TTK4250

Week 7

Advanced topics in target tracking

Edmund Førland Brekke

3. October 2024

Recap from earlier

The IMM method

- Hybrid state vector $\mathbf{y}_k$ consists of mode s_k and kinematic state $\mathbf{x}_k$.
- Approximate $p(\mathbf{x}_{k-1}|s_k, \mathbf{z}_{1_k-1})$ as a single Gaussian.

Single target tracking

- The target may or may not be detected.
- A measurement may or may not be from the target.

Probabilistic data association filter

- $\Pr\{a_k = j\}$ is the probability that $\mathbf{z}_k^j$ comes from the target.

$$\beta_k^j \propto \begin{cases} \lambda \dfrac{1 - P_{\mathrm{D}}}{P_{\mathrm{D}}} & \text{if} \quad i = 0 \\ l^j & \text{if} \quad i > 0 \end{cases}$$

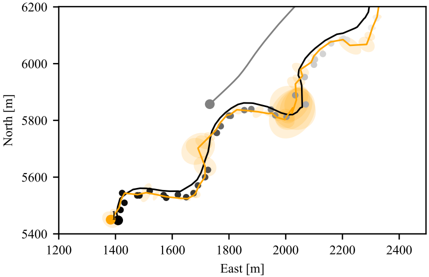
- Obtain $p(\mathbf{x}_k|Z_{1:k})$ by mixture reduction over all values of a_k.

Point processes

- Clutter is modeled as a Poisson point process with intensity function $\lambda(\mathbf{z})$.
- The expected number of clutter measurements in a region A is $\int_A \lambda(\mathbf{z}) \mathrm{d}\mathbf{z}$.

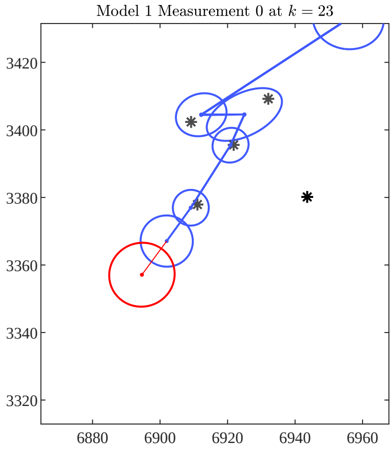
Outline

Motivation for IMM-PDAF: The Joyride scenario



- Experiment that was conducted in Trondheimsfjorden in May 2017.
- A Simrad 4G radar is placed on top of the Maritime Robotics sport vessel "Telemetron".
- A motorboat is making all sorts of crazy maneuvers that it can manage.
- Limited detection probability, occasional false alarms and the large sampling time (2.8 s) together with the maneuvers make this a challenging tracking problem.
- Go-pro cameras provided visual documentation, but the video is not suitable for sensor fusion due to lack of synchronization.

The combined challenge of target maneuvers and clutter



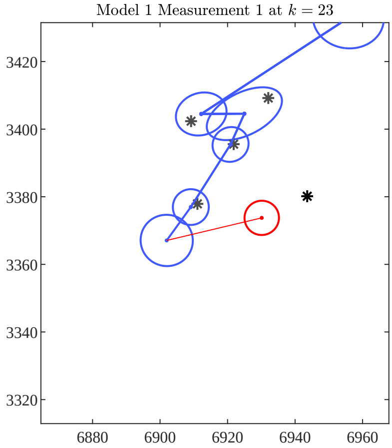
	Prior	$a_k = 0$	$a_k = 1$
$s_k = 1$	0.748	**0.380**	0.005
$s_k = 2$	0.138	0.070	0.005
$s_k = 3$	0.113	0.058	0.482

Explanation for the models:

$s_k = 1$	CV model with low process noise
$s_k = 2$	CT model
$s_k = 3$	CV model with high process noise

We have to discuss the relationships between $\Pr\{s_k, a_k \mid Z_{1:k}\}$, $\Pr\{s_k \mid a_k, Z_{1:k}\}$, $\Pr\{a_k \mid s_k, Z_{1:k}\}$, $\Pr\{a_k \mid Z_{1:k}\}$ and $\Pr\{s_k \mid Z_{1:k}\}$.

The combined challenge of target maneuvers and clutter



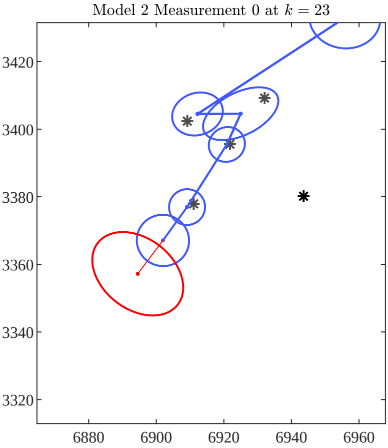
	Prior	$a_k = 0$	$a_k = 1$
$s_k = 1$	0.748	0.380	**0.005**
$s_k = 2$	0.138	0.070	0.005
$s_k = 3$	0.113	0.058	0.482

Explanation for the models:

$s_k = 1$	CV model with low process noise
$s_k = 2$	CT model
$s_k = 3$	CV model with high process noise

We have to discuss the relationships between $\Pr\{s_k, a_k \mid Z_{1:k}\}$, $\Pr\{s_k \mid a_k, Z_{1:k}\}$, $\Pr\{a_k \mid s_k, Z_{1:k}\}$, $\Pr\{a_k \mid Z_{1:k}\}$ and $\Pr\{s_k \mid Z_{1:k}\}$.

