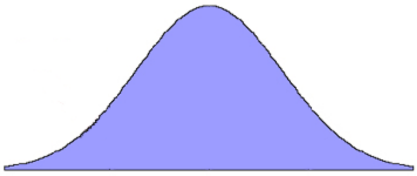
## APPLICATIONS ON OTHER BRANCHES OF MATHEMATICS

Throughout Calculus, we learned different applications of differential and integral calculus.

- Finding the Slope of a curve – Calculus gave us a generalized method of finding the slope of a curve. Calculus allows us to find out how steeply a curve will tilt at any given time. This can be very useful in any area of study.

- Calculating the Area of Any Shape – We have standard methods to calculate the area of some shapes. Calculus allows us to do much more. Trying to find the area of a shape like this would be very difficult if it weren't for calculus.



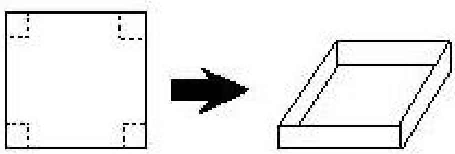
https://calculus.nipissingu.ca/calc_app.html

- Calculate Complicated $x$-intercepts – Without the Intermediate Value Theorem, it would be exceptionally hard to find or even know that a root existed in some functions. Using tools in Calculus, we can also calculate an irrational root to any degree of accuracy, something our calculator would not be able to tell us if it wasn't for calculus.

- Visualizing Graphs – Using calculus, we can practically graph any function or equation we would like. We can find out the maximum and minimum values, where it increases and decreases, and much more without even graphing a point, all using calculus.

- Finding the Average of a Function – A function can represent many things. One example is the path of an airplane. Using calculus, you can calculate its average cruising altitude, velocity, and acceleration. Same goes for a car, bus, or anything else that moves along a path.

- Calculating Optimal Values – By using the optimization of functions in just a few steps, you can answer very practical and useful questions such as: "You have a square piece of cardboard, with sides 1 meter in length. Using that piece of cardboard, you can make a box, what are the dimensions of a box containing the maximal volume?" These types of problems are a wonderful result of what calculus can do for us.



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- Indefinite integrals also have applications in mathematical economics.

**Definition 4.1**

**Marginal cost** is the change in total cost resulting from one unit change of production. If $C(x)$ is the total cost function for producing $x$ units, then the marginal cost function is given by $C'(x)$. **Overhead cost** is the cost when no units are produced, that is, $C(0)$.

*Example 4.1*

The marginal cost function for a commodity is given by $C'(x) = 3x^2 + 4x + 5$, where $x$ is the number of units produced. Suppose the overhead cost is P 1,000. Find

a. the total cost function, and
b. the cost of producing ten (10) units.

*Solution:*

a. Recall by definition of antiderivative that since $C'(x) = 3x^2 + 4x + 5$, or $d\big(C(x)\big) = (3x^2 + 4x + 5)dx$, where $C(x)$ is the total cost function, then
$$C(x) = \int (3x^2 + 4x + 5)dx = x^3 + 2x^2 + 5x + k,$$
for some constant $k$. Now, the overhead cost is $C(0) = k$. But it is given that the overhead cost is P 1,000. Therefore, $k = 1{,}000$ and
$$C(x) = x^3 + 2x^2 + 5x + 1{,}000.$$

b. Let $x = 10$, then
$$C(10) = 10^3 + 2(10^2) + 5(10) + 1{,}000 = 2{,}250.$$
Therefore, the cost of producing ten (10) units is P 2,250.

- In life sciences, calculus is applied in predicting population growth.

*Example 4.2*

The growth rate of a city's population after $t$ years is modeled by $P'(t) = 500t^{1.06}$. If the city's current population is 50,000, what will be its population after 10 years?

