Quadratic Programming

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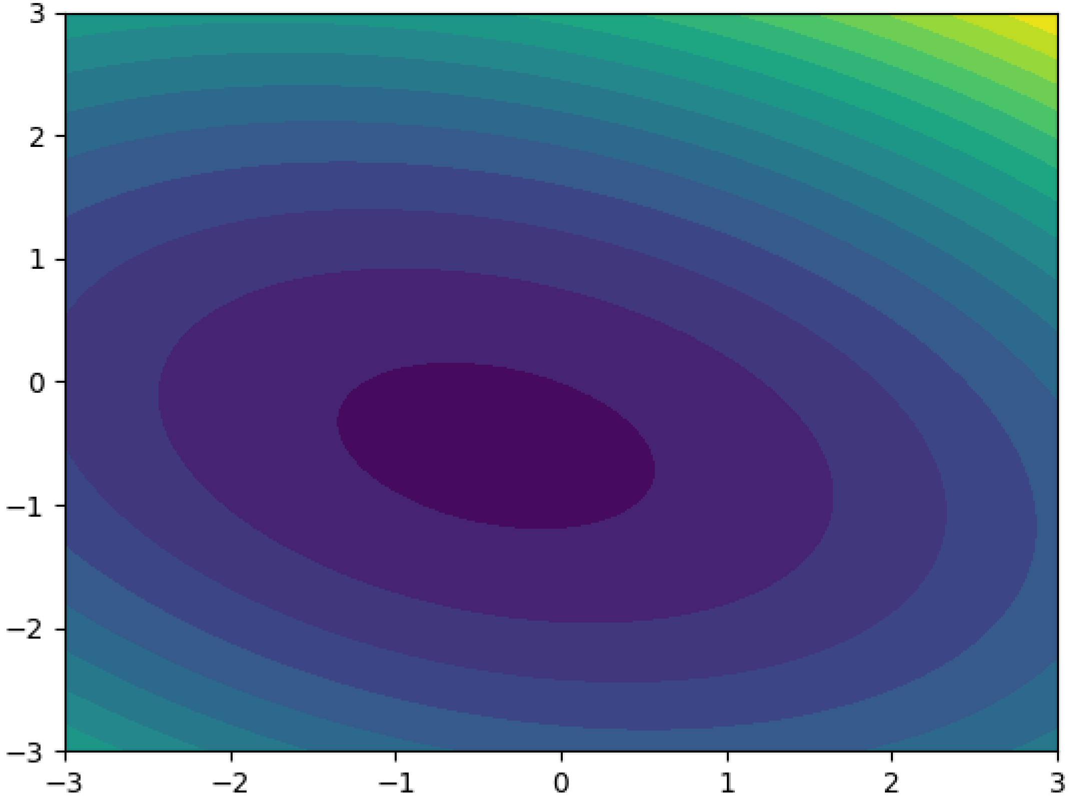
Quadratic programming

- Quadratic programming (QP) is the process of solving certain mathematical optimization problems involving quadratic functions subject to linear constraints on the variables.

$$\begin{aligned} \min_x \ & \frac{1}{2} x^T P x + q^T x \\ \text{subject to } & Ax = b \\ & Gx \leq h \end{aligned}$$

