TTK4250

Week 2

The multivariate Gaussian

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Recap from last week

Key probability results

$$\Pr\{A|B\} = \frac{\Pr\{B|A\}\Pr\{A\}}{\Pr\{B\}}$$

$$\Pr\{A\} = \sum_{B_i} \Pr\{A|B_i\}\Pr\{B_i\}$$

Random variables and distributions

If X is a random variable we can quantify $\Pr\{X \in A\}$ for any subset A of the outcome space.

- cdf = $\Pr\{X \leq x\}$
- pdf = derivative of cdf.
- pmf = $\Pr\{X = x_i\}$ if X is discrete.

Manipulations of random variables

We can use . . .

- Moment- or probability-generating functions.
- The theorem for nonlinear transformations of random variables.

Estimators

An estimator $\hat{\mathbf{x}}$ is a function that for the given data $\mathbf{z} \in \mathcal{Z}$ provides a guess of the quantity $\mathbf{x}$ that we want to estimate.

$$\hat{\mathbf{x}} : \mathcal{Z} \to \mathbb{R}^n, \quad \mathbf{x} \in \mathbb{R}^n$$

Estimators are random variables

Example 1: ML estimator of Rayleigh distribution parameter.

Let $\mathbf{z} = [z_1, \ldots, z_M]^\mathsf{T}$ consist of IID samples from a Rayleigh distribution:

$$p(\mathbf{z} \mid \eta) = \prod_{i=1}^{M} \frac{z_i}{\eta} \exp\left(-\frac{z_i^2}{\eta^2}\right).$$

By differentiating the logarithm of $p(\mathbf{z} \mid \eta)$ and equating the derivative to zero, we get the ML estimator

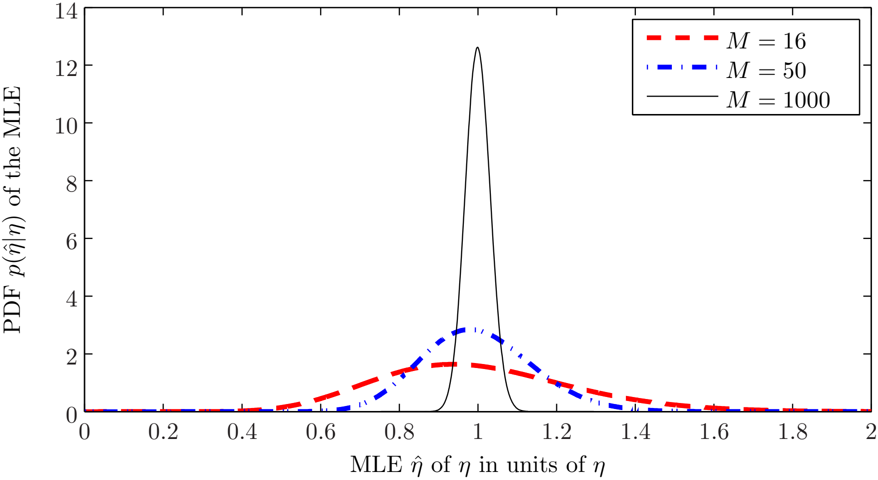
$$\hat{\eta} = \frac{1}{2M} \sum_{i=1}^{M} z_i^2.$$

This entity (a sum of IID exponential random variables) has a Gamma distribution:[a]

$$p(\hat{\eta} \mid \eta) = \mathrm{Gamma}\left(\hat{\eta}\,;\, M, \frac{2\eta}{M}\right) = \frac{M^M}{\Gamma(M)} \left(\frac{\hat{\eta}}{\eta}\right)^M \exp\left(-M\frac{\hat{\eta}}{\eta}\right). \tag{1}$$

[a] See Papoulis & Pillai (2002): "Probability, Random Variables and Stochastic Processes" for more on Gamma, Rayleigh, Exponential and all kinds of distributions.

Estimators are random variables



LS and MMSE estimators

The least squares (LS) estimator.

For any estimation problem on the form $\mathbf{z} = \mathbf{h}(\mathbf{x}) + \mathbf{w}$ the least squares estimator is

$$\hat{\mathbf{x}}_{\text{LS}} = \arg\min_{\mathbf{x}} \|\mathbf{z} - \mathbf{h}(\mathbf{x})\|_2^2$$

- The LS estimator does not make any assumptions about the measurement "noise" $\mathbf{w}$. It is therefore of a non-probabilistic nature.
- If $\mathbf{w}$ is IID multivariate Gaussian, then the LS estimator is identical to the MLE.

The minimum mean square error (MMSE) estimator.

This is the probabilistic counterpart of the LS estimator. It is given by

$$\hat{\mathbf{x}}_{\text{MMSE}} = \arg\min_{\hat{\mathbf{x}}} E\left[(\hat{\mathbf{x}} - \mathbf{x})^{\mathsf{T}}(\hat{\mathbf{x}} - \mathbf{x}) \mid \mathbf{z}\right] = E[\mathbf{x}|\mathbf{z}] = \int \mathbf{x} p(\mathbf{x}|\mathbf{z}) \mathrm{d}\mathbf{x}$$

MMSE estimator versus MAP estimator.