*Solution:*

We first want to find the function $P(t)$ which gives the population after $t$ years. Since $P'(t) = 500t^{1.06}$, then
$$P(t) = 500 \int t^{1.06}\, dt = \frac{500}{2.06} t^{2.06} + C.$$
To solve for $C$, we use the fact that the city's current population is 50,000, that is, $P(0) = 50{,}000$. But
$$P(0) = \frac{500}{2.06} \cdot (0)^{2.06} + C = C.$$
Hence, it follows that $C = 50{,}000$ and
$$P(t) = \frac{500}{2.06} t^{2.06} + 50{,}000.$$
Therefore, after ten (10) years, the population of the city will be
$$P(10) = \frac{500}{2.06} \cdot 10^{2.06} + 50{,}000 \approx 77{,}868.$$

*Note: The answer is rounded off since population is a counting number.*

## PROBABILITY THEORY

The $z$-table is a very useful tool in probability theory. There are times when we are interested to know the probability of randomly selecting a data $X$ which is at most $x$, that is, $\mathrm{Prob}(X < x)$. This value is equal to the probability that the $z$-score of the randomly selected data is at most $z = \frac{x-\mu}{\sigma}$, where $\mu$ is the mean and $\sigma$ is the standard deviation of the data set. Equivalently, $\mathrm{Prob}(X < x) = \mathrm{Prob}(Z < z)$. This is where the $z$-table becomes useful.

To use the $z$-table, we firstly compute for the $z$-score that corresponds to $x$, using the formula
$$z = \frac{x - \mu}{\sigma}.$$
Secondly, we locate $\mathrm{Prob}(Z < z)$ using the $z$-table. This then is value of $\mathrm{Prob}(X < x)$.

| $z$ | .00 | .01 | .02 |
|---|---|---|---|
| −3.4 | .0003 | .0003 | .0003 |
| −3.3 | .0005 | .0005 | .0005 |
| −3.2 | .0007 | .0007 | .0006 |
| ⋮ | ⋮ | ⋮ | ⋮ |
| 0.3 | .6179 | .6217 | .6255 |
| 0.4 | .6554 | .6591 | .6628 |
| 0.5 | .6915 | .6950 | .6985 |

*Figure 4.1*

*Example 4.3*

Suppose $\mu = 3$ and $\sigma = 4$. Find the probability that we select a data $X$ that is less than 5.

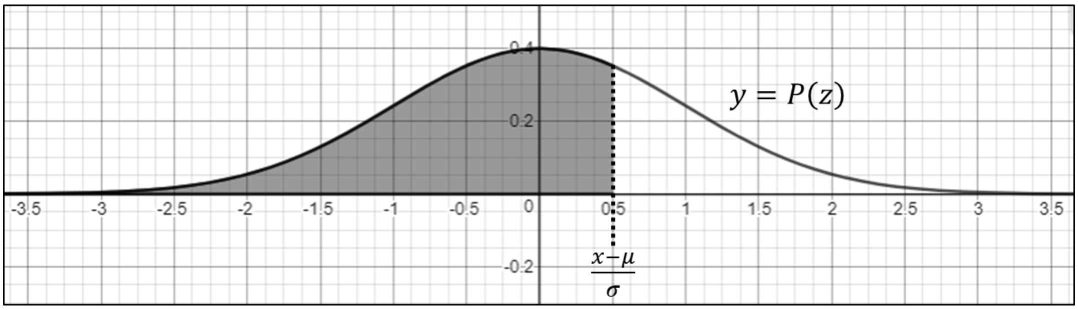
*Solution:*

Solving for the $z$-score,
$$z = \frac{5 - 3}{4} = 0.5.$$
Locating 0.5 in the $z$-table (see *Figure 4.1*), we see that $\mathrm{Prob}(Z < z) = 0.6915$. Hence,
$$\mathrm{Prob}(X < 5) = 0.6915.$$

But how were the entries in the table derived? Every set of data produces different mean and standard deviation. For easier analysis, we standardize them into what we know as the standard normal distribution

$$P(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}},$$

where $z = \frac{x-\mu}{\sigma}$ is the corresponding $z$-score for the value $x$ within the range of the data set having mean $\mu$ and standard deviation $\sigma$. The probability of randomly selecting a data $X$ that is at most $x$ is equal to $\text{Prob}(Z < z)$, which is actually equal to the area of the shaded region below.




