Homework, MA 213 **Due: 2014-Nov-10**

You may work with others to figure out how to do questions, and you are welcome to look for answers in the book, online, by talking to someone who had the course before, etc. However, you must write the answers on your own. You must also show your work (you may, of course, quote any result from the book).

1. Consider the two linear functions $h\colon \mathbb{R}^3 \to \mathcal{P}_2$ and $g\colon \mathcal{P}_2 \to \mathcal{M}_{2\times 2}$ given as here.

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \mapsto (a+b)x^2 + (2a+2b)x + c \qquad px^2 + qx + r \mapsto \begin{pmatrix} p & p-2q \\ q & 0 \end{pmatrix}$$

Use these bases for the spaces.

$$B = \langle \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \rangle \quad C = \langle 1+x, 1-x, x^2 \rangle \quad D = \langle \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 3 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 4 \end{pmatrix} \rangle$$

(a) Give the formula composition map $g \circ h\colon \mathbb{R}^3 \to \mathcal{M}_{2\times 2}$ directly from the above definition.
Following the definitions gives this.

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \mapsto (a+b)x^2 + (2a+2b)x + c$$

$$\mapsto \begin{pmatrix} a+b & (a+b) - 2(2a+2b) \\ 2a+2b & 0 \end{pmatrix} = \begin{pmatrix} a+b & -3a-3b \\ 2a+2b & 0 \end{pmatrix}$$

(b) Represent h and g with respect to the appropriate bases.
Because

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \mapsto 2x^2 + 4x + 1 \qquad \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \mapsto x^2 + 2x + 1 \qquad \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \mapsto 0x^2 + 0x + 1$$

we get this representation for h.

$$\mathrm{Rep}_{B,C}(h) = \begin{pmatrix} 5/2 & 3/2 & 1/2 \\ -3/2 & -1/2 & 1/2 \\ 2 & 1 & 0 \end{pmatrix}$$

Similarly, because

$$1+x \mapsto \begin{pmatrix} 0 & -2 \\ 1 & 0 \end{pmatrix} \quad 1-x \mapsto \begin{pmatrix} 0 & 2 \\ -1 & 0 \end{pmatrix} \quad x^2 \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

this is the representation of g.

$$\mathrm{Rep}_{C,D}(g) = \begin{pmatrix} 0 & 0 & 1 \\ -1 & 1 & 1/2 \\ 1/3 & -1/3 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

(c) Represent $g \circ h$ with resepct to the appropriate bases.
The action of $g \circ h$ on the domain basis is this.

$$\begin{pmatrix}1\\1\\1\end{pmatrix} \mapsto \begin{pmatrix}2&-6\\4&0\end{pmatrix} \quad \begin{pmatrix}0\\1\\1\end{pmatrix} \mapsto \begin{pmatrix}1&-3\\2&0\end{pmatrix} \quad \begin{pmatrix}0\\0\\1\end{pmatrix} \mapsto \begin{pmatrix}0&0\\0&0\end{pmatrix}$$

We have this.

$$\mathrm{Rep}_{B,D}(g \circ h) = \begin{pmatrix}2&1&0\\-3&-3/2&0\\4/3&2/3&0\\0&0&0\end{pmatrix}$$

(d) Check that the two matrices from the second part multiply to the matrix from the third part.
The matrix multiplication is routine, taking care with the order.

$$\begin{pmatrix}0&0&1\\-1&1&1/2\\1/3&-1/3&0\\0&0&0\end{pmatrix}\begin{pmatrix}5/2&3/2&1/2\\-3/2&-1/2&1/2\\2&1&0\end{pmatrix} = \begin{pmatrix}2&1&0\\-3&-3/2&0\\4/3&2/3&0\\0&0&0\end{pmatrix}$$

2. Use these matrices.

$$A = \begin{pmatrix}1&3&-1\\0&1&2\end{pmatrix} \quad B = \begin{pmatrix}2&2\\-1&0\end{pmatrix} \quad C = \begin{pmatrix}0&1&1\\1&1&1\\2&-2&3\end{pmatrix} \quad D = \begin{pmatrix}0&0\\4&0\end{pmatrix}$$

(a) Find $-3A$ and $2B - 5D$, or state "not defined."

$$-3A = \begin{pmatrix}-3&-9&3\\0&-3&-6\end{pmatrix} \qquad 2B - 5D = \begin{pmatrix}4&4\\-22&0\end{pmatrix}$$

(b) Which matrix products are defined?
The only ones that are defined are AC, BA, $BB = B^2$, BD, DA, DB, and $DD = D^2$.

(c) Compute AB and AC, or state "not defined."
The product AB is not defined.

$$AC = \begin{pmatrix}1&6&1\\5&-3&7\end{pmatrix}$$

3. Show how to use matrix multiplication to bring this matrix to echelon form.

$$\begin{pmatrix}1&2&1&0\\2&3&1&-1\\7&11&4&-3\end{pmatrix}$$

Multiply by $C_{1,2}(-2)$, then by $C_{1,3}(-7)$, and then by $C_{2,3}(-3)$, paying attention to the right-to-left order.

$$\begin{pmatrix}1&0&0\\0&1&0\\0&-3&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&0\\-7&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\-2&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&2&1&0\\2&3&1&-1\\7&11&4&-3\end{pmatrix} = \begin{pmatrix}1&2&1&0\\0&-1&-1&-1\\0&0&0&0\end{pmatrix}$$