Parametric Curves and Surfaces

Quantitative Engineering Analysis

Overview and Orientation

In Shapes I we discussed the representation of curves and surfaces using explicit or implicit functions. In Shapes II we discussed how to compute derivatives and integrals of functions of many variables using explicit or implicit functions to define the regions of integration. In Shapes III we will generalize all of these ideas to curves and surfaces defined *parametrically*. Parametric curves and surfaces is the most general, and in many cases the most useful, way to define these entities. Generally speaking, you can open up pretty much any multivariable calculus book and read about parametric curves and surfaces. You will also find plenty of readings and videos on the web.

I estimate (very very roughly) that it will take about 3 hours on the parametric curves (8 questions), and about 3 hours on the parametric surfaces (2 questions) - there is an additional optional question on surfaces for those of you who are interested AND have the time. This estimate is based on my assumption that you've probably "seen" a parametric curve before, but not a parametric surface. I'm therefore assuming that it will take you much longer to wrap your head around parametric surfaces. I therefore suggest that you do the first 4 questions on parametric curves, and then go on to look at the surfaces before coming back to the curves later.

Parametric Curves and their Length

In Shapes I we considered curves in the plane, represented by either explicit functions ($y = f(x)$ or $x = f(y)$) or an implicit function $f(x, y) = 0$. A more general representation involves defining x and y in terms of a third variable (or parameter) as follows,

$$x = f(u), y = g(u)$$

Each value of u defines a point $(f(u), g(u))$ which we can plot. If we collect all the points defined by $u \in [a, b]$, then we get a parametric curve.

There is no reason to limit ourselves to curves in the plane. For example, in 3D we simply define

$$x = f(u), y = g(u), z = h(u)$$

and the collection of points so defined trace out a curve in 3D.

An alternative to these coordinate definitions involves representing each point with its position vector, $\mathbf{r}(u)$. Since the position vector depends on a single parameter u, the end of the position vector traces out a curve in space. We will prefer to use the following notation

$$\mathbf{r}(u) = x(u)\mathbf{i} + y(u)\mathbf{j} + z(u)\mathbf{k},\ u \in [a, b]$$

which defines a parametric function $\mathbf{r}(u)$. One major advantage of this notation is that we can take derivatives of this vector function

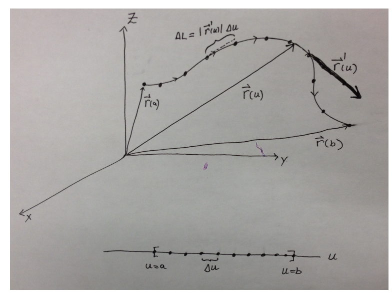
$$\mathbf{r}'(u) = x'(u)\mathbf{i} + y'(u)\mathbf{j} + z'(u)\mathbf{k}$$

which can be interpreted as follows: for any given value of u this vector is tangent to the parametric curve. At times we might be more interested in a unit tangent vector $\mathbf{T}$, which we can obtain by normalizing

$$\mathbf{T} = \frac{\mathbf{r}'}{|\mathbf{r}'|}$$

Since we can also interpret this vector function as a position vector, taking the derivative should produce a vector tangent to the tangent vector, which we will define as the normal vector. The unit normal vector $\mathbf{N}$ is therefore

$$\mathbf{N} = \frac{\mathbf{T}'}{|\mathbf{T}'|}$$



Now that we know how to define a general parametric curve, we are ready to compute with it. For example, we could compute the length of the curve. In order to do so, let's lay down a set of points in the u-domain separated by Δu. Each point is mapped to the space curve, and the approximate length of each section of the curve is

$$\Delta L = |\mathbf{r}'(u)|\Delta u$$

Refining this for smaller Δu and then summing up the pieces results in the integral

$$L = \int_a^b |\mathbf{r}'(u)|\, du$$

which defines the length of the curve.

