Digital Signal Processing

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Contents

1 Software Installation

Run the following commands

```
sudo apt-get update
sudo apt-get install libffi-dev libsndfile1 python3
    -scipy python3-numpy python3-matplotlib
sudo pip install cffi pysoundfile
```

2 Digital Filter

2.1 Download the sound file from

```
wget https://github.com/HARI-donk-EY/
    sig_pros/blob/main/codes/2/sound_files/
    Sound_Noise.wav
```

2.2 You will find a spectrogram at https://academo.org/demos/spectrum-analyzer.
Upload the sound file that you downloaded in Problem 2.1 in the spectrogram and play. Observe the spectrogram. What do you find?
Solution: There are a lot of yellow lines between 440 Hz to 5.1 KHz. These represent the synthesizer key tones. Also, the key strokes are audible along with background noise.

2.3 Write the python code for removal of out of band noise and execute the code.
Solution:

```
import soundfile as sf
from scipy import signal

#read .wav file

input_signal,fs = sf.read('sound_files/
    Sound_Noise.wav')

#sampling frequency of Input signal

sampl_freq=fs

#order of the filter

order=4

#cutoff frquency 4kHz

cutoff_freq=4000.0

#digital frequency

Wn=2*cutoff_freq/sampl_freq

# b and a are numerator and
    denominatorpolynomials respectively

b, a = signal.butter(order,Wn, 'low')

#filter the input signal with butterworth filter

output_signal = signal.filtfilt(b, a,
    input_signal)

#output signal = signal.lfilter(b, a, input
    signal)
#write the output signal into .wav file

sf.write('sound_files/
    Sound_With_ReducedNoise.wav',
    output_signal, fs)
```

2.4 The output of the python script in Problem 2.3 is the audio file Sound_With_ReducedNoise.wav.
Play the file in the spectrogram in Problem 2.2. What do you observe?
Solution: The key strokes as well as background noise is subdued in the audio. Also, the signal is blank for frequencies above 5.1 kHz.

3 Difference Equation

3.1 Let

$$x(n) = \left\{ \underset{\uparrow}{1}, 2, 3, 4, 2, 1 \right\} \tag{3.1}$$

Sketch $x(n)$.

3.2 Let

$$y(n) + \frac{1}{2}y(n-1) = x(n) + x(n-2), \qquad y(n) = 0, n < 0 \tag{3.2}$$

Sketch $y(n)$.
Solution: The following code yields Fig. 3.3.

```
wget https://github.com/HARI-donk-EY/
    sig_pros/blob/main/codes/3/xnyn.py
```

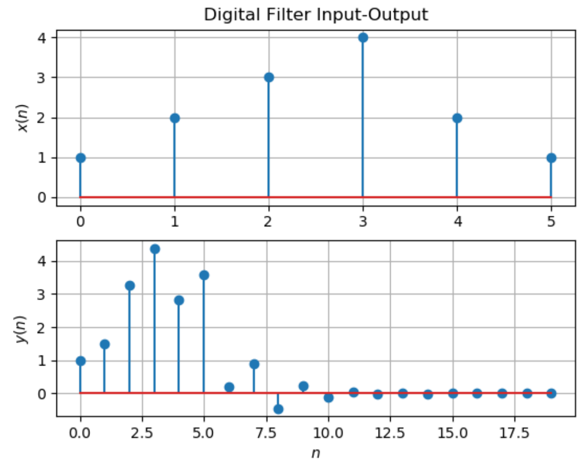

Fig. 3.3

3.3 Repeat the above exercise using C code.
Solution: The following codes can be used, the C-code used to generate $y(n)$ and the plot yields the same Figure as that of 3.3

```
wget https://github.com/HARI-donk-EY/
    sig_pros/blob/main/codes/3/xnyn.c

wget https://github.com/HARI-donk-EY/
    sig_pros/blob/main/codes/3/xnyn2.py
```

