Shapes III

Setup

1: circle

```
Begin["One`"]
```

```
One`
```

Defining `r[u]`

```
r[θ_] := {R Sin[θ] + x0, R Cos[θ] + y0}
```

Tangent vector

```
t[u_] := r'[u] / Norm[r'[u]];
TraditionalForm@FullSimplify[t[u], {R > 0, u ∈ Reals}]
```

$\{\cos(u), -\sin(u)\}$

Normal vector

```
n[u_] := t'[u] / Norm[t'[u]]
TraditionalForm@FullSimplify[n[u], {R > 0, u ∈ Reals}]
```

$\{-\sin(u), -\cos(u)\}$

Visualization

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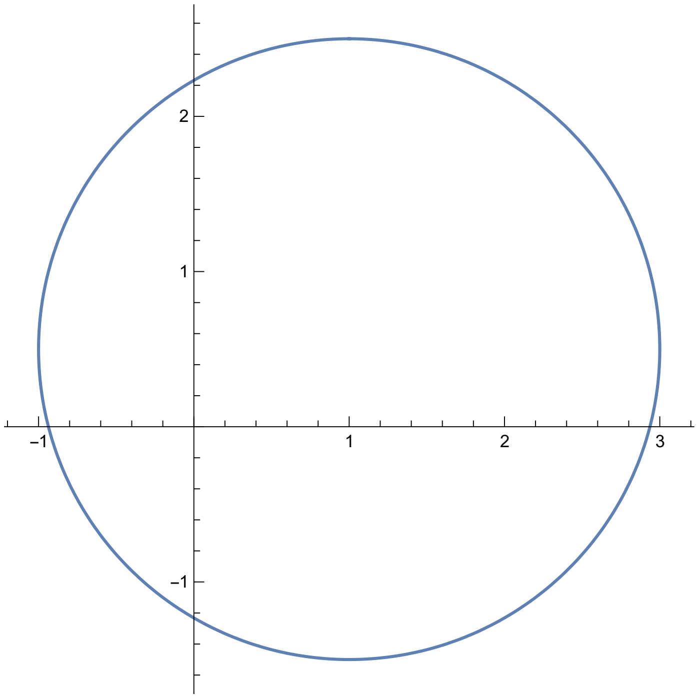
```
ParametricPlot[r[θ] /. {x0 → 1, y0 → .5, R → 2}, {θ, 0, 2 π}]
```



Curve length

```
l = FullSimplify[Integrate[Norm[r'[u]], {u, 0, 2 π}], {R > 0}]
```

$2\,\pi\,R$

```
End[]
```

One`

2: ellipse

```
Begin["Two`"]
```

Two`

Defining `r[u]`

```
r[θ_] := {a Sin[θ] + x0, b Cos[θ] + y0}
```

Tangent vector

```
t[u_] := r'[u] / Norm[r'[u]];
TraditionalForm@FullSimplify[t[u], {a > b > 0, u ∈ Reals}]
```

$$\left\{\frac{a\cos(u)}{\sqrt{|a\cos(u)|^2+|b\sin(u)|^2}},\ -\frac{b\sin(u)}{\sqrt{|a\cos(u)|^2+|b\sin(u)|^2}}\right\}$$

Normal vector

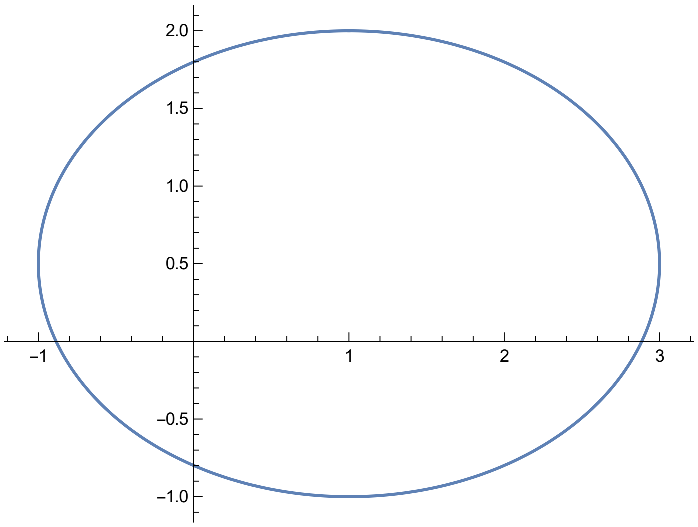
```
n[u_] := t'[u] / Norm[t'[u]]
TraditionalForm@FullSimplify[n[u], {a > b > 0, u ∈ Reals}]
```

