Supplementary Material

I. MODEL

The model used is given as a set of individual traps (microtraps) arranged in a single line. Each microtrap is simply a Paul trap that contains only a single ion, it is thus generally not elongated along an axis as linear Paul traps are. The potential energy in this situation is then given as the sum of the trappinging potentials and the Coulomb potentials between the ions, as given in Eq. 1.

$$V = \frac{e^2}{4\pi\varepsilon_0}\sum_{i=1}^{N-1}\sum_{j=i+1}^{N}\frac{1}{((j-i)d + x_j - x_i)} + \frac{1}{2}M\omega^2\sum_{i=1}^{N}x_i^2 \tag{1}$$

where x_i is the position of the i^{th} ion in the chain relative to the centre of its own trap, d is the separation between each microtrap, ω is the angular trapping frequency of each microtrap, M is the ion mass and N the number of ions in the chain.

For efficient simulation of the motion of ions, we use a normal mode expansion. This approximates the motion in terms of N oscillatory modes, each mode described by some frequency of oscillation ω_p and coupling to the ions $\vec{b}_p$. This is done by linearising the potential around the ions stationary points and is valid for sufficiently small displacements of the ions around their stationary points. The motion of the i^{th} ion is then given by

$$x_i = \sum_{p=1}^{N} A_p b_p^i \sin\left(\omega_p t + \phi_p\right), \tag{2}$$

with the amplitude of each mode A_p and the phase of each mode ϕ_p determined by initial conditions.

The error associated with this approximation can be estimated by the upper limit of the size of the next order term in the expansion at maximum displacement, which is of the order of 10^{-5} for the parameters used in this analysis ($\omega = 2\pi \times 10^6$ Hz, $d = 10^{-4}$ m). This approximation could not be made for significantly lower trapping frequencies and smaller ion spacings.

II. GATE SCHEME

We analyse fast gate schemes that use a series of broadband counter-propagating π-pulses, incident on the two ions to which the gate is to be applied. These π-pulses can be simple square pulses of the appropriate height, or shaped for convenience of production or robustness of the change of state. They are always used in counter-propagating pairs so that they do not change the internal state of the ions, but give them a state-dependent momentum kick. These kicks are significant due to the Lamb-Dicke parameter, which is typically of the order of $\eta \approx 0.16$. These state-dependent kicks then have a state-dependent effect on the energy (and phase evolution) of those modes through interaction via the Coulomb potential. Correctly chosen kicks can ultimately return the motional state of the ions to their initial state, leaving the net effect of a controlled phase gate:

$$\hat{U}_{\text{CPhase}} = e^{i\frac{\pi}{4}\sigma_1^z\sigma_2^z} \tag{3}$$

A general fast gate can be described by a set of pulse timings $\vec{t}$ and pulse-group intensities $\vec{z}$. Different pulse-group intensities are generated by having different numbers of single pulses comprising a pulse-group. These single pulses then arrive at, or symmetrically around, the pulse time given for that group, hence the pulse intensities are given as as integer multiples. Multiple fast gate schemes have been proposed in the literature, such as: GZC [1], Duan [2] and FRAG [3]. Each scheme imposes a different set of restrictions on the number of distinct pulses, symmetry and different ratios of pulse numbers in pulse groups. For the FRAG scheme, the timings of theses pulses and the number of pulses in each pulse group are given by the vectors $\underline{t}$ and $\underline{z}$ respectively:

$$\begin{aligned}\underline{t} &= (-\tau_1, -\tau_2, -\tau_3, \tau_3, \tau_2, \tau_1),\\ \underline{z} &= (-n, 2n, -2n, 2n, -2n, n).\end{aligned} \tag{4}$$

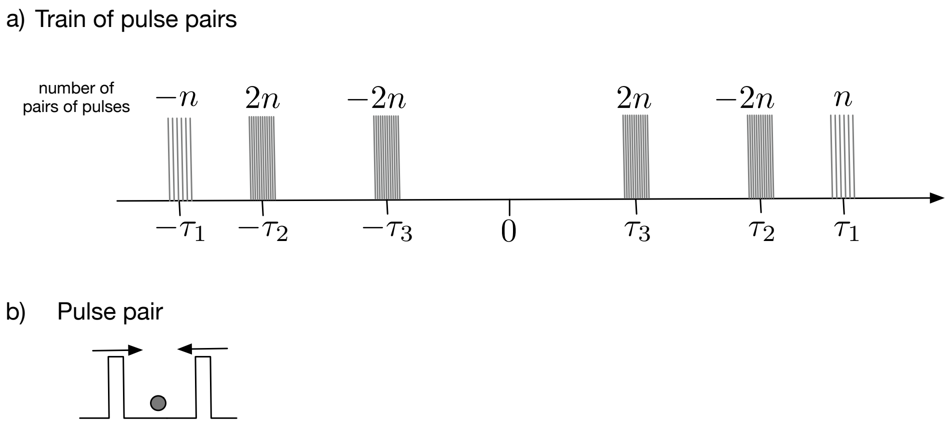

FIG. 1. a) Diagram with the pulse timing for the FRAG scheme. The components z_j of the z vector indicate the number of pairs of pulses that hit the ion at each time τ_j. The sign in z_j indicates which pulse within each pair (shown in b) reaches the ion first. This gives the sign of the momentum kick imprinted on the ion.

