BB8: SVD

Eric Miller - QEA - April 13, 2016

Definitions

In[1]:=
```
SetDirectory[NotebookDirectory[]];
```

Eventually, these functions may want to use the Notation` package to actually make the subscripted letters have downvalues. For now, I'm just being careful.

In[2]:=
```
Clear@ClearSubscripts
ClearSubscripts[top_] := (
   (Unset[Evaluate[#[[1]]]];
      Evaluate[#[[1]]]) & /@ Select[DownValues[Subscript], #[[1, 1, 1]] == top &]
  )
Clear@unknownArray
unknownArray[top_, dim_ /; NumberQ[dim]] := (
   Table[top_i, {i, dim}]
  )
unknownArray[top_, {dim1_ /; NumberQ[dim1], dim2_ /; NumberQ[dim2]}] := (
   Table[top_{i,j}, {i, dim1}, {j, dim2}]
  )
```

Eigenvectors of Data

Because we are defining the eigenvectors (including v_M) to be normalized, $\| v_M \| = \| w \| = 1$.

$$u = \frac{w}{\| w \|} = w$$

Equation 7 in the packet states that

$$\sigma_w{}^2 = \frac{w^T R\, w}{w^T\, w} \tag{1}$$

The fact that w is an eigenvector means $R\, w = \lambda_M\, w$.

$$\sigma_w{}^2 = \frac{w^T \lambda_M\, w}{w^T\, w} \tag{2}$$

The commutivity of scalar multiplication implies

$$\sigma_w{}^2 = \lambda_M \frac{w^T\, w}{w^T\, w} = \lambda_M \tag{3}$$

QED

SVD Image Compression

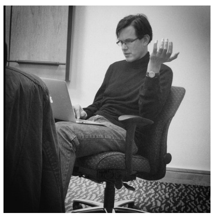
Compression

Without looking at the provided sample code at all, I decided to write my own in *Mathematica*. It turns out we were *supposed* to do that. Oops.

In[41]:=
```
mark = ColorConvert[
```

```
, "Grayscale"]
```

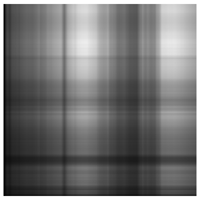
Out[41]=

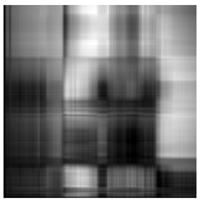


In[44]:=
```
compress[n_, image_] := Module[{u, w, v, u2, w2, v2},
  {u, w, v} = SingularValueDecomposition[ImageData@image];
  u2 = u[[All, 1 ;; n]];
  w2 = w[[1 ;; n, 1 ;; n]];
  v2 = v[[All, 1 ;; n]];
  Image[u2.w2.Transpose@v2]
 ]
```

In[486]:=
```
Grid[Table[{i, Image[compress[i, mark], ImageSize → Tiny]},
  {i, {1, 3, 5, 10, 20, 50, 100, 200}}]]
```

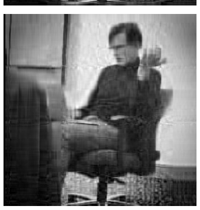
Out[486]=

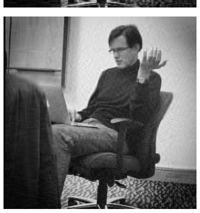
1 

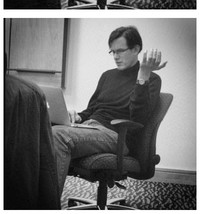
3 

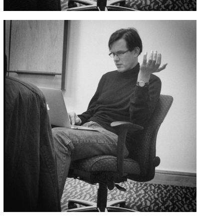
5 

10 

20 

50 

100 

200 

Exercises

In[152]:=
```
A = {{3, 2}, {2, 3}, {2, -2}};
Grid@Eigensystem[A.Transpose@A]
Grid@Eigensystem[Transpose[A].A]
```

$$A = \begin{pmatrix} 3 & 2 \\ 2 & 3 \\ 2 & -2 \end{pmatrix};$$

Out[153]=

25	9	0
{1, 1, 0}	{1, -1, 4}	{-2, 2, 1}

Out[154]=

25	9
{1, 1}	{-1, 1}

In[155]:=
```
MatrixForm[vt = Normalize /@ Eigenvectors[Transpose@A.A]]
vt[[All, 1]].vt[[All, 2]]
sigma = Sqrt@{{25, 0}, {0, 9}};
MatrixForm[u = Transpose[Normalize /@ Eigenvectors[A.Transpose@A][[1 ;; 2]] * {1, -1}]]
u[[All, 1]].u[[All, 2]]
```