The combined challenge of target maneuvers and clutter



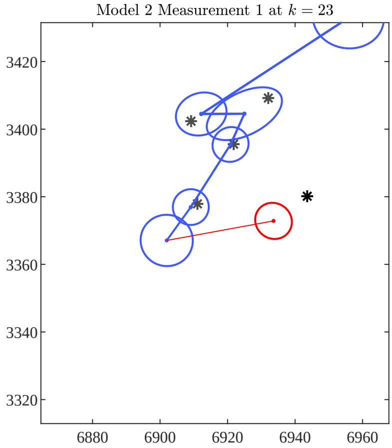
	Prior	$a_k = 0$	$a_k = 1$
$s_k = 1$	0.748	0.380	0.005
$s_k = 2$	0.138	**0.070**	0.005
$s_k = 3$	0.113	0.058	0.482

Explanation for the models:

$s_k = 1$	CV model with low process noise
$s_k = 2$	CT model
$s_k = 3$	CV model with high process noise

We have to discuss the relationships between $\Pr\{s_k, a_k \mid Z_{1:k}\}$, $\Pr\{s_k \mid a_k, Z_{1:k}\}$, $\Pr\{a_k \mid s_k, Z_{1:k}\}$, $\Pr\{a_k \mid Z_{1:k}\}$ and $\Pr\{s_k \mid Z_{1:k}\}$.

The combined challenge of target maneuvers and clutter



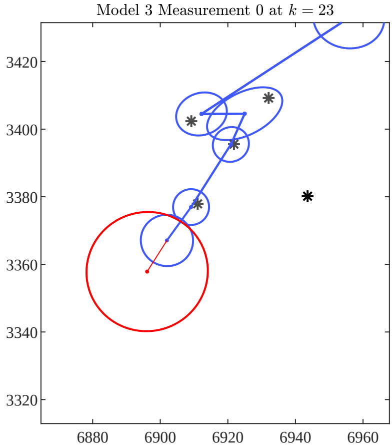
	Prior	$a_k = 0$	$a_k = 1$
$s_k = 1$	0.748	0.380	0.005
$s_k = 2$	0.138	0.070	**0.005**
$s_k = 3$	0.113	0.058	0.482

Explanation for the models:

$s_k = 1$	CV model with low process noise
$s_k = 2$	CT model
$s_k = 3$	CV model with high process noise

We have to discuss the relationships between $\Pr\{s_k, a_k \mid Z_{1:k}\}$, $\Pr\{s_k \mid a_k, Z_{1:k}\}$, $\Pr\{a_k \mid s_k, Z_{1:k}\}$, $\Pr\{a_k \mid Z_{1:k}\}$ and $\Pr\{s_k \mid Z_{1:k}\}$.

The combined challenge of target maneuvers and clutter



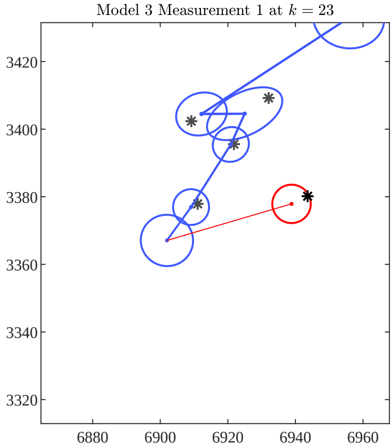
	Prior	$a_k = 0$	$a_k = 1$
$s_k = 1$	0.748	0.380	0.005
$s_k = 2$	0.138	0.070	0.005
$s_k = 3$	0.113	**0.058**	0.482

Explanation for the models:

- $s_k = 1$ CV model with low process noise
- $s_k = 2$ CT model
- $s_k = 3$ CV model with high process noise

We have to discuss the relationships between $\Pr\{s_k, a_k \mid Z_{1:k}\}$, $\Pr\{s_k \mid a_k, Z_{1:k}\}$, $\Pr\{a_k \mid s_k, Z_{1:k}\}$, $\Pr\{a_k \mid Z_{1:k}\}$ and $\Pr\{s_k \mid Z_{1:k}\}$.

The combined challenge of target maneuvers and clutter



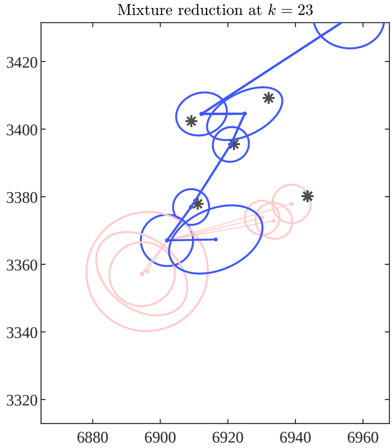
	Prior	$a_k = 0$	$a_k = 1$
$s_k = 1$	0.748	0.380	0.005
$s_k = 2$	0.138	0.070	0.005
$s_k = 3$	0.113	0.058	**0.482**

Explanation for the models:

$s_k = 1$	CV model with low process noise
$s_k = 2$	CT model
$s_k = 3$	CV model with high process noise

We have to discuss the relationships between $\Pr\{s_k, a_k \mid Z_{1:k}\}$, $\Pr\{s_k \mid a_k, Z_{1:k}\}$, $\Pr\{a_k \mid s_k, Z_{1:k}\}$, $\Pr\{a_k \mid Z_{1:k}\}$ and $\Pr\{s_k \mid Z_{1:k}\}$.

