# Lecture 11
# Type Systems

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# Module II

**Behavioral constraints**

assertions
types
pre/post-conditions

+ bounded guarantees
+ unbounded guarantees

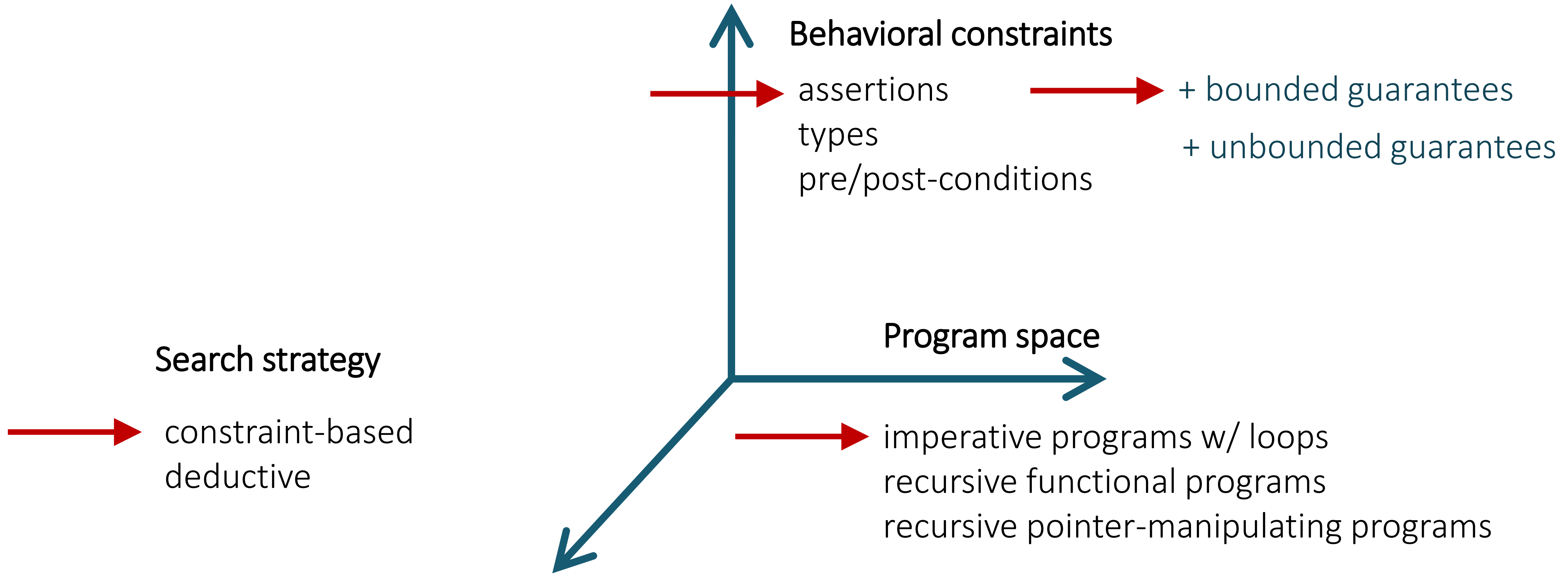
**Program space**

imperative programs w/ loops
recursive functional programs
recursive pointer-manipulating programs

