

PINGALA ASSIGNMENTS

J Sai Sri Hari Vamshi
AI21BTECH11014

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1 JEE 2019

Let α and β ($\alpha > \beta$) be the roots of the equation $z^2 - z - 1 = 0$. Define,

$$a_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad n \geq 1 \quad (1.1)$$

$$b_n = a_{n-1} - a_{n+1}, \quad n \geq 2, \quad b_1 = 1 \quad (1.2)$$

Verify the following using a python code.

1.1

$$\sum_{k=1}^n a_k = a_{n+2} - 1, \quad n \geq 1 \quad (1.3)$$

Solution:

Download the Python code using

```
$ wget https://github.com/HARI-donk-
-EY/sig_pros/tree/main/pingala/codes/1
_1.py
```

and run it using,

```
$ python3 1_1.py
```

From Fig. 1.1, both the graphs are similar for *LHS* and *RHS*.

Hence 1.1 is true.

1.2

$$\sum_{k=1}^{\infty} \frac{a_k}{10^k} = \frac{10}{89} \quad (1.4)$$

Solution: Download the Python code using

```
$ wget https://github.com/HARI-donk-
-EY/sig_pros/tree/main/pingala/codes/1
_2.py
```

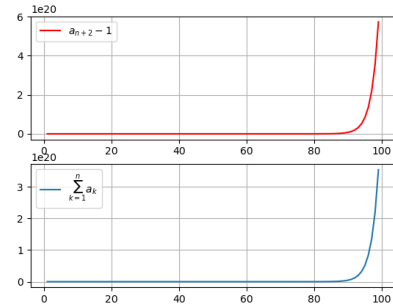


Fig. 1.1

and run it using,

```
$ python3 1_2.py
```

The Fig. 1.2 shows that the difference between *LHS* and *RHS* tends to zero as the value of k increases.

It shows that for a large value of k , the

$$LHS \rightarrow RHS$$

Hence 1.2 is true.

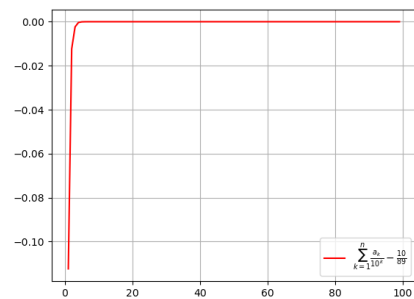


Fig. 1.2

1.3

$$b_n = \alpha^n + \beta^n, \quad n \geq 1 \quad (1.5)$$

Solution: Download the Python code using

```
$ wget https://github.com/HARI-donk-
-EY/sig_pros/tree/main/pingala/codes/1_
_3.py
```

and run it using,

```
$ python3 1_3.py
```

From Fig. 1.3, both the graphs are similar for *LHS* and *RHS*.

Hence 1.3 is true.

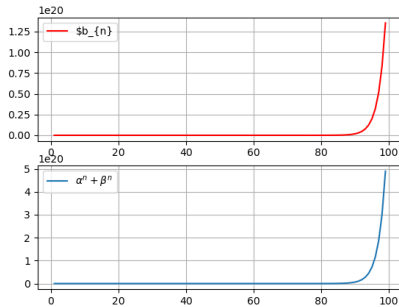


Fig. 1.3

1.4

$$\sum_{k=1}^{\infty} \frac{b_k}{10^k} = \frac{8}{89} \quad (1.6)$$

Solution:

Download the Python code using

```
$ wget https://github.com/HARI-donk-
-EY/sig_pros/tree/main/pingala/codes/1_
_4.py
```

and run it using,

```
$ python3 1_4.py
```

The Fig. 1.4 shows that the difference between *LHS* and *RHS* tends to $\frac{12}{89}$ as the value of k increases.

It shows that for a large value of k , the

$$LHS \rightarrow RHS$$

Hence 1.4 is false.

