TTK4250

Week 6

Single-target tracking

Edmund Førland Brekke

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Recap from last week

Hybrid systems

$$
\begin{aligned}
\mathbf{y}_k &= \begin{bmatrix} \mathbf{x}_k \\ s_k \end{bmatrix} \\
\mathbf{x}_k &= \mathbf{f}^{(s_k)}(\mathbf{x}_{k-1}) + \mathbf{v}_k, , \quad \mathbf{v}_k \sim \mathcal{N}(\mathbf{0}, \mathbf{Q}^{(s_k)}) \\
\mathbf{z}_k &= \mathbf{h}^{(s_k)}(\mathbf{x}_k) + \mathbf{w}_k, \quad \mathbf{w}_k \sim \mathcal{N}(\mathbf{0}, \mathbf{R}^{(s_k)}) \\
\pi^{ij} &= \Pr\{s_k = j \mid s_{k-1} = i\}
\end{aligned}
$$

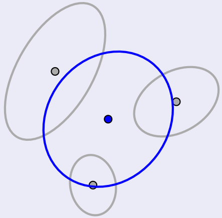
Generic Bayesian procedure

- The models involved.
- The previous posterior.
- The prediction.
- The posterior.

The IMM method

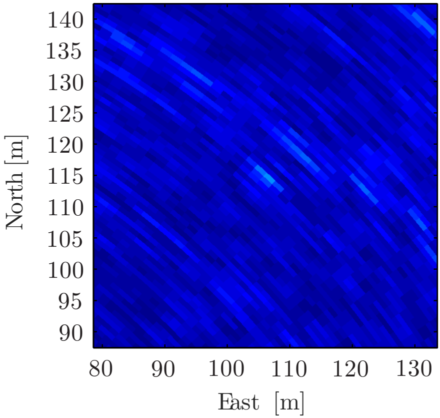
- M filters in parallel.
- Approximate $p(\mathbf{x}_{k-1}|s_k, \mathbf{z}_{1:k-1})$ as a single Gaussian.
- Perform an (E)KF cycle for each value of s_k.

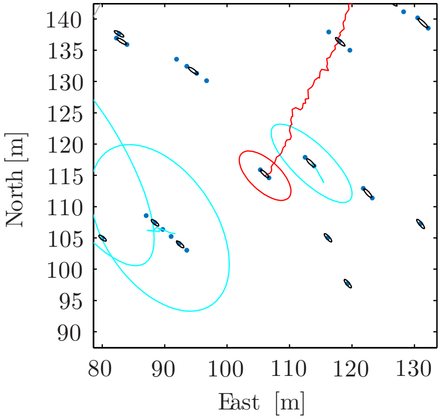
Mixture reduction



An example of target tracking

Tracking a human rebreather diver in 90kHz active sonar data.[1]





[1] Brekke et al. (2010) "The Signal-to-Noise Ratio of Human Divers", Proc. OCEANS.

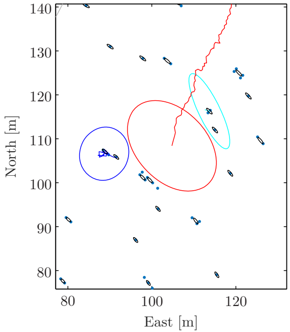
Why is single-target tracking more than just filtering?

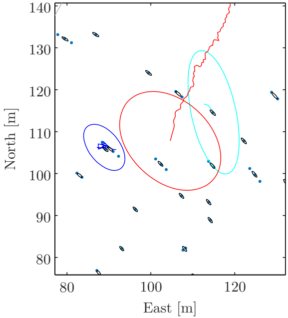
Misdetections.

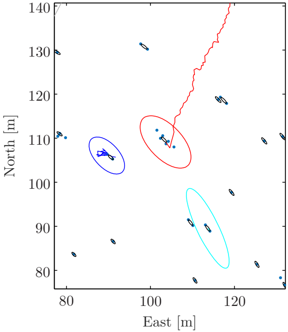
A target does not necessarily give a measurement in each sensor scan.

Clutter.

A measurement does not necessarily come from a target.







As a consequence, **data association** is an issue: We need to make inference on which, if any, measurement comes from the target.

Assumptions of (standard) single-target tracking

1. At time step $k-1$ one and only one target exists in the surveillance region $\mathcal{S}$ with state vector $\mathbf{x}_{k-1}$.
2. The prior density of $\mathbf{x}_{k-1}$ is given as $p_{k-1}(\mathbf{x}_{k-1})$.
3. The state vector of the target evolves from time step $k-1$ to k according to a Markov model of the form $f_{\mathbf{x}}(\mathbf{x}_k|\mathbf{x}_{k-1}) = p(\mathbf{x}_k|\mathbf{x}_{k-1})$.
4. A measurement of the target is detected with probability P_{D}.
5. If a target-originating measurement exists, then it is related to $\mathbf{x}_k$ according to a likelihood of the form $f_{\mathbf{z}}(\mathbf{z}_k|\mathbf{x}_k) = p(\mathbf{z}_k|\mathbf{x}_k)$.
6. An unknown number of φ_k measurements originate from clutter, where the discrete-valued random variable φ_k is distributed according to $\mu(\varphi)$.
7. If $\mathbf{z}$ is a clutter measurement, then it is distributed according to a pdf $c(\mathbf{z})$, independently of all other measurements.