Consider again *Example 4.3*. This time, let us use integral calculus and not the $z$-table. Since $z = 0.5$, then

$$P(Z < 0.5) = \int_{-\infty}^{0.5} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz \approx 0.6914624612740131 \approx 0.6915.$$

The indefinite integral cannot be expressed in terms of finite additions, subtractions, multiplications, and root extractions. Hence, 0.6915 is computed through numerical approximations. This makes it clearer that the entries of the $z$-table are actually computed using calculus.

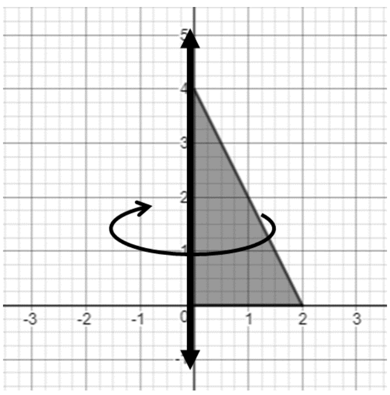
## SOLID GEOMETRY

**Definition 4.2**

If a region on a plane is revolved about a line, the resulting solid is called a **solid of revolution,** and the line is called its **axis of revolution**.

*Example 4.4*

The images below show some solids of revolution with their corresponding regions on a plane and axes of revolution.

| Region and Axis of Symmetry | Resulting Solid of Revolution |
| --- | --- |
|  |  |

| | |
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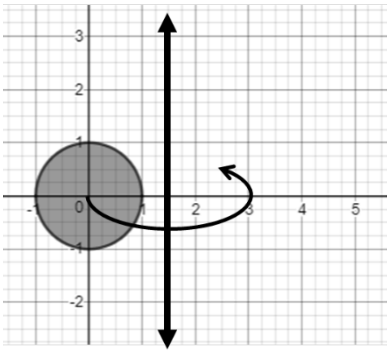
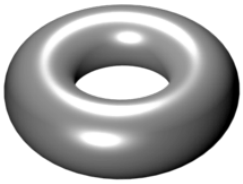
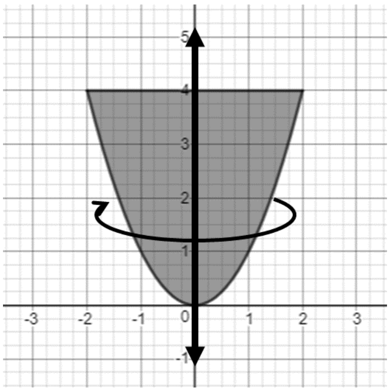
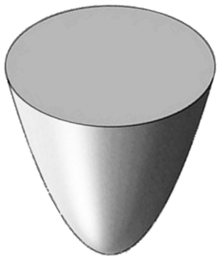
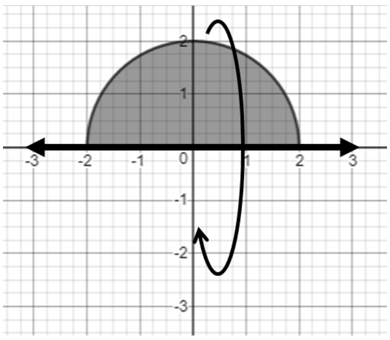
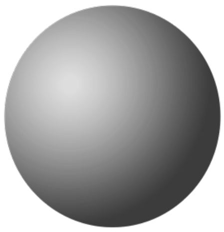
|  |  |
|  |  |
|  |  |

The methods of solving the volume of a solid of revolution are all founded on integral calculus.

**Theorem 4.1 (The Disc Method)**

Let $f(x)$ be a nonnegative continuous function and $a$ and $b$ be real numbers such that $a < b$. Let $R$ be the plane region bounded by the graph of $y = f(x)$, the lines $x = a$ and $x = b$, and the $x$-axis. Suppose we revolve $R$ about the $x$-axis. The volume of the resulting solid of revolution is given by

$$\pi \int_a^b f^2(x)dx.$$

*Proof:*

Let us visualize the region $R$ using *Figure 4.2a*. When we revolve $R$ about the $x$-axis, the resulting solid of revolution is shown in *Figure 4.2b*.

