Notes and Reflections

Unfortunately, I got sick the night this bset was due, and ended up falling ~3 days behind in QEA as a result. I'm going to be caught back up soon, but it's not a course of action I would recommend.

As such, this is a Bset I am less proud of than I'd like, but still glad I finished.

Lingering points of confusion include:

- **When it is time to plug in F=ma, what reference frames do I need to define the forces in?**
- **What are the advantages of simulating something like a spring pendulum in cylindrical vs. cartesian coordinates?**
- **Is there a nice way to represent derivatives of compound reference frames in one operation? I find myself doing a lot of math by hand, which feels painful after the beauty that is handling cartesian vector equations.**

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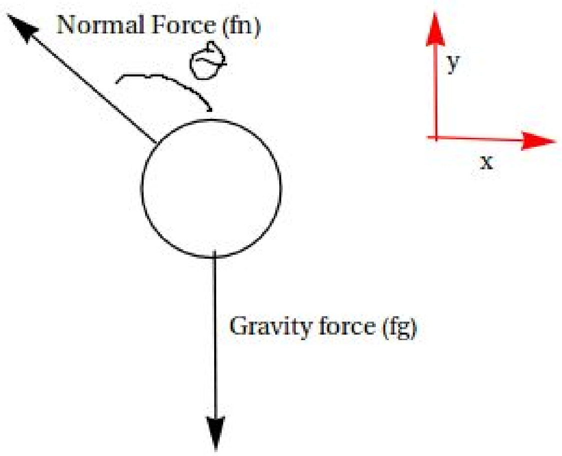
Planar Pendulum

1. Cartesian Equations

Step 1: Define FBDs



Step 2: Define vector force equations

```
ClearAll[θ, fn, fg];
θ[t_] := ArcTan[y[t], x[t]];
fn = tension[t] * {-Sin[θ[t]], Cos[θ[t]]};
fg = m g {0, -1};
```

Step 3: Determine equations of motion

```
eq1 = fn + fg == m {x''[t], y''[t]}
```

$$\left\{-\frac{\text{tension}[t]\, x[t]}{\sqrt{x[t]^2 + y[t]^2}},\ -g\, m + \frac{\text{tension}[t]\, y[t]}{\sqrt{x[t]^2 + y[t]^2}}\right\} == \{m\, x''[t],\ m\, y''[t]\}$$

```
constraint = Norm@{x[t], y[t]} == l
```

$$\sqrt{\text{Abs}[x[t]]^2 + \text{Abs}[y[t]]^2} == l$$

Unfortunately, attempting to solve these directly is really nasty because tension(t) acts as a constraint force. *Mathematica* is capable of doing it, but there are some subtleties that I haven't been able to work out.

2. Cylindrical coordinates

Step 1: FBD

Step 2: Componentwise forces

$$F_r = F_T + -mg\cos\theta$$
$$F_\theta = -mg\sin\theta$$

Step 3: Acceleration equation

$$a_{P/O} = (\ddot{r} - r\dot{\theta}^2)\hat{e}_r + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{e}_\theta$$

(from Bset 1)

Step 4: Equations of Motion

$$(F_T - mg\sin\theta) = (\ddot{r} - r\dot{\theta})m$$

$$-mg\sin\theta = m(2\dot{r}\dot{\theta} + r\ddot{\theta})$$

Step 5: Constraint equation

$$r = l$$

3. Compare equations of motion

Unfortunately, because of the way I formatted the cartesian equations of motion, finding this equality is nontrivial. I will come back to this problem if I have time.

4. Computational simulation

```
sol = NDSolveValue[{eqr, eqθ, constraint, initialConditions} /. parameters,
   {r[t], tension[t], θ[t]}, {t, 0, 10}];

In[225]:= pe[t_] := (-m g Cos[sol[[3, 0]][t]] + m g l) /. parameters;
ke[t_] := 1/2 m (sol[[1, 0]][t] sol[[3, 0]]'[t])^2 /. parameters;
Plot[Evaluate@sol, {t, 0, 10}, PlotRange → Full,
 PlotLegends → StringSplit@"r tension θ"]
Plot[Evaluate@{pe[t], ke[t], pe[t] + ke[t]}, {t, 0, 10},
 PlotLegends → StringSplit@"pe ke total"]
```

