Lecture 6
Introduction to SAT and SMT

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Logistics

Project proposals

- Due Friday (Oct 20) by the end of the day
- Upload to the Proposals directory inside the shared Google folder
- Can be a Google Doc or a PDF
- File name must be "Team-N", where N is your team ID

Why do we care?

1. Synthesis is combinatorial search, and so is SAT
2. SAT solvers are really good these days
3. ??? ← this week
4. Profit!!!

The SAT problem

Input: propositional formula in CNF

literal
= atom or its negation

formula
= conjunction (set) of clauses

$$\boxed{(\neg \boxed{a} \vee \boxed{\neg b}) \wedge (\boxed{b} \vee c) \wedge \boxed{(\neg a \vee \neg c \vee d)} \wedge (b \vee \neg c \vee \neg d) \wedge (a \vee d)}$$

atom
= propositional variable

clause
= disjunction of literals

The SAT problem

Problem: find a *satisfying assignment* (also called a *model*)

- or determine that the formula is *unsatisfiable*

$$(\neg a \vee \neg b) \wedge (b \vee c) \wedge (\neg a \vee \neg c \vee d) \wedge (b \vee \neg c \vee \neg d) \wedge (a \vee d)$$

a satisfying assignment:

$\{a \mapsto 0, b \mapsto 1, c \mapsto 0, d \mapsto 1\}$

can be written as a set of literals:

$\{\neg a, b, \neg c, d\}$

or as a formula:

$\neg a \wedge b \wedge \neg c \wedge d$

Naive solution

$$(\neg a \lor \neg b) \land (b \lor c) \land (\neg a \lor \neg c \lor d) \land (b \lor \neg c \lor \neg d) \land (a \lor d)$$

Build a truth table!

- We can’t do fundamentally better: it’s an NP-complete problem
- But we can do way better in practice for common instances

$2^{|P|}$

0000	0
0001	0
0010	0
0011	0
0100	0
0101	1
0110	0
0111	1
…	

Intuition: Sudoku

Easy vs hard: what’s the difference?

7	9					3		
					6	9		
8				3			7	6
			9	6	5			2
		5	4	1	8	7		
4			7	2	3			
6	1			9				8
		2	3					
		9					5	4

		9	7	4	8			
7								
	2		1		9			
		7				2	4	
	6	4		1		5	9	
	9	8				3		
			8		3		2	
								6
			2	7	5	9		

Most real-world SAT instances allow a lot of inference

DPLL algorithm

[Davis, Putnam '60]
[Davis, Logemann, Loveland '62]

State: current model *M* (a sequence of annotated literals)

decision literal

$$M = \boxed{a^d} \neg b\ c$$

Transitions:

- decide $\quad M \longrightarrow M\ l^d \quad$ if *l* undefined in *M*
- unit-propagate $\quad M \longrightarrow M\ l \quad$ if there is a clause where are literals are false except *l*, which is undefined
- backtrack $\quad M l^d M' \longrightarrow M\ \neg l \quad$ if there is a conflicting clause and *M'* has no decision literals
- fail $\quad M \longrightarrow Unsat \quad$ if there is a conflicting clause and no decision literals

DPLL: example

$$(\neg a \lor \neg b) \land (b \lor c) \land (\neg a \lor \neg c \lor d) \land \boxed{(b \lor \neg c \lor \neg d)} \land (a \lor d)$$

$M =$	$\emptyset$	decide
	a^d	unit-propagate
	$a^d\ \neg b$	unit-propagate
	$a^d\ \neg b\ c$	unit-propagate
	$a^d\ \neg b\ c\ d$	backtrack
	$\neg a$	unit-propagate
	$\neg a\ d$	decide
	$\neg a\ d \neg c^d$	unit-propagate
	$\neg a\ d \neg c^d\ b$	SAT!

DPLL + clause learning

$$(\neg a \vee b) \wedge (\neg c \vee d) \wedge (\neg e \vee \neg f) \wedge \boxed{(f \vee \neg e \vee \neg b)}$$

$M =$	$\emptyset$	decide	
	a^d	unit-propagate	
	$a^d\ b$	decide	
	$a^d\ b\ c^d$	unit-propagate	
	$a^d\ b\ c^d\ d$	decide	
	$a^d\ b\ c^d\ d\ e^d$	unit-propagate	
	$a^d\ b\ c^d\ d\ e^d\ \neg f$	backtrack	Wait, but why?
	$a^d\ b\ c^d\ d\ \neg e$		

DPLL + clause learning

$$(\neg a \vee b) \wedge (\neg c \vee d) \wedge (\neg e \vee \neg f) \wedge \boxed{(f \vee \neg e \vee \neg b)} \wedge (\neg a \vee \neg e)$$

$M = \emptyset$	decide	
a^d	unit-propagate	
$a^d\ b$	decide	
$a^d\ b\ c^d$	unit-propagate	
$a^d\ b\ c^d\ d$	decide	
$a^d\ b\ c^d\ d\ e^d$	unit-propagate	
$a^d\ b\ c^d\ d\ e^d\ \neg f$	backtrack	Wait, but why?
$a^d\ b\ c^d\ d\ \neg e$		

Bad decision!