Unconstrained QP

- $\min_x \ \frac{1}{2}x^T P x + q^T x$



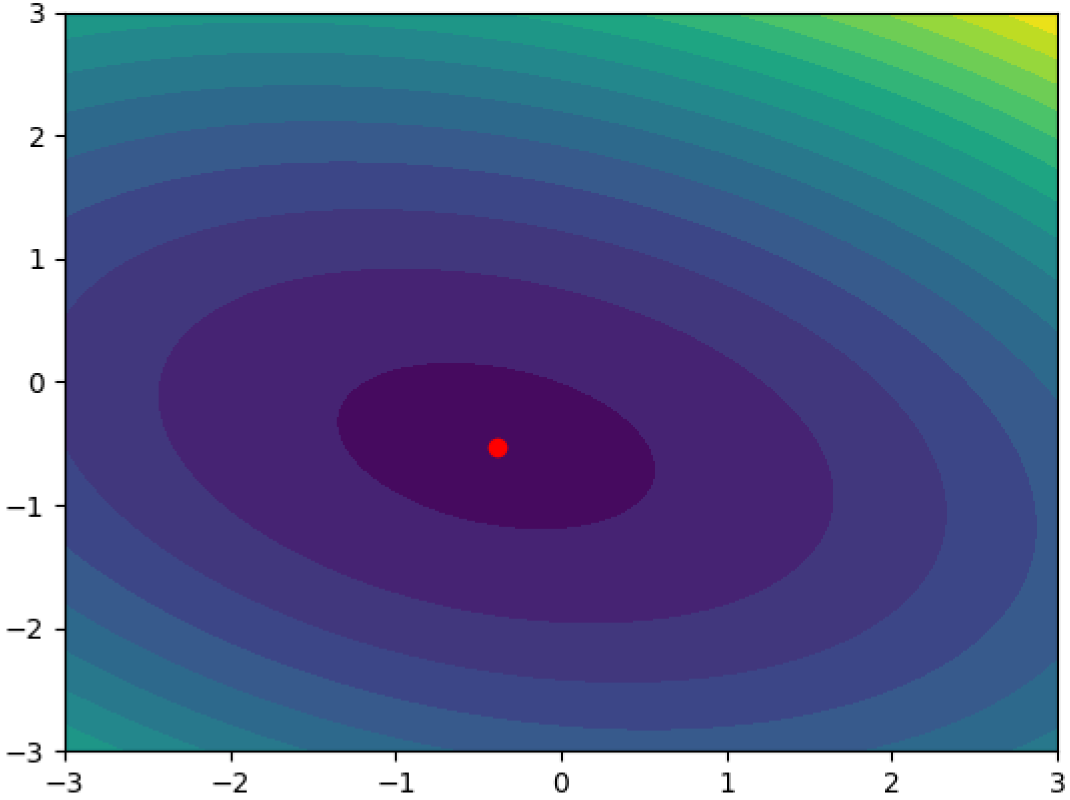
Unconstrained QP

- if $x_* = \arg\min_x \ \frac{1}{2}x^T P x + q^T x$, then $\frac{\partial}{\partial x}(\frac{1}{2}x_*^T P x_* + q^T x_*) = 0$
- Using $\frac{\partial}{\partial x}(\frac{1}{2}x^T P x + q^T x) = Px + q$:

$$
\begin{aligned}
Px_* + q &= 0 \\
Px_* &= -q \\
x_* &= -P^{-1}q
\end{aligned}
$$

Unconstrained QP

- $x_* = -P^{-1}q$



Lagrangian multiplier

- Consider the optimization problem:

$$\min_{x} f(x) \quad \text{s. t. } Ax = b;\ Gx \leq h$$

- Lagrangian multiplier $\mathcal{L}$:

$$\mathcal{L}(x, u, v) = f(x) + u^T(Ax - b) + v^T(Gx - h)$$

- The optimization problem avove can be rewritten as follows:

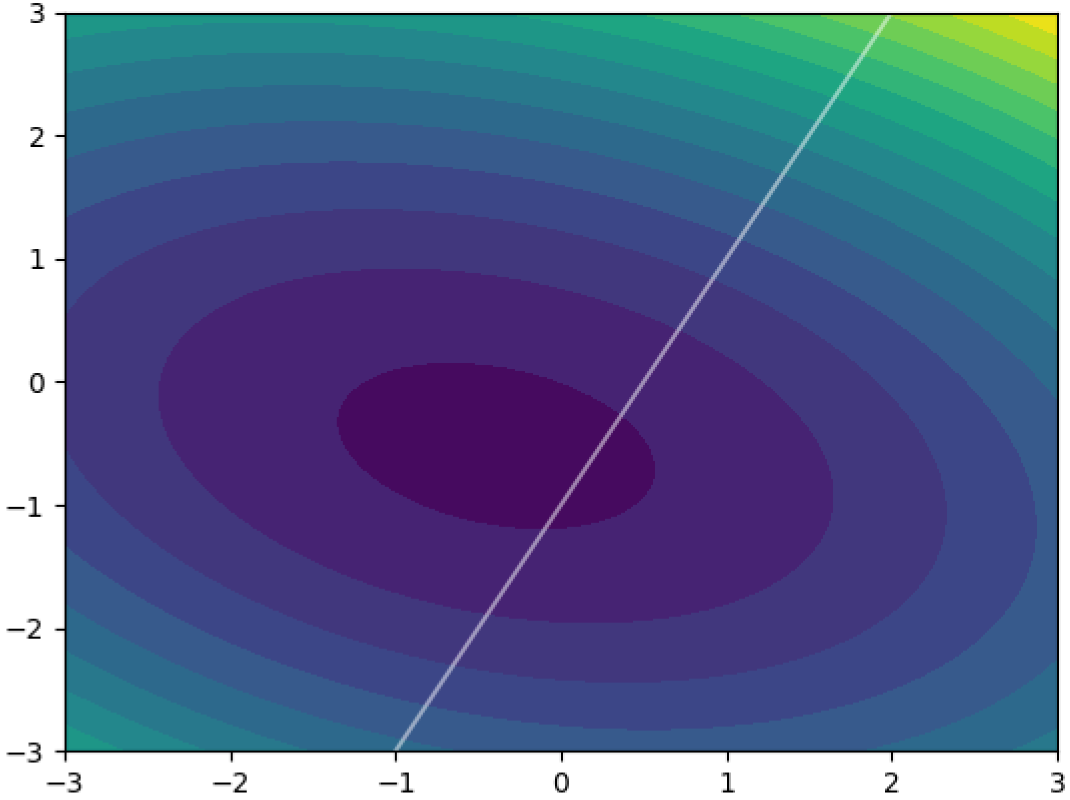
$$\max_{u,v} \min_{x} \mathcal{L}(x, u, v) \quad \text{s. t. } v \geq 0$$

