#### 1

# Digital Signal Processing

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#### 1 Software Installation

Run the following commands

sudo apt-get update sudo apt-get install libffi-dev libsndfile1 python3 -scipy python3-numpy python3-matplotlib sudo pip install cffi pysoundfile

### 2 DIGITAL FILTER

2.1 Download the sound file from

wget https://github.com/HARI-donk-EY/ sig\_pros/blob/main/codes/2/sound\_files/ Sound\_Noise.wav

- 2.2 You will find a spectrogram at https://academo.org/demos/spectrum-analyzer.
  - Upload the sound file that you downloaded in Problem 2.1 in the spectrogram and play. Observe the spectrogram. What do you find? **Solution:** There are a lot of yellow lines between 440 Hz to 5.1 KHz. These represent the synthesizer key tones. Also, the key strokes are audible along with background noise.
- 2.3 Write the python code for removal of out of band noise and execute the code.

**Solution:** 

```
import soundfile as sf
from scipy import signal
#read .wav file
input signal, fs = sf.read('sound files/
    Sound Noise.wav')
#sampling frequency of Input signal
sampl freq=fs
#order of the filter
order=4
#cutoff frquency 4kHz
cutoff freq=4000.0
#digital frequency
Wn=2*cutoff freq/sampl freq
# b and a are numerator and
    denominator polynomials respectively
b, a = signal.butter(order, Wn, 'low')
#filter the input signal with butterworth filter
output signal = signal.filtfilt(b, a,
    input signal)
\#output\ signal = signal.lfilter(b,\ a,\ input)
    signal)
#write the output signal into .wav file
sf.write('sound files/
    Sound With ReducedNoise.wav',
    output signal, fs)
```

2.4 The output of the python script in Problem 2.3 is the audio file Sound With ReducedNoise.wav.

Play the file in the spectrogram in Problem 2.2. What do you observe?

**Solution:** The key strokes as well as background noise is subdued in the audio. Also, the signal is blank for frequencies above 5.1 kHz.

## 3 Difference Equation

3.1 Let

$$x(n) = \left\{ 1, 2, 3, 4, 2, 1 \right\} \tag{3.1}$$

Sketch x(n).

3.2 Let

$$y(n) + \frac{1}{2}y(n-1) = x(n) + x(n-2),$$
  
$$y(n) = 0, n < 0 \quad (3.2)$$

Sketch y(n).

**Solution:** The following code yields Fig. 3.3.

wget https://github.com/HARI-donk-EY/sig pros/blob/main/codes/3/xnyn.py

3.3 Repeat the above exercise using C code.

**Solution:** The following codes can be used, the C-code used to generate y(n) and the plot yields the same Figure as that of 3.3

wget https://github.com/HARI-donk-EY/sig\_pros/blob/main/codes/3/xnyn.c

wget https://github.com/HARI-donk-EY/sig pros/blob/main/codes/3/xnyn2.py

## 4 Z-Transform

4.1 The Z-transform of x(n) is defined as

$$X(z) = \mathcal{Z}\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n)z^{-n}$$
 (4.1)

Show that

$$Z{x(n-1)} = z^{-1}X(z)$$
 (4.2)

and find

$$\mathcal{Z}\{x(n-k)\}\tag{4.3}$$

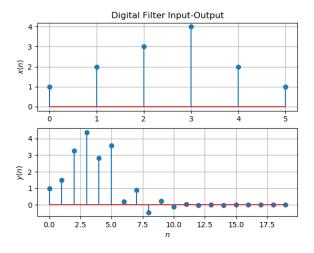


Fig. 3.3

**Solution:** From (4.1),

$$Z\{x(n-1)\} = \sum_{n=-\infty}^{\infty} x(n-1)z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} x(n)z^{-n-1} = z^{-1} \sum_{n=-\infty}^{\infty} x(n)z^{-n}$$
(4.4)
$$(4.5)$$

resulting in (4.2). Similarly, it can be shown that

$$\mathcal{Z}\{x(n-k)\} = z^{-k}X(z) \tag{4.6}$$

4.2 Obtain X(z) for x(n) defined from 3.1.

**Solution:** From 3.1,

$$x(n) = \left\{ 1, 2, 3, 4, 2, 1 \right\}$$

and from (4.1), and for given x(n) we have,

$$X(n) = \mathcal{Z}\{x(n)\}$$

$$= 1 + 2z^{-1} + 3z^{-2} + 4z^{-3} + 2z^{-4} + z^{-5}$$

$$(4.8)$$

4.3 Find

$$H(z) = \frac{Y(z)}{X(z)} \tag{4.9}$$

from (3.2) assuming that the Z-transform is a linear operation.

