Circuits and Transforms

EE3900: Linear Systems and Signal Processing

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Contents

1. Definitions

1.1 The unit step function is defined as

$$u(t) = \begin{cases} 1 & t > 0 \\ \frac{1}{2} & t = 0 \\ 0 & t < 0 \end{cases} \tag{1.1}$$

1.2 The Laplace transform of $g(t)$ is defined as

$$G(s) = \int_{-\infty}^{\infty} g(t)e^{-st}\,\mathrm{d}t \tag{1.2}$$

2. Laplace Transform

2.1 In the circuit, the switch S is connected to position P for a long time so that the charge on the capacitor becomes q_1 µC. Then S is switched to position Q. After a long time, the charge on the capacitor is q_2 µC

2.2 Draw the circuit using latex-tikz
Solution:

2.3 Find q_1.
Solution:
After a long time when a steady state is achieved, a capacitor behaves like an open circuit, that is, current passing through it is zero.

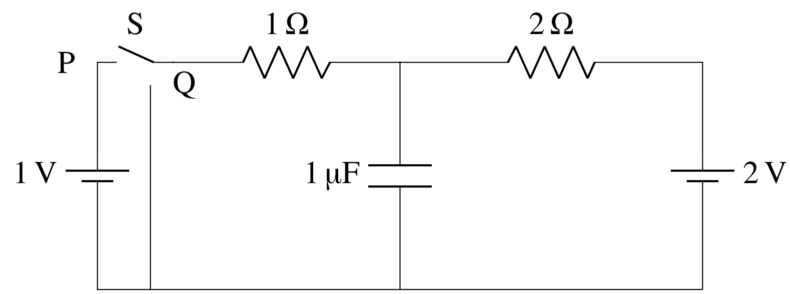

Fig. 2.2. Circuit diagram of the circuit in question

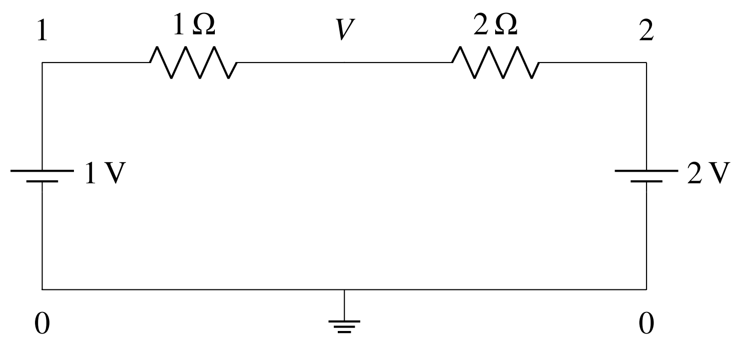

Fig. 2.3. Circuit diagram at steady state before flipping the switch

By Kirchoff's Law we can get,

$$\frac{V-1}{1} - \frac{V-2}{2} = 0 \tag{2.1}$$

$$\implies V = \frac{4}{3}\,\mathrm{V} \tag{2.2}$$

$$\implies q_1 = CV = \frac{4}{3}\,\mu\mathrm{C} \tag{2.3}$$

2.4 Show that the Laplace Transform of $u(t)$ is $\frac{1}{s}$ and find the ROC.

Solution:
The Laplace Transform of $u(t)$ is given by,

$$\mathcal{L}\{u(t)\} = \int_{-\infty}^{\infty} u(t)e^{-st}dt \tag{2.4}$$
$$= \int_{0}^{\infty} e^{-st}dt \tag{2.5}$$
$$= \lim_{R\to\infty} \frac{1 - e^{-sR}}{s} \tag{2.6}$$

This limit is finite only if $\Re(s) > 0$, which is going to be its *ROC*.

Therefore,

$$u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s} \qquad \Re(s) > 0 \tag{2.7}$$

2.5 Show that,

$$e^{-at}u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+a} \qquad a > 0 \tag{2.8}$$

and find *ROC*.
Solution:
The Laplace Transform of $e^{-at}u(t)$ for $a > 0$ is given by,

$$\mathcal{L}\{u(t)\} = \int_{-\infty}^{\infty} e^{-at}u(t)e^{-st}dt \tag{2.9}$$
$$= \int_{0}^{\infty} e^{-(s+a)t}dt \tag{2.10}$$
$$= \lim_{R\to\infty} \frac{1 - e^{-(s+a)R}}{s+a} \tag{2.11}$$

This limit is finite only if $\Re(s + a) > 0$, which is going to be its ROC
Therefore

$$e^{-at}u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+a} \qquad \Re(s) > -a \tag{2.12}$$

since a is real.

