LINEAR TRANSFORMATIONS WHICH PRESERVE TRACE AND POSITIVE SEMIDEFINITENESS OF OPERATORS

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This work may be considered a completion of the paper by J. de Pillis: *Linear transformations which preserve Hermitian and positive semidefinite operators*, published in 1967 [2]: necessary conditions have been formulated.

Let $\mathscr{A}_1$ be the full algebra of linear operators on the n-dimensional Hilbert space $\mathscr{H}_1$, and let $\mathscr{A}_2$ be the full algebra of linear operators on the m-dimensional Hilbert space $\mathscr{H}_2$. Let $\mathscr{L}(\mathscr{A}_1, \mathscr{A}_2)$ denote the complex space of linear maps from $\mathscr{A}_1$ to $\mathscr{A}_2$ and S denotes the cone of all $T \in \mathscr{L}(\mathscr{A}_1, \mathscr{A}_2)$ which send positive semidefinite operators from $\mathscr{A}_1$ to positive semidefinite operators from $\mathscr{A}_2$. The aim of this paper is to present a necessary and sufficient condition for a transformation in $\mathscr{L}(\mathscr{A}_1, \mathscr{A}_2)$ to be in the cone S, and to preserve trace of the operators.

1. Preliminaries and notation

Let $\mathscr{L}(\mathscr{H}_1, \mathscr{H}_2)$ be the vector space of linear transformations from the Hilbert space $\mathscr{H}_1$ with the inner product $(\cdot,\cdot)_1$, to the Hilbert space $\mathscr{H}_2$ with the inner product $(\cdot,\cdot)_2 (\cdot,\cdot)_1$. The symbol $[\cdot,\cdot]_1$ stands for the inner product on $\mathscr{A}_1$ (resp. $[\cdot,\cdot]_2$ on $\mathscr{A}_2$) defined by the equality $[A, B]_1 = \mathrm{tr}\,(B^*A)$ for all $A, B \in \mathscr{A}_1$ (resp. $[A, B]_2 = \mathrm{tr}\,(B^*A)$ on $\mathscr{A}_2$).

The symbol $(x \times y)$ will denote an element of $\mathscr{L}(\mathscr{H}_1, \mathscr{H}_2)$ defined for fixed $x \in \mathscr{H}_2$ and $y \in \mathscr{H}_1$ by $(x \times y)\, z = (z, y)_1 x$ for all $z \in \mathscr{H}_1$.

Let $\mathscr{H}_1 \otimes \mathscr{H}_2$ and $\mathscr{A}_1 \otimes \mathscr{A}_2$ denote the tensor product of spaces $\mathscr{H}_1$, $\mathscr{H}_2$ and of algebras $\mathscr{A}_1$, $\mathscr{A}_2$, respectively. The inner product in the vector space $\mathscr{H}_1 \otimes \mathscr{H}_2$ is defined by (cf. e.g. [2])

$$((x_1 \otimes y_1, x_2 \otimes y_2)) = (x_1, x_2)_1 \cdot (y_1, y_2)_2, \tag{1.1}$$

for all $x_1, x_2 \in \mathscr{H}_1$ and all $y_1, y_2 \in \mathscr{H}_2$. We define the inner product which gives the algebra $\mathscr{A}_1 \otimes \mathscr{A}_2$ a Hilbert space structure analogously:

$$[[A_1 \otimes B_1, A_2 \otimes B_2]] = [A_1, A_2]_1 \cdot [B_1, B_2]_2, \tag{1.2}$$

for all $A_1, A_2 \in \mathscr{A}_1$ and all $B_1, B_2 \in \mathscr{A}_2$.

