Circuits and Transforms

EE3900: Linear Systems and Signal Processing

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1. Definitions

1.1 The unit step function is defined as

$$u(t) = \begin{cases} 1 & t > 0 \\ \frac{1}{2} & t = 0 \\ 0 & t < 0 \end{cases} \tag{1.1}$$

1.2 The Laplace transform of $g(t)$ is defined as

$$G(s) = \int_{-\infty}^{\infty} g(t)e^{-st}\,\mathrm{d}t \tag{1.2}$$

2. Laplace Transform

2.1. In the circuit, the switch S is connected to position P for a long time so that the charge on the capacitor becomes q_1 µC. Then S is switched to position Q. After a long time, the charge on the capacitor is q_2 µC

2.2. Draw the circuit using latex-tikz
Solution:

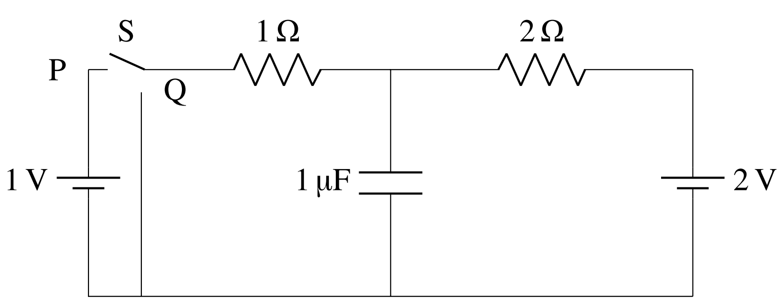

Fig. 2.2. Circuit diagram of the circuit in question

2.3. Find q_1.
Solution:
After a long time when a steady state is achieved, a capacitor behaves like an open circuit, that is, current passing through it is zero.

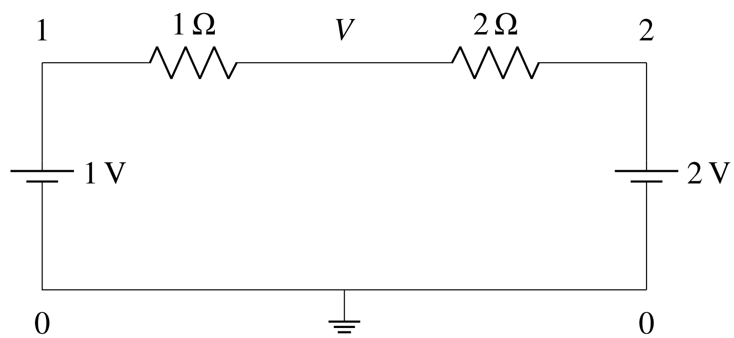

Fig. 2.3. Circuit diagram at steady state before flipping the switch

By Kirchoff's Law we can get,

$$\frac{V-1}{1} - \frac{V-2}{2} = 0 \tag{2.1}$$

$$\implies V = \frac{4}{3}\,\mathrm{V} \tag{2.2}$$

$$\implies q_1 = CV = \frac{4}{3}\,\mu\mathrm{C} \tag{2.3}$$

2.4. Show that the Laplace Transform of $u(t)$ is $\frac{1}{s}$ and find the ROC.

Solution:

The Laplace Transform of $u(t)$ is given by,

$$\mathcal{L}\{u(t)\} = \int_{-\infty}^{\infty} u(t)e^{-st}dt \tag{2.4}$$
$$= \int_{0}^{\infty} e^{-st}dt \tag{2.5}$$
$$= \lim_{R\to\infty} \frac{1-e^{-sR}}{s} \tag{2.6}$$

This limit is finite only if $\Re(s) > 0$, which is going to be its ROC.

Therefore,

$$u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s} \qquad \Re(s) > 0 \tag{2.7}$$

2.5. Show that,

$$e^{-at}u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s+a} \qquad a > 0 \tag{2.8}$$

and find ROC.

Solution:

The Laplace Transform of $e^{-at}u(t)$ for $a > 0$ is given by,

$$\mathcal{L}\{u(t)\} = \int_{-\infty}^{\infty} e^{-at}u(t)e^{-st}dt \tag{2.9}$$
$$= \int_{0}^{\infty} e^{-(s+a)t}dt \tag{2.10}$$
$$= \lim_{R\to\infty} \frac{1-e^{-(s+a)R}}{s+a} \tag{2.11}$$

This limit is finite only if $\Re(s+a) > 0$, which is going to be its ROC

Therefore

$$e^{-at}u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s+a} \qquad \Re(s) > -a \tag{2.12}$$

since a is real.

2.6. Now consider the following resistive circuit transformed from Fig. 2.2

where

$$u(t) \xleftrightarrow{\mathcal{L}} V_1(s) \tag{2.13}$$
$$2u(t) \xleftrightarrow{\mathcal{L}} V_2(s) \tag{2.14}$$

Find the voltage across the capacitor $V_c(s)$

Solution:

$$V_1(s) = \frac{1}{s} \qquad \Re(s) > 0 \tag{2.15}$$
$$V_2(s) = \frac{2}{s} \qquad \Re(s) > 0 \tag{2.16}$$
$$\tag{2.17}$$

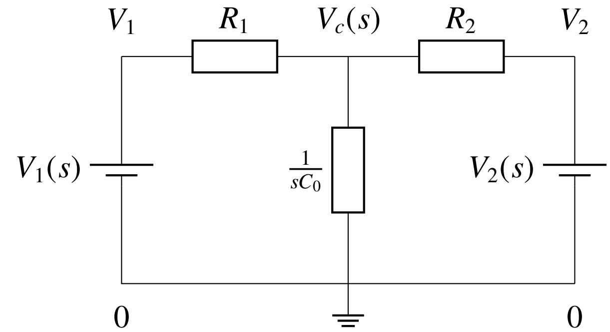

Fig. 2.6. Circuit diagram in s-domain before flipping the switch

By Kirchoff's Junction Law, we get,

$$\frac{V_c - V_1}{R_1} + \frac{V_c - V_2}{R_2} + \frac{V_c - 0}{\frac{1}{sC_0}} = 0 \tag{2.18}$$
$$V_c\left(\frac{1}{R_1} + \frac{1}{R_2} + sC_0\right) = \frac{V_1}{R_1} + \frac{V_2}{R_2} \tag{2.19}$$
$$V_c(s) = \frac{\frac{1}{sR_1} + \frac{2}{sR_2}}{\frac{1}{R_1} + \frac{1}{R_2} + sC_0} \tag{2.20}$$
$$V_c(s) = \frac{\frac{1}{R_1C_0} + \frac{2}{R_2C_0}}{s\left(s + \frac{1}{R_1C_0} + \frac{1}{R_2C_0}\right)} \tag{2.21}$$

2.7. Find $v_c(t)$. Plot the graph using Python.

Solution:

Upon performing the partial fraction decomposition upon the above obtained result, we get,

$$V_c(s) = \frac{\frac{1}{R_1C_0} + \frac{2}{R_2C_0}}{\frac{1}{R_1C_0} + \frac{1}{R_2C_0}}\left(\frac{1}{s} - \frac{1}{s} + \frac{1}{R_1C_0} + \frac{1}{R_2C_0}\right), \quad \Re(s) > 0 \tag{2.22}$$

On taking the inverse Laplace Transform we get,

$$v_c(t) = \frac{2R_1 + R_2}{R_1 + R_2}\left(u(t) - e^{-\left(\frac{1}{R_1}+\frac{1}{R_2}\right)\frac{1}{C_0}}\right) \tag{2.23}$$