Eigenvalues and Eigenvectors

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What is this about?

Many problems of analysis in engineering can be formulated as a question about the eigenvalues and eigenvectors of a square matrix. We will first examine eigenvalues and eigenvectors for small matrices by hand. Then we will introduce the tools and techniques used to compute eigenvalues and eigenvectors for square matrices of any size. This is a standard topic in Linear Algebra, and can be found in any standard textbook or online reference. See for example the page *eigenvalue* at WolframMathWorld.

Definition and Notation

If there exists a vector $\mathbf{v} \neq \mathbf{0}$ such that

$$\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$$

for some scalar λ, then λ is called an eigenvalue of $\mathbf{A}$ with corresponding eigenvector $\mathbf{v}$. An $n \times n$ matrix has n eigenvalues (if you include repeated eigenvalues). If we treat $\mathbf{A}$ as a transformation matrix then we are seeking the vector $\mathbf{v}$ which is simply scaled when acted on by the matrix $\mathbf{A}$.

Eigenvalues and eigenvectors of a Diagonal matrix

Recall from our earlier work that a scaling matrix is a diagonal matrix such as

$$\mathbf{A} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$

We know that this matrix scales geometrical objects by a factor of 2 in the x-direction and by a factor of 3 in the y-direction. It seems reasonable to assume then that $\lambda_1 = 2$ is an eigenvalue with corresponding eigenvector $\mathbf{v}_1 = \begin{bmatrix} 1 & 0 \end{bmatrix}^T$, and that $\lambda_2 = 3$ is an eigenvalue with corresponding eigenvector $\mathbf{v}_2 = \begin{bmatrix} 0 & 1 \end{bmatrix}^T$. Let's check if the first one is true:

$$\mathbf{A}\mathbf{v}_1 = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 2\mathbf{v}_1$$

Therefore $\lambda_1 = 2$ is an eigenvalue with corresponding eigenvector $\mathbf{v}_1 = \begin{bmatrix} 1 & 0 \end{bmatrix}^T$.

1. Confirm that $\lambda_2 = 3$ is an eigenvalue with corresponding eigenvector $\mathbf{v}_2 = \begin{bmatrix} 0 & 1 \end{bmatrix}^T$.

Based on this example, here is the first result about eigenvalues which we will prove later.

Theorem: The eigenvalues of a diagonal matrix are the entries on the diagonal.

2. What are the eigenvalues and eigenvectors of the following diagonal matrices

(a)

$$\mathbf{A} = \begin{bmatrix} -3 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

(b)

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

(c)

$$\mathbf{A} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The eigenvalues of a Triangular matrix

Recall from our earlier work that a skew matrix is a triangular matrix such as

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

We know that this matrix doesn't scale geometrical objects in any direction. Both eigenvalues are therefore likely to be 1. What about the corresponding eigenvector? Let's go ahead and try to find $\mathbf{v}_1$, assuming that $\lambda_1 = 1$. Let's assume that $\mathbf{v}_1 = \begin{bmatrix} x & y \end{bmatrix}^T$ and set $\mathbf{A}\mathbf{v}_1 = \mathbf{v}_1$ to give

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

Let's subtract the unknown vector from both sides

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

and then factor the left hand side to produce

$$\begin{bmatrix} 1-1 & 1 \\ 0 & 1-1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

or more simply

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The corresponding LSAE has the solution $y = 0$ and x is arbitrary. The eigenvector is therefore

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

and there is only one eigenvector in this case.

What about a more general triangular matrix such as

$$\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 0 & -3 \end{bmatrix}$$

Let's check whether the diagonal entries are the eigenvalues. Let's assume $\lambda_1 = 2$ and $\mathbf{v}_1 = \begin{bmatrix} x & y \end{bmatrix}^T$ and set $\mathbf{A}\mathbf{v}_1 = 2\mathbf{v}_1$ to give

$$\begin{bmatrix} 2 & 1 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 2 \begin{bmatrix} x \\ y \end{bmatrix}$$

Let's subtract the unknown vector from both sides

$$\begin{bmatrix} 2 & 1 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} - 2 \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

and then factor the left hand side to produce

$$\begin{bmatrix} 2-2 & 1 \\ 0 & -3-2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

or more simply

$$\begin{bmatrix} 0 & 1 \\ 0 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The corresponding LSAE has solution $y = 0$ and x is arbitrary. The eigenvector is therefore

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

3. Confirm that $\lambda_2 = -3$ and $\mathbf{v}_2 = \begin{bmatrix} 1/5 & -1 \end{bmatrix}^T$ is an eigenvalue and corresponding eigenvector of $\mathbf{A}$.

Based on this example, here is the second result about eigenvalues that we will prove later

Theorem: The eigenvalues of a triangular matrix are the entries on the diagonal.