**Search strategy**

constraint-based
deductive

# Last week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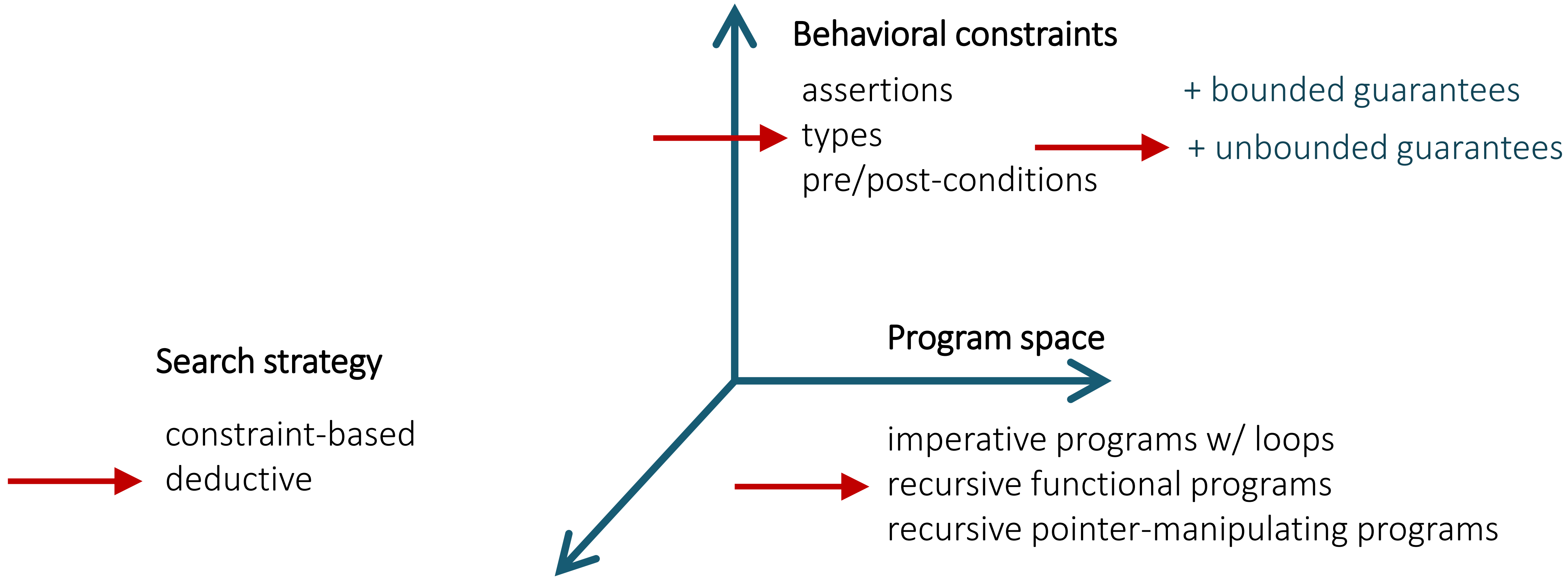
# This week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# Example: insert into a sorted list

Input:

x | 5
---|---

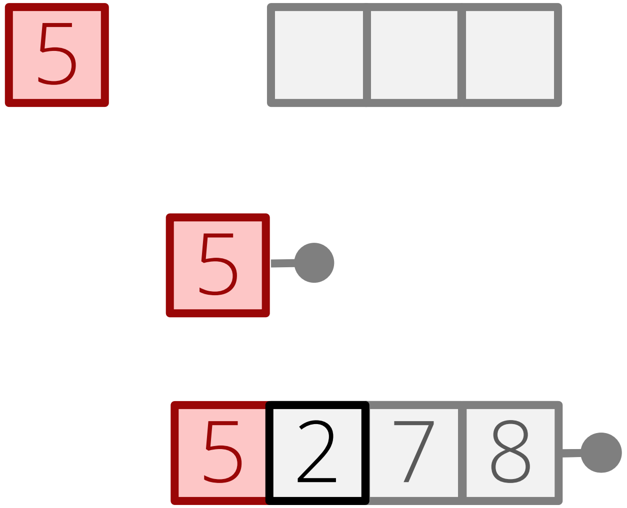
| xs | 1 | 2 | 7 | 8 |
|---|---|---|---|---|

Output:

| ys | 1 | 2 | 5 | 7 | 8 |
|---|---|---|---|---|---|

# In a functional language

```
insert x xs =
  match xs with
    Nil →
      Cons x Nil
    Cons h t →
      if x ≤ h
        then Cons x xs
        else Cons h (insert x t)
```



# Specification for insert

Input:

x

xs: sorted list

Output:

ys: sorted list

elems ys = elems xs ∪ {x}

How can I express this formally?

Types!

How can I verify this for all inputs?

Type checking!

# Agenda

Today:

- Simple types and how to check them
- Refinement types and how to check them

Thursday:

- Specification for insert as a refinement type
- How to use refinement type checking for synthesis?