The combined challenge of target maneuvers and clutter



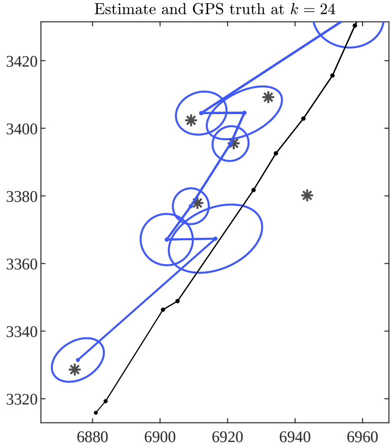
	Prior	$a_k = 0$	$a_k = 1$
$s_k = 1$	0.748	0.380	0.005
$s_k = 2$	0.138	0.070	0.005
$s_k = 3$	0.113	0.058	0.482

Explanation for the models:

$s_k = 1$	CV model with low process noise
$s_k = 2$	CT model
$s_k = 3$	CV model with high process noise

We have to discuss the relationships between $\Pr\{s_k, a_k \mid Z_{1:k}\}$, $\Pr\{s_k \mid a_k, Z_{1:k}\}$, $\Pr\{a_k \mid s_k, Z_{1:k}\}$, $\Pr\{a_k \mid Z_{1:k}\}$ and $\Pr\{s_k \mid Z_{1:k}\}$.

The combined challenge of target maneuvers and clutter



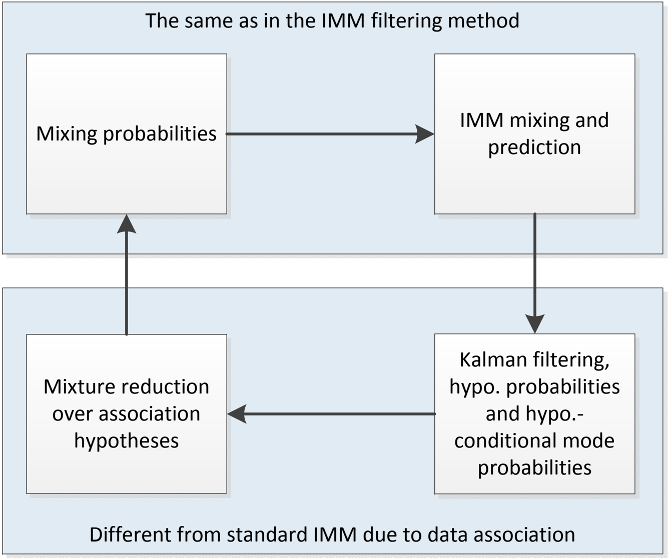
	Prior	$a_k = 0$	$a_k = 1$
$s_k = 1$	0.748	0.380	0.005
$s_k = 2$	0.138	0.070	0.005
$s_k = 3$	0.113	0.058	0.482

Explanation for the models:

$s_k = 1$	CV model with low process noise
$s_k = 2$	CT model
$s_k = 3$	CV model with high process noise

We have to discuss the relationships between $\Pr\{s_k, a_k \mid Z_{1:k}\}$, $\Pr\{s_k \mid a_k, Z_{1:k}\}$, $\Pr\{a_k \mid s_k, Z_{1:k}\}$, $\Pr\{a_k \mid Z_{1:k}\}$ and $\Pr\{s_k \mid Z_{1:k}\}$.

Workflow of an IMM-PDAF



Outline

Track management

The tracking system must itself make a decision whether a sequence of measurements are likely to originate from a new target, and if so establish a track on that target.

The M/N logic

- We establish a **preliminary track** whenever we have received sufficiently nearby detections in two consecutive scans.
- After we have received detections inside its validation gate $M \leq N$ times we declare it a **confirmed track**.
- The preliminary track is terminated if it has received detections inside its gate in less than M of the first N scans.

We can also use M/N logic (with different values of M and N) to terminate confirmed tracks which have not received enough measurements over the last N scans.

The IPDA

We want an alternative to the M/N logic that is more systematic.

Key idea of the IPDA

- We calculate the probability that a target exists, and use this existence probability as a score function for track confirmation or termination.
- To do this we need to introduce a model for target existence, and link it to the PDAF framework.
  - How does existence evolve in time?
  - What can we say on existence based on the measurements?

Markov chain model for target existence

- $r_k =$ the probability that the target exists at time k.
- $P_{\mathrm{S}} =$ the probability that an existing target survives to the next time step.

