UNIT-1

Vector Differentiation

Vector function: Let 's' be the set of real numbers for each scalar $t \in s$, if there exists a unique vector $\bar{f}$ then $\bar{f}$ is said to be a vector function.

The vector function of a scalar variable 't'. denoted by $\bar{f}(t)$, which can be expressed in terms of mutually orthogonal unit vectors $\bar{i}, \bar{j}$ & $\bar{k}$ as

$$\bar{f}(t) = f_1(t)\bar{i} + f_2(t)\bar{j} + f_3(t)\bar{k}$$

Derivative of a Vector function

If $\lim_{t \to a} \frac{\bar{f}(t) - \bar{f}(a)}{t-a}$, $t \neq a$ and $a \in I$, exists then the limit is called as derivative of $\bar{f}$ at $t = a$ and is denoted by $\left(\frac{d\bar{f}}{dt}\right)_{t=a}$ or $\bar{f}^{1}(a)$

Properties

If $\bar{A}$ and $\bar{B}$ are differentiable vector functions of a scalar variable 't', then.

(i) $\frac{d}{dt}(\bar{A} \pm \bar{B}) = \frac{d\bar{A}}{dt} \pm \frac{d\bar{B}}{dt}$

(ii) $\frac{d}{dt}(\bar{A} \cdot \bar{B}) = \frac{d\bar{A}}{dt} \cdot \bar{B} + \bar{A} \cdot \frac{d\bar{B}}{dt}$

(iii) $\frac{d}{dt}(\bar{A} \times \bar{B}) = \frac{d\bar{A}}{dt} \times \bar{B} + \bar{A} \times \frac{d\bar{B}}{dt}$

(iv) $\frac{d}{dt}(\phi \bar{A}) = \frac{d\phi}{dt}\bar{A} + \phi \frac{d\bar{A}}{dt}$

differentiable scalar function of 't'.

(v) If $\bar{c}$ is any constant vector, then

$$\frac{d}{dt}(\bar{c}) = \bar{0}$$

Note: If $\bar{f} = f_1(t)\bar{i} + f_2(t)\bar{j} + f_3(t)\bar{k}$

then

$$\frac{d\bar{f}}{dt} = \bar{i}\frac{df_1}{dt} + \bar{j}\frac{df_2}{dt} + \bar{k}\frac{df_3}{dt}$$

Scalar and Vector point function

Consider a region in 3-dimensional space. If to each point $p(x, y, z)$, we associate a unique scalar ϕ then $\phi(x, y, z)$ is called a scalar point function.

Similarly if to each point $p(x, y, z)$, we associate a unique vector $\bar{f}$, then $\bar{f}(x, y, z)$ is called a vector point function.

Eg: i) At each point $p(x, y, z)$ of a heated solid, there will be temperature $T(x, y, z)$, which is a scalar point fn.

ii) Consider a particle moving in space. At each point $p(x, y, z)$ on its path, the particle will be having a velocity $\bar{v}(x, y, z)$ be a vector point fn

Tangent Vector

Let $x = x(t)$, $y = y(t)$, $z = z(t)$ be continuous and derivable in $a \le t \le b$

Then the set of all points $(x(t), y(t), z(t))$, is called a curve in space

If $\bar{r}(t)$ represents a curve, then $\frac{d\bar{r}}{dt}$ is a tangent vector to the curve at a point 'p'.

Vector Differential Operator

The vector differential operator denoted by '∇' and is defined as $\nabla \equiv \bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}$.

Gradient of a scalar point function

If ϕ is a scalar point function defined in some region of space, then the vector function

$\bar{i}\frac{\partial \phi}{\partial x} + \bar{j}\frac{\partial \phi}{\partial y} + \bar{k}\frac{\partial \phi}{\partial z}$ is defined as gradient of ϕ

and is denoted by grad ϕ or $\nabla \phi$

ie grad $\phi = \nabla \phi = \bar{i}\frac{\partial \phi}{\partial x} + \bar{j}\frac{\partial \phi}{\partial y} + \bar{k}\frac{\partial \phi}{\partial z}$

$= \sum \bar{i}\frac{\partial \phi}{\partial x}$

Properties: If f and g are two-scalar point function. then

(i) $grad(f \pm g) = grad f \pm grad g$

(ii) $grad(fg) = f(grad g) + g(grad f)$

(iii) $grad\left(\frac{f}{g}\right) = \frac{g(grad f) - f(grad g)}{g^2}$

Results

1. The directional derivative of a scalar function ϕ at $p(x,y,z)$ in the direction of a unit vector $\hat{e}$ is $\hat{e}\cdot \text{grad}\,\phi$

