

# Digital Signal Processing

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## 1 SOFTWARE INSTALLATION

Run the following commands

```
sudo apt-get update
sudo apt-get install libffi-dev libsndfile1 python3
    -scipy python3-numpy python3-matplotlib
sudo pip install cffi pysoundfile
```

## 2 DIGITAL FILTER

### 2.1 Download the sound file from

```
wget https://github.com/HARI-donk-EY/
    sig_pros/blob/main/codes/2/sound_files/
    Sound_Noise.wav
```

### 2.2 You will find a spectrogram at <https://academo.org/demos/spectrum-analyzer>.

Upload the sound file that you downloaded in Problem 2.1 in the spectrogram and play. Observe the spectrogram. What do you find?

**Solution:** There are a lot of yellow lines between 440 Hz to 5.1 KHz. These represent the synthesizer key tones. Also, the key strokes are audible along with background noise.

### 2.3 Write the python code for removal of out of band noise and execute the code.

**Solution:**

```
import soundfile as sf
from scipy import signal

#read .wav file

input_signal,fs = sf.read('sound_files/
    Sound_Noise.wav')

#sampling frequency of Input signal

sampl_freq=fs

#order of the filter

order=4

#cutoff frequency 4kHz

cutoff_freq=4000.0

#digital frequency

Wn=2*cutoff_freq/sampl_freq

# b and a are numerator and
    denominatorpolynomials respectively

b, a = signal.butter(order,Wn, 'low')

#filter the input signal with butterworth filter

output_signal = signal.filtfilt(b, a,
    input_signal)

#output signal = signal.lfilter(b, a, input
    signal)
#write the output signal into .wav file

sf.write('sound_files/
    Sound_With_ReducedNoise.wav',
    output_signal, fs)
```

2.4 The output of the python script in Problem 2.3 is the audio file Sound\_With\_ReducedNoise.wav.

Play the file in the spectrogram in Problem 2.2. What do you observe?

**Solution:** The key strokes as well as background noise is subdued in the audio. Also, the signal is blank for frequencies above 5.1 kHz.

### 3 DIFFERENCE EQUATION

3.1 Let

$$x(n) = \left\{ \underset{\uparrow}{1}, 2, 3, 4, 2, 1 \right\} \quad (3.1)$$

Sketch  $x(n)$ .

3.2 Let

$$y(n) + \frac{1}{2}y(n-1) = x(n) + x(n-2),$$

$$y(n) = 0, n < 0 \quad (3.2)$$

Sketch  $y(n)$ .

**Solution:** The following code yields Fig. 3.3.

```
wget https://github.com/HARI-donk-EY/
sig_pros/blob/main/codes/3/xnyn.py
```

3.3 Repeat the above exercise using C code.

**Solution:** The following codes can be used, the C-code used to generate  $y(n)$  and the plot yields the same Figure as that of 3.3

```
wget https://github.com/HARI-donk-EY/
sig_pros/blob/main/codes/3/xnyn.c
```

```
wget https://github.com/HARI-donk-EY/
sig_pros/blob/main/codes/3/xnyn2.py
```

### 4 Z-TRANSFORM

4.1 The Z-transform of  $x(n)$  is defined as

$$X(z) = \mathcal{Z}\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n)z^{-n} \quad (4.1)$$

Show that

$$\mathcal{Z}\{x(n-1)\} = z^{-1}X(z) \quad (4.2)$$

and find

$$\mathcal{Z}\{x(n-k)\} \quad (4.3)$$

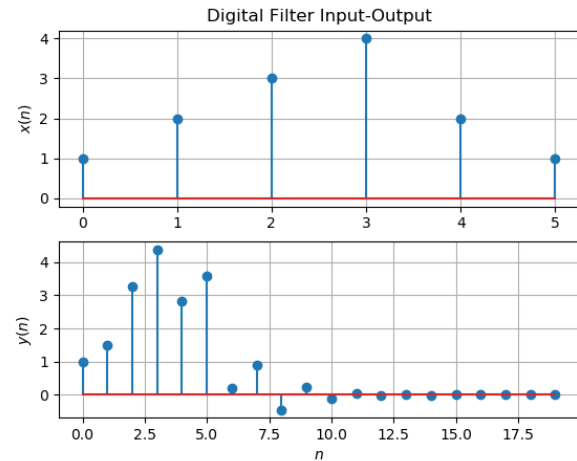


Fig. 3.3

**Solution:** From (4.1),

$$\begin{aligned} \mathcal{Z}\{x(n-1)\} &= \sum_{n=-\infty}^{\infty} x(n-1)z^{-n} \quad (4.4) \\ &= \sum_{n=-\infty}^{\infty} x(n)z^{-n-1} = z^{-1} \sum_{n=-\infty}^{\infty} x(n)z^{-n} \quad (4.5) \end{aligned}$$