4 Z-Transform

4.1 The Z-transform of $x(n)$ is defined as

$$X(z) = \mathcal{Z}\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n)z^{-n} \tag{4.1}$$

Show that

$$\mathcal{Z}\{x(n-1)\} = z^{-1}X(z) \tag{4.2}$$

and find

$$\mathcal{Z}\{x(n-k)\} \tag{4.3}$$

Solution: From (4.1),

$$\mathcal{Z}\{x(n-1)\} = \sum_{n=-\infty}^{\infty} x(n-1)z^{-n} \tag{4.4}$$

$$= \sum_{n=-\infty}^{\infty} x(n)z^{-n-1} = z^{-1} \sum_{n=-\infty}^{\infty} x(n)z^{-n} \tag{4.5}$$

resulting in (4.2). Similarly, it can be shown that

$$\mathcal{Z}\{x(n-k)\} = z^{-k}X(z) \tag{4.6}$$

4.2 Obtain $X(z)$ for $x(n)$ defined from 3.1.

Solution: From 3.1,

$$x(n) = \left\{ \underset{\uparrow}{1}, 2, 3, 4, 2, 1 \right\}$$

and from (4.1), and for given $x(n)$ we have,

$$X(n) = \mathcal{Z}\{x(n)\} \tag{4.7}$$

$$= 1 + 2z^{-1} + 3z^{-2} + 4z^{-3} + 2z^{-4} + z^{-5} \tag{4.8}$$

4.3 Find

$$H(z) = \frac{Y(z)}{X(z)} \tag{4.9}$$

from (3.2) assuming that the Z-transform is a linear operation.

Solution: Applying (4.6) in (3.2),

$$Y(z) + \frac{1}{2}z^{-1}Y(z) = X(z) + z^{-2}X(z) \tag{4.10}$$

$$\implies \frac{Y(z)}{X(z)} = \frac{1 + z^{-2}}{1 + \frac{1}{2}z^{-1}} \tag{4.11}$$

4.4 Find the Z transform of

$$\delta(n) = \begin{cases} 1 & n = 0 \\ 0 & \text{otherwise} \end{cases} \tag{4.12}$$

and show that the Z-transform of

$$u(n) = \begin{cases} 1 & n \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{4.13}$$

is

$$U(z) = \frac{1}{1 - z^{-1}}, \quad |z| > 1 \tag{4.14}$$

Solution: It is easy to show that

$$\delta(n) \overset{\mathcal{Z}}{\rightleftharpoons} 1 \tag{4.15}$$

and from (4.13),

$$U(z) = \sum_{n=0}^{\infty} z^{-n} \tag{4.16}$$

$$= \frac{1}{1 - z^{-1}}, \quad |z| > 1 \tag{4.17}$$

using the fomula for the sum of an infinite geometric progression.

4.5 Show that

$$a^n u(n) \overset{\mathcal{Z}}{\rightleftharpoons} \frac{1}{1 - az^{-1}} \quad |z| > |a| \tag{4.18}$$

Solution: From (4.13), we get,

$$a^n u(n) = \begin{cases} a^n & n \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{4.19}$$

from above,

$$U_a(z) = \sum_{n=0}^{\infty} a^n z^{-n} \tag{4.20}$$

$$= \sum_{n=0}^{\infty} (az^{-1})^n = \frac{1}{1 - az^{-1}}, \quad |z| > 1 \tag{4.21}$$

using the fomula for the sum of an infinite geometric progression.