Based on all this the goal is to estimate the predicted pdf $p_{k|k-1}(\mathbf{x}_k) = p(\mathbf{x}_k|Z_{1:k-1})$ and the posterior pdf $p_k(\mathbf{x}_k) = p(\mathbf{x}_k|Z_{1:k})$.

Here Z_k is the set of measurement at time k, and $Z_{1:k}$ is the collection of all such sets so far received.

The prior density

Gaussian prior

In most cases, the prior density, a.k.a. previous posterior density, is a Gaussian

$$p_{k-1}(\mathbf{x}_{k-1}) = \mathcal{N}(\mathbf{x}_{k-1}\,;\,\hat{\mathbf{x}}_{k-1}, \mathbf{P}_{k-1}).$$

But most of the general formulas that we will derive also hold if $p_{k-1}(\mathbf{x}_{k-1})$ is something more complicated.

Gaussian mixture prior

Towards the end of Chapter 7 we shall encounter a situation where the prior density is a Gaussian mixture

$$p_{k-1}(\mathbf{x}_{k-1}) = \sum_{s_k} w_k^{(s_k)} \mathcal{N}(\mathbf{x}_{k-1}\,;\,\hat{\mathbf{x}}_{k-1}^{(s_k)}, \mathbf{P}_{k-1}^{(s_k)}).$$

The process model and the predicted density

We have covered this already, but let us repeat it.

Prediction and the product identity

- Let the process model be Gaussian-linear:

$$f_{\mathbf{x}}(\mathbf{x}_k \mid \mathbf{x}_{k-1}) = \mathcal{N}(\mathbf{x}_k\,;\,\mathbf{F}\mathbf{x}_{k-1}, \mathbf{Q})$$

- The predicted density then becomes

$$\begin{aligned}
p_{k|k-1}(\mathbf{x}_k) &= \int f_{\mathbf{x}}(\mathbf{x}_k \mid \mathbf{x}_{k-1}) p_{k-1}(\mathbf{x}_{k-1}) \mathrm{d}\mathbf{x}_{k-1} \\
&= \int \mathcal{N}(\mathbf{x}_k\,;\,\mathbf{F}\mathbf{x}_{k-1}, \mathbf{Q}) \mathcal{N}(\mathbf{x}_{k-1}\,;\,\hat{\mathbf{x}}_{k-1}, \mathbf{P}_{k-1}) \mathrm{d}\mathbf{x}_{k-1} \\
&= \int \mathcal{N}(\mathbf{x}_k\,;\,\mathbf{F}\hat{\mathbf{x}}_{k-1}, \mathbf{F}\mathbf{P}_{k-1}\mathbf{F}^{\mathsf{T}} + \mathbf{Q})\, \mathcal{N}(\mathbf{x}_{k-1}\,;\,\ldots) \mathrm{d}\mathbf{x}_{k-1} \\
&= \mathcal{N}(\mathbf{x}_k\,;\,\mathbf{F}\hat{\mathbf{x}}_{k-1}, \mathbf{F}\mathbf{P}_{k-1}\mathbf{F}^{\mathsf{T}} + \mathbf{Q}) \int \mathcal{N}(\mathbf{x}_{k-1}\,;\,\ldots) \mathrm{d}\mathbf{x}_{k-1} \\
&= \mathcal{N}(\mathbf{x}_k\,;\,\mathbf{F}\hat{\mathbf{x}}_{k-1}, \mathbf{F}\mathbf{P}_{k-1}\mathbf{F}^{\mathsf{T}} + \mathbf{Q}).
\end{aligned}$$

Detection probability

Why detection probability is an issue

No detector can in general guarantee target detection.

- The target could be occluded.
- Noise from the surroundings (e.g., wave crests) could be detected instead.
- The target could be near invisible (e.g., optical camera at night or stealth airplane in radar data).

Modeling the misdetection problem

- We can model target detection as a Bernoulli random variable δ with the distribution

$$\Pr\{\delta\} = \begin{cases} P_{\mathrm{D}} & \text{if } \delta = 1 \\ 1 - P_{\mathrm{D}} & \text{if } \delta = 0. \end{cases}$$

- For now we assume that the detection probability P_{D} is fixed, but sometimes it is modeled as something that depends on the target state, i.e., $P_{\mathrm{D}}(\mathbf{x}_k)$.
- Typical values for P_{D} may lie in the area between 0.5 and 0.95.

Clutter

Causes of false alarms

No detector can in general guarantee that false alarms will not occur.

- Electric noise in the sensor itself can cause false alarms.
- Electromagnetic or acoustic beams may be reflected by arbitrary stones, waves, birds etc.
- Interference effects or deliberate jammings can cause false alarms.

The false alarm rate P_{FA} and its relationship to P_{D}

- We define P_{FA} as the probability that a given sensor cell contains a false alarm.
- Most detector can be tuned to decrease or increase P_{FA}, ...
- but unfortunately this leads to a corresponding decrease or increase of P_{D}.

Beware: A single sensor scan should never be used to declare the presence or absence of an object. This is a task for a complete tracking system (e.g., M/N-logic or IPDA). P_{D} and P_{FA} on this slide on refers to the measurement extraction process.