Example

A common example is the parametric representation for a circle of radius R, centered at the origin in the xy-plane. If we define

$$\begin{aligned} x &= R\cos(u) \\ y &= R\sin(u) \end{aligned}$$

for $u \in [0, 2\pi]$ then the circle is traced out once in the counterclockwise direction starting at $(R, 0)$. In this way, we can identify the parameter u as being the angle from the x-axis to a point on the circle. Our more compact notation would simply read

$$\mathbf{r}(u) = R\cos(u)\mathbf{i} + R\sin(u)\mathbf{j},\ u \in [0, 2\pi]$$

which has derivative

$$\mathbf{r}'(u) = -R\sin(u)\mathbf{i} + R\cos(u)\mathbf{j}$$

The unit tangent vector is

$$\begin{aligned} \mathbf{T} &= \frac{\mathbf{r}'}{|\mathbf{r}'|} \\ &= -\sin(u)\mathbf{i} + \cos(u)\mathbf{j} \end{aligned}$$

and the unit normal vector is

$$\begin{aligned} \mathbf{N} &= \frac{\mathbf{T}'}{|\mathbf{T}'|} \\ &= -\cos(u)\mathbf{i} - \sin(u)\mathbf{j} \end{aligned}$$

For the length of the curve, we have

$$|\mathbf{r}'(u)| = R$$

and the integral becomes

$$L = \int_0^{2\pi} R du = 2\pi R$$

which is the circumference of a circle of radius R.

1. Find a parameterized function $\mathbf{r}(u)$ in the plane whose trace is a circle centered at (x_0, y_0) with radius R. What is the unit tangent vector? What is the unit normal vector? Choose values for (x_0, y_0) and R, and use Matlab or Mathematica to visualize the circle and to compute its length.

2. Find a parameterized function $\mathbf{r}(u)$ in the plane whose trace is an ellipse centered at (x_0, y_0) with semi-major axis a, and semi-minor axis b, $b < a$. What is the unit tangent vector? What is the unit normal vector? Choose values for (x_0, y_0), a, and b, and use Matlab or Mathematica to visualize the ellipse and to compute its length.

3. A logarithmic spiral in the plane can be defined by the parameterized function

$$\mathbf{r}(u) = ae^{bu}\cos(u)\mathbf{i} + ae^{bu}\sin(u)\mathbf{j},\ a > 0, b < 0, u \in [0, \infty)$$

(a) Use Matlab or Mathematica to visualize this curve for different values of a and b. What happens as $u \to \infty$? What is the unit tangent vector? What is the unit normal vector?

(b) Interpret the variables a and b, and the parameter u.

(c) Choose values for a and b, and use Matlab or Mathematica to compute the length of the spiral for $u \in [0, \infty]$.

4. A helix in 3D can be defined by the parameterized function

$$\mathbf{r}(u) = a\cos(u)\mathbf{i} + a\sin(u)\mathbf{j} + bu\mathbf{k}, a > 0, b > 0, u \geq 0$$

(a) Use Matlab or Mathematica to visualize this curve for different values of a and b. What is the unit tangent vector? Should I ask you about the unit normal vector in this case?

(b) Interpret the variables a and b, and the parameter u.

(c) Choose values for a and b, and use Matlab or Mathematics to compute the length of the helix corresponding to 5 complete turns.

5. Recall the boat hull definition, $z = |y|^n - 1$, for $-1 < y < 1$.

(a) Propose a parametric hull definition, and use it to visualize the hull in Matlab or Mathematica for different values of n. What is the unit tangent vector? What is the unit normal vector?

(b) Set up the integral for the length of the curve that defines the hull and evaluate it using Matlab or Mathematica. Plot the length as a function of n and explain.

(c) Assuming that the boat floats flat, find the depth of the water line assuming that the density of the boat is half of that of water (you already did this - just remind yourself).

(d) Using your parametric hull definition, compute the length of the hull underwater. Again plot as a function of n and explain. (The amount of the boat underwater plays a critical role in considering drag.)

Bézier Curves

A Bézier curve is a parameterized curve often used in CAD and related fields. It is defined by a set of control points $\mathbf{a}_0, \mathbf{a}_1, \ldots, \mathbf{a}_n$. The first and last control points lie on the curve, but in general the intermediate control points do not. The shape of a curve can be tuned by moving the intermediate control points.