- We call $g(u, v) = \min_x \mathcal{L}(x, u, v)$ as a dual function, and Lagrangian dual problem is:

$$\max_{u,v} g(u, v) \quad \text{s. t. } v \geq 0$$

Equality constrained QP

- $\min_x \ \frac{1}{2}x^T P x + q^T x \quad \text{s. t. } Ax = b$



Equality constrained QP

- Dual function: $g(u) = \min_x \mathcal{L}(x,u) = \frac{1}{2}x^T P x + q^T x + u^T(Ax - b)$
- Using $\frac{\partial}{\partial x}\mathcal{L}(x,u) = Px + q + A^T u$, $\arg\min_x \mathcal{L}(x,u) = -P^{-1}(A^T u + q)$. then:

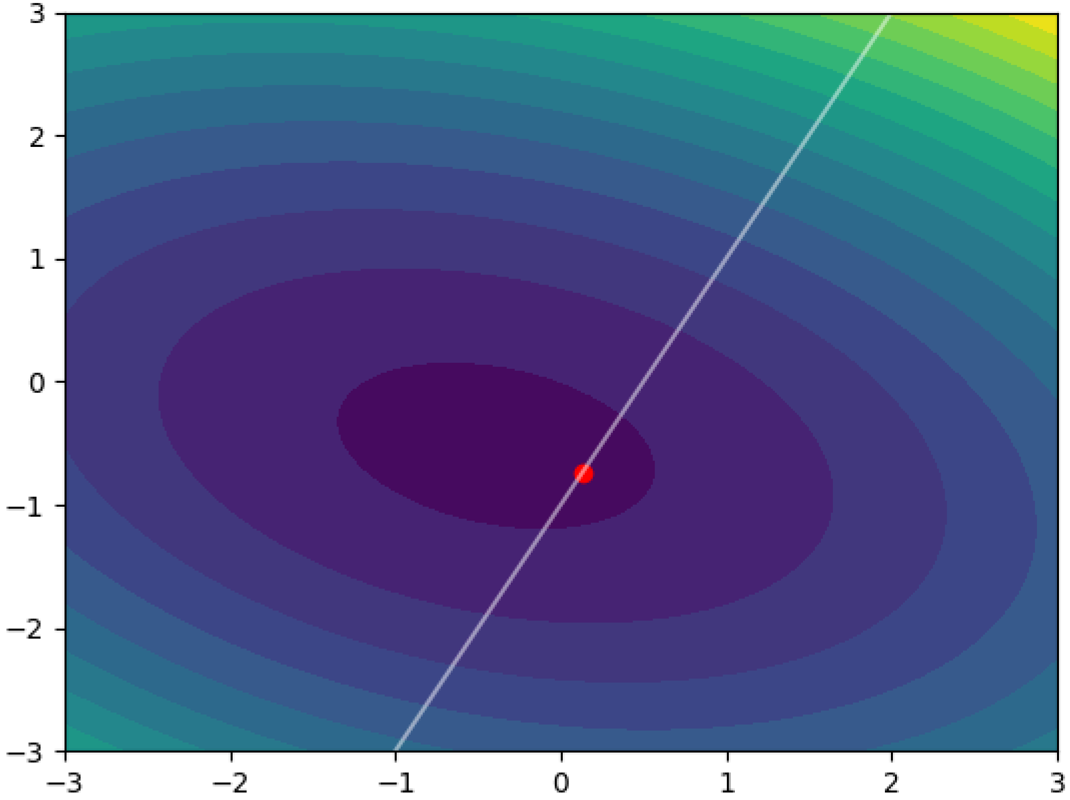
$$
\begin{aligned}
g(u) &= \frac{1}{2}(A^T u + q)^T P^{-1}(A^T u + q) - q^T P^{-1}(A^T u + q) - u^T(AP^{-1}(A^T u + q) + b) \\
&= -\frac{1}{2}(u^T A^T P^{-1} A^T u + q^T P^{-1} q) - (AP^{-1}q + b)^T u
\end{aligned}
$$