**Solution:** Applying (4.6) in (3.2),

$$Y(z) + \frac{1}{2}z^{-1}Y(z) = X(z) + z^{-2}X(z)$$
 (4.10)

$$\implies \frac{Y(z)}{X(z)} = \frac{1 + z^{-2}}{1 + \frac{1}{2}z^{-1}} \tag{4.11}$$

4.4 Find the Z transform of

$$\delta(n) = \begin{cases} 1 & n = 0 \\ 0 & \text{otherwise} \end{cases}$$
 (4.12)

and show that the Z-transform of

$$u(n) = \begin{cases} 1 & n \ge 0 \\ 0 & \text{otherwise} \end{cases}$$
 (4.13)

is

$$U(z) = \frac{1}{1 - z^{-1}}, \quad |z| > 1 \tag{4.14}$$

**Solution:** It is easy to show that

$$\delta(n) \stackrel{\mathcal{Z}}{\rightleftharpoons} 1 \tag{4.15}$$

and from (4.13),

$$U(z) = \sum_{n=0}^{\infty} z^{-n}$$
 (4.16)

$$=\frac{1}{1-z^{-1}}, \quad |z| > 1 \tag{4.17}$$

using the fomula for the sum of an infinite geometric progression.

#### 4.5 Show that

$$a^n u(n) \stackrel{\mathcal{Z}}{\rightleftharpoons} \frac{1}{1 - az^{-1}} \quad |z| > |a| \tag{4.18}$$

**Solution:** From (4.13), we get,

$$a^{n}u(n) = \begin{cases} a^{n} & n \ge 0\\ 0 & \text{otherwise} \end{cases}$$
 (4.19)

from above,

$$U_a(z) = \sum_{n=0}^{\infty} a^n z^{-n}$$
 (4.20)

$$= \sum_{n=0}^{\infty} (az^{-1})^n = \frac{1}{1 - az^{-1}}, \quad |z| > 1$$
(4.21)

using the fomula for the sum of an infinite

geometric progression.

4.6 Let

$$H(e^{j\omega}) = H(z = e^{j\omega}).$$
 (4.22)

Plot  $|H(e^{j\omega})|$ . Comment.  $H(e^{j\omega})$  is known as the *Discret Time Fourier Transform* (DTFT) of x(n).

**Solution:** The following code plots Fig. 4.6.

wget https://github.com/HARI-donk-EY/sig\_pros/blob/main/codes/4/dtft.py

We can note that the graph in Fig. 4.6 is an even periodic function.

From the equations and we get,

$$|H(e^{j\omega})| = \left| \frac{1 + e^{-2j\omega}}{1 + \frac{1}{2}e^{-j\omega}} \right|$$
(4.23)  
$$= \left| \frac{1 + \cos(-2\omega) + j\sin(-2\omega)}{1 + \frac{1}{2}(\cos(-\omega) + j\sin(-\omega))} \right|$$
(4.24)  
$$= \sqrt{\frac{(1 + \cos 2\omega)^2 + (\sin 2\omega)^2}{\left(1 + \frac{1}{2}\cos \omega\right)^2 + \left(\frac{1}{2}\sin \omega\right)^2}}$$
(4.25)

$$=\sqrt{\frac{2(1+\cos 2\omega)}{\frac{5}{4}+\cos \omega}}\tag{4.26}$$

$$=\sqrt{\frac{2(2\cos^2\omega)}{\frac{5}{4}+\cos\omega}}\tag{4.27}$$

$$=\frac{4|\cos\omega|}{\sqrt{5+4\cos\omega}}\tag{4.28}$$

and so its fundamental period is  $2\pi$ .