The Poisson clutter model

Physical motivation.

We are given a sensor array with a large number N of resolution cells, and assume that no target is present.

- For each resolution cell, there is a probability P_{FA} that it will yield a false alarm.
- The spatial distribution of any given false alarm is therefore uniform over the sensor image.
- The number φ of false alarms will have a binomial distribution:

$$\Pr(\varphi) = \binom{N}{\varphi} P_{\mathrm{FA}}^{\varphi} (1 - P_{\mathrm{FA}})^{N-\varphi}$$

- Let $\Lambda = NP_{\mathrm{FA}}$. If N is sufficiently large and P_{FA} is sufficiently small, then the binomial is well approximated by the Poisson distribution

$$\mu(\varphi) = \Pr(\varphi) = \mathrm{Poiss}(\varphi; \Lambda) = \exp(-\Lambda)\frac{\Lambda^{\varphi}}{\varphi!}.$$

We refer to Λ as the rate parameter. Also notice that Λ is the expectation of the Poisson distribution.

The Poisson clutter model continued

The Poisson model described on the previous slide is an example of a **point process**.

Definition: Point process.

A point process describes how a random number of points are distributed randomly in some space (say, $\mathbb{R}^2$).

Point processes are often referred to as random finite sets in the tracking literature.[2]

The Poisson process

- It is characterized by the following property: Given two disjoint regions $A \subset \mathbb{R}^2$ and $B \subset \mathbb{R}^2$, the number of points in A is independent of the number of points in B.
- In general, it is specified by an **intensity function** $\lambda(\mathbf{z}) = \Lambda c(\mathbf{z})$.
  Here $c(\mathbf{z})$ is the pdf of an individual point of the point process.
- We can find the expected number of points according to $\Lambda = \int \lambda(\mathbf{z})\,\mathrm{d}\mathbf{z}$.

[2] Mahler (2007): "Statistical Multisource-Multitarget Information Fusion".

An alternative: The diffuse clutter model

- In theory, P_{FA} is a tunable parameter, and the Poisson intensity λ will be directly given by P_{FA} and the sensor resolution.
- In practice, the actual false alarm rate is likely to be significantly different from the desired false alarm rate due to blurring, target extent and unmodeled noise artefacts.
- It may therefore be necessary to estimate λ, but such estimates are likely to be very inaccurate.

The diffuse clutter model

We let all clutter cardinalities have the same probability

$$\mu(\phi = 0) = \mu(\phi = 1) = \mu(\phi = 2) = \ldots = \epsilon \tag{1}$$

- The spatial density of clutter measurements is again assumed uniform within the region of interest.
- To ensure that (1) is a valid pmf, simply say that all cardinalities above some upper limit have probability zero.

The data association problem

- We receive m_k measurements $Z_k = \{\mathbf{z}_k^1, \ldots, \mathbf{z}_k^{m_k}\}$ at time k.
- Denote the cumulative measurement set by $Z_{1:k} = (Z_1, \ldots, Z_k)$.
- **At most one of the measurements in Z_k can come from the target. The other ones must be clutter.**
- The outcome space can be divided into $m_k + 1$ mutually exclusive events:

$$\begin{aligned} a_k &= 0 && \text{No measurement originates from the target} \\ a_k &= 1 && \text{Measurement 1 originates from the target} \\ &\vdots \\ a_k &= m_k && \text{Measurement } m_k \text{ originates from the target.} \end{aligned} \tag{2}$$

- The total probability theorem gives the posterior

$$p_k(\mathbf{x}_k) = \sum_{a_k} p(\mathbf{x}_k \mid a_k, Z_{1:k}) \Pr\{a_k \mid Z_{1:k}\}. \tag{3}$$

The goal in single-target tracking is typically to find $p(\mathbf{x}_k \mid a_k, Z_{1:k})$ and $\Pr\{a_k \mid Z_{1:k}\}$, and then simplify the mixture (3) so that we get a tractable approximation of the true posterior.

The measurement update in general terms

The event-conditional posteriors

The pdf terms in (3) are found according to Bayes rule:

$$p(\mathbf{x}_k \mid a_k, Z_{1:k}) \propto \begin{cases} p_{k|k-1}(\mathbf{x}_k) & \text{if} \quad a_k = 0 \\ f_{\mathbf{z}}(\mathbf{z}_k^i \mid \mathbf{x}_k) p_{k|k-1}(\mathbf{x}_k) & \text{if} \quad a_k > 0 \end{cases}$$

Under Gaussian-linear assumptions this boils down to a Kalman filter.

The posterior association probabilities.

