

# Advanced Deep Learning

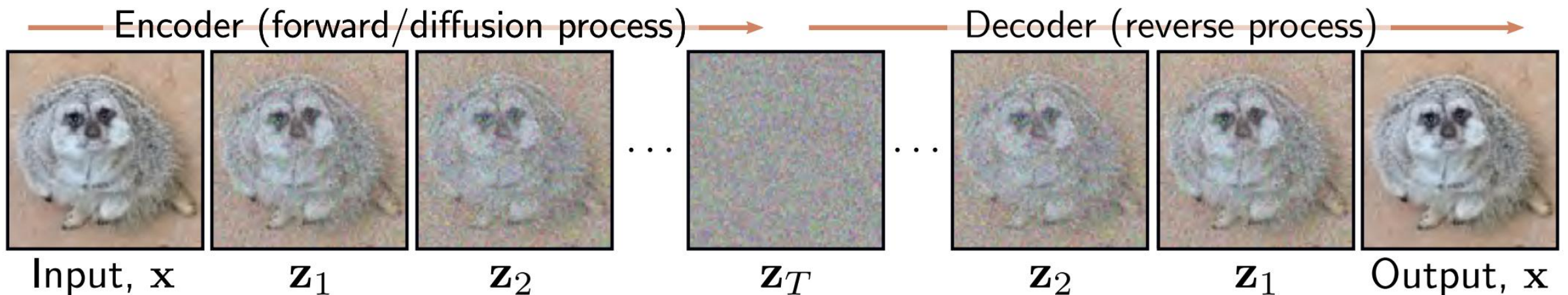
DATA.ML.230

## Diffusion models

From: Simon J. D. Prince, Chapter 18 – Understanding Deep Learning, MIT Press (19 May 2025)

# Diffusion models

- Approximate the data likelihood using a lower bound based on an encoder that maps to the latent variable:
  - The encoder is predetermined (noising process)
- Probabilistic models that define a nonlinear mapping from latent variables to observed data (decoder)
  - The decoder learns to inverse the encoding process



# Diffusion models

Encoder (diffusion or forward process)

- Map a data example  $\mathbf{x}$  through a series of variables  $\mathbf{z}_1, \mathbf{z}_2, \dots, \mathbf{z}_T$

$$\begin{aligned}\mathbf{z}_1 &= \sqrt{1 - \beta_1} \cdot \mathbf{x} + \sqrt{\beta_1} \cdot \boldsymbol{\epsilon}_1 \\ \mathbf{z}_t &= \sqrt{1 - \beta_t} \cdot \mathbf{z}_{t-1} + \sqrt{\beta_t} \cdot \boldsymbol{\epsilon}_t \quad \forall t \in 2, \dots, T\end{aligned}$$

How quickly the noise is blended  
(noise schedule)

Noise drawn from a standard  
Normal distribution

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Forward process

$$q(\mathbf{z}_1 | \mathbf{x}) = \text{Norm}_{\mathbf{z}_1} \left[ \sqrt{1 - \beta_1} \mathbf{x}, \beta_1 \mathbf{I} \right]$$

$$q(\mathbf{z}_t | \mathbf{z}_{t-1}) = \text{Norm}_{\mathbf{z}_t} \left[ \sqrt{1 - \beta_t} \mathbf{z}_{t-1}, \beta_t \mathbf{I} \right] \quad \forall t \in \{2, \dots, T\}$$

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Markov chain:  
Each probability of  $\mathbf{z}_t$  is  
determined by the  
preceding variable  $\mathbf{z}_{t-1}$

# Diffusion models

## Diffusion kernel

- To train the decoder we use multiple samples  $\mathbf{z}_t$  at time  $t$  for the same input  $\mathbf{x}$

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time-consuming process



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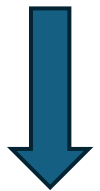
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Substitute  $\mathbf{z}_1$  in  $\mathbf{z}_2$

$$\mathbf{z}_2 = \sqrt{(1 - \beta_2)(1 - \beta_1)} \cdot \mathbf{x} + \sqrt{1 - (1 - \beta_2)(1 - \beta_1)} \cdot \boldsymbol{\epsilon}$$