$$\left\{-\frac{b\sin(u)}{\sqrt{a^2\cos^2(u)+b^2\sin^2(u)}},\ -\frac{a\cos(u)}{\sqrt{a^2\cos^2(u)+b^2\sin^2(u)}}\right\}$$

Visualization

```
case = {x0 → 1, y0 → .5, a → 2, b → 1.5};
```

```
ParametricPlot[r[θ] /. case, {θ, 0, 2 π}]
```



Curve length

```
l = FullSimplify[Integrate[Norm[r'[u]] /. case, {u, 0, 2 π}], {a > b > 0}]
11.0517
```

```
End[]
Two`
```

3: spiral

Visualization

```
Begin["Three`"]
Three`
```

Defining `r[u]`

```
r[u_] := {a E^(b u) Cos[u], a E^(b u) Sin[u]}
$Assumptions = {a > 0, b < 0, u ≥ 0};
```

Tangent vector

```
t[u_] := r'[u] / Norm[r'[u]];
TraditionalForm@FullSimplify[t[u]]
```

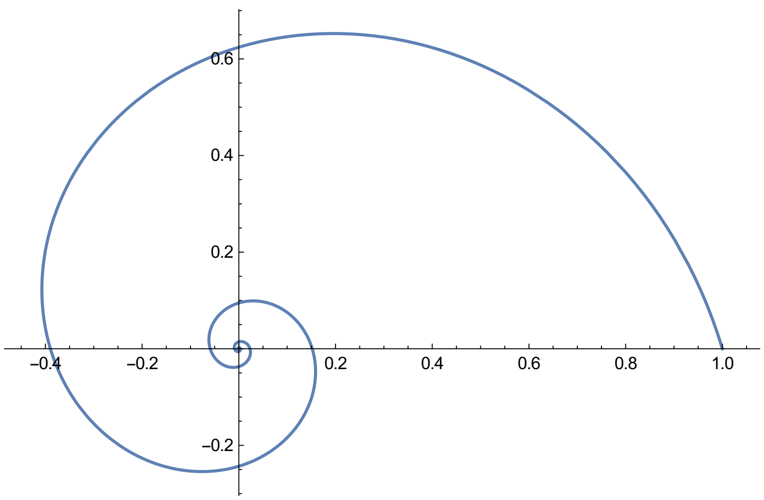
$$\left\{\frac{b\cos(u)-\sin(u)}{\sqrt{b^2+1}}, \frac{b\sin(u)+\cos(u)}{\sqrt{b^2+1}}\right\}$$

Normal vector

```
n[u_] := t'[u] / Norm[t'[u]] /. case
(*TraditionalForm@Simplify[n[u]]*)
```

Visualization

```
case = {a → 1, b → -.3};
ParametricPlot[r[θ] /. case, {θ, 0, 100}, PlotRange → All]
```



Analysis

The parameter a seems to control the x-intercept of the starting point, while b controls how quickly the spiral converges.

Curve length

```
l = Integrate[Norm[r'[u]] /. case, {u, 0, ∞}]
```

```
3.4801
```

```
End[]
```

```
Three`
```

4: 3D spiral

Setup

```
Begin["Four`"]
```

```
Four`
```

Defining `r[u]`

```
r[u_] := {a Cos[u], a Sin[u], b u}
$Assumptions = {a > 0, b > 0, u ≥ 0};
```

Tangent vector

```
t[u_] := r'[u] / Norm[r'[u]];
TraditionalForm@FullSimplify[t[u]]
```

$$\left\{-\frac{a\sin(u)}{\sqrt{a^2+b^2}}, \frac{a\cos(u)}{\sqrt{a^2+b^2}}, \frac{b}{\sqrt{a^2+b^2}}\right\}$$