2.6 Now consider the following resistive circuit transformed from Fig. 2.2
where

$$u(t) \overset{\mathcal{L}}{\longleftrightarrow} V_1(s) \tag{2.13}$$
$$2u(t) \overset{\mathcal{L}}{\longleftrightarrow} V_2(s) \tag{2.14}$$

Find the voltage across the capacitor $V_c(s)$

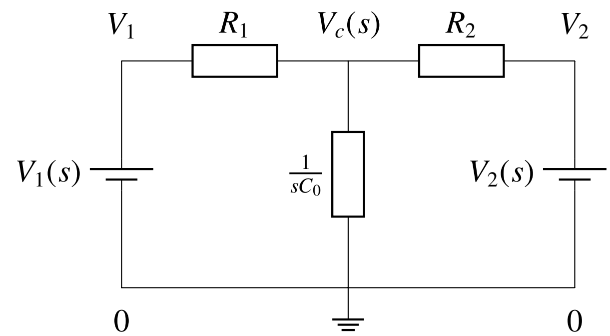

Fig. 2.6. Circuit diagram in s-domain before flipping the switch

Solution:

$$V_1(s) = \frac{1}{s} \qquad \Re(s) > 0 \tag{2.15}$$
$$V_2(s) = \frac{2}{s} \qquad \Re(s) > 0 \tag{2.16}$$
$$\tag{2.17}$$

By Kirchoff's Junction Law, we get,

$$\frac{V_c - V_1}{R_1} + \frac{V_c - V_2}{R_2} + \frac{V_c - 0}{\frac{1}{sC_0}} = 0 \tag{2.18}$$
$$V_c\left(\frac{1}{R_1} + \frac{1}{R_2} + sC_0\right) = \frac{V_1}{R_1} + \frac{V_2}{R_2} \tag{2.19}$$
$$V_c(s) = \frac{\frac{1}{sR_1} + \frac{2}{sR_2}}{\frac{1}{R_1} + \frac{1}{R_2} + sC_0} \tag{2.20}$$
$$V_c(s) = \frac{\frac{1}{R_1C_0} + \frac{2}{R_2C_0}}{s\left(s + \frac{1}{R_1C_0} + \frac{1}{R_2C_0}\right)} \tag{2.21}$$

2.7 Find $v_c(t)$. Plot the graph using Python.

Solution:
Upon performing the partial fraction decomposition upon the above obtained result, we get,

$$V_c(s) = \frac{\frac{1}{R_1C_0} + \frac{2}{R_2C_0}}{\frac{1}{R_1C_0} + \frac{1}{R_2C_0}}\left(\frac{1}{s} - \frac{1}{s} + \frac{1}{R_1C_0} + \frac{1}{R_2C_0}\right), \quad \Re(s) > 0 \tag{2.22}$$

On taking the inverse Laplace Transform we get,

$$v_c(t) = \frac{2R_1 + R_2}{R_1 + R_2}\left(u(t) - e^{-\left(\frac{1}{R_1}+\frac{1}{R_2}\right)\frac{1}{C_0}}u(t)\right) \tag{2.23}$$

$$= \frac{2R_1 + R_2}{R_1 + R_2}\left(1 - e^{-\left(\frac{1}{R_1}+\frac{1}{R_2}\right)\frac{1}{C_0}}\right)u(t) \tag{2.24}$$

Substitute the values $R_1 = 1\,\Omega$, $R_2 = 2\,\Omega$ and $C_0 = 1\,\mu\text{F}$

$$v_c(t) = \frac{4}{3}\left(1 - e^{-\frac{3}{2}\times 10^6 t}\right) \tag{2.25}$$

The code for plotting the graph can be obtained by using the following commands

```
$ wget https://raw.githubusercontent.com/
    HARI-donk-EY/sig_pros/main/circuit/
    codes/2_7.py
```

Run the code using.

```
$ python3 2_7.py
```

We get the Fig. 2.7.

