TTK4250

Week 8

Inertial navigation: Attitude representations

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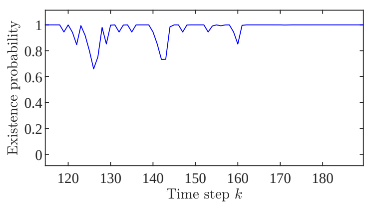
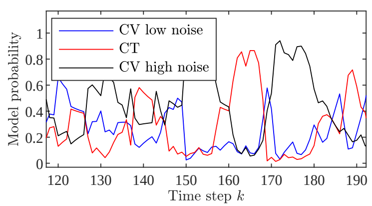
Recap from last week

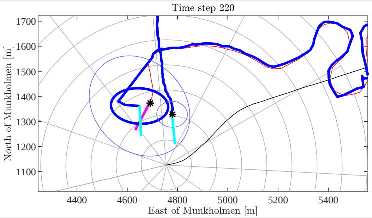
IPDA



IMM-PDAF



JPDA



2D assignment and Murty

We can solve the assignment problem as a virtual auction.

	$t = 1$	$t = 2$
$a^t = 1$	0.07	-18.84
$a^t = 2$	-0.74	5.69
$a^1 = 0$	-0.35	$-\infty$
$a^2 = 0$	$-\infty$	-1.70

Inertial navigation

Previously, we wanted to estimate the motion of objects seen by a sensor.

Now, we want to estimate our own motion.

Inertial and complementary sensors

To solve this problem, we can use both sensors that measure changes in velocity (inertial sensors) and sensors that measure position, velocity and attitude relative to the external world (complementary sensors).

- The inertial sensors include accelerometers and gyros, typically combined in an Inertial Measurement Unit (IMU).
- Estimation of own motion based on IMU measurements is the task of an Inertial Navigation System (INS).
- Fusion with complementary measurements must be part of the INS if drift is to be avoided.

Challenges and characteristics of inertial navigation

- ☺ We have more information available about the motion than in target tracking.
- ☹ To utilize that information, we need a larger state vector (9 states or more).
- 😐 We must choose an attitude representation.
  - Angle-axis
  - Euler angles
  - Rotation matrices
  - Quaternions
- ☹ There can potentially be a long time between every time we get a complementary measurement.
  $\Rightarrow$ We may need precise dead reckoning capabilities.
- ☺ It is fairly obvious how to make a suitable kinematic model for INS.
  But that does not mean that the mathematics is trivial.
- 😐 Filter divergence is not acceptable.
  Insofar as we have sufficient complementary measurements.
  Because the INS may be a direct input to the control system.
- ☺ There is no need for data association (unless we include SLAM as part of the INS).

Standard approaches to inertial navigation

Complementary filter

Add together attitude estimates from different sensors (e.g. accelerometer and gyro) propagated through filters whose transfer functions add to an all-pass transfer function.

Nonlinear observers

Estimate the states so that the estimation error, viewed as a dynamical system, follows a prescribed filter characteristic (e.g., asymptotic convergence towards zero).

The error-state Kalman filter (ESKF, a.k.a. the multiplicative EKF)

- Treat IMU measurements as control inputs.
- Covariance corresponds to a hypothetical error state of smaller dimension than the nominal state maintained by the filter.

Pre-integration (On-manifold pre-integration)

- Fuse together all IMU measurements between two complementary measurements into a single control input.
- Necessary to enable efficient smoothing over long time horizons.

Attitude representations

The space of all possible attitudes is the manifold $\mathrm{SO}(3)$.

Euler angles

+ Intuitive and convenient for displaying and interpreting attitude.
+ Has the same dimension as $\mathrm{SO}(3)$.
- Suffers from singularities at 90° pitch.
- Composition of rotations is difficult.

Rotation matrices

+ No singularities.
+ Rotation matrices are anyway going to be used a lot in a navigation system.
- Has significantly higher dimension (9) than $\mathrm{SO}(3)$.

Quaternions

+ No singularities.
+ Closer to Euler angles than to rotation matrices in terms of dimension (4).
+ Renormalization and interpolation is easier than for rotation matrices.
- The least intuitive representation.

Attitude representations

Angle-axis representation

- + Intuitive and perhaps the most fundamental representation.
- + Same dimension as quaternions (4).
- - Composition of rotations is difficult.
- - Ill defined for zero rotation angle.

