Lecture 15
Refinement Types and Type-Driven Synthesis

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Motivation

Goal: use deductive reasoning for top-down propagation

- prune unverifiable candidates early
- need synthesis-friendly verification technique!

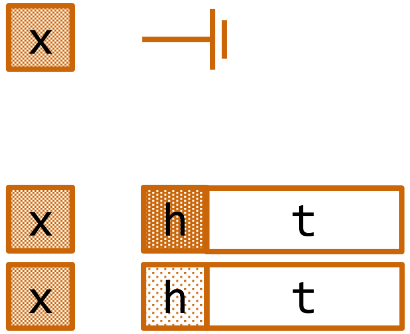
Observation: type checkers are good at rejecting incomplete programs!

```
L
L + L
```

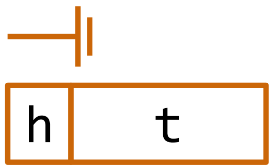
~~[N] + L~~

Running example

```
// Insert x into a sorted list xs
insert :: x:e  →  xs:List e  →  List e
insert x xs =
  match xs with
    Nil →
    Cons h t →
      if x ≤ h
        then Cons x xs
        else Cons h (insert x t)
```




```
data List e where
  Nil :: List e
  Cons :: h:e  →  t:List e  →  List e
```



Rejecting incomplete programs

[Pierce, Turner. TPLS'00]

```
// Insert x into a sorted list xs
insert :: x:e  →  xs:List e  →  List e
insert x xs =
  match xs with
    Nil → Cons xs ...
    ...
```

bidirectional type-checking!

Expected
e
and got
List e

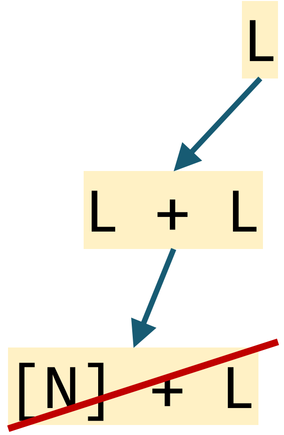
Motivation

Goal: use deductive reasoning for top-down propagation

- prune unverifiable candidates early
- need synthesis-friendly verification technique!

Observation: type checkers are good at rejecting incomplete programs!

Idea: can we use types as behavioral constraints for synthesis?



Conventional types are not enough

```
// Insert x into a sorted list xs
insert :: x:e  →  xs:List e  →  List e
insert x xs =
  match xs with
    Nil → Nil
    Cons h t →
      if x ≤ h
        then Cons x xs
        else Cons h (insert x t)
```

Refinement types

[Rondon et al.'08, Kawaguchi et al.'09]

```
Nat
```

base types

```
max :: x: Int → y: Int → { v: Int | x ≤ v ∧ y ≤ v }
```

dependent function types

```
xs :: { v: List Nat }
```

polymorphic datatypes

```
data List α where
  Nil  :: { List α | len v = 0 }
  Cons :: x: α → { List α | len v = len xs + 1 }
```

```
measure len :: List α → Int
  len Nil = 0
  len (Cons _ xs) = len xs + 1
```

Refinement types

$$e ::= \text{true} \mid \text{false} \mid n \mid e + e \mid x \mid e\ e \mid \lambda x{:}T.\,e$$

Terms

$$T ::= \{\nu{:}\ \mathrm{B} \mid \mathrm{e}\}$$ (basic types)

$$\mid x{:}T_1 \to T_2$$ (function types)

$$\mid \alpha$$ (type variables)

Types

$$S ::= T \mid \forall \alpha.\,S$$

Type schemas

T-num
$$\frac{(n = 0, 1, \dots)}{\Gamma \vdash n :: \{\nu{:}\,\mathrm{Int} \mid \nu = n\}}$$

T-var
$$\frac{(x{:}T \in \Gamma)}{\Gamma \vdash x :: \{\nu{:}\,T \mid \nu = x\}}$$