# What is a type system?

Formalization of a typing discipline of a language

- independently of a particular type checking algorithm (more or less)
- if a type checking algorithm exists, type system is *decidable*

Deductive system for proving facts about programs and types

- defined using *inference rules* over *judgments*

environment / context
(declares free variables of $\mathfrak{I}$)

$$\Gamma \vdash \mathfrak{I}$$

assertion
for example:

typically:

$$x_1 : T_1, \dots, x_n : T_n$$

| | |
|---|---|
| $e :: T$ | “e has type T” |
| $T$ | “T is well-formed” |
| $T' <: T$ | “T’ is a subtype of T” |

# Simple type system

$$e \ ::= \text{true} \mid \text{false} \mid n \mid e + e$$

Syntax of terms (programs)

$$T \ ::= \text{Bool} \mid \text{Int}$$

Syntax of types

Inference Rules

$$\text{T-true} \ \frac{}{\Gamma \vdash \text{true} :: \text{Bool}} \qquad \text{T-false} \ \frac{}{\Gamma \vdash \text{false} :: \text{Bool}} \qquad \text{T-num} \ \frac{(n = 0, 1, \dots)}{\Gamma \vdash n :: \text{Int}}$$

label $\longrightarrow$

$$\text{T-plus} \ \frac{\Gamma \vdash e_1 :: \text{Int} \qquad \Gamma \vdash e_2 :: \text{Int}}{\Gamma \vdash e_1 + e_2 :: \text{Int}}$$

$\longleftarrow$ premises

$\longleftarrow$ conclusion

# Type derivations

$\emptyset \vdash 1 + 2 :: \mathrm{Int}$ is a valid judgment, because....

$$\dfrac{\text{T-num}\ \dfrac{}{\emptyset \vdash 1 :: \mathrm{Int}} \qquad \text{T-num}\ \dfrac{}{\emptyset \vdash 2 :: \mathrm{Int}}}{\emptyset \vdash 1 + 2 :: \mathrm{Int}}\ \text{T-plus}$$

We say that $1 + 2$ is *well-typed* (and has type $\mathrm{Int}$)

# Type derivations

$\emptyset \vdash 1 + \mathrm{true} :: \mathrm{Int}$ is not a valid judgment, because....

$$\text{T-plus}\ \dfrac{\text{T-num}\ \dfrac{}{\emptyset \vdash 1 :: \mathrm{Int}} \qquad \emptyset \vdash \mathrm{true} :: \mathrm{Int}}{\emptyset \vdash 1 + \mathrm{true} :: \mathrm{Int}}$$

We say that $1 + \mathrm{true}$ is *ill-typed* (or *not typable*)

# Type checking vs inference

The problem of discovering the derivation of $\Gamma \vdash e :: T$ is called *type checking*

The problem of discovering the type $T$ such that there exists a derivation of $\Gamma \vdash e :: T$ is called *type inference*

If we have a mechanism for inference, we can also do checking

- How?
- But: can also leverage top-level type to make checking more efficient

# Function types

$$e \ ::= \text{true} \mid \text{false} \mid n \mid e + e$$

Syntax of terms (programs)

$$\mid x \mid e\ e \mid \lambda x.\, e$$

(variable, application, lambda abstraction)

$$T \ ::= \text{Bool} \mid \text{Int}$$

(basic types)

Syntax of types

$$\mid T_1 \to T_2$$

(function types)

T-var $$\frac{(x : T \in \Gamma)}{\Gamma \vdash x :: T}$$

T-abs $$\frac{\Gamma; x : T \vdash e :: T'}{\Gamma \vdash \lambda x.\, e :: T \to T'}$$

T-app $$\frac{\Gamma \vdash e_1 :: T \to T' \qquad \Gamma \vdash e_2 :: T}{\Gamma \vdash e_1\ e_2 :: T'}$$

# Exercise 1

Infer the type of $\lambda x.\, inc\ x$ in $\Gamma = [inc\colon \mathrm{Int} \to \mathrm{Int}]$ using the rules

T-num $\dfrac{(n = 0, 1, \ldots)}{\Gamma \vdash n :: \mathrm{Int}}$

T-plus $\dfrac{\Gamma \vdash e_1 :: \mathrm{Int} \qquad \Gamma \vdash e_2 :: \mathrm{Int}}{\Gamma \vdash e_1 + e_2 :: \mathrm{Int}}$

T-var $\dfrac{(x\colon T \in \Gamma)}{\Gamma \vdash x :: T}$

T-abs $\dfrac{\Gamma; x\colon T \vdash e :: T'}{\Gamma \vdash \lambda x\colon T.\, e :: T \to T'}$