In the absence of measurements the target existence probability is the supposed to follow this Markov chain:

$$\begin{bmatrix} r_k \\ 1 - r_k \end{bmatrix} = \begin{bmatrix} P_{\mathrm{S}} & 0 \\ 1 - P_{\mathrm{S}} & 1 \end{bmatrix} \begin{bmatrix} r_{k-1} \\ 1 - r_{k-1} \end{bmatrix} = \begin{bmatrix} P_{\mathrm{S}} r_{k-1} \\ 1 - P_{\mathrm{S}} r_{k-1} \end{bmatrix}.$$

Existence and association events in the IPDA

Existence events

$$
\begin{aligned}
E &: \quad \text{The target exists.} \\
D &: \quad \text{The target is detected.}
\end{aligned}
$$

Association events

These are as before $a_k = 0, a_k = 1, \ldots, a_k = m_k$.

Combined events

- If $i > 0$ then $a_k = i$ implies existence E.
- $a_k = 0$ implies $\neg D$, but does not yield any implications for E.

We must therefore distinguish the association probabilities from the event probabilities according to

$$\beta_k^{a_k} = P\{a_k \,|\, E, Z_{1:k}\} = \frac{P\{a_k, E \,|\, Z_{1:k}\}}{P\{E \,|\, Z_{1:k}\}}.$$

Existence and association events in the IPDA

Association probabilities for the IPDA

- The association probabilities $\Pr\{a_k \mid E, Z_{1:k}\}$ for the IPDA are exactly the same as for the PDAF.
- The probabilities $\Pr\{a_k \mid Z_{1:k}\}$ are the same if we replace P_{D} with $P_{\mathrm{D}} r$.

Existence update for the IPDA

Assume Poisson clutter, Gaussian-linear kinematics, etc. The posterior existence probability in the IPDA is then

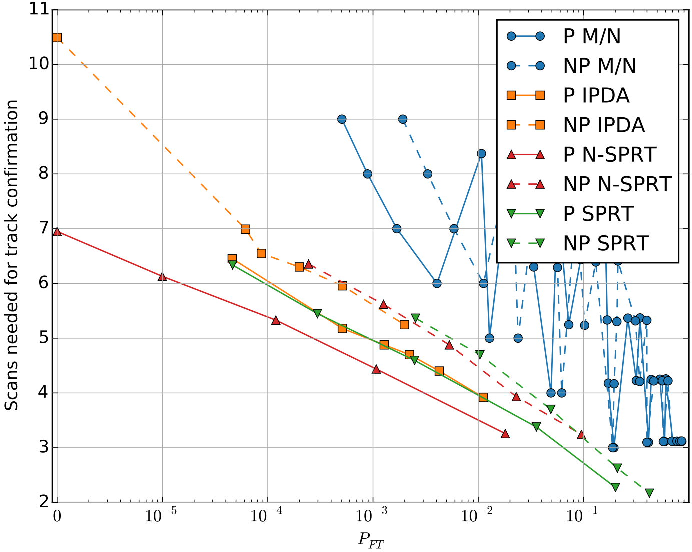
$$r_k = \frac{\mathcal{L}_k}{1 - (1 - \mathcal{L}_k) r_{k|k-1}} r_{k|k-1}$$

where

$$\mathcal{L}_k = 1 - P_{\mathrm{D}} + \frac{P_{\mathrm{D}}}{\lambda} \sum_{a_k=1}^{m_k} l^{a_k} = 1 - P_{\mathrm{D}} + \frac{P_{\mathrm{D}}}{\lambda} \sum_{a_k=1}^{m_k} \mathcal{N}(\mathbf{z}_k^{a_k}\,;\, \hat{\mathbf{z}}_{k|k-1}, \mathbf{S}_k).$$

and $r_{k|k-1}$ is the predicted existence probability.

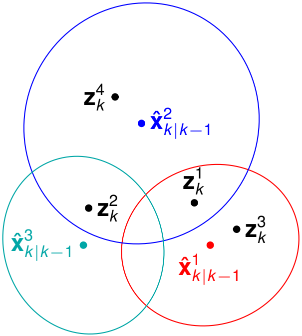
Initialization times for different techniques



Wilthil et al.: “Track Initiation for Maritime Radar Tracking with and without Prior Information”, Proc. Fusion, 2018.

Outline

The multi-target tracking problem



Given a scan of sensor detections $\mathbf{z}_k^1, \mathbf{z}_k^2, \ldots ,, \mathbf{z}_k^{m_k}$ and given a list of existing tracks with predictions

$$\hat{\mathbf{x}}_{k|k-1}^1, \hat{\mathbf{x}}_{k|k-1}^2, \ldots , \hat{\mathbf{x}}_{k|k-1}^n$$

we define the multi-target association hypotheses as

$$a_k = \begin{bmatrix} a_k^1 & a_k^2 & \ldots & a_k^n \end{bmatrix}$$

where

$$a_k^t = \begin{cases} j & \text{if measurement } j \text{ is claimed by target } t \\ 0 & \text{if no measurement is claimed by target } t. \end{cases}$$

- Possible data association hypotheses a_k for the above scenario could be $[1, 2, 0]$, $[3, 1, 2]$, $[0, 4, 2]$, etc.
- We want to evaluate the probabilities $\Pr\{a_k \mid Z_{1:k}\}$ and the association-conditional posterior pdfs $p(\mathbf{x}_k^1, \ldots , \mathbf{x}_k^n \mid a_k, Z_{1:k})$.