- MMSE and MAP are equal for all symmetric posterior PDFs.
- The MMSE estimator is a Bayes estimator which minimizes expected Bayes risk.
- Care should be exercised in choosing estimator if $p(\mathbf{x}|\mathbf{z})$ is multimodal or skewed.

Unbiasedness, MSE

Definition of unbiased estimator.

Let $\tilde{\mathbf{x}} = \mathbf{x} - \hat{\mathbf{x}}$ be the estimation error. We then require that $E[\tilde{\mathbf{x}}] = 0$.

- The MMSE estimator is always unbiased.
- The ML and MAP estimators are in general biased.

Unbiasedness is desirable. However, there may exist biased estimators with lower MSE, and there exist estimation problems where the requirement of unbiasedness will lead to unacceptable degradation in MSE.

Variance and MSE: Vector case.

The variance is given by

$$\mathrm{Cov}(\hat{\mathbf{x}}) = E\left[(\hat{\mathbf{x}} - E[\hat{\mathbf{x}}])(\hat{\mathbf{x}} - E[\hat{\mathbf{x}}])^{\mathsf{T}}\right]$$

The MSE is given by

$$\mathrm{MSE}(\hat{\mathbf{x}}) = E\left[(\hat{\mathbf{x}} - \mathbf{x})^{\mathsf{T}}(\hat{\mathbf{x}} - \mathbf{x})\right] = \mathrm{tr}\left(E[(\hat{\mathbf{x}} - \mathbf{x})(\hat{\mathbf{x}} - \mathbf{x})^{\mathsf{T}}]\right)$$

LMMSE estimators

When the MMSE is too complicated we may settle for the best linear estimator. This entails finding the best estimator $\hat{\mathbf{x}}$ on the form

$$\hat{\mathbf{x}} = \mathbf{A}\mathbf{z} + \mathbf{b}$$

that mimimizes

$$\mathrm{MSE}(\hat{\mathbf{x}}) = E\left(\|\mathbf{x} - \hat{\mathbf{x}}\|_2^2\right).$$

This minimization problem has the solution

$$\hat{\mathbf{x}} = E[\mathbf{x}] + \mathrm{Cov}(\mathbf{x}, \mathbf{z})\mathrm{Cov}(\mathbf{z})^{-1}(\mathbf{z} - E[\mathbf{z}]). \tag{2}$$

To use the LMMSE estimator we need to know the first two moments (expectation and covariance) of the joint PDF $p(\mathbf{x}, \mathbf{z}) = p(\mathbf{z}|\mathbf{x})p(\mathbf{x})$.

The MSE of the LMMSE estimator.

Let $\tilde{\mathbf{x}} = \mathbf{x} - \hat{\mathbf{x}}$ be the estimation error. The MSE is then given by the matrix

$$E[\tilde{\mathbf{x}}\tilde{\mathbf{x}}^{\mathsf{T}}] = \mathrm{Cov}(\mathbf{x}) - \mathrm{Cov}(\mathbf{x}, \mathbf{z})\mathrm{Cov}(\mathbf{z})^{-1}\mathrm{Cov}(\mathbf{x}, \mathbf{z})^{\mathsf{T}} \tag{3}$$

- Thus, the LMMSE estimator provides a simple measure of its own performance.
- You recognize (2) and (3) as the update step of the Kalman filter.
- (3) may be misleading if we have wrong values of $\mathrm{Cov}(\mathbf{x})$, $\mathrm{Cov}(\mathbf{x}, \mathbf{z})$ or $\mathrm{Cov}(\mathbf{z})$.

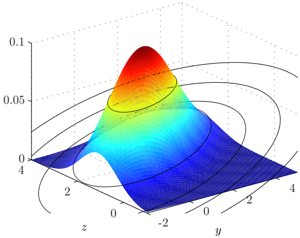
The multivariate Gaussian

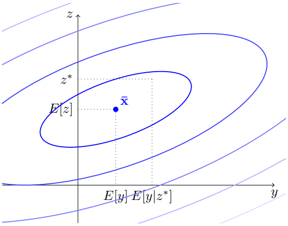
This is the probability distribution given by

$$\mathcal{N}(\mathbf{x}\,;\,\boldsymbol{\mu},\mathbf{P}) = \frac{1}{(2\pi)^{\frac{n}{2}}|\mathbf{P}|^{\frac{1}{2}}}\exp\left(-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^{\mathsf{T}}\mathbf{P}^{-1}(\mathbf{x}-\boldsymbol{\mu})\right)$$

where $\boldsymbol{\mu}$ is the expectation of **x** and **P** is the covariance of **x**.

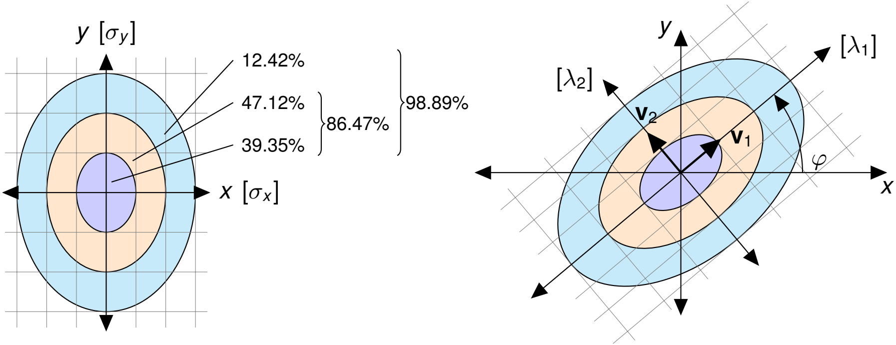
- The matrix **P** must be symmetric positive definite (SPD).
- All the dependence on **x** is encapsulated by a **quadratic form**.