4.6 Let

$$H\left(e^{\jmath\omega}\right) = H\left(z = e^{\jmath\omega}\right). \tag{4.22}$$

Plot $|H\left(e^{\jmath\omega}\right)|$. Comment. $H(e^{\jmath\omega})$ is known as the *Discret Time Fourier Transform* (DTFT) of $x(n)$.
Solution: The following code plots Fig. 4.6.

```
wget https://github.com/HARI−donk−EY/
    sig_pros/blob/main/codes/4/dtft.py
```

We can note that the graph in Fig. 4.6 is an even periodic function.
From the equations and we get,

$$|H\left(e^{\jmath\omega}\right)| = \left|\frac{1 + e^{-2\jmath\omega}}{1 + \frac{1}{2}e^{-\jmath\omega}}\right| \tag{4.23}$$

$$= \left|\frac{1 + \cos(-2\omega) + j\sin(-2\omega)}{1 + \frac{1}{2}(\cos(-\omega) + j\sin(-\omega))}\right| \tag{4.24}$$

$$= \sqrt{\frac{(1 + \cos 2\omega)^2 + (\sin 2\omega)^2}{\left(1 + \frac{1}{2}\cos\omega\right)^2 + \left(\frac{1}{2}\sin\omega\right)^2}} \tag{4.25}$$

$$= \sqrt{\frac{2\left(1 + \cos 2\omega\right)}{\frac{5}{4} + \cos\omega}} \tag{4.26}$$

$$= \sqrt{\frac{2\left(2\cos^2\omega\right)}{\frac{5}{4} + \cos\omega}} \tag{4.27}$$

$$= \frac{4|\cos\omega|}{\sqrt{5 + 4\cos\omega}} \tag{4.28}$$

and so its fundamental period is 2π.

4.7 Express $h(n)$ in terms of $H(e^{jw})$.
Solution: $h(n)$ is given by the inverse DTFT (IDTFT) of $H\left(e^{\jmath\omega}\right)$

$$h(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H\left(e^{j\omega}\right)e^{j\omega n}\, d\omega \tag{4.29}$$

We can prove this from (4.1),

$$H(e^{jw}) = \sum_{k=-\infty}^{\infty} h(k)e^{-j\omega k} \tag{4.30}$$

$$\tag{4.31}$$

multiplying both sides with $e^{-j\omega k}$ and

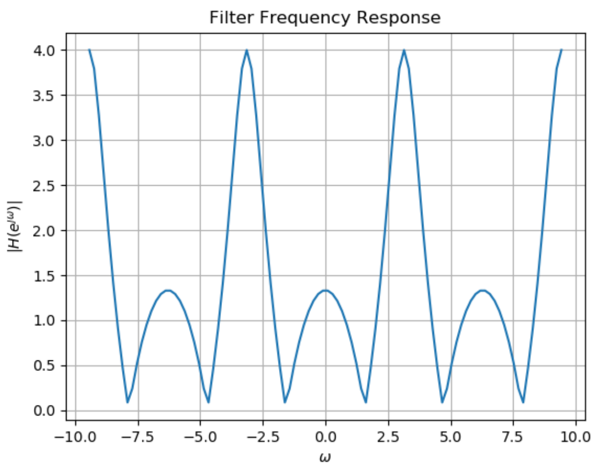

Fig. 4.6: $|H(e^{j\omega})|$

integrating w.r.t. ω.

$$\int_{-\pi}^{\pi} H(e^{jw})e^{j\omega k}\mathrm{d}\omega = \sum_{k=-\infty}^{\infty} h(k)\int_{-\pi}^{\pi} e^{-j\omega n}e^{j\omega k}\mathrm{d}\omega \tag{4.32}$$

$$= \sum_{k=-\infty}^{\infty} h(k)\int_{-\pi}^{\pi} e^{-j\omega(n-k)}\mathrm{d}\omega \tag{4.33}$$

we know that $e^{-j\omega(n-k)}$ is an odd function so integration would be zero for all values of $\omega(n-k) \neq 0$, but 1 if $\omega(n-k) = 0$.

$$\int_{-\pi}^{\pi} e^{-j\omega(n-k)}\mathrm{d}\omega = \begin{cases} 0 & if n-k \neq 0 \\ 1 & if n-k = 0 \end{cases}$$

From this we can complete the summation.

$$\int_{-\pi}^{\pi} H(e^{jw})e^{j\omega k}\mathrm{d}\omega = \sum_{k=-\infty}^{\infty} h(k)\int_{-\pi}^{\pi} e^{-j\omega(n-k)}\mathrm{d}\omega \tag{4.34}$$

$$= h(n)\int_{-\pi}^{\pi}\mathrm{d}\omega \tag{4.35}$$

$$= h(n)2\pi \tag{4.36}$$

$$h(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{jw})e^{j\omega k}\mathrm{d}\omega \tag{4.37}$$

Hence proved.