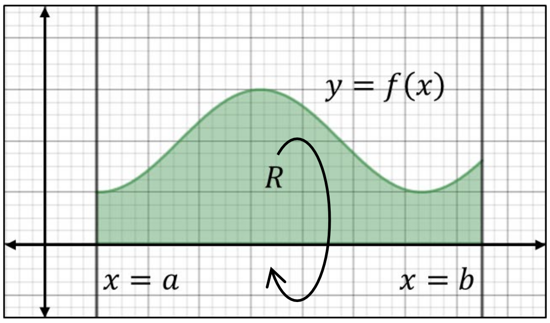



Figure 4.2a

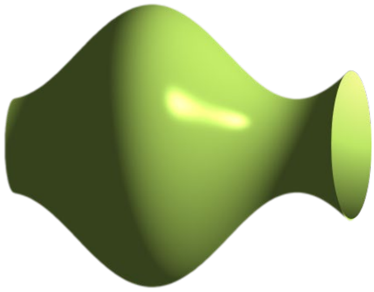
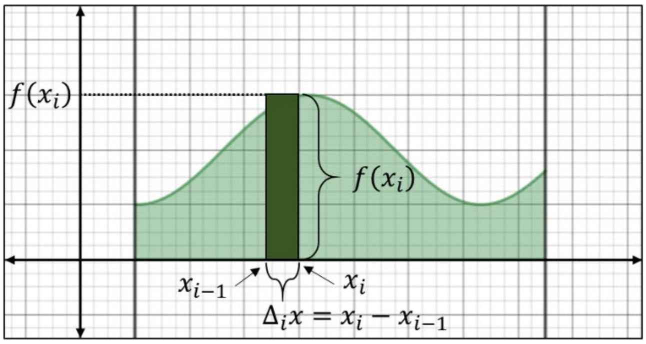
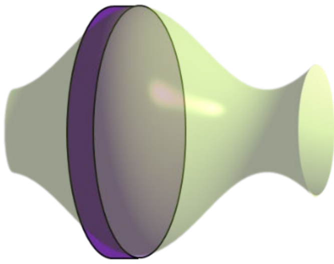


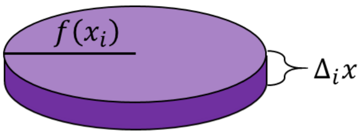
Figure 4.2b

Consider the $i^{\text{th}}$ rectangular strip which is perpendicular to the axis of revolution, which is the $x$-axis. Along with the revolution, a corresponding disc is formed from this rectangular strip.

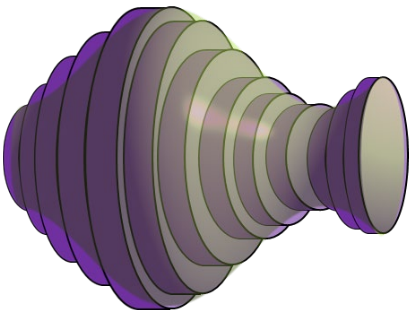


The disc formed is a cylinder of height $\Delta_\text{i} x = x_i - x_{i-1}$ and with base of radius $f(x_i)$.




Thus, the volume of the cylinder is $V_i = \pi f^2(x_i)\Delta_i x$.

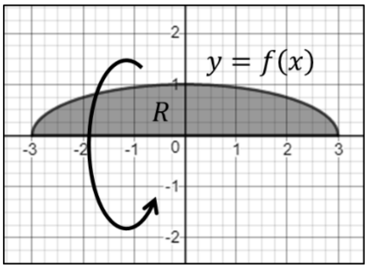


By the concept of the Riemann sums, the volume of *Figure 4.2b* is given by

$$V = \lim_{n \to +\infty} \sum_{i=1}^{n} \pi f^2(x_i)\Delta_i x = \pi \int_a^b f^2(x)dx.$$

*Example 4.5*

Let $f(x) = \frac{\sqrt{9-x^2}}{3}$ and let $R$ be the region bounded by $y = f(x)$ and the $x$-axis. Find the volume of the solid obtained by revolving the region $R$ about the $x$-axis using disc method.