resulting in (4.2). Similarly, it can be shown that

$$\mathcal{Z}\{x(n-k)\} = z^{-k}X(z) \quad (4.6)$$

4.2 Obtain  $X(z)$  for  $x(n)$  defined from 3.1.

**Solution:** From 3.1,

$$x(n) = \left\{ \underset{\uparrow}{1}, 2, 3, 4, 2, 1 \right\}$$

and from (4.1), and for given  $x(n)$  we have,

$$X(z) = \mathcal{Z}\{x(n)\} \quad (4.7)$$

$$= 1 + 2z^{-1} + 3z^{-2} + 4z^{-3} + 2z^{-4} + z^{-5} \quad (4.8)$$

4.3 Find

$$H(z) = \frac{Y(z)}{X(z)} \quad (4.9)$$

from (3.2) assuming that the Z-transform is a linear operation.

**Solution:** Applying (4.6) in (3.2),

$$Y(z) + \frac{1}{2}z^{-1}Y(z) = X(z) + z^{-2}X(z) \quad (4.10)$$

$$\Rightarrow \frac{Y(z)}{X(z)} = \frac{1 + z^{-2}}{1 + \frac{1}{2}z^{-1}} \quad (4.11)$$

4.4 Find the Z transform of

$$\delta(n) = \begin{cases} 1 & n = 0 \\ 0 & \text{otherwise} \end{cases} \quad (4.12)$$

and show that the Z-transform of

$$u(n) = \begin{cases} 1 & n \geq 0 \\ 0 & \text{otherwise} \end{cases} \quad (4.13)$$

is

$$U(z) = \frac{1}{1 - z^{-1}}, \quad |z| > 1 \quad (4.14)$$

**Solution:** It is easy to show that

$$\delta(n) \stackrel{Z}{=} 1 \quad (4.15)$$

and from (4.13),

$$U(z) = \sum_{n=0}^{\infty} z^{-n} \quad (4.16)$$

$$= \frac{1}{1 - z^{-1}}, \quad |z| > 1 \quad (4.17)$$

using the formula for the sum of an infinite geometric progression.

4.5 Show that

$$a^n u(n) \stackrel{Z}{=} \frac{1}{1 - az^{-1}} \quad |z| > |a| \quad (4.18)$$

**Solution:** From (4.13), we get,

$$a^n u(n) = \begin{cases} a^n & n \geq 0 \\ 0 & \text{otherwise} \end{cases} \quad (4.19)$$

from above,

$$U_a(z) = \sum_{n=0}^{\infty} a^n z^{-n} \quad (4.20)$$

$$= \sum_{n=0}^{\infty} (az^{-1})^n = \frac{1}{1 - az^{-1}}, \quad |z| > 1 \quad (4.21)$$

using the formula for the sum of an infinite

geometric progression.

4.6 Let

$$H(e^{j\omega}) = H(z = e^{j\omega}). \quad (4.22)$$

Plot  $|H(e^{j\omega})|$ . Comment.  $H(e^{j\omega})$  is known as the *Discrete Time Fourier Transform* (DTFT) of  $x(n)$ .

**Solution:** The following code plots Fig. 4.6.

```
wget https://github.com/HARI-donk-EY/
sig_pros/blob/main/codes/4/dtft.py
```

We can note that the graph in Fig. 4.6 is an even periodic function.