Around a unit vector $\mathbf{n}$ by an angle α, a vector $\mathbf{v}$ is rotated into $\mathbf{v}'$ as follows:

$$\mathbf{v}' = (1 - \cos\alpha)(\mathbf{n} \cdot \mathbf{v})\mathbf{n} + \cos\alpha\mathbf{v} + \sin\alpha\,(\mathbf{n} \times \mathbf{v}). \tag{1}$$

From angle-axis to rotation matrix

The rotation matrix that does the rotation operation in (1) is given by

$$\mathrm{R}(\alpha, \mathbf{n}) = (1 - \cos\alpha) \cdot \mathbf{n}\mathbf{n}^{\mathrm{T}} + \cos\alpha \cdot \mathbf{I} + \sin\alpha \cdot \mathrm{S}(\mathbf{n})$$

where $\mathrm{S}(\mathbf{n})$ is the skew-symmetric matrix given by $\mathrm{S}(\mathbf{n})\mathbf{a} = \mathbf{n} \times \mathbf{a}$.

We can also define the rotation vector $\mathbf{k} = \alpha\mathbf{n}$. Then it can be shown that

$$\mathrm{R}(\alpha, \mathbf{n}) = \exp(\mathrm{S}(\mathbf{k})).$$

Euler angles

Which axes do the Euler angles rotate around?

- "Classic Euler angles": $Z \to X \to Z, \;\; X \to Y \to X, \;\; Y \to Z \to Y, \ldots$
- Tait-Bryan-angles: $X \to Y \to Z, \;\; Y \to Z \to X, \;\; Z \to X \to Y, \ldots$

Is the sequence of rotations *intrinsic* or *extrinsic*?

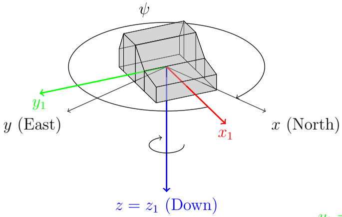
- Intrinsic: During each rotation the axes are also rotated, so that the next rotation are carried out around its corresponding axis as rotated by previous rotations.
- Extrinsic: All the rotations are applied around the original axes.
- Intrinsic sequence $A \to B \to C$ = extrinsic sequence $C \to B \to A$.
- For small angles, the difference between intrinsic and extrinsic tends towards zero.

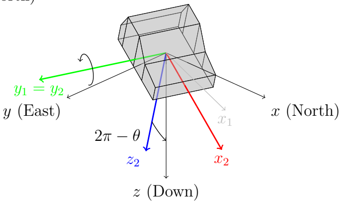
Conventional choice in navigation:

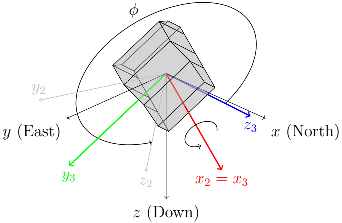
- We use the intrinsic sequence $Z \to Y \to X$.
- Yaw ψ is then unaffected by subsequent changes in pitch θ and roll ϕ.
- The corresponding extrinsic sequence is $X \to Y \to Z$ and we get

$$\mathrm{R}(\phi, \theta, \psi) \triangleq \mathrm{R}_{\boldsymbol{e}_z, \psi} \mathrm{R}_{\boldsymbol{e}_y, \theta} \mathrm{R}_{\boldsymbol{e}_x, \phi}.$$

Euler angles







Intrinsic sequence of rotations defining the Euler angles following the NED convention.

Conventions for rotation matrices / quaternions

We also encounter different conventions with regard to the interpretation of rotations in general.

- Is the rotation active or passive? Does it rotate an object or does it rotate the coordinate system in which the object is described?
- Are (passive) rotations local-to-global or global-to-local?

The passive local-to-global convention.

The vector $\mathbf{v}^b$ is decomposed in the body frame of a vehicle whose attitude we want to represent. The same vector has the representation $\mathbf{v}^w$ in the (body-fixed) global coordinate system. **The rotation matrix of the attitude then tells us how we must rotate $\mathbf{v}^b$ to make it coincide with $\mathbf{v}^w$:**

$$\mathbf{v}^w = \mathrm{R}(\phi, \theta, \psi)\mathbf{v}^b = \mathrm{R}_b^w \mathbf{v}^b$$

This is the same as active rotation of the vehicle, as illustrated on the previous slide.

Quaternions

Following Hamilton's convention[a] I write the quaternion as

$$\boldsymbol{q} = \begin{bmatrix} \eta \\ \boldsymbol{\epsilon} \end{bmatrix}$$

where η is the scalar part and $\boldsymbol{\epsilon}$ is the vector part. The concept of quaternions can be defined by the following properties:

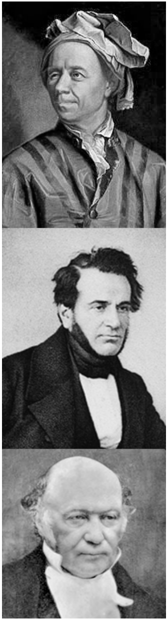
- The sum of two quaternions is straightforward.
- The product of two quaternions is given by

$$\boldsymbol{q}_a \otimes \boldsymbol{q}_b = \begin{bmatrix} \eta_a \eta_b - \boldsymbol{\epsilon}_a^\top \boldsymbol{\epsilon}_b \\ \eta_b \boldsymbol{\epsilon}_a + \eta_a \boldsymbol{\epsilon}_b + \boldsymbol{\epsilon}_a \times \boldsymbol{\epsilon}_b \end{bmatrix}$$

  Quaternion multiplication is associative and distributive, but not commutative.