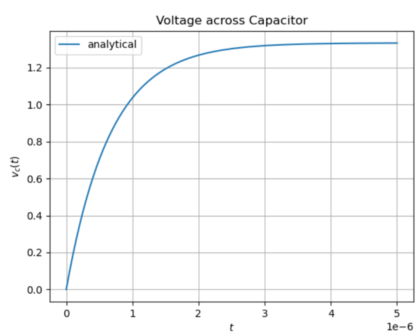

Fig. 2.7. Plot of $v_c(t)$ before flipping the switch

2.8 Verify the results using ngspice.

Solution:
The code for plotting the graph can be obtained by using the following commands

```
$ wget https://raw.githubusercontent.com/
    HARI-donk-EY/sig_pros/main/circuit/
    codes/2_8.py
$ wget https://raw.githubusercontent.com/
    HARI-donk-EY/sig_pros/main/circuit/
    codes/2_8.cir
```

Run the code using.

```
$ ngspice 2_8.cir
$ python3 2_8.py
```

We get the Fig. 2.8.

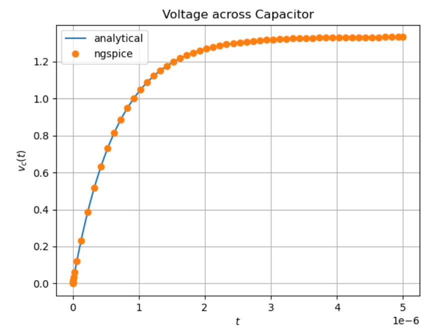

Fig. 2.8. Plot of $v_c(t)$ before flipping the switch, comparing it along side data obtained from ngspice

2.9 Obtain Fig. 2.6 using equivalent differential equation.

Solution:
Using Kirchoff's Junction Law,

$$\frac{v_c(t) - v_1(t)}{R_1} + \frac{v_c(t) - v_2(t)}{R_2} + \frac{\mathrm{d}q}{\mathrm{d}t} = 0 \tag{2.26}$$

where $q(t)$ is the charge on the capacitor
On taking the Laplace transform on both sides of this equation

$$\frac{V_c(s) - V_1(s)}{R_1} + \frac{V_c(s) - V_2(s)}{R_2} + \left(sQ(s) - q(0^-)\right) = 0 \tag{2.27}$$

But $q(0^-) = 0$ and

$$q(t) = C_0 v_c(t) \tag{2.28}$$

$$\implies Q(s) = C_0 V_c(s) \tag{2.29}$$

Thus

$$\frac{V_c(s) - V_1(s)}{R_1} + \frac{V_c(s) - V_2(s)}{R_2} + sC_0V_c(s) = 0 \tag{2.30}$$

$$\implies \frac{V_c(s) - V_1(s)}{R_1} + \frac{V_c(s) - V_2(s)}{R_2} + \frac{V_c(s) - 0}{\frac{1}{sC_0}} = 0 \tag{2.31}$$

which is the same equation as the one we obtained from Fig. 2.6

3. Initial Conditions

3.1 Find q_2 in Fig. 2.2

Solution:

After a long time, when steady state is achieved, a capacitor behaves like an open circuit, i.e., current passing through it is zero

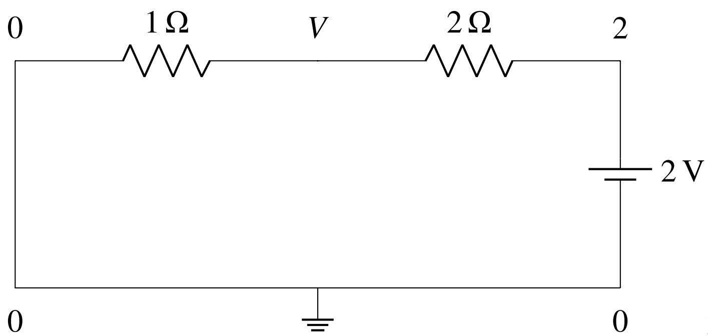

Fig. 3.1. Circuit diagram at steady state after flipping the switch

By Kirchoff's junction law, we get

$$\frac{V-0}{1}+\frac{V-2}{2}=0 \tag{3.1}$$

$$\implies V=\frac{2}{3}\,\text{V} \tag{3.2}$$

$$\implies q_2=CV=\frac{2}{3}\,\mu\text{C} \tag{3.3}$$

3.2 Draw the equivalent s-domain resistive circuit when S is switched to position Q. Use variables R_1, R_2, C_0 for the passive elements. Use latex-tikz

