Spatial Evolutionary Games and their Applications to Bacterial Communities

Hannah Riley

10300295

School of Physics and Astronomy

University of Manchester

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Abstract

Evolutionary game theory is the product of applying classical game theory to biological systems, it can be used to model interacting populations of different species. This dissertation begins with an introduction to evolutionary game theory and the replicator equations. There are a number of phenomena in real biological systems that can't be explained by replicator dynamics alone. We explore mechanisms for the survival of cooperative behaviour using the prisoners dilemma as a framework. Application of a spatial distribution to a population allows cooperating individuals to form clusters to protect themselves from exploitation by defectors. A method of modelling a population with a spatial distribution is discussed, and the effect of applying such a distribution to the prisoners dilemma and three-strategy cyclic games is explored. Finally, some examples of game theoretic interaction in real populations of *Escherichia coli* are investigated.

Contents

1   Introduction

The origin of game theory dates back to work done by John Von Neumann and Oskar Morgenstern in the early 20th century to predict the decisions made by rational individuals [1, 2]. Game theory has numerous applications from economics to ethics to animal behaviour, it was developed with the purpose of helping an individual choose the optimal strategy (in a specified situation) based on the actions of their opponents [3]. In the 1970's, evolutionary game theory was introduced by John Maynard Smith and George Price in an effort to apply game theory to biology [1]. In contrast to the self-serving actions chosen by rational individuals in classical game theory, we now consider the dynamics of populations of different species, which are dictated by their Darwinian fitness [1, 2]. Evolutionary dynamics can be applied to a vast range of real biological systems from competition between bacterial strains to the interaction between a parasite and its host [1, 4]. The replicator equations only offer idealised models of real biological systems, application of additional mechanisms is required to simulate certain phenomena observed in the natural world, such as cooperative behaviour or stable biodiversity [5].

In this dissertation, the main pillars of game theory are introduced, which are then used to describe the central ideas of evolutionary dynamics with a focus on a famous game called the *prisoners dilemma* and cyclic three-strategy games. After exploring various mechanisms that allow for the survival of cooperative behaviour, we then move on to investigate the effect of spatial distributions on evolutionary games. We study the a method of modelling a spatial distribution and its effect on the prisoners dilemma and three-strategy cyclic games. Finally, evidence of the effect of spatial distributions on real biological systems are presented in the context of game theory.

2   Background Theory

2.1   The Payoff Matrix

Classical game theory focuses on rational decisions made by individuals to maximise their personal gain [6]. Imagine a situation where you and an opponent are placed in a predicament in which you are each faced with a choice between two courses of action; you may either cooperate with your opponent, or defect and work against them. Depending on your choice of strategy and the choice of your opponent, you would receive a different 'payoff' [6]. The different payoffs that you might receive can be summarised in a 'payoff matrix', see Table 1. This is an describes a 2x2 normal form symmetric game, there are two players who can each choose between two strategies, it is only necessary to tabulate the payoffs for one of the players as they are identical for the other player [1, 6]. Note the strategies available to players are not always either cooperation or defection, the payoff matrix has been introduced in this way as the choice between these strategies is a central topic of this dissertation.

	C (Cooperate)	D (Defect)
C	R	S
D	T	P

Table 1: Basic payoff matrix, by selecting the row corresponding to your choice of strategy and the column corresponding to your opponent's choice of strategy, you can find the payoff you would receive. For instance, if both you and your opponent chose to cooperate with each other, you would each gain a *reward R* for working towards a common good, while if you both chose to defect you would each receive a *punishment P*. If you defected while your opponent cooperated, you would receive a *temptation T* while your opponent would get the *sucker's payoff S* for being so naively trusting [6].

To further illustrate this point, let's now consider the *prisoners dilemma* as an example. In this scenario, two suspects have been arrested and are being questioned by police. The suspects have two options available to them, they may either refuse to answer police questions (thus cooperating with the other suspect) or testify against the other suspect (defecting). Depending on the strategies chosen the two suspects will be sentenced to different times in prison, this information can be organised in a payoff matrix, see Table 2 [6].

	C (Cooperate)	D (Defect)
C	1 year	10 years
D	0 years	5 years

Table 2: Pay off matrix for prisoners dilemma [6].

Clearly, the most mutually beneficial choice is for both suspects to cooperate with one another and remain silent, resulting in a combined stay in prison of 2 years. However, the two suspects are not allowed to communicate with one another while they are questioned so they cannot coordinate their decisions. Furthermore, if one suspect decided to cooperate while the other defected, the cooperating suspect would receive the worst sentence while the defector would walk free. From the point of view of one of the suspects, it is as if the decision of the other has already been made and thus they must act selfishly to try to reduce their own sentence as much as possible. We can notice that the best strategy is to always defect, since the payoff is always less than if you were to cooperate given a predefined decision on the part of the other suspect. Therefore the optimal strategy for the two suspects is where they both defect, resulting in a combined sentence of 10 years. This outcome is an example of what is known as a *Nash equilibrium*, where neither suspect has any advantage to gain by deviating from their current strategy on their own (unilaterally) as this would result in that suspect receiving the 10 year prison sentence [6].

2.2 Evolutionary Game Theory and the Replicator Equations

Transitioning from classical to evolutionary game theory, individuals are now born with a set strategy encoded in their genes which they cannot change [1]. Let's now suppose that we have a large number of individuals N who form a *well-mixed* population, meaning that any pair of players can interact [6]. Say the population N is comprised of two species which exhibit either cooperative behaviour (species C) or defective behaviour (species D), dividing the whole population into two parts, N_C and N_D where $N_C + N_D = N$ [7]. We may non-dimensionalize this system by setting $c = \frac{N_C}{N}$ and $d = \frac{N_D}{N}$ [6]. As the population is of a fixed size, the system only has one degree of freedom c, we can write $d = 1 - c$ [7]. Using the payoffs defined in Table 1, we can define the *fitness* ($f_C(c)$, $f_D(c)$) of each strategy as,