Normal vector

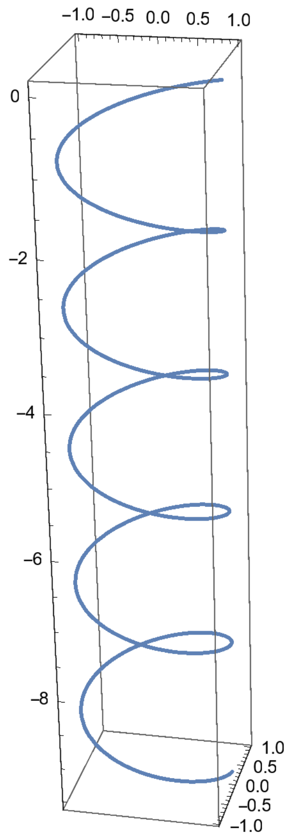
Note that the normal vector shown here is actually one of many possible normal vectors. In 2d, it was one of the two perpendicular vectors available, but in 3D, a whole plane of perpendicular vectors are available.

```
n[u_] := t'[u] / Norm[t'[u]] /. case
TraditionalForm@FullSimplify[n[u]]
```

$$\{-1.\cos(u), -1.\sin(u), 0.\}$$

Visualization

```
case = {a → 1, b → -.3};
ParametricPlot3D[r[θ] /. case, {θ, 0, 10 π}, PlotRange → All]
```



Analysis

The parameter a seems to control the x-intercept of the starting point, while b controls how quickly the spiral converges.

Curve length

```
l = Integrate[Norm[r'[u]], {u, 0, 10 π}]
l /. case
```

$10\sqrt{a^2+b^2}\ \pi$

32.7992

```
End[]
```

Four`

5: Boat hull

Setup

In[379]:=
```
Begin["Five`"]
```

Out[379]=
```
Five`
```

Defining `r[u]`

In[380]:=
```
r[u_] := {u, Abs[u]^n - 1}
$Assumptions = {n > 0, -1 < u < 1};
```

Tangent vector

In[382]:=
```
t[u_] := FullSimplify@Normalize[r'[u]];
TraditionalForm@FullSimplify[t[u]]
FullSimplify[%, u > 0]
```

Out[383]//TraditionalForm=

$$\left\{\frac{1}{\sqrt{n^2\left(u^2\right)^{n-1}+1}}, \frac{n\,\mathrm{sgn}(u)\,|u|^n}{\sqrt{n^2\left(u^2\right)^n\,\mathrm{sgn}(u)^2+u^2}}\right\}$$

Out[384]=

$$\left\{\frac{1}{\sqrt{1+n^2\,u^{-2+2\,n}}}, \frac{n\,u^n}{\sqrt{u^2+n^2\,u^{2\,n}}}\right\}$$

Normal vector

In[385]:=
```
norm[u_] := Normalize[t'[u]] /. n → 2
TraditionalForm@FullSimplify[norm[u]]
FullSimplify[%, u > 0]
```

Out[386]//TraditionalForm=

$$\left\{-\left(\left(2\,u\,\mathrm{sgn}(u)^3\,(u\,\mathrm{Abs}''(u)+\mathrm{sgn}(u))\right)\middle/\left(\sqrt{4\,u^2+1}\;\left|\mathrm{sgn}(u)^2+|u|\,\mathrm{Abs}''(u)\right|\right)\right), \frac{\mathrm{sgn}\left(u\,\mathrm{Abs}''(u)+\mathrm{sgn}(u)^3\right)}{\sqrt{4\,u^2+1}\;\mathrm{sgn}(u)}\right\}$$