Now let $\mathscr{J}$ denote the linear transformation, sending the space $\mathscr{L}(\mathscr{A}_1, \mathscr{A}_2)$ to the space $\mathscr{A}_1 \otimes \mathscr{A}_2$

$$\mathscr{J}: \mathscr{L}(\mathscr{A}_1, \mathscr{A}_2) \to \mathscr{A}_1 \otimes \mathscr{A}_2, \tag{1.3}$$

where $\mathscr{J}(\boldsymbol{T})$ – the element of $\mathscr{A}_1 \otimes \mathscr{A}_2$ – is defined for each $\boldsymbol{T} \in \mathscr{L}(\mathscr{A}_1, \mathscr{A}_2)$ by the equality

$$[[\mathscr{J}(\boldsymbol{T}), A^* \otimes B]] = [\boldsymbol{T}(A), B]_2 \tag{1.4}$$

for all $A \in \mathscr{A}_1$, $B \in \mathscr{A}_2$. It may easily be verified (cf. [2]) that

$$\mathscr{J}(\boldsymbol{T}) = \sum_i E_i^* \otimes \boldsymbol{T}(E_i) \tag{1.5}$$

for any orthonormal basis $\{E_i\}$ in $\mathscr{A}_1$. The proof of the following theorem may be found in [2]: *A transformation $\boldsymbol{T} \in \mathscr{L}(\mathscr{A}_1, \mathscr{A}_2)$ sends Hermitian operators in $\mathscr{A}_1$ to Hermitian operators in $\mathscr{A}_2$ if and only if $\mathscr{J}(\boldsymbol{T})$ is Hermitian.*

2. The main results

Now we present a necessary and sufficient condition for a transformation in $\mathscr{L}(\mathscr{A}_1, \mathscr{A}_2)$ to be in the cone S.

THEOREM 1. *A transformation $\boldsymbol{T} \in \mathscr{L}(\mathscr{A}_1, \mathscr{A}_2)$ is in S if and only if $((\mathscr{J}(\boldsymbol{T})x \otimes y, x \otimes y)) \geqslant 0$, for all $x \in \mathscr{H}_1$ and $y \in \mathscr{H}_2$.*

Proof: It follows from the spectral theorem that $\boldsymbol{T}$ preserves positive semidefinite operators ($\boldsymbol{T} \in S$) if and only if $\boldsymbol{T}$ satisfies the inequality

$$(\boldsymbol{T}(P_x)y, y)_2 \geqslant 0 \tag{2.1}$$

for all $x \in \mathscr{H}_1$ and $y \in \mathscr{H}_2$, where P_x denotes the orthogonal projection onto the subspace spanned by x; i.e., $(x, x)_1 = 1$ and $P_x = (x \times x)$. It may easily be verified that

$$\boldsymbol{T}(A) = \sum_i \boldsymbol{T}(E_i)[A, E_i]_1, \tag{2.2}$$

for any orthonormal basis $\{E_i\}$ in $\mathscr{A}_1$. (2.2) implies the following for P_x

$$\begin{aligned}
\boldsymbol{T}(P_x) &= \sum_i \boldsymbol{T}(E_i)[P_x, E_i]_1 = \sum \boldsymbol{T}(E_i)\,\mathrm{tr}(E_i^* x \times x) \\
&= \sum_{ij} \boldsymbol{T}(E_i)\big(E_i^*(x \times x)e_j, e_j\big)_1 \\
&= \sum_{ij} \boldsymbol{T}(E_i)(E_i^* x, e_j)_1 (e_j, x)_1 = \sum_i \boldsymbol{T}(E_i)(E_i^* x, x)_1,
\end{aligned} \tag{2.3}$$

where $\{e_j\}$ is any orthonormal basis in $\mathscr{H}_1$. Substituting into (2.1) $\boldsymbol{T}(P_x)$ given by (2.3) we get

$$\sum (E_i^* x, x)_1 (\boldsymbol{T}(E_i)y, y)_2 \geqslant 0, \tag{2.4}$$

for all $x \in \mathscr{H}_1$, $y \in \mathscr{H}_2$ and any orthonormal basis $\{E_i\}$ in $\mathscr{A}_1$.

Inequality (2.4) is equivalent to

$$((\sum_i E_i^* \otimes T(E_i)\, x \otimes y,\ x \otimes y)) \geqslant 0. \tag{2.5}$$

Now, according to (1.5) we can write condition (2.4) in the form

$$((\mathscr{J}(T)\, x \otimes y,\ x \otimes y)) \geqslant 0, \tag{2.6}$$

for all $x \in \mathscr{H}_1$ and $y \in \mathscr{H}_2$. This completes the proof.