4.7 Express h(n) in terms of  $H(e^{jw})$ .

**Solution:** h(n) is given by the inverse DTFT (IDTFT) of  $H(e^{j\omega})$ 

$$h(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H(e^{j\omega}) e^{j\omega n} d\omega \qquad (4.29)$$

We can prove this from (4.1),

$$H(e^{jw}) = \sum_{k=-\infty}^{\infty} h(k)e^{-j\omega k}$$
 (4.30)

(4.31)

multiplying both sides with  $e^{-j\omega k}$  and

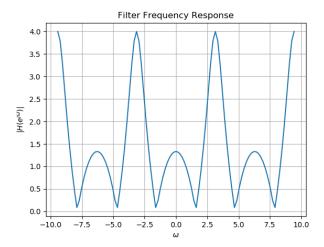


Fig. 4.6:  $|H(e^{j\omega})|$ 

integrating w.r.t.  $\omega$ .

$$\int_{-\pi}^{\pi} H(e^{jw})e^{j\omega k} d\omega = \sum_{k=-\infty}^{\infty} h(k) \int_{-\pi}^{\pi} e^{-j\omega n} e^{j\omega k} d\omega$$

$$= \sum_{k=-\infty}^{\infty} h(k) \int_{-\pi}^{\pi} e^{-j\omega(n-k)} d\omega$$

$$(4.33)$$

we know that  $e^{-j\omega(n-k)}$  is an odd function so integration would be zero for all values of  $\omega(n-k) \neq 0$ , but 1 if  $\omega(n-k) = 0$ .

$$\int_{-\pi}^{\pi} e^{-j\omega(n-k)} d\omega = \begin{cases} 0 & if n-k \neq 0 \\ 1 & if n-k = 0 \end{cases}$$

From this we can complete the summation.

$$\int_{-\pi}^{\pi} H(e^{jw})e^{j\omega k}d\omega = \sum_{k=-\infty}^{\infty} h(k) \int_{-\pi}^{\pi} e^{-j\omega(n-k)}d\omega$$

$$= h(n) \int_{-\pi}^{\pi} d\omega \qquad (4.34)$$

$$= h(n)2\pi \qquad (4.36)$$

$$h(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H(e^{jw})e^{j\omega k}d\omega \qquad (4.37)$$

Hence proved.

5 IMPULSE RESPONSE

5.1 Using long division, find

$$h(n), \quad n < 5$$
 (5.1)

for H(z) in (4.6).

**Solution:** 

$$H(z) = \frac{1 + z^{-2}}{1 + \frac{1}{2}z^{-1}}$$
 (5.2)

Substitute  $z^{-1} = x$ 

So,

$$H(z) = -4 + 2z^{-1} + \frac{5}{1 + \frac{1}{2}z^{-1}}$$

$$= -4 + 2z^{-1} + 5\sum_{n=0}^{\infty} \left(\frac{-1}{2}\right)^n z^{-n}$$

$$= 1 - \frac{1}{2}z^{-1} + 5\sum_{n=2}^{\infty} \left(\frac{-1}{2}\right)^n z^{-n}$$

$$= \sum_{n=0}^{\infty} \left(\frac{-1}{2}\right)^n z^{-n} + 4\sum_{n=2}^{\infty} \left(\frac{-1}{2}\right)^n z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} u(n) \left(\frac{-1}{2}\right)^n z^{-n} + \sum_{n=-\infty}^{\infty} u(n-2) \left(\frac{-1}{2}\right)^{n-2} z^{-n}$$

$$(5.7)$$

Therefore from (4.1),

$$h(n) = \left(\frac{-1}{2}\right)^n u(n) + \left(\frac{-1}{2}\right)^{n-2} u(n-2)$$
 (5.8)

5.2 Find an expression for h(n) using H(z), given that

$$h(n) \stackrel{\mathcal{Z}}{\rightleftharpoons} H(z) \tag{5.9}$$

and there is a one to one relationship between h(n) and H(z). h(n) is known as the *impulse response* of the system defined by (3.2).

Take H(z) from (4.6)

**Solution:** From (4.6),

$$H(z) = \frac{1}{1 + \frac{1}{2}z^{-1}} + \frac{z^{-2}}{1 + \frac{1}{2}z^{-1}}$$
 (5.10)

$$\implies h(n) = \left(-\frac{1}{2}\right)^n u(n) + \left(-\frac{1}{2}\right)^{n-2} u(n-2)$$
(5.1)

using (4.18) and (4.6). Let,

$$h(n) = h_1(n) + h_2(n)$$
 (5.12)

Where,

$$h_1(n) = \left(-\frac{1}{2}\right)^n u(n) \tag{5.13}$$

$$h_1(n) = \left(-\frac{1}{2}\right)^{n-2} u(n-2) \tag{5.14}$$

Then ROC of  $h_1(n)$  and  $h_2(n)$  will both be  $|z| > |-\frac{1}{2}|$ .

