Definitions

Eigenvalues and Eigenvectors of a Diagonal Matrix

Eigenvalues of a Triangular Matrix

```
Solve[(1 1
       0 1).{x, y} == {x, y}, {x, y}]
```

Solve::svars : Equations may not give solutions for all "solve" variables. ≫

```
{{y → 0}}
```

```
Roots[Det[(a b
           0 c) - λ IdentityMatrix[2]] == 0, λ]
```

```
λ == c || λ == a
```

3.

$$\begin{bmatrix} 2 & 1 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} 1/5 \\ -1 \end{bmatrix} = \begin{bmatrix} 2/5 - 1 \\ 0 + 3 \end{bmatrix} = \begin{bmatrix} -3/5 \\ 3 \end{bmatrix} = -3 \cdot \begin{bmatrix} 1/5 \\ -1 \end{bmatrix}$$

4.

I decided to automate here, and gained a ton of insight from doing so. In particular, thinking about solving the equation $A.x = \lambda.x$ for known λ as looking for the null space of a matrix is really cool.

```
In[7]:= triangleEigens[A_] := Module[{},
          values = Diagonal[A];
          vectors = Table[MatrixForm@
              Transpose@NullSpace[A - val IdentityMatrix[Length@A]], {val, values}];
          Grid[Transpose[{values,
              vectors}], Frame → All]
        ]
```

```
triangleEigens[(1 0
                1 2)]
```

1	$\begin{pmatrix} -1 \\ 1 \end{pmatrix}$
2	$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

```
triangleEigens[{{1, 0, 0}, {-1, 2, 0}, {2, 1, 3}}]
```

$$\text{triangleEigens}\left[\begin{pmatrix} 1 & 0 & 0 \\ -1 & 2 & 0 \\ 2 & 1 & 3 \end{pmatrix}\right]$$

1	$\begin{pmatrix} -2 \\ -2 \\ 3 \end{pmatrix}$
2	$\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$
3	$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$$\text{triangleEigens}\left[\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 4 \end{pmatrix}\right]$$

2	$\begin{pmatrix} 0 & 1 \\ -2 & 0 \\ 1 & 0 \end{pmatrix}$
2	$\begin{pmatrix} 0 & 1 \\ -2 & 0 \\ 1 & 0 \end{pmatrix}$
4	$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

This last problem is particularly interesting because the value 2 appears multiple times on the diagonal, and is thus a repeated eigenvalue. There *are* still three distinct eigenvectors, but pairing them up nicely is difficult.

The Characteristic Equation

Nice explanation! When I first learned this, it seemed unmotivated, but seeing the derivation this plainly makes it obvious.

```
In[8]:= myCharacteristicEqn[A_] := Module[{},
          Factor@Det[A - λ IdentityMatrix[Length@A]] == 0
        ]
```

Diagonal matrices

```
myCharacteristicEqn[DiagonalMatrix[{-3, -1, 4}]]
```

$-(-4+\lambda)\ (1+\lambda)\ (3+\lambda) == 0$

I don’t know why exactly the leading - snuck in, but it isn’t particularly important.

```
myCharacteristicEqn[DiagonalMatrix[{1, 1, 2}]]
```

$-(-2+\lambda)\ (-1+\lambda)^2 == 0$

As expected, there is a squared term representing the duplicate eigenvalue.

```
myCharacteristicEqn[DiagonalMatrix[{2, 4, 0}]]
```

$-(-4+\lambda)\ (-2+\lambda)\ \lambda == 0$

And bare eigenvalues of 0 work too!

Triangular Matrices

```
myCharacteristicEqn[(1 0
                     1 2)]
```

$(-2+\lambda)\ (-1+\lambda) == 0$

```
myCharacteristicEqn[( 1 0 0
                     -1 2 0
                      2 1 3)]
```

$-(-3+\lambda)\ (-2+\lambda)\ (-1+\lambda) == 0$

```
myCharacteristicEqn[(2 0 0
                     0 2 0
                     0 1 4)]
```

$-(-4+\lambda)\ (-2+\lambda)^2 == 0$

8.