2 PINGALA SERIES

2.1 The *one sided* Z-transform of $x(n)$ is defined as

$$X^+(z) = \sum_{n=0}^{\infty} x(n)z^{-n}, \quad z \in \mathbb{C} \quad (2.1)$$

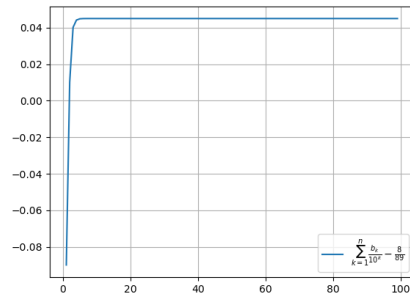


Fig. 1.4

2.2 The *Pingala* series is generated using the difference equation

$$x(n+2) = x(n+1) + x(n) \quad (2.2)$$

$$x(0) = x(1) = 1, \quad n \geq 0 \quad (2.3)$$

Generate a stem plot for $x(n)$.

Solution:

Obtain the python code to generate the plot using

```
$ wget https://github.com/HARI-donk-EY/
sig_pros/tree/main/pingala/codes/2_2.py
```

Run the code using

```
$ python3 2_2.py
```

The following Fig. 2.2 is obtained

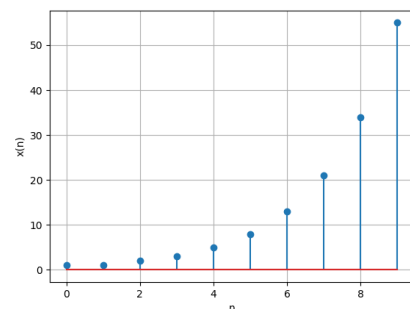


Fig. 2.2

2.3 Find $X^+(z)$.

Solution:

$$x(n+2) = x(n+1) + x(n) \quad (2.4)$$

Applying positive Z-transform on both sides as we know that Z-transform is a linear operator.

$$\sum_{k=0}^{\infty} x(k+2)z^{-k} = \sum_{k=0}^{\infty} x(k+1)z^{-k} + \sum_{k=0}^{\infty} x(k)z^{-k} \quad (2.5)$$

$$z^2 (X^+(z) - x(0) - x(1)) = X^+(z) + z(X^+(z) - x(0)) \quad (2.6)$$

$$X^+(z) = \frac{z^2}{z^2 - z - 1} \quad (2.7)$$

$$X^+(z) = \frac{1}{1 - z^{-1} - z^{-2}} \quad (2.8)$$

2.4 Find $x(n)$.

Solution:

$$X^+(z) = \frac{1}{(1 - \alpha z)(1 - \beta z)} \quad (2.9)$$

where α, β are the roots of the equation

$$z^2 - z - 1 = 0 \quad (2.10)$$

Co-efficient of z^{-k} in the above expression is $x(k)$, so by comparing co-efficients.

$$X^+(z) = \frac{1}{\alpha - \beta} \left(\frac{\alpha}{1 - \alpha z^{-1}} - \frac{\beta}{1 - \beta z^{-1}} \right) \quad (2.11)$$

Using binomial theorem, we get

$$x(k) = \frac{\alpha^{k+1} - \beta^{k+1}}{\alpha - \beta} \quad (2.12)$$

2.5 Sketch

$$y(n) = x(n-1) + x(n+1), \quad n \geq 0 \quad (2.13)$$

Solution:

Obtain the python code to generate the plot using

```
$ wget https://github.com/HARI-donk-EY/sig_pros/tree/main/pingala/codes/2_5.py
```

Run the code using

```
$ python3 2_5.py
```

The following Fig. 2.5 is obtained

2.6 Find $Y^+(z)$. **Solution:**

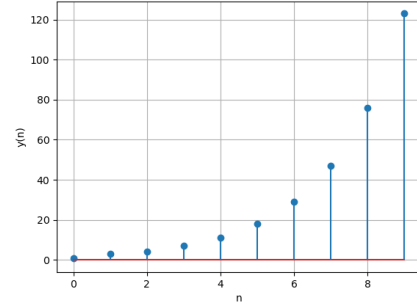


Fig. 2.5

Take +ve Z-transform on both sides of (2.13).