The covariance matrix **P** is the inverse of the Hessian of $-\ln\mathcal{N}(\mathbf{x}\,,\,\boldsymbol{\mu},\mathbf{P})$.

The shape and probability mass of covariance ellipses

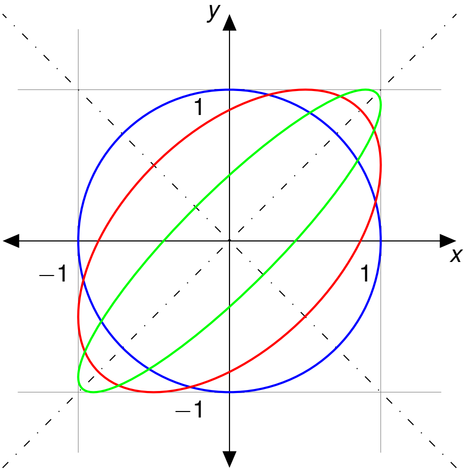


- The probability that $\mathbf{x}$ is within the ellipse $(\mathbf{x}-\boldsymbol{\mu})^{\mathsf{T}}\mathbf{P}^{-1}(\mathbf{x}-\boldsymbol{\mu}) = g^2$ is given by

$$\texttt{chi2cdf}(g^2, n)$$

- The shape of the ellipses is given by the eigenvectors and eigenvalues of $\mathbf{P}$.

The role of correlations



$$\begin{bmatrix} x \\ y \end{bmatrix} \sim \mathcal{N}\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & a \\ a & 1 \end{bmatrix} \right)$$

$$\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 1 & a \\ a & 1 \end{bmatrix}^{-1} \begin{bmatrix} x \\ y \end{bmatrix} = 1$$

$a = 0.0$

$a = 0.5$

$a = 0.9$

- Correlations make the covariance ellipses narrower.

$\Rightarrow$ Correlations can be exploited to achieve accurate state estimation.

Key rules: Independence and Linearity

Independence

Two random vectors $\mathbf{x}$ and $\mathbf{y}$ with probability density functions $\mathcal{N}(\mathbf{x}\,;\,\mathbf{a},\mathbf{A})$ and $\mathcal{N}(\mathbf{y}\,;\,\mathbf{b},\mathbf{B})$ are independent if and only if

$$p(\mathbf{x},\mathbf{y}) = p\left(\begin{bmatrix}\mathbf{x}\\ \mathbf{y}\end{bmatrix}\right) = \mathcal{N}\left(\begin{bmatrix}\mathbf{x}\\ \mathbf{y}\end{bmatrix};\begin{bmatrix}\mathbf{a}\\ \mathbf{b}\end{bmatrix},\begin{bmatrix}\mathbf{A} & 0\\ 0 & \mathbf{B}\end{bmatrix}\right).$$

Linearity

If $p(\mathbf{x}) = \mathcal{N}(\mathbf{x}\,;\,\mathbf{a},\mathbf{A})$ and $\mathbf{y} = \mathbf{F}\mathbf{x}$, then $p(\mathbf{y}) = \mathcal{N}(\mathbf{y}\,;\,\mathbf{F}\mathbf{a},\mathbf{F}\mathbf{A}\mathbf{F}^\mathsf{T})$.

Example: Cholesky factorization

Let $\mathbf{x} \sim \mathcal{N}(\mathbf{0},\mathbf{A})$. Since $\mathbf{A}$ is symmetric positive definite it has a Cholesky factorization $\mathbf{L}$ so that $\mathbf{A} = \mathbf{L}\mathbf{L}^\mathsf{T}$. Define the transformed RV $\mathbf{y} = \mathbf{L}^{-1}\mathbf{x}$. The expectation of $\mathbf{y}$ is then obviously $\mathbf{0}$ and its covariance is

$$\mathrm{Cov}[\mathbf{y}] = \mathbf{L}^{-1}\mathbf{A}(\mathbf{L}^{-1})^\mathsf{T} = \mathbf{L}^{-1}\mathbf{L}\mathbf{L}^\mathsf{T}(\mathbf{L}^{-1})^\mathsf{T} = \mathbf{I}.$$

Key rules: Marginalization and conditioning

Let $\mathbf{x}$ and $\mathbf{y}$ have the joint distribution

$$p(\mathbf{x}, \mathbf{y}) = \mathcal{N}\left(\begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix}; \begin{bmatrix} \mathbf{a} \\ \mathbf{b} \end{bmatrix}, \begin{bmatrix} \mathbf{P}_{xx} & \mathbf{P}_{xy} \\ \mathbf{P}_{xy}^{\mathsf{T}} & \mathbf{P}_{yy} \end{bmatrix}\right)$$

Marginalization

The marginal distribution of $\mathbf{y}$ is $p(\mathbf{y}) = \mathcal{N}(\mathbf{y}\,;\, \mathbf{b}, \mathbf{P}_{yy})$.