Hence ROC of H(z) is  $|z| > \frac{1}{2}$ 

5.3 Sketch h(n). Is it bounded? Justify theoretically.

**Solution:** The following code plots Fig. 5.3.

wget https://github.com/HARI-donk-EY/sig pros/blob/main/codes/5/hn.py

We know,

$$h(n) = \left(-\frac{1}{2}\right)^n u(n) + \left(-\frac{1}{2}\right)^{n-2} u(n-2)$$

then,

$$\lim_{n\to\infty}h(n)=\lim_{n\to\infty}\left(-\frac{1}{2}\right)^nu(n)+\lim_{n\to\infty}\left(-\frac{1}{2}\right)^{n-2}u(n-2)$$

(5.15)

$$h(n) = \begin{cases} 5\left(\frac{-1}{2}\right)^n & n \ge 2\\ \left(\frac{-1}{2}\right)^n & 0 \le n < 2\\ 0 & n < 0 \end{cases}$$
 (5.16)

5.4 Is h(n) convergent? Justify using ratio test.

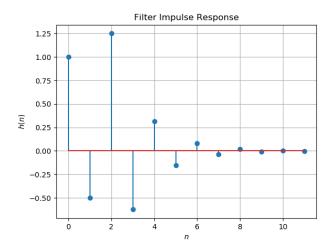


Fig. 5.3: h(n) as the inverse of H(z)

### **Solution:**

$$h(n) = \left(-\frac{1}{2}\right)^n u(n) + \left(-\frac{1}{2}\right)^{n-2} u(n-2) \quad (5.17)$$

If  $L = \lim_{n \to \infty} \left| \frac{h(n+1)}{h(n)} \right| < 1$ , then h(n) is convergent.

$$L = \lim_{n \to \infty} \left| \frac{h(n+1)}{h(n)} \right|$$

$$= \lim_{n \to \infty} \left| \frac{\left(-\frac{1}{2}\right)^{n+1} u(n+1) + \left(-\frac{1}{2}\right)^{n-1} u(n-1)}{\left(-\frac{1}{2}\right)^{n} u(n) + \left(-\frac{1}{2}\right)^{n-2} u(n-2)} \right|$$
(5.18)

$$= \left| \frac{-\frac{1}{2} + \left(-\frac{1}{2}\right)^{-1}}{1 + \left(-\frac{1}{2}\right)^{-2}} \right| \tag{5.20}$$

$$= \left| \frac{-\frac{5}{2}}{5} \right| \tag{5.21}$$

$$=\frac{1}{2}$$
 (5.22)

Since L < 1, we can say that h(n) is convergent.

5.5 The system with h(n) is defined to be stable if

$$\sum_{n=-\infty}^{\infty} h(n) < \infty \tag{5.23}$$

Is the system defined by (3.2) stable for the impulse response in (5.9)?

#### **Solution:**

$$\sum_{n=-\infty}^{\infty} h(n) = \sum_{n=-\infty}^{\infty} \left( \left( -\frac{1}{2} \right)^n u(n) + \left( -\frac{1}{2} \right)^{n-2} u(n-2) \right)$$

$$= \sum_{n=0}^{\infty} \left( -\frac{1}{2} \right)^n u(n) + \sum_{(n-2)=0}^{\infty} \left( -\frac{1}{2} \right)^{n-2} u(n-2)$$

$$= 2 \left( \frac{1}{1+\frac{1}{2}} \right)$$

$$= \frac{4}{2}$$

$$(5.27)$$

As  $\sum_{n=-\infty}^{\infty} h(n) = \frac{4}{3}$  is less than  $\infty$ , the system defined by (3.2) is stable for the impulse response in (5.9).

5.6 Verify the above result using a python code. **Solution:** The following code determines if it is convergent or not:

wget https://github.com/HARI-donk-EY/sig pros/blob/main/codes/5/new.py

5.7 Compute and sketch h(n) using

$$h(n) + \frac{1}{2}h(n-1) = \delta(n) + \delta(n-2),$$
 (5.28)

This is the definition of h(n).

**Solution:** The following code plots Fig. 5.7. Note that this is the same as Fig. 5.3.

wget https://github.com/HARI-donk-EY/sig\_pros/blob/main/codes/5/hndef.py

Computing,

$$h(0) = 1$$

$$h(1) = -\frac{1}{2}h(0)$$

$$h(2) = -\frac{1}{2}h(1) + 1$$

$$Parallely, h(n) = -\frac{1}{2}h(n-1)$$

5.8 Compute

$$y(n) = x(n) * h(n) = \sum_{n=-\infty}^{\infty} x(k)h(n-k)$$
 (5.29)

Comment. The operation in (5.29) is known as *convolution*.