T-app
$$\frac{\Gamma \vdash e_1 :: x{:}T \to T' \qquad \Gamma \vdash e_2 :: T}{\Gamma \vdash e_1\ e_2 :: T'[x \mapsto e_2]}$$

Example

Let's check that $\Gamma \vdash (\lambda y\colon \text{Int}.\, \text{double } y)\ 5 :: \text{Nat}$

- $\text{Nat} = \{\nu\colon \text{Int} \mid \nu \geq 0\}$
- $\Gamma = [\text{double}\colon\ x\colon \text{Int} \to\ \{\nu\colon \text{Int} \mid \nu = 2 * x\}]$

T-num
$$\frac{(n = 0, 1, \dots)}{\Gamma \vdash n :: \{\nu\colon \text{Int} \mid \nu = n\}}$$

T-var
$$\frac{(x\colon T \in \Gamma)}{\Gamma \vdash x :: \{\nu\colon T \mid \nu = x\}}$$

T-abs
$$\frac{\Gamma; x\colon T \vdash e :: T'}{\Gamma \vdash \lambda x\colon T.\, e :: T \to T'}$$

T-app
$$\frac{\Gamma \vdash e_1 :: x\colon T \to T' \quad \Gamma \vdash e_2 :: T}{\Gamma \vdash e_1\ e_2 :: T'[x \mapsto e_2]}$$

We need subtyping!

Subtyping

Intuitively, T' is a subtype of T if all values of type T' also belong to T

- written $T' <: T$
- e.g. $\mathrm{Nat} <: \mathrm{Int}$ or $\{\nu\colon \mathrm{Int} \mid \nu = 5\} <: \mathrm{Nat}$

Defined via inference rules:

$$\text{Sub-base}\ \frac{[\![\Gamma]\!] \wedge e' \Rightarrow e}{\Gamma \vdash \{v\colon B \mid e'\} <: \{v\colon B \mid e\}}$$

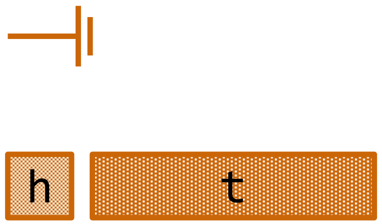
$$\text{Sub-fun}\ \frac{\Gamma \vdash T_1 <: T_1' \qquad \Gamma; x\colon T_1 \vdash T_2' <: T_2}{\Gamma \vdash x\colon T'_1 \to T_2' <: x\colon T_1 \to T_2}$$

Conventional types are not enough

```
// Insert x into a sorted list xs
insert :: x:e  →  xs:List e  →  List e
insert x xs =
  match xs with
    Nil → Nil
    Cons h t →
      if x ≤ h
        then Cons x xs
        else Cons h (insert x t)
```

Refinement types

```
data SList e where          sorted lists
  Nil :: SList e
  Cons :: h:e →
          t:SList {ν:e | ν ≥ h} →
          SList e
```



Refinement types as specs

[Rondon et al. PLDI'08]

```
// Insert x into a sorted list xs
insert :: x:e  →  xs:SList e  →
           {ν:SList e | elems ν = elems xs ∪ {x}}
insert x xs =
  match xs with
    Nil → Nil
    Cons h t →
      if x ≤ h
        then Cons x xs
        else Cons h (insert x t)
```

Expected
{ν:SList e|elems ν = elems xs ∪ {x}}
and got
{ν:SList e|elems xs ⊆ elems ν}

Incomplete programs?

```
// Insert x into a sorted list xs
insert :: x:e  →  xs:SList e  →
            {v:SList e | elems v = elems xs ∪ {x}}
insert x xs =
  match xs with
    Nil → Nil
    Cons h t → ...
```

Bidirectional type checking

```
{ν:SList e | elems ν = {x}}
```

```
insert x xs =
  match xs with
    Nil → Nil
    Cons h t → ...
```