The probabilities $P\{a_k \mid Z_{1:k}\}$ require some more work. Let us first define the "measurement likelihood"

$$l^{a_k} = \int f_{\mathbf{z}}(\mathbf{z}_k^{a_k} \mid \mathbf{x}_k) p_{k|k-1}(\mathbf{x}_k) \, \mathrm{d}\mathbf{x}_k.$$

It can then be shown (Theorem 7.3.1 in the book) that the posterior association probabilities are

$$\Pr\{a_k \mid Z_{1:k}\} \propto \begin{cases} (1 - P_{\mathrm{D}})\mu(m_k) & \text{if} \quad a_k = 0 \\ l^{a_k} P_{\mathrm{D}} \mu(m_k - 1)/m_k c(\mathbf{z}_k^{a_k}) & \text{if} \quad a_k > 0. \end{cases} \tag{4}$$

The PDAF: Assuming Gaussian-linearity

The Probabilistic Data Association Filter (PDAF) is one of the simplest and most popular tracking algorithms. The key underlying assumption is

$$p_{k|k-1}(\mathbf{x}_k) = \mathcal{N}(\mathbf{x}_k; \hat{\mathbf{x}}_{k|k-1}, \mathbf{P}_{k|k-1}), \tag{5}$$

together with Gaussian-linearity of the process model and measurement model:

$$\begin{aligned} f_{\mathbf{x}}(\mathbf{x}_k|\mathbf{x}_{k-1}) &= \mathcal{N}(\mathbf{x}_k\,;\,\mathbf{F}\mathbf{x}_{k-1}, Q) \\ f_{\mathbf{z}}(\mathbf{z}_k|\mathbf{x}_k) &= \mathcal{N}(\mathbf{z}_k\,;\,\mathbf{H}\mathbf{x}_k, \mathbf{R}). \end{aligned}$$

The event-conditional posteriors

For each measurement $\mathbf{z}_k^{a_k}$ we have $p(\hat{\mathbf{x}}_k \mid a_k, Z_{1:k}) = \mathcal{N}(\mathbf{x}_k\,;\,\hat{\mathbf{x}}_k^{a_k}, \mathbf{P}_k^{a_k})$ where

$$\begin{aligned} \hat{\mathbf{x}}_k^{a_k} &= \hat{\mathbf{x}}_{k|k-1} + \mathbf{W}_k(\mathbf{z}_k^{a_k} - \mathbf{H}\hat{\mathbf{x}}_{k|k-1}) \\ \mathbf{P}_k^{a_k} &= (\mathbf{I} - \mathbf{W}_k\mathbf{H})\mathbf{P}_{k|k-1} \\ \mathbf{W}_k &= \mathbf{P}_{k|k-1}\mathbf{H}^\mathsf{T}(\mathbf{H}\mathbf{P}_{k|k-1}\mathbf{H}^\mathsf{T} + \mathbf{R})^{-1} \end{aligned}$$

For $a_k = 0$ we have $p(\mathbf{x}_k \mid a_k, Z_{1:k}) = p_{k|k-1}(\mathbf{x})$.

The validation gate

We also assume that only measurements within some gate $\mathcal{G}$, given g standard deviations around $\hat{\mathbf{z}}_{k|k-1} = \mathbf{H}\hat{\mathbf{x}}_{k|k-1}$ according to $\mathbf{S}_k = \mathbf{H}\mathbf{P}_{k|k-1}\mathbf{H}^\mathsf{T} + \mathbf{R}$ are validated:

$$\mathcal{G} = \left\{ \mathbf{z} \;\text{ such that }\; (\mathbf{z} - \hat{\mathbf{z}}_{k|k-1})^\mathsf{T}\mathbf{S}_k^{-1}(\mathbf{z} - \hat{\mathbf{z}}_{k|k-1}) < g^2 \right\}.$$

Clutter measurements are assumed spatially uniform with pdf $c(\mathbf{z}) = 1/V_k$ where **V_k is the volume of the gate.** Since there is a chance the true measurement is outside the gate, we introduce the gate probability P_G and replace P_D with $P_\mathrm{D}P_\mathrm{G}$, and also divide the likelihood by $f_\mathbf{z}(\cdot)$ by P_G in the weight calculation.

- We have the following expression for the gate volume:

  $$V_k = c_{n_z}|g^2\mathbf{S}_k|^{1/2}, \;\; c_1 = 2, \;\; c_2 = \pi, \;\; c_3 = 4\pi/3, \ldots$$

  where c_{n_z} is the dimension of the measurement space (e.g., 2 for range-bearing measurements).
- The gate probability P_G is more difficult to calculate.[3] It is as good as 1 if $g \geq 3$.

[3] Bar-Shalom & Li (1995): “Multitarget-Multisensor Tracking: Principles and Techniques”, p. 96.

Association weights for parametric and diffuse clutter models

Association weights with the Poisson model.

$$\beta_k^i = \Pr\{a_k = i \mid Z_{1:k}\} \propto \begin{cases} \lambda \dfrac{1 - P_\mathrm{D} P_\mathrm{G}}{P_\mathrm{D}} & \text{if} \quad i = 0 \\ \mathcal{N}(\mathbf{z}_k^i\,;\, \hat{\mathbf{z}}_{k|k-1}, \mathbf{S}_k) & \text{if} \quad i > 0 \end{cases}$$

Association weights with the diffuse model.

$$\beta_k^i = \Pr\{a_k = i \mid Z_{1:k}\} \propto \begin{cases} \dfrac{m_k}{V_k} \dfrac{1 - P_\mathrm{D} P_\mathrm{G}}{P_\mathrm{D}} & \text{if} \quad i = 0 \\ \mathcal{N}(\mathbf{z}_k^i\,;\, \hat{\mathbf{z}}_{k|k-1}, \mathbf{S}_k) & \text{if} \quad i > 0 \end{cases}$$

We see that m_k / V_k takes the role of a clutter estimate when prior knowledge about the clutter density is not provided.