The sign of the components of $\underline{z}$ corresponds to changing the direction of the initially incident pulse, and the factor of n is an integer that characterizes the overall scale of numbers of pulses at each time. With an infinite repetition rate laser, To produce an effective C-Phase gate the timings (τ_1, τ_2, τ_3) are chosen to give the desired gate. In the original FRAG scheme proposal there was a strict ordering on the magnitude of (τ_1, τ_2, τ_3). In this implementation we do not impose a strict ordering of the times (τ_1, τ_2, τ_3), effectively resulting in a set of six possible pulse schemes. The total gate time is therefore twice the maximum of the values of τ_1, τ_2, and τ_3.

III. OPTIMISATION

We use numerical searches to find pulse timings that produce high quality gate operations, with the state-averaged fidelity F, given as the integral of the square of the norm of the overlap between the post-gate state with the target state integrated over all initial states. This is efficient to compute and it is strongly related to other distance measures for high-fidelity gates. As we examine fidelities extremely close to unity, we report the infidelity $1-F$. This is a function of the phase mismatch $\Delta\phi$ around the target $\pi/4$ phase, and the population changes of the motional modes, as given by:

$$\Delta\phi = |\sum_p 8\eta^2 \frac{\omega}{\omega_p} b_p^1 b_p^2 \sum_{i \neq j} z_i z_j \sin\left(\omega_n |t_i - t_j|\right) - \frac{\pi}{4}|$$

and

$$\Delta P_p = 4\eta \sqrt{\frac{\omega}{\omega_p}} \sum_k z_k \sin\left(\omega_p t_k\right)$$

where ΔP_p is the population changes of the p-th motional mode.

For efficient computation of two-ion gates, we further simplify this measure by using a truncated expansion of the infidelity in these variables:

$$1 - F \approx \frac{2}{3}(\Delta\phi)^2 + 0.8(\Delta P_1)^2 + 0.8(\Delta P_2)^2. \tag{5}$$

While this approximate form is efficient for generating gate schemes, we use the full form when reporting achievable fidelities, for example, in the presence of multiple ions.

We can see from Eq. (5) that the infidelity for a two-ion system, $1-F$, depends on the Lamb-Dicke parameter η, the angular frequencies collective motional modes ω_n, the coupling of the k-th ion to the p-th mode, b_p^k, and the number of pulses in the i-th pulse train z_i. The collective mode frequencies ω_n can be calculated from the mass of the ions M, the separation of the microtraps d, and the trapping frequency ω of the individual microtraps.

We search for pulse timings that produce optimal gate fidelity within a given time bound. This optimisation is run as a set of local gradient searches in the three-dimensional parameter space of the pulse timings, over a large set of initial gate sequences. The highest fidelity of these local optimisations is then taken to be the optimal gate for that cap in the gate time. Note that the optimal gate occasionally takes less time than the maximum allowed. By increasing the cap in total gate time and repeating this process, we map out the optimal fidelity for fast gates as a function of gate time.

IV. DIMENSIONLESS PARAMETER

We see from Eq. 5 that the system behaviour depends on the ratios of the frequencies of the collective modes. These are in turn functions of the geometry, and the dimensionless parameter $\xi = \frac{d^3\omega^2}{\alpha}$, where $\alpha = \frac{e^2}{4\pi\varepsilon_0}\frac{1}{M}$. Here e is the electron charge, M the mass of the ions, and ε_0 the vacuum permittivity. For a two-ion system, there is only one ratio, so it entirely characterises the behaviour.

We define χ as the normalised difference between the breathing mode frequency and the common motional mode frequency $\chi = \frac{\omega_{\mathrm{BR}}-\omega}{\omega}$ which can be expressed in terms of the more fundamental parameter ξ as given in Eq. 6.

$$\chi = \sqrt{\frac{1}{3}(9 - \beta\gamma^{\frac{1}{3}} + \beta\gamma^{\frac{2}{3}})} - 1 \tag{6}$$

where

$$\gamma = 1 + \frac{3(9 + \sqrt{3}\sqrt{27 + 2\xi})}{\xi}$$

and

$$\beta = 9 - \sqrt{3}\sqrt{27 + 2\xi}$$

Even for three or more ions, the system is still well characterised by χ, which is the normalised gap to the lowest energy excitation in the system. This is because it defines the rate of relative acquisition of phase between the excited and unexcited modes. Its value lies in the range between 0 and $\sqrt{3} - 1$. The upper bound corresponds to the limit where both microtraps are merged, which is the case for standard linear trap geometries.

We also observe a factor of n^2 appearing in all fidelity terms, as z_i scales with n. Assuming a fixed value for η we then have that, together, n and χ give a full description of $\Delta\phi$ and ΔP_n in conjunction with the gate times. Therefore, they completely specify the optimal infidelities as a function of the dimensionless gate time τ_G expressed in trap periods $\tau_G = \frac{\omega t_G}{2\pi}$.

[1] J. J. Garcia-Ripoll, P. Zoller, and J. I. Cirac, Physical Review Letters **91**, 157901 (2003).
[2] L. M. Duan, Physical Review Letters **93**, 100502 (2004).
[3] C. D. B. Bentley, A. R. R. Carvalho, and J. J. Hope, New Journal of Physics **17** (2015), 10.1088/1367-2630/17/10/103025.