Round-trip type checking

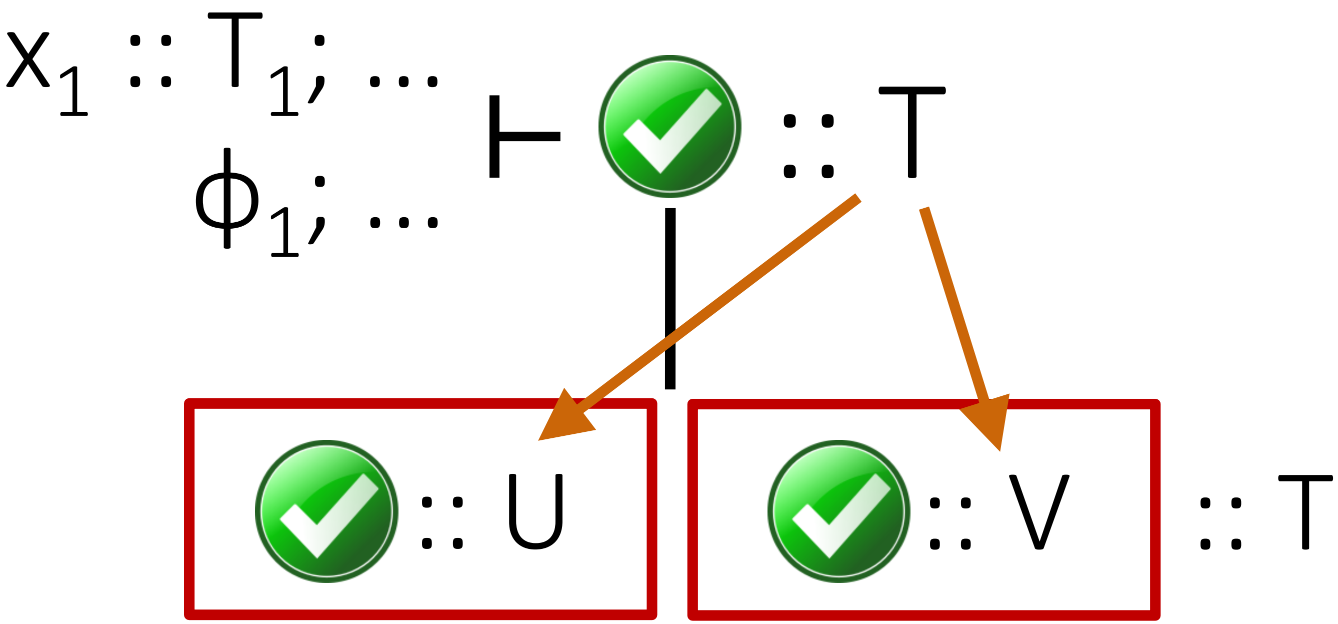
$\{\nu{:}e \mid \nu \geq h\}$

```
insert x xs =
  match xs with
    Nil → Cons x Nil
    Cons h t →
      Cons h (insert x ...)      h  (insert x ...)
```