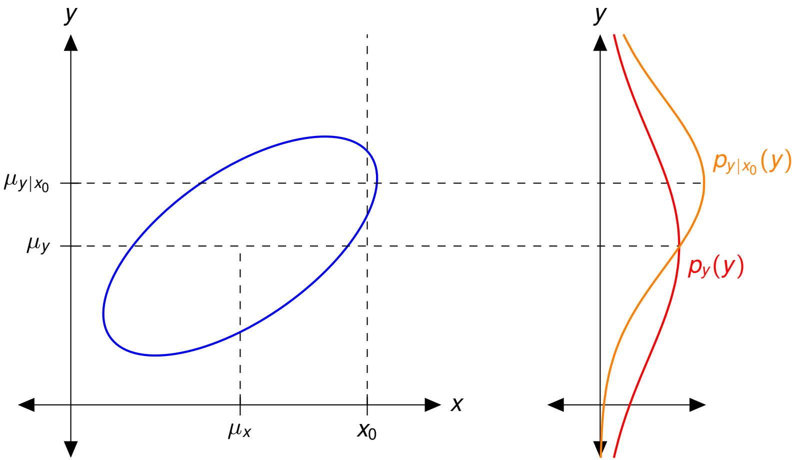
Conditioning

The conditional distribution pf $\mathbf{x}$ given $\mathbf{y}$ is $p(\mathbf{x}|\mathbf{y}) = \mathcal{N}(\mathbf{x}\,;\, \boldsymbol{\mu}_{x|y}, \mathbf{P}_{x|y})$ where

$$\boldsymbol{\mu}_{x|y} = \mathbf{a} + \mathbf{P}_{xy}\mathbf{P}_{yy}^{-1}(\mathbf{y} - \mathbf{b}) \qquad \text{and} \qquad \mathbf{P}_{x|y} = \mathbf{P}_{xx} - \mathbf{P}_{xy}\mathbf{P}_{yy}^{-1}\mathbf{P}_{xy}^{\mathsf{T}}.$$

Notice that the formulas for conditioning are very similar to the formulas we derived for the LMMSE estimator.

Illustration: Marginalization and conditioning



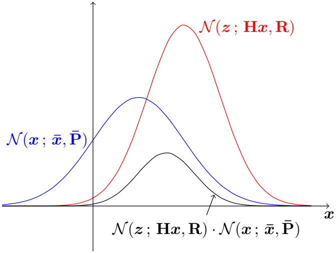
Key rules: The product identity

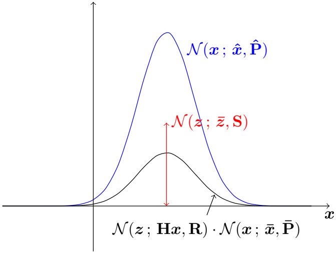
The product of a Gaussian in $\mathbf{x}$ with a Gaussian that depends linearly on $\mathbf{x}$ is proportional to another Gaussian in $\mathbf{x}$:

$$\mathcal{N}(\mathbf{z}\,;\,\mathbf{H}\mathbf{x},\mathbf{R})\mathcal{N}(\mathbf{x}\,;\,\bar{\mathbf{x}},\bar{\mathbf{P}}) = \mathcal{N}(\mathbf{z}\,;\,\bar{\mathbf{z}},\mathbf{S})\mathcal{N}(\mathbf{x}\,;\,\hat{\mathbf{x}},\hat{\mathbf{P}}).$$

The Gaussians on the right-hand side are given by

$$\begin{aligned} \bar{\mathbf{z}} &= \mathbf{H}\bar{\mathbf{x}} \\ \mathbf{S} &= \mathbf{R} + \mathbf{H}\bar{\mathbf{P}}\mathbf{H}^\mathsf{T} \end{aligned} \qquad \begin{aligned} \hat{\mathbf{x}} &= \bar{\mathbf{x}} + \mathbf{W}(\mathbf{z} - \mathbf{H}\bar{\mathbf{x}}) \\ \hat{\mathbf{P}} &= (\mathbf{I} - \mathbf{W}\mathbf{H})\bar{\mathbf{P}} \\ \mathbf{W} &= \bar{\mathbf{P}}\mathbf{H}^\mathsf{T}\mathbf{S}^{-1}. \end{aligned}$$





Overview of a proof of the product identity

If we assume that the rules for conditioning and marginalization are proved, we can prove the product identity in the following three steps:[1]

1. We construct a joint Gaussian over $\mathbf{z}$ and $\mathbf{x}$ which can be factorized in two manners:

$$p(\mathbf{z}, \mathbf{x}) = p(\mathbf{z}|\mathbf{x})p(\mathbf{x}) = p(\mathbf{x}|\mathbf{z})p(\mathbf{z}) \tag{4}$$

   We define $p(\mathbf{z}, \mathbf{x})$ by letting the first factorization in (4) be identical to the left-hand-side of the product identity.