$$\sum_{k=0}^{\infty} y(k)z^{-k} = \sum_{k=0}^{\infty} x(k+1)z^{-k} + \sum_{k=0}^{\infty} x(k)z^{-k} \quad (2.14)$$

$$Y^+(z) = z(X^+(z) - x(0)) + X^+(z) \quad (2.15)$$

$$\because x(-1) = 0$$

$$Y^+(z) = \frac{z + z^{-1}}{1 - z^{-1} - z^{-2}} - z \quad (2.16)$$

$$\therefore Y^+(z) = \frac{1 + 2z^{-1}}{1 - z^{-1} - z^{-2}} \quad (2.17)$$

2.7 Find $y(n)$.

Solution:

Co-efficient of z^{-n} in $Y^+(z)$ will be $y(n)$.

$$Y^+(z) = \frac{1}{1 - z^{-1} - z^{-2}} + \frac{2z^{-2}}{1 - z^{-1} - z^{-2}} \quad (2.18)$$

$$y(k) = \frac{\alpha^{k+1} - \beta^{k+1}}{\alpha - \beta} + 2 \frac{\alpha^k - \beta^k}{\alpha - \beta} \quad (2.19)$$

$$y(k) = \frac{\alpha^{k+2} + \alpha^k - \beta^k - \beta^{k+2}}{\alpha - \beta} \quad (2.20)$$

$$y(k) = \frac{\alpha^{k+2} - \beta \alpha^{k+1} + \alpha \beta^{k+1} - \beta^{k+2}}{\alpha - \beta} \quad (2.21)$$

$$[\because \alpha\beta = -1]$$

$$\therefore y(k) = \alpha^{k+1} + \beta^{k+1} \quad (2.22)$$

3 POWER OF THE Z TRANSFORM

$$X^+(z) = \sum_{k=0}^{\infty} x(k)z^{-k} = z \sum_{k=1}^{\infty} a(k)z^{-k} \quad (3.17)$$

3.1 Show that.

$$\sum_{k=1}^n a_k = \sum_{k=0}^{n-1} = x(n) * u(n-1) \quad (3.1)$$

Solution:

$$x(k) = a(k+1) \quad (3.2)$$

$$\Rightarrow \sum_{k=0}^{n-1} x(k) = \sum_{k=0}^{n-1} a(k+1) \quad (3.3)$$

$$\Rightarrow \sum_{k=1}^n a_k = \sum_{k=0}^{n-1} x(k) \quad (3.4)$$

$$x(n) * u(n-1) = \sum_{k=-\infty}^{\infty} x(k)u(n-k-1) \quad (3.5)$$

$$u(n-k-1) = \begin{cases} 0 & k < n-1 \\ 1 & k \geq n-1 \end{cases} \quad (3.6)$$

$$x(k) = 0, \quad \forall k < 0 \quad (3.7)$$

$$\therefore \sum_{k=0}^{n-1} x(k) = x(n) * u(n-1) \quad (3.8)$$

3.2 show that,

$$a_{n+2} - 1, \quad n \geq 1 \quad (3.9)$$

can be expressed as,

$$[x(n+1) - 1]u(n) \quad (3.10)$$

Solution:

$$x(k) = a(k+1) \quad (3.11)$$

$$\Rightarrow x(k+1) = a(k+2) \quad (3.12)$$

$$a(k+2) - 1 = x(k+1) - 1 \quad (3.13)$$

$$\therefore a(k+2) - 1 = [x(k+1) - 1]u(k) \quad (3.14)$$

$$[\because \forall n \geq 1] \quad (3.15)$$

3.3 show that,

$$\sum_{k=1}^{\infty} \frac{a_k}{10^k} = \frac{1}{10} \sum_{x(k)}^{10^k} = \frac{1}{10} X^+(10) \quad (3.16)$$

Solution:

$$z = 10 \quad (3.18)$$

$$\Rightarrow 10 \sum_{k=1}^{\infty} \frac{a_k}{10^k} = \sum_{k=0}^{\infty} \frac{x(k)}{10^k} \quad (3.19)$$

$$= X^+(10) \quad (3.20)$$

$$\sum_{k=1}^{\infty} \frac{a_k}{10^k} = \frac{1}{10} \sum_{k=0}^{\infty} \frac{x(k)}{10^k} \quad (3.21)$$

$$\therefore \sum_{k=1}^{\infty} \frac{a_k}{10^k} = \frac{1}{10} X^+(10) \quad (3.22)$$

3.4 Show that,

$$\alpha^n + \beta^n, \quad n \geq 1 \quad (3.23)$$

can be expressed as

$$w(n) = (\alpha^{n+1} + \beta^{n+1})u(n) \quad (3.24)$$

and find $W(z)$.