- Using $\frac{\partial g(u)}{\partial u} = -(A^T P^{-1} A)u - (AP^{-1}q + b)$:

$$
\begin{aligned}
u_* &= -(A^T P^{-1} A)^{-1}(AP^{-1}q + b) \\
x_* &= -P^{-1}(A^T u_* + q) \\
&= P^{-1}(A^T (A^T P^{-1} A)^{-1}(AP^{-1}q + b) - q)
\end{aligned}
$$

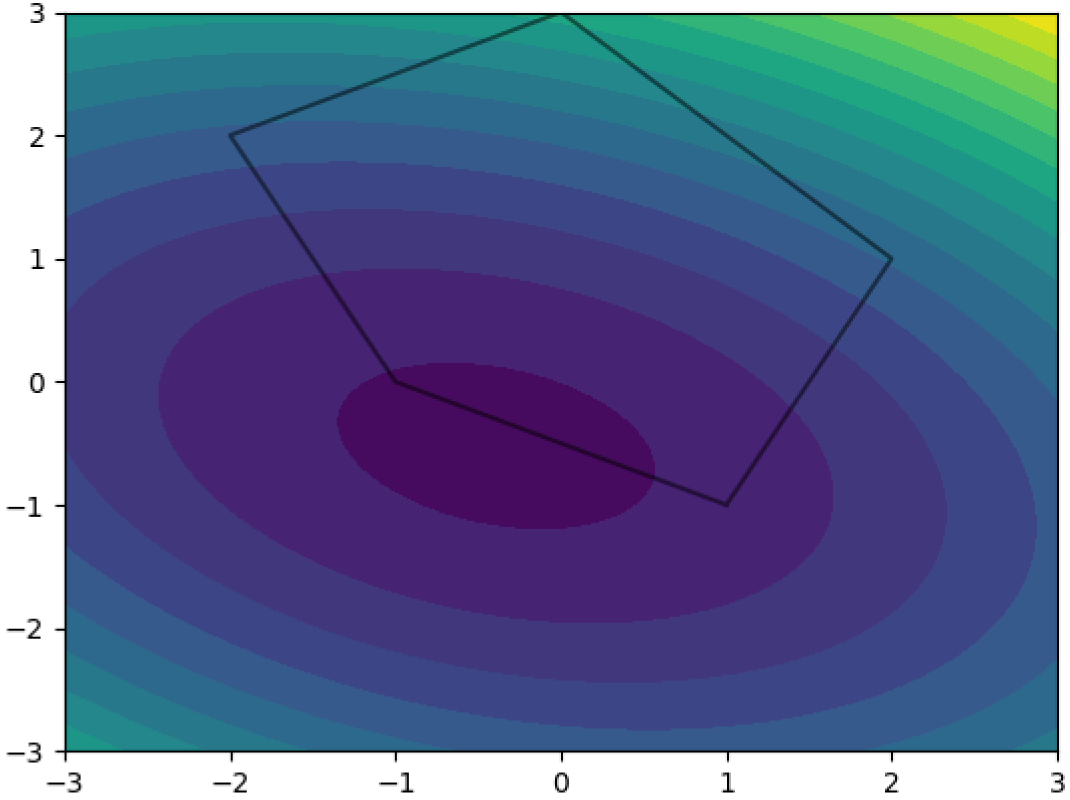
Equality constrained QP

- $x_* = P^{-1}(A^T(A^TP^{-1}A)^{-1}(AP^{-1}q + b) - q)$



Inequality constrained QP

- $\min_x \ \frac{1}{2}x^T P x + q^T x \quad \text{s. t. } Gx \le h,$



Inequality constrained QP

- Penalty method

  $\min_x \ \frac{1}{2}x^T Px + q^T x + \sum_{i=0}^{n} \max(0, g_i^T x - h)^2$

- Barrier method

  $\min_x \ \frac{1}{2}x^T Px + q^T x + \frac{1}{t}\phi(x), \text{ where } \phi(x) := \begin{cases} \sum_{i=0}^{n} -\log(h - g_i^T x) & \text{for } Gx < h \\ \infty & \text{otherwise} \end{cases}$