From the equations and we get,

$$|H(e^{j\omega})| = \left| \frac{1 + e^{-2j\omega}}{1 + \frac{1}{2}e^{-j\omega}} \right| \quad (4.23)$$

$$= \left| \frac{1 + \cos(-2\omega) + j \sin(-2\omega)}{1 + \frac{1}{2}(\cos(-\omega) + j \sin(-\omega))} \right| \quad (4.24)$$

$$= \sqrt{\frac{(1 + \cos 2\omega)^2 + (\sin 2\omega)^2}{\left(1 + \frac{1}{2} \cos \omega\right)^2 + \left(\frac{1}{2} \sin \omega\right)^2}} \quad (4.25)$$

$$= \sqrt{\frac{2(1 + \cos 2\omega)}{\frac{5}{4} + \cos \omega}} \quad (4.26)$$

$$= \sqrt{\frac{2(2 \cos^2 \omega)}{\frac{5}{4} + \cos \omega}} \quad (4.27)$$

$$= \frac{4|\cos \omega|}{\sqrt{5 + 4 \cos \omega}} \quad (4.28)$$

and so its fundamental period is  $2\pi$ .

4.7 Express  $h(n)$  in terms of  $H(e^{j\omega})$ .

**Solution:**  $h(n)$  is given by the inverse DTFT (IDTFT) of  $H(e^{j\omega})$

$$h(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H(e^{j\omega}) e^{j\omega n} d\omega \quad (4.29)$$

We can prove this from (4.1),

$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h(k) e^{-j\omega k} \quad (4.30)$$

$$(4.31)$$

multiplying both sides with  $e^{-j\omega k}$  and

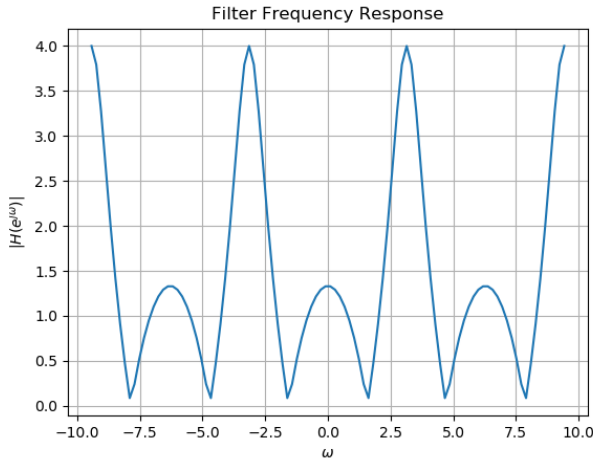


Fig. 4.6:  $|H(e^{j\omega})|$

integrating w.r.t.  $\omega$ .

$$\int_{-\pi}^{\pi} H(e^{j\omega}) e^{j\omega k} d\omega = \sum_{k=-\infty}^{\infty} h(k) \int_{-\pi}^{\pi} e^{-j\omega n} e^{j\omega k} d\omega \quad (4.32)$$

$$= \sum_{k=-\infty}^{\infty} h(k) \int_{-\pi}^{\pi} e^{-j\omega(n-k)} d\omega \quad (4.33)$$

we know that  $e^{-j\omega(n-k)}$  is an odd function so integration would be zero for all values of  $\omega(n-k) \neq 0$ , but 1 if  $\omega(n-k) = 0$ .

$$\int_{-\pi}^{\pi} e^{-j\omega(n-k)} d\omega = \begin{cases} 0 & \text{if } n-k \neq 0 \\ 1 & \text{if } n-k = 0 \end{cases}$$

From this we can complete the summation.

$$\int_{-\pi}^{\pi} H(e^{j\omega}) e^{j\omega k} d\omega = \sum_{k=-\infty}^{\infty} h(k) \int_{-\pi}^{\pi} e^{-j\omega(n-k)} d\omega \quad (4.34)$$

$$= h(n) \int_{-\pi}^{\pi} d\omega \quad (4.35)$$

$$= h(n) 2\pi \quad (4.36)$$

$$h(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H(e^{j\omega}) e^{j\omega k} d\omega \quad (4.37)$$

Hence proved.

## 5 IMPULSE RESPONSE

5.1 Using long division, find

$$h(n), \quad n < 5 \quad (5.1)$$

for  $H(z)$  in (4.6).

