Pendulums, Laplace Transform

October 30, 2016

Pendulums. NOT AGAIN!!!

As we were going through the dynamics module, you got to figure out the equation of motion for a few different pendulums, simulate and take some simple qualitative data on a point mass pendulum. For this one, we are going to revisit a specific pendulum.....your very first ISIM lab from last year! We are going to ask you to analytically solve the governing equation for this system and then do a quantitative comparison of experimental data to the model in order to extract the damping coefficient of the system, which is an important, empirically determined system characteristic.

The governing equation for this system is a second order differential equation, so you'll get to use all the stuff you've learned on that to understand this system.

1. First, we need to find the governing equation for this system.
   (a) The ISIM pendula are what are called 'physical' pendula, which is to say, they are real three dimensional objects, not point masses. You will therefore need to be thinking in terms of moment of inertia about the pivot point. Calculate the moment of inertia by Solidworks, by full calculation of the geometry, or by making a justifiable approximation to the geometry. Please note that the ISIM pendula are not all the same size, so pick one and hold onto it until you are done with the whole experiment.
   (b) What would be the equation of motion for this system if it were completely conservative? (No friction = no damping.)
   (c) How would the equation of motion change if we were to add in linear damping: in other words, we are going to assume the damping force is linearly proportional to angular velocity of the pendulum. (What other possible damping models are there?).
   (d) This equation of motion is non-linear!!! We have a hard time solving non-linear differential equations. We are therefore going to linearize this equation by assuming the small angle approximation for $\sin\theta$ which is $\sin\theta \approx \theta$. What does your governing equation look like now?
2. Interpret this governing equation qualitatively, what behavior do you expect for a given initial condition? Draw the phase plane and interpret that in terms of your expected physical behavior.

3. What is the characteristic equation for this differential equation? What are the roots of the characteristic equation? For what conditions are the roots real? For what conditions are the roots complex?

4. Use these roots to write the expression for the position of the pendulum as a function of time.
   (a) This equation has two unknown parameters in it. How do we find values for these two parameters (hint: think about initial conditions).
   (b) How does this equation behave if the roots are real? How does this equation behave if the roots are complex? Graph your solution for a few different values of the damping parameter. For what value of the damping coefficient does the pendulum come to rest the fastest?

5. Ok, now to set up and take data on the ISIM pendulum!
   (a) Attach the pendulum to the knob on the potentiometer, hanging down off the edge of a desk. Hook your Analog Discovery ground up to one end of the potentiometer, the +5V supply to the other end of the potentiometer, and Ch1+ to the wiper (middle pin). Make sure to connect your Ch1- to ground so you don't get a noisy signal. In Waveforms, make sure to turn on the +5V supply and open up the SCOPE. Verify that you see the voltage level change as you move the pendulum.
   (b) You will need to calibrate the voltage to angle conversion, just like you did your first day in ISIM. You don't need more than about 5 data points spanning about 20 degrees on either side of zero. Measure the voltage at each pendulum position and record the position with the protractor. Use this data to figure out a linear function which converts the voltage to angle.
   (c) Now, increase the time range on the Waveforms scope, start collecting data, raise the pendulum to a small initial angle (why does it need to be small?) and release. Collect data in a single sweep until the pendulum comes to rest, then stop the scope and save the data.

6. Import your data into Matlab, truncate the data to preserve only the data after the release of the pendulum and convert voltage to angle using your calibration curve. Then use the curve fitting tool to fit to your solution from your governing equation. How well does your model match the data? What is the frequency of the oscillations, what is the predicted frequency? What is the damping coefficient?

7. You should turn in your answers to the questions in 1-4, and a plot of your experimental data plotted along with your fitting function and the fit values for the frequency and damping coefficient.

Introduction to Laplace Transform

Goals

In this part of the assignment, you will do the following

- Compute the Laplace transform for some simple functions.
- Compute the inverse Laplace transform of some simple functions.

There are two problems that you will need to answer and upload for this part of the assignment, and a brief reading diagnostic will be put up.
Key ideas/terms you should look out for are

- Region of convergence
- Partial fraction expansion
- Laplace transform of a derivative

Why another transform?