$$\text{sigma} = \text{Sqrt@}\begin{pmatrix} 25 & 0 \\ 0 & 9 \end{pmatrix};$$

Out[155]//MatrixForm=
$$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

Out[156]= 0

Out[158]//MatrixForm=
$$\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{3\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{3\sqrt{2}} \\ 0 & -\frac{2\sqrt{2}}{3} \end{pmatrix}$$

Out[159]= 0

In[160]:=
```
MatrixForm /@ {A, u.sigma.vt}
```

Out[160]=
$$\left\{ \begin{pmatrix} 3 & 2 \\ 2 & 3 \\ 2 & -2 \end{pmatrix}, \begin{pmatrix} 3 & 2 \\ 2 & 3 \\ 2 & -2 \end{pmatrix} \right\}$$

SVD

1.a.

In[173]:=
```
A = {{1, 1}, {0, 1}, {-1, 1}};
Grid@Eigensystem[A.Transpose@A]
Grid@Eigensystem[Transpose@A.A]
```

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \end{pmatrix};$$

Out[174]=

3	2	0
{1, 1, 1}	{-1, 0, 1}	{1, -2, 1}

Out[175]=

3	2
{0, 1}	{1, 0}

Now, find the major components.

```
In[285]:= eig1 = Normalize /@ Eigenvectors[A.Transpose@A, 2] * {1, -1};
          u = Transpose@eig1;
          HoldForm@u == (u // MatrixForm)

          lambda = DiagonalMatrix@Sqrt@Eigenvalues[Transpose@A.A];
          HoldForm@Λ == (lambda // MatrixForm)

          eig2 = Normalize /@ Eigenvectors[Transpose@A.A, 2];
          v = Transpose@eig2;
          HoldForm@v == (v // MatrixForm)
```

Out[287]= $u == \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$

Out[289]= $\Lambda == \begin{pmatrix} \sqrt{3} & 0 \\ 0 & \sqrt{2} \end{pmatrix}$

Out[292]= $v == \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

Finally, verify that the solution works.

```
In[294]:= HoldForm@A == MatrixForm@A == MatrixForm[u.lambda.Transpose[v]]
          A == u.lambda.Transpose[v]
```

Out[294]= $A == \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \end{pmatrix} == \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \end{pmatrix}$

Out[295]= True

1.b.

In[315]:= $A = \begin{pmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{pmatrix}$;

```
          Grid@Eigensystem[A.Transpose@A]
          Grid@Eigensystem[Transpose@A.A]
```

Out[316]=

25	9
{1, 1}	{-1, 1}

Out[317]=

25	9	0
{1, 1, 0}	{1, -1, 4}	{-2, 2, 1}

Now, find the major components.

```
In[318]:= eig1 = Normalize /@ Eigenvectors[A.Transpose@A, 2] * {1, -1};
          u = Transpose@eig1;
          HoldForm@u == (u // MatrixForm)

          lambda = DiagonalMatrix@Sqrt@Eigenvalues[Transpose@A.A, 2];
          HoldForm@Λ == (lambda // MatrixForm)

          eig2 = Normalize /@ Eigenvectors[Transpose@A.A, 2];
          v = Transpose@eig2;
          HoldForm@v == (v // MatrixForm)
```

Out[320]= $u == \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$

Out[322]= $\Lambda == \begin{pmatrix} 5 & 0 \\ 0 & 3 \end{pmatrix}$

Out[325]= $v == \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{3\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{3\sqrt{2}} \\ 0 & \frac{2\sqrt{2}}{3} \end{pmatrix}$

Finally, verify that the solution works.

```
In[326]:= HoldForm@A == MatrixForm@A == MatrixForm[u.lambda.Transpose[v]]
          A == u.lambda.Transpose[v]
```

Out[326]= $A == \begin{pmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{pmatrix} == \begin{pmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{pmatrix}$

Out[327]= True

2.

When I apply SingularValueDecomposition[], the result tends to have an extra zero row floating around. I think it's from the *reduced* thing hinted at earlier.