In this course, we restrict ourselves to the problem where n and only n targets are known to exist.

The at most-one-assumptions

Most multi-target tracking methods build upon the following two fundamental assumptions.

At most one measurement comes from the target

- Also known as the point target assumption.
- It rules out so-called extended targets, which generate several measurements.
- Makes modeling much easier that for extended targets: No need to specify models for extent and for how many measurement the target is likely to generate.

At most one target is the source of any measurement

- Also known as no merged measurement assumption.
- Automatically satisfied if the measurements of all targets are Gaussian.
- Helps make the tracking problem well-posed. Without it, perhaps a single measurement was generated by a hundred targets together?

Since both assumptions are physically dubious, preprocessing techniques (e.g., detection and blob extraction) should be designed in order to make these assumptions as valid as possible.

The standard model for multi-target tracking

General assumptions:

M1 New targets are born according to a Poisson process with intensity $\mu(\mathbf{x})$.
M2 Existing targets survive from time step $k-1$ to k with probability $P_{\mathrm{S}}(\mathbf{x}_{k-1})$.
M3 The motion of a surviving target is given by $f_{\mathbf{x}}(\mathbf{x}_k|\mathbf{x}_{k-1})$.
M4 A target with state $\mathbf{x}_k$ generates a measurement $\mathbf{z}_k$ with probability $P_{\mathrm{D}}(\mathbf{x}_k)$.
M5 Clutter measurements occur according to a Poisson process with intensity $\lambda(\mathbf{z})$.
M6 The measurement of a detected target is related to the state according to $f_{\mathbf{z}}(\mathbf{z}_k|\mathbf{x}_k)$.

The standard model for multi-target tracking

General assumptions:

- M1 New targets are born according to a Poisson process with intensity $\mu(\mathbf{x})$.
- M2 Existing targets survive from time step $k-1$ to k with probability $P_{\mathrm{S}}(\mathbf{x}_{k-1})$.
- M3 The motion of a surviving target is given by $f_{\mathbf{x}}(\mathbf{x}_k|\mathbf{x}_{k-1})$.
- M4 A target with state $\mathbf{x}_k$ generates a measurement $\mathbf{z}_k$ with probability $P_{\mathrm{D}}(\mathbf{x}_k)$.
- M5 Clutter measurements occur according to a Poisson process with intensity $\lambda(\mathbf{z})$.
- M6 The measurement of a detected target is related to the state according to $f_{\mathbf{z}}(\mathbf{z}_k|\mathbf{x}_k)$.

Additional assumtions for the **Joint Probabilistic Data Association (JPDA):**

- M7 The number n of targets is constant and known.
- M8 The single target motion model and the likelihood are Gaussian-linear:

$$f_{\mathbf{x}}(\mathbf{x}_k \mid \mathbf{x}_{k-1}) = \mathcal{N}(\mathbf{x}_k\,;\,\mathbf{F}\mathbf{x}_{k-1}, Q)$$
$$f_{\mathbf{z}}(\mathbf{z}_k\,;\,\mathbf{x}_k) = \mathcal{N}(\mathbf{z}_k \mid \mathbf{H}\mathbf{x}_k, \mathbf{R})$$

- M9 At the end of the previous estimation cycle, the posterior densities of the targets are independent and Gaussian

$$p^t_{k-1}(\mathbf{x}^t_{k-1}) = \mathcal{N}(\mathbf{x}^t_{k-1}\,;\,\hat{\mathbf{x}}^t_{k-1}, \mathbf{P}^t_{k-1}).$$

- M10 Target t has constant detection probability P^t_{D}.
- M11 The clutter Poisson process has constant intensity λ.

Mixtures in the JPDA

Based on the assumptions, the predicted density for all the targets is

$$p(\mathbf{x}_k^1, \ldots, \mathbf{x}_k^n \mid Z_{1:k-1}) = \prod_t p_{k|k-1}^t(\mathbf{x}_k^t) = \prod_t \mathcal{N}(\mathbf{x}_k^t \,;\, \hat{\mathbf{x}}_{k|k-1}^t, \mathbf{P}_{k|k-1}^t)$$

where $\hat{\mathbf{x}}_{k|k-1}^t$ and $\mathbf{P}_{k|k-1}^t$ are found by KF prediction starting from $\hat{\mathbf{x}}_{k-1}^t$ and $\mathbf{P}_{k-1}^t$.

Since the association hypotheses are mutual exhaustive and exclusive, the total probability theorem yields

$$p(\mathbf{x}_k^1, \ldots, \mathbf{x}_k^n \mid Z_{1:k}) = \sum_{a_k} p(\mathbf{x}_k^1, \ldots, \mathbf{x}_k^n \mid a_k, Z_{1:k}) \Pr\{a_k \mid Z_{1:k}\}$$

The hypothesis-conditional pdfs are products of Gaussians of the form

$$p(\mathbf{x}_k^1, \ldots, \mathbf{x}_k^n \mid a_k, Z_{1:k}) = \prod_{t=1}^{n} \mathcal{N}(\mathbf{x}_k^t \,;\, \hat{\mathbf{x}}_k^{t,a_k^t}, \mathbf{P}_k^{t,a_k^t}).$$