2. The quadratic form in $p(\mathbf{z}, \mathbf{x})$ will then be a sum of two contributions from $p(\mathbf{z}|\mathbf{x})$ and $p(\mathbf{x})$. We manipulate this sum so that it becomes a single quadratic form describing $p(\mathbf{z}, \mathbf{x})$ as a Gaussian in the stacked vector $[\mathbf{z}^\mathsf{T}, \mathbf{x}^\mathsf{T}]^\mathsf{T}$.
3. We obtain the second factorization in (4) by means of the conditioning and marginalization rules. This factorization is identical to the right-hand-side of the product identity.

[1] Based on L.-C. Tokle (2018): "Multi target tracking using random finite sets with a hybrid state space and approximations."

The canonical form

- We recall that the inverse of the covariance $\mathbf{P}$ of a Gaussian $\mathcal{N}(\mathbf{x}\,;\,\boldsymbol{\mu},\mathbf{P})$ is the Hessian of $-\ln\mathcal{N}(\mathbf{x}\,,\,\boldsymbol{\mu},\mathbf{P})$.
- We also refer to this matrix, typically denoted $\boldsymbol{\Lambda}$, as the **information matrix** or **precision matrix**.
- In some problems (e.g., SLAM) the information matrix may have a neater structure than the covariance matrix.

The canonical form

We can parameterize the multivariate Gaussian as

$$\mathcal{N}(\mathbf{x}\,;\,\boldsymbol{\mu},\mathbf{P}) = \exp\left(a + \boldsymbol{\eta}^{\mathsf{T}}\mathbf{x} - \frac{1}{2}\mathbf{x}^{\mathsf{T}}\boldsymbol{\Lambda}\mathbf{x}\right)$$

where

$$\begin{aligned}
\boldsymbol{\Lambda} &= \mathbf{P}^{-1} \\
\boldsymbol{\eta} &= \boldsymbol{\Lambda}\boldsymbol{\mu} \quad \textbf{= information state, potential vector} \\
a &= -(1/2)\left(n\ln(2\pi) - \ln|\boldsymbol{\Lambda}| + \boldsymbol{\eta}^{\mathsf{T}}\boldsymbol{\Lambda}^{-1}\boldsymbol{\eta}\right)
\end{aligned}$$

Marginalization and conditioning in canonical form

Theorem

Let $\mathbf{x}$ and $\mathbf{y}$ have the joint distribution

$$p(\mathbf{x}, \mathbf{y}) \propto \exp\left(\begin{bmatrix} \boldsymbol{\eta}_a^\mathsf{T} & \boldsymbol{\eta}_b^\mathsf{T} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} - \frac{1}{2} \begin{bmatrix} \mathbf{x}^\mathsf{T} & \mathbf{y}^\mathsf{T} \end{bmatrix} \begin{bmatrix} \boldsymbol{\Lambda}_{xx} & \boldsymbol{\Lambda}_{xy} \\ \boldsymbol{\Lambda}_{xy}^\mathsf{T} & \boldsymbol{\Lambda}_{yy} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} \right)$$

- The **marginal distribution** of $\mathbf{y}$ is has potential vector $\boldsymbol{\eta}_* = \boldsymbol{\eta}_b - \boldsymbol{\Lambda}_{xy}^\mathsf{T} \boldsymbol{\Lambda}_{xx}^{-1} \boldsymbol{\eta}_a$ and information matrix $\boldsymbol{\Lambda}_* = \boldsymbol{\Lambda}_{yy} - \boldsymbol{\Lambda}_{xy}^\mathsf{T} \boldsymbol{\Lambda}_{xx}^{-1} \boldsymbol{\Lambda}_{xy}$
- The **conditional distribution** of $\mathbf{x}$ given $\mathbf{y}$ has potential vector $\boldsymbol{\eta}_{x|y} = \boldsymbol{\eta}_a - \boldsymbol{\Lambda}_{xy} \boldsymbol{\eta}_b$ and information matrix $\boldsymbol{\Lambda}_{x|y} = \boldsymbol{\Lambda}_{xx}$.

Implications

- Conditioning is easy in canonical form, but difficult in covariance form.
- Marginalization is easy in covariance form, but difficult in canonical form.

Therefore, if we plan to only introduce more information in a sensor fusion system, without forgetting any information, then canonical form may be preferable.

Probabilistic state estimation

- So far we have only discussed estimation in static systems.
- In the remainder of the course we want to do estimation in **dynamic systems**.

Continuous time vs discrete time.

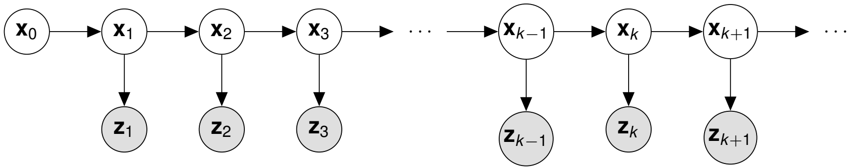
- Continuous time: $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}, \mathbf{v}), \;\; \mathbf{z} = \mathbf{h}(\mathbf{x}, \mathbf{w})$.
  - Often most closely related to the underlying physics.
  - Conceptually challenging (continuous-time white noise is a mathematical abstraction).
  - Impossible to implement on a computer.
- Discrete time: $\mathbf{x}_k = \mathbf{f}(\mathbf{x}_{k-1}, \mathbf{u}_k, \mathbf{v}_k), \;\; \mathbf{z} = \mathbf{h}(\mathbf{x}, \mathbf{w})$.