5 Impulse Response

5.1 Using long division, find

$$h(n), \quad n < 5 \tag{5.1}$$

for H(z) in (4.6).
Solution:

$$H(z) = \frac{1+z^{-2}}{1+\frac{1}{2}z^{-1}} \tag{5.2}$$

Substitute $z^{-1} = x$

$$\begin{array}{r@{\,}l} & \;\; 2x - 4 \\ \frac{1}{2}x + 1 & \overline{)\; x^2 \qquad + 1} \\ & \underline{-x^2 - 2x} \\ & \qquad -2x + 1 \\ & \qquad \underline{\;\; 2x + 4} \\ & \qquad\qquad 5 \end{array}$$

So,

$$H(z) = -4 + 2z^{-1} + \frac{5}{1+\frac{1}{2}z^{-1}} \tag{5.3}$$

$$= -4 + 2z^{-1} + 5\sum_{n=0}^{\infty}\left(\frac{-1}{2}\right)^n z^{-n} \tag{5.4}$$

$$= 1 - \frac{1}{2}z^{-1} + 5\sum_{n=2}^{\infty}\left(\frac{-1}{2}\right)^n z^{-n} \tag{5.5}$$

$$= \sum_{n=0}^{\infty}\left(\frac{-1}{2}\right)^n z^{-n} + 4\sum_{n=2}^{\infty}\left(\frac{-1}{2}\right)^n z^{-n} \tag{5.6}$$

$$= \sum_{n=-\infty}^{\infty} u(n)\left(\frac{-1}{2}\right)^n z^{-n} + \sum_{n=-\infty}^{\infty} u(n-2)\left(\frac{-1}{2}\right)^{n-2} z^{-n} \tag{5.7}$$

Therefore from (4.1),

$$h(n) = \left(\frac{-1}{2}\right)^n u(n) + \left(\frac{-1}{2}\right)^{n-2} u(n-2) \tag{5.8}$$

5.2 Find an expression for $h(n)$ using $H(z)$, given that

$$h(n) \overset{\mathcal{Z}}{\rightleftharpoons} H(z) \tag{5.9}$$

and there is a one to one relationship between $h(n)$ and $H(z)$. $h(n)$ is known as the *impulse response* of the system defined by (3.2).

Take $H(z)$ from (4.6)
Solution: From (4.6),

$$H(z) = \frac{1}{1+\frac{1}{2}z^{-1}} + \frac{z^{-2}}{1+\frac{1}{2}z^{-1}} \tag{5.10}$$

$$\implies h(n) = \left(-\frac{1}{2}\right)^n u(n) + \left(-\frac{1}{2}\right)^{n-2} u(n-2) \tag{5.11}$$

using (4.18) and (4.6). Let,

$$h(n) = h_1(n) + h_2(n) \tag{5.12}$$

Where,

$$h_1(n) = \left(-\frac{1}{2}\right)^n u(n) \tag{5.13}$$

$$h_1(n) = \left(-\frac{1}{2}\right)^{n-2} u(n-2) \tag{5.14}$$

Then ROC of $h_1(n)$ and $h_2(n)$ will both be $|z| > |-\frac{1}{2}|$.
Hence ROC of $H(z)$ is $|z| > \frac{1}{2}$

5.3 Sketch $h(n)$. Is it bounded? Justify theoretically.
Solution: The following code plots Fig. 5.3.

```
wget https://github.com/HARI−donk−EY/
    sig_pros/blob/main/codes/5/hn.py
```

