PINGALA ASSIGNMENTS

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AI21BTECH11014

Contents

1 JEE 2019

Let α and β $(\alpha > \beta)$ be the roots of the equation $z^2 - z - 1 = 0$. Define,

$$a_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad n \geq 1 \tag{1.1}$$

$$b_n = a_{n-1} - a_{n+1}, \quad n \geq 2, \quad b_1 = 1 \tag{1.2}$$

Verify the following using a python code.

1.1
$$\sum_{k=1}^{n} a_k = a_{n+2} - 1, \quad n \geq 1 \tag{1.3}$$

Solution:
Download the Python code using

```
$ wget https://https://github.com/HARI−donk
    −EY/sig_pros/tree/main/pingala/codes/1
    _1.py
```

and run it using,

```
$ python3 1_1.py
```

From Fig. 1.1, both the graphs are similar for *LHS* and *RHS*.
Hence 1.1 is true.

1.2
$$\sum_{k=1}^{\infty} \frac{a_k}{10^k} = \frac{10}{89} \tag{1.4}$$

Solution: Download the Python code using

```
$ wget https://https://github.com/HARI−donk
    −EY/sig_pros/tree/main/pingala/codes/1
    _2.py
```

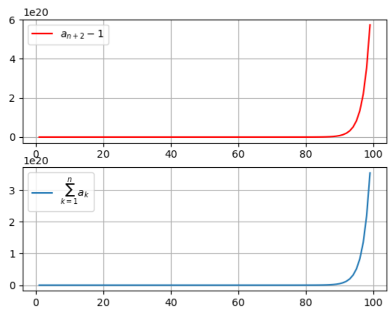

Fig. 1.1

and run it using,

```
$ python3 1_2.py
```

The Fig. 1.2 shoes that the difference between *LHS* and *RHS* tens to zero as the value of k increases.
It shows that for a large value of k, the

$$LHS \rightarrow RHS$$

Hence 1.2 is true.

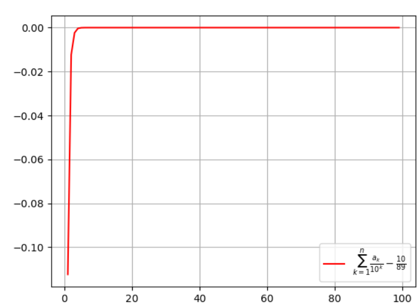

Fig. 1.2

1.3
$$b_n = \alpha^n + \beta^n, \quad n \geq 1 \tag{1.5}$$

Solution: Download the Python code using

```
$ wget https://https://github.com/HARI−donk
    −EY/sig_pros/tree/main/pingala/codes/1
    _3.py
```

and run it using,

```
$ python3 1_3.py
```

From Fig. 1.3, both the graphs are similar for *LHS* and *RHS*.
Hence 1.3 is true.

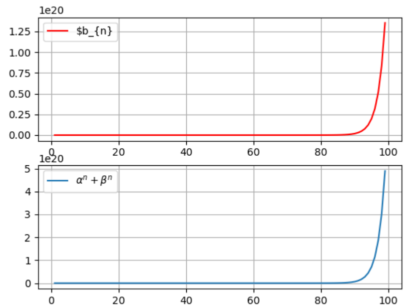

Fig. 1.3

1.4

$$\sum_{k=1}^{\infty} \frac{b_k}{10^k} = \frac{8}{89} \tag{1.6}$$

Solution:
Download the Python code using

```
$ wget https://https://github.com/HARI−donk
    −EY/sig_pros/tree/main/pingala/codes/1
    _4.py
```

and run it using,

```
$ python3 1_4.py
```

The Fig. 1.4 shows that the difference between *LHS* and *RHS* tends to $\frac{12}{89}$ as the value of k increases.
It shows that for a large value of k, the

$$LHS \nrightarrow RHS$$

Hence 1.4 is false.