2. grad ϕ is a vector along the normal to the level surface at $p(x,y,z)$

3. If θ is an angle b/w two level surfaces f and g, then $\cos\theta = \dfrac{(\text{grad} f)\cdot(\text{grad} g)}{|\text{grad} f|\,|\text{grad} g|}$

4. The ~~maximum~~ greatest value of the directional derivative is in the direction of grad ϕ, its max value is $|\text{grad}\,\phi|$

① Find the directional derivative of $f = xy + yz + zx$ in the direction of $\bar{i} + 2\bar{j} + 2\bar{k}$ at $(1,2,0)$

Sol: Given that $f = xy + yz + zx$.

$\dfrac{\partial f}{\partial x} = y + z$, $\quad \dfrac{\partial f}{\partial y} = x + z$, $\quad \dfrac{\partial f}{\partial z} = y + x$.

Now, $\text{grad} f = i\,(y+z) + j\,(x+z) + \bar{k}\,(y+x)$

$(\text{grad} f)_{(1,2,0)} = i(2) + j(1) + k(2+1)$

$= 2\bar{i} + \bar{j} + 3\bar{k}$

Let $\bar{a} = \bar{i} + 2\bar{j} + 2\bar{k}$

unit vector in the direction of

$$\hat{e} = \frac{\bar{a}}{|\bar{a}|} = \frac{i+2j+2k}{\sqrt{1+4+4}} = \frac{i+2j+2k}{3}$$

$\therefore$ Directional derivative in the direction of $\hat{e}$ is

$$\hat{e} \cdot \text{grad} f = \frac{(i+2j+2k)}{3} \cdot (2i+2j+2k)$$

$$= \frac{1}{3}\left(1(2)+2(2)+2(2)\right)$$

$$= \frac{10}{3}$$

(2) Find the directional derivative of the function $f = x^2 - y^2 + 2z^2$ at $P(1,2,3)$ in the direction of $\overrightarrow{PQ}$ where $Q = (5,0,4)$

Sol Given that $f = x^2 - y^2 + 2z^2$

$$\frac{\partial f}{\partial x} = 2x \quad \frac{\partial f}{\partial y} = -2y \quad \frac{\partial f}{\partial z} = 4z.$$

Now, $\text{grad} f = \bar{i}(2x) + \bar{j}(-2y) + \bar{k}(4z)$

$$(\text{grad} f)_{(1,2,3)} = i(2) + j(-2\times 2) + k(4\times 3)$$

$$= 2\bar{i} - 4j + 12k.$$

Given,

In the direction of $\overrightarrow{PQ}$ and $Q = (5,0,4)$

$$\hat{e} = \frac{\overrightarrow{PQ}}{|\overrightarrow{PQ}|} = \frac{4i-2j+k}{\sqrt{16+4+1}} = \frac{4i-2j+k}{\sqrt{21}}$$

$\overrightarrow{PQ} = Q - P = 4\bar{i} - 2\bar{j} + \bar{k}$

$\therefore$ Directional derivative in the direction of $\hat{e}$ is

$$\hat{e} \cdot \text{grad} f = \frac{8+8+12}{\sqrt{21}} = \frac{28}{\sqrt{21}}$$

3) Find the ~~direction~~ al unit normal vector to the surface $x^3+y^3+3xyz=3$ at the point $(1,2,-1)$

Sol: Let $\phi = x^3+y^3+3xyz-3$

Now,

$\frac{\partial \phi}{\partial x} = 3x^2+3yz$; $\frac{\partial \phi}{\partial y} = 3y^2+3xz$; $\frac{\partial \phi}{\partial z} = 3xy$

$\text{grad}\,\phi = \Sigma i \frac{\partial \phi}{\partial x}$

$= \bar{i}(3x^2+3yz) + \bar{j}(3y^2+3xz) + \bar{k}(3xy)$

$(\text{grad}\,\phi)_{(1,2,-1)} = i(3-6) + j(12-3) + \bar{k}(6)$

$= -3i+9j+6k$

Since grad ϕ is normal vector to the given surface then unit normal, $\bar{N} = \frac{\text{grad}\,\phi}{|\text{grad}\,\phi|}$