The PDAF: Mixture reduction

- In order to ensure Gaussianity at the next cycle we need to reduce the Gaussian mixture to a single Gaussian.
- Moment matching: We construct a Gaussian with the same first- and second-order moments (i.e., expectation and covariance) as the mixture.

The expectation.

$$\hat{\mathbf{x}}_k = \beta_k^0 \, \hat{\mathbf{x}}_{k|k-1} + \sum_{i=1}^{m_k} \beta_k^i \, \hat{\mathbf{x}}_k^i \tag{6}$$

The covariance.

First we find the "spread-of-the-innovations":

$$\tilde{\mathbf{P}}_k = \mathbf{W}_k \left[\sum_{i=1}^{m_k} \beta_k^i \boldsymbol{\nu}_k^i (\boldsymbol{\nu}_k^i)^\mathsf{T} - \boldsymbol{\nu}_k \boldsymbol{\nu}_k^\mathsf{T} \right] \mathbf{W}_k^\mathsf{T}. \tag{7}$$

The covariance of the mixture is then

$$\mathbf{P}_k = \mathbf{P}_{k|k-1} - (1 - \beta_k^0) \mathbf{W}_k \mathbf{S}_k \mathbf{W}_k^\mathsf{T} + \tilde{\mathbf{P}}_k. \tag{8}$$

Illustration of the PDAF

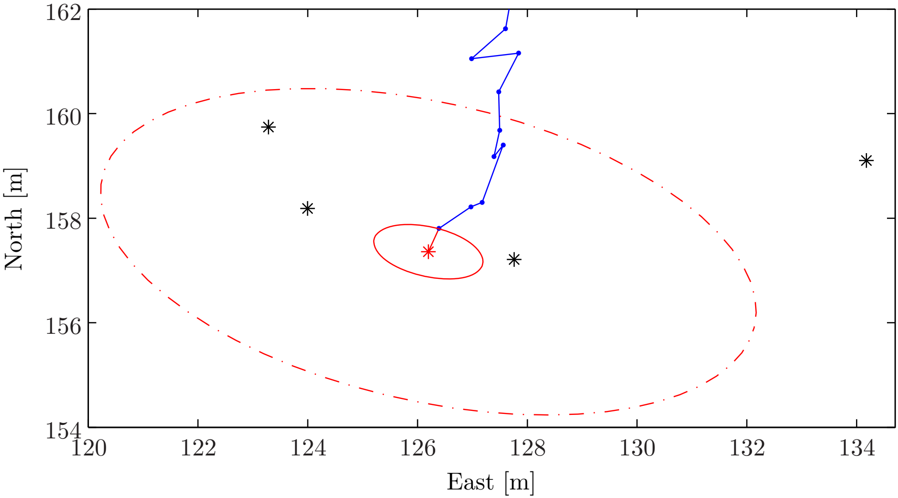
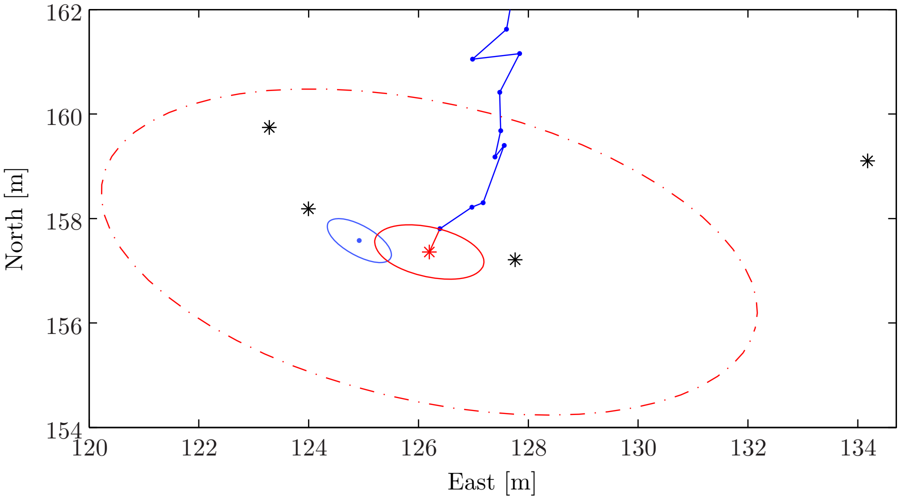