T-app $\dfrac{\Gamma \vdash e_1 :: T \to T' \quad \Gamma \vdash e_2 :: T}{\Gamma \vdash e_1\ e_2 :: T'}$

# Inference algorithm

**Goal:** compute the type of term from the types of subterms

$$\dfrac{\text{T-var}\ \dfrac{}{\Gamma \vdash \mathit{inc}\ ::} \qquad \dfrac{}{\Gamma \vdash 1\ ::}\ \text{T-num}}{\Gamma \vdash \mathit{inc}\ 1\ ::}\ \text{T-app}$$

$$\Gamma = [\mathit{inc} \colon \mathrm{Int} \to \mathrm{Int}]$$

# Inference algorithm

**Problem:** to compute the types of this term, we had to *guess* the type of x:

$$\dfrac{\dfrac{}{\Gamma, x : ? \vdash inc ::}\,\text{T-var} \qquad \dfrac{}{\Gamma, x : ? \vdash x ::}\,\text{T-var}}{\dfrac{\Gamma, x : ? \vdash inc\ x ::}{\Gamma \vdash \lambda x.\, inc\ x ::}\,\text{T-abs}}\,\text{T-app}$$

**Solution:** constraint-based type inference

- aka Hindley-Milner type inference

# Constraint-based type inference

[Hindley'69][Milner'78]

**Idea:** separate inference into constraint generation and constraint solving

1. Whenever you need to guess a type, generate a *type variable*
2. Whenever two types must match, generate a *unification constraint*
3. Solve unification constraints to assign types to type variables

$$\Gamma \vdash \lambda x.\, inc\ x :: ?$$

# Example

Type derivation

$$\dfrac{\dfrac{}{\Gamma, x{:}\,\alpha \vdash inc :: \mathrm{Int} \to \mathrm{Int}}\ \text{T-var} \qquad \dfrac{}{\Gamma, x{:}\,\alpha \vdash x :: \alpha}\ \text{T-var}}{\dfrac{\Gamma, x{:}\,\alpha \vdash inc\ x :: \mathrm{Int}}{\Gamma \vdash \lambda x.\,inc\ x :: \alpha \to \mathrm{Int}}\ \text{T-abs}}\ \text{T-app}$$

Type assignment

$$\alpha \ \to \ \mathrm{Int}$$

Unification constraints

$$\alpha \sim \mathrm{Int}$$

$$\Gamma = [inc{:}\,\mathrm{Int} \to \mathrm{Int}]$$

# What about type checking?

Type derivation

$$
\dfrac{\dfrac{\overline{\Gamma, x{:}\,\alpha \vdash inc :: \mathrm{Int} \to \mathrm{Int}}\;\text{T-var} \qquad \overline{\Gamma, x{:}\,\alpha \vdash x :: \alpha}\;\text{T-var}}{\Gamma, x{:}\,\alpha \vdash inc\ x :: \mathrm{Int}}\;\text{T-app}}{\Gamma \vdash \lambda x.\, inc\ x :: \alpha \to \mathrm{Int}}\;\text{T-abs}
$$

$$\mathrm{Int} \to \mathrm{Int}$$

Type assignment

$$\alpha \;\to\; \mathrm{Int}$$

Unification constraints

$$\alpha \sim \mathrm{Int}$$

$$\alpha \to \mathrm{Int} \sim \mathrm{Int} \to \mathrm{Int}$$

$$\Gamma = [inc{:}\,\mathrm{Int} \to \mathrm{Int}]$$

# Can we do better?

Type derivation

$$\dfrac{\dfrac{\overline{\Gamma, x: \mathrm{Int} \vdash inc :: \mathrm{Int} \to \mathrm{Int}}\ \text{T-var} \qquad \overline{\Gamma, x: \mathrm{Int} \vdash x :: \mathrm{Int}}\ \text{T-var}}{\Gamma, x: \mathrm{Int} \vdash inc\ x :: \mathrm{Int}}\ \text{T-app}}{\Gamma \vdash \lambda x. inc\ x :: \mathrm{Int} \to \mathrm{Int}}\ \text{T-abs}$$