Solution:

Applying Z-transform on both sides,

$$W(z) = \sum_{n=-\infty}^{\infty} (\alpha^{n+1} + \beta^{n+1})u(n)z^{-n} \quad (3.25)$$

$$= \sum_{n=0}^{\infty} (\alpha^{n+1} + \beta^{n+1})z^{-n} \quad (3.26)$$

$$= \alpha \sum_{n=0}^{\infty} (\alpha z^{-1})^n + \beta \sum_{n=0}^{\infty} (\beta z^{-1})^n \quad (3.27)$$

$$\text{ROC: } |z| > \max(\alpha, \beta)$$

$$= \frac{\alpha}{1 - \alpha z^{-1}} + \frac{\beta}{1 - \beta z^{-1}} \quad (3.28)$$

$$= \frac{1 + 2z^{-1}}{1 - z^{-1} - z^{-2}} \quad (3.29)$$

3.5 Show that

$$\sum_{k=1}^{\infty} \frac{b_k}{10^k} = \frac{1}{10} \sum_{k=0}^{\infty} \frac{y(k)}{10^k} = \frac{1}{10} Y^+(10) \quad (3.30)$$

Solution:

$$y(k) = b(k+1) \quad (3.31)$$

$$\sum_{k=0}^{\infty} y(k)z^{-k} = \sum_{k=0}^{\infty} b(k+1)z^{-k} \quad (3.32)$$

$$= Y^+(z) \quad (3.33)$$

$$\sum_{k=0}^{\infty} y(k)z^{-k} = z \sum_{k=1}^{\infty} b(k)z^{-k} \quad (3.34)$$

$$= Y^+(z) \quad (3.35)$$

$$\text{Assume: } z = 10 \quad (3.36)$$

$$\therefore \sum_{k=1}^{\infty} \frac{b_k}{10^k} = \frac{1}{10} \sum_{k=0}^{\infty} \frac{y(k)}{10^k} \quad (3.37)$$

$$= \frac{1}{10} Y^+(10) \quad (3.38)$$

3.6 Solve the JEE 2019 problem.

Solution:

1.2

$$X^+(z) = z \sum_{k=1}^{\infty} a(k)z^{-k} \quad (3.39)$$

$$= \frac{1}{1 - z^{-1} - z^{-2}} \quad (3.40)$$

$$\text{Assume: } z = 10 \quad (3.41)$$

$$\sum_{k=1}^{\infty} \frac{a_k}{10^k} = \frac{1}{10 \left(1 - \frac{1}{10} - \frac{1}{100}\right)} \quad (3.42)$$

$$= \frac{10}{89} \quad (3.43)$$

1.3

$$y(k) = \alpha^{k+1} + \beta^{k+1} \quad (3.44)$$

$$y(k) = b(k+1) \quad (3.45)$$

$$\Rightarrow b(k) = \alpha^k + \beta^k \quad (3.46)$$

1.4

$$\sum_{k=1}^{\infty} \frac{b_k}{10^k} = \frac{1}{10} Y^+(10) \quad (3.47)$$

$$= \frac{1}{10} \left[\frac{1 + \frac{2}{10}}{1 - \frac{1}{10} - \frac{1}{100}} \right] \quad (3.48)$$

$$\therefore Y^+(z) = \frac{1 + 2z^{-1}}{1 - z^{-1} - z^{-2}} \quad (3.49)$$

$$= \frac{12}{89} \quad (3.50)$$

Run the following code to get the expressions of $x(n)$ and $y(n)$

```
$ https://github.com/HARI-donk-EY/
sig_pros/tree/main/pingala/codes/Xz.py
```

Use the following command in the terminal to run the code

```
$ python3 Xz.py
```