$$f_C(c, d) = Rc + Sd \Rightarrow f_C(c) = Rc + S(1 - c) \tag{1}$$

$$f_D(c, d) = Tc + Pd \Rightarrow f_D(c) = Tc + P(1 - c). \tag{2}$$

This illustrates the average payoff received for the respective populations C and D, the higher the fitness of a population, the more successful it will be [8]. Notice that the fitness is not constant, instead it is dependent on the frequencies of species C and D in the population [9]. The more prevalent a particular species is, the more likely it is to succeed. We also denote the average fitness $\langle f \rangle$ as,

$$\langle f \rangle = cf_C(c) + (1 - c)f_D(c), \tag{3}$$

where each fitness is weighted by the frequency of the corresponding species [6]. Using the fitness, we can construct equations to describe the time evolution of the two species in the population [6]. The equations,

$$\dot{c} = c(f_C(c) - \langle f \rangle) \tag{4}$$

$$\dot{d} = (1 - c)(f_D(c) - \langle f \rangle) \tag{5}$$

are known as the Replicator Equations [7]. It is clear that the abundance of a particular species in the population will increase if the fitness of that species exceeds the average fitness of the entire population. The replicator equations can be extended to describe populations containing an arbitrary number of species [6].

2.3 Possible Outcomes of Two-Strategy Games

The dynamics of the species in the population is dependent on the relationships between the payoffs in the payoff matrix. The generic cases for populations containing two species will now be outlined.
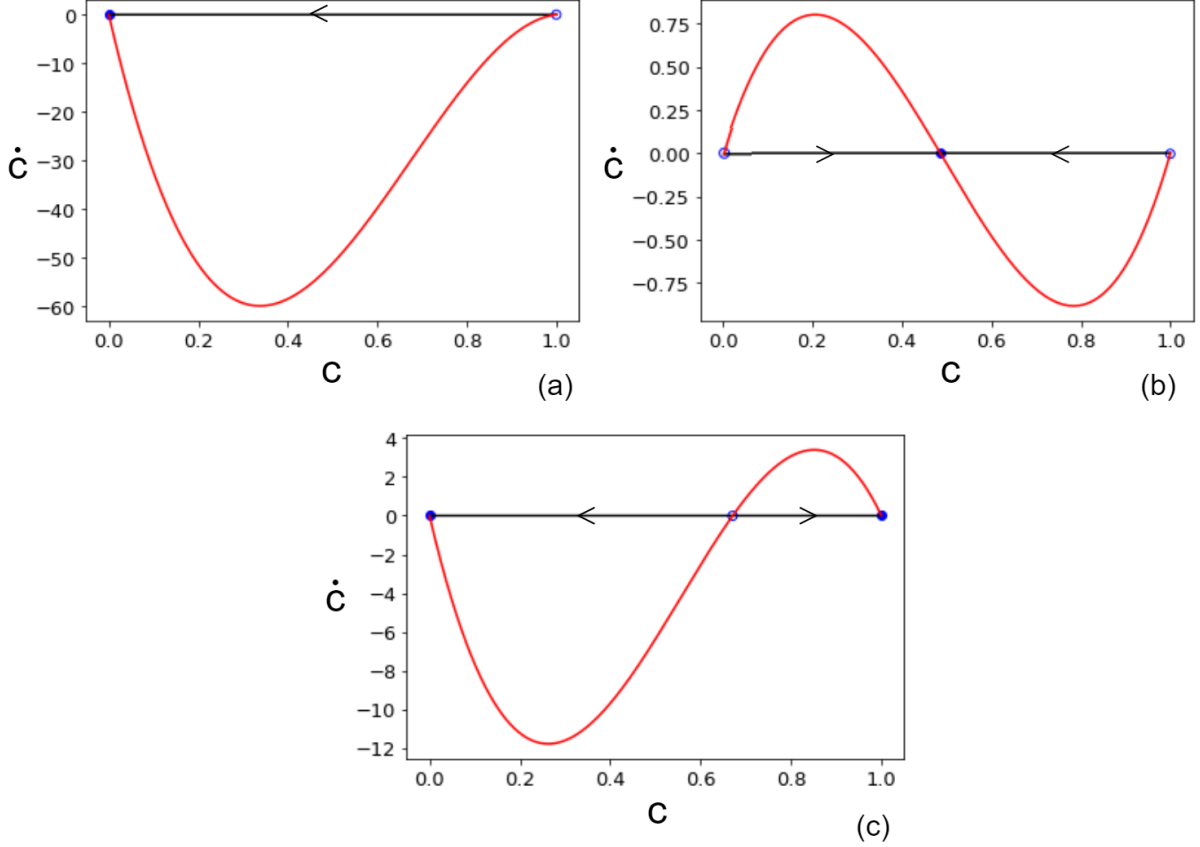
Figure 1: Phase portraits for the cases of (a) dominance (the prisoners dilemma), (b) the snow-drift game, and (c) the coordination game. Filled circles represent stable equilibria, while unfilled circles represent unstable equilibria. The arrows represent the direction of the trajectory of the changing concentration of cooperators.

2.3.1 Dominance

Dominance occurs when the strategy of one species is always preferable to the other, whether they are interacting with one of their own kind or the other species. The prisoners dilemma is an example of this (where $T > R$ and $P > S$), it represents a situation where species D is always 'fitter' than species C, resulting in the extinction of the latter. In terms of the classical description of the game, it is always best for the prisoner to defect resulting in the 'extinction' of the cooperative strategy. This can bee seen in Figure 1(a) where $\dot{c}$ is always negative, resulting in a stable fixed point at $c = 0$ [1,6]. In this situation, we call D an 'evolutionary stable strategy' meaning that if the population N is comprised solely of individuals of species D, it cannot be invaded by any individuals of species C [10].

2.3.2 The Snowdrift Game

The snowdrift game represents a scenario where it is preferable for the two populations to coexist. Here $T > R$ and $S > P$, each species receives a higher payoff when interacting with individuals of the other species, see Figure 1(b) [1,6]. In the context of rational individuals playing a game against each other, this coexistence would be known as a *mixed strategy*, where the player might adopt a strategy where they choose to either defect or cooperate with some probability [6].