Illustration of the PDAF



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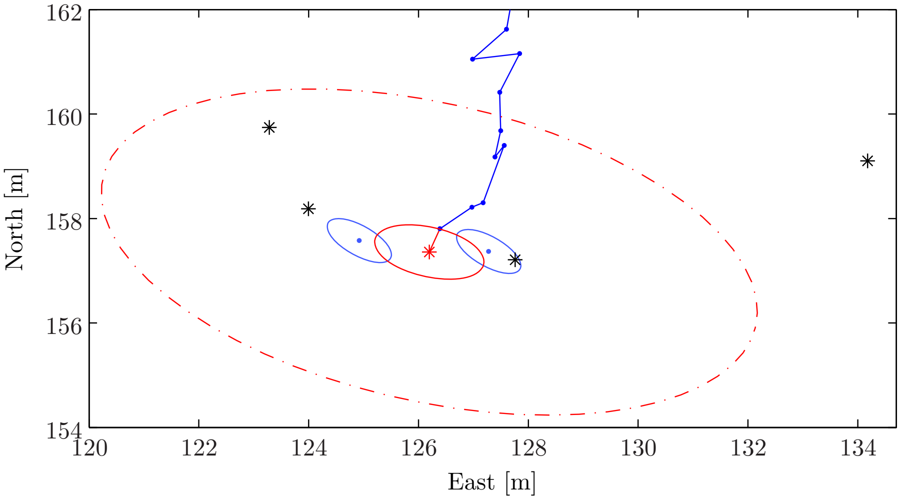

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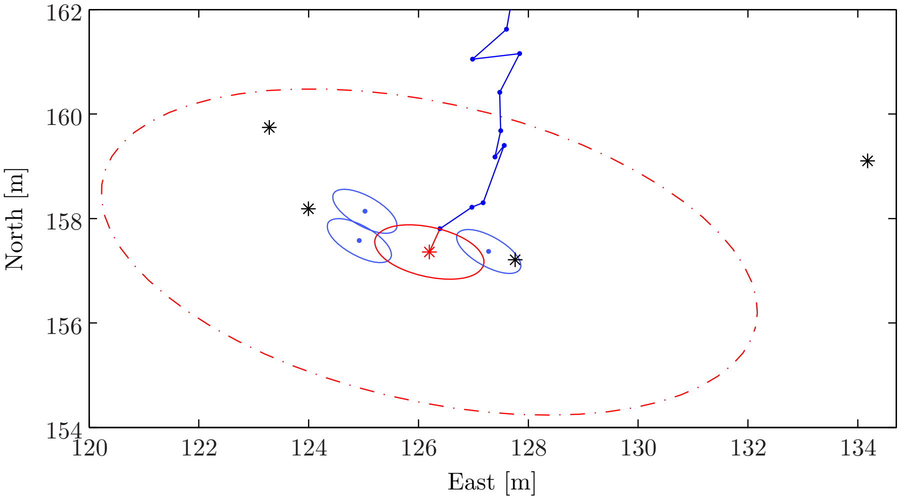
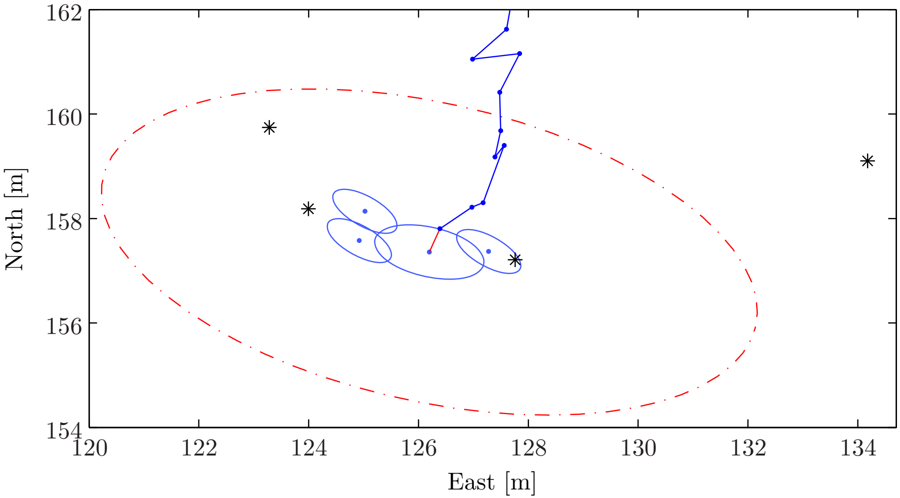


Illustration of the PDAF



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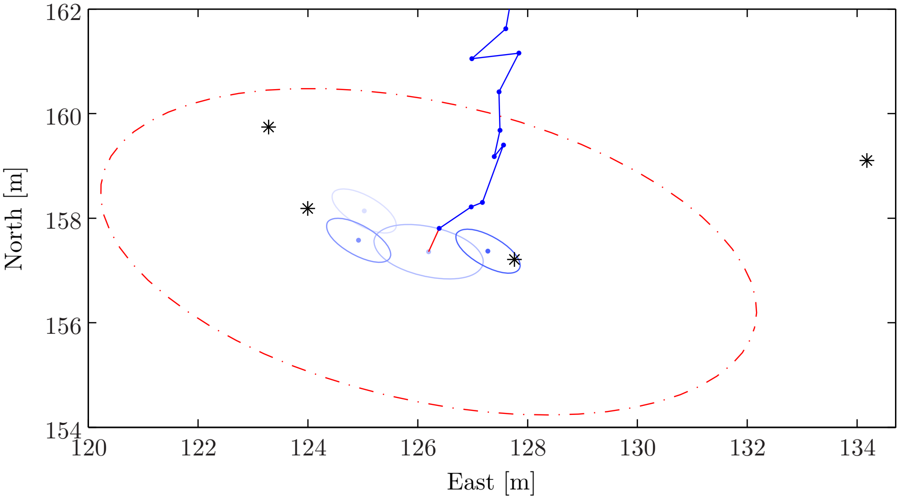