*Solution:*

We see that $y = f(x)$ is the upper portion of the ellipse $x^2 + 9y^2 = 9$. Revolving the region about the $x$-axis therefore produces a solid with a surface called ellipsoid. Let $V$ be the volume of the solid, then

$$\begin{aligned} V &= \pi \int_a^b f^2(x)dx \\ &= \pi \int_a^b \frac{9-x^2}{9} dx \\ &= \pi \int_a^b \left(1 - \frac{1}{9}x^2\right) dx \end{aligned}$$

To determine the limits of integration $a$ and $b$, we solve for the intersection of $y = f(x)$ and the $x$-axis, that is, the line $y = 0$. When $y = 0$, then

$$0 = f(x) = \frac{\sqrt{9-x^2}}{3}.$$

Solving for $x$,

$$\begin{gathered} 0 = \sqrt{9-x^2} \\ 0 = 9 - x^2 \\ x^2 = 9 \\ x = \pm 3. \end{gathered}$$

Hence, it follows that $y = f(x)$ and $y = 0$ intersect at $x = -3$ and $x = 3$. Taking these as the limits of integration,

$$\begin{aligned} V &= \pi \int_{-3}^{3} \left(1 - \frac{1}{9}x^2\right) dx \\ &= \pi \left(x - \frac{1}{27}x^3\right)\Big|_{-3}^{3} \\ &= \pi \left[\left(3 - \frac{1}{27} \cdot 3^3\right) - \left(-3 - \frac{1}{27}(-3)^3\right)\right] \\ &= 4\pi. \end{aligned}$$

Therefore, the solid has a volume of $4\pi$ cubic units.

**Theorem 4.2 (The Washer Method)**

Let $f(x)$ and $g(x)$ be nonnegative continuous functions such that for all $x \in [a, b]$, $g(x) \leq f(x)$. Let $R$ be the plane region bounded by the graphs of $y = f(x)$ and $y = g(x)$, and the lines $x = a$ and $x = b$. Suppose we revolve $R$ about the $x$-axis. The volume of the resulting solid of revolution is given by

$$\pi \int_a^b [f^2(x) - g^2(x)]dx.$$

*Proof:*

Let us visualize the $R$ using *Figure 4.3a*. When we revolve $R$ about the $x$-axis, the resulting solid of revolution is shown in *Figure 4.3b*.

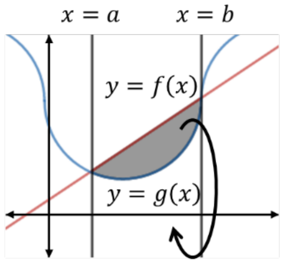



*Figure 4.3a*

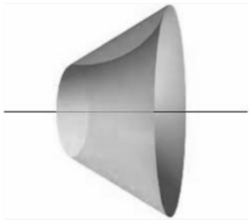


*Figure 4.3b*

Let $R_1$ be the region bounded by $y = f(x)$, the $x$-axis, and the lines $x = a$ and $x = b$. Let $R_2$ be the region bounded by $y = g(x)$, the $x$-axis, and the lines $x = a$ and $x = b$. When revolved about the $x$-axis, by *Theorem 4.3*, the solids generated will have the volume

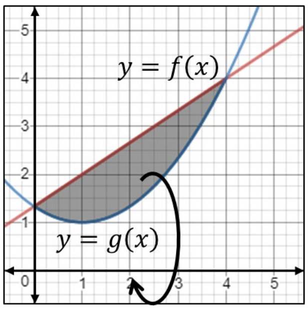
$$\pi \int_a^b f^2(x)dx \quad \text{and} \quad \pi \int_a^b g^2(x)dx.$$

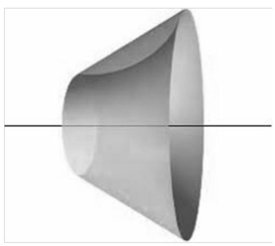
When the solid obtained from $R_2$ is removed from the solid obtained from $R_1$, the solid in *Figure 4.3b* is therefore formed. Thus, its volume is

$$\begin{aligned} V &= \pi \int_a^b f^2(x)dx - \pi \int_a^b g^2(x)dx \\ &= \pi \left[ \int_a^b f^2(x)dx - \int_a^b g^2(x)dx \right] \\ &= \pi \int_a^b [f^2(x) - g^2(x)]dx. \end{aligned}$$