2.3.3 The Coordination Game

When $R > T$ and $P > S$ it is possible for either species to dominate while the other becomes extinct, this situation is referred to as the coordination game, see Figure 1(c) [6]. The result of this evolutionary game depends on the initial conditions of the system. There is also an unstable equilibrium where the two populations may coexist if their respective sizes are precisely balanced, though this is a incredibly unlikely scenario in a real biological system.

2.4 Cyclic Three-Strategy Games

Lets now consider a game with three possible strategies, R, P, and S. If strategy S wins against strategy P, P wins against R and R wins against S we have what is known as a cyclic (or rock-paper-scissors) game [11]. The payoffs associated with this game are illustrated in table 3.

	R (Rock)	P (Paper)	S (Scissors)
R	0	-a	b
P	b	0	-a
S	-a	b	0

Table 3: Payoff matrix for a cyclic 3-strategy game where $a, b > 0$ [11].

If $a < b$, we can see from Figure 2(b) that the trajectories lead to a mixed strategy Nash equilibrium, where in the case of evolutionary game theory, the three different species coexist [1,10]. If $a > b$, the interior fixed point now becomes unstable causing the trajectories to spiral outwards until it follows a heteroclinic cycle along the edge of the simplex, see Figure 2(c) [1, 10]. In the context of three competing populations, once the system made contact with the edge of the simplex, at least one of the species would become extinct [6, 11, 12]. Figure 2(a) represents the case where $a = b$, similarly for the case where $a < b$ all three species are able to survive and coexist, however in this case the sizes of the three populations are not constant and actually oscillate about the mixed strategy Nash equilibrium at the centre of the simplex [10].

It is important to note that the description of population dynamics offered by the replicator equations is idealised in comparison to real systems [5]. The replicator equations work well in the deterministic limit for infinitely large populations, however in reality populations are finite and can exhibit stochastic behaviour [5]. If we include such stochasticity in our cyclic game
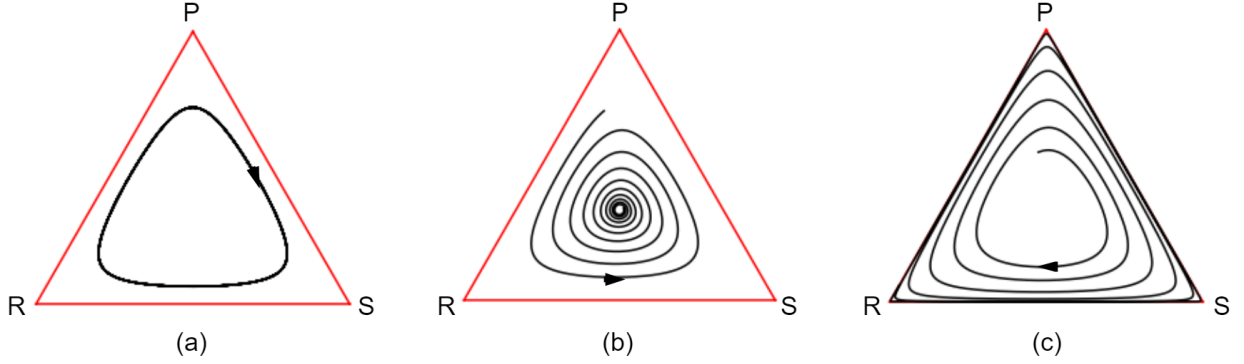
Figure 2: The replicator equations for a three-strategy cyclic game were solved using the fourth order Runge-Kutta method, for more detail see the appendix. The solutions were plotted on a simplex. The corners of the simplex indicate uniform populations made up of the indicated species. The edges of the simplex correspond to some mixture of the two species that label the ends of the edge, while the species that labels the opposite vertex is extinct. Points inside the simplex indicate mixed populations including all three species, the centre of the simplex indicates a state where all three species have the same frequency [8]. The sub-figures represent the situations where (a) $a = b$ (b) $a < b$ (c) $a > b$, refer to Table 3.

where $a = b$, random fluctuations could cause the trajectory of the system to touch the edge of the simplex, eliminating one of the species [6]. Despite its apparent vulnerability to noise, cyclic behaviour is often observed in real biological systems, such as the mating strategies of lizards and non-hierarchical competition between bacterial strains, the later will be explored later [1, 4, 13].

3   Mechanisms for the Survival of Cooperation

In the natural world, it is expected that individuals will choose to act selfishly with the goal of self-preservation [14]. However, cooperative behaviour, otherwise known as altruism, is actually observed in a rich variety of forms in many natural systems [15, 16]. The question we might ask is: how might cooperation emerge in a situation where it is most beneficial for an individual to act to serve their own self-interest? Exhibiting altruistic behaviour can make an organism vulnerable to exploitation. This is a conundrum has puzzled many, even the likes of Charles Darwin [14]. This problem is reminiscent of the situation in the prisoners dilemma where the optimum strategy is to defect, despite the fact that mutual cooperation is most beneficial for all. For this reason the prisoners dilemma has often been used as a framework to discuss the mystery of the evolution and maintenance of cooperative behaviour [17]. Much thought has been dedicated to this problem and a number of different mechanisms have been proposed to allow for the survival of cooperation, some of which have been described in Martin Nowak's paper 'Five rules for the evolution of cooperation' which will be briefly discussed [16].

3.1   Kin Selection

In order for an organism to be successful, it is not only important for it to reproduce to spread its genes, they may choose to cooperate with their relatives to promote their survival [9, 14, 16]. Hamilton's rule describes how individuals will choose to cooperate if the degree of *relatedness* between them and their opponent (chance that a gene is shared) is greater than the ratio of the payoffs $\frac{S}{T}$ [16].

3.2   Direct Reciprocity

Individuals in a population have a chance of interacting with the same opponent more than once. If an individual shows good faith and cooperates with their partner, upon their next meeting their opponent may choose to cooperate. Presenting altruistic behaviour may be beneficial for an individual in the long-run. Direct reciprocity, otherwise known as 'tit-for-tat', is implemented as follows; an individual starts with the strategy of cooperation, then in subsequent meetings with the same opponent they will play the strategy that their opponent played in the previous round [16].