The Laplace transform is useful in a number of engineering applications, including to solve differential equations with initial conditions, for which the Fourier transform is very cumbersome to use, to solve driven differential equations, and to provide insight into the frequency domain behavior of systems. The Laplace transform takes differential equations and turns them into algebraic equations. Moreover, the Laplace transform is useful in analyzing things like the stability of systems and used in a number of places in controls.

We never take Laplace transforms of arbitrary signals, such as when we took the DFT of audio signals in Module 1. In some simple cases, we evaluate the Laplace transform of signals defined by equations (as opposed to an audio sample). In these cases, we use a few known transforms (i.e. by looking up a table of known transforms, or from memory).

Definition of the Laplace Transform

The Laplace transform of a function $f(t)$ is a function of a new variable s which is complex in general. It is defined as follows

$$\mathcal{L}\{f(t)\} = F(s) = \int_{0^-}^{\infty} f(t)e^{-st}dt \tag{1}$$

The notation 0^- refers to a value infinitesimally less than zero. For a given $f(t)$ this integral may not converge for all values of s, and the values of s for which the integral does converge is called the Region of Convergence (ROC).

Readings and Videos

- From Paul's Online Math Notes: Laplace Transform, read sections on
  - The Definition.
  - The Laplace transform.
- Please watch the following video which illustrates the computation of the Laplace transform for an example signal $f(t) = e^{-at}$. Laplace Transform Example
- If you are interested in another example, please watch the following video from Professor Katherine Kim of Ulsan National Institute of Science and Technology (UNIST), and an Olin Alum! Another Laplace Transform Example

Problem

1. Consider the unit step function $u(t-a)$, for $a \geq 0$. This function is defined as follows

$$u(t-a) = \begin{cases} 1 \text{ if } t \geq a \\ 0 \text{ otherwise.} \end{cases} \tag{2}$$

This function is illustrated in Figure 1 and can be used as an idealized model for a switch being turned on, for instance.

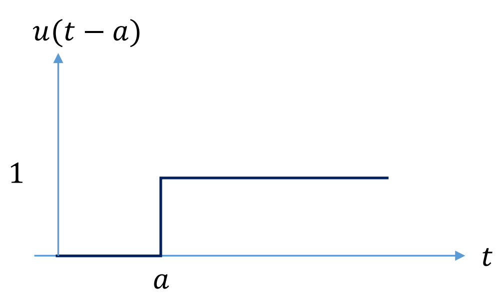

Figure 1: Unit step function.

(a) Show that

$$\mathcal{L}\{u(t)\} = \frac{1}{s} \tag{3}$$

with the ROC being $s > 0$, if s is restricted to being real.

(b) Using the definition of the Laplace transform, show that

$$\mathcal{L}\{f(t-a)u(t-a)\} = e^{-as}F(s) \tag{4}$$

(c) Show that

$$\mathcal{L}\{u(t-a)\} = \frac{e^{-as}}{s} \tag{5}$$

with the ROC being $s > 0$, if s is restricted to being real.

The Inverse Laplace transform

The inverse Laplace transform recovers a time-domain signal from its Laplace transform. The formula for this is not pretty, and we are going to spare you from looking at it. It turns out that in practice, we usually find the inverse transform by manipulating Laplace transforms using their properties, into a form which we can look up in a table.

Please read the following regarding the inverse Laplace transform: Inverse Laplace transform.

In manipulating Laplace transforms, we will find partial-fraction expansion useful. The following example illustrates the partial fraction expansion of a function $\frac{s}{(s+1)(s+2)}$

$$\frac{s}{(s-1)(s+2)} = \frac{1}{3(s-1)} + \frac{2}{3(s+2)} \tag{6}$$

You can compute this using WolframAlpha using `PartialFractionExpansion`, but if you want a formula for the case that a, b and c are all different values and $q(s)$ is some polynomial in s, here it is.

$$\frac{q(s)}{(s-a)(s-b)} = \frac{q(a)}{(a-b)(s-a)}$$
$$\frac{q(s)}{(s-a)(s-b)(s-c)} = \frac{q(a)}{(a-b)(a-c)(s-a)} + \frac{q(b)}{(b-a)(b-c)(s-b)} + \frac{q(c)}{(c-a)(c-b)(s-c)}$$

Problem

2. Find the inverse Laplace transform of the following

(a)

$$Y(s) = \frac{s+1}{s^2+5s+4} \tag{7}$$

(b)

$$Y(s) = \frac{1}{s^2+2s+2} \tag{8}$$

Note that even though the polynomial in the denominator of the expression has complex roots, you can still use partial fraction expansion. Please use the identity $\sin(\theta) = \frac{1}{2j}e^{j\theta} - \frac{1}{2j}e^{-j\theta}$ to simplify $y(t)$.