*Example 4.6*

Let $f(x) = \frac{2}{3}(x + 2)$ and $g(x) = \frac{x^2 - 2x + 4}{3}$. Find the volume of the solid generated by revolving about the $x$-axis the region bounded by the parabola $y = f(x)$ and the line $y = g(x)$. To which real number does it



*Solution:*

To obtain the limits of integration, we locate the value/s of $x$ where $f(x) = g(x)$, that is,

$$\begin{gathered} \frac{x^2 - 2x + 4}{3} = \frac{2}{3}(x + 2) \\ x^2 - 2x + 4 = 2(x + 2) \\ x^2 - 2x + 4 = 2x + 4 \\ x^2 - 4x = 0 \\ x(x - 4) = 0. \end{gathered}$$

This implies that $x = 0$ and $x - 4 = 0$ or $x = 4$, which can be verified by the graphs of $f$ and $g$. Therefore, the volume of the solid obtained is

$$V = \pi \int_0^4 [f^2(x) - g^2(x)]dx$$

$$= \pi \int_0^4 \left[\frac{4}{9}(x+2)^2 - \frac{(x^2-2x+4)^2}{9}\right] dx$$
$$= \frac{\pi}{9}\int_0^4 (-x^4 + 4x^3 - 8x^2 + 32x)dx$$
$$= \frac{\pi}{9}\left(-\frac{1}{5}x^5 + x^4 - \frac{8}{3}x^3 + 16x^2\right)\Big|_0^4$$
$$= \frac{2{,}048}{135}\pi.$$

Therefore, the volume of the solid is $\frac{2{,}048}{135}\pi$ cubic units.

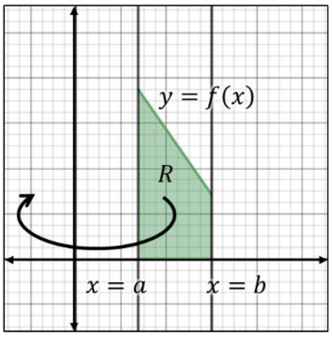
**Theorem 4.3 (The Shell Method)**

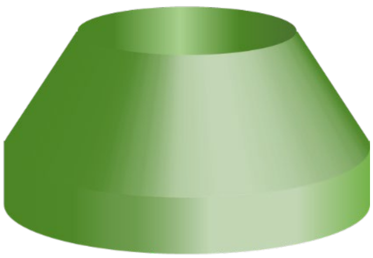
Let $f(x)$ be a nonnegative continuous function. Let $R$ be the plane region bounded by the graphs of $y = f(x)$, the $x$-axis, and the lines $x = a$ and $x = b$ with $a \leq b$. Suppose we revolve $R$ about the $y$-axis. The volume of the resulting solid of revolution is given by

$$2\pi \int_a^b x f(x) dx.$$

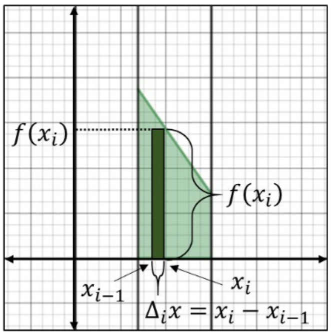
*Proof:*

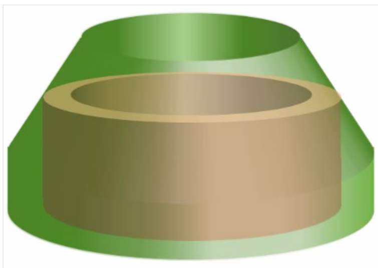
Consider the similar way of visualization below.



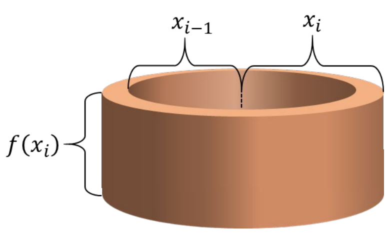



Consider the $i^{\text{th}}$ rectangular strip perpendicular to the $x$-axis. Along with the revolution about the $y$-axis, a corresponding cylindrical shell is formed from this rectangular strip.