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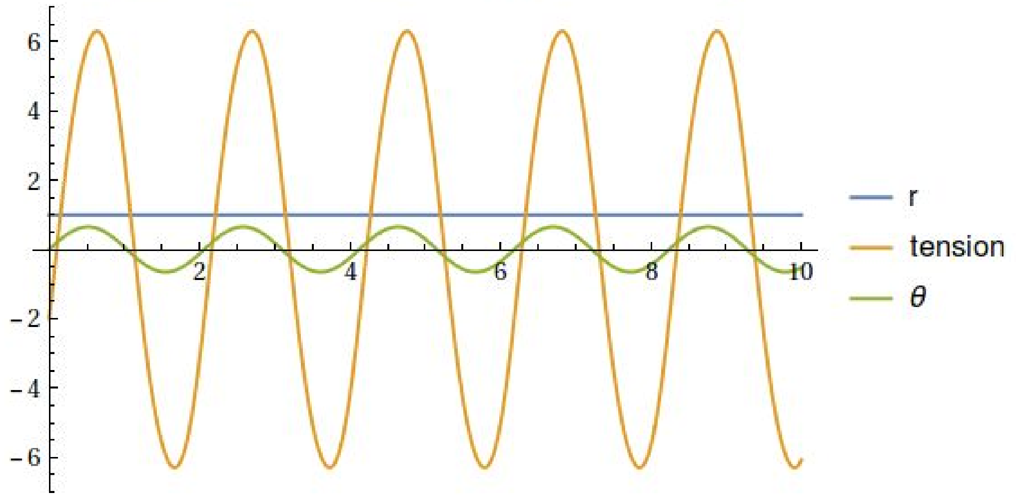