$\bar{N} = \frac{-3i+9j+6k}{\sqrt{9+81+36}} = \frac{3(-i+3j+2k)}{\sqrt{126}}$

$= \frac{-\bar{i}+3j+2\bar{k}}{\sqrt{14}}$

4) Find the angle b/w the surfaces $xy^2z = 3x+z^2$ and $3x^2-y^2+2z=1$ at $(1,-2,1)$

Let $f: xy^2z-3x-z^2$ and

$g: 3x^2-y^2+2z-1$

Now, $\frac{\partial f}{\partial x} = y^2z-3$; $\frac{\partial f}{\partial y} = 2xyz$; $\frac{\partial f}{\partial z} = xy^2-2z$

$$\text{grad } f = \sum \bar{i} \frac{\partial f}{\partial x}$$

$$= \bar{i}(y^2 z - 3) + \bar{j}(2xyz) + \bar{k}(xy^2 - 2z)$$

$$(\text{grad } f)_{(1,-2,1)} = i(4-3) + j(-4) + \bar{k}(4-2)$$

$$= \bar{i} - 4\bar{j} + 2\bar{k}$$

Now,

$$\frac{\partial g}{\partial x} = 6x \qquad \frac{\partial g}{\partial y} = -2y \qquad \frac{\partial g}{\partial z} = 2.$$

$$\text{grad } g = \sum \bar{i} \frac{\partial f}{\partial x}$$

$$= \bar{i}(6x) + \bar{j}(-2y) + \bar{k}(2)$$

$$(\text{grad } g)_{(1,-2,1)} = i(6) + j(4) + k(2)$$

$$= 6\bar{i} + 4\bar{j} + 2\bar{k}$$

Let θ be the angle b/w the two surfaces then

$$\cos\theta = \frac{(\text{grad} f)(\text{grad } g)}{|\text{grad} f||\text{grad} g|}$$

$$= \frac{(\bar{i} - 4\bar{j} + 2\bar{k}) \cdot (6\bar{i} + 4\bar{j} + 2\bar{k})}{\sqrt{1+16+4}\sqrt{36+16+4}}$$

$$= \frac{1(6) + (-4)(4) + 2(2)}{\sqrt{21}\sqrt{56}}$$

$$= \frac{6 - 16 + 4}{\sqrt{7 \times 3}\sqrt{14 \times 4}} = \frac{-6}{2 \times 7\sqrt{6}} = \frac{-3}{7\sqrt{6}}$$

$$\sqrt{7 \times 3} \times 2\sqrt{7 \times 2}$$

$$\theta = \cos^{-1}\left(\frac{-3}{7\sqrt{6}}\right) \qquad = 2 \times 7\sqrt{3 \times 2}$$

5) Find the maximum value of the directional derivative of $\phi = x^2yz$ at $(1,4,1)$

Given that $\phi = x^2yz$

$$\frac{\partial\phi}{\partial x} = 2xyz \; ; \; \frac{\partial\phi}{\partial y} = x^2z \quad \frac{\partial\phi}{\partial z} = x^2y$$

Now,

$$\text{grad}\,\phi = \Sigma i \frac{\partial\phi}{\partial x} = i(2xyz) + j(x^2z) + k(x^2y)$$

$$(\text{grad}\,\phi)_{(1,4,1)} = i(8) + j(1) + \bar{k}(4) = 8\bar{i} + \bar{j} + 4\bar{k}$$

Since the greatest value of the directional derivative is in the direction of grad ϕ, then its max value $= |\text{grad}\,\phi| = \sqrt{64+1+16} = 9$.

6) Find the directional derivative of the function $xy^2 + yz^2 + 2x^2$ along the tangent to the curve $x=t$, $y=t^2$, $z=t^3$ at $(1,1,1)$

Sol: position vector $\bar{r} = t\bar{i} + t^2\bar{j} + t^3\bar{k}$

$$\frac{\partial\bar{r}}{\partial t} = \bar{i} + 2t\bar{j} + 3t^2\bar{k}$$

tangent to the curve $\left(\frac{\partial r}{\partial t}\right)_{(1,1,1)} = (i + 2j + 3k)_{t=1}$

$$\therefore \bar{a} = \bar{i} + 2\bar{j} + 3\bar{k}$$

Given $f = xy^2 + yz^2 + 2x^2$

$\frac{\partial f}{\partial x} = y^2 + 2xz \quad \frac{\partial f}{\partial y} = 2xy + z^2 \quad \frac{\partial f}{\partial z} = 2yz + x^2$