The cylindrical shell has inner radius $x_{i-1}$, outer radius $x_i$, and height $f(x_i)$.



Thus, the volume of the $i^{\text{th}}$ cylindrical shell is

$$\begin{aligned} V_i &= \pi x_i^2 f(x_i) - \pi x_{i-1}^2 f(x_i) \\ &= \pi f(x_i)\left(x_i^2 - x_{i-1}^2\right) \\ &= \pi f(x_i)(x_i - x_{i-1})(x_i + x_{i-1}). \end{aligned}$$

Let $m_i = \frac{x_i + x_{i-1}}{2}$, which is the midpoint of the interval $[x_{i-1}, x_i]$. We see that $2m = x_i + x_{i-1}$ and since $x_i - x_{i-1} = \Delta_i x$,

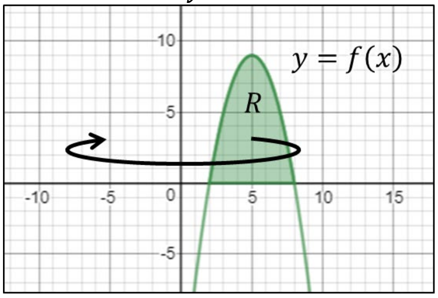
$$V_i = 2\pi m_i f(x_i)\Delta_i x.$$

By the concept of the Riemann sums, the volume of the solid is

$$V = \lim_{n\to+\infty} \sum_{i=1}^{n} 2\pi m_i f(x_i)\Delta_i x = 2\pi \int_a^b x f(x)\,dx.$$

*Example 4.7*

Let $f(x) = -x^2 + 10x - 16$. Find the volume of the solid obtained when the region bounded above by $y = f(x)$ and below by the $x$-axis is revolved about the $y$-axis.




*Solution:*

First, we find the limits of integration by finding the values of $x$ where $y = f(x)$ and $y = 0$ intersect, that is,

$$-x^2 + 10x - 16 = 0.$$

By the quadratic formula,

$$\begin{aligned} x &= \frac{-10 \pm \sqrt{10^2 - 4(-1)(-16)}}{2(-1)} \\ &= -\frac{-10 \pm \sqrt{36}}{2} \\ &= \frac{10 \pm 6}{2} \\ &= 5 \pm 3. \end{aligned}$$

Thus, the limits of integration are $5 - 3 = 2$ and $5 + 3 = 8$. Hence, by the shell method, the volume $V$ of the solid is

$$\begin{aligned} V &= 2\pi \int_2^8 x f(x)\,dx \\ &= 2\pi \int_2^8 (-x^3 + 10x^2 - 16x)\,dx \\ &= 2\pi \left(-\frac{1}{4}x^4 + \frac{10}{3}x^3 - 8x^2\right)\Bigg|_2^8 \\ &= 2\pi \left[\left(-\frac{8^4}{4} + \frac{10(8^3)}{3} - 8(8^2)\right) - \left(-\frac{2^4}{4} + \frac{10(2^3)}{3} - 8(2^2)\right)\right] \\ &= 360\pi. \end{aligned}$$

Therefore, the solid is $360\pi$ cubic units in volume.

## PHYSICS (KINEMATICS)

An application of calculus in physics, particularly in kinematics, comes in the study of **rectilinear motion**, which is the motion of a particle on a line. Recall that if a particle moves along a coordinate line and its distance traveled at time $t$ is given by $s(t)$, then its velocity and acceleration functions are

$$v(t) = s'(t) \text{ and } a(t) = v'(t),$$

respectively. Hence, it follows that

$$s(t) = \int v(t)dt \text{ and } v(t) = \int a(t)dt.$$

Note that when acceleration is negative, this means that the particle is decelerating (slowing down).

*Example 4.8*

Determine the distance function (in inches) of a particle moving along a straight line with velocity $v(t) = 4t + 1$ at $t$ seconds given that after a second, the particle moved a distance of 5 inches.