- Active-set method

  $\min_x \ \frac{1}{2}x^T Px + q^T x \quad \text{s. t. } g_i^T x = h_i, \forall i \in \{i | g_i^T x > h_i, i = 1, \ldots, n\}$

Inequality constrained QP (Active-set method)

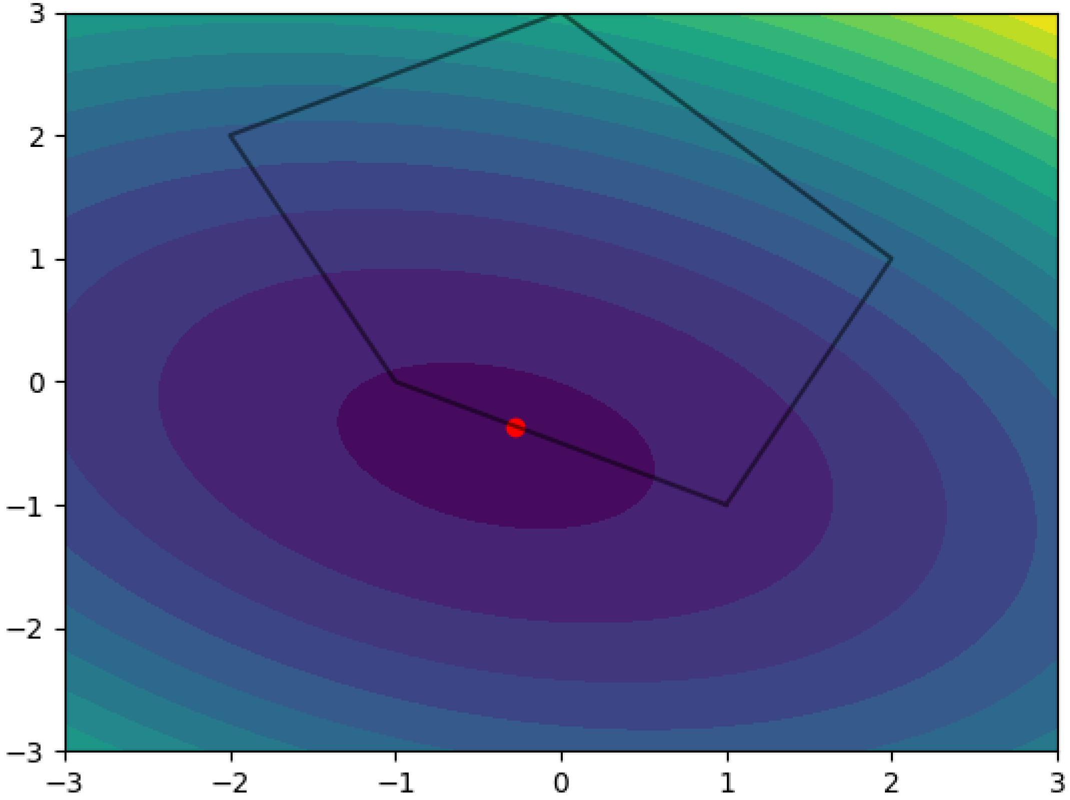
- Find unconstrained problem solution $x_{\text{unc}} = \arg\min_x \ \frac{1}{2}x^T P x + q^T x$
- Using $G = \begin{bmatrix} \vdots \\ g_i^T \\ \vdots \end{bmatrix}, h = \begin{bmatrix} \vdots \\ h_i \\ \vdots \end{bmatrix}$ and x_{unc}, find active inequality constiraints:

$$G_a = \begin{bmatrix} \vdots \\ g_i^T \\ \vdots \end{bmatrix}, h_a = \begin{bmatrix} \vdots \\ h_i \\ \vdots \end{bmatrix}, \forall i \in \{i | g_i^T x_{\text{unc}} > h_i, i = 1, \ldots, n\}$$