4. What are the eigenvalues and eigenvectors of the following triangular matrices?

(a)
$$\begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}$$

(b)
$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 2 & 0 \\ 2 & 1 & 3 \end{bmatrix}$$

(c)
$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 4 \end{bmatrix}$$

The Characteristic Equation

So far we've dealt with matrices for which it is possible to think your way to the eigenvalues. For general matrices, this is rarely the case, and we need a method that is foolproof. The method most widely adopted involves the determination of an algebraic equation for the eigenvalues, usually known as the *characteristic* equation. For this reason, eigenvalues are often known as characteristic values. Here is how it works.

We seek an eigenvalue λ and corresponding eigenvector $\mathbf{v}$ which satisfies
$$\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$$
We subtract $\lambda\mathbf{v}$ from both sides
$$\mathbf{A}\mathbf{v} - \lambda\mathbf{v} = \mathbf{0}$$
and then factor the left hand side to give
$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{v} = \mathbf{0}$$
Notice that in order to factor we introduced the identity matrix. The left hand side is nothing but the original matrix $\mathbf{A}$ with λ subtracted

from the diagonal terms. Since we are seeking a non-zero solution to this equation, we require that the determinant of the matrix on the left hand side is zero

$$det(\mathbf{A} - \lambda\mathbf{I}) = 0$$

(If it is not zero, then the only solution is the zero vector which isn't very useful.) As we will see shortly, setting the determinant of this matrix equal to zero will result in a polynomial in λ, which we know how to solve from days gone by.

Consider the diagonal matrix from earlier,

$$\mathbf{A} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$

First, let's figure out $\mathbf{A} - \lambda\mathbf{I}$: we subtract λ from the diagonal entries to give

$$\mathbf{A} - \lambda\mathbf{I} = \begin{bmatrix} 2-\lambda & 0 \\ 0 & 3-\lambda \end{bmatrix}$$

Now we compute the determinant of this matrix, which in this case is

$$det(\mathbf{A} - \lambda\mathbf{I}) = (2-\lambda)(3-\lambda)$$

and then we set it equal to zero

$$(2-\lambda)(3-\lambda) = 0$$

This equation is the *characteristic* equation. Notice that it is a polynomial in λ: in fact it is a second-order polynomial in λ in this case. Since it is already factored, it is trivial to solve and we find two solutions

$$\lambda_1 = 2,\ \lambda_2 = 3$$

which are the eigenvalues of the original matrix $\mathbf{A}$. The corresponding eigenvectors are determined as before.

5. What is the characteristic equation for each of the following diagonal matrices?

(a)

$$\mathbf{A} = \begin{bmatrix} -3 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

(b)

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

(c)

$$\mathbf{A} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

6. What is the characteristic equation for each of the following triangular matrices?

(a)

$$\begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}$$

(b)

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 2 & 0 \\ 2 & 1 & 3 \end{bmatrix}$$

(c)

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 4 \end{bmatrix}$$

7. Prove that the eigenvalues of an $n \times n$ diagonal matrix are the entries on the diagonal. (You will need to use properties about determinants of diagonal matrices that we previously considered).

8. Prove that the eigenvalues of an $n \times n$ triangular matrix are the entries on the diagonal. (You will need to use properties about determinants of triangular matrices that we previously considered).

The eigenvalues and eigenvectors of a 2x2 matrix

To find the eigenvalues of general 2x2 matrices we use the characteristic equation. Let's start with an example. Consider the matrix

$$\mathbf{A} = \begin{bmatrix} 18 & -2 \\ 12 & 7 \end{bmatrix}$$

We are seeking λ and $\mathbf{v}$ which satisfy

$$\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$$

or equivalently

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{v} = \mathbf{0}$$

For this example we have

$$\mathbf{A} - \lambda\mathbf{I} = \begin{bmatrix} 18 - \lambda & -2 \\ 12 & 7 - \lambda \end{bmatrix}$$

which has determinant

$$det(\mathbf{A} - \lambda \mathbf{I}) = (18 - \lambda)(7 - \lambda) + 24$$

and the characteristic equation is

$$(18 - \lambda)(7 - \lambda) + 24 = 0$$

Expanding this equation results in a second-order polynomial in λ

$$\lambda^2 - 25\lambda + 150 = 0$$

which can be factored to produce

$$(\lambda - 15)(\lambda - 10) = 0$$

so that the eigenvalues are $\lambda_1 = 15$ and $\lambda_2 = 10$. (We could use the quadratic formula as necessary.)