$\text{grad } f = i\frac{\partial f}{\partial x} + \bar{j}\frac{\partial f}{\partial y} + \bar{k}\frac{\partial f}{\partial z}$

$= (y^2 + 2xz)i + (2xy + z^2)j + (2yz + x^2)k$

$(\text{grad} f)_{(1,1,1)} = (1+2)\bar{i} + (1+2)j + 3\bar{j}$

$= 3\bar{i} + 3\bar{j} + 3\bar{k}$

unit vector along the $\bar{a}$ is $\hat{e}$

$\hat{e} = \frac{\bar{a}}{|\bar{a}|} = \frac{i+2j+3\bar{k}}{\sqrt{1+4+9}} = \frac{(i+2j+3k)}{\sqrt{14}}$

$\therefore$ Directional derivative in the direction of $\hat{e}$ is

$\hat{e} \text{ graf} = \frac{i+2j+3k}{\sqrt{14}} (3\bar{i} + 3\bar{j} + 3\bar{k})$

$= \frac{1}{\sqrt{14}} (3 + 3(2) + 3(3))$

$= \frac{18}{\sqrt{14}}$

7) Find the directional derivative of $xyz^2 + xz$ at $(1,1,1)$ in the direction of the normal to the surface $3xy^2 + y = 2$ at $(0,1,1)$

Sol. grad ϕ

$\frac{\partial \phi}{\partial x} = 3y^2 \quad \frac{\partial \phi}{\partial y} = 6xy + 1 \quad \frac{\partial \phi}{\partial z} = -1$

$\text{grad } \phi = \Sigma \bar{i} \frac{\partial \phi}{\partial x}$

$= i(3yz) + j(6xy + 1) + k(-1)$

$(\text{grad} \phi)_{(0,1,1)} = 3\bar{i} + j - k$

$$\hat{e} = \frac{\text{grad}\,\phi}{|\text{grad}\,\phi|} = \frac{3i+j-k}{\sqrt{9+1+1}} = \frac{3i+j-k}{\sqrt{11}}$$

Given

grad f

$$\frac{\partial f}{\partial x} = z^2y+z \quad \frac{\partial f}{\partial y} = z^2x \quad \frac{\partial f}{\partial z} = 2xy+x$$

$$\text{grad } f = \Sigma \bar{i}\frac{\partial f}{\partial x}$$

$$= i(z^2y+z) + j(z^2x) + \bar{k}(2xy+x)$$

$$(\text{grad} f)_{(1,1,1)} = 2\bar{i}+\bar{j}+3\bar{k}$$

∴ Directional derivative in the direction of $\hat{e}$ is

$$\hat{e}\cdot\text{grad}f = \frac{3i+j-k}{\sqrt{11}}(2i+j+3k)$$

$$= \frac{6-3+1}{\sqrt{11}} = \frac{4}{\sqrt{11}}$$

Divergence of a vector

If $\bar{f}$ is any continuously differenceable vector point function, then

$$\bar{i}\frac{\partial \bar{f}}{\partial x} + j\frac{\partial \bar{f}}{\partial y} + k\cdot\frac{\partial \bar{f}}{\partial z}$$ is called as

divergence of $\bar{f}$ and is denoted by div $\bar{f}$

$$\text{div}\,\bar{f} = i\frac{\partial f}{\partial x} + j\frac{\partial f}{\partial y} + k\frac{\partial f}{\partial z}$$

$$= \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)\bar{f} = \nabla\cdot\bar{f}$$

we can take $\text{div}\,\bar{f} = \Sigma i\cdot\frac{\partial \bar{f}}{\partial x}$.

Result: (i) If $\bar{f}$

then div $\bar{f}$

since div $\bar{f}$

(ii) div ($\bar{f}$

since d

(iii) A vec

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Result:- (i) If $\bar{f} = f_1\bar{i} + f_2\bar{j} + f_3\bar{k}$

Then $\operatorname{div}\bar{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$.

since $\operatorname{div}\bar{f} = \sum \bar{i} \cdot \frac{\partial \bar{f}}{\partial x}$

$= \sum \bar{i} \frac{\partial}{\partial x}(f_1\bar{i} + f_2\bar{j} + f_3\bar{k})$

$= \sum (\bar{i}\cdot\bar{i}) \frac{\partial f_1}{\partial x}$

$= \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$.