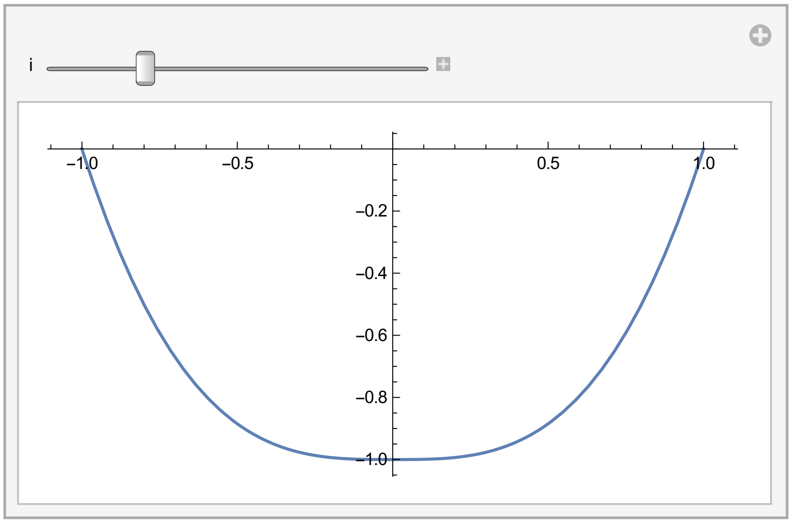
Out[387]=

$$\left\{-\frac{2\,u\,\mathrm{Sign}[1+u\,\mathrm{Abs}''[u]]}{\sqrt{1+4\,u^2}}, \frac{\mathrm{Sign}[1+u\,\mathrm{Abs}''[u]]}{\sqrt{1+4\,u^2}}\right\}$$

Visualization

In[388]:=
```
Manipulate[case = {n → i};
 ParametricPlot[r[u] /. case, {u, -1, 1}, PlotRange → All], {i, 1, 10}]
```

Out[388]=

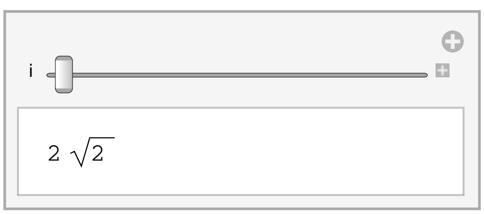


Analysis

Curve length

In[389]:=
```
Manipulate[case = {n → i};
 2 * Integrate[Norm[r'[u]] /. case, {u, 0, 1}], {i, 1, 10}]
```

Out[389]=



In[390]:=
```
l = Integrate[Norm[r'[u]] /. case, {u, 0, 10 π}]
l /. case
```

Out[390]= $10\sqrt{2}\,\pi$

Out[391]= $10\sqrt{2}\,\pi$

Underwater Hull

In[392]:=
```
End[]
```

Out[392]= Five`

6: Linear Bézier

```
Begin["Six`"]
```

```
Seven`
```

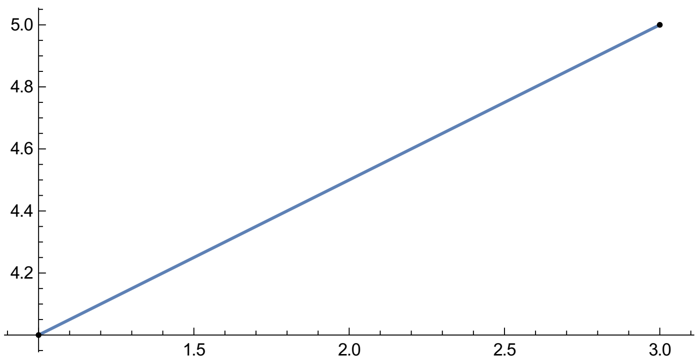
Setup

```
r[u_] := (1 - u) pts[[1]] + u pts[[2]]
$Assumptions = {u ∈ Reals};
```

Plotting

```
case = pts → {{1, 4}, {3, 5}}
Show[ParametricPlot[r[u] /. case, {u, 0, 1}], Graphics@Point[pts /. case]]
```

```
pts → {{1, 4}, {3, 5}}
```




```
End[]
```

```
Seven`
```