Solution:

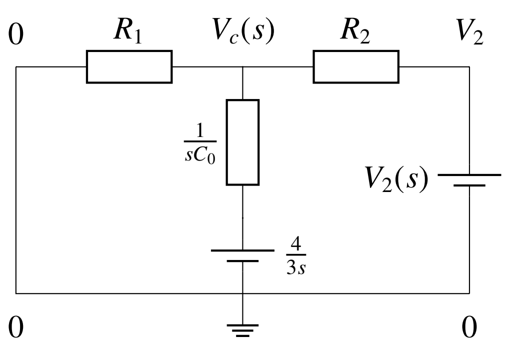

Fig. 3.2. Circuit diagram in s-domain after flipping the switch

The battery $\frac{4}{3s}$ corresponds to the intial potential difference of $\frac{4}{3}$ V across the capacitor just before switching it to Q

3.3 Find $V_c(s)$

Solution:
By Kirchoff's junction law, we get

$$\frac{V_c-0}{R_1}+\frac{V_c-V_2}{R_2}+\frac{V_c-\frac{4}{3s}}{\frac{1}{sC_0}}=0 \tag{3.4}$$

$$\implies V_c\left(\frac{1}{R_1}+\frac{1}{R_2}+sC_0\right)=\frac{V_2}{R_2}+\frac{4}{3}C_0 \tag{3.5}$$

$$\implies V_c(s)=\frac{\frac{2}{sR_2}+\frac{4}{3}C_0}{\frac{1}{R_1}+\frac{1}{R_2}+sC_0} \tag{3.6}$$

$$=\frac{\frac{2}{R_2C_0}+\frac{4}{3}s}{s\left(s+\frac{1}{R_1C_0}+\frac{1}{R_2C_0}\right)} \tag{3.7}$$

3.4 Find $v_c(t)$. Plot using Python

Solution: On performing partial fraction decomposition

$$V_c(s)=\frac{4}{3}\left(\frac{1}{s+\frac{1}{R_1C_0}+\frac{1}{R_2C_0}}\right)+\frac{\frac{2}{R_2C_0}}{\frac{1}{R_1C_0}+\frac{1}{R_2C_0}}\left(\frac{1}{s}-\frac{1}{s+\frac{1}{R_1C_0}+\frac{1}{R_2C_0}}\right) \tag{3.8}$$

for $\Re(s)>0$
On taking the inverse Laplace transform, we get

$$v_c(t)=\frac{4}{3}e^{-\left(\frac{1}{R_1}+\frac{1}{R_2}\right)\frac{t}{C_0}}u(t)+\frac{2R_1}{R_1+R_2}\left(u(t)-e^{-\left(\frac{1}{R_1}+\frac{1}{R_2}\right)\frac{t}{C_0}}u(t)\right) \tag{3.9}$$

Substitute the values $R_1=1\,\Omega, R_2=2\,\Omega, C_0=1\,\mu\text{F}$

$$v_c(t)=\frac{4}{3}e^{-\frac{3}{2}\times10^6t}u(t)+\frac{2}{3}\left(1-e^{-\frac{3}{2}\times10^6t}\right)u(t) \tag{3.10}$$

$$=\frac{2}{3}\left(1+\mathrm{e}^{-\frac{3}{2}\times10^6\mathrm{t}}\right)\mathrm{u(t)}\,\mathrm{V} \tag{3.11}$$

The code for plotting the graph can be obtained by using the following commands

```
$ wget https://raw.githubusercontent.com/
    HARI-donk-EY/sig_pros/main/circuit/
    codes/3_4.py
```

Run the code using.

```
$ python3 3_4.py
```

We get the Fig. 3.4.