2 Pingala Series

2.1 The *one sided* Z-transform of $x(n)$ is defined as

$$X^{+}(z) = \sum_{n=0}^{\infty} x(n) z^{-n}, \quad z \in \mathbb{C} \tag{2.1}$$

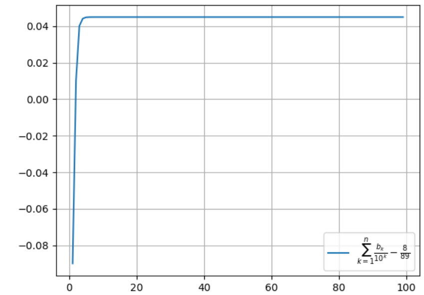

Fig. 1.4

2.2 The *Pingala* series is generated using the difference equation

$$x(n+2) = x(n+1) + x(n) \tag{2.2}$$
$$x(0) = x(1) = 1, \; n \geq 0 \tag{2.3}$$

Generate a stem plot for $x(n)$.

Solution:
Obtain the python code to generate the plot using

```
$ wget https://github.com/HARI−donk−EY/
    sig_pros/tree/main/pingala/codes/2_2.py
```

Run the code using

```
$ python3 2_2.py
```

The following Fig. 2.2 is obtained

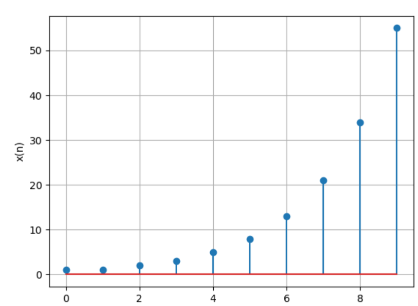

Fig. 2.2

2.3 Find $X^{+}(z)$.
Solution:

$$x(n+2) = x(n+1) + x(n) \tag{2.4}$$

Applying positive Z-transform on both sides as w know that Z-transform is a linear operator.

$$\sum_{k=0}^{\infty} x(k+2)z^{-k} = \sum_{k=0}^{\infty} x(k+1) + \sum_{k=0}^{\infty} x(k) \tag{2.5}$$

$$z^2\left(X^+(z) - x(0) - x(1)\right) = X^+(z) + z\left(X^+(z) - x(0)\right) \tag{2.6}$$

$$X^+(z) = \frac{z^2}{z^2 - z - 1} \tag{2.7}$$

$$X^+(z) = \frac{1}{1 - z^{-1} - z^{-2}} \tag{2.8}$$

2.4 Find $x(n)$.
Solution:

$$X^+(z) = \frac{1}{(1-\alpha z)(1-\beta z)} \tag{2.9}$$

where α, β are the roots of the equation

$$z^2 - z - 1 = 0 \tag{2.10}$$

Co-efficient os z^{-k} in the above expresson is $x(k)$, so by comparing co-efficients.

$$X^+(z) = \frac{1}{\alpha - \beta}\left(\frac{\alpha}{1-\alpha z^{-1}} - \frac{\beta}{1-\beta z^{-1}}\right) \tag{2.11}$$

Using binomial theorem, we get

$$x(k) = \frac{\alpha^{k+1} - \beta^{k+1}}{\alpha - \beta} \tag{2.12}$$

2.5 Sketch

$$y(n) = x(n-1) + x(n+1), \quad n \geq 0 \tag{2.13}$$

Solution:
Obtain the python code to generate the plot using

```
$ wget https://github.com/HARI−donk−EY/
    sig_pros/tree/main/pingala/codes/2_5.py
```

Run the code using

```
$ python3 2_5.py
```

The following Fig. 2.5 is obtained

2.6 Find $Y^+(z)$. **Solution:**

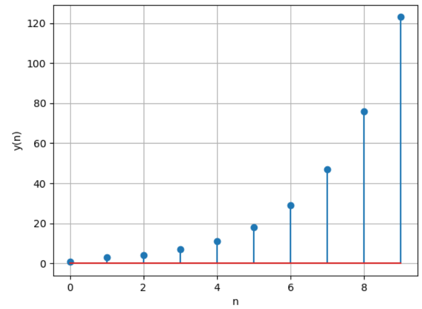

Fig. 2.5

Take +ve Z-transform on both sides of (2.13).