In[328]:= `MatrixForm /@ SingularValueDecomposition@` $\begin{pmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \end{pmatrix}$

`MatrixForm /@ SingularValueDecomposition@` $\begin{pmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{pmatrix}$

Out[328]= $\left\{ \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & -\sqrt{\frac{2}{3}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{pmatrix}, \begin{pmatrix} \sqrt{3} & 0 \\ 0 & \sqrt{2} \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}$

Out[329]= $\left\{ \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}, \begin{pmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{3\sqrt{2}} & -\frac{2}{3} \\ \frac{1}{\sqrt{2}} & \frac{1}{3\sqrt{2}} & \frac{2}{3} \\ 0 & -\frac{2\sqrt{2}}{3} & \frac{1}{3} \end{pmatrix} \right\}$

Applications

Fibbonacci

Temperature Compression

In[392]:=
```
Clear[bTr, wTr, sTr, bNew, sNew, wNew]
```

In[396]:=
```
{wTr, wNew, bTr, bNew, sNew, sTr, pNew, pTr} =
  Import["avg_temperatures_pt2.mat"][[All, All, 1]];
```

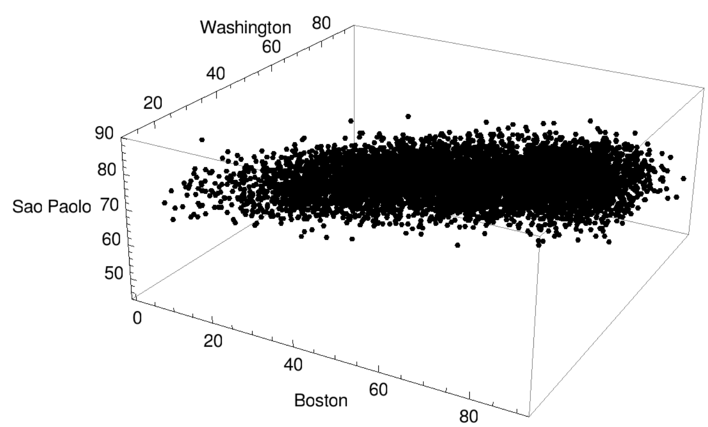
In[391]:=
```
temps[[All, 1]]
```

Out[391]=
```
{w_tr, w_new, b_tr, b_new, s_new, s_tr, p_new, p_tr}
```

Analyze the input data

In[405]:=
```
ListPointPlot3D[Transpose@{bTr, wTr, sTr},
 AxesLabel → {"Boston", "Washington", "Sao Paolo"}]
```

Out[405]=



Build a covariance matrix

In[424]:=
```
R = Covariance[Transpose@{bTr, wTr, sTr}];
R // MatrixForm
Grid@N@Round[Eigensystem@R, 10^-2]
```

Out[425]//MatrixForm=

$$\begin{pmatrix} 281.993 & 267.413 & -57.3231 \\ 267.413 & 282.566 & -57.5062 \\ -57.3231 & -57.5062 & 39.7308 \end{pmatrix}$$

Out[426]=

562.31	27.12	14.87
{-0.7, -0.7, 0.15}	{0.11, 0.1, 0.99}	{0.71, -0.71, -0.01}

In[427]:=
```
Vp = Transpose@Eigenvectors[R, 2];
Vp // MatrixForm
```

Out[428]//MatrixForm=

$$\begin{pmatrix} -0.698331 & 0.11341 \\ -0.699114 & 0.103721 \\ 0.153535 & 0.988119 \end{pmatrix}$$

Start working toward compression

```
In[434]:= T = # - Mean@# & /@ {bTr, wTr, sTr};
          T[[All, 1 ;; 5]] // MatrixForm
```

Out[435]//MatrixForm=

$$\begin{pmatrix} -13.2821 & -14.5821 & -22.0821 & -21.1821 & -31.1821 \\ -17.7643 & -18.5643 & -29.0643 & -25.3643 & -37.4643 \\ 3.37712 & 5.07712 & 4.27712 & 4.27712 & 6.87712 \end{pmatrix}$$

Do the compression

```
In[436]:= compressed = Transpose[Vp].T;
          compressed[[All, 1 ;; 5]] // MatrixForm
```

Out[437]//MatrixForm=

$$\begin{pmatrix} 22.2131 & 23.9413 & 36.3966 & 33.1814 & 49.0232 \\ -0.0118712 & 1.43752 & -1.29262 & -0.806785 & -0.626801 \end{pmatrix}$$