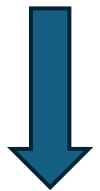
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$$\mathbf{z}_t = \sqrt{\alpha_t} \cdot \mathbf{x} + \sqrt{1 - \alpha_t} \cdot \boldsymbol{\epsilon}$$

where  $\alpha_t = \prod_{s=1}^t 1 - \beta_s$

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$$q(\mathbf{z}_t|\mathbf{x}) = \text{Norm}_{\mathbf{z}_t} \left[ \sqrt{\alpha_t} \cdot \mathbf{x}, (1 - \alpha_t)\mathbf{I} \right]$$

# Diffusion models

## Decoder (reverse process)

- Learns a series of probabilistic mappings back from latent variable  $\mathbf{z}_t$  to  $\mathbf{z}_{t-1}$ , starting from  $t = T$ , until we reach the data  $\mathbf{x}$
- We approximate the reverse distributions by normal distributions:

$$Pr(\mathbf{z}_T) = \text{Norm}_{\mathbf{z}_T}[\mathbf{0}, \mathbf{I}]$$

$$Pr(\mathbf{z}_{t-1}|\mathbf{z}_t, \phi_t) = \text{Norm}_{\mathbf{z}_{t-1}}[\mathbf{f}_t[\mathbf{z}_t, \phi_t], \sigma_t^2 \mathbf{I}]$$

$$Pr(\mathbf{x}|\mathbf{z}_1, \phi_1) = \text{Norm}_{\mathbf{x}}[\mathbf{f}_1[\mathbf{z}_1, \phi_1], \sigma_1^2 \mathbf{I}],$$


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Neural network mapping  $\mathbf{z}_t$  to  $\mathbf{z}_{t-1}$




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Neural network mapping  $\mathbf{z}_t$  to  $\mathbf{z}_{t-1}$



- Data generation using ancestral sampling

# Diffusion models

Decoder (reverse process)

- Training by maximizing the log-likelihood of the training data  $\{\mathbf{x}_i\}$ :

$$\hat{\phi}_{1...T} = \operatorname{argmax}_{\phi_{1...T}} \left[ \sum_{i=1}^I \log \left[ \operatorname{Pr}(\mathbf{x}_i | \phi_{1...T}) \right] \right]$$

where  $\operatorname{Pr}(\mathbf{x} | \phi_{1...T}) = \int \operatorname{Pr}(\mathbf{x}, \mathbf{z}_{1...T} | \phi_{1...T}) d\mathbf{z}_{1...T}$



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where  $\operatorname{Pr}(\mathbf{x} | \phi_{1...T}) = \int \operatorname{Pr}(\mathbf{x}, \mathbf{z}_{1...T} | \phi_{1...T}) d\mathbf{z}_{1...T}$

← Intractable integral

→ Use ELBO

$$\operatorname{ELBO}[\phi_{1...T}] = \int q(\mathbf{z}_{1...T} | \mathbf{x}) \log \left[ \frac{\operatorname{Pr}(\mathbf{x}, \mathbf{z}_{1...T} | \phi_{1...T})}{q(\mathbf{z}_{1...T} | \mathbf{x})} \right] d\mathbf{z}_{1...T}$$

# Diffusion models

## Decoder (reverse process)

$$\text{ELBO}[\phi_{1...T}]$$

$$\begin{aligned} &= \int q(\mathbf{z}_{1...T}|\mathbf{x}) \log \left[ \frac{Pr(\mathbf{x}, \mathbf{z}_{1...T}|\phi_{1...T})}{q(\mathbf{z}_{1...T}|\mathbf{x})} \right] d\mathbf{z}_{1...T} \\ &\approx \int q(\mathbf{z}_{1...T}|\mathbf{x}) \left( \log [Pr(\mathbf{x}|\mathbf{z}_1, \phi_1)] + \sum_{t=2}^T \log \left[ \frac{Pr(\mathbf{z}_{t-1}|\mathbf{z}_t, \phi_t)}{q(\mathbf{z}_{t-1}|\mathbf{z}_t, \mathbf{x})} \right] \right) d\mathbf{z}_{1...T} \\ &= \mathbb{E}_{q(\mathbf{z}_1|\mathbf{x})} \left[ \underbrace{\log [Pr(\mathbf{x}|\mathbf{z}_1, \phi_1)]}_{\text{Reconstruction accuracy}} \right] - \sum_{t=2}^T \mathbb{E}_{q(\mathbf{z}_t|\mathbf{x})} \left[ D_{KL} \left[ q(\mathbf{z}_{t-1}|\mathbf{z}_t, \mathbf{x}) || Pr(\mathbf{z}_{t-1}|\mathbf{z}_t, \phi_t) \right] \right] \end{aligned}$$