3.3   Indirect Reciprocity

Indirect reciprocity relies on the concept of reputation, if an individual has shown good will and cooperated with an opponent, this improves their social status with those who witnessed the act, making the bystanders more willing to cooperate with that individual [16].

3.4   Network Reciprocity

So far we have only considered well-mixed populations, where individuals have high mobility and may interact with any member of the population. In a real population, some pairs of individuals will interact more often than others. Network reciprocity takes this into account by having members of the population occupy vertices of a graph, the vertices are connected by edges indicating which individuals interact with one another. Cooperators are able to form clusters on the network where they assist each other without being extorted by the defectors [5, 16].

3.5   Group Selection

Similarly to network reciprocity, group selection preserves cooperation by allowing individuals to form clusters, however there is a subtle difference between the two mechanisms [5]. Group selection organises the population into smaller subpopulations, which then compete with each other [5]. The subpopulations also exhibit different growth rates depending on the proportions of cooperators and defectors it is made up of [16].

Here we will explore a special case of network reciprocity known as spatial reciprocity where the specific effect of a spatial distribution on the survival of altruism can be investigated [16].

A spatial game can be relatively simple, without any strategic complexity such as memory of previous interactions and conscious decisions that have to be made in tit-for-tat [17].

4   Spatial Games

The *cellular automaton* was developed by Neumann around the same time as he was working on game theory [4, 18]. Cellular automata are generally made up of a grid of cells which are each assigned some value. The cells follow a set of rules which dictate how interactions with their neighbours in the grid cause their values to evolve. Though multiple iterations applying these rules, the whole system evolves [19]. Spatial evolutionary games combine these two fields which, discounting the fact that they share a creator, were previously unrelated [4, 18]. There are a number of different variations of spatial prisoners dilemma, here we will summarise the model developed by Martin Nowak and Robert May [17, 18, 20].

4.1   Details of the Spatial Prisoners Dilemma Model

A model of a spatial game was set up as a square lattice, where each cell of the lattice was occupied by a different member of the population [16, 21]. Periodic boundary conditions were imposed by connecting the edges of the lattice so that every cell of the grid had eight neighbours, see Figure 3(b) [22]. A simplified form of the prisoners dilemma was used where $R = 1, T = b > 1$, $S = 0, P = \epsilon < 1$. To further reduce the complexity the problem, the payoff $\epsilon = 0$ was chosen so that the system is only dependent on the parameter b (which characterises the magnitude of the advantage defectors have over cooperators) [18]. This simplification didn't change the outcome of the experiment compared to the results found if ϵ were an arbitrarily small positive number, thus it did not matter that the condition $P > S$ was technically violated [17].

4.1.1   Deterministic vs. Stochastic Dynamics

Cell updating rules define methods to determine which strategy occupies a particular cell in succeeding generations [8]. One such method used in the spatial games played by Nowak and May is explained here. The payoff received by each cell was the sum of the payoffs resulting from pairwise games played with each of the cells 8 nearest neighbours and itself (this model allowed for self-interaction) [8, 22]. The strategy of a particular cell was updated in a new generation by replacing it with the strategy of the nearest neighbour cell with the largest payoff (unless the cell that was being updated had the highest payoff in the neighbourhood) [8, 22]. This means that the new strategy of the cell was only dependent on the 24 surrounding cells as well as itself, as the local neighbourhoods of each cell dictate their respective payoffs, see figure 3 [8, 22].

The cell strategy update rule previously described is deterministic, meaning that future states of the system are completely dependent on the initial conditions of the system [6, 23]. As previously stated in section 2.4 this is not usually the case in real biological systems, for instance, there
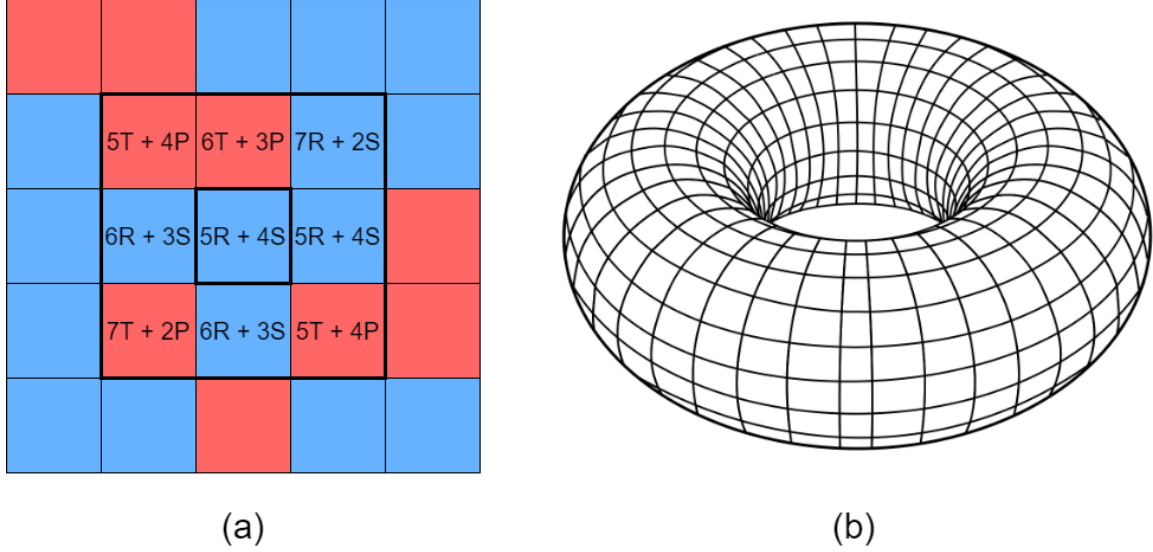
Figure 3: (a) Diagram representing the deterministic update rule with self-interaction. This figure was created with reference to a diagram in [8]. (b) Square lattice deformed so that forms a torus and its edges meet.

may be fluctuations in the availability of food or nutrients, effecting the success of particular individuals [14]. To include this in the model, a stochastic cell updating rule was formulated. The probability P_j that a cell j, which has a set of neighbouring cells N_j, contains cooperative individual after the next update ws given by

$$P_j = \frac{\Sigma_{i \in N_j} A_i s_i}{\Sigma_{i \in N_j} A_i}, \tag{6}$$

where A_i is the payoff of cell i. $s_i = 1$ if i was a cooperator and $s_i = 0$ if it was a defector [23]. This method considers the ratio of the sum of the payoffs of the neighbouring cooperator cells to the total payoff earned by all surrounding cells. It is clear that the more high-scoring cooperators there are in the immediate neighbourhood, the more likely the next cell is to be occupied by a cooperator.

