Homework, MA 213 **Due: 2014-Sep-29**

1. Find a basis for each space. Verify that it is a basis.

   (a) The subspace $M = \{ a + bx + cx^2 + dx^3 \mid a - 2b + c - d = 0 \}$ of $\mathcal{P}_3$.
   Parametrize $a - 2b + c - d = 0$ as $a = 2b - c + d$, $b = b$, $c = c$, and $d = d$ to get this description of M as the span of a set of three vectors.

   $$M = \{ (2 + x) \cdot b + (-1 + x^2) \cdot c + (1 + x^3) \cdot d \mid b, c, d \in \mathbb{R} \}$$

   To show that this three-vector set is a basis, what remains is for us to verify that it is linearly independent.

   $$0 + 0x + 0x^2 + 0x^3 = (2 + x) \cdot c_1 + (-1 + x^2) \cdot c_2 + (1 + x^3) \cdot c_3$$

   From the x terms we see that $c_1 = 0$. From the x^2 terms we see that $c_2 = 0$. The x^3 terms give that $c_3 = 0$.

   (b) This subspace of $\mathcal{M}_{2\times 2}$.

   $$W = \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a - c = 0 \}$$

   First parametrize the description (note that the fact that b and d are not mentioned in the description of W does not mean they are zero or absent, it means that they are unrestricted).

   $$W = \{ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \cdot b + \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \cdot c + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \cdot d \mid b, c, d \in \mathbb{R} \}$$

   That gives W as the span of a three element set. We will be done if we show that the set is linearly independent.

   $$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \cdot c_1 + \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \cdot c_2 + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \cdot c_3$$

   Using the upper right entries we see that $c_1 = 0$. The upper left entries give that $c_2 = 0$, and the lower left entries show that $c_3 = 0$.

2. Give two different bases for $\mathbb{R}^3$. Verify that each is a basis.

   Obviously there are many different correct choices of bases. The natural basis for $\mathbb{R}^3$ is this.

   $$\mathcal{E}_3 = \langle \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \rangle$$

   The verification that it spans $\mathbb{R}^3$ is easy: for any $x, y, z \in \mathbb{R}$ this equation has a solution,

   $$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \cdot c_1 + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \cdot c_2 + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \cdot c_3 \qquad (*)$$

namely, $c_1 = x$, $c_2 = y$, and $c_3 = z$. Further, the set is linearly independent since the relationship

$$\begin{pmatrix}0\\0\\0\end{pmatrix} = \begin{pmatrix}1\\0\\0\end{pmatrix} \cdot c_1 + \begin{pmatrix}0\\1\\0\end{pmatrix} \cdot c_2 + \begin{pmatrix}0\\0\\1\end{pmatrix} \cdot c_3$$

obviously has only the trivial solution. (*Comment.* We could have done the argument in one step by observing that equation $(*)$ shows that each vector from $\mathbb{R}^3$ is represented with respect to this basis $\mathcal{E}_3$ in one and only one way.)

This is a second basis for $\mathbb{R}^3$.

$$B = \langle \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\1\\1\end{pmatrix} \rangle$$

To verify that that it spans $\mathbb{R}^3$ suppose $x, y, z \in \mathbb{R}$, then

$$\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}1\\0\\0\end{pmatrix} \cdot c_1 + \begin{pmatrix}1\\1\\0\end{pmatrix} \cdot c_2 + \begin{pmatrix}1\\1\\1\end{pmatrix} \cdot c_3$$

has a solution, namely $c_1 = x - y$, $c_2 = y - z$, and $c_3 = z$. Further, the set B is linearly independent since in the relationship

$$\begin{pmatrix}0\\0\\0\end{pmatrix} = \begin{pmatrix}1\\0\\0\end{pmatrix} \cdot c_1 + \begin{pmatrix}1\\1\\0\end{pmatrix} \cdot c_2 + \begin{pmatrix}1\\1\\1\end{pmatrix} \cdot c_3$$

the third components give that $c_3 = 0$, then the second components give that $c_2 = 0$, and with those two the first components give that $c_1 = 0$.

3. Represent the vector with respect to each of the two bases.

$$\vec{v} = \begin{pmatrix}3\\-1\end{pmatrix} \qquad B_1 = \langle \begin{pmatrix}1\\-1\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix} \rangle,\ B_2 = \langle \begin{pmatrix}1\\2\end{pmatrix}, \begin{pmatrix}1\\3\end{pmatrix} \rangle$$

Solving

$$\begin{pmatrix}3\\-1\end{pmatrix} = \begin{pmatrix}1\\-1\end{pmatrix} \cdot c_1 + \begin{pmatrix}1\\1\end{pmatrix} \cdot c_2$$

gives $c_1 = 2$ and $c_2 = 1$.

$$\mathrm{Rep}_{B_1}(\begin{pmatrix}3\\-1\end{pmatrix}) = \begin{pmatrix}2\\1\end{pmatrix}_{B_1}$$

Similarly, solving

$$\begin{pmatrix}3\\-1\end{pmatrix} = \begin{pmatrix}1\\2\end{pmatrix} \cdot c_1 + \begin{pmatrix}1\\3\end{pmatrix} \cdot c_2$$

gives this.

$$\mathrm{Rep}_{B_2}(\begin{pmatrix}3\\-1\end{pmatrix}) = \begin{pmatrix}10\\-7\end{pmatrix}_{B_2}$$