Because A is triangular, $A-\lambda$I must also be diagonal. We showed earlier that this implies that det($A-\lambda$I) is the product of the diagonal elements of that matrix, which are first-degree polynomials of the form ($a-\lambda$) where $a \in$ Diagonal(A). Thus, their product is a degree-n polynomial with roots consisting of the diagonal elements of A.

7.

Diagonal matrices are triangular. See problem 8.

Eigenvalues and Eigenvectors of 2x2

9.

```
In[22]:= A = (18 -2
              12  7);
         15 → MatrixForm@NullSpace[A - 15 IdentityMatrix[2]][[1]]
         10 → MatrixForm@NullSpace[A - 10 IdentityMatrix[2]][[1]]
```

Out[23]= $15 \to \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

Out[24]= $10 \to \begin{pmatrix} 1 \\ 4 \end{pmatrix}$

10.

In[34]:=
```
A = (3  -2
     4  -1);
HoldForm@Det[A] == A[[1, 1]] * A[[2, 2]] - A[[1, 2]] * A[[2, 1]]
HoldForm@Tr[A] == Total@Diagonal@A
TraditionalForm[λ^2 - Tr[A] λ + Det[A] == 0]
```

Out[35]= Det[A] == 5

Out[36]= Tr[A] == 2

Out[37]//TraditionalForm=

$$\lambda^2 - 2\,\lambda + 5 = 0$$

Relatively standard

In[38]:=
```
A = (0  -1
     1   0);
HoldForm@Det[A] == A[[1, 1]] * A[[2, 2]] - A[[1, 2]] * A[[2, 1]]
HoldForm@Tr[A] == Total@Diagonal@A
TraditionalForm[λ^2 - Tr[A] λ + Det[A] == 0]
```

Out[39]= Det[A] == 1

Out[40]= Tr[A] == 0

Out[41]//TraditionalForm=

$$\lambda^2 + 1 = 0$$

This equation has no real roots.

In[46]:=
```
A = (Cos[θ]  -Sin[θ]
     Sin@θ    Cos@θ);
HoldForm@Det[A] == A[[1, 1]] * A[[2, 2]] - A[[1, 2]] * A[[2, 1]]
HoldForm@Tr[A] == Total@Diagonal@A
TraditionalForm[Simplify[λ^2 - Tr[A] λ + Det[A]] == 0]
```

Out[47]= Det[A] == Cos[θ]^2 + Sin[θ]^2

Out[48]= Tr[A] == 2 Cos[θ]

Out[49]//TraditionalForm=

$$-2\,\lambda\cos(\theta) + \lambda^2 + 1 = 0$$

After throwing on a simplify, lots of the trig goes away, which is nice.

11.

In[140]:=
```
myEigenValues[A_] := (
  Clear@λ;
  List @@ Roots[myCharacteristicEqn[A], λ][[All, 2]]
 )
```

In[163]:=
```
myEigenValues@(3 -2
               4 -1)
```
Out[163]= {1 - 2 i, 1 + 2 i}

In[164]:=
```
myEigenValues@(0 -1
               1  0)
```
Out[164]= {i, - i}

In[170]:=
```
myEigenValues@(Cos[θ]  -Sin[θ]
               Sin@θ    Cos@θ)
% == myEigenValues@RotationMatrix@θ
```
Out[170]= {Cos[θ] - i Sin[θ], Cos[θ] + i Sin[θ]}

Out[171]= True

12.

There is an awesome theorem showing that the sum of the roots of any polynomial in standard form is always $-\frac{b}{a}$ where b is the coefficient on the x^{n-1} term and a is the coefficient on the x^n term. In this case, $b = -\det(A) \wedge a = 1$, so the sum of the roots (the sum of the eigenvalues) must be $\det(A)$.

13.