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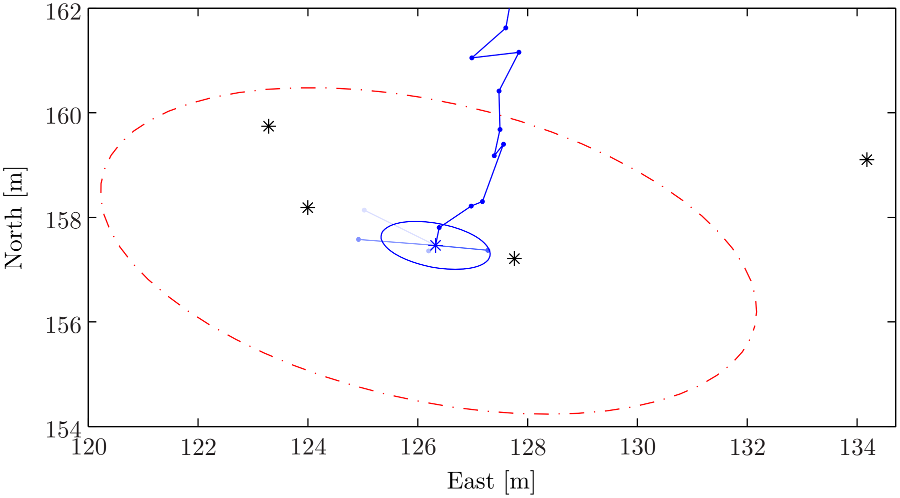

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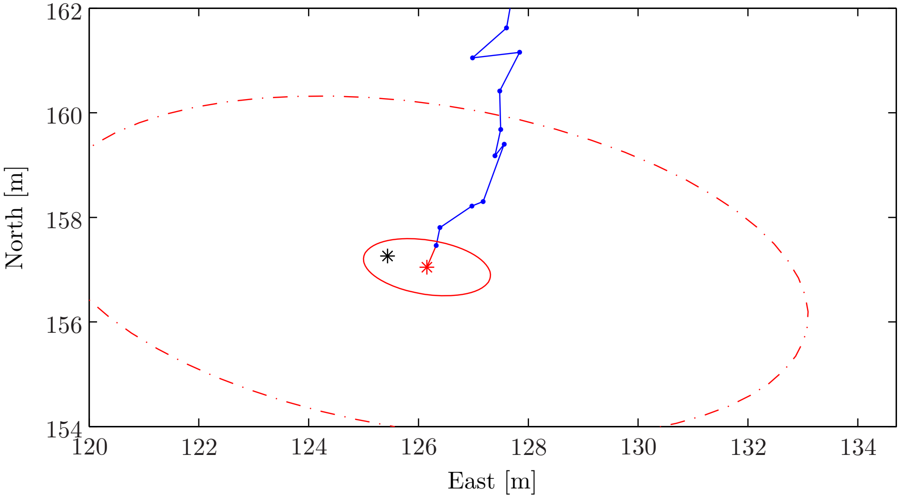
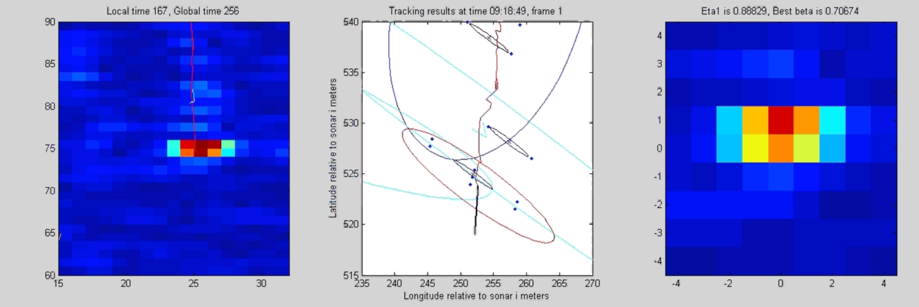


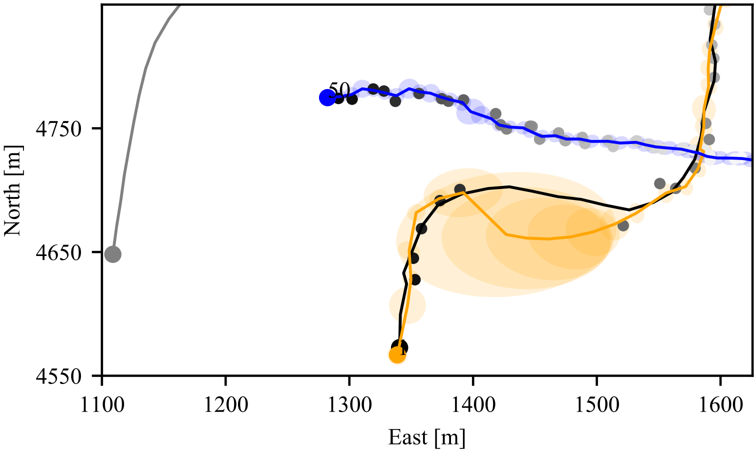
Visualization ideas

- Display measurements, tracks, covariance ellipses and validation gates.
- It can be nice to superpose the tracks on the raw sensor data.
- Experiment with different colors for tracks at different stages or of different quality.
- Zoom.