7: Quadratic Bézier

```
Begin["Seven`"]
```

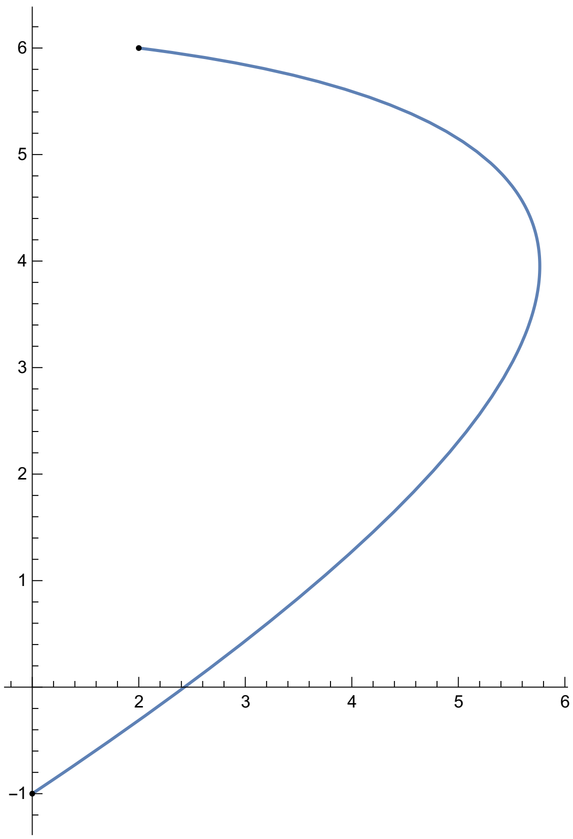
```
Eight`
```

Setup

```
r[u_] := (1 - u)^2 pts[[1]] + 2 u (1 - u) pts[[2]] + u^2 pts[[3]]
$Assumptions = {u ∈ Reals};
```

Plotting

```
case = pts → {{2, 6}, {10, 5}, {1, -1}};
Show[ParametricPlot[r[u] /. case, {u, 0, 1}], Graphics@Point[pts /. case]]
```




```
End[]
```

```
Eight`
```

8: Cubic Bézier

```
Begin["Eight`"]
```

```
Nine`
```

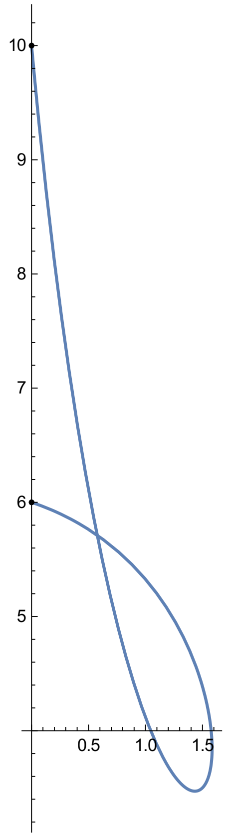
Setup

```
r[u_] := (1 - u) ^3 pts[[1]] + 3 u (1 - u) ^2 pts[[2]] + 3 u^2 (1 - u) pts[[3]] + u^3 pts[[4]]
$Assumptions = {u ∈ Reals};
```

Plotting

```
case = pts → {{0, 6}, {3, 5}, {1, -1}, {0, 10}}
Show[ParametricPlot[r[u] /. case, {u, 0, 1}], Graphics@Point[pts /. case]]
```

```
pts → {{0, 6}, {3, 5}, {1, -1}, {0, 10}}
```




```
End[]
```

```
Nine`
```

9: Donut

```
Begin["Nine`"]
```

```
Nine`
```

Setup

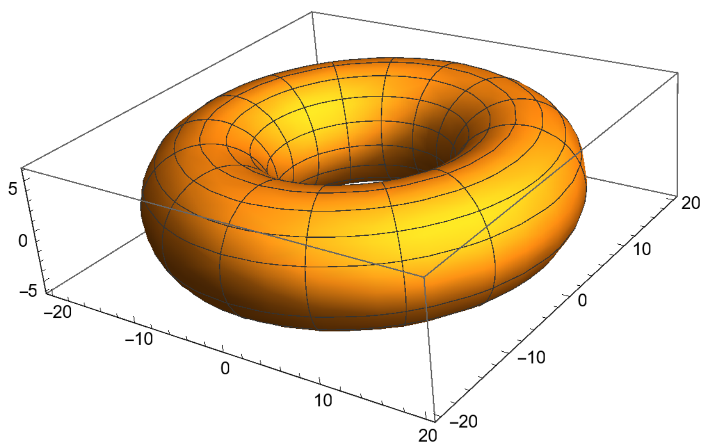
```
r[u_, v_] :=
  (a + radius Cos[u]) Cos[v] ihat3 + (a + radius Cos[u]) Sin[v] jhat3 + radius Sin[u] khat3
$Assumptions = {radius < a, u ∈ Reals, v ∈ Reals, radius > 0}
```

```
{radius < a, u ∈ Reals, v ∈ Reals, radius > 0}
```