Type-driven Synthesis

http://tiny.cc/synquid

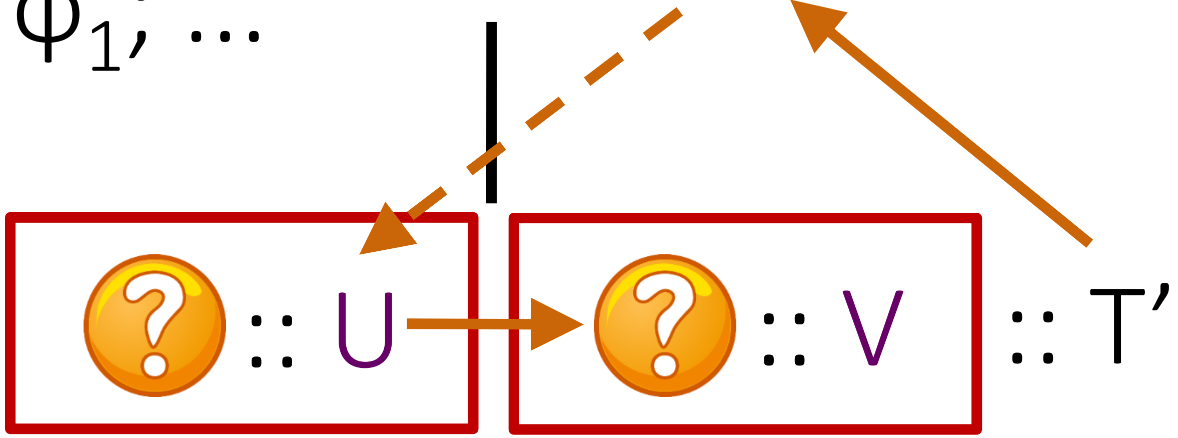
Synthesis from refinement types



I. top-down enumerative search

Synthesis from refinement types

$$x_1 :: T_1; \ldots \quad \phi_1; \ldots \vdash \textcolor{red}{??} :: T$$



I. top-down enumerative search

II. round-trip type checking

Synthesis from refinement types

$$\begin{array}{r} x_1 :: T_1;\ \dots \\ \phi_1;\ \dots \end{array} \vdash \; ?? :: T$$

$$\textbf{if}\ \boxed{??::\{\mathrm{Bool} \mid \nu = P\}}\ \textbf{then}\ \boxed{P \vdash \checkmark :: T}\ \textbf{else}\ \boxed{\neg P \vdash ??::T}$$

- I. top-down enumerative search
- II. round-trip type checking
- III. condition abduction