**Solution:**

$$H(z) = \frac{1 + z^{-2}}{1 + \frac{1}{2}z^{-1}} \quad (5.2)$$

Substitute  $z^{-1} = x$

$$\begin{array}{r} 2x - 4 \\ \frac{1}{2}x + 1 \overline{) x^2 + 1} \\ \underline{-x^2 - 2x} \phantom{+ 1} \\ -2x + 1 \\ \underline{2x + 4} \\ 5 \end{array}$$

So,

$$H(z) = -4 + 2z^{-1} + \frac{5}{1 + \frac{1}{2}z^{-1}} \quad (5.3)$$

$$= -4 + 2z^{-1} + 5 \sum_{n=0}^{\infty} \left(\frac{-1}{2}\right)^n z^{-n} \quad (5.4)$$

$$= 1 - \frac{1}{2}z^{-1} + 5 \sum_{n=2}^{\infty} \left(\frac{-1}{2}\right)^n z^{-n} \quad (5.5)$$

$$= \sum_{n=0}^{\infty} \left(\frac{-1}{2}\right)^n z^{-n} + 4 \sum_{n=2}^{\infty} \left(\frac{-1}{2}\right)^n z^{-n} \quad (5.6)$$

$$= \sum_{n=-\infty}^{\infty} u(n) \left(\frac{-1}{2}\right)^n z^{-n} + \sum_{n=-\infty}^{\infty} u(n-2) \left(\frac{-1}{2}\right)^{n-2} z^{-n} \quad (5.7)$$

Therefore from (4.1),

$$h(n) = \left(\frac{-1}{2}\right)^n u(n) + \left(\frac{-1}{2}\right)^{n-2} u(n-2) \quad (5.8)$$

5.2 Find an expression for  $h(n)$  using  $H(z)$ , given that

$$h(n) \stackrel{Z}{\rightleftharpoons} H(z) \quad (5.9)$$

and there is a one to one relationship between  $h(n)$  and  $H(z)$ .  $h(n)$  is known as the *impulse response* of the system defined by (3.2).

Take  $H(z)$  from (4.6)

**Solution:** From (4.6),

$$H(z) = \frac{1}{1 + \frac{1}{2}z^{-1}} + \frac{z^{-2}}{1 + \frac{1}{2}z^{-1}} \quad (5.10)$$

$$\Rightarrow h(n) = \left(-\frac{1}{2}\right)^n u(n) + \left(-\frac{1}{2}\right)^{n-2} u(n-2) \quad (5.11)$$

using (4.18) and (4.6). Let,

$$h(n) = h_1(n) + h_2(n) \quad (5.12)$$

Where,

$$h_1(n) = \left(-\frac{1}{2}\right)^n u(n) \quad (5.13)$$

$$h_2(n) = \left(-\frac{1}{2}\right)^{n-2} u(n-2) \quad (5.14)$$

Then ROC of  $h_1(n)$  and  $h_2(n)$  will both be  $|z| > \frac{1}{2}$ .

Hence ROC of  $H(z)$  is  $|z| > \frac{1}{2}$

5.3 Sketch  $h(n)$ . Is it bounded? Justify theoretically.

**Solution:** The following code plots Fig. 5.3.

```
wget https://github.com/HARI-donk-EY/
sig_pros/blob/main/codes/5/hn.py
```

We know,

$$h(n) = \left(-\frac{1}{2}\right)^n u(n) + \left(-\frac{1}{2}\right)^{n-2} u(n-2)$$

then,

$$\lim_{n \rightarrow \infty} h(n) = \lim_{n \rightarrow \infty} \left(-\frac{1}{2}\right)^n u(n) + \lim_{n \rightarrow \infty} \left(-\frac{1}{2}\right)^{n-2} u(n-2) \quad (5.15)$$

$$h(n) = \begin{cases} 5\left(\frac{-1}{2}\right)^n & n \geq 2 \\ \left(\frac{-1}{2}\right)^n & 0 \leq n < 2 \\ 0 & n < 0 \end{cases} \quad (5.16)$$