### Reconstruction accuracy

- ELBO will be larger if the model prediction matches the data
- We use Monte Carlo estimate as in VAE

# Diffusion models

## Decoder (reverse process)

$$\text{ELBO}[\phi_{1...T}]$$

$$= \int q(\mathbf{z}_{1...T}|\mathbf{x}) \log \left[ \frac{Pr(\mathbf{x}, \mathbf{z}_{1...T}|\phi_{1...T})}{q(\mathbf{z}_{1...T}|\mathbf{x})} \right] d\mathbf{z}_{1...T}$$

$$\approx \int q(\mathbf{z}_{1...T}|\mathbf{x}) \left( \log [Pr(\mathbf{x}|\mathbf{z}_1, \phi_1)] + \sum_{t=2}^T \log \left[ \frac{Pr(\mathbf{z}_{t-1}|\mathbf{z}_t, \phi_t)}{q(\mathbf{z}_{t-1}|\mathbf{z}_t, \mathbf{x})} \right] \right) d\mathbf{z}_{1...T}$$

$$= \mathbb{E}_{q(\mathbf{z}_1|\mathbf{x})} \left[ \log [Pr(\mathbf{x}|\mathbf{z}_1, \phi_1)] \right] - \sum_{t=2}^T \mathbb{E}_{q(\mathbf{z}_t|\mathbf{x})} \left[ \underbrace{D_{KL} \left[ q(\mathbf{z}_{t-1}|\mathbf{z}_t, \mathbf{x}) \parallel Pr(\mathbf{z}_{t-1}|\mathbf{z}_t, \phi_t) \right]}_{\text{Distance between normal distributions}} \right]$$

Distance between normal distributions

$$\frac{1}{2\sigma_t^2} \left\| \frac{(1 - \alpha_{t-1})}{1 - \alpha_t} \sqrt{1 - \beta_t} \mathbf{z}_t + \frac{\sqrt{\alpha_{t-1}} \beta_t}{1 - \alpha_t} \mathbf{x} - \mathbf{f}_t[\mathbf{z}_t, \phi_t] \right\|^2 + C$$

# Diffusion models

## Diffusion loss function

$$L[\phi_{1...T}] = \sum_{i=1}^I \left( \overbrace{-\log \left[ \text{Norm}_{\mathbf{x}_i} \left[ \mathbf{f}_1[\mathbf{z}_{i1}, \phi_1], \sigma_1^2 \mathbf{I} \right] \right]}^{\text{reconstruction term}} + \sum_{t=2}^T \frac{1}{2\sigma_t^2} \left\| \underbrace{\frac{1 - \alpha_{t-1}}{1 - \alpha_t} \sqrt{1 - \beta_t} \mathbf{z}_{it} + \frac{\sqrt{\alpha_{t-1}} \beta_t}{1 - \alpha_t} \mathbf{x}_i}_{\text{target, mean of } q(\mathbf{z}_{t-1} | \mathbf{z}_t, \mathbf{x})} - \underbrace{\mathbf{f}_t[\mathbf{z}_{it}, \phi_t]}_{\text{predicted } \mathbf{z}_{t-1}} \right\|^2 \right)$$

$L[\phi_{1...T}]$  is used to train a network for each diffusion time step

# Diffusion models

Reparameterization of loss function

**Idea:** Learn to predict the noise that was mixed with the original data  
example to create the current variable

$$\mathbf{z}_t = \sqrt{\alpha_t} \cdot \mathbf{x} + \sqrt{1 - \alpha_t} \cdot \epsilon \quad \longrightarrow \quad \mathbf{x} = \frac{1}{\sqrt{\alpha_t}} \cdot \mathbf{z}_t - \frac{\sqrt{1 - \alpha_t}}{\sqrt{\alpha_t}} \cdot \epsilon$$