Round-trip type checking

Nil :: {List a | l

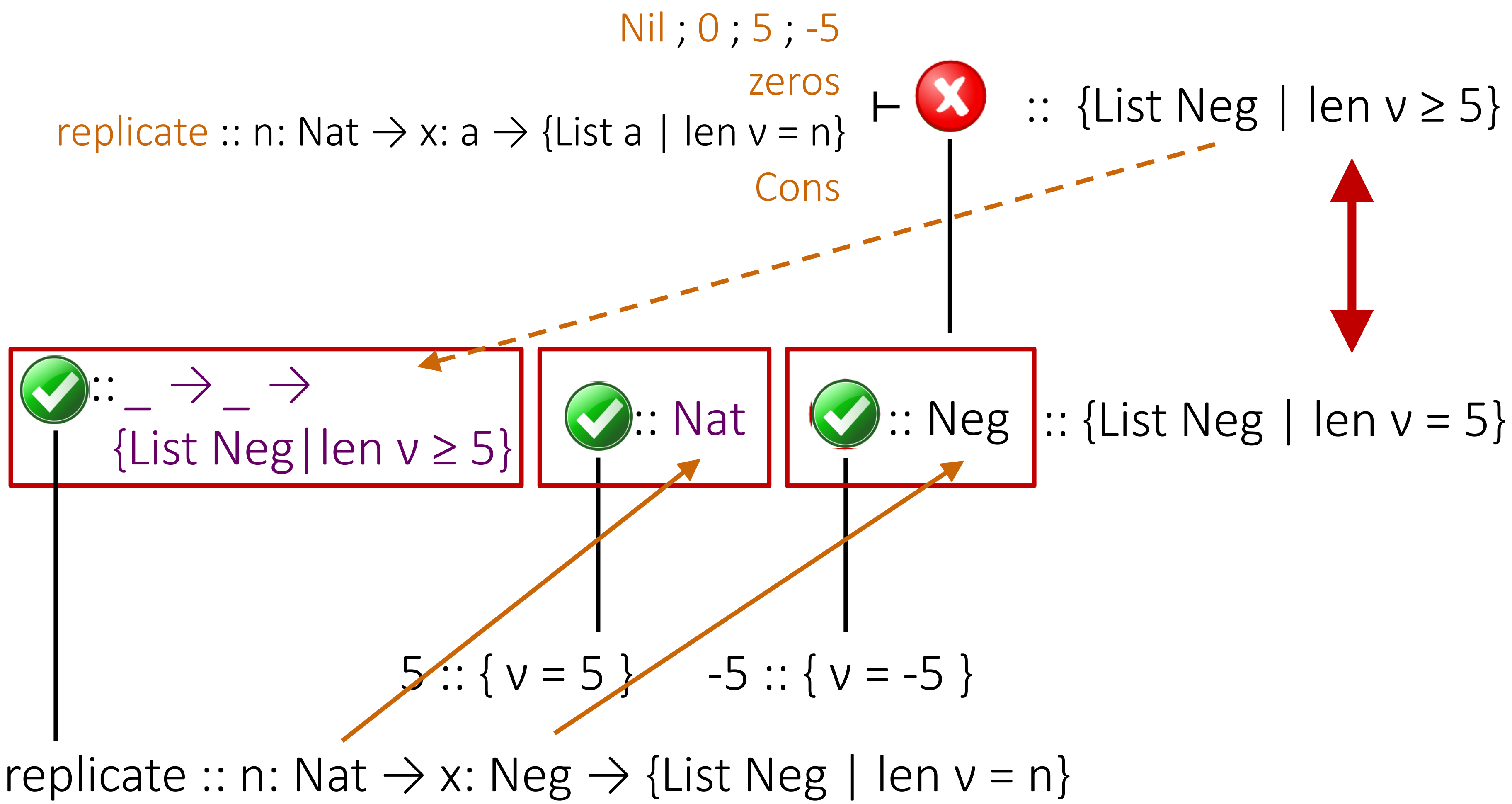
zeros :: n:Nat → {l

Γ ⊢ ✗ :: {List Neg | len v ≥ 5}

✗ :: _ → {List Neg | len v ≥ 5} √ l :: {List Neg|len v = 0}

zeros :: n:Nat → {List Zero | len v = n}

Round-trip type checking



Q3: Can RTTC reject these terms?

`inc ?? :: {Int | v = 5}`

- where `inc :: x:Int → {Int | v = x + 1}`
- NO! don't know if we can find `?? :: {Int | v + 1 = 5}`

`nats ?? :: List Pos`

- where `nats :: n:Nat → {List Nat| len v = n}`
  `Nat = {Int | v >= 0}`, `Pos = {Int | v > 0}`
- YES! `n:Nat → {List Nat| len v = n}` not a subtype of
  `_ → List Pos`

`duplicate ?? :: {List Int | len v = 5}`

- where `duplicate :: xs:List a → {List a | len v = 2*(len xs)}`
- YES! using a consistency check (len v = 2*(len xs) ∧ len v = 5 → UNSAT)

Condition abduction

Nil ; 0 ; -5 ; n :: Nat

(≤) ; (≠) ⊢ :: {List Neg | len v = n}

n ≤ 0

if n ≤ 0 **then** Nil **else** $\boxed{\Gamma;\neg(n \leq 0) \vdash ?? :: \{\text{List Neg} \mid \text{len } v = n\}}$

Liquid abduction

UNSAT core

$$\left[\ \boxed{n \geq 0} \land \boxed{\text{len } v = 0} \land P \land \boxed{\neg(\text{len } v = n)}\ \right]$$

$\boxed{n \leq 0}$

$n \leq -5$

$-5 \leq n$

$\boxed{n \neq 0}$

$n \neq -5$

n :: Nat

Nil :: {List a | len v = 0}

Next week

We start with Unit III (applications)

- We will discuss several applications per week
- There will be several papers to pick from (no questions)
- In class, we'll focus more on discussing the papers