The deterministic and stochastic cases can be written generally with one formula,

$$P_j = \frac{\Sigma_{i \in N_j} A_i^m s_i}{\Sigma_{i \in N_j} A_i^m} \tag{7}$$

where the parameter m is a weighting factor for how much the neighbouring cell with the highest payoff is favoured in the update [23]. When $m \to \infty$, the most successful neighbour is infinitely weighted and we recover the deterministic case. When $m = 1$, we return to the stochastic update rule in Equation 6 [23]. Alternative stochastic update rules have been formulated in other models that include a noise parameter, this makes it possible to vary the stochasticity of the system and investigate the effect this has on its behaviour [18, 20].

4.1.2 Synchronous vs. Asynchronous Cell Updating

Synchronous updating, otherwise known as discrete time dynamics, means that all of the cells in the lattice are updated at the same time. Asynchronous updating or continuous time dynamics is where a player is chosen at random to interact with its neighbours and update its strategy while the rest of the cells remain the same [21, 24]. Arguments can be made for the use of either method, due to the great number of biological systems this model could be used to describe. While seasonal mating and gestation periods in some species may lead to regular, synchronised updates of the population, this is not always the case, therefore it is important to investigate both cases [21, 24].

4.2 Results and Observations of the Spatial Prisoners Dilemma

Through many numerical simulations, it was found that cooperative behaviour was able to survive for a considerable range of values of the parameter b for each update rule, see Figure 4 [18, 23]. Each variation of the system exhibited quite different spatial patterns and dynamics [17].
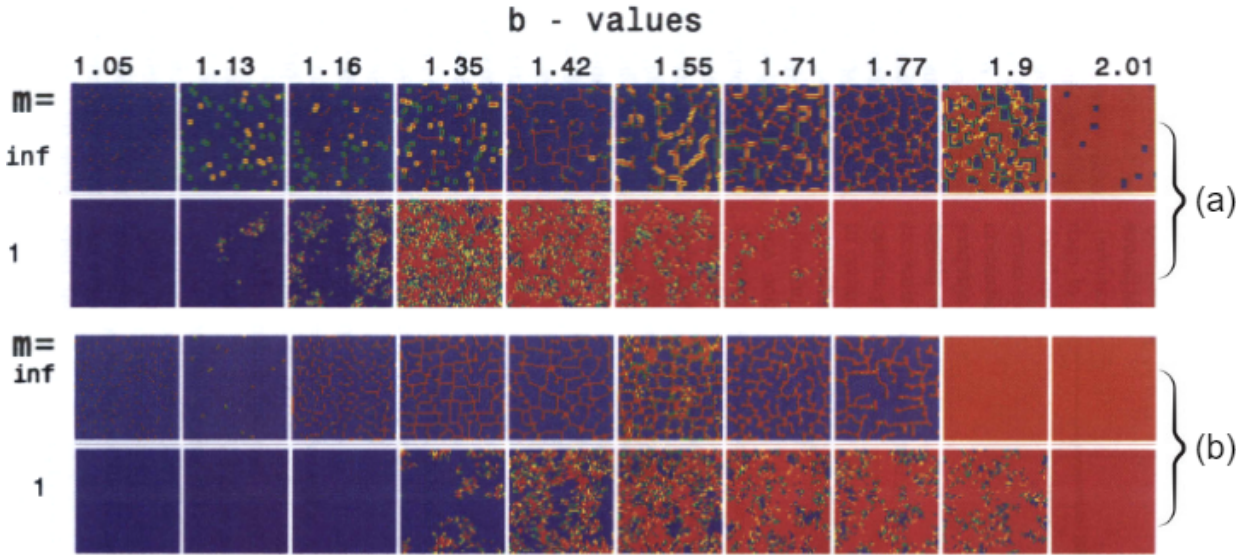

Figure 4: Comparison of results for deterministic ($m = \infty$) and stochastic ($m = 1$) update rules for (a) discrete time and (b) continuous time. Cooperators and defectors are colour coded as follows: a C site that was also a C in the previous generation is blue, a D that was also a D in the previous generation is red, a C site that was previously a D is green, and a D site following a C is yellow [17]. All of these spatial patterns are independent of their initial conditions [21]. Figures taken from [23].

The most interesting dynamics occurred in the deterministic synchronous system for $1.8 < b < 2$ [23]. In this regime, groups of cooperators grew, shrank, and moved across the lattice in a disordered fashion as they battled against a sea of defectors [23]. When $b < 1.8$, the concentration of cooperators oscillated regularly as groups of cooperators successfully resisted invasion