Substituting  $\mathbf{x}$  to the loss function:

$$\begin{aligned} L[\phi_{1...T}] = & \sum_{i=1}^I \left( -\log \left[ \text{Norm}_{\mathbf{x}_i} \left[ \mathbf{f}_1[\mathbf{z}_{i1}, \phi_1], \sigma_1^2 \mathbf{I} \right] \right) \right. \\ & \left. + \sum_{t=2}^T \frac{1}{2\sigma_t^2} \left\| \left( \frac{1}{\sqrt{1 - \beta_t}} \mathbf{z}_{it} - \frac{\beta_t}{\sqrt{1 - \alpha_t} \sqrt{1 - \beta_t}} \epsilon_{it} \right) - \mathbf{f}_t[\mathbf{z}_{it}, \phi_t] \right\|^2 \right) \end{aligned}$$

# Diffusion models

Replace the model  $\hat{\mathbf{z}}_{t-1} = \mathbf{f}_t[\mathbf{z}_t, \phi_t]$  with a new model  $\hat{\epsilon} = \mathbf{g}_t[\mathbf{z}_t, \phi_t]$  predicting the noise  $\epsilon$  mixed with  $\mathbf{x}$  to create  $\mathbf{z}_t$ :

$$\mathbf{f}_t[\mathbf{z}_t, \phi_t] = \frac{1}{\sqrt{1 - \beta_t}} \mathbf{z}_t - \frac{\beta_t}{\sqrt{1 - \alpha_t} \sqrt{1 - \beta_t}} \mathbf{g}_t[\mathbf{z}_t, \phi_t]$$

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Substituting to the loss function leads to:

$$L[\phi_{1...T}] = \sum_{i=1}^I \sum_{t=1}^T \underbrace{\frac{\beta_t^2}{(1 - \alpha_t)(1 - \beta_t)2\sigma_t^2}} \left\| \mathbf{g}_t[\mathbf{z}_{it}, \phi_t] - \epsilon_{it} \right\|^2$$

In practice the scaling  
factors are ignored

# Diffusion models

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**Algorithm 18.1:** Diffusion model training

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**Input:** Training data  $\mathbf{x}$

**Output:** Model parameters  $\phi_t$

**repeat**

**for**  $i \in \mathcal{B}$  **do**

        // For every training example index in batch

$t \sim \text{Uniform}[1, \dots, T]$

        // Sample random timestep

$\epsilon \sim \text{Norm}[\mathbf{0}, \mathbf{I}]$

        // Sample noise

$\ell_i = \left\| \mathbf{g}_t \left[ \sqrt{\alpha_t} \mathbf{x}_i + \sqrt{1 - \alpha_t} \epsilon, \phi_t \right] - \epsilon \right\|^2$

        // Compute individual loss

    Accumulate losses for batch and take gradient step

**until** converged

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# Diffusion models

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**Algorithm 18.2:** Sampling

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**Input:** Model,  $\mathbf{g}_t[\bullet, \phi_t]$

**Output:** Sample,  $\mathbf{x}$

$\mathbf{z}_T \sim \text{Norm}_{\mathbf{z}}[\mathbf{0}, \mathbf{I}]$

// Sample last latent variable

**for**  $t = T \dots 2$  **do**

$\hat{\mathbf{z}}_{t-1} = \frac{1}{\sqrt{1-\beta_t}}\mathbf{z}_t - \frac{\beta_t}{\sqrt{1-\alpha_t}\sqrt{1-\beta_t}}\mathbf{g}_t[\mathbf{z}_t, \phi_t]$     // Predict previous latent variable

