TTK4250 Sensor Fusion
Assignment 6

Hand in: *Friday 18. October 23:59* on Blackboard.

This assignment should be handed in on Blackboard, as a single PDF file, before the deadline. You are supposed to show how you got to each answer unless told otherwise. If you struggle, we encourage you to ask for help from a classmate or come to the exercise class on Monday.

Task 1: *Number of association events in JPDA*

Take the data association example in figure 8.1 in the book. We are wondering how many data association events there are here. There are 3 targets and 4 measurements, where no target gates all the measurements.

(a) If we disregard the validation gates, there is a formula for calculating the number of association events in JPDA. If we have N targets that is going to be associated to m out of a total of M measurements ($m \leq \min(N, M)$), it becomes the number of ways to pick m out of N targets unordered and m out of M measurements unordered before assigning these m targets to the m measurements. This number is given by $\binom{N}{m}\binom{M}{m}m! = \frac{N!M!}{m!(N-m)!(M-m)!}$. However, m can be any number between 0 and $\min(N, M)$ so we have to sum over this range to find the total number of association events:

$$\sum_{m=0}^{\min(N,M)} \frac{N!M!}{m!(N-m)!(M-m)!} \tag{1}$$

What is this number for the example in figure 8.1?

(b) We now include the validation gates. There is no simple formula for the number of events in this case. The simplest option is to enumerate the possibilities and count them. What is the number of association events in figure 8.1 considering the validation gates? How many percent of the non gated events is this? Would you say gating is a helpful approximation for data association?

(c) How many of the gated events does not have any missed detections?

(d) If the unassociated measurements could also be new targets, there are 2 possible events for each of these. It is therefore also 2^{M-m} events for new targets or false alarms in addition to the possible target to measurement associations, when m out of M measurements are known to be associated to already established targets, The total number of events when not considering validation gates then becomes

$$\sum_{m=0}^{\min(N,M)} \frac{2^{M-m}N!M!}{m!(N-m)!(M-m)!} \tag{7}$$

What is this number for the example? Is there a lot to save by not having to consider these hypotheses (do not consider the validation gate)?

Task 2: *Finding the most probable associations in JPDA*

You are here to investigate the auction algorithm (Algorithm 5) and Murty's method (Algorithm 6) on the association problem in figure 8.1. A reward matrix (log likelihood ratio matrix) for this association problem is given in subsection 8.3.2. You can use $\epsilon = 0.01$ for this task.

(a) Go through the auction algorithm (by hand) for the problem to show that the most likely association is $a(1) = 3$, $a(2) = 1$ and $a(3) = 2$, where $a(t) = j$ indicates that target t is associated to measurement j.

(b) The last part basically solved lines 2-4 of Murty's method. Go through lines the rest of Murty's method to find the second-best association. State the solution and reward for each solution. You do not need to go through more than one more run of the auction algorithm as you should be able to see the solutions straight away in this problem.

In our case, the first partition will make the first subproblem by setting $R_1(3,1) = -\infty$ and solve that problem. The second subproblem is found by enforcing $a_1(1) = 3$ and set $R_1(2,1) = -\infty$.

(c) Murty's algorithm works by enforcing some associations (fix certain a(t)) while making others impossible (setting certain elements in the reward matrix to $-\infty$) and then solving the resulting assignment problem, over and over in a structured manner. Say that the second best solution to the the assignement problem was found to be the solution of making the association $a(2) = 1$ impossible using Murty's algorithm. What would the reward matrix look like for the selected problem-solution pair in line 6?

Task 3: *Pure quaternion powers and exponential*

For simplicity, we let the quaternion $\begin{bmatrix}\eta & \epsilon^T\end{bmatrix}^T$ be equivalently written as $\eta + \epsilon$ in this task. Here η and ϵ is, of course, the real and imaginary (or hyper imaginary, if you prefer) parts of the quaternion, respectively.

All the rules for complex numbers can be shown to hold for quaternions, except commutativity of the imaginary parts in the product of two quaternions. For instance, we have that the quaternion product can be written as $(\eta_1 + \epsilon_1)(\eta_2 + \epsilon_2) = \eta_1\eta_2 + \eta_1\epsilon_2 + \eta_2\epsilon_1 + \epsilon_1\epsilon_2$ using the current notation. We can write the product of the imaginary parts in terms of more familiar vector products, $\epsilon_1\epsilon_2 = -\epsilon_1^T\epsilon_2 + \epsilon_1 \times \epsilon_2$. The first term is a scalar and therefore the real part while the latter term is a vector and therefore the imaginary part. Since the cross product is anti commutative ($v \times w = -w \times v$), the imaginary product, and thereby the quaternion product, is not commutative unless $\epsilon_1 \times \epsilon_2 = 0$. $v \times w = 0$ only holds when the vectors are parallel or at least one is zero.

We state two handy facts of imaginary numbers, ie. complex numbers with zero real part.

- The powers of the imaginary numbers can be written as

$$
\begin{aligned}
(\alpha i)^{2n+1} &= \alpha^{2n+1} i^{2n} i = \alpha^{2n+1}(-1)^n i, && \text{(odd powers)}\\
(\alpha i)^{2n} &= \alpha^{2n} i^{2n} = \alpha^{2n}(-1)^n. && \text{(even powers)}
\end{aligned}
$$

- The exponential function relates imaginary numbers to rotations through

$$
\begin{aligned}
e^{\alpha i} &= \sum_{n=0}^{\infty} \frac{(\alpha i)^n}{n!} = \left[\sum_{n=0}^{\infty} \frac{(\alpha i)^{2n}}{(2n)!}\right] + \left[\sum_{n=0}^{\infty} \frac{(\alpha i)^{2n+1}}{(2n+1)!}\right] = \left[\sum_{n=0}^{\infty} \frac{(-1)^n \alpha^{2n}}{(2n)!}\right] + \left[\sum_{n=0}^{\infty} \frac{(-1)^n \alpha^{2n+1}}{(2n+1)!}\right] i\\
&= \cos(\alpha) + \sin(\alpha) i.
\end{aligned}
$$

(a) Show that the powers of a pure quaternion (zero real part), αv with $|v| = 1$, acts like the powers of imaginary numbers, only with v substituted for i. That is, show that $(\alpha v)^{2n+1} = \alpha^{2n+1}(-1)^n v$ and $(\alpha v)^{2n} = \alpha^{2n}(-1)^n$.