Plotting

```
case = {radius → 6, a → 14}
ParametricPlot3D[r[u, v] /. case, {u, 0, 2 π}, {v, 0, 2 π}]
```

{radius → 6, a → 14}



Normal Vector

```
norm = FullSimplify@Normalize[D[r[u, v], u] × D[r[u, v], v]];
niceForm[norm]
```

$\hat{i}\,(-\cos(u))\cos(v) - \hat{j}\cos(u)\sin(v) - \hat{k}\sin(u)$

Surface Area

```
TraditionalForm@Integrate[Norm[D[r[u, v], u] × D[r[u, v], v]], {u, 0, 2 π}, {v, 0, 2 π}]
```

$4\,\pi^2\,a\,\mathrm{radius}$

```
End[]
```

Nine`

10: Spray

```
Begin["Ten`"]
```

Ten`

Setup

$$x = \frac{u\,L}{2};$$

$$z = v\left(\frac{16\,H\,x^4}{L^4} - H\right);$$

$$y = \frac{4\,W\,x^2}{2\,L^2} - \frac{1}{2}\,W\,\sqrt{\frac{H+z}{H}};$$

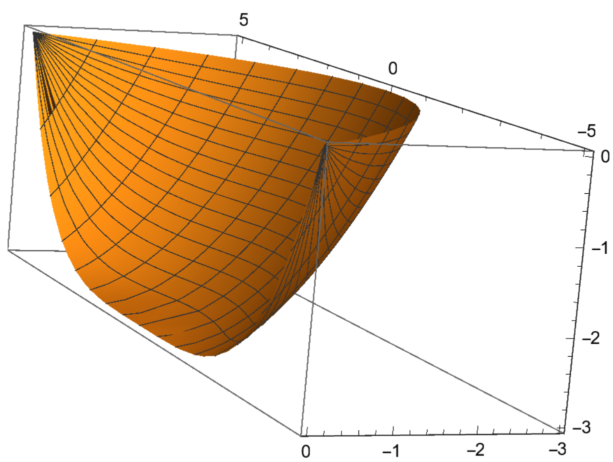
```
r[u_, v_] := {x, y, z}
$Assumptions = {H > 0, L > 0, W > 0, u ∈ Reals, v ∈ Reals}
```

```
{H > 0, L > 0, W > 0, u ∈ Reals, v ∈ Reals}
```

Plotting

```
case = {H → 3, W → 6, L → 10}
ParametricPlot3D[r[u, v] /. case, {u, -1, 1}, {v, 0, 1}]
```

```
{H → 3, W → 6, L → 10}
```



Normal Vector

Surface Area

```
integrand = FullSimplify@Norm[D[r[u, v], u] × D[r[u, v], v]] /. case;
result = Integrate[integrand, {u, -1, 1}, {v, 0, 1}]
N@result
```

$$\int_{-1}^{1} -\frac{1}{8\sqrt{25+36u^2}} 3\left(-4\sqrt{3125+8100u^2+5184u^4}+4u^2\sqrt{\left(25+36u^2\right)\left(25+100u^4+144u^6\right)} - 25\,\mathrm{Log}\left[225+288u^2+4\sqrt{3125+8100u^2+5184u^4}\right] + 25\,\mathrm{Log}\left[25+4u^2\left(50u^2+72u^4+\sqrt{\left(25+36u^2\right)\left(25+100u^4+144u^6\right)}\right)\right]\right) du$$

```
35.1718
```

```
End[]
```

```
Ten`
```