Recursive vs batch.

- In **recursive estimation (filtering),** we go through the steps as new data arrive:
  - Given some information about $\mathbf{x}_{k-1}$ ...
  - we **predict** $\mathbf{x}_k$ ...
  - we adjust our prediction of $\mathbf{x}_k$ based on the data $\mathbf{z}_k$ ...
  - and so on.
- In **batch estimation,** we estimate all the state variables $\mathbf{x}_{1:k} = [\mathbf{x}_1; \ldots; \mathbf{x}_k]$ simultaneously.
- Batch estimation is related to **smoothing,** where one filters both forward and backwards in time, in order to exploit future data to improve past estimates.

Recursive Bayesian estimation: Model and key concepts

We study systems whose structure fits the **graphical model** below:



- The horizontal arrows represent a **process model** of the form $p(\mathbf{x}_k \mid \mathbf{x}_{k-1})$
- The vertical arrows represent a **measurement model** of the form $p(\mathbf{z}_k \mid \mathbf{x}_k)$.

This structure reflects the following **Markov assumptions**

$$p(\mathbf{x}_k \mid \mathbf{x}_1, \ldots, \mathbf{x}_{k-2}, \mathbf{x}_{k-1}, \mathbf{z}_1, \ldots, \mathbf{z}_{k-2}, \mathbf{z}_{k-1}) = p(\mathbf{x}_k \mid \mathbf{x}_{k-1})$$
$$p(\mathbf{z}_k \mid \mathbf{x}_1, \ldots, \mathbf{x}_{k-2}, \mathbf{x}_{k-1}, \mathbf{x}_k, \mathbf{z}_1, \ldots, \mathbf{z}_{k-2}, \mathbf{z}_{k-1}) = p(\mathbf{z}_k \mid \mathbf{x}_k)$$

Recursive Bayesian estimation: The Bayes filter

In the Bayesian philosophy we want a pdf as our solution. This pdf may or may not be given by parameters such as expectation, covariance etc.

What do we know about $\mathbf{x}_k$ after observing $\mathbf{z}_{1:k} = (\mathbf{z}_1, \mathbf{z}_2, \dots, \mathbf{z}_k)$?

- The total probability theorem yields the predicted density

$$p(\mathbf{x}_k|\mathbf{z}_{1:k-1}) = \int p(\mathbf{x}_k|\mathbf{x}_{k-1})p(\mathbf{x}_{k-1}|\mathbf{z}_{1:k-1})\mathrm{d}\mathbf{x}_{k-1}.$$

- Bayes' rule yields the posterior density

$$p(\mathbf{x}_k|\mathbf{z}_{1:k}) = \frac{p(\mathbf{z}_k|\mathbf{x}_k)p(\mathbf{x}_k|\mathbf{z}_{1:k-1})}{p(\mathbf{z}_k \mid \mathbf{z}_{1:k-1})} \propto p(\mathbf{z}_k|\mathbf{x}_k)p(\mathbf{x}_k|\mathbf{z}_{1:k-1}).$$

Remark: Violations of the Markov assumptions can be handled by replacing the Markov chain by a higher order Markov chain that models the temporal correlations. We must then extend the state vector with corresponding states.

Linearity, Gaussianity and the Kalman filter

"Everything should be made as simple as possible, but not simpler."

- In general, we cannot find a closed-form solution to the Bayes filter.
- If the posterior can be described with reasonable accuracy by a few parameters (e.g., expectation and covariance), then we should look for a compact representation.

Closed-form solution to the Bayes filter = Kalman filter

When does a closed-form solution to the Bayes filter exist?

- When the initial density is Gaussian $\mathcal{N}(\mathbf{x}_0\,;\,\hat{\mathbf{x}}_0, \mathbf{P}_0)$
- ... and the Markov model is Gaussian-linear $\mathcal{N}(\mathbf{x}_k\,;\,\mathbf{F}\mathbf{x}_{k-1}, Q)$
- ... and the likelihood is Gaussian-linear $\mathcal{N}(\mathbf{z}_k\,;\,\mathbf{H}\mathbf{x}_k, \mathbf{R})$
- ... and standard independence assumptions apply.

The prediction step of the Kalman filter

The predicted density is given by

$$\begin{aligned}
p(\mathbf{x}_k|\mathbf{z}_{1:k-1}) &= \int p(\mathbf{x}_k|\mathbf{x}_{k-1})p(\mathbf{x}_{k-1}|\mathbf{z}_{1:k-1})\mathrm{d}\mathbf{x}_{k-1} \\
&= \int \mathcal{N}(\mathbf{x}_k\,;\,\mathbf{F}\mathbf{x}_{k-1}, Q)\mathcal{N}(\mathbf{x}_{k-1}\,;\,\hat{\mathbf{x}}_{k-1}, \mathbf{P}_{k-1})\mathrm{d}\mathbf{x}_{k-1} \\
&= \mathcal{N}(\mathbf{x}_k\,;\,\mathbf{F}\hat{\mathbf{x}}_{k-1}, \mathbf{F}\mathbf{P}_{k-1}\mathbf{F}^\mathsf{T} + Q) \\
&\quad \cdot \int \mathcal{N}(\mathbf{x}_{k-1}\,;\,\text{some vector}\,,\,\text{some covariance matrix}\,)\mathrm{d}\mathbf{x}_{k-1} \\
&= \mathcal{N}(\mathbf{x}_k\,;\,\hat{\mathbf{x}}_{k|k-1}, \mathbf{P}_{k|k-1}).
\end{aligned}$$

- $\hat{\mathbf{x}}_{k-1}$ is the previous state estimate.
- $\mathbf{P}_{k-1}$ is the previous covariance.
- $\hat{\mathbf{x}}_{k|k-1} = \mathbf{F}\hat{\mathbf{x}}_{k-1}$ is the predicted state estimate.
- $\mathbf{P}_{k|k-1} = \mathbf{F}\mathbf{P}_{k-1}\mathbf{F}^\mathsf{T} + Q$ is the predicted covariance.