- Solve equality constrained problem $\min_x \ \frac{1}{2}x^T P x + q^T x \quad \text{s. t. } G_a x = h_a$

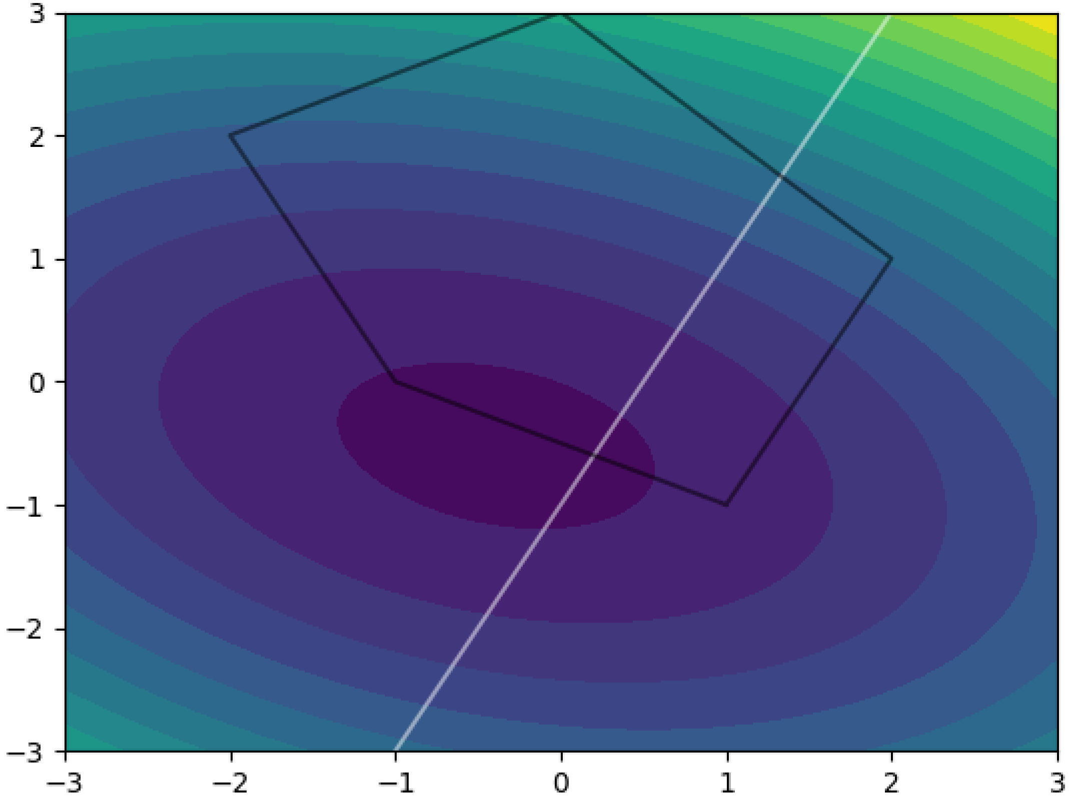
Inequality constrained QP (Active-set method)

- $x_* = \arg\min_x \ \frac{1}{2}x^T P x + q^T x \quad \text{s. t. } G_a x = h_a$



General QP

- $\min_x \ \frac{1}{2}x^T P x + q^T x \quad \text{s. t. } Ax = b;\ Gx \le h$



General QP

- Penalty method

  $\min_x \ \frac{1}{2}x^T P x + q^T x + \sum_{i=0}^{n} \max(0, g_i^T x - h)^2 \quad \text{s. t. } Ax = b$

- Barrier method

  $\min_x \ \frac{1}{2}x^T P x + q^T x + \frac{1}{t}\phi(x) \quad \text{s. t. } Ax = b$

- Active-set method

  $\min_x \ \frac{1}{2}x^T P x + q^T x \quad \text{s. t. } \begin{bmatrix} A \\ G_a \end{bmatrix} x = \begin{bmatrix} b \\ h_a \end{bmatrix}$

General QP

- $x_* = \begin{bmatrix} A \\ G_a \end{bmatrix}^{-1} \begin{bmatrix} b \\ h_a \end{bmatrix}$ (active-set method)