DPLL + clause learning

$$(\neg a \vee b) \wedge (\neg c \vee d) \wedge (\neg e \vee \neg f) \wedge (f \vee \neg e \vee \neg b) \wedge (\neg a \vee \neg e)$$

$M =$	$\emptyset$	decide
	a^d	unit-propagate
	$a^d\ b$	decide
	$a^d\ b\ c^d$	unit-propagate
	$a^d\ b\ c^d\ d$	decide
	$a^d\ b\ c^d\ d\ e^d$	unit-propagate
	$a^d\ b\ c^d\ d\ e^d\ \neg f$	**backjump**
	$a^d\ b\ \neg e$	

Beyond propositional logic

What if our formula looks like this?

$$(p \land \neg q \lor a = f(b - c)) \land (g(g(b) \neq c \lor a - c \leq 7)$$

- talks about integers, functions, sets, lists...

One idea: bit-blast everything and use SAT

- can only find solutions within bounds
- very inefficient, so bounds are small

Better idea: combine SAT with special **solvers** for **theories**

- they “natively understand” integers, functions, etc

First-order theories

theory = <function symbols, predicate symbols, axioms>

ground first-order formulas over functions and predicates

Example: theory of Equality and Uninterpreted Functions

EUF = <{f, g, h, ...}, {=}, {

$$\forall x.\, x = x$$

$$\forall x\, y.\, x = y \;\Rightarrow\; y = x$$

$$\forall x\, y\, z.\, x = y \land y = z \;\Rightarrow\; x = z$$

$$\forall x\, y.\, x = y \;\Rightarrow\; f(x) = f(y)$$

}>

The SMT problem

literal
= atom or its negation

formula
= conjunction (set) of clauses

theory +

$$\boxed{\boxed{g(a) = c} \wedge \boxed{\left(\boxed{\mathrm{f}\big(\mathrm{g}(\mathrm{a})\big) \neq f(c)} \vee \boxed{g(a) = \mathrm{d}}\right)} \wedge c \neq d}$$

atom
= predicate
application

clause
= disjunction of
literals

Theories for our purpose

theory = <function symbols, predicate symbols, ~~axioms~~>

solver

can decide consistency of conjunctions of literals

$$f(a) = c$$
$$f(b) \neq d$$
$$c = d$$
$$a = b$$

$\xrightarrow{\text{EUF solver}}$ Inconsistent!

Basic DPLL(T)

$$g(a) = c \ \wedge \ \big(f(g(a)) \neq f(c) \vee g(a) = d\big) \ \wedge \ c \neq d$$

abstract atoms to propositional variables

$$p \quad \wedge \quad (\neg q \quad \vee \quad r) \quad \wedge \quad \neg s$$

$$p \wedge (\neg q \ \vee \ r) \wedge \neg s \xrightarrow{\text{SAT solver}} p \ \neg q \ \neg s$$

$$\text{Inconsistent!} \xleftarrow{\text{EUF solver}} \boxed{g(a) = c} \quad \boxed{f(g(a)) \neq f(c)} \quad c \neq d$$

$$p \wedge (\neg q \ \vee \ r) \wedge \neg s \ \wedge \ (\neg p \vee q) \xrightarrow{\text{SAT solver}} p \ q \ r \ \neg s$$

$$\text{Inconsistent!} \xleftarrow{\text{EUF solver}} \boxed{g(a) = c} \quad f(g(a)) = f(c) \quad \boxed{g(a) = d} \quad \boxed{c \neq d}$$

$$p \wedge (\neg q \ \vee \ r) \wedge \neg s \ \wedge \ (\neg p \vee q) \wedge (\neg p \wedge \neg r \wedge s) \xrightarrow{\text{SAT solver}} \text{Unsat}$$

DPLL(T) optimizations

Basic

Check consistency of full propositional models

Upon inconsistency, add clause and restart

Check consistency after adding a literal

Advanced

Check consistency of partial assignment while being built

Upon inconsistency, do conflict analysis and backjump

Add a theory-propagate rule to DPLL

- like unit-propagate, but infers all literals that follow from the theory

Popular theories

Equality and Uninterpreted Functions

EUF = <{f, g, h, ...}, {=}, axioms of equality & congruence>

Linear Integer Arithmetic

LIA = <{0, 1, ..., +, -}, {=, ≤}, axioms of arithmetic>

Arrays

Arrays = <{sel, store}, {=}, $\forall a\ i\ v.\ \mathrm{sel}(\mathrm{store}(a, i, v), i) = v$

$\forall a\ i\ j\ v.\ i \neq j \ \Rightarrow\ \mathrm{sel}(\mathrm{store}(a, i, v), j) = \mathrm{sel}(a, j)$ >

Theories can be combined!

Nelson-Oppen combination

Popular SMT solvers

Z3 (Microsoft): https://github.com/Z3Prover/z3/wiki

CVC4 (Stanford): http://cvc4.cs.stanford.edu/web/

Yices (SRI): http://yices.csl.sri.com/

SMT-LIB

Uniform format for SMT problems understood by all solvers

```
(declare-fun x () Int)
(declare-fun y () Int)
(declare-fun z () Int)
(assert (>= (* 2 x) (+ y z)))
(declare-fun f (Int) Int)
(declare-fun g (Int Int) Int)
(assert (< (f x) (g x x)))
(assert (> (f y) (g x x)))
(check-sat)
(get-model)
```

Z3 demo

https://rise4fun.com/Z3/tutorial

Why do we care?

If we can encode a synthesis problem as SAT/SMT, we can use solvers to do the search for us

Get some inspiration from how solvers search

- Unit propagation similar to top-down propagation (pruning through inference of consequences of a guess)
- Backjumping / clause learning?
- Coarse-grained reasoning and gradual refinement like in DPLL(T)?