$$\sum_{k=0}^{\infty} y(k)z^{-k} = \sum_{k=0}^{\infty} x(k+1)z^{-k} + \sum_{k=0}^{\infty} z^{-k} \tag{2.14}$$

$$Y^+(z) = z\left(X^+(z) - x(0)\right) + z^{-1}X^+(z) \tag{2.15}$$

$\because x(-1) = 0$

$$Y^+(z) = \frac{z + z^{-1}}{1 - z^{-1} - z^{-2}} - z \tag{2.16}$$

$$\therefore Y^+(z) = \frac{1 + 2z^{-1}}{1 - z^{-1} - z^{-2}} \tag{2.17}$$

2.7 Find $y(n)$.
Solution:
Co-efficient of z^{-n} in $Y^+(z)$ will be $y(n)$.

$$Y^+(z) = \frac{1}{1 - z^{-1} - z^{-2}} + \frac{2z^{-2}}{1 - z^{-1} - z^{-2}} \tag{2.18}$$

$$y(k) = \frac{\alpha^{k+1} - \beta^{k+1}}{\alpha - \beta} + 2\frac{\alpha^k - \beta^k}{\alpha - \beta} \tag{2.19}$$

$$y(k) = \frac{\alpha^{k+2} + \alpha^k - \beta^k - \beta^{k+2}}{\alpha - \beta} \tag{2.20}$$

$$y(k) = \frac{\alpha^{k+2} - \beta\alpha^{k+1} + \alpha\beta^{k+1} - \beta^{k+2}}{\alpha - \beta} \tag{2.21}$$

$[\because \alpha\beta = -1]$

$$\therefore y(k) = \alpha^{k+1} + \beta^{k+1} \tag{2.22}$$

3.1 Show that.

$$\sum_{k=1}^{n} a_k = \sum_{k=0}^{n-1} = x(n) * u(n-1) \tag{3.1}$$

Solution:

$$x(k) = a(k+1) \tag{3.2}$$

$$\implies \sum_{k=0}^{n-1} x(k) = \sum_{k=0}^{n-1} a(k+1) \tag{3.3}$$

$$\implies \sum_{k=1}^{n} a_k = \sum_{k=0}^{n-1} x(k) \tag{3.4}$$

$$x(n) * u(n-1) = \sum_{k=-\infty}^{\infty} x(k)u(n-k-1) \tag{3.5}$$

$$u(n-k-1) = \begin{cases} 0 & k < n-1 \\ 1 & k \geq n-1 \end{cases} \tag{3.6}$$

$$x(k) = 0, \quad \forall k < 0 \tag{3.7}$$

$$\therefore \sum_{k=0}^{n-1} x(k) = x(n) * u(n-1) \tag{3.8}$$

3.2 show that,

$$a_{n+2} - 1, \quad n \geq 1 \tag{3.9}$$

can be expressed as,

$$[x(n+1) - 1]u(n) \tag{3.10}$$

Solution:

$$x(k) = a(k+1) \tag{3.11}$$

$$\implies xk+1) = a(k+2) \tag{3.12}$$

$$a(k+2) - 1 = x(k+1) - 1 \tag{3.13}$$

$$\therefore a(k+2) - 1 = [x(k+2) - 1]u(k) \tag{3.14}$$

$$[\because \forall \quad n \geq 1] \tag{3.15}$$

3.3 show that,

$$\sum_{k=1}^{\infty} \frac{a_k}{10^k} = \frac{1}{10} \sum_{x(k)}^{10^k} = \frac{1}{10} X^+(10) \tag{3.16}$$

Solution:

$$X^+(z) = \sum_{k=0}^{\infty} x(k) z^{-k} = z \sum_{k=1}^{\infty} a(k) z^{-k} \tag{3.17}$$

$$z = 10 \tag{3.18}$$

$$\implies 10 \sum_{k=1}^{\infty} \frac{a_k}{10^k} = \sum_{k=0}^{\infty} \frac{x(k)}{10^k} \tag{3.19}$$

$$= X^+(10) \tag{3.20}$$

$$\sum_{k=1}^{\infty} \frac{a_k}{10^k} = \frac{1}{10} \sum_{k=0}^{\infty} \frac{x(k)}{10^k} \tag{3.21}$$

$$\therefore \sum_{k=1}^{\infty} \frac{a_k}{10^k} = \frac{1}{10} X^+(10) \tag{3.22}$$

3.4 Show that,

$$\alpha^n + \beta^n, \quad n \geq 1 \tag{3.23}$$

can be expressed as

$$w(n) = \left(\alpha^{n+1} + \beta^{n+1}\right) u(n) \tag{3.24}$$

and find $W(z)$.