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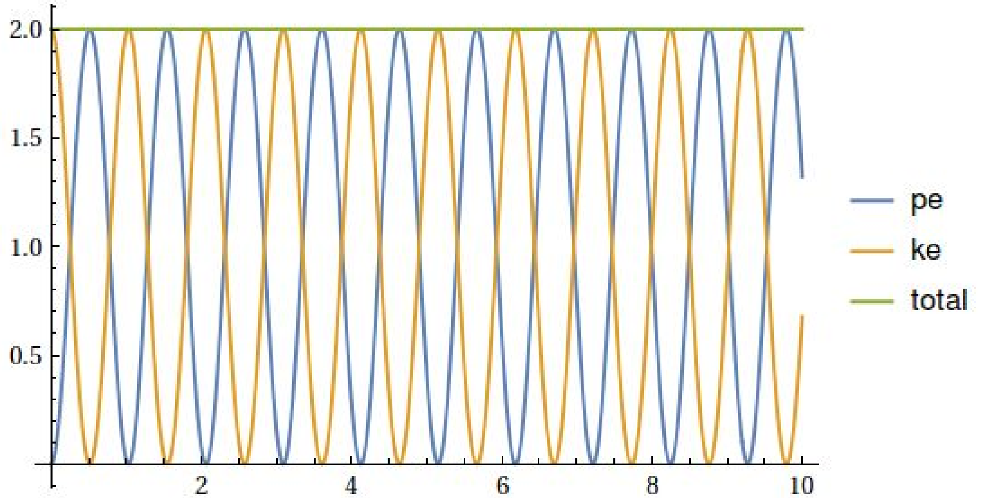
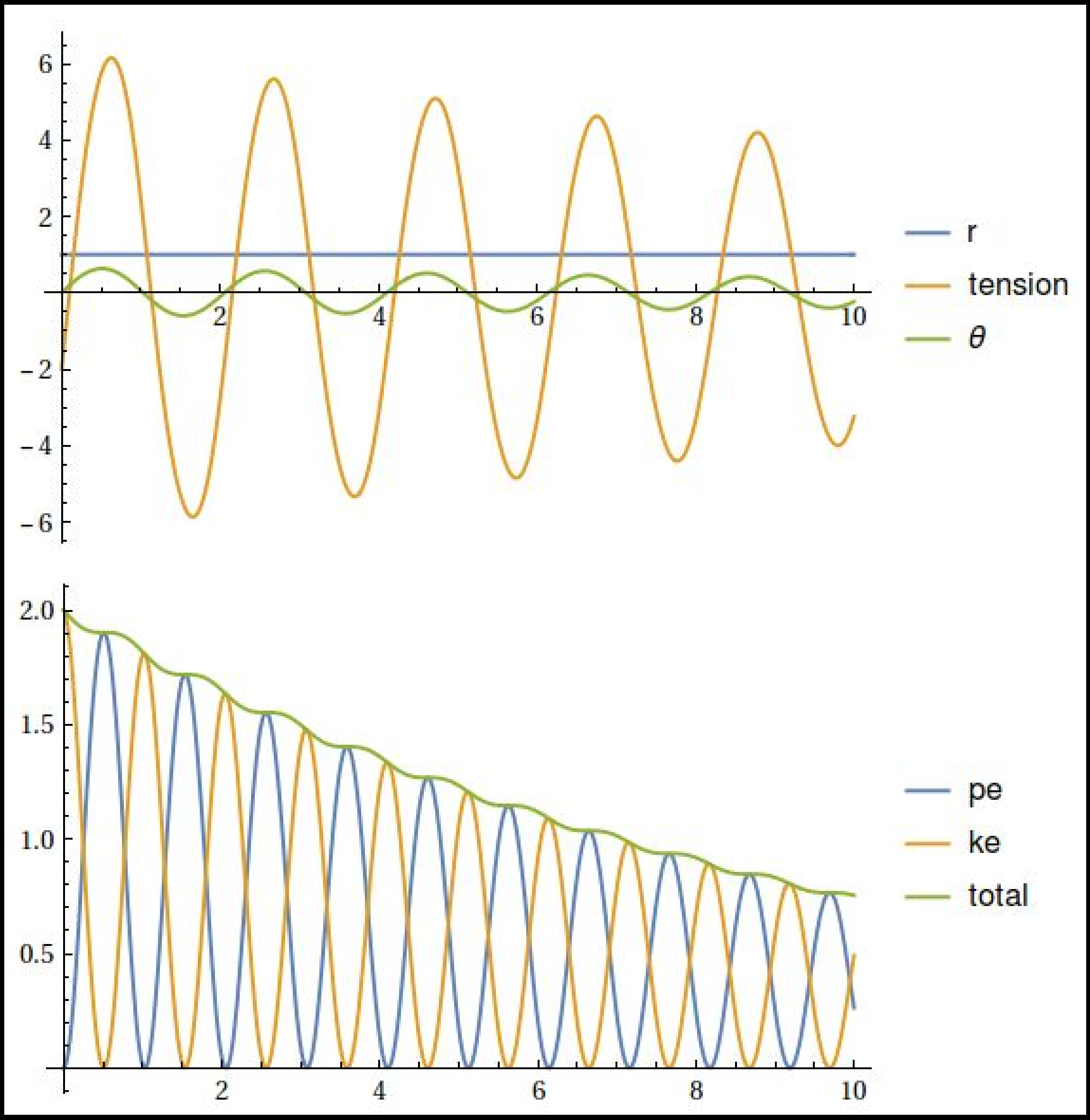