Rotations are represented by the set of unit quaternions satisfying

$$\|\boldsymbol{q}\| = \sqrt{\eta^2 + \epsilon_1^2 + \epsilon_2^2 + \epsilon_3^2} = 1.$$



[a] The same convention is used in Egeland & Gravdahl (2002). This differs from the JPL convention, which is used by many American authors (Markley, Roumeliotis).

Quaternions

Matrix formulations of the quaternion product

It is sometimes convenient to write the quaternion product as a 4×4 matrix given by one of the quaternions, multiplied with the other quaternion:

$$\boldsymbol{q}_a \otimes \boldsymbol{q}_b = \left(\eta_a \mathrm{I} + \begin{bmatrix} 0 & -\boldsymbol{\epsilon}_a^\mathsf{T} \\ \boldsymbol{\epsilon}_a & \mathrm{S}(\boldsymbol{\epsilon}_a) \end{bmatrix} \right) \begin{bmatrix} \eta_b \\ \boldsymbol{\epsilon}_b \end{bmatrix}$$

$$\boldsymbol{q}_a \otimes \boldsymbol{q}_b = \left(\eta_b \mathrm{I} + \begin{bmatrix} 0 & -\boldsymbol{\epsilon}_b^\mathsf{T} \\ \boldsymbol{\epsilon}_b & -\mathrm{S}(\boldsymbol{\epsilon}_b) \end{bmatrix} \right) \begin{bmatrix} \eta_a \\ \boldsymbol{\epsilon}_a \end{bmatrix}$$

Identity quaternion

The identity quaternion is $\boldsymbol{q}_\mathrm{I} = \begin{bmatrix} 1 \\ \mathbf{0} \end{bmatrix}$ and satisfies $\boldsymbol{q} \otimes \boldsymbol{q}_\mathrm{I} = \boldsymbol{q}_\mathrm{I} \otimes \boldsymbol{q} = \boldsymbol{q}$ for any $\boldsymbol{q}$.

Quaternions

Conjugate and inverse of a quaternion

- The conjugate of $\boldsymbol{q}$ is $\boldsymbol{q}^* = \begin{bmatrix} \eta \\ -\boldsymbol{\epsilon} \end{bmatrix}$.
- The inverse of $\boldsymbol{q}$ is given by $\boldsymbol{q} \otimes \boldsymbol{q}^{-1} = \boldsymbol{q}^{-1} \otimes \boldsymbol{q} = \boldsymbol{q}_\mathrm{I}$ and can be found according to $\boldsymbol{q}^{-1} = \boldsymbol{q}^* / \|\boldsymbol{q}\|^2$. For unit quaternions we have $\boldsymbol{q}^{-1} = \boldsymbol{q}^*$.

Axis-angle $\longleftrightarrow$ Unit quaternions

Let the unit quaternion be given by

$$\boldsymbol{q} = \begin{bmatrix} \eta \\ \boldsymbol{\epsilon} \end{bmatrix} = \begin{bmatrix} \cos\frac{\alpha}{2} \\ \mathbf{n}\sin\frac{\alpha}{2} \end{bmatrix}$$

and assume that $\|\mathbf{n}\| = 1$. Let the vector $\mathbf{v}'$ be related to the vector $\mathbf{v}$ according to $\mathbf{v}' = \mathrm{R}_{\mathbf{n},\alpha}\mathbf{v}$. In terms of quaternions, the same rotation is given by

$$\begin{bmatrix} 0 \\ \mathbf{v}' \end{bmatrix} = \boldsymbol{q} \otimes \begin{bmatrix} 0 \\ \mathbf{v} \end{bmatrix} \otimes \boldsymbol{q}^*$$

Quaternion conversions

From quaternions to Euler angles.

$$
\begin{aligned}
\phi &= \operatorname{atan2}\left(2(\epsilon_3\epsilon_2 + \eta\epsilon_1), \eta^2 - \epsilon_1^2 - \epsilon_2^2 + \epsilon_3^2\right) \\
\theta &= \operatorname{asin}\left(2(\eta\epsilon_2 - \epsilon_1\epsilon_3)\right) \\
\psi &= \operatorname{atan2}\left(2(\epsilon_1\epsilon_2 + \eta\epsilon_3), \eta^2 + \epsilon_1^2 - \epsilon_2^2 - \epsilon_3^2\right)
\end{aligned}
$$

- This conversion is useful to display results and check whether the quaternions behave sensibly.
- Notice the usage of $\operatorname{atan2}$ to avoid singularities.

