Homework, MA 213 **Due: 2014-Nov-17**

You may work with others to figure out how to do questions, and you are welcome to look for answers in the book, online, by talking to someone who had the course before, etc. However, you must write the answers on your own. You must also show your work (you may, of course, quote any result from the book).

1. For each space find the matrix changing a vector representation with respect to B to one with respect to D.

   (a) $V = \mathbb{R}^3$, $B = \mathcal{E}_3$, $D = \langle \begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}0\\1\\-1\end{pmatrix} \rangle$

   Start by computing the effect of the identity function on each element of the starting basis B. Obviously this is the effect.

$$\begin{pmatrix}1\\0\\0\end{pmatrix} \overset{\text{id}}{\longmapsto} \begin{pmatrix}1\\0\\0\end{pmatrix} \quad \begin{pmatrix}0\\1\\0\end{pmatrix} \overset{\text{id}}{\longmapsto} \begin{pmatrix}0\\1\\0\end{pmatrix} \quad \begin{pmatrix}0\\0\\1\end{pmatrix} \overset{\text{id}}{\longmapsto} \begin{pmatrix}0\\0\\1\end{pmatrix}$$

   Now represent the three outputs with respect to the ending basis.

$$\text{Rep}_D(\begin{pmatrix}1\\0\\0\end{pmatrix}) = \begin{pmatrix}-2/3\\5/3\\-1/3\end{pmatrix} \quad \text{Rep}_D(\begin{pmatrix}0\\1\\0\end{pmatrix}) = \begin{pmatrix}1/3\\-1/3\\2/3\end{pmatrix} \quad \text{Rep}_D(\begin{pmatrix}0\\0\\1\end{pmatrix}) = \begin{pmatrix}1/3\\-1/3\\-1/3\end{pmatrix}$$

   Concatenate them into a basis.

$$\text{Rep}_{B,D}(\text{id}) = \begin{pmatrix}-2/3 & 1/3 & 1/3\\ 5/3 & -1/3 & -1/3\\ -1/3 & 2/3 & -1/3\end{pmatrix}$$

   (b) $V = \mathbb{R}^3$, $B = \langle \begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}0\\1\\-1\end{pmatrix} \rangle$, $D = \mathcal{E}_3$

   One way to find this is to take the inverse of the prior matrix, since it converts bases inthe other direction. Alternatively, we can compute these three

$$\text{Rep}_{\mathcal{E}_3}(\begin{pmatrix}1\\2\\3\end{pmatrix}) = \begin{pmatrix}1\\2\\3\end{pmatrix} \quad \text{Rep}_{\mathcal{E}_3}(\begin{pmatrix}1\\1\\1\end{pmatrix}) = \begin{pmatrix}1\\1\\1\end{pmatrix} \quad \text{Rep}_{\mathcal{E}_3}(\begin{pmatrix}0\\1\\-1\end{pmatrix}) = \begin{pmatrix}0\\1\\-1\end{pmatrix}$$

   and put them in a matrix.

$$\text{Rep}_{B,D}(\text{id}) = \begin{pmatrix}1 & 1 & 0\\ 2 & 1 & 1\\ 3 & 1 & -1\end{pmatrix}$$

   (c) $V = \mathcal{P}_2$, $B = \langle x^2, x^2 + x, x^2 + x + 1\rangle$, $D = \langle 2, -x, x^2\rangle$

Representing $\mathrm{id}(x^2)$, $\mathrm{id}(x^2+x)$, and $\mathrm{id}(x^2+x+1)$ with respect to the ending basis gives this.

$$\mathrm{Rep}_D(x^2) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad \mathrm{Rep}_D(x^2+x) = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \quad \mathrm{Rep}_D(x^2+x+1) = \begin{pmatrix} 1/2 \\ -1 \\ 1 \end{pmatrix}$$

Put them together.

$$\mathrm{Rep}_{B,D}(\mathrm{id}) = \begin{pmatrix} 0 & 0 & 1/2 \\ 0 & -1 & -1 \\ 1 & 1 & 1 \end{pmatrix}$$

2. Find the P and Q to express H via PHQ as a block partial identity matrix.

$$H = \begin{pmatrix} 2 & 1 & 1 \\ 3 & -1 & 0 \\ 1 & 3 & 2 \end{pmatrix}$$

Gauss's Method gives this.

$$\begin{pmatrix} 2 & 1 & 1 \\ 3 & -1 & 0 \\ 1 & 3 & 2 \end{pmatrix} \xrightarrow[-(1/2)\rho_1+\rho_3]{-(3/2)\rho_1+\rho_2} \xrightarrow{\rho_2+\rho_3} \xrightarrow[-(2/5)\rho_2]{(1/2)\rho_1} \begin{pmatrix} 1 & 1/2 & 1/2 \\ 0 & 1 & 3/5 \\ 0 & 0 & 0 \end{pmatrix}$$

Column operations complete the job of reaching the canonical form for matrix equivalence.

$$\xrightarrow{-(3/5)\mathrm{col}_2+\mathrm{col}_3} \xrightarrow[-(1/5)\mathrm{col}_1+\mathrm{col}_3]{-(1/2)\mathrm{col}_1+\mathrm{col}_2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Then these are the two matrices.

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2/5 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1/2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1/2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -3/2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1/2 & 0 & 0 \\ 3/5 & -2/5 & 0 \\ -2 & 1 & 1 \end{pmatrix}$$

$$Q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -3/5 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1/2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -1/5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1/2 & -1/5 \\ 0 & 1 & -3/5 \\ 0 & 0 & 1 \end{pmatrix}$$

3. Project the vector to the line.

$$\vec{v} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} \qquad L = \{c\begin{pmatrix} 1 \\ -1 \end{pmatrix} \mid c \in \mathbb{R}\}$$

The formula is straightforward.

$$\frac{\begin{pmatrix} 3 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix}}{\begin{pmatrix} 1 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix}} \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1/2 \\ -1/2 \end{pmatrix}$$

4. Express this nonsingular matrix as a product of elementary reduction matrices.

$$T = \begin{pmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 3 & 1 & 2 \end{pmatrix}$$

The Gauss-Jordan reduction is routine.

$$\begin{pmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 3 & 1 & 2 \end{pmatrix} \xrightarrow[-3\rho_1+\rho_3]{-2\rho_1+\rho_2} \xrightarrow{-\rho_2+\rho_3} \xrightarrow[(1/2)\rho_3]{-(1/5)\rho_2} \xrightarrow{-2\rho_2+\rho_1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Thus we know elementary reduction matrices $R_1, \ldots, R_6$ such that $R_6 \cdot R_5 \cdots R_1 \cdot T = I$. Move the matrices to the other side (paying attention to order; you first multiply both sides from the left by R_6^{-1}, etc.).

$$\begin{aligned} T &= \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1/5 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{pmatrix}^{-1} \begin{pmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{aligned}$$