You may work with others to figure out how to do questions, and you are welcome to look for answers in the book, online, by talking to someone who had the course before, etc., but you must write the answers on your own. *You must show your work.*

1. Give the solution set of each system.

(a)
$$3x + 2y + z = 1$$
$$x - y + z = 2$$
$$5x + 5y + z = 0$$

$$\begin{pmatrix} 3 & 2 & 1 & | & 1 \\ 1 & -1 & 1 & | & 2 \\ 5 & 5 & 1 & | & 0 \end{pmatrix} \xrightarrow[\substack{-(5/3)\rho_1+\rho_3}]{-(1/3)\rho_1+\rho_2} \begin{pmatrix} 3 & 2 & 1 & | & 1 \\ 0 & -5/3 & 2/3 & | & 5/3 \\ 0 & 5/3 & -2/3 & | & -5/3 \end{pmatrix} \xrightarrow{\rho_2+\rho_3} \begin{pmatrix} 3 & 2 & 1 & | & 1 \\ 0 & -5/3 & 2/3 & | & 5/3 \\ 0 & 0 & 0 & | & 0 \end{pmatrix}$$

The solution set is this.

$$\left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + \begin{pmatrix} -3/5 \\ 2/5 \\ 1 \end{pmatrix} z \mid z \in \mathbb{R} \right\}$$

(b)
$$x + y - 2z = 0$$
$$x - y = -3$$
$$3x - y - 2z = -6$$
$$2y - 2z = 3$$

$$\begin{pmatrix} 1 & 1 & -2 & | & 0 \\ 1 & -1 & 0 & | & 3 \\ 3 & -1 & -2 & | & -6 \\ 0 & 2 & -2 & | & 3 \end{pmatrix} \xrightarrow[\substack{-3\rho_1+\rho_3}]{-\rho_1+\rho_2} \begin{pmatrix} 1 & 1 & -2 & | & 0 \\ 0 & -2 & 2 & | & -3 \\ 0 & -4 & 4 & | & -6 \\ 0 & 2 & -2 & | & 3 \end{pmatrix} \xrightarrow[\substack{\rho_2+\rho_4}]{-2\rho_2+\rho_3} \begin{pmatrix} 1 & 1 & -2 & | & 0 \\ 0 & -2 & 2 & | & -3 \\ 0 & 0 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{pmatrix}$$

The solution set is this.

$$\left\{ \begin{pmatrix} -3/2 \\ 3/2 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} z \mid z \in \mathbb{R} \right\}$$

(c)
$$2x - y - z + w = 4$$
$$x + y + z = -1$$

$$\begin{pmatrix} 2 & -1 & -1 & 1 & | & 4 \\ 1 & 1 & 1 & 0 & | & -1 \end{pmatrix} \xrightarrow{-(1/2)\rho_1+\rho_2} \begin{pmatrix} 2 & -1 & -1 & 1 & | & 4 \\ 0 & 3/2 & 3/2 & -1/2 & | & -3 \end{pmatrix}$$

Here is the solution set.

$$\left\{ \begin{pmatrix} 1 \\ -2 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \end{pmatrix} z + \begin{pmatrix} -1/3 \\ 1/3 \\ 0 \\ 1 \end{pmatrix} w \mid z, w \in \mathbb{R} \right\}$$

(d)
$$x + y - 2z = 0$$
$$x - y = -3$$
$$3x - y - 2z = 0$$

$$\begin{pmatrix} 1 & 1 & -2 & | & 0 \\ 1 & -1 & 0 & | & -3 \\ 3 & -1 & -2 & | & 0 \end{pmatrix} \xrightarrow[\substack{-3\rho_1+\rho_3}]{-\rho_1+\rho_2} \begin{pmatrix} 1 & 1 & -2 & | & 0 \\ 0 & -2 & 2 & | & -3 \\ 0 & -4 & 4 & | & 0 \end{pmatrix} \xrightarrow{-2\rho_2+\rho_3} \begin{pmatrix} 1 & 1 & -2 & | & 0 \\ 0 & -2 & 2 & | & -3 \\ 0 & 0 & 0 & | & 6 \end{pmatrix}$$

The solution set is empty.