Visualization ideas[4]

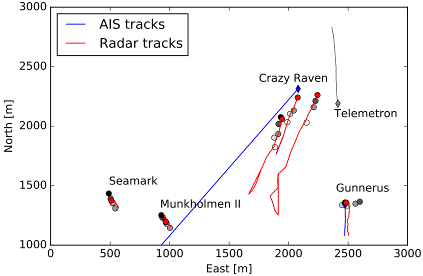
- Make movies.
- Alternatively: Display temporal information using color shades.



[4] Brekke, Hem & Tokle (2020): "Multitarget Tracking With Multiple Models and Visibility: Derivation and Verification on Maritime Radar Data", IEEE-JOE, 2021.

Visualization ideas[5]

- Compare with actual video footage.
- “Ground truth” (AIS) and radar tracks in different colors.



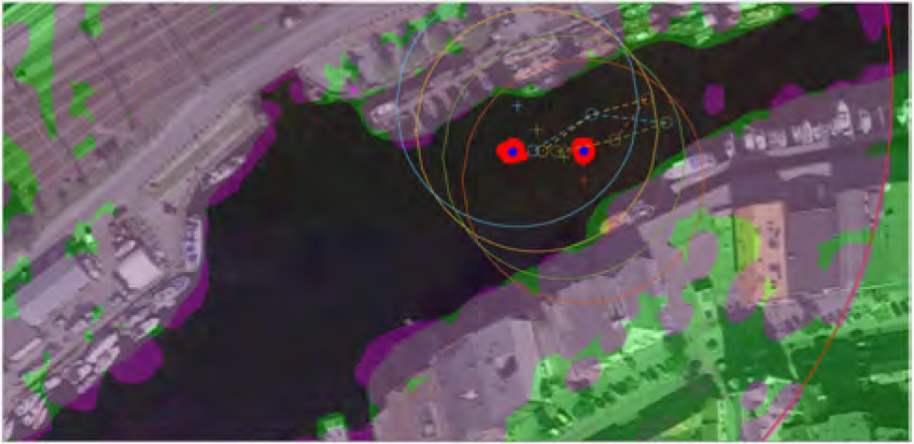
(a) Radar tracks from PDAF.

(b) The same scenario seen from drone video.

[5]Brekke, Wilthil, Eriksen, Kufoalor, Helgesen, Hagen, Breivik & Johansen (2019): “The Autosea project: Developing closed-loop target tracking and collision avoidance systems”, Proc. ICMASS.

Visualization ideas[6]

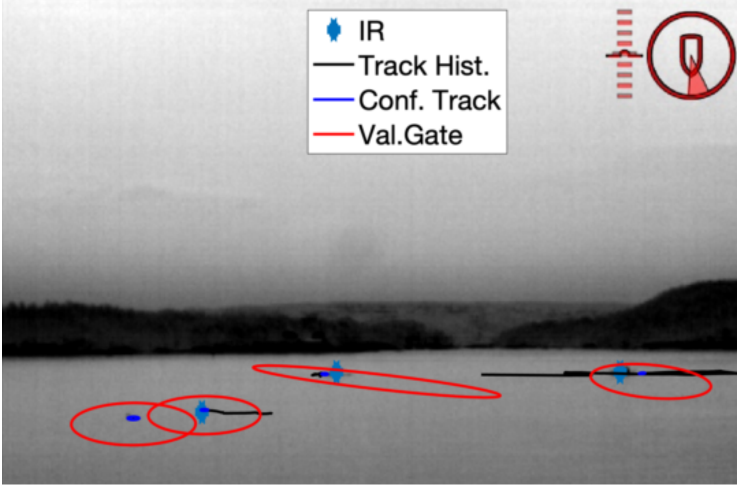
- Visualize together with geographical surroundings.
- Visualize together with processed sensor data.



[6] Jesper Pedersen (2018): "Harbor Surveillance with a K-best, Track Terminating, Hypothesis-Oriented MHT", MSc thesis, NTNU.

Visualization ideas[7]

- Compare with video footage (IR in this case).
- Project the tracking results into the video.



[7] Helgesen, Brekke, Stahl & Engelhardtsen (2020): "Low Altitude Georeferencing for Imaging Sensors in Maritime Tracking", Proc. IFAC World Congress.

Reflections

Roles of tracking: Safety critical or not so critical?

- Collision avoidance. **Please don't miss a target.**
- Defence applications. **Please don't establish false tracks.**
- Surveillance. **Please don't make so many mistakes that operator gets jaded.**
- Post-processing and data interpretation. **Please don't learn the wrong things.**

Alternatives to the Bayesian paradigm

- Nearest neighbor techniques (e.g. SORT).
- Counting techniques (e.g. Recursive RANSAC).
- Signal processing techniques (e.g. Hough transform for track-before-detect).
- Machine learning.
  - Typically used to extract detections (e.g. bounding boxes) that are fed to a tracker.
  - Can also be used to learn how to do data association (transformers).
  - Can learn to recognize motion and distance.
- Occupancy grid + vector field.

The Bayesian approach typically works well insofar as it is easy to make simple and reliable models.