We know,

$$h(n) = \left(-\frac{1}{2}\right)^n u(n) + \left(-\frac{1}{2}\right)^{n-2} u(n-2)$$

then,

$$\lim_{n\to\infty} h(n) = \lim_{n\to\infty}\left(-\frac{1}{2}\right)^n u(n) + \lim_{n\to\infty}\left(-\frac{1}{2}\right)^{n-2} u(n-2) \tag{5.15}$$

$$h(n) = \begin{cases} 5\left(\frac{-1}{2}\right)^n & n \geq 2 \\ \left(\frac{-1}{2}\right)^n & 0 \leq n < 2 \\ 0 & n < 0 \end{cases} \tag{5.16}$$

5.4 Is $h(n)$ convergent? Justify using ratio test.

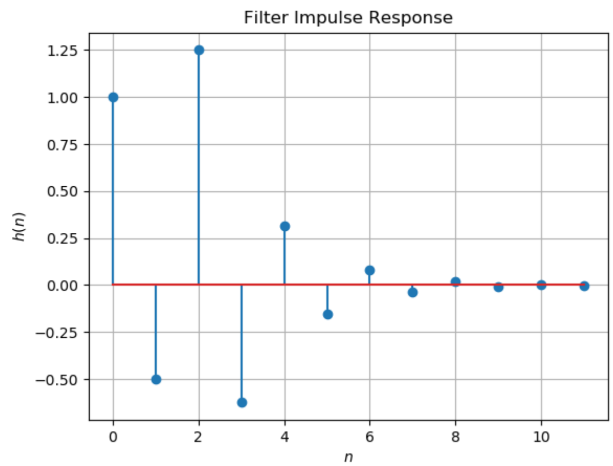

Fig. 5.3: $h(n)$ as the inverse of $H(z)$

Solution:

$$h(n) = \left(-\frac{1}{2}\right)^n u(n) + \left(-\frac{1}{2}\right)^{n-2} u(n-2) \tag{5.17}$$

If $L = \lim_{n\to\infty}\left|\frac{h(n+1)}{h(n)}\right| < 1$, then $h(n)$ is convergent.

$$L = \lim_{n\to\infty}\left|\frac{h(n+1)}{h(n)}\right| \tag{5.18}$$

$$= \lim_{n\to\infty}\left|\frac{\left(-\frac{1}{2}\right)^{n+1} u(n+1) + \left(-\frac{1}{2}\right)^{n-1} u(n-1)}{\left(-\frac{1}{2}\right)^{n} u(n) + \left(-\frac{1}{2}\right)^{n-2} u(n-2)}\right| \tag{5.19}$$

$$= \left|\frac{-\frac{1}{2} + \left(-\frac{1}{2}\right)^{-1}}{1 + \left(-\frac{1}{2}\right)^{-2}}\right| \tag{5.20}$$

$$= \left|\frac{-\frac{5}{2}}{5}\right| \tag{5.21}$$

$$= \frac{1}{2} \tag{5.22}$$

Since $L < 1$, we can say that $h(n)$ is convergent.

5.5 The system with $h(n)$ is defined to be stable if

$$\sum_{n=-\infty}^{\infty} h(n) < \infty \tag{5.23}$$

Is the system defined by (3.2) stable for the impulse response in (5.9)?

Solution:

$$\sum_{n=-\infty}^{\infty} h(n) = \sum_{n=-\infty}^{\infty} \left(\left(-\frac{1}{2}\right)^{n} u(n) + \left(-\frac{1}{2}\right)^{n-2} u(n-2) \right) \tag{5.24}$$

$$= \sum_{n=0}^{\infty} \left(-\frac{1}{2}\right)^{n} u(n) + \sum_{(n-2)=0}^{\infty} \left(-\frac{1}{2}\right)^{n-2} u(n - \tag{5.25}$$

$$= 2\left(\frac{1}{1+\frac{1}{2}}\right) \tag{5.26}$$

$$= \frac{4}{3} \tag{5.27}$$

As $\sum_{n=-\infty}^{\infty} h(n) = \frac{4}{3}$ is less than ∞, the system defined by (3.2) is stable for the impulse response in (5.9).