5.4 Is  $h(n)$  convergent? Justify using ratio test.

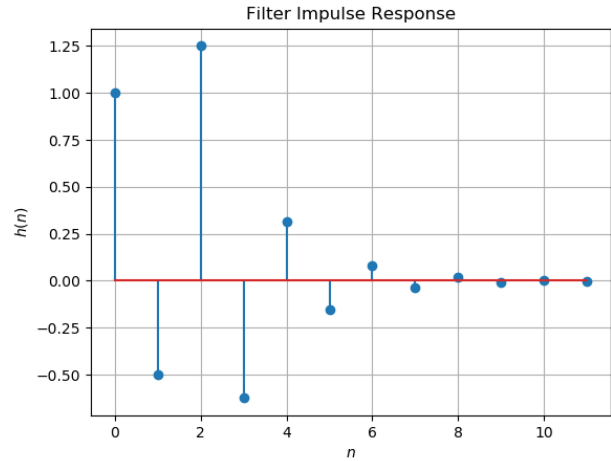


Fig. 5.3:  $h(n)$  as the inverse of  $H(z)$

**Solution:**

$$h(n) = \left(-\frac{1}{2}\right)^n u(n) + \left(-\frac{1}{2}\right)^{n-2} u(n-2) \quad (5.17)$$

If  $L = \lim_{n \rightarrow \infty} \left| \frac{h(n+1)}{h(n)} \right| < 1$ , then  $h(n)$  is convergent.

$$L = \lim_{n \rightarrow \infty} \left| \frac{h(n+1)}{h(n)} \right| \quad (5.18)$$

$$= \lim_{n \rightarrow \infty} \left| \frac{\left(-\frac{1}{2}\right)^{n+1} u(n+1) + \left(-\frac{1}{2}\right)^{n-1} u(n-1)}{\left(-\frac{1}{2}\right)^n u(n) + \left(-\frac{1}{2}\right)^{n-2} u(n-2)} \right| \quad (5.19)$$

$$= \left| \frac{-\frac{1}{2} + \left(-\frac{1}{2}\right)^{-1}}{1 + \left(-\frac{1}{2}\right)^{-2}} \right| \quad (5.20)$$

$$= \left| \frac{-\frac{5}{2}}{5} \right| \quad (5.21)$$

$$= \frac{1}{2} \quad (5.22)$$

Since  $L < 1$ , we can say that  $h(n)$  is convergent.

5.5 The system with  $h(n)$  is defined to be stable if

$$\sum_{n=-\infty}^{\infty} h(n) < \infty \quad (5.23)$$

Is the system defined by (3.2) stable for the impulse response in (5.9)?

**Solution:**

$$\sum_{n=-\infty}^{\infty} h(n) = \sum_{n=-\infty}^{\infty} \left( \left(-\frac{1}{2}\right)^n u(n) + \left(-\frac{1}{2}\right)^{n-2} u(n-2) \right) \quad (5.24)$$

$$= \sum_{n=0}^{\infty} \left(-\frac{1}{2}\right)^n u(n) + \sum_{(n-2)=0}^{\infty} \left(-\frac{1}{2}\right)^{n-2} u(n-2) \quad (5.25)$$

$$= 2 \left( \frac{1}{1 + \frac{1}{2}} \right) \quad (5.26)$$

$$= \frac{4}{3} \quad (5.27)$$

As  $\sum_{n=-\infty}^{\infty} h(n) = \frac{4}{3}$  is less than  $\infty$ , the system defined by (3.2) is stable for the impulse response in (5.9).

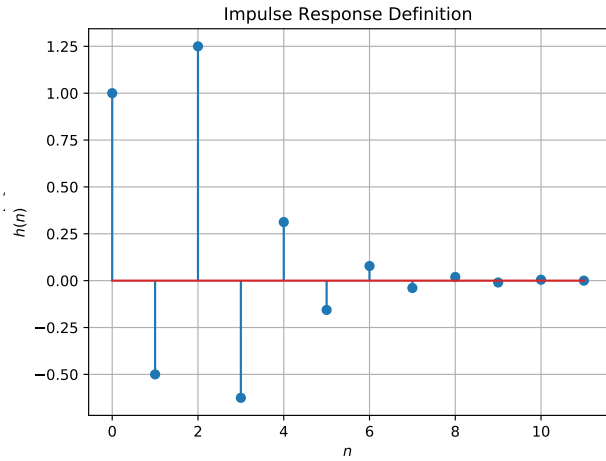


Fig. 5.7:  $h(n)$  from the definition

5.6 Verify the above result using a python code.

**Solution:** The following code determines if it is convergent or not:

```
wget https://github.com/HARI-donk-EY/
sig_pros/blob/main/codes/5/new.py
```

5.7 Compute and sketch  $h(n)$  using

$$h(n) + \frac{1}{2}h(n-1) = \delta(n) + \delta(n-2), \quad (5.28)$$

This is the definition of  $h(n)$ .