$\epsilon \sim \text{Norm}_{\epsilon}[\mathbf{0}, \mathbf{I}]$

// Draw new noise vector

$\mathbf{z}_{t-1} = \hat{\mathbf{z}}_{t-1} + \sigma_t \epsilon$

// Add noise to previous latent variable

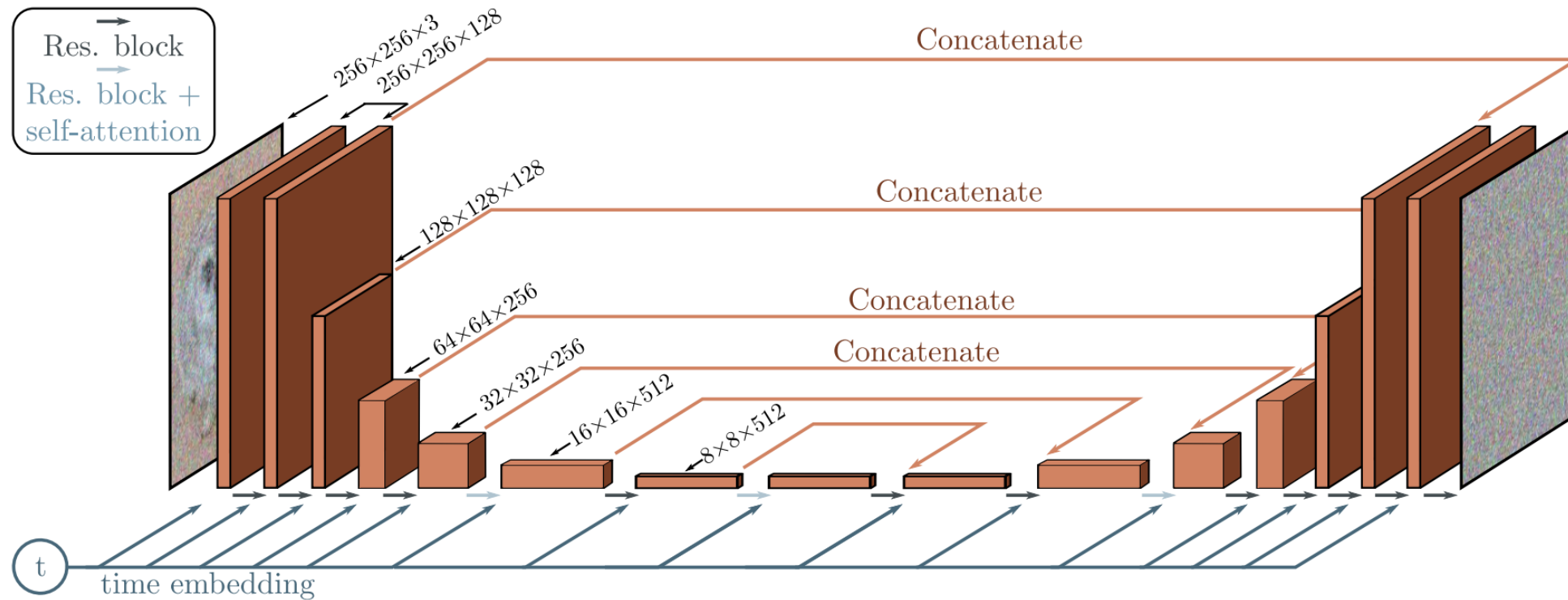
$\mathbf{x} = \frac{1}{\sqrt{1-\beta_1}}\mathbf{z}_1 - \frac{\beta_1}{\sqrt{1-\alpha_1}\sqrt{1-\beta_1}}\mathbf{g}_1[\mathbf{z}_1, \phi_1]$

// Generate sample from  $\mathbf{z}_1$  without noise

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# Applications of Diffusion models

## Image generation

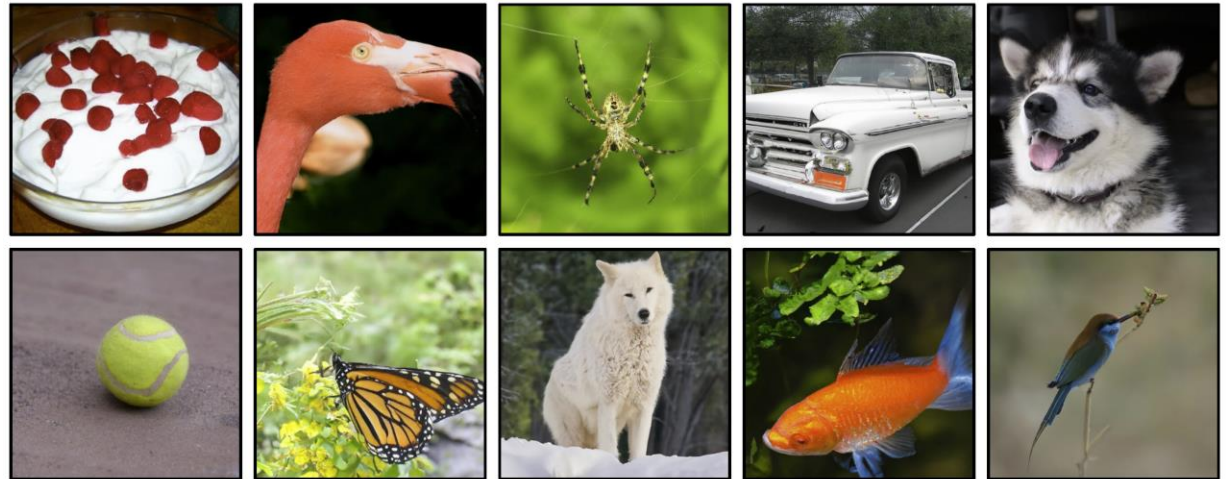


Instead of training multiple networks, train one and enhance its input with a (predetermined) vector representing the time step