6. The *linear* Bézier curve is defined by

$$\mathbf{r}(u) = (1-u)\mathbf{a}_0 + u\mathbf{a}_1, \ u \in [0,1]$$

Define two arbitrary points in the plane, and use Matlab or Mathematics to visualize the linear Bézier curve that connects them. What kind of curve is this?

7. The *quadratic* Bézier curve is defined by

$$\mathbf{r}(u) = (1-u)^2\mathbf{a}_0 + 2u(1-u)\mathbf{a}_1 + u^2\mathbf{a}_2, \ u \in [0,1]$$

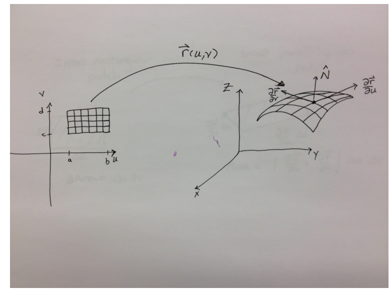
Define three arbitrary points in the plane, and use Matlab or Mathematics to visualize the quadratic Bézier curve that connects the first and last control point. What happens when you move the second control point around? Show that the tangent lines to the curve at $\mathbf{a}_0$ and $\mathbf{a}_2$ intersect at $\mathbf{a}_1$.

8. The *cubic* Bézier curve is defined by

$$\mathbf{r}(u) = (1-u)^3\mathbf{a}_0 + 3u(1-u)^2\mathbf{a}_1 + 3u^2(1-u)\mathbf{a}_2 + u^3\mathbf{a}_3, \ u \in [0,1]$$

Define four arbitrary points in the plane, and use Matlab or Mathematics to visualize the cubic Bézier curve that connects the first and last control point. What happens when you move the second and third control points around? Show that the tangent line to the curve at $\mathbf{a}_0$ intersects $\mathbf{a}_1$ and the tangent line to the curve at $\mathbf{a}_3$ intersects $\mathbf{a}_2$.

Parametric Surfaces and their Area

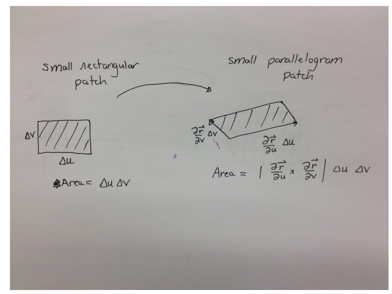


A parametric surface can be defined with a position vector that depends on two parameters,

$$\mathbf{r}(u,v) = x(u,v)\mathbf{i} + y(u,v)\mathbf{j} + z(u,v)\mathbf{k},\ u \in [a,b],\ v \in [c,d].$$

In a sense the parametric function $\mathbf{r}$ lifts the rectangle from the uv-domain, and deforms it in order to produce a surface. Notice in the figure how the grid in the uv-plane becomes a set of coordinate curves on the surface.

What about each of the partial derivatives, $\frac{\partial \mathbf{r}}{\partial u}$ and $\frac{\partial \mathbf{r}}{\partial v}$? Each one defines a tangent vector to the surface, and taken together they belong to the tangent plane. The normal vector can be obtained by taking their cross-product, so the unit normal vector is



$$\mathbf{N} = \frac{\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}}{|\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}|}$$

What about the surface area? Each small rectangular patch in the uv-plane gets lifted onto the surface and is well approximated by a parallelogram. One side of the parallelogram is $\frac{\partial \mathbf{r}}{\partial u}\Delta u$ and the other is $\frac{\partial \mathbf{r}}{\partial v}\Delta v$. The area of this parallelogram is then

$$\Delta A = |\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}|\ \Delta u \Delta v$$