**Solution:** The following code plots Fig. 5.7. Note that this is the same as Fig. 5.3.

```
wget https://github.com/HARI-donk-EY/
sig_pros/blob/main/codes/5/hndef.py
```

Computing,

$$h(0) = 1$$

$$h(1) = -\frac{1}{2}h(0)$$

$$h(2) = -\frac{1}{2}h(1) + 1$$

$$\text{Parallely, } h(n) = -\frac{1}{2}h(n-1)$$

5.8 Compute

$$y(n) = x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k) \quad (5.29)$$

Comment. The operation in (5.29) is known as *convolution*.

**Solution:** The following code plots Fig. 5.8. Note that this is the same as  $y(n)$  in Fig. 3.3.

```
wget https://github.com/HARI-donk-EY/
sig_pros/blob/main/codes/5/ynconv.py
```

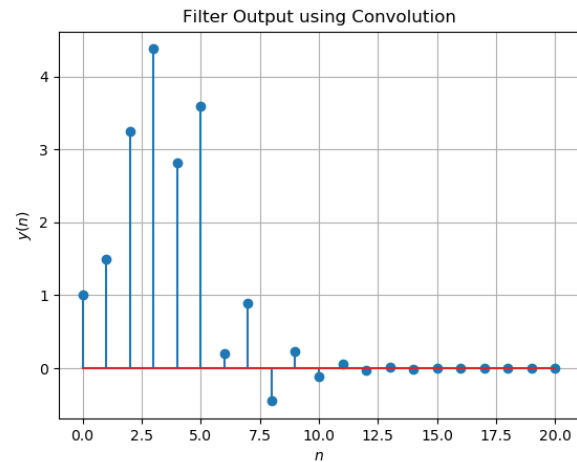


Fig. 5.8:  $y(n)$  from the definition of convolution

5.9 Express the above convolution using a toeplitz

matrix. **Solution:**

$$\mathbf{y} = \mathbf{x} \otimes \mathbf{h}$$

(5.30) 6.1 Compute

$$\mathbf{y} = \begin{pmatrix} h_1 & 0 & \cdot & \cdot & \cdot & 0 \\ h_2 & h_1 & \cdot & \cdot & \cdot & 0 \\ h_3 & h_2 & h_1 & \cdot & \cdot & 0 \\ h_{m-1} & \cdot & \cdot & \cdot & h_2 & h_1 \\ h_m & h_{m-1} & \cdot & \cdot & \cdot & h_2 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & \cdot & h_m \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{pmatrix} \quad (5.31)$$

$$\mathbf{y} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ \frac{-1}{2} & 1 & 0 & 0 & 0 & 0 \\ \frac{3}{4} & \frac{-1}{2} & 1 & 0 & 0 & 0 \\ \frac{-5}{8} & \frac{3}{4} & \frac{-1}{2} & 1 & 0 & 0 \\ \frac{7}{16} & \frac{-5}{8} & \frac{3}{4} & \frac{-1}{2} & 1 & 0 \\ \frac{-9}{32} & \frac{7}{16} & \frac{-5}{8} & \frac{3}{4} & \frac{-1}{2} & 1 \\ \frac{11}{64} & \frac{-9}{32} & \frac{7}{16} & \frac{-5}{8} & \frac{3}{4} & \frac{-1}{2} \\ 0 & \frac{11}{64} & \frac{-9}{32} & \frac{7}{16} & \frac{-5}{8} & \frac{3}{4} \\ 0 & 0 & \frac{11}{64} & \frac{-9}{32} & \frac{7}{16} & \frac{-5}{8} \\ 0 & 0 & 0 & \frac{11}{64} & \frac{-9}{32} & \frac{7}{16} \\ 0 & 0 & 0 & 0 & \frac{11}{64} & \frac{-9}{32} \\ 0 & 0 & 0 & 0 & 0 & \frac{11}{64} \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1. \\ 1.5 \\ 3.25 \\ 4.375 \\ 2.8125 \\ 3.59375 \\ 0.203125 \\ 0.9375 \\ -0.39062 \\ 0.3125 \\ 0. \\ 0.078125 \end{pmatrix} \quad (5.32)$$