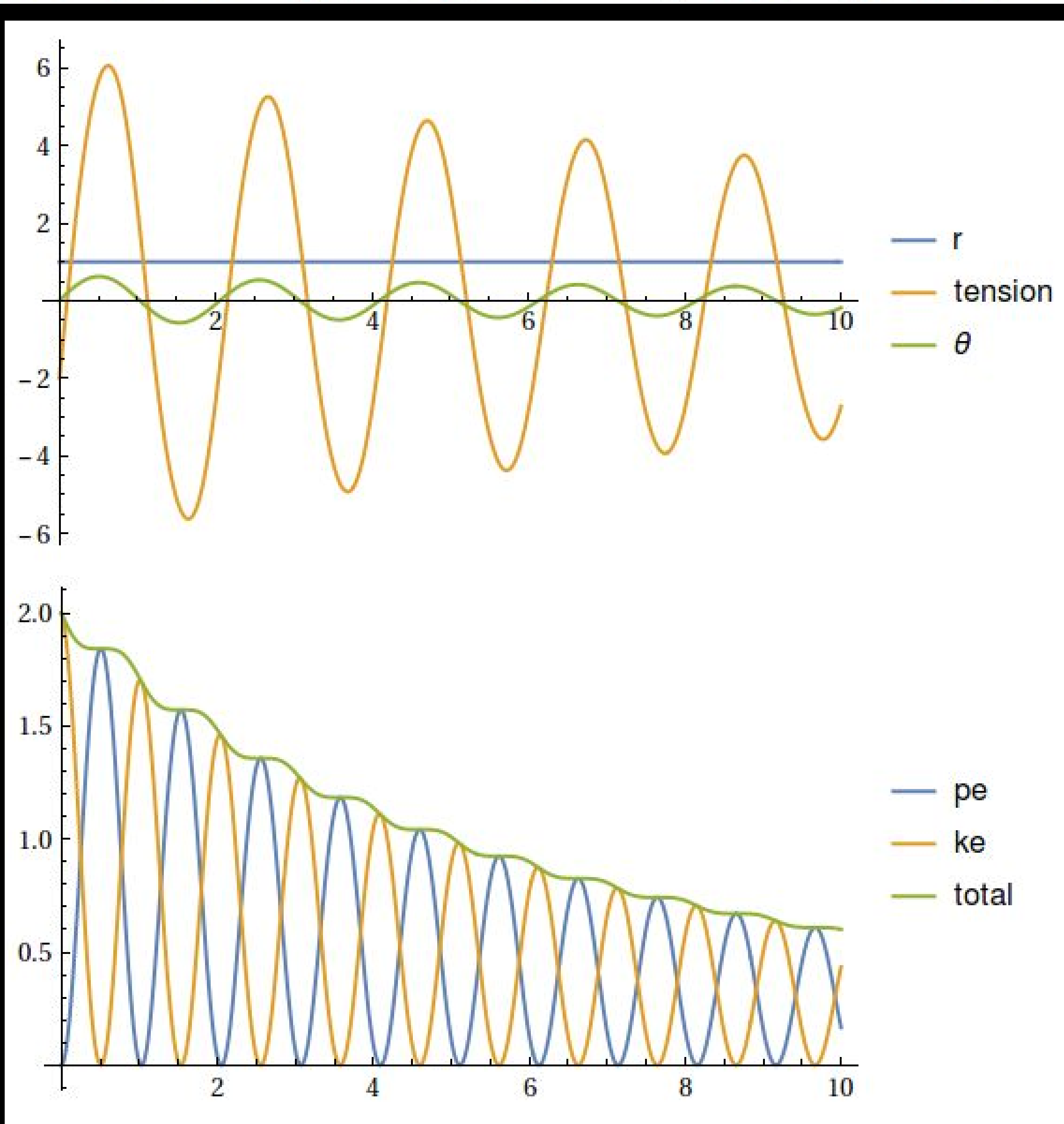
6a. Adding viscous drag

$$-g\, m\, \mathrm{Sin}[\theta[t]] - 0.1\, \theta'[t] == m\,(2\, r'[t]\, \theta'[t] + r[t]\, \theta''[t])$$



6b. Adding air drag

$$-g\, m\, \mathrm{Sin}[\theta[t]] - 0.1\, \mathrm{Sign}[\theta'[t]]\, \theta'[t]^2 == m\,(2\, r'[t]\, \theta'[t] + r[t]\, \theta''[t])$$



7. Hardware

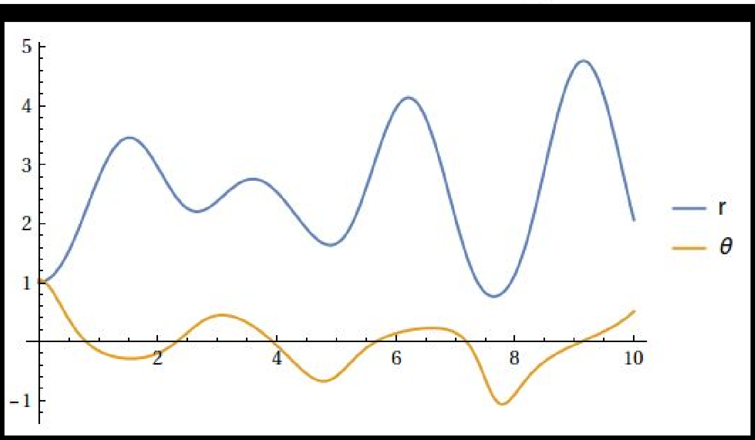
I decided to skip this part for now, in service of catching back up. I’d love to do it sometime, and I did sink 15min into installing and learning Tracker.