5.6 Verify the above result using a python code.
Solution: The following code determines if it is convergent or not:

```
wget https://github.com/HARI−donk−EY/
    sig_pros/blob/main/codes/5/new.py
```

5.7 Compute and sketch $h(n)$ using

$$h(n) + \frac{1}{2}h(n-1) = \delta(n) + \delta(n-2), \tag{5.28}$$

This is the definition of $h(n)$.
Solution: The following code plots Fig. 5.7. Note that this is the same as Fig. 5.3.

```
wget https://github.com/HARI−donk−EY/
    sig_pros/blob/main/codes/5/hndef.py
```

Computing,

$$h(0) = 1$$
$$h(1) = -\frac{1}{2}h(0)$$
$$h(2) = -\frac{1}{2}h(1) + 1$$
$$Parallely, h(n) = -\frac{1}{2}h(n-1)$$

5.8 Compute

$$y(n) = x(n) * h(n) = \sum_{n=-\infty}^{\infty} x(k)h(n-k) \tag{5.29}$$

Comment. The operation in (5.29) is known as *convolution*.

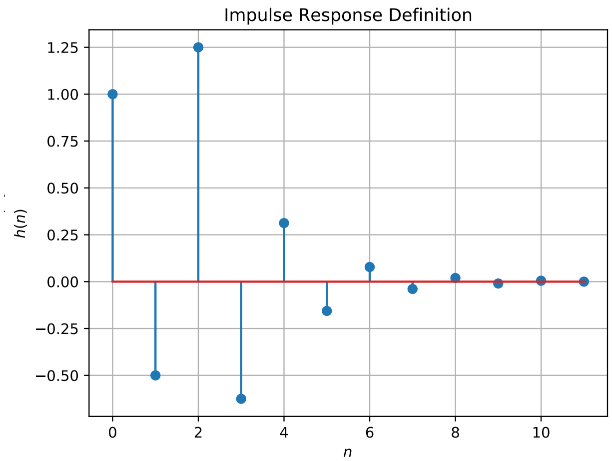

Fig. 5.7: $h(n)$ from the definition

Solution: The following code plots Fig. 5.8. Note that this is the same as $y(n)$ in Fig. 3.3.

```
wget https://github.com/HARI−donk−EY/
    sig_pros/blob/main/codes/5/ynconv.py
```

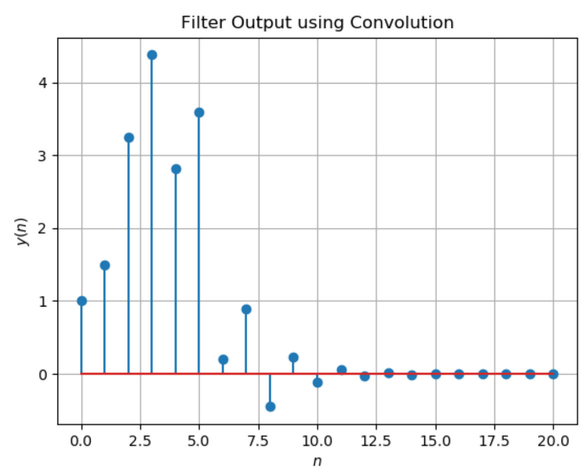

Fig. 5.8: $y(n)$ from the definition of convolution

5.9 Express the above convolution using a toeplitz matrix. **Solution:**

$$\mathbf{y} = \mathbf{x} \circledast \mathbf{h} \tag{5.30}$$

$$\mathbf{y} = \begin{pmatrix} h_1 & 0 & . & . & . & 0 \\ h_2 & h_1 & . & . & . & 0 \\ h_3 & h_2 & h_1 & . & . & 0 \\ h_{m-1} & . & . & . & h_2 & h_1 \\ h_m & h_{m-1} & . & . & . & h_2 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & 0 & . & . & . & h_m \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ . \\ . \\ . \\ . \\ x_n \end{pmatrix} \tag{5.31}$$