And this is what we got in (5.29)

5.10 Show that

$$y(n) = \sum_{n=-\infty}^{\infty} x(n-k)h(k) \quad (5.33)$$

**Solution:** From (5.29), we substitute  $k := n-k$  to get

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k) \quad (5.34)$$

$$= \sum_{n-k=-\infty}^{\infty} x(n-k)h(k) \quad (5.35)$$

$$= \sum_{k=-\infty}^{\infty} x(n-k)h(k) \quad (5.36)$$

$$X(k) \triangleq \sum_{n=0}^{N-1} x(n)e^{-j2\pi kn/N}, \quad k = 0, 1, \dots, N-1 \quad (6.1)$$

and  $H(k)$  using  $h(n)$ .

**Solution:** The following code generates the data required for plotting.

```
wget https://github.com/HARI-donk-EY/
sig_pros/blob/main/codes/6/header.h
wget https://github.com/HARI-donk-EY/
sig_pros/blob/main/codes/6/6.1/
XkHk_dat.c
```

The following code plots Fig. 6.1.

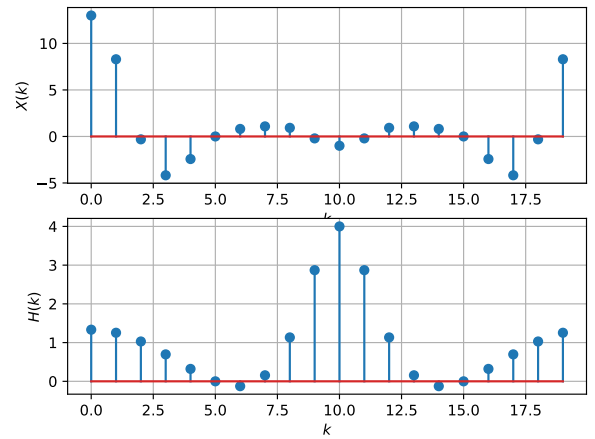


Fig. 6.1:  $X(k), H(k)$  from the DFT

```
wget https://github.com/HARI-donk-EY/
sig_pros/blob/main/codes/6/6.1/XkHk.py
```

6.2 Compute

$$Y(k) = X(k)H(k) \quad (6.2)$$

**Solution:** The following code plots Fig. 6.2. The data can be obtained from

```
wget https://github.com/HARI-donk-EY/
sig_pros/blob/main/codes/6/header.h
wget https://github.com/HARI-donk-EY/
sig_pros/blob/main/codes/6/6.2/y(k)dat.c
```

The plotting is done by

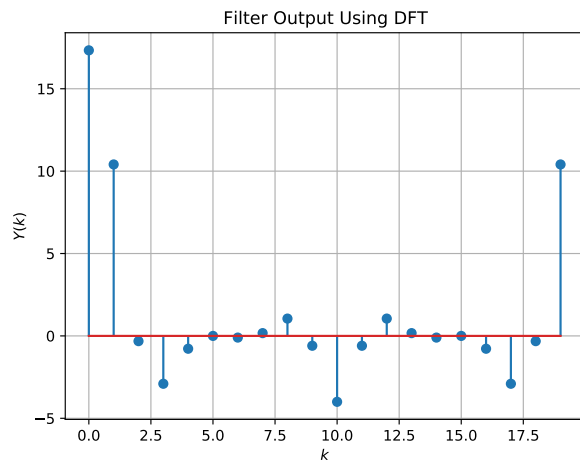


Fig. 6.2:  $Y(k)$  from the DFT

```
wget https://github.com/HARI-donk-EY/
sig_pros/blob/main/codes/6/6.2/yk.py
```

### 6.3 Compute

$$y(n) = \frac{1}{N} \sum_{k=0}^{N-1} Y(k) \cdot e^{j2\pi kn/N}, \quad n = 0, 1, \dots, N-1 \quad (6.3)$$

**Solution:** The following code plots Fig. ??.  
Note that this is the same as  $y(n)$  in Fig.