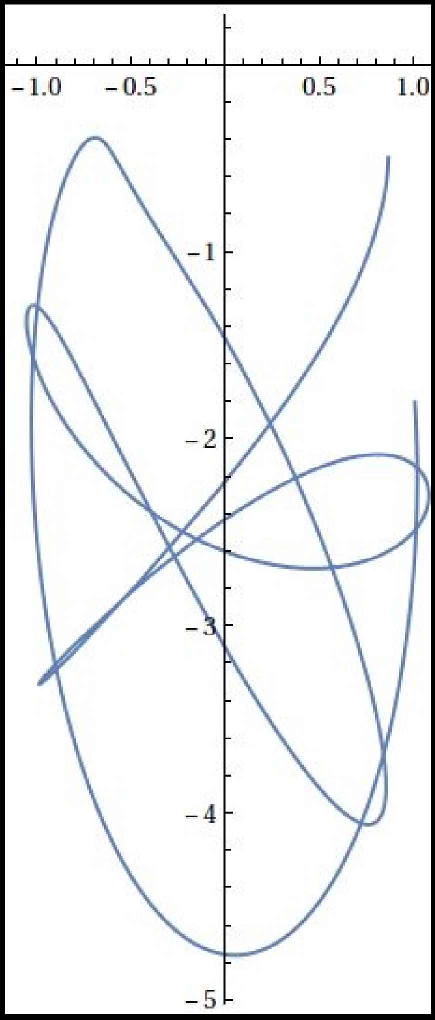
Springy Pendulum

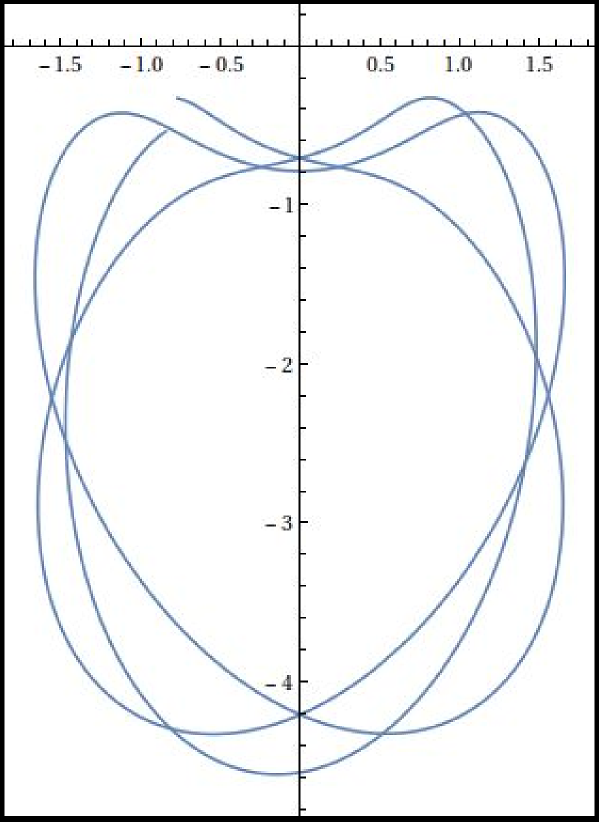
$$g\,m\,\mathrm{Cos}[\theta[t]] - k\,(-l + r[t]) == m\,(-r[t]\,\theta'[t] + r''[t])$$

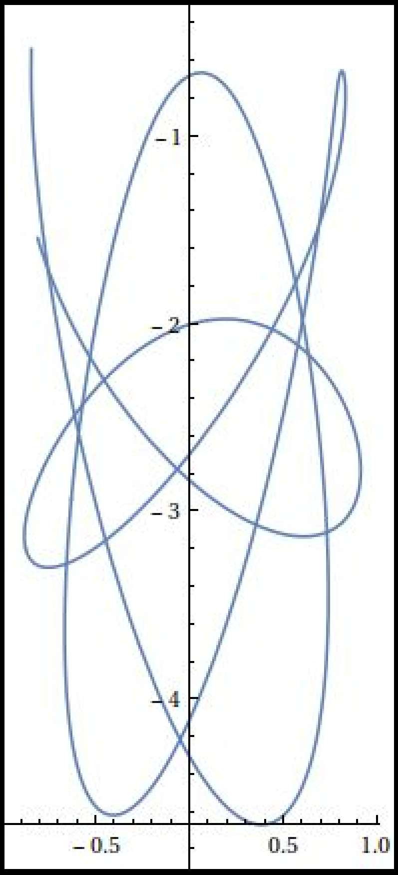
$$-g\,m\,\mathrm{Sin}[\theta[t]] == m\,(2\,r'[t]\,\theta'[t] + r[t]\,\theta''[t])$$