Hypothesis-conditional posteriors

The statistics $\hat{\mathbf{x}}_k^{t,a_k^t}$ and $\mathbf{P}_k^{t,a_k^t}$ are found by means of the usual KF formulas

$$
\begin{aligned}
\hat{\mathbf{x}}_k^{t,a_k^t} &= \begin{cases} \hat{\mathbf{x}}_{k|k-1}^t & \text{if } a_k^t = 0 \\ \hat{\mathbf{x}}_{k|k-1}^t + \mathbf{W}_k^t(\mathbf{z}_k^{a_k^t} - \mathbf{H}\hat{\mathbf{x}}_{k|k-1}^t) & \text{if } a_k^t > 0 \end{cases} \\
\mathbf{P}_k^{t,a_k^t} &= \begin{cases} \mathbf{P}_{k|k-1}^t & \text{if } a_k^t = 0 \\ (\mathbf{I} - \mathbf{W}_k^t\mathbf{H})\mathbf{P}_{k|k-1}^t & \text{if } a_k^t > 0. \end{cases} \\
\hat{\mathbf{x}}_{k|k-1}^t &= \mathbf{F}\hat{\mathbf{x}}_{k-1}^t \\
\mathbf{P}_{k|k-1}^t &= \mathbf{F}\mathbf{P}_{k-1}^t\mathbf{F}^\mathsf{T} + \mathbf{Q}
\end{aligned}
$$

Notice that the evaluation of $\hat{\mathbf{x}}_k^{t,a_k^t}$ and $\mathbf{P}_k^{t,a_k^t}$ only has to be done for each feasible target-measurement assignment (t, j). It does not have to be done for every target t and association hypothesis a_k, which in general would involve a much larger number of evaluations.

Hypothesis probabilities in the JPDA

The JPDA formula for hypothesis probabilities

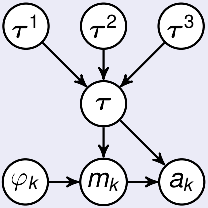
The probability of an association hypothesis a_k is given by

$$\Pr\{a_k | Z_{1:k}\} \propto \lambda^{\varphi_k} \prod_{t\,:\,a_k^t=0} (1 - P_{\mathrm{D}}^t) \prod_{t\,:\,a_k^t>0} P_{\mathrm{D}}^t \, l^{t,a_k^t}$$

where

$$l^{t,a_k^t} = \int f_{\mathbf{z}}(\mathbf{z}_k^{a_k^t} \,|\, \mathbf{x}_k^t) p_{k|k-1}^t(\mathbf{x}_k^t)\, \mathrm{d}\mathbf{x}_k^t = \mathcal{N}(\mathbf{z}_k^{a_k^t}\, ; \, \mathbf{H}\hat{\mathbf{x}}_{k|k-1}^{t,a_k^t}, \mathbf{H}\mathbf{P}_{k|k-1}^{t,a_k^t}\mathbf{H}^{\mathsf{T}} + \mathbf{R}).$$

Outline of proof



- $\Pr\{a_k \,|\, Z_{1:k}\} \propto p(Z_k \,|\, m_k, a_k, Z_{1:k-1})\Pr\{a_k \,|\, m_k, Z_{1:k-1}\}$
- Let $\boldsymbol{\tau} = [\tau^1, \ldots, \tau^n]$ be a target detection indicator.
- $\Pr\{a_k \,|\, m_k, Z_{1:k-1}\} \propto \Pr\{a_k \,|\, \boldsymbol{\tau}, m_k\}\Pr\{m_k \,|\, \boldsymbol{\tau}\}\Pr\{\boldsymbol{\tau}\}$

$$\Pr\{m_k \,|\, \boldsymbol{\tau}\} = e^{-V\lambda}\frac{(V\lambda)^{\varphi_k}}{\varphi_k!}, \quad \Pr\{a_k \,|\, \boldsymbol{\tau}, m_k\} = \frac{\varphi_k!}{m_k!}$$

Marginal association probabilities in the JPDA

What we want is not primarily the probabilities $\Pr\{a_k \mid Z_{1:k}\}$, but rather the probabilities

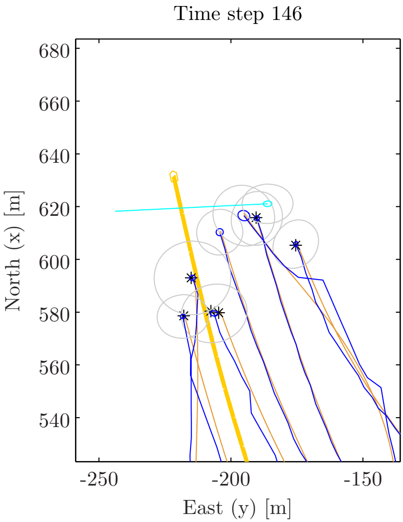
$$\beta_k^{t,j} = \Pr\{\mathbf{z}_k^j \text{ is assigned to target } t \mid Z_{1:k}\} = \sum_{a_k \text{ s.t. } a_k^t = j} \Pr\{a_k \mid Z_{1:k}\}.$$