1

2. For the second system in the first question, give the associated homogeneous system and give its solution set.

$$
\begin{pmatrix}
1 & 1 & -2 & \big| & 0 \\
1 & -1 & 0 & \big| & 0 \\
3 & -1 & -2 & \big| & 0 \\
0 & 2 & -2 & \big| & 0
\end{pmatrix}
\xrightarrow[-3\rho_1+\rho_3]{-\rho_1+\rho_2}
\begin{pmatrix}
1 & 1 & -2 & \big| & 0 \\
0 & -2 & 2 & \big| & 0 \\
0 & -4 & 4 & \big| & 0 \\
0 & 2 & -2 & \big| & 0
\end{pmatrix}
\xrightarrow[\rho_2+\rho_4]{-2\rho_2+\rho_3}
\begin{pmatrix}
1 & 1 & -2 & \big| & 0 \\
0 & -2 & 2 & \big| & 0 \\
0 & 0 & 0 & \big| & 0 \\
0 & 0 & 0 & \big| & 0
\end{pmatrix}
$$

The solution set is this.

$$
\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} z \;\middle|\; z \in \mathbb{R} \right\}
$$

3. Do Gauss-Jordan reduction.

(a)
$$
\begin{aligned}
x + y - z &= 3 \\
2x - y - z &= 1 \\
3x + y + 2z &= 0
\end{aligned}
$$

$$
\begin{pmatrix}
1 & 1 & -1 & \big| & 3 \\
2 & -1 & -1 & \big| & 1 \\
3 & 1 & 2 & \big| & 0
\end{pmatrix}
\xrightarrow[-3\rho_1+\rho_3]{-2\rho_1+\rho_2}
\begin{pmatrix}
1 & 1 & -1 & \big| & 3 \\
0 & -3 & -1 & \big| & 1 \\
0 & -2 & 5 & \big| & -9
\end{pmatrix}
\xrightarrow{-(2/3)\rho_2+\rho_3}
\begin{pmatrix}
1 & 1 & -1 & \big| & 3 \\
0 & -3 & -1 & \big| & 1 \\
0 & 0 & 13/3 & \big| & -17/3
\end{pmatrix}
$$

$$
\xrightarrow[(3/13)\rho_3]{-(1/3)\rho_2}
\begin{pmatrix}
1 & 1 & -1 & \big| & 3 \\
0 & 1 & -1/3 & \big| & 5/3 \\
0 & 0 & 1 & \big| & -17/13
\end{pmatrix}
\xrightarrow[(1/3)\rho_3+\rho_2]{\rho_3+\rho_1}
\begin{pmatrix}
1 & 1 & 0 & \big| & 22/13 \\
0 & 1 & 0 & \big| & 16/13 \\
0 & 0 & 1 & \big| & -17/13
\end{pmatrix}
\xrightarrow{-\rho_2+\rho_1}
\begin{pmatrix}
1 & 0 & 0 & \big| & 6/13 \\
0 & 1 & 0 & \big| & 16/13 \\
0 & 0 & 1 & \big| & -17/13
\end{pmatrix}
$$

(b)
$$
\begin{aligned}
x + y + 2z &= 0 \\
2x - y + z &= 1 \\
4x + y + 5z &= 1
\end{aligned}
$$

$$
\begin{pmatrix}
1 & 1 & 2 & \big| & 0 \\
2 & -1 & 1 & \big| & 1 \\
4 & 1 & 5 & \big| & 1
\end{pmatrix}
\xrightarrow[-4\rho_1+\rho_3]{-2\rho_1+\rho_2}
\begin{pmatrix}
1 & 1 & 2 & \big| & 0 \\
0 & -3 & -3 & \big| & 1 \\
0 & -3 & -3 & \big| & 1
\end{pmatrix}
\xrightarrow{-\rho_2+\rho_3}
\begin{pmatrix}
1 & 1 & 2 & \big| & 0 \\
0 & -3 & -3 & \big| & 1 \\
0 & 0 & 0 & \big| & 0
\end{pmatrix}
$$

$$
\xrightarrow{-(1/3)\rho_2}
\begin{pmatrix}
1 & 1 & 2 & \big| & 0 \\
0 & 1 & 1 & \big| & -1/3 \\
0 & 0 & 0 & \big| & 0
\end{pmatrix}
\xrightarrow{-\rho_2+\rho_1}
\begin{pmatrix}
1 & 0 & 1 & \big| & 1/3 \\
0 & 1 & 1 & \big| & -1/3 \\
0 & 0 & 0 & \big| & 0
\end{pmatrix}
$$