9. Determine the corresponding eigenvectors of $\lambda_1 = 15$ and $\lambda_2 = 10$ for this example.

Since this process will work for any matrix, let's apply it to the most general 2x2 matrix

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Again we seek λ and $\mathbf{v}$ so that

$$\mathbf{Av} = \lambda \mathbf{v}$$

or equivalently

$$(\mathbf{A} - \lambda \mathbf{I})\mathbf{v} = \mathbf{0}$$

Non-zero solutions for $\mathbf{v}$ exist when

$$det(\mathbf{A} - \lambda \mathbf{I}) = 0$$

so in the general case the characteristic equation is

$$(a - \lambda)(d - \lambda) - bc = 0$$

Expanding and simplifying gives

$$\lambda^2 - (a + d)\lambda + (ad - bc) = 0$$

which is a second-order polynomial in λ with two coefficients. Notice that the last one is just $det(\mathbf{A})$ and the middle one involves the sum of the diagonal entries of $\mathbf{A}$, which is known as the trace of $\mathbf{A}$ or $tr(\mathbf{A})$ for short. The characteristic equation is therefore

$$\lambda^2 - tr(\mathbf{A})\lambda + det(\mathbf{A}) = 0$$

Right away we see that the eigenvalues do not depend on the four entries in the matrix $\mathbf{A}$: they are determined by two numbers associated with the matrix, the trace and the determinant.

10. Determine the trace and determinant of the following 2x2 matrices and then write down the corresponding characteristic equation.

(a)

$$\mathbf{A} = \begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix}$$

(b)

$$\mathbf{A} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

(c)

$$\mathbf{A} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Finally, let's consider the solutions of the characteristic equation for a 2x2 matrix. Using the quadratic formula we have

$$\lambda = \frac{tr(\mathbf{A}) \pm \sqrt{tr(\mathbf{A})^2 - 4det(\mathbf{A})}}{2}$$

If you recall all the work you did in school with the solutions to the quadratic, you will notice that there are two solutions as expected, one for each eigenvalue. Furthermore, the solutions may be *complex* if

$$tr(\mathbf{A})^2 - 4det(\mathbf{A}) < 0$$

For example, let's find the eigenvalues of the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

The trace and determinant are

$$tr(\mathbf{A}) = 1 + 1 = 2$$

$$det(\mathbf{A}) = (1)(1) - (-1)(1) = 2$$

so the characteristic equation is

$$\lambda^2 - 2\lambda + 2 = 0$$

which has solutions

$$\lambda = \frac{2 \pm \sqrt{4-8}}{2} = 1 \pm i$$

so $\lambda_1 = 1 + i$ and $\lambda_2 = 1 - i$. Alternatively, we could write these complex eigenvalues using *Euler's* formula

$$Ae^{i\phi} = A(\cos\phi + i\sin\phi)$$

where A is the amplitude and ϕ is the phase. The amplitude is determined by

$$A = \sqrt{1^2 + 1^2} = \sqrt{2}$$

and the phase is determined by

$$\tan\phi = \frac{1}{1}$$

or simply

$$\phi = \pi/4$$

so that the eigenvalues are

$$\lambda_1 = \sqrt{2}e^{i\pi/4},\ \lambda_2 = \sqrt{2}e^{-i\pi/4}$$

11. Use the solutions of the characteristic equation to find the eigenvalues of the following 2x2 matrices.

    (a)
    $$\mathbf{A} = \begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix}$$

    (b)
    $$\mathbf{A} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

    (c)
    $$\mathbf{A} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

12. Use the solutions of the characteristic equation to prove that $\lambda_1 + \lambda_2 = tr(\mathbf{A})$.

13. Use the solutions of the characteristic equation to prove that $\lambda_1\lambda_2 = det(\mathbf{A})$.

14. Use the solutions of the characteristic equation to prove that the eigenvalues of a symmetric 2x2 matrix are real.

What about the eigenvectors corresponding to complex eigenvalues? Let's consider our previous example,

$$\mathbf{A} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

which has eigenvalues $\lambda_1 = 1 + i$, and $\lambda_2 = 1 - i$. Let's find the eigenvector corresponding to λ_1. We seek a non-zero solution to $\mathbf{Av} = \lambda_1\mathbf{v}_1$. Set $\mathbf{v}_1 = \begin{bmatrix} x & y \end{bmatrix}^T$ to give

$$\begin{bmatrix} 1-(1+i) & -1 \\ 1 & 1-(1+i) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

which simplifies to

$$\begin{bmatrix} -i & 1 \\ 1 & -i \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

If we multiply the second row by $-i$ we obtain the first row so both of the equations are equivalent. The solution to the first equation is obtained from

$$-ix + y = 0$$

which has solution

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ i \end{bmatrix}$$

So, perhaps not surprisingly, the eigenvector $\mathbf{v}_1$ is complex.

15. Show that the eigenvector corresponding to $\lambda_2 = 1 - i$ is

$$\mathbf{v}_2 = \begin{bmatrix} 1 \\ -i \end{bmatrix}$$

Notice that the eigenvalues come in a complex conjugate pair and so do the eigenvectors.

16. Find the eigenvectors of the following 2x2 matrices.

(a)

$$\mathbf{A} = \begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix}$$

(b)

$$\mathbf{A} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

(c)

$$\mathbf{A} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$