Decompress

```
In[438]:= reconstructed = Vp.compressed;
          reconstructed[[All, 1 ;; 5]] // MatrixForm
```

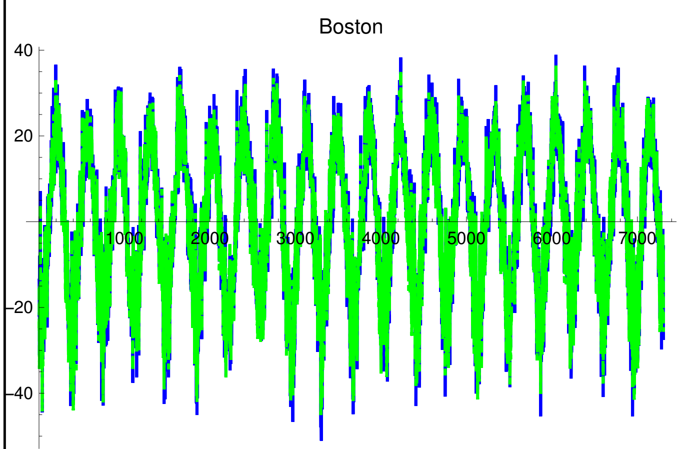
Out[439]//MatrixForm=

$$\begin{pmatrix} -15.5135 & -16.5559 & -25.5635 & -23.2631 & -34.3055 \\ -15.5308 & -16.5886 & -25.5795 & -23.2813 & -34.3378 \\ 3.39876 & 5.09626 & 4.31089 & 4.2973 & 6.90741 \end{pmatrix}$$

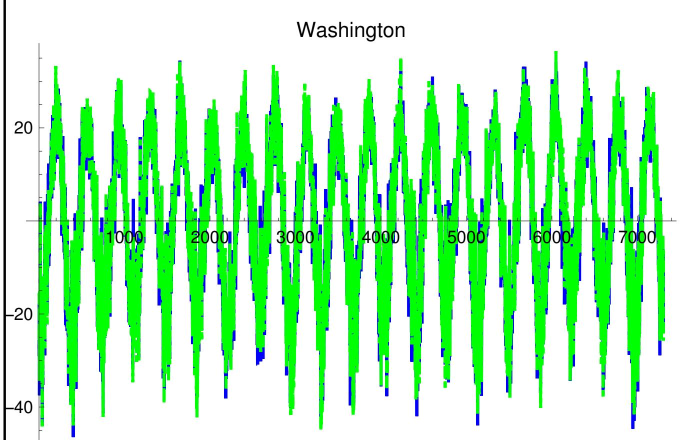
Analyze the results

```
In[487]:= (* Many apologies for the cluttered appearance of this code. I promise it
           isn't that bad. *)
          cities = {"Boston", "Washington", "Sao Paolo"};
          Grid[
           Table[{ListLinePlot[{T[[i]], reconstructed[[i]]}, PlotStyle → {Blue, Green},
               PlotMarkers → None, PlotLabel → cities[[i]], ImageSize → 350],
              CorrelationDistance[T[[i]], reconstructed[[i]]]}, {i, 3}], Spacings → {0, 3},
           Frame → {False, All}]
```

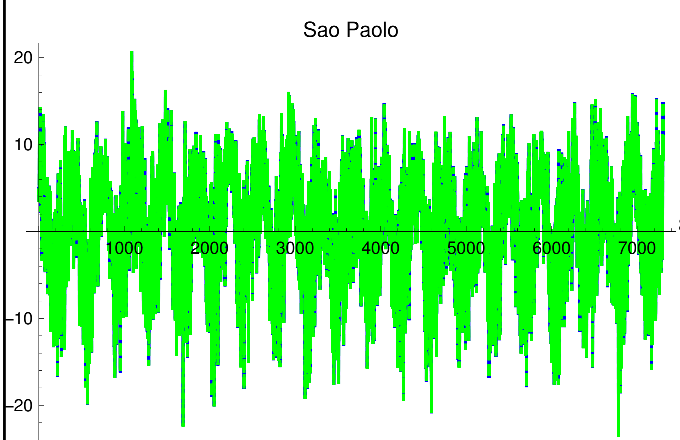
Out[488]=



0.0132532



0.013253



8.79109×10^{-6}

Misc