Type assignment

Unification constraints

$$\mathrm{Int} \sim \mathrm{Int}$$

$$\Gamma = [inc: \mathrm{Int} \to \mathrm{Int}]$$

# Bidirectional type-system

[Pierce, Turner'00]

Rules differentiate between type inference and checking

$$\Gamma \vdash e \uparrow T \qquad\qquad \Gamma \vdash e \downarrow T$$

"$e$ generates $T$ in $\Gamma$" "$e$ checks against $T$ in $\Gamma$"

I-var $\dfrac{(x : T \in \Gamma)}{\Gamma \vdash x \uparrow T}$ C-abs $\dfrac{\Gamma; x : T_1 \vdash e \downarrow T_2}{\Gamma \vdash \lambda x.\, e \downarrow T_1 \to T_2}$ C-I $\dfrac{\Gamma \vdash e \uparrow T' \quad \Gamma \vdash T \sim T'}{\Gamma \vdash e \downarrow T}$

# Polymorphism (aka “generics”)

$$
\begin{aligned}
e \;&::= \text{true} \mid \text{false} \mid n \mid e + e \\
&\quad\mid x \mid e\ e \mid \lambda x.\, e
\end{aligned}
$$

Terms

$$
\begin{aligned}
T \;&::= \text{Bool} \mid \text{Int} && \text{(basic types)} \\
&\quad\mid T_1 \to T_2 && \text{(function types)} \\
&\quad\mid \alpha && \text{(type variables)}
\end{aligned}
$$

Types

$$S \;::= T \mid \forall \alpha.\, S$$

Type schemas

T-gen
$$\frac{\Gamma; \alpha \vdash e :: S}{\Gamma \vdash e :: \forall \alpha.\, S}$$

T-inst
$$\frac{\Gamma \vdash e :: \forall \alpha.\, S \qquad \Gamma \vdash T}{\Gamma \vdash e :: S[\alpha \mapsto T]}$$

# Exercise 3

Let's infer the type of $id\ 5$ in $\Gamma$
where $\Gamma = [\mathrm{id} : \forall\alpha.\alpha \ \to \alpha]$

using the following rules:

T-num $$\frac{(n = 0, 1, \ldots)}{\Gamma \vdash n :: \mathrm{Int}}$$

T-var $$\frac{(x{:}\,T \in \Gamma)}{\Gamma \vdash x :: T}$$

T-app $$\frac{\Gamma \vdash e_1 :: T \to T' \qquad \Gamma \vdash e_2 :: T}{\Gamma \vdash e_1\ e_2 :: T'}$$

T-gen $$\frac{\Gamma; \alpha \vdash e :: S}{\Gamma \vdash\ e :: \forall\alpha.S}$$

T-inst $$\frac{\Gamma \vdash e :: \forall\alpha.S \qquad \Gamma \vdash T}{\Gamma \vdash\ e :: S[\alpha \mapsto T]}$$

# Agenda

Today:

- Simple types and how to check them
- Refinement types and how to check them
- Specification for insert as a refinement type

Thursday:

- How to use refinement type checking for synthesis?

# Types as specifications

```
insert :: ∀ a . a → List a → List a
```

# Conventional types are not enough

```
// Insert x into a sorted list xs
insert :: x:a  →  xs:List a  →  List a
insert x xs =
  match xs with
    Nil → Nil
    Cons h t →
      if x ≤ h
        then Cons x xs
        else Cons h (insert x t)
```

# Refinement types

[Rondon et al.'08]

```
Nat
```

base types

```
max :: x: Int → y: Int → { v: Int | x ≤ v ∧ y ≤ v }
```

dependent function types

```
xs :: { v: List Nat }
```

polymorphic datatypes

```
data List α where
    Nil  ::  { List α | len v = 0 }
    Cons  ::  x: α  → { List α | len v = len
                       xs + 1 }
```

```
measure len :: List α → Int
    len Nil = 0
    len (Cons _ xs) = len xs + 1
```

# Refinement types

$$e \ ::= \text{true} \mid \text{false} \mid n \mid e + e \\ \mid x \mid e\ e \mid \lambda x.\, e$$

Terms

$$T \ ::= \{\nu : \mathrm{B} \mid \mathrm{e}\}$$ (basic types)

$$\mid x : T_1 \to T_2$$ (function types)

$$\mid \alpha$$ (type variables)