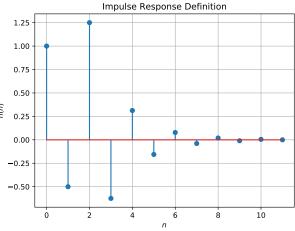


Fig. 5.7: h(n) from the definition

**Solution:** The following code plots Fig. 5.8. Note that this is the same as y(n) in Fig. 3.3.

wget https://github.com/HARI-donk-EY/ sig pros/blob/main/codes/5/ynconv.py

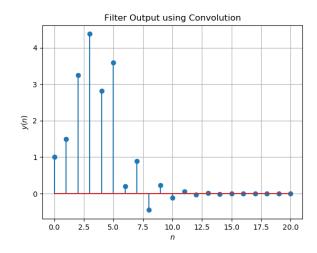


Fig. 5.8: y(n) from the definition of convolution

5.9 Express the above convolution using a toeplitz

matrix. Solution:

$$\mathbf{y} = \mathbf{x} \circledast \mathbf{h} \tag{5.30}$$

$$\mathbf{y} = \begin{pmatrix} h_1 & 0 & \cdot & \cdot & \cdot & 0 \\ h_2 & h_1 & \cdot & \cdot & \cdot & 0 \\ h_3 & h_2 & h_1 & \cdot & \cdot & 0 \\ h_{m-1} & \cdot & \cdot & \cdot & h_2 & h_1 \\ h_m & h_{m-1} & \cdot & \cdot & \cdot & h_2 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & \cdot & h_m \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{pmatrix}$$
(5.31)

And this is what we got in (5.29)

## 5.10 Show that

$$y(n) = \sum_{n = -\infty}^{\infty} x(n - k)h(k)$$
 (5.33)

**Solution:** From (5.29), we substitute k := n - k to get

$$y(n) = \sum_{k=-\infty}^{\infty} x(k) h(n-k)$$
 (5.34)

$$=\sum_{n-k=-\infty}^{\infty}x\left(n-k\right)h\left(k\right)\tag{5.35}$$

$$=\sum_{k=-\infty}^{\infty}x(n-k)h(k)$$
 (5.36)

6 DFT

## 6.1 Compute

$$X(k) \stackrel{\triangle}{=} \sum_{n=0}^{N-1} x(n)e^{-j2\pi kn/N}, \quad k = 0, 1, \dots, N-1$$
(6.1)

and H(k) using h(n).

**Solution:** The following code generates the data required for plotting.

wget https://github.com/HARI-donk-EY/sig\_pros/blob/main/codes/6/header.h wget https://github.com/HARI-donk-EY/sig\_pros/blob/main/codes/6/6.1/XkHk\_dat.c

The following code plots Fig. 6.1.

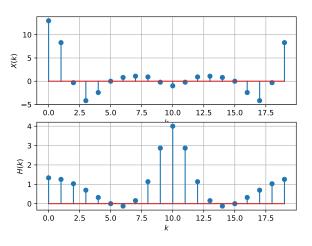


Fig. 6.1: X(k), H(k) from the DFT

wget https://github.com/HARI-donk-EY/sig pros/blob/main/codes/6/6.1/XkHk.py

## 6.2 Compute

$$Y(k) = X(k)H(k) \tag{6.2}$$

**Solution:** The following code plots Fig. 6.2. The data can be obtained from

wget https://github.com/HARI-donk-EY/ sig\_pros/blob/main/codes/6/header.h wget https://github.com/HARI-donk-EY/ sig\_pros/blob/main/codes/6/6.2/y(k)dat.c

The plotting is done by

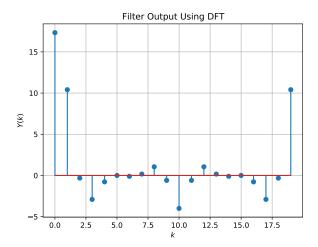


Fig. 6.2: Y(k) from the DFT

wget https://github.com/HARI-donk-EY/sig\_pros/blob/main/codes/6/6.2/yk.py

## 6.3 Compute

$$y(n) = \frac{1}{N} \sum_{k=0}^{N-1} Y(k) \cdot e^{j2\pi kn/N}, \quad n = 0, 1, \dots, N-1$$
(6.3)

**Solution:** The following code plots Fig. ??. Note that this is the same as y(n) in Fig.