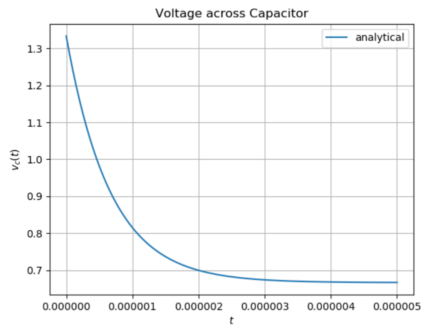

Fig. 3.4. Plot of $v_c(t)$ after flipping the switch

3.5 Verify the results using ngspice.

Solution:
The code for plotting the graph can be obtained by using the following commands

```
$ wget https://raw.githubusercontent.com/
    HARI-donk-EY/sig_pros/main/circuit/
    codes/3_5.py
$ wget https://raw.githubusercontent.com/
    HARI-donk-EY/sig_pros/main/circuit/
    codes/3_5.cir
```

Run the code using.

```
$ ngspice 3_5.cir
$ python3 3_5.py
```

We get the Fig. 3.5.

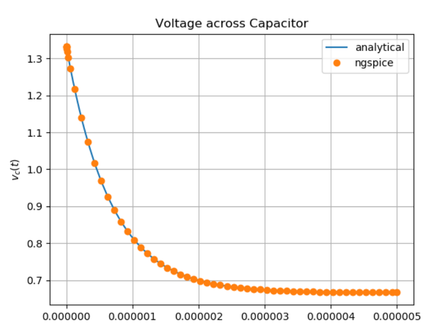

Fig. 3.5. Plot of $v_c(t)$ after flipping the switch, comparing it along side data obtained from ngspice

3.6 Find $v_c(0^-)$, $v_c(0^+)$ and $v_c(\infty)$

Solution:

At $t = 0^-$, the switch still hasn't been switched to Q and the circuit is in steady state

$$v_c(0^-) = \frac{4}{3}\,\text{V} \tag{3.12}$$

For $t \geq 0$, we can use the above formula

$$v_c(0^+) = \lim_{t\to 0^+} v_c(t) = \frac{4}{3}\,\text{V} \tag{3.13}$$

$$v_c(\infty) = \lim_{t\to\infty} v_c(t) = \frac{2}{3}\,\text{V} \tag{3.14}$$

3.7 Obtain Fig. 3.2 using the equivalent differential equation

Solution:

Using Kirchoff's junction law

$$\frac{v_c(t) - 0}{R_1} + \frac{v_c(t) - v_2(t)}{R_2} + \frac{dq}{dt} = 0 \tag{3.15}$$

where $q(t)$ is the charge on the capacitor
On taking the Laplace transform on both sides of this equation

$$\frac{V_c(s) - 0}{R_1} + \frac{V_c(s) - V_2(s)}{R_2} + (sQ(s) - q(0^-)) = 0 \tag{3.16}$$

But $q(0^-) = \frac{4}{3}C_0$ and

$$q(t) = C_0 v_c(t) \tag{3.17}$$

$$\implies Q(s) = C_0 V_c(s) \tag{3.18}$$

Thus

$$\frac{V_c(s) - 0}{R_1} + \frac{V_c(s) - V_2(s)}{R_2} + \left(sC_0 V_c(s) - \frac{4}{3}C_0\right) = 0 \tag{3.19}$$

$$\implies \frac{V_c(s) - 0}{R_1} + \frac{V_c(s) - V_2(s)}{R_2} + \frac{V_c(s) - \frac{4}{3s}}{\frac{1}{sC_0}} = 0 \tag{3.20}$$

which is the same equation as the one we obtained from Fig. 3.2

4. Bilinear Transform

4.1 In Fig. 2.2, consider the case when S is switched to Q right in the beginning. Formulate the differential equation

Solution:
The differential equation is the same as before

$$\frac{v_c(t)-0}{R_1}+\frac{v_c(t)-v_2(t)}{R_2}+\frac{dq}{dt}=0 \tag{4.1}$$

$$\text{i.e., } \frac{v_c(t)}{R_1}+\frac{v_c(t)-v_2(t)}{R_2}+C_0\frac{dv_c}{dt}=0 \tag{4.2}$$