Types

$$S \ ::= T \mid \forall \alpha.\, S$$

Type schemas

T-num $$\frac{(n = 0, 1, \dots)}{\Gamma \vdash n :: \{\nu : \mathrm{Int} \mid \nu = n\}}$$

T-var $$\frac{(x : T \in \Gamma)}{\Gamma \vdash x :: \{\nu : T \mid \nu = x\}}$$

T-app $$\frac{\Gamma \vdash e_1 :: x : T \to T' \qquad \Gamma \vdash e_2 :: T}{\Gamma \vdash e_1\ e_2 :: T'[x \mapsto e_2]}$$

# Example

Let's check that $\Gamma \vdash \mathrm{inc}\ 5 :: \mathrm{Nat}$

- $\mathrm{Nat} = \{v{:}\,\mathrm{Int} \mid v \geq 0\}$
- $\Gamma = [\mathrm{inc}{:}\, y{:}\,\mathrm{Int} \to \{v{:}\,\mathrm{Int} \mid v = y + 1\}]$

T-num $$\frac{(n = 0, 1, \dots)}{\Gamma \vdash n :: \{v{:}\,\mathrm{Int} \mid v = n\}}$$

T-abs $$\frac{\Gamma; x{:}\,T \vdash e :: T'}{\Gamma \vdash \lambda x{:}\,T.\, e :: T \to T'}$$

T-var $$\frac{(x{:}\,T \in \Gamma)}{\Gamma \vdash x :: \{v{:}\,T \mid v = x\}}$$

T-app $$\frac{\Gamma \vdash e_1 :: x{:}\,T \to T' \qquad \Gamma \vdash e_2 :: T}{\Gamma \vdash e_1\ e_2 :: T'[x \mapsto e_2]}$$

We need subtyping!

# Subtyping

Intuitively, $T'$ is a subtype of $T$ if all values of type $T'$ also belong to $T$

- written $T' <: T$
- e.g. $\mathrm{Nat} <: \mathrm{Int}$ or $\{\nu: \mathrm{Int} \mid \nu = 5\} <: \mathrm{Nat}$

Defined via inference rules:

$$\text{Sub-base}\ \frac{[\![\Gamma]\!] \land e' \Rightarrow e}{\Gamma \vdash \{\nu : B \mid e'\} <: \{\nu : B \mid e\}}$$

$$\text{Sub-fun}\ \frac{\Gamma \vdash T_1 <: T_1' \qquad \Gamma; x: T_1 \vdash T_2' <: T_2}{\Gamma \vdash x: T'_1 \to T_2' <: x: T_1 \to T_2}$$

# Refinement type inference

[Rondon et al.'08]

**Idea:** separate inference into (subtyping) constraint generation and (subtyping) constraint solving

1. Whenever you need to guess a type, generate a *type variable*
2. Whenever two types must match, generate a *subtyping constraint*
3. Solve subtyping constraints to assign refined types to type variables

$$\Gamma \vdash \lambda x.\, \mathrm{inc}\; x :: \mathrm{Nat} \rightarrow \mathrm{Nat}$$

# Example

Type derivation

$$\dfrac{\dfrac{\dfrac{}{\Gamma, x{:}\,\alpha \vdash inc :: \{y{:}\,\text{Int} \rightarrow \{\nu{:}\,\text{Int} \mid \nu = y+1\}} \;\text{T-var} \qquad \dfrac{}{\Gamma, x{:}\,\alpha \vdash x :: \alpha}\;\text{T-var}}{\Gamma, x{:}\,\alpha \vdash inc\ x :: \{\nu{:}\,\text{Int} \mid \nu = x+1\}}\;\text{T-app}}{\Gamma \vdash \lambda x.\, inc\ x :: \text{x:}\alpha \rightarrow \{\nu{:}\,\text{Int} \mid \nu = x+1\}}\;\text{T-abs}$$

$$\text{Nat} \rightarrow \text{Nat}$$

Type assignment

$$\alpha \rightarrow \{\nu{:}\,\text{Int} \mid P\}$$
$$P \rightarrow true$$

Horn clauses

$$P \Rightarrow true$$
$$\nu \geq 0 \Rightarrow P$$
$$x \geq 0 \wedge \nu = x + 1 \Rightarrow \nu \geq 0$$