by defectors [22]. In contrast, when $1.8 < b < 2$, the proportion of cooperators and defectors fluctuated randomly as each strategy competed for space [17]. Despite this disordered behaviour, after a sufficient amount of time, the system settled to fluctuate about a fractional frequency of cooperators ≈ 0.318, see Figure 5 [17]. Even when the two strategies competed so fiercely that no static spatial structure could be formed, cooperative behaviour still managed to persist at a constant proportion [22].
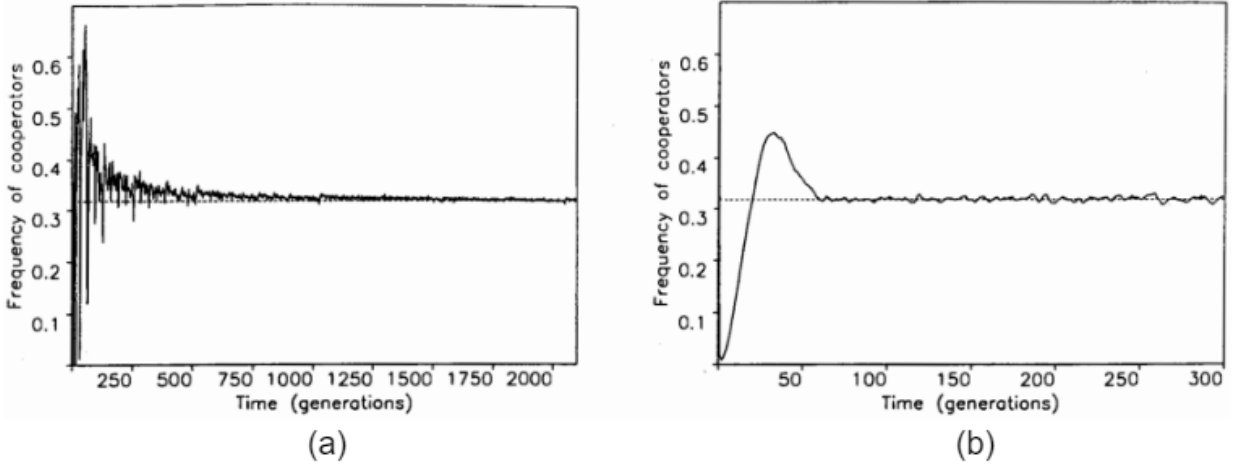

(a)　　　　　　　　　　　　　　　　　(b)

Figure 5: In the parameter region $1.8 < b < 2$ with a deterministic synchronous update rule, the proportion of cooperators eventually settled to fluctuate randomly about the value 0.318. This behaviour was independent of initial conditions, whether the initial state of the system had (a) a single defector, or (b) a random mix of cooperators and defectors [17,22]. Figure taken from [22].

These spatial games provide a mechanism for the survival of cooperation without the requirement of complex rules or reasoning on the part of individuals in the population, this means they can be applied to a wider range of biological systems than mechanisms like direct reciprocity [14,17,21,23]. However, the boundaries between the different mechanisms of the maintenance of cooperation are actually slightly blurred. These spatial games could be thought of as a way of formulating kin selection. In real-world biological systems, individuals are likely to stay close to their genetic relatives. Where cooperators form clusters in these spatial games, one could consider this to be cooperation amongst kin [5,22].

The fact that we observe coexistence between defectors and cooperators for multiple types of cell updating rules indicates that adding a spatial distribution to a population offers a robust explanation for such coexistence. Though the specific details of these systems are unlikely to relate directly to any real biological system, the main principle persists: provided cooperators are able to form clusters to prevent themselves from being taken advantage of by defectors, cooperation may survive [14].

4.3 Cyclic Spatial Games

Recall the stochastic nature of biological systems can destroy the coexistence of species in three-strategy cyclic games. In addition to promoting the existence of cooperation in the prisoners dilemma, application of a spatial distribution can also promote biodiversity [6].

Nowak and May briefly investigated the effect of spatial distributions on cyclic three-strategy games and found that the populations organised themselves into rotating spiral waves [23]. This behaviour has also been observed in a model developed by Tobias Reichenbach, Mauro Mobilia and Erwin Frey [12]. The model has a similar set-up to Nowak and May's spatial games, it has a square lattice with periodic boundary conditions and the game follows an asynchronous, stochastic updating rule [12]. The three subpopulations exhibited the cyclic dominance displayed in Figure 2(c), where the system spirals away from coexistence until extinction occurs [12]. This model investigated the effect of increasing mobility on the structure of the system. For very low mobility, the three species organised themselves into rotating spiral patterns. By chasing each other across the lattice each subpopulation is able to avoid extinction. As the mobility of the individuals was increased, bringing the system closer to a well-mixed state, these patterns grew in size until they were eventually destroyed as one of the three species triumphed over the rest leaving the system in a uniform state containing a single species, this can be seen in Figure 6 [6].
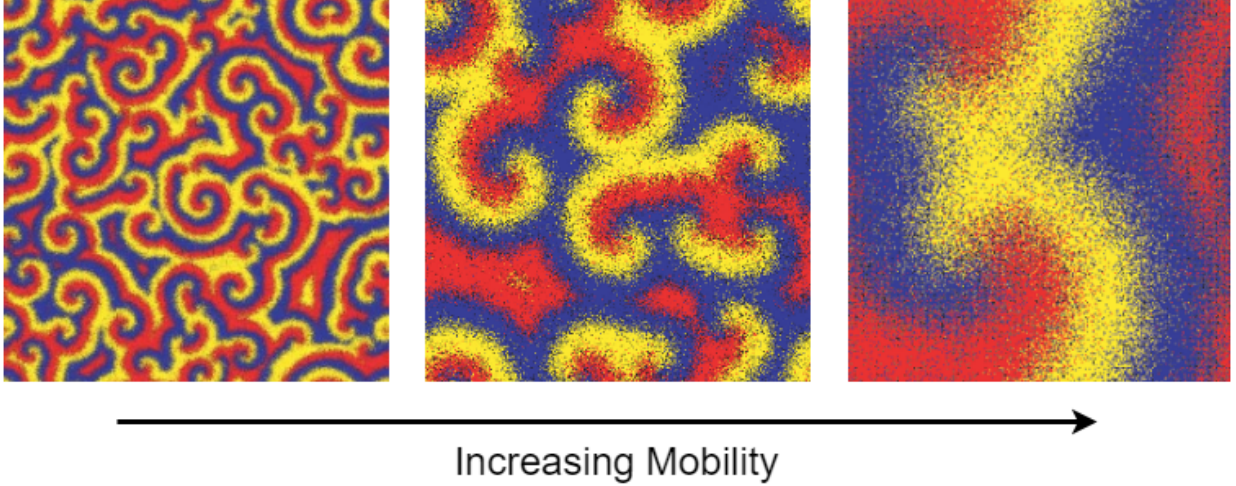

Figure 6: Spiral patterns created on a lattice in a cyclic three-strategy game. As the mobility of the individuals increases the spiral patterns break down [12]. Figure taken from [12].