but with a different initial condition

$$q(0^-)=q(0)=0 \tag{4.3}$$

4.2 Find $H(s)$ considering the ouput voltage at the capacitor

Solution:
On taking the Laplace transform on both sides of this equation

$$\frac{V_c(s)}{R_1}+\frac{V_c(s)-V_2(s)}{R_2}+sQ(s)-0=0 \tag{4.4}$$

$$\implies V_c(s)\left(\frac{1}{R_1}+\frac{1}{R_2}\right)+sC_0V_c(s)=\frac{V_2(s)}{R_2} \tag{4.5}$$

$$\implies \frac{V_c(s)}{V_2(s)}=\frac{\frac{1}{R_2}}{\frac{1}{R_1}+\frac{1}{R_2}+sC_0} \tag{4.6}$$

The transfer function is thus

$$H(s)=\frac{\frac{1}{R_2C_0}}{s+\frac{1}{R_1C_0}+\frac{1}{R_2C_0}} \tag{4.7}$$

On substituting the values, we get

$$H(s)=\frac{5\times10^5}{s+1.5\times10^6} \tag{4.8}$$

4.3 Plot $H(s)$. What kind of filter is it?
Solution:
Download the following Python code that plots Fig. 4.3

```
$ wget https://raw.githubusercontent.com/
    HARI-donk-EY/sig_pros/main/circuit/
    codes/4_3.py
```

Run the codes by executing

```
$ python3 4_3.py
```

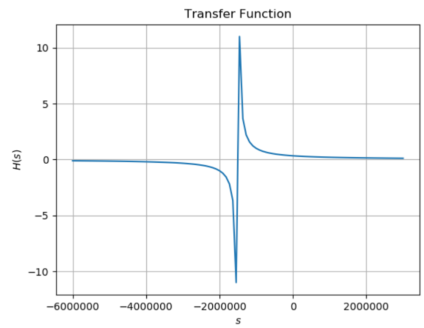

Fig. 4.3. Plot of $H(s)$

Consider the frequency-domain transfer function by putting $s=\jmath\omega$

$$H(\jmath\omega)=\frac{5\times10^5}{\jmath\omega+1.5\times10^6} \tag{4.9}$$

$$\implies \left|H(\jmath\omega)\right|=\frac{5\times10^5}{\sqrt{\omega^2+2.25\times10^{12}}} \tag{4.10}$$

As ω increases, $\left|H(\jmath\omega)\right|$ decreases.

In other words, the amplitude of high-frequency signals gets diminished and they get filtered out.

Therefore, this is a low-pass filter.

4.4 Using trapezoidal rule for integration, formulate the difference equation by considering

$$y(n)=y(t)|_{t=n} \tag{4.11}$$

Solution:

$$\frac{v_c(t)}{R_1}+\frac{v_c(t)-v_2(t)}{R_2}+C_0\frac{dv_c}{dt}=0 \tag{4.12}$$

$$\implies C_0\frac{dv_c}{dt}=\frac{2u(t)-v_c(t)}{R_2}-\frac{v_c(t)}{R_1} \tag{4.13}$$

$$\implies v_c(t)|_{t=n}^{n+1}=\int_n^{n+1}\left(\frac{2u(t)-v_c(t)}{R_2C_0}-\frac{v_c(t)}{R_1C_0}\right)dt \tag{4.14}$$

By the trapezoidal rule of integration

$$\int_a^b f(t)dt\approx\frac{b-a}{2}(f(a)+f(b)) \tag{4.15}$$

Consider $y(t) = v_c(t)$

$$y(n+1) - y(n) = \frac{1}{R_2C_0}(u(n) + u(n+1)) - \frac{1}{2}(y(n+1) + y(n))\left(\frac{1}{R_1C_0} + \frac{1}{R_2C_0}\right) \quad (4.16)$$

Thus, the difference equation is

$$y(n+1)\left(1 + \frac{1}{2R_1C_0} + \frac{1}{2R_2C_0}\right) = y(n)\left(1 - \frac{1}{2R_1C_0} - \frac{1}{2R_2C_0}\right) + \frac{1}{R_2C_0}(u(n) + u(n+1)) \quad (4.17)$$