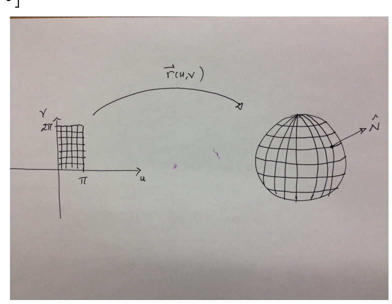
Refining the size of the rectangular patch in the uv-plane gives the surface area as the following double integral

$$A = \iint |\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}|\ dudv$$

Example

A good example is the parametric equations for the surface of a sphere. A very common choice for a sphere of radius R centered at $(0,0,0)$ is

$$\mathbf{r}(u,v) = R\sin(u)\cos(v)\mathbf{i} + R\sin(u)\sin(v)\mathbf{j} + R\cos(u)\mathbf{k}, u \in [0,\pi], v \in [0,2\pi]$$



With this choice of parameterization, we can interpret u and v as the angles shown in the figure. Notice how the grid in the uv-plane becomes a set of curves on the surface which resemble the lines of latitude and longitude. Each derivative is

$$\begin{aligned}
\frac{\partial \mathbf{r}}{\partial u} &= R\cos(u)\cos(v)\mathbf{i} + R\cos(u)\sin(v)\mathbf{j} - R\sin(u)\mathbf{k} \\
\frac{\partial \mathbf{r}}{\partial v} &= -R\sin(u)\sin(v)\mathbf{i} + R\sin(u)\cos(v)\mathbf{j}
\end{aligned}$$

and the cross-product is

$$\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} = R^2 \cos(v) \sin^2(u)\mathbf{i} + R^2 \sin(v) \sin^2(u)\mathbf{j} + R^2 \cos(u) \sin(u)\mathbf{k}$$

with magnitude

$$\left|\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\right| = R^2 \sin(u)$$

so that the normal vector becomes

$$\mathbf{N} = \cos(v) \sin(u)\mathbf{i} + \sin(v) \sin(u)\mathbf{j} + \cos(u)\mathbf{k}$$

and the surface area of the sphere is

$$\begin{aligned} \text{Area} &= \int_0^{2\pi} \int_0^{\pi} R^2 \sin(u)\; du dv \\ &= \int_0^{2\pi} -R^2 \cos(u)|_0^{\pi}\; dv \\ &= \int_0^{2\pi} 2R^2\; dv \\ &= 4\pi R^2 \end{aligned}$$

which agrees with the well known result!

9. If you love donuts you will want to use Matlab or Mathematica to visualize the following parametric surface, and compute the unit normal vector and its surface area

   $$\mathbf{r}(u,v) = (a + r\cos(u))\cos(v)\mathbf{i} + (a + r\cos(u))\sin(v)\mathbf{j} + r\sin(u)\mathbf{k}$$

   where $r < a$.

10. Recall the explicit hull definition that we proposed for the *Spray*,

    $$y = \frac{W}{2}\frac{4}{L^2}x^2 - \frac{W}{2}\sqrt{\frac{z+H}{H}}$$

    where W is the width, L is the length, and H is the height. The domain of definition of the surface was also given explicitly by

    $$z = -H + H\frac{16}{L^4}x^4$$

    for $x \in [-L/2, L/2]$. Propose a parametric representation for this surface and use Matlab or Mathematica to visualize it and to compute the surface area.

Bézier Surfaces – OPTIONAL

Bézier curves can be generalized in order to build a Bézier surface. A Bézier surface of degree (n,m) is defined by a set of $(n+1)(m+1)$ control $\mathbf{a}_{ij}$. The general formulation is

$$\mathbf{r}(u,v) = \sum_{i=0}^{n} \sum_{j=0}^{m} B_i^n(u) B_j^m(v), u \in [0,1], v \in [0,1]$$

where

$$B_i^n(u) = \frac{n!}{i!(n-i)!} u^i (1-u)^{n-i}$$

is a polynomial in u. A bi-cubic Bézier surface corresponds to $n = m = 3$, and requires 16 control points. Four of these lie on the surface, and the other 12 can be used to control the surface. More complicated surfaces can be built by gluing together patches built from Bézier surfaces.

11. Build a visualization in Matlab or Mathematica of a bi-cubic Bézier surface, and use the control points to play with the surface. Which four points lie on the surface? Which eight points control the end slopes of the boundary curves? What do the other four points control?