Hint: Write out the powers up to 2 and see if you can write the higher powers in terms if these, and note that $q^0 = 1$. Equations (10.21) and (10.34) can be useful.

(b) Show that similarly to the imaginary numbers, the relationship $e^{\alpha v} = \cos(\alpha) + \sin(\alpha) v$ also holds for a pure quaternion αv, with $|v| = 1$ and α real. Briefly discuss how this relates to theorem 10.1.2.

(c) (Optional for those with interest) What will the exponential of a general quaternion be. Based on this, can you find what the logarithm of a quaternion must be?

Hint: That the real part of the quaternion commutes with the imaginary part simplifies finding the exponential a lot, as $e^{a+b} = e^a e^b$ whenever $ab = ba$ (which is verified through the power series).

Task 4: *Quaterion kinematics*

We are here going to derive equation (10.43) while showing that the first order approximation done in (10.41) is actually exact in this limit. (10.43) states that $\dot{q} = \frac{1}{2} q\omega$, where the factor $\frac{1}{2}$ can be attributed to the quaternion only representing half the rotation (its conjugate the other half), whereas ω is the full rotational rate. Similarly to the book we are going to use

$$q(t+\Delta t) = q(t)\Delta q(t, t+\Delta t).$$

However, we are going to use the result of the previous task to state that

$$\Delta q(t, t+\Delta t) = \cos\left(\frac{\Delta\alpha(t, t+\Delta t)}{2}\right) + \sin\left(\frac{\Delta\alpha(t, t+\Delta t)}{2}\right)\Delta v(t, t+\Delta t) = e^{\frac{\Delta\theta(t, t+\Delta t)}{2}}$$

for the pure quaternion $\Delta\theta(t, t+\Delta t) = \Delta\alpha(t, t+\Delta t)\Delta v(t, t+\Delta t)$. Here $\alpha(t, t+\Delta t)$ is the angle of rotation between time t and $t+\Delta t$, and $v(t, t+\Delta t)$ is the pure quaternion of unit length (ie. a unit vector) which specifies the axis this rotation is about. We also use (10.42) in the sense that $\lim_{\Delta t \to 0} \frac{\Delta\theta(t, t+\Delta t)}{\Delta t} = \omega(t)$.

(a) What is the interpretation of $\omega(t) = \lim_{\Delta t \to 0} \frac{\Delta\theta(t, t+\Delta t)}{\Delta t} = \omega(t)$, as it is stated here and in equation (10.42) in the book? Specifically mention which coordinate systems it relates and is specified in.

(b) Show that the quaternion time derivative $\dot{q}(t) = \lim_{\Delta t \to 0} \frac{q(t+\Delta t) - q(t)}{\Delta t}$ can be written as $q(t) \lim_{\Delta t \to 0} \frac{e^{\frac{\Delta\theta(t, t+\Delta t)}{2}} - 1}{\Delta t}$, and that taking the limit gives equation (10.43).

Hint: The quaternions are distributive, ie. $a(b+c) = ab + bc$ for quaternions a, b, and c. Constants (here a quaternion) in terms of the limiting variable (here Δt) can always be taken outside the limit. Also, using the series representation of the exponential function one can write

$$\begin{aligned}
\Delta q(t, t+\Delta t) = e^{\frac{\Delta\theta(t, t+\Delta t)}{2}} &= 1 + \frac{\Delta\theta(t, t+\Delta t)}{2} + \sum_{n=2}^{\infty} \frac{\left(\frac{\Delta\theta(t, t+\Delta t)}{2}\right)^n}{n!} \\
&= 1 + \frac{\Delta\theta(t, t+\Delta t)}{2} + \frac{\Delta\theta(t, t+\Delta t)}{2} \sum_{n=2}^{\infty} \frac{\left(\frac{\Delta\theta(t, t+\Delta t)}{2}\right)^{n-1}}{n!} \\
&= 1 + \frac{\Delta\theta(t, t+\Delta t)}{2}[1 + O(\Delta\theta(t, t+\Delta t))],
\end{aligned}$$

where $O(\Delta\theta(t, t+\Delta t)) \overset{\Delta t \to 0}{\longrightarrow} 0$.

Task 5: *The error state dynamics*

In this task we are going to take a closer look at the proof of theorem 10.3.1 for the position, velocity and orientation error dynamics.

(a) Which, if any, approximations are made to arrive at the linearized position error state dynamics $\delta\dot{\rho} = \dot{\rho}_t - \dot{\rho} = \delta v$?

(b) Derive the linearized velocity error state dynamics $\delta\dot{v} = \dot{v}_t - \dot{v} \approx -R(q)S(a_m - a_b)\delta\theta - R(q)\delta a_b - R(q)a_n$ by using $R(\delta q) = R(e^{\frac{\delta\theta}{2}}) = e^{S(\delta\theta)}$ and its series. Which, if any, approximations are made?

(c) Which, if any, approximations are made in order to derive the linearized orientation error state dynamics $\delta\dot{\theta} = -S(\omega_m - \omega_b)\delta\theta - \delta\omega_b - \omega_n$?