(ii) $\operatorname{div}(\bar{f} \pm \bar{g}) = \operatorname{div}\bar{f} \pm \operatorname{div}\bar{g}$

since $\operatorname{div}(\bar{f} \pm \bar{g}) = \sum \bar{i} \frac{\partial}{\partial x}(\bar{f} \pm \bar{g})$

$= \sum \bar{i} \cdot \left(\frac{\partial \bar{f}}{\partial x} \pm \frac{\partial \bar{g}}{\partial x}\right)$

$= \sum \left(i \cdot \frac{\partial \bar{f}}{\partial x} \pm i \cdot \frac{\partial \bar{g}}{\partial x}\right)$

$= \sum \bar{i} \cdot \frac{\partial \bar{f}}{\partial x} \pm \sum i \cdot \frac{\partial \bar{g}}{\partial x}$

$= \nabla \bar{f} \pm \nabla \bar{g}$

(iii) A vector point function $\bar{f}$ is said to be solenoidal if $\operatorname{div}\bar{f} = 0$

Physical Interpretation of divergence

Suppose $\bar{f}(x, y, z, t)$ is the velocity of a fluid at a point (x, y, z) and time t.

Imagine a small rectangular box within the fluid we would like to measure the rate per unit volume at which the fluid flows out at any given time t. The divergence of $\bar{f}$ measures.

the outward flow or expansion of the fluid from their point at any time 't'. This gives the physical interpretation of divergence

1) If $\bar{f} = xy^2\bar{i} + 2x^2yz\bar{j} - 3yz^2\bar{k}$ find div $\bar{f}$ at $(1,-1,1)$

Sol: Here $f_1 = xy^2$; $f_2 = 2x^2yz$; $f_3 = -3yz^2$

$$\text{div } \bar{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

$$= \frac{\partial}{\partial x}(xy^2) + \frac{\partial}{\partial y}(2x^2yz) + \frac{\partial}{\partial z}(-3yz^2)$$

$$= y^2(1) + (2x^2z)(1) + (-3y)(2z)$$

$$= y^2 + 2x^2z - 6yz$$

$$\therefore (\text{div } \bar{f})_{(1,-1,1)} = (-1)^2 + 2(1)^2(1) - 6(-1)(1)$$

$$= 9$$

2) S.T $\bar{f} = (x+3y)\bar{i} + (y-2z)\bar{j} + (x-2z)\bar{k}$ is solenoidal.

Sol: $$\text{div } \bar{f} = \frac{\partial}{\partial x}(x+3y) + \frac{\partial}{\partial y}(y-2z) + \frac{\partial}{\partial z}(x-2z)$$

$$= (1+0) + (1-0) + (0-2) = 0$$

$\bar{f}$ is a solenoidal vector

3) find di

Let ϕ =

Now,

$\therefore$ grad ϕ

Thus,

div $\bar{f}$

4) S.T

div

5) If

-d

Sol

3) find div $\bar{f}$, when $\bar{f} = grad(x^3+y^3+z^3-3xyz)$

Let $\phi = x^3+y^3+z^3-3xyz$

Now, $\frac{\partial\phi}{\partial x} = 3x^2-3yz$; $\frac{\partial\phi}{\partial y} = 3y^2-3xz$ and

$\frac{\partial\phi}{\partial z} = 3z^2-3xy$

$\therefore grad\,\phi = \sum \bar{i}\frac{\partial\phi}{\partial x}$

$\bar{f} = \bar{i}(3x^2-3yz)+\bar{j}(3y^2-3xz)+\bar{k}(3z^2-3xy)$

Thus,

$div\,\bar{f} = \frac{\partial}{\partial x}(3x^2-3yz)+\frac{\partial}{\partial y}(3y^2-3xz)+\frac{\partial}{\partial z}(3z^2-3xy)$