Solution:

Applying Z-transform on both sides,

$$W(z) = \sum_{n=-\infty}^{\infty} \left(\alpha^{n+1} + \beta^{n+1}\right) u(n) z^{-n} \tag{3.25}$$

$$= \sum_{n=0}^{\infty} \left(\alpha^{n+1} + \beta^{n+1}\right) z^{-n} \tag{3.26}$$

$$= \alpha \sum_{n=0}^{\infty} (\alpha z^{-1})^n + \beta \sum_{n=0}^{\infty} (\beta z^{-1})^n \tag{3.27}$$

ROC: $\quad |z| > \max(\alpha, \beta)$

$$= \frac{\alpha}{1 - \alpha z^{-1}} + \frac{\beta}{1 - \beta z^{-1}} \tag{3.28}$$

$$= \frac{1 + 2z^{-1}}{1 - z^{-1} - z^{-2}} \tag{3.29}$$

3.5 Show that

$$\sum_{k=1}^{\infty} \frac{b_k}{10^k} = \frac{1}{10} \sum_{k=0}^{\infty} \frac{y(k)}{10^k} = \frac{1}{10} Y^+(10) \tag{3.30}$$

Solution:

$$y(k) = b(k+1) \tag{3.31}$$

$$\sum_{k=0}^{\infty} y(k) z^{-k} = \sum_{k=0}^{\infty} b(k+1) z^{-k} \tag{3.32}$$

$$= Y^{+}(z) \tag{3.33}$$

$$\sum_{k=0}^{\infty} y(k) z^{-k} = z \sum_{k=1}^{\infty} b(k) z^{-k} \tag{3.34}$$

$$= Y^{+}(z) \tag{3.35}$$

$$\text{Assume:} \quad z = 10 \tag{3.36}$$

$$\therefore \sum_{k=1}^{\infty} \frac{b_k}{10^k} = \frac{1}{10} \sum_{k=0}^{\infty} \frac{y(k)}{10^k} \tag{3.37}$$

$$= \frac{1}{10} Y^{+}(10) \tag{3.38}$$

3.6 Solve the JEE 2019 problem.
Solution:
1.2

$$X^{+}(z) = z \sum_{k=1}^{\infty} a(k) z^{-k} \tag{3.39}$$

$$= \frac{1}{1 - z^{-1} - z^{-2}} \tag{3.40}$$

$$\text{Assume:} \quad z = 10 \tag{3.41}$$

$$\sum_{k=1}^{\infty} \frac{a_k}{10^k} = \frac{1}{10\left(1 - \frac{1}{10} - \frac{1}{100}\right)} \tag{3.42}$$

$$= \frac{10}{89} \tag{3.43}$$

1.3

$$y(k) = \alpha^{k+1} + \beta^{k+1} \tag{3.44}$$

$$y(k) = b(k+1) \tag{3.45}$$

$$\implies b(k) = \alpha^{k} + \beta^{k} \tag{3.46}$$

1.4

$$\sum_{k=1}^{\infty} \frac{b_k}{10^k} = \frac{1}{10} Y^{+}(10) \tag{3.47}$$

$$= \frac{1}{10} \left[\frac{1 + \frac{2}{10}}{1 - \frac{1}{10} - \frac{1}{100}} \right] \tag{3.48}$$

$$\because Y^{+}(z) = \frac{1 + 2z^{-1}}{1 - z^{-1} - z^{-2}} \tag{3.49}$$

$$= \frac{12}{89} \tag{3.50}$$

Run the following code to get the expressions of $x(n)$ and $y(n)$

```
$ https://github.com/HARI−donk−EY/
    sig_pros/tree/main/pingala/codes/Xz.py
```

Use the following command in the terminal to run the code

```
$ python3 Xz.py
```