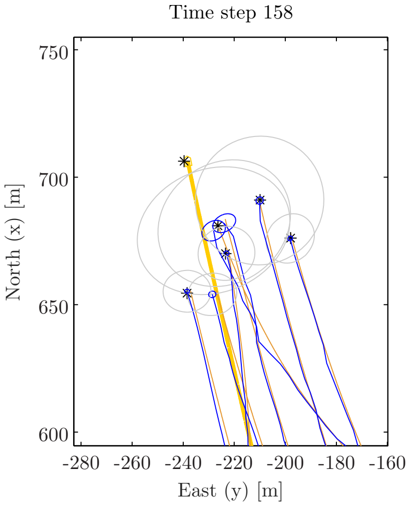
For each track we can find its posterior single-Gaussian approximation according to

$$p_k^t(\mathbf{x}_k^t) = \sum_{j=0}^{m_k} \beta_k^{t,j} \mathcal{N}(\mathbf{x}_k^t\,;\,\hat{\mathbf{x}}_k^{t,j}, \mathbf{P}_k^{t,j}) \approx \mathcal{N}(\mathbf{x}_k^t\,;\,\hat{\mathbf{x}}_k^t, \mathbf{P}_k^t)$$

where the merged expectation $\hat{\mathbf{x}}_k^t$ and covariance $\mathbf{P}_k^t$ are found in exactly the same manner as we did for the PDAF.

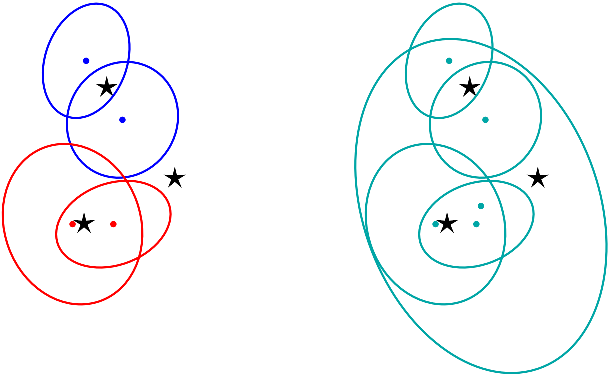
Illustration of MTT in action





Track clustering

- To mitigate complexity we only use the multi-target machinery for clusters of tracks that share measurements.
- If a track does not share measurements with other tracks we use a single-target tracker.



Accelerated implementation of the JPDA

In principle it is straightforward to evaluate the sum required to find $\beta_k^{t,j}$, but since its complexity is exponential, approximations or smart strategies may be needed:

1. Only generate the most probable a_k's. "Reformulating Reid's MHT method with generalised Murty K-best ranked linear assignment algorithm", IEE Proceedings.
   The easiest to implement and it works well for typical scenarios. The hypothesis search can be done by means of **Murty's method**, which again will call a method for solving the optimal assignment problem, such as the **Auction method**.
2. Exploit the fact that a lot of structure is repeated among the different a_k's.[1]
   This kind of solutions can be developed with basis in **junction tree** or **Bayes tree** techniques. Whether this can be done for MHT as well is an open research topic.
3. Loopy belief propagation.[2]
   This is an example of **variational inference**. Tends to give best performance for runtime in dense environments, but suboptimal in de-facto single-target scenarios.[3]

---

[1] Horridge & Maskell (2006): "Real-Time Tracking Of Hundreds Of Targets With Efficient Exact JPDAF Implementation", Proc. ICIF.

[2] Williams & Lau (2014): "Approximate evaluation of marginal association probabilities with belief propagation", IEEE-TAES.

[3] Gaglione, Braca, Soldi, Meyer, Hem, Brekke & Hlawatsch (2023): "Comments on "Variations of Joint Integrated Data Association with Radar and Target-Provided Measurements"", JAIF.

The 2-D assignment problem

By taking the logarithm of the posterior probability we can define the hypothesis score

$$s(a_k) = \sum_{t \text{ s.t. } a_k(t)=0} \ln \underbrace{(1 - P_{\mathrm{D}}^t)}_{\text{Depends only on } t} + \sum_{t \text{ s.t. } a_k(t)>0} \underbrace{\ln P_{\mathrm{D}}^t + \ln \tilde{l}^{t,a_k^t}}_{\substack{\text{Depends only on } t \text{ and} \\ \text{measurement number } j=a_k(t)}}$$

where $\tilde{l}^{t,a_k^t} = l^{t,a_k^t}/\lambda$. The score can be found by picking the relevant item in each column in a gain matrix. Columns correspond to tracks, while rows correspond to measurements.

$$\begin{bmatrix} \ln P_{\mathrm{D}}^1 + \ln \tilde{l}^{1,1} & \ln P_{\mathrm{D}}^2 + \ln \tilde{l}^{2,1} & -\infty \\ -\infty & \ln P_{\mathrm{D}}^2 + \ln \tilde{l}^{2,2} & \ln P_{\mathrm{D}}^3 + \ln \tilde{l}^{3,2} \\ \ln P_{\mathrm{D}}^1 + \ln \tilde{l}^{1,3} & -\infty & -\infty \\ -\infty & \ln P_{\mathrm{D}}^2 + \ln \tilde{l}^{2,4} & -\infty \\ \ln(1 - P_{\mathrm{D}}^1) & -\infty & -\infty \\ -\infty & \ln(1 - P_{\mathrm{D}}^2) & -\infty \\ -\infty & -\infty & \ln(1 - P_{\mathrm{D}}^3) \end{bmatrix} = \begin{bmatrix} -5.69 & 5.37 & -\infty \\ -\infty & -3.80 & 6.58 \\ 4.78 & -\infty & -\infty \\ -\infty & 5.36 & -\infty \\ -0.46 & -\infty & -\infty \\ -\infty & -0.52 & -\infty \\ -\infty & -\infty & -0.60 \end{bmatrix}$$

The 3 best hypotheses are $[3, 1, 2]$ (16.72), $[3, 4, 2]$ (16.71) and $[5, 1, 2]$ (11.49).