The update step of the Kalman filter

The posterior density is given by

$$
\begin{aligned}
p(\mathbf{x}_k|\mathbf{z}_{1:k}) &\propto p(\mathbf{z}_k|\mathbf{x}_k)\, p(\mathbf{x}_k|\mathbf{z}_{1:k-1}) \\
&= \mathcal{N}(\mathbf{z}_k\,;\,\mathbf{H}\mathbf{x}_k,\mathbf{R})\,\mathcal{N}(\mathbf{x}_k\,;\,\hat{\mathbf{x}}_{k|k-1},\mathbf{P}_{k|k-1}) \\
&= \mathcal{N}(\mathbf{z}_k\,;\,\mathbf{H}\hat{\mathbf{x}}_{k|k-1},\mathbf{H}\mathbf{P}_{k|k-1}\mathbf{H}^{\mathsf{T}}+\mathbf{R})\mathcal{N}(\mathbf{x}_k\,;\,\hat{\mathbf{x}}_k,\mathbf{P}_k) \\
&\propto \mathcal{N}(\mathbf{x}_k\,;\,\hat{\mathbf{x}}_k,\mathbf{P}_k).
\end{aligned}
$$

- $\hat{\mathbf{x}}_k = \hat{\mathbf{x}}_{k|k-1} + \mathbf{W}_k(\mathbf{z}_k - \mathbf{H}\hat{\mathbf{x}}_{k|k-1})$ is the posterior state estimate.
- $\mathbf{P}_k = (\mathbf{I} - \mathbf{W}_k\mathbf{H})\mathbf{P}_{k|k-1}$ is the posterior covariance.
- $\mathbf{W}_k = \mathbf{P}_{k|k-1}\mathbf{H}^{\mathsf{T}}(\mathbf{H}\mathbf{P}_{k|k-1}\mathbf{H}^{\mathsf{T}} + \mathbf{R})^{-1}$ is the Kalman gain.

More about the covariance

Joseph form

$$\mathbf{P}_k = (\mathbf{I} - \mathbf{W}_k\mathbf{H})\mathbf{P}_{k|k-1}(\mathbf{I} - \mathbf{W}_k\mathbf{H})^\mathsf{T} + \mathbf{W}\mathbf{R}\mathbf{W}^\mathsf{T}$$

Information form

$$\mathbf{P}_k^{-1} = \mathbf{H}^\mathsf{T}\mathbf{R}^{-1}\mathbf{H} + \mathbf{P}_{k|k-1}^{-1}$$

Orthogonality properties

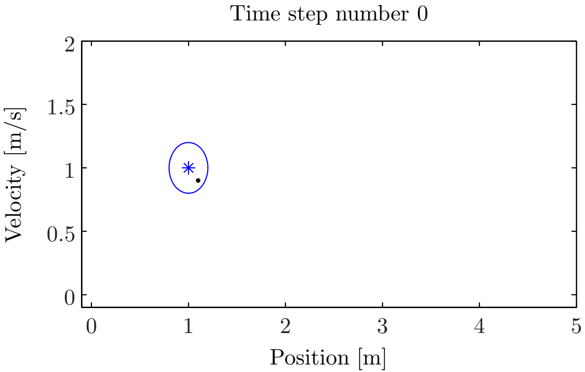
- The estimation errors $\tilde{\mathbf{x}}_k = \hat{\mathbf{x}}_k - \mathbf{x}_k$ do not constitute a white sequence:

$$E[\tilde{\mathbf{x}}_k\tilde{\mathbf{x}}_{k-1}^\mathsf{T}] = (\mathbf{I} - \mathbf{W}_k\mathbf{H})\mathbf{F}\mathbf{P}_k.$$

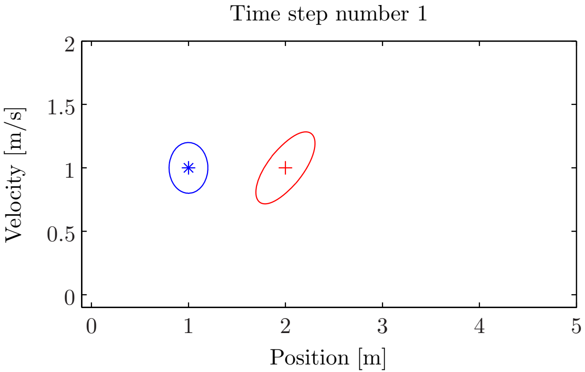
- The innovations $\boldsymbol{\nu}_k = \mathbf{z}_k - \mathbf{H}\hat{\mathbf{x}}_{k|k-1}$ on the other hand are a white sequence:

$$E[\boldsymbol{\nu}_k\boldsymbol{\nu}_j^\mathsf{T}] = \mathbf{0} \text{ if } k \neq j \quad \Leftrightarrow \quad p(\mathbf{z}_{1:k}) = \prod_{j=1}^{k} p(\boldsymbol{\nu}_j).$$