$$\mathbf{y} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ \frac{-1}{2} & 1 & 0 & 0 & 0 & 0 \\ \frac{5}{4} & \frac{-1}{2} & 1 & 0 & 0 & 0 \\ \frac{-5}{8} & \frac{5}{4} & \frac{-1}{2} & 1 & 0 & 0 \\ \frac{5}{16} & \frac{-5}{8} & \frac{5}{4} & \frac{-1}{2} & 1 & 0 \\ \frac{-5}{32} & \frac{5}{16} & \frac{-5}{8} & \frac{5}{4} & \frac{-1}{2} & 1 \\ \frac{5}{64} & \frac{-5}{32} & \frac{5}{16} & \frac{-5}{8} & \frac{5}{4} & \frac{-1}{2} \\ 0 & \frac{5}{64} & \frac{-5}{32} & \frac{5}{16} & \frac{-5}{8} & \frac{5}{4} \\ 0 & 0 & \frac{5}{64} & \frac{-5}{32} & \frac{5}{16} & \frac{-5}{8} \\ 0 & 0 & 0 & \frac{5}{64} & \frac{-5}{32} & \frac{5}{16} \\ 0 & 0 & 0 & 0 & \frac{5}{64} & \frac{-5}{32} \\ & & & & & \frac{5}{64} \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1. \\ 1.5 \\ 3.25 \\ 4.375 \\ 2.8125 \\ 3.59375 \\ 0.203125 \\ 0.9375 \\ -0.39062 \\ 0.3125 \\ 0. \\ 0.078125 \end{pmatrix} \tag{5.32}$$

And this is what we got in (5.29)

5.10 Show that

$$y(n) = \sum_{n=-\infty}^{\infty} x(n-k)h(k) \tag{5.33}$$

Solution: From (5.29), we substitute $k := n-k$ to get

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k) \tag{5.34}$$

$$= \sum_{n-k=-\infty}^{\infty} x(n-k)\, h(k) \tag{5.35}$$

$$= \sum_{k=-\infty}^{\infty} x(n-k)\, h(k) \tag{5.36}$$

6 DFT

6.1 Compute

$$X(k) \triangleq \sum_{n=0}^{N-1} x(n) e^{-\jmath 2\pi kn/N}, \quad k = 0, 1, \dots, N-1 \tag{6.1}$$

and $H(k)$ using $h(n)$.

Solution: The following code plots Fig. 6.1.

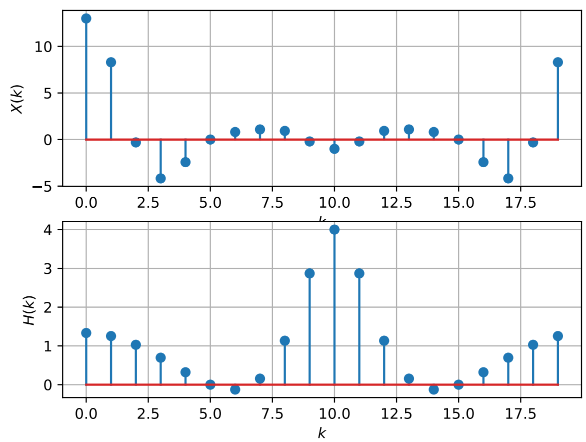

Fig. 6.1: $X(k), H(k)$ from the DFT

```
wget https://github.com/LokeshBadisa/
    EE3900-Linear-Systems-and-Signal-
    Processing/blob/main/codes/xkhkdft.py
```