From Euler angles to quaternions.

$$
\boldsymbol{q} = \begin{bmatrix}
\cos\frac{\phi}{2}\cos\frac{\theta}{2}\cos\frac{\psi}{2} + \sin\frac{\phi}{2}\sin\frac{\theta}{2}\sin\frac{\psi}{2} \\
\sin\frac{\phi}{2}\cos\frac{\theta}{2}\cos\frac{\psi}{2} - \cos\frac{\phi}{2}\sin\frac{\theta}{2}\sin\frac{\psi}{2} \\
\cos\frac{\phi}{2}\sin\frac{\theta}{2}\cos\frac{\psi}{2} + \sin\frac{\phi}{2}\cos\frac{\theta}{2}\sin\frac{\psi}{2} \\
\cos\frac{\phi}{2}\cos\frac{\theta}{2}\sin\frac{\psi}{2} - \sin\frac{\phi}{2}\sin\frac{\theta}{2}\cos\frac{\psi}{2}
\end{bmatrix}
$$

Quaternions, rotation matrices and rotation vectors

From quaternion to rotation matrix.

$$R = \mathbf{I} + 2\eta S(\epsilon) + 2S(\epsilon)S(\epsilon)$$

Proof:

If $\mathbf{n}$ is a unit vector then $S(\mathbf{n})S(\mathbf{n}) = \mathbf{n}\mathbf{n}^{\mathsf{T}} - \mathbf{I}$. Then axis-angle-to-rotation-matrix formula becomes

$$\begin{aligned} R_{\mathbf{n},\alpha} &= \cos\alpha\mathbf{I} + (1-\cos\alpha)\mathbf{n}\mathbf{n}^{\mathsf{T}} + \sin\alpha S(\mathbf{n}) \\ &= \mathbf{I} + (1-\cos\alpha)S(\mathbf{n})S(\mathbf{n}) + \sin\alpha S(\mathbf{n}). \end{aligned}$$

We use that $1-\cos\alpha = 2\sin^2\frac{\alpha}{2}$ and $\sin\alpha = \sin\frac{\alpha}{2}\cos\frac{\alpha}{2}$ and furthermore that

$$\eta = \cos\frac{\alpha}{2}, \quad \epsilon = \mathbf{n}\sin\frac{\alpha}{2}.$$

From rotation matrix to axis-angle.

Several formulas exists.[a][b]

[a] Forster (2017) “On-Manifold Preintegration for Real-Time Visual-Inertial Odometry”, IEEE-TR.
[b] Visser (2006): “Cayley-Hamilton for roboticists”, Proc. IROS.

Perturbations and angular velocity

Perturbations.

The ESKF is based on the concept of perturbations. A perturbed attitude $\tilde{\boldsymbol{q}}$ can be expressed as

$$\tilde{\boldsymbol{q}} = \boldsymbol{q} \otimes \Delta \boldsymbol{q}_L$$

The perturbation $\Delta \boldsymbol{q}_L$ can alternatively be described by means angle-axis representation $\Delta \boldsymbol{\theta}_L$. If we recall the axis-angle-to-quaternion formula, and take the limit for $\alpha \rightarrow 0$, then we get the following expression

$$\Delta \boldsymbol{q}_L = \begin{bmatrix} 1 \\ \frac{1}{2}\Delta \boldsymbol{\theta}_L \end{bmatrix} + \mathrm{HOT}(\Delta \boldsymbol{\theta}_L) \tag{2}$$

Letting the perturbation be a time-dependent function, we define the angular velocity (decomposed in local frame) as

$$\boldsymbol{\omega}_L(t) = \frac{\mathrm{d}\Delta \boldsymbol{\theta}_L}{\mathrm{d}t}$$

Notice that angular velocity is based most directly on the angle-axis representation.

Time derivative of attitude

We need a kinematic model that connects angular velocity $\boldsymbol{\omega}$ with the evolution of the attitude.

Rotation matrix formulation

Let ω be the angular velocity decomposed in body frame, and let R be the passive body-to-world rotation matrix. Then

$$\dot{\mathrm{R}} = \mathrm{R}\mathrm{S}(\boldsymbol{\omega})$$

Quaternion formulation

Let ω be the angular velocity decomposed in body frame, and let $\boldsymbol{q}$ be the passive body-to-world unit quaternion. Then

$$\dot{\boldsymbol{q}} = \frac{1}{2}\boldsymbol{q} \otimes \begin{bmatrix} 0 \\ \boldsymbol{\omega} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -\boldsymbol{\epsilon}^{\mathsf{T}} \\ \eta\mathbf{I} + \mathrm{S}(\boldsymbol{\epsilon}) \end{bmatrix}\boldsymbol{\omega}$$