theil's estimatiom method for posterior distribution

Prior distribution for return

$$\Pi = I\mu + \epsilon^{(e)}$$

Additional information:

$$Q = P\mu + \epsilon^{(v)}$$

Combine two equations:

$$\begin{bmatrix} \Pi \\ Q \end{bmatrix} = \begin{bmatrix} I \\ P \end{bmatrix} \mu + \begin{bmatrix} \epsilon^{(e)} \\ \epsilon^{(v)} \end{bmatrix}$$

Apply GLS:

$$\mu^* = \left(\begin{bmatrix} I \\ P \end{bmatrix}^T \begin{bmatrix} \tau\Sigma & \\ & \Omega \end{bmatrix}^{-1} \begin{bmatrix} I \\ P \end{bmatrix} \right)^{-1} \begin{bmatrix} I \\ P \end{bmatrix}^T \begin{bmatrix} \tau\Sigma & \\ & \Omega \end{bmatrix}^{-1} \begin{bmatrix} \Pi \\ Q \end{bmatrix}$$

After algebraic manipulations:

Posterior mean of expected returns:

$$\mu^* = \left[(\tau\Sigma)^{-1} + P^T \Omega^{-1} P \right]^{-1} \left[(\tau\Sigma)^{-1} \Pi + P^T \Omega^{-1} Q \right]$$

Posterior covariance of expected returns:

$$M^{-1} = \left[(\tau\Sigma)^{-1} + P^T \Omega^{-1} P \right]^{-1}$$

Consequence: the posterior distribution of returns is

$$r \sim \mathcal{N}(\mu^*, \Sigma^*)$$

with $\Sigma^* = \Sigma + M^{-1}$.

Portfolio optimization

One can now find the optimal weights by solving the classical mean-variance problem:

$$\max w^T \mu^* - \frac{\delta}{2} w^T \Sigma^* w$$

the solution being:

$$w^* = \frac{1}{\delta} \Sigma^{*-1} \mu^*$$

See paper by He and Litterman for various manipulations of this last equation.

Calculation

Code freely adapted from https://github.com/systematicinvestor/SIT, but using the notation of the paper.

Market data from He & Litterman:

```
 data =
'1,0.4880,0.4780,0.5150,0.4390,0.5120,0.4910
 0.4880,1,0.6640,0.6550,0.3100,0.6080,0.7790
 0.4780,0.6640,1,0.8610,0.3550,0.7830,0.6680
 0.5150,0.6550,0.8610,1,0.3540,0.7770,0.6530
 0.4390,0.3100,0.3550,0.3540,1,0.4050,0.3060
 0.5120,0.6080,0.7830,0.7770,0.4050,1,0.6520
 0.4910,0.7790,0.6680,0.6530,0.3060,0.6520,1'

  Corrmat = matrix( as.double(spl( gsub('\n', ',', data), ',')),
                    nrow = length(spl(data, '\n')), byrow=TRUE)

  stdevs = c(16.0, 20.3, 24.8, 27.1, 21.0,  20.0, 18.7)/100
  w.eq = c(1.6, 2.2, 5.2, 5.5, 11.6, 12.4, 61.5)/100
  # Prior covariance of returns
  Sigma = Corrmat * (stdevs %*% t(stdevs))
```

Equilibrium risk premium

```
# risk aversion parameter
delta = 2.5
Pi = delta * Sigma %*% w.eq
```

Summary market data

Assets	Std Dev	Weq	PI
Australia	16	1.6	3.9
Canada	20.3	2.2	6.9
France	24.8	5.2	8.4
Germany	27.1	5.5	9
Japan	21	11.6	4.3
UK	20	12.4	6.8
USA	18.7	61.5	7.6

Table 1: Solution with View 1. P: view matrix, $\bar{\mu}$: ex-post expected return, w^*: optimal weights, $\frac{W_{eq}}{1+\tau}$: scaled equilibrium weights

	P	$\bar{\mu}$	w^*	$w^* - \frac{W_{eq}}{1+\tau}$
Australia	0.0	4.3	1.5	0.0
Canada	0.0	7.6	2.1	0.0
France	-29.5	9.3	-3.9	-8.9
Germany	100.0	11.0	35.4	30.2
Japan	0.0	4.5	11.0	0.0
UK	-70.5	7.0	-9.5	-21.3
USA	0.0	8.1	58.6	0.0

View 1: is The German equity market will outperform the rest of European Markets by 5% a year.

```
P = matrix(c(0, 0, -29.5, 100, 0, -70.5, 0)/100, nrow=1)
Q = 5/100
tau = 0.05

Omega = as.matrix(diag(tau * P %*% Sigma %*% t(P)))
tau.Sigma.inv = solve(tau*Sigma)
M.inverse = solve(tau.Sigma.inv + (t(P) %*% solve(Omega) %*% P))
mu.bar = M.inverse %*% (tau.Sigma.inv %*% Pi + t(P) %*% solve(Omega) %*% Q)
Sigma.bar = M.inverse + Sigma

w.star = (1/delta) * solve(Sigma.bar) %*% mu.bar

df = data.frame(100*cbind(t(P), mu.bar, w.star, w.star-w.eq/(1+tau)))
row.names(df) = AssetNames
names(df) =  c('P', "$\\bar{\\mu}$", '$w^*$','$w^* - \\frac{W_{eq}}{1+\\tau}$')
kable(df, digits = 1, format="latex", booktabs=T, escape=F,
      caption="Solution with View 1. P: view matrix, $\\bar{\\mu}$: ex-post expected return,
      $w^*$: optimal weights, $\\frac{W_{eq}}{1+\\tau}$: scaled equilibrium weights")
```