Using the Laplace transform to solve initial-value problems.

We illustrate the use of the Laplace transform to solve initial-value problems through two examples. Both these examples are "un-driven", in the sense that there is no input to the system. We are just observing the natural behavior of the system over time given some inital conditions.

We will do much more of these in coming days, including examples for which other techniques (other than Laplace Transform) are much more cumbersome to solve the differential equations. **Note that there are no problems in this section, but we have 2 worked examples.**

First, we need to see the relationship between the Laplace transform of the derivative of a function and the Laplace transform of the original function. Consider a function $f(t)$. The following property of the Laplace transform is derived here. You do not need to understand how this was derived but the link is given in case you are interested. Also, please ignore the comment in the note about generalized derivative – for our purposes, this is just the derivative.

$$\mathcal{L}\left\{\frac{d}{dt}f(t)\right\} = sF(s) - f(0^-). \tag{9}$$

Additionally, we have

$$\begin{aligned}\mathcal{L}\left\{\frac{d^2}{dt^2}f(t)\right\} &= s^2F(s) - sf'(0^-) - f(0^-),\\ \mathcal{L}\left\{\frac{d^3}{dt^3}f(t)\right\} &= s^3F(s) - s^2f''(0^-) - sf'(0^-) - f(0^-),\end{aligned} \tag{10}$$

and so on.

Example 1:

As an example, consider a system governed by the following differential equation, with the initial conditions $y(0^-) = 2$.

$$\frac{dy(t)}{dt} + 4y(t) = 0. \tag{11}$$

Take the Laplace transform of each term in the above expression.

$$\begin{aligned}sY(s) - y(0^-) + 4Y(s) &= 0\\ (s+4)Y(s) - 2 &= 0\\ Y(s) &= \frac{2}{s+4}\end{aligned} \tag{12}$$

To get $y(t)$ from the above expression, we need to invert the Laplace transform. From the example in the previous section (in the video), we see that the inverse Laplace transform of $\frac{1}{s+4}$ is e^{-4t}, for $t \geq 0$.

$$y(t) = 2e^{-4t}. \tag{13}$$

Example 2:

As another example, consider a system governed by the following differential equation, with the initial conditions $y(0-) = 1$, $y'(0^-) = 0$.

$$\frac{d^2y(t)}{dt^2} + 4\frac{dy(t)}{dt} + 5y(t) = 0. \tag{14}$$

Take the Laplace transform of each term in the above expression.

$$\begin{aligned} s^2Y(s) - sy'(0^-) - y(0^-) + 4sY(s) - 4y(0^-) + 5Y(s) &= 0 \\ s^2Y(s) - 1 + 4sY(s) - 4 + 5Y(s) &= 0 \\ s^2Y(s) + 4sY(s) + 5Y(s) &= 5 \end{aligned}$$

Collecting terms, we have

$$\begin{aligned} Y(s) &= \frac{5}{s^2 + 4s + 5} \\ &= \frac{5}{(s+2-j)(s+2+j)} \end{aligned} \tag{15}$$

Using partial fraction expansion,

$$\begin{aligned} Y(s) &= \frac{5}{2j(s+2-j)} - \frac{5}{2j(s+2+j)} \\ y(t) &= \frac{5}{2j}e^{-2+j}t - \frac{5}{2j}e^{-2-j}t \\ &= 5e^{-2t}\left(\frac{1}{2j}e^{jt} - \frac{1}{2j}e^{-jt}\right) \end{aligned} \tag{16}$$

Applying the identity $\sin(\theta) = \frac{1}{2j}e^{j\theta} - \frac{1}{2j}e^{-j\theta}$, yields

$$y(t) = 5e^{-2t}\sin(t) \tag{17}$$

for $t \geq 0$. This expression for $y(t)$ is a sine wave which decays with time.