5 Game Theoretic Interaction in Populations of E. coli

We will now consider the effect of a spatial distribution on real bacterial systems. Bacterial systems are very complex, their behaviour depends on a wide realm of factors including their spa-

tial distribution, the mobility of the bacteria, environmental conditions, and competition among bacterial strains. There are a number of studies which investigate the evolutionary dynamics of spatially distributed bacterial populations [14]. The results of two experiments which explore the behaviour of these systems will be discussed, the first of which concerns the maintenance of cooperation in the prisoners dilemma [25], the second, biodiversity in cyclic three-strategy games [13].

5.1 Maintenance of Cooperation in E. coli

The effect of spatial structure on the survival of cooperation in the prisoners dilemma was investigated in a real system of two different strains of *Escherichia coli* [25]. E. coli communities are known to restrict their own growth in order to conserve resources and therefore avoid population collapse, this is known as the stationary phase. Instead of an individual acting selfishly and continuing to multiply to increase their number of relatives without concern for the others around them, the E. coli act with the survival of the whole bacterial colony in mind, this is a form of cooperation. Here a wild-type (WT) E. coli strain acts cooperatively, while a mutant strain, which does not follow the mutual decision to restrict growth when resources run low, instead receives a growth advantage in the stationary phase (we refer to these bacteria using the acronym *GASP* from now on). The defecting GASP mutants continue to multiply using up resources while the WT E. coli restricts its growth to preserve resources [25].

First, the behaviour of the two strains of E. coli in unstructured environments was investigated. This was done by placing mixed populations of E. coli containing different starting proportions of each strain into continuously shaken flasks. The changing fraction of WT and GASP bacteria was measured over the course of 8 days, see Figure 7(a). After a couple of days, all samples were becoming increasingly dominated by the GASP population, this is in agreement with what we observe when the replicator equations are applied to the prisoners dilemma.
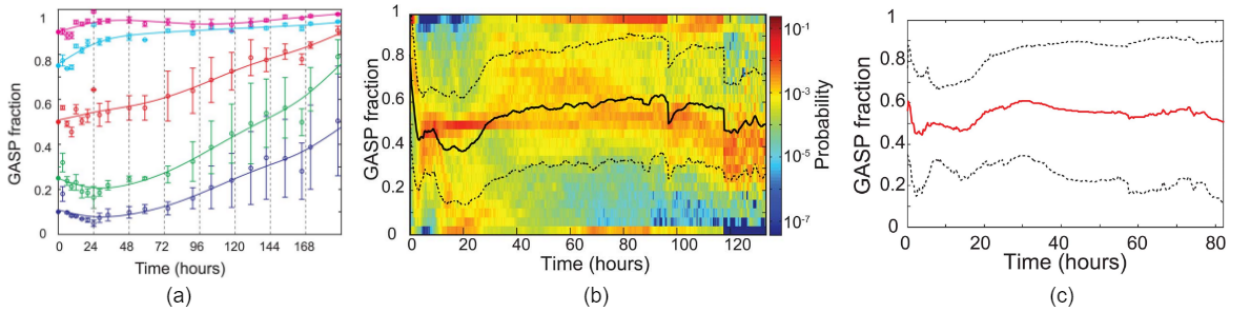

Figure 7: Fraction of total population made up of GASP E. coli at different times in (a) continuously shaken flasks with different starting fractions of the GASP mutant, (b) 85 microhabitats, (c) a single large habitat [25]. Figures taken from [25].

Next, a series of 85 connected micro-habitats were created to house different subpopulations of the E. coli. The system exhibited dynamic behaviour, with the communities inhabiting each microhabitat constantly changing as the WT and GASP E. coli competed with one another. Despite the continuous variation in the proportion of the total population made up of each type of E. coli, the fraction of GASP cells was observed to fluctuate about 0.5, indicating a steady coexistence between the WT and GASP strains. In fact, this coexistence was observed not only on a global scale, but also within the individual microhabitats themselves. The same experiment was then repeated in a single large habitat with a volume equal to that of the 85-patch micro-habitat system. It was shown that the WT and GASP cells organised themselves into segregated patches, which allowed both strains to survive together, see Figure 8. The fraction of GASP bacteria that made up the population were observed to fluctuate about 0.5 (see Figure 7(c)), as was observed in the experiment with 85 microhabitats, thus providing evidence of the survival of cooperative behaviour in a spatially extended biological prisoners dilemma system.
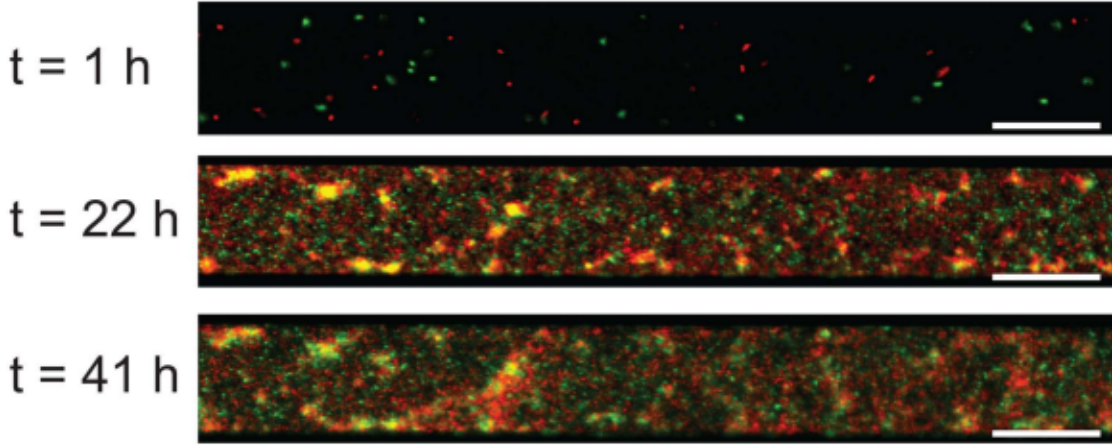

Figure 8: Microscopy photos taken of WT (green) and GASP (red) E. coli strains in the large habitat at different time intervals. The white line represents a length of 50μm [25]. Figure taken from [25].