4.5 Find $H(z)$

Solution:
Let $\mathcal{Z}\{y(n)\} = Y(z)$
On taking the Z-transform on both sides of the difference equation

$$zY(z)\left(1 + \frac{1}{2R_1C_0} + \frac{1}{2R_2C_0}\right) = Y(z)\left(1 - \frac{1}{2R_1C_0} - \frac{1}{2R_2C_0}\right) + \frac{1}{R_2C_0}\left(\frac{1}{1-z^{-1}} + \frac{z}{1-z^{-1}}\right) \quad (4.18)$$

$$Y(z)\left(z + \frac{z}{2R_1C_0} + \frac{z}{2R_2C_0} - 1 + \frac{1}{2R_1C_0} + \frac{1}{2R_2C_0}\right) = \frac{1}{R_2C_0}\frac{1+z}{1-z^{-1}} \quad (4.19)$$

Also

$$v_2(t) = 2 \qquad \forall t \geq 0 \quad (4.20)$$
$$\implies x(n) = 2u(n) \quad (4.21)$$
$$\implies X(z) = \frac{2}{1-z^{-1}} \qquad |z| > 1 \quad (4.22)$$

Thus, the transfer function in z-domain is

$$H(z) = \frac{Y(z)}{X(z)} \quad (4.23)$$
$$= \frac{\frac{1+z}{2R_2C_0}}{z + \frac{z}{2R_1C_0} + \frac{z}{2R_2C_0} - 1 + \frac{1}{2R_1C_0} + \frac{1}{2R_2C_0}} \quad (4.24)$$
$$= \frac{\frac{1+z^{-1}}{2R_2C_0}}{1 + \frac{1}{2R_1C_0} + \frac{1}{2R_2C_0} - z^{-1} + \frac{z^{-1}}{2R_1C_0} + \frac{z^{-1}}{2R_2C_0}} \quad (4.25)$$

On substituting the values

$$H(z) = \frac{2.5 \times 10^5(1+z^{-1})}{7.5 \times 10^5 + 1 + (7.5 \times 10^5 - 1)z^{-1}} \quad (4.26)$$

with the ROC being

$$|z| > \max\left(1, \left|\frac{7.5 \times 10^5 - 1}{7.5 \times 10^5 + 1}\right|\right) \quad (4.27)$$
$$\implies |z| > 1 \quad (4.28)$$

4.6 How can you obtain $H(z)$ from $H(s)$?

Solution:
The Z-transform can be obtained from the Laplace transform by the substitution

$$s = \frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}} \quad (4.29)$$

where T is the step size of the trapezoidal rule (1 in our case)
This is known as the bilinear transform
Thus

$$H(z) = \frac{\frac{1}{R_2C_0}}{2\frac{1-z^{-1}}{1+z^{-1}} + \frac{1}{R_1C_0} + \frac{1}{R_2C_0}} \quad (4.30)$$
$$= \frac{\frac{1+z^{-1}}{2R_2C_0}}{1 - z^{-1} + \left(\frac{1}{2R_1C_0} + \frac{1}{2R_2C_0}\right)(1+z^{-1})} \quad (4.31)$$
$$= \frac{\frac{1+z^{-1}}{2R_2C_0}}{1 + \frac{1}{2R_1C_0} + \frac{1}{2R_2C_0} - z^{-1} + \frac{z^{-1}}{2R_1C_0} + \frac{z^{-1}}{2R_2C_0}} \quad (4.32)$$
$$= \frac{2.5 \times 10^5(1+z^{-1})}{7.5 \times 10^5 + 1 + (7.5 \times 10^5 - 1)z^{-1}} \quad (4.33)$$

which is the same as what we obtained earlier
In order to find $h(n)$ we represent $H(z)$ as

$$H(z) = \frac{c(1+z^{-1})}{a + bz^{-1}} \quad (4.34)$$
$$= \frac{c}{b} + \frac{c(1-\frac{a}{b})}{a + bz^{-1}} \quad (4.35)$$

Taking the the inverse z transform

$$h(n) = \frac{c}{b}\delta(n) + \frac{c}{a}\left(1 - \frac{a}{b}\right)\left(\frac{a}{b}\right)^n u(n) \quad (4.36)$$