*Solution:*

Let $s(t)$ be the distance function at time $t$. Then

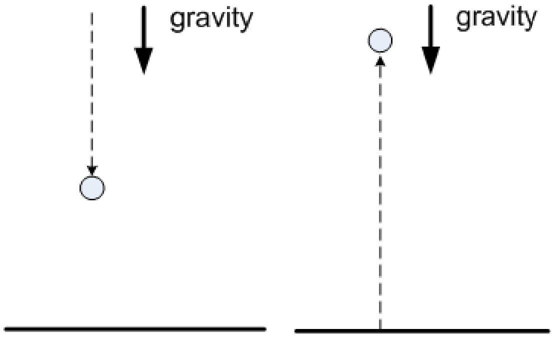
$$s(t) = \int (4t + 1)dt = 2t^2 + t + C.$$

Solving for $C$, since $s(1) = 5$, then

$$2(1^2) + 1 + C = 5$$
$$3 + C = 5$$
$$C = 2.$$

Therefore, $s(t) = 2t^2 + t + 2$.

Motion along a vertical line can also be considered. A falling object is subject to the force of the earth's gravity and speeds up with a constant acceleration of $9.8 \text{ m/s}^2$ or $32 \text{ ft/s}^2$. On the other hand, an object thrown directly upward from a certain height slows down because of the pull.




*Example 4.9*

A ball is thrown vertically upward from the ground with an initial velocity of $49 \text{ m/s}$. Assuming that gravity is the only force acting on the body, find the maximum height attained by the ball.

*Solution:*

Let $s(t)$ be the distance of the ball from the ground $t$ seconds after being thrown vertically. Let $v(t)$ be the velocity of the ball at $t$ seconds. Since the acceleration of the object due to gravity is $a(t) = -9.8 \text{ m/s}^2$, the velocity at time $t$ is given by

$$v(t) = \int a(t)dt = -9.8 \int dt = -9.8t + C_1.$$

To find the constant $C_1$, we use the fact that the initial velocity is $49 \text{ m/s}$. Hence,

$$v(0) = -9.8(0) + C_1 = C_1 = 49.$$

Therefore, $v(t) = -9.8t + 49$. Thus, the distance function is

$$s(t) = \int v(t)dt = -4.9t^2 + 49t + C_2.$$

To find the constant $C_2$, we use the fact that the object is thrown from the ground. Hence, at $t = 0$, its distance from the ground is 0, that is,

$$s(0) = -4.9(0^2) + 49(0) + C_2 = C_2 = 0.$$

Therefore, $s(t) = -4.9t^2 + 49t$. The ball reaches maximum height when $v(t) = 0$, that is

$$-9.8t + 49 = 0$$
$$-9.8t = -49$$
$$t = 5.$$

At $t = 5$, the height is

$$s(5) = -4.9(5^2) + 49(5) = 122.5.$$

Therefore, the ball attains a maximum height of $122.5$ m.

*Example 4.10*

A stone is dropped from a height of $400$ ft. If the stone is initially at rest and assuming gravity is the only force acting on the stone, how long will it take the stone to reach the ground and at what speed will it hit the ground?

*Solution:*

Since the gravity is the only force acting on the stone, then $a(t) = 32$ ft/s$^2$. Therefore, velocity (ft/s) at time $t$ is

$$v(t) = \int a(t)dt = 32 \int dt = 32t + C_1.$$

Solving for the constant $C_1$, since the stone is initially at rest, then

$$v(0) = 32(0) + C_1 = C_1 = 0.$$

Hence, $v(t) = 32t$. Therefore, the distance (ft) travelled after $t$ seconds is

$$s(t) = \int v(t)dt = 32 \int tdt = 16t^2 + C_2.$$

Solving for $C_2$,

$$s(0) = 16(0^2) + C_2 = C_2 = 0.$$

Therefore, $s(t) = 16t^2$. The stone hits the ground at $s(t) = 400$, that is,

$$16t^2 = 400$$
$$t^2 = 25$$
$$t = 5.$$

Hence, the stone hits the ground after $5$ seconds. At $t = 5$, velocity is

$$v(5) = 32(5) = 160,$$

that is, the speed of impact is $160$ ft/s.

**REFERENCES**

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