5.2 Maintenance of Biodiversity in E. coli

The effect of a spatial distribution on the maintenance of biodiversity in a cyclic three-strain bacterial community was investigated in an experiment in reference [13]. One strain of the E. coli bacteria (referred to as strain C) produces a toxin called colicin, which damages colicin sensitive cells (labelled S). colicin resistant cells (R) are able to reproduce faster than the C cells as they don't have to sustain the production of colicin. Finally, S cells have a faster growth rate than R cells as the resistance of R cells to the colicin acts as a double edged sword, impeding other cell functions such as nutrient uptake [13]. We can see that each strain of E.coli dominates one of the others, while being vulnerable to competition from the other, forming a cyclic game.

In a similar fashion to the experiment with GASP and WT E. coli in the previous subsection, the behaviour of the E. coli was studied in various environments including a shaken flask and a 'static plate' which provided a spatial distribution [13]. In the flask environment, the S and C strains were driven to extinction, leaving a population made up exclusively of the R strain, see Figure 9(a). This situation is similar to that described in Figure 2(c), where in a well-mixed environment, the three strains are not able to coexist.
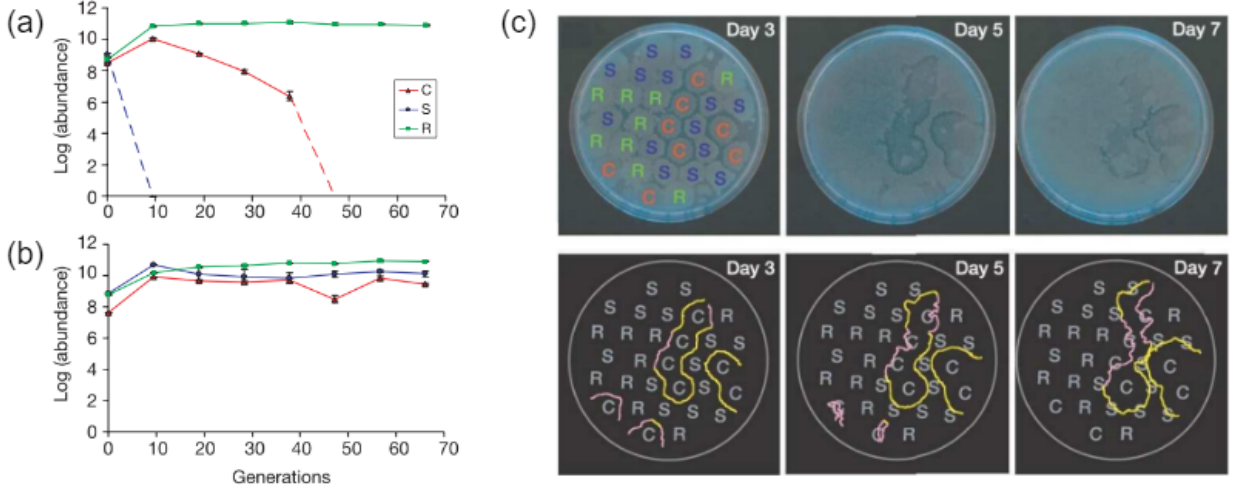

Figure 9: The time evolution of the abundance of each strain in the experiments with (a) the shaken flask, and (b) the static plate. Data points are separated by a time period of 24hrs. (c) Images of the static plate taken at different times during the experiment showing the dynamic coexistence of the three strains of E. coli [13]. Figures taken from [13].

All three strains were able to survive for an extended period of time when placed on the static plate, see Figure 9(b). In fact, as shown in Figure 9(c) it was possible to observe the different strains 'chasing' each other across the plate in accordance with their cyclic dominance, this behaviour is similar to that observed in the cyclic lattice simulation discussed in Section 4.3. This experiment provides empirical evidence of how a spatial distribution can promote biodiversity [6, 13].

6   Conclusion

Beginning with the replicator equations, we have studied the behaviour of well-mixed populations. We have seen that cooperative behaviour and biodiversity can be maintained by limiting the which individuals that can interact with each other. From its roots as a method of modelling socio-economic behaviour, the versatility of game theory has only grown over the past century. The diversity of systems that can be modelled using game theory is reflected in the numerous mechanisms that are employed to reproduce their behaviour. It has been fascinating to learn about how nonlinear physics can be applied to biological systems. Given more time to

do this project, it would have been interesting to produce some original lattice based simulations, perhaps with spatial irregularities such as non-uniform distributions of nutrients.

7 Appendix: Creation of Figure 2

The replicator equations for the three strategies were solved numerically using the fourth order Runge-Kutta method. Staring from the initial conditions $\mathbf{x}_n = (R_n, P_n, S_n)$, after a time interval Δt the system is in state $\mathbf{x}_{n+1}$ given by,

$$\mathbf{x}_{n+1} = \mathbf{x}_n + \frac{1}{6}(\mathbf{k}_1 + 2\mathbf{k}_2 + 2\mathbf{k}_3 + \mathbf{k}_4). \tag{8}$$

The parameters $\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4$ are calculated as follows,

$$\mathbf{k}_1 = \mathbf{f}(\mathbf{x}_n)\Delta t, \quad \mathbf{k}_2 = \mathbf{f}(\mathbf{x}_n + \frac{1}{2}\mathbf{k}_1)\Delta t, \quad \mathbf{k}_3 = \mathbf{f}(\mathbf{x}_n + \frac{1}{2}\mathbf{k}_2)\Delta t, \quad \mathbf{k}_4 = \mathbf{f}(\mathbf{x}_n + \mathbf{k}_2)\Delta t, \tag{9}$$

where the function $\mathbf{f}$ is a vector containing the replicator equations for each population [26]. The resulting dynamics of a number of different cases of a three-strategy cyclic game were then plotted on a simplex.

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