# Applications of Diffusion models

Conditional generation using classifier guidance:

- Modifies the denoising update to using class information  $c$
- Uses a classifier on the latent variable  $\mathbf{z}_t$
- Gradients of the classifier affect the parameters of the generating neural network

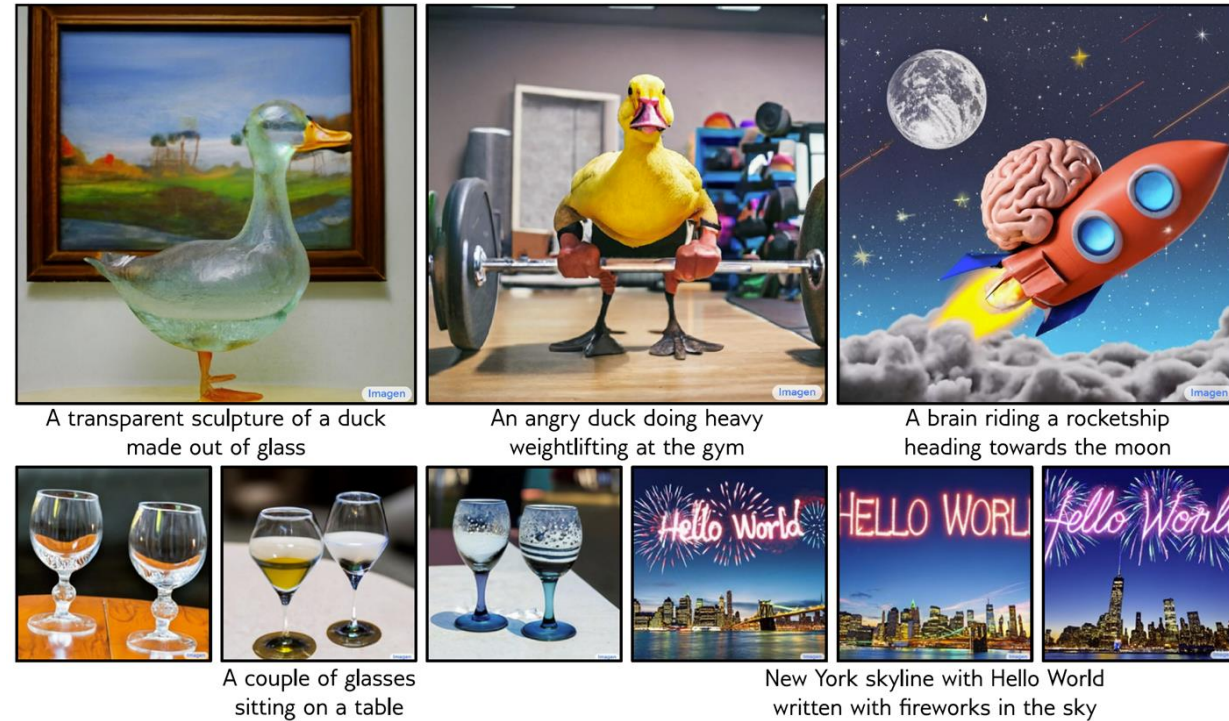


**Figure 18.12** Conditional generation using classifier guidance. Image samples conditioned on different ImageNet classes. The same model produces high quality samples of highly varied image classes. Adapted from Dhariwal & Nichol (2021).