Remark. Condition (2.6) is weaker than the condition for positive semidefiniteness in $\mathscr{A}_1 \otimes \mathscr{A}_2$ (because elements of the form $x \otimes y$ do not constitute the full vector space $\mathscr{H}_1 \otimes \mathscr{H}_2$). Let $I \in \mathscr{L}(\mathscr{A}_1, \mathscr{A}_1)$ be the identity transformation $I(A) = A$ for all $A \in \mathscr{A}_1$. Surely $I \in S$. Let $\{e_j\}$ be any orthonormal basis in $\mathscr{H}_1$; then $\{(e_i \otimes e_j)\}$ $(i, j = 1, \ldots, n)$ is an orthonormal basis for $\mathscr{A}_1$, and thus (1.5) implies ([2])

$$\mathscr{J}(I) = \sum_{ij} (e_i \times e_j)^* \otimes (e_i \times e_j) = \sum_{ij} (e_j \times e_i) \otimes (e_i \times e_j). \tag{2.7}$$

It is easily seen that vectors from $\mathscr{H}_1 \otimes \mathscr{H}_1$ of the form $(e_p \otimes e_q - e_q \otimes e_p)$ are eigenvectors of $\mathscr{J}(I)$ corresponding to the eigenvalue -1. Therefore $\mathscr{J}(I)$ is not positive semidefinite on the space $\mathscr{H}_1 \otimes \mathscr{H}_1$, but it does satisfy inequality (2.6). Indeed, we have

$$\begin{aligned} ((\mathscr{J}(I)\, x \otimes y,\ x \otimes y)) &= ((\sum_{ij} (e_j \times e_i) \otimes (e_i \times e_j)\, x \otimes y,\ x \otimes y)) \\ &= \sum_{ij} ((e_j \times e_i)\, x,\ x)_1 \cdot ((e_i \times e_j)\, y,\ y)_1 \\ &= \sum_{ij} (x, e_i)_1 (e_j, x)_1 (y, e_j)_1 (e_i, y)_1 \\ &= (x, y)_1 (y, x)_1 = |(x, y)_1|^2 \geqslant 0 \end{aligned} \tag{2.8}$$

for all $x, y \in \mathscr{H}_1$.

THEOREM 2. *Let $T \in \mathscr{L}(\mathscr{A}_1, \mathscr{A}_2)$ and $T(A^*) = T(A)^*$ for all $A \in \mathscr{A}_1$ (T sends Hermitian operators in $\mathscr{A}_1$ to Hermitian operators in $\mathscr{A}_2$). The transformation T preserves trace of the operators if and only if*

$$\sum_k ((\mathscr{J}(T)\, e_i \otimes f_k,\ e_j \otimes f_k)) = \delta_{ij}, \tag{2.9}$$

for any orthonormal basis $\{e_k\}$ in $\mathscr{H}_1$ and any orthonormal basis $\{f_i\}$ in $\mathscr{H}_2$.

Proof: The equality $\operatorname{tr} T(A) = \operatorname{tr} A$ for all $A \in \mathscr{A}_1$ may be written as

$$[T(A), I_2]_2 = [A, I_1]_1, \tag{2.10}$$

where I_1 is the identity operator in $\mathscr{A}_1$ and I_2 is identity operator in $\mathscr{A}_2$. This equality is equivalent to

$$T^*(I_2) = I_1, \tag{2.11}$$

or according to (2.2), to

$$T^*(I_2) = \sum_i \operatorname{tr} F_i^* T^*(F_i) = I_1 \tag{2.12}$$

for any orthonormal basis $\{F_i\}$ in $\mathscr{A}_2$. On the other hand it may easily be verified ([2]), that if $T(A)^* = T(A^*)$ for all $A \in \mathscr{A}_1$, then $\mathscr{I}(T) = \sum T^*(F_i) \otimes F_i^*$ for any orthonormal basis $\{F_i\}$ in $\mathscr{A}_2$. Thus condition (2.12) may be written as

$$\sum_k \left((\mathscr{I}(T)\, e_i \otimes f_k,\, e_j \otimes f_k) \right) = \delta_{ij}. \tag{2.13}$$

This completes the proof.

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