$$\left\{r[0] == l,\ r'[0] == 0,\ \theta[0] == \frac{\pi}{3},\ \theta'[0] == 0\right\}$$









Spinny Pendulum

Attempt 1: Reference frame transformations

In this problem, many of the equations derived in the last one still hold, but the non-inertial nature of frame O2 means that F=ma cannot be applied in that frame. As such, we need to convert to a static frame O1 before putting together F=ma. We can do this using equation (19) from inclass 2, copied below.

$$\begin{aligned}
\vec{a}_{P/o_1} &= \left.\frac{d^2}{dt^2}\right|_{o_1} \vec{r}_{P/o_1} = \left.\frac{d}{dt}\right|_{o_1} \left(\left.\frac{d}{dt}\right|_{o_1} \vec{r}_{P/o_1} \right) = \cdots \\
&= \left.\frac{d^2}{dt^2}\right|_{o_1} {}^{o_1}\vec{r}_{o_2/o_1} + \left.\frac{d^2}{dt^2}\right|_{o_2} {}^{o_2}\vec{r}_{P/o_2} + \left.\frac{d}{dt}\right|_{o_2} {}^{o_1}\vec{\omega}^{o_2} \times {}^{o_2}\vec{r}_{P/o_2} \\
&+ 2\ {}^{o_1}\vec{\omega}^{o_2} \times \left.\frac{d}{dt}\right|_{o_2} {}^{o_2}\vec{r}_{P/o_2} + {}^{o_1}\vec{\omega}^{o_2} \times \left({}^{o_1}\vec{\omega}^{o_2} \times {}^{o_2}\vec{r}_{P/o_2} \right) \qquad (19)
\end{aligned}$$

To be clear, O1 is centered at the center of the pendulum's travel.

Unfortunately, as I started solving this for the spinny pendulum, I ended up very confused by all the cross products and vector components. In particular, expressing ω in the cylindrical coordinates got nasty very quickly, and I didn't like it.

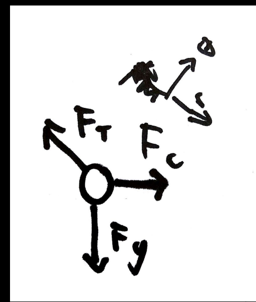
Attempt 2: Pseudo-forces

Instead, I'm going to use pseudo-forces to solve the problem (please don't kill me, Chris).

I want to solve the problem in the rotating cylindrical reference frame with z going through the pivot of the pendulum.

When solving problems in a rotating reference frame, three pseudo-forces need to be taken into account, Centrifugal force, Coriolis force, and Euler force. Of the three, only Centrifugal force applies in this problem because the Coriolis force is perpendicular to the direction of motion (and thus in the z direction, which is blocked by a constraint) and the Euler force only applies if the rate of rotation of the reference frame isn't constant.

Thus, my FBD becomes...



With corresponding equations of motion of...