# Applications of Diffusion models

Conditional generation using text:

- Condition the generating network with text embeddings

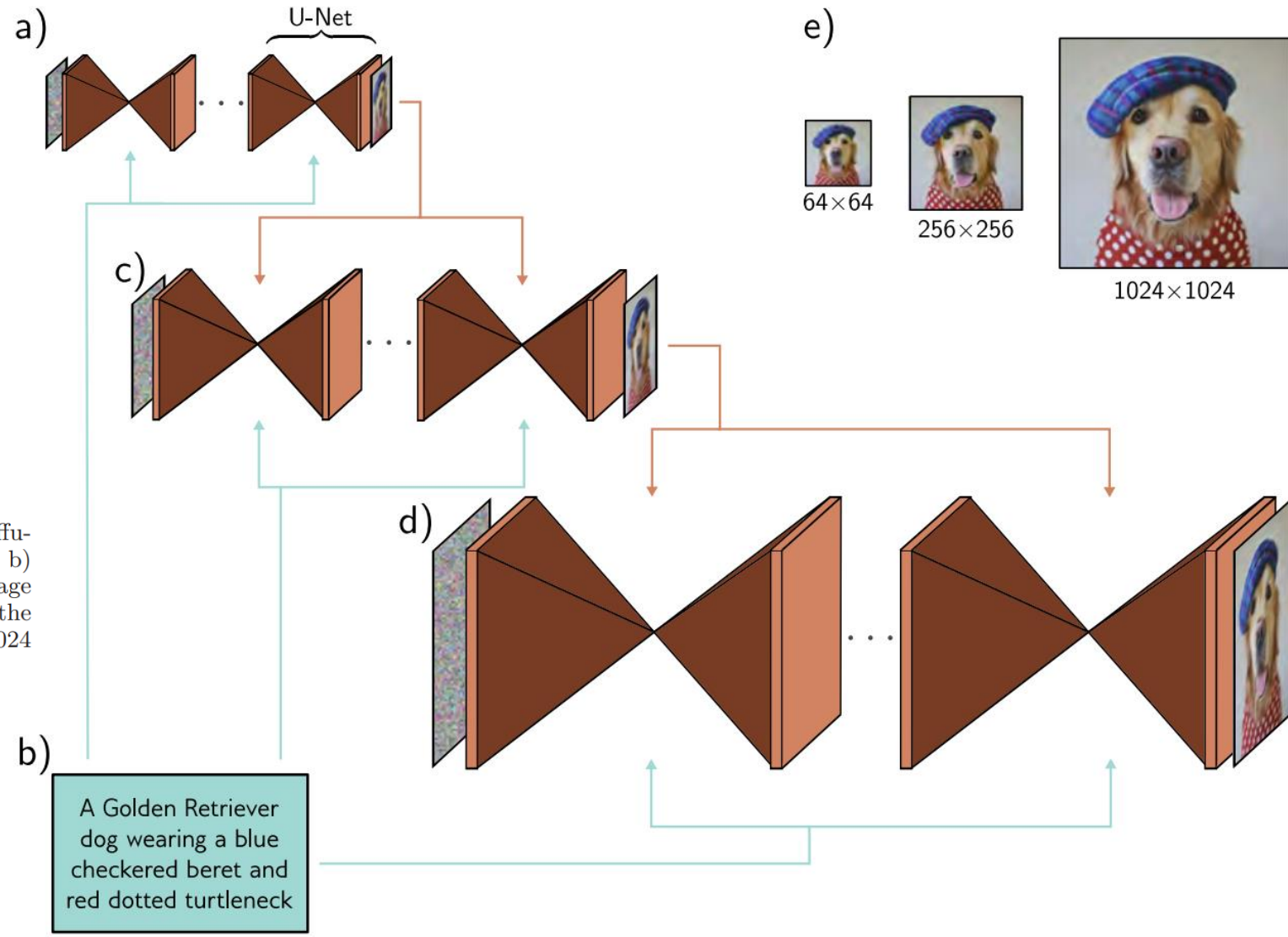


**Figure 18.13** Conditional generation using text prompts. Synthesized images from a cascaded generation framework, conditioned on a text prompt encoded by a large language model. The stochastic model can produce many different images compatible with the prompt. The model can count objects and incorporate text into images. Adapted from Saharia et al. (2022b).



# Applications of Diffusion models

## Conditional generation of higher resolution images



**Figure 18.11** Cascaded conditional generation based on a text prompt. a) A diffusion model consisting of a series of U-Nets is used to generate a  $64 \times 64$  image. b) This generation is conditioned on a sentence embedding computed by a language model. c) A higher resolution  $256 \times 256$  image is generated and conditioned on the smaller image *and* the text encoding. d) This is repeated to create a  $1024 \times 1024$  image. e) Final image sequence. Adapted from Saharia et al. (2022b).