Solving the 2D assignment problem: the Auction method

The underlying idea.

A number n of customers are bidding for $m \geq n$ items. The algorithm terminates when all customers are satisfied with their items.

---

procedure AUCTION(**A**) ▷ Reward matrix **A** is of size $(m+n) \times n$
  `unassigned queue` $\leftarrow$ customers $1:n$
  `prices` $\leftarrow \mathbf{0}_{1\times m}$
  while unassigned customers exist **do**
    $t^* \leftarrow$ pop first unassigned customer in `unassigned queue`
    for each customer **do**
      Register the item i that maximizes value, i.e., reward minus price
    end for
    $i^* \leftarrow$ preferred item of customer t^*
    Take item i^* from its current customer and give it to t^*
    Add the customer who had i^* to `unassigned queue`
    Find values for all items available for customer t^*
    $y \leftarrow$ how much customer t^* gains from chosing best item over next best item
    Increase the price of item i^* corrrespondingly $+\epsilon$
  end while
end procedure

Murty's method

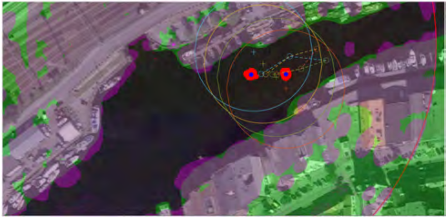
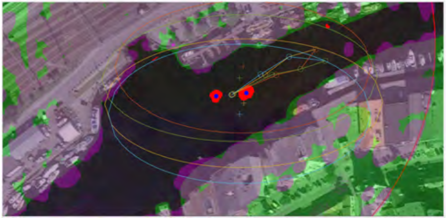
The method maintains a list Q of problem-solution pairs $\langle \mathrm{P}, \mathrm{S} \rangle$ where the problem P is in the form of a score matrix, and the solution S is in the form of an assignment of measurements to tracks.

```
procedure MURTY(P_1, M)
    Q[1] ← SOLVE( P_1)
    R ← [ ]
    while |R| < M and Q is non-empty do
        [Q[∗], Q] ← POPMAX(Q)
        R ← [R, Q[∗]]
        Q′ ← PARTITION(Q[∗].P, Q[∗].S)
        Q ← [Q, Q′]
    end while
    return R
end procedure
```

```
procedure PARTITION(P, S)
    Q′ ← [ ]
    for T = ⟨y, z, . . .⟩ ∈ S do
        P′ ← P
        Remove T from P′
        q ← SOLVE( P′ )
        if q.S is valid then
            Q′ ← [Q′, q]
        end if
        Enforce T in P
    end for
    return Q′
end procedure
```

Weaknesses of PDAF/JPDA

- Single-scan: Misassociations cannot be amended at a later time.[4]
- No in-built track initiation capability. The M-out-of-N principle is often used, but JIPDA is a more elegant alternative.[5]
- Track-coalescence: MMSE estimator may end up between two targets. The images below show this phenomena in radar tracking at Ravnkloa/Brattøra.[6]

---

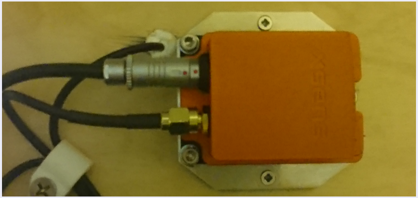
[4] For this reason, Multiple Hypothesis Tracking (Chapter 9) or Poisson Multi-Bernoulli Mixture (PMBM) filter (Chapter 14) should be considered for more challenging tracking problems.

[5] Brekke, Hem & Tokle (2021): "Multitarget Tracking With Multiple Models and Visibility: Derivation and Verification on Maritime Radar Data", IEEE-JOE.

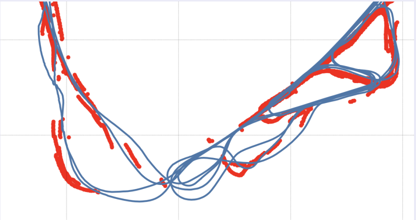
[6] Pedersen, Jesper (2017): "Surveillance of the Channel", 5th year project, NTNU.

The road ahead

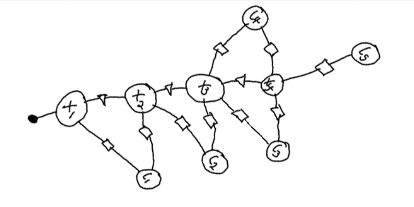
Inertial navigation



SLAM: filtering solutions



Probabilistic graphical models



SLAM: Smoothing solutions