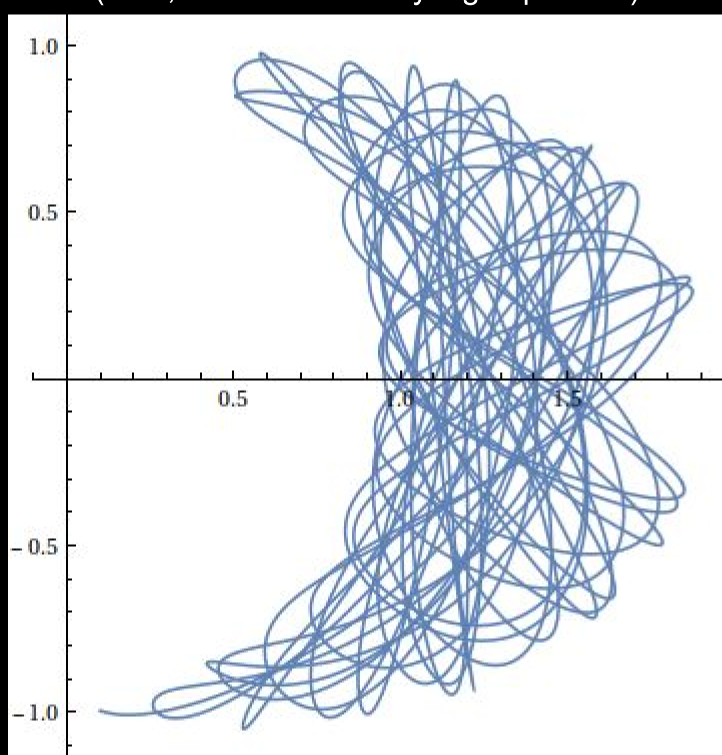
```
fg = m g;
ft = k (r[t] - l);
fc = m Ω r[t] Sin[θ[t]];
eqr = -ft + fg Cos[θ[t]] + fc Sin[θ[t]] == m (r''[t] - r [t] θ'[t])
eqθ = -fg Sin[θ[t]] + fc Cos[θ[t]] == m (2 r'[t] θ'[t] + r[t] θ''[t])
```

$$g\, m \cos(\theta(t)) - k\,(r(t) - l) + m\,\Omega\, r(t) \sin^2(\theta(t)) = m\,(r''(t) - r(t)\,\theta'(t))$$

$$m\,\Omega\, r(t) \sin(\theta(t)) \cos(\theta(t)) - g\, m \sin(\theta(t)) = m\,(2\, r'(t)\,\theta'(t) + r(t)\,\theta''(t))$$

Running the simulation:

(note, this is with a very high spin rate)



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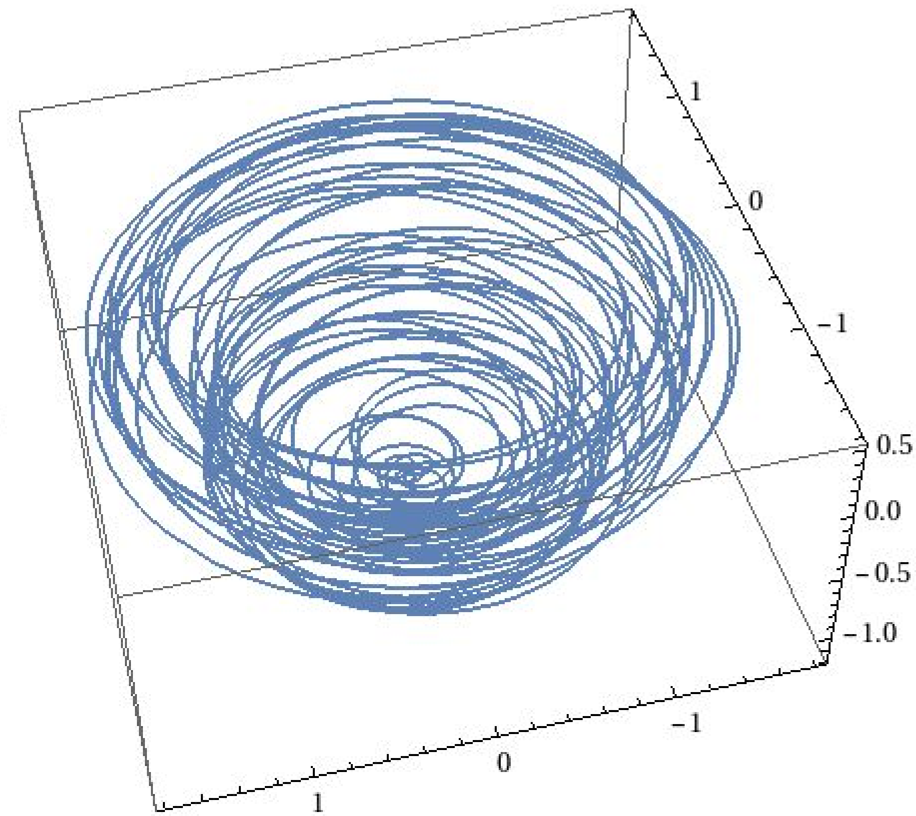

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