A second related theorem shows that the product of the roots of any polynomial is $(-1)^n * \frac{z}{a}$ where n is the degree of the polynomial, and z is the constant term. In this case, $n = 2 \wedge z = \det(A)$, so the product of the roots is $\det(A)$.

14.

Consider $A = \begin{pmatrix} a & b \\ b & d \end{pmatrix}$. $\det(A) = a\,d - b^2$. $\text{tr}(a) = a + d$. In order for the eigenvalues to be complex, it is necessarily true that $\text{tr}(A)^2 - 4\det(A) < 0$. Expanding, $(a+d)^2 - 4\left(-b^2 + a\,d\right) < 0$, or $a^2 + 4\,b^2 - 2\,a\,d + d^2 < 0$. In the worst case, $b = 0$, so $a^2 - 2\,a\,d + d^2 = (a-d)^2 < 0$. Because $a - d$ is real, its square must be non-negative, so the discriminant of the equation must be nonnegative, and the eigenvalue(s) must be real.

In[57]:=
```
A = {{a, b}, {b, d}};
```

In[83]:=
```
Tr[A]^2 - 4 Det[A] < 0
Expand@%
% /. b → 0
Factor@%
```
Out[83]= $(a + d)^2 - 4\left(-b^2 + a\,d\right) < 0$

Out[84]= $a^2 + 4\,b^2 - 2\,a\,d + d^2 < 0$

Out[85]= $a^2 - 2\,a\,d + d^2 < 0$

Out[86]= $(a - d)^2 < 0$

Or, the lame way,

In[88]:= `Reduce[Tr[A]^2 - 4 Det[A] < 0]`

Out[88]= `False`

15.

As discovered Nathan and Sam, the problem here should read $v_2 = \begin{pmatrix} 1 \\ i \end{pmatrix}$.

In[123]:=
```
A = (1  -1
     1   1);
λ2 = 1 - ⅈ;
v2 = (1
      ⅈ);
A.v2 // MatrixForm
λ2 v2 // MatrixForm
```

Out[126]//MatrixForm=
$\begin{pmatrix} 1 - i \\ 1 + i \end{pmatrix}$

Out[127]//MatrixForm=
$\begin{pmatrix} 1 - i \\ 1 + i \end{pmatrix}$

16.

This uses definitions from problem 11.

In[151]:=
```
myEigenVectors[A_] := (
  DeleteDuplicates@
   Flatten[Map[NullSpace[A - # IdentityMatrix[Length@A]] &, myEigenValues[A]], 1]
 )
```

In[155]:= `MatrixForm /@ myEigenVectors@(3 -2; 4 -1)`

Out[155]= $\left\{\begin{pmatrix} 1 - i \\ 2 \end{pmatrix}, \begin{pmatrix} 1 + i \\ 2 \end{pmatrix}\right\}$

In[156]:= `MatrixForm /@ myEigenVectors@(0 -1; 1 0)`

Out[156]= $\left\{\begin{pmatrix} i \\ 1 \end{pmatrix}, \begin{pmatrix} -i \\ 1 \end{pmatrix}\right\}$

In[160]:=
```
MatrixForm /@ myEigenVectors@(Cos[θ]  -Sin[θ]
                              Sin@θ    Cos@θ)
MatrixForm /@ myEigenVectors@RotationMatrix[θ]
myEigenValues@RotationMatrix[θ]
```

Out[160]= $\left\{\begin{pmatrix} -i \\ 1 \end{pmatrix}, \begin{pmatrix} i \\ 1 \end{pmatrix}\right\}$

Out[161]= $\left\{\begin{pmatrix} -i \\ 1 \end{pmatrix}, \begin{pmatrix} i \\ 1 \end{pmatrix}\right\}$

Out[162]= $\{\mathrm{Cos}[\theta] - i\,\mathrm{Sin}[\theta], \mathrm{Cos}[\theta] + i\,\mathrm{Sin}[\theta]\}$

Misc