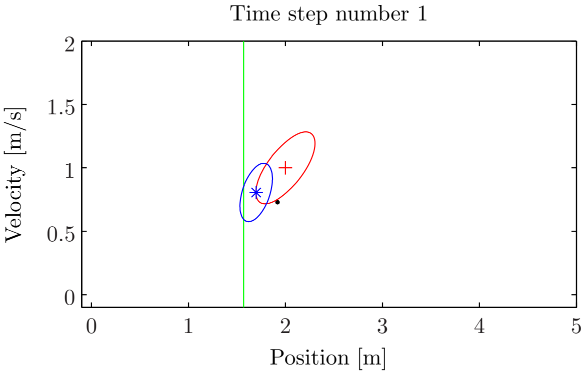
Example run of the Kalman filter



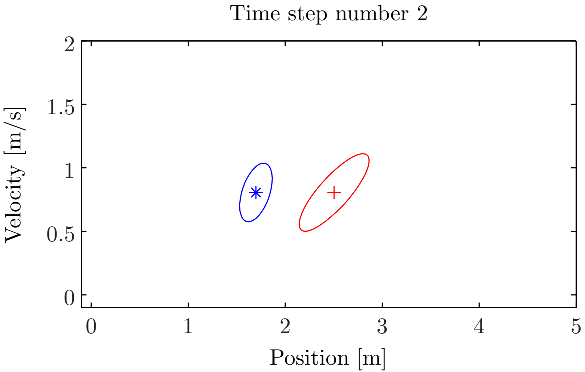
Example run of the Kalman filter



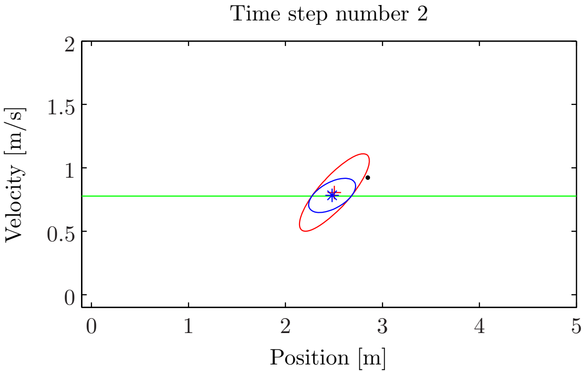
Example run of the Kalman filter



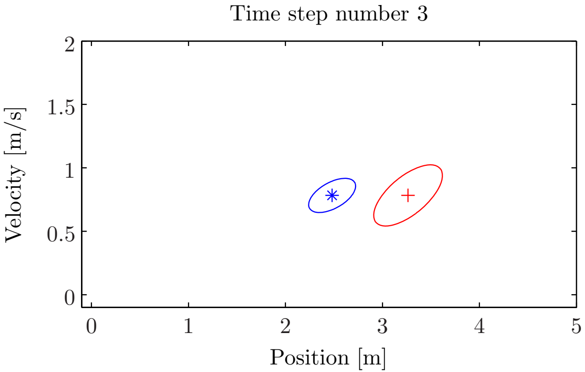
Example run of the Kalman filter



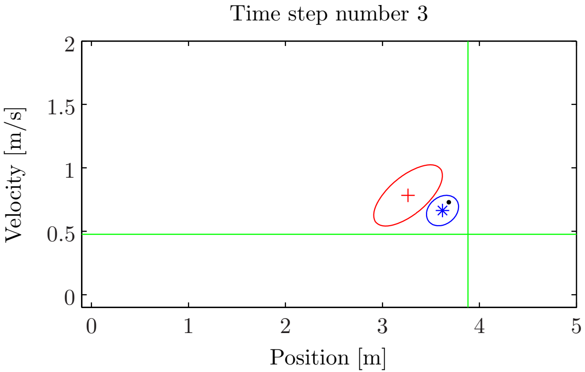
Example run of the Kalman filter



Example run of the Kalman filter

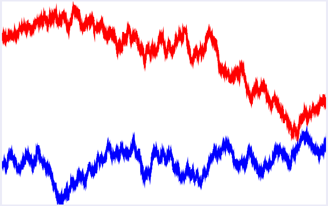


Example run of the Kalman filter

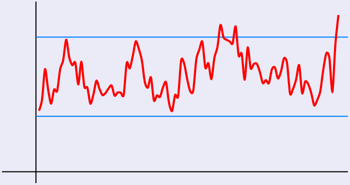


The road ahead

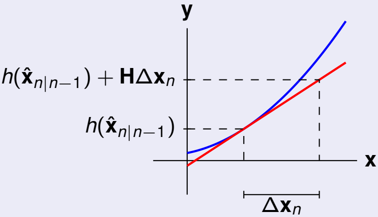
Stochastic processes



Tuning of the Kalman filter



The EKF



Particle filters