Subtyping constraints

$$\alpha <: \text{Int}$$
$$\cancel{\text{x:}\alpha \rightarrow \{\nu{:}\,\text{Int} \mid \nu = x+1\} <: \text{Nat} \rightarrow \text{Nat}}$$
$$\text{Nat} <: \alpha$$
$$x{:}\,\text{Nat} \vdash \{\nu{:}\,\text{Int} \mid \nu = x+1\} <: \text{Nat}$$

$$\Gamma = [\text{inc}{:}\, y{:}\,\text{Int} \rightarrow \{\nu{:}\,\text{Int} \mid \nu = y+1\}]$$

# Bidirectional type-checking

[Polikarpova et al.'16]

Type derivation

$$
\dfrac{\dfrac{\dfrac{}{\Gamma, x\!:\mathrm{Nat} \vdash inc :: y\!:\mathrm{Int} \to \{\nu\!:\mathrm{Int} \mid \nu = y + 1\}}\;\text{T-var} \qquad \dfrac{}{\Gamma, x\!:\mathrm{Nat} \vdash x :: \mathrm{Nat}}\;\text{T-var}}{\Gamma, x\!:\mathrm{Nat} \vdash inc\ x :: \mathrm{Nat}}\;\text{T-app}}{\Gamma \vdash \lambda x.\, inc\ x :: \mathrm{Nat} \to \mathrm{Nat}}\;\text{T-abs}
$$

Horn clauses

$$\nu \geq 0 \Rightarrow true$$

$$x \geq 0 \wedge \nu = x + 1 \Rightarrow \nu \geq 0$$

Subtyping constraints

$$\mathrm{Nat} <: \mathrm{Int}$$

$$x\!:\mathrm{Nat} \vdash \{\nu\!:\mathrm{Int} \mid \nu = x + 1\} <: \mathrm{Nat}$$

$$\Gamma = [\mathrm{inc}\!: y\!:\mathrm{Int} \to \{\nu\!:\mathrm{Int} \mid \nu = y + 1\}]$$

# Recursion

$$
\begin{aligned}
e \;::=&\; \text{true} \mid \text{false} \mid n \mid e + e \\
&\mid x \mid e\ e \mid \lambda x.\, e \mid \text{fix}\ f.\, e
\end{aligned}
$$

Terms

$$
\begin{aligned}
T \;::=&\; \{\mathrm{v}\colon \mathrm{B} \mid \mathrm{e}\} && \text{(basic types)} \\
&\mid x\colon T_1 \to T_2 && \text{(function types)} \\
&\mid \alpha && \text{(type variables)}
\end{aligned}
$$

Types

$$S \;::=\; T \mid \forall \alpha.\, S$$

Type schemas

T-fix
$$\frac{\Gamma, f\colon S \vdash e :: S}{\Gamma \vdash \text{fix}\ f.\, e :: S}$$

# Example: factorial

$$\text{fix } f.\, \lambda n.\, \text{if } n \leq 1 \text{ then } 1 \text{ else } n * (f\ (n-1))$$

$$\text{T-app}\ \frac{\ldots \vdash f :: \text{Nat} \to \text{Nat} \qquad \ldots \vdash n - 1 :: \{\text{Int} \mid \nu = n - 1\}}{\ldots \vdash f\ (n - 1) :: \text{Nat}}$$

$$\ldots$$

$$\text{T-abs}\ \dfrac{\text{T-if}\ \dfrac{\ldots \vdash n \leq 1 :: \text{Bool} \qquad \ldots \vdash 1 :: \text{Nat} \qquad \ldots, n > 1 \vdash n * \big(f\ (n-1)\big) :: \text{Nat}}{f : \text{Nat} \to \text{Nat}, n : \text{Nat} \vdash \text{if } n \leq 1 \text{ then } 1 \text{ else } n * \big(f\ (n-1)\big) :: \text{Nat}}}{\text{T-fix}\ \dfrac{f : \text{Nat} \to \text{Nat} \vdash \lambda n \ldots :: \text{Nat} \to \text{Nat}}{\emptyset \vdash \text{fix } f.\, \lambda n \ldots :: \text{Nat} \to \text{Nat}}}$$

$$n > 1 \vdash \{\text{Int} \mid \nu = n - 1\} <: \text{Nat}$$