$= (6x-0)+(6y-0)+(6z-0)$

$= 6(x+y+z)$

4) S.T div $\bar{r} = 3$

$div\,\bar{r} = \sum \bar{i}\cdot\frac{\partial\bar{r}}{\partial x}$

$= \sum(\bar{i}\cdot\bar{i})$

$= \bar{i}\cdot\bar{i}+\bar{j}\cdot\bar{j}+\bar{k}\cdot\bar{k}$

$= 1+1+1 = 3$

$\because \bar{r} = x\bar{i}+y\bar{j}+z\bar{k}$

then $\frac{\partial\bar{r}}{\partial x} = \bar{i}$

5) If $\bar{f} = (x+3y)\bar{i}+(y-2z)\bar{j}+(x+pz)\bar{k}$ is solenoidal, fin p

Sol

$div\,\bar{f} = \frac{\partial}{\partial x}(x+3y)+\frac{\partial}{\partial y}(y-2z)+\frac{\partial}{\partial z}(x+pz)$

$= (1+0)+(1-0)+(0+p) = p+2$

$\because \bar{f}$ is solenoidal, then

$div\,\bar{f} = 0 \Rightarrow 2+p = 0$

$\boxed{p = -2}$

Curl of a vector

If $\bar{f}$ is any continuously differentiable vector point function. Then the vector function

$\bar{i} \times \frac{\partial \bar{f}}{\partial x} + \bar{j} \times \frac{\partial \bar{f}}{\partial y} + \bar{k} + \frac{\partial \bar{f}}{\partial z}$ is defined as curl of $\bar{f}$ and is denoted by curl $\bar{f}$ or $\nabla \times \bar{f}$

$$\text{curl } \bar{f} = i \times \frac{\partial \bar{f}}{\partial x} + j \times \frac{\partial \bar{f}}{\partial y} + \bar{k} \times \frac{\partial \bar{f}}{\partial z}.$$

$$= \left(i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z}\right) \times \bar{f}$$

$$= \nabla \times \bar{f}$$

we can take $\text{curl } \bar{f} = \Sigma i \times \frac{\partial \bar{f}}{\partial x}$.

Results (i) If $\bar{f} = f_1 \bar{i} + f_2 \bar{j} + f_3 \bar{k}$, then

$$\text{curl } \bar{f} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial f} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$$

By def, $\text{curl } \bar{f} = \Sigma \bar{i} \times \frac{\partial \bar{f}}{\partial x}$

$$= \Sigma i \times \frac{\partial}{\partial x}(f_1 \bar{i} + f_2 \bar{j} + f_3 \bar{k})$$

$$= \Sigma \left(\frac{\partial f_2}{\partial x} \bar{k} - \frac{\partial f_3}{\partial x} \bar{j}\right)$$

$$= \left(\frac{\partial f_2}{\partial x} \bar{k} - \frac{\partial f_3}{\partial x} \bar{j}\right) + \left(\frac{\partial f_3}{\partial y} \bar{i} - \frac{\partial f_1}{\partial y} \bar{k}\right) + \left(\frac{\partial f_1}{\partial z} \bar{j} - \frac{\partial f_2}{\partial z} \ldots\right.$$

$$= \bar{i}\left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right) - \bar{j}\left(\frac{\partial f_3}{\partial x} - \frac{\partial f_1}{\partial z}\right) + \bar{k}\left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \ldots\right.$$

$$= \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ f_1 & f_2 & f_3 \end{vmatrix}.$$

ii) $\text{curl}(\bar{f} \pm g) = \text{curl}\bar{f} \pm \text{curl}g$

Physical Interpretation of curl

If $\bar{\omega}$ is the angular velocity of a rigid body rotating about a fixed axis and $\bar{v}$ is the velocity of any point $p(x, y, z)$ on the body, then

$$\boxed{\bar{\omega} = \frac{1}{2}\text{curl}\,\bar{v}}$$

Thus, the angular velocity of rotation of any point ~~poir~~ is equal to half the curl of velocity vector.

Irrotational Motion

Any motion in which curl of the velocity vector is a null vector

$\text{curl}\,\bar{v} = \bar{0}$ is said to be irrotational

Def A vector $\bar{f}$ is said to be irrotational, if

$\text{curl}\,\bar{f} = \bar{0}$

If $\bar{f}$ is irrotational, then $\exists$ a scalar fn ϕ such that $\bar{f} = \text{grad}\,\phi$

1) If $\bar{f} = xy^2\bar{i} + 2x^2yz\bar{j} - 3yz^2\bar{k}$, find curl $\bar{f}$ at $(1, -1, 1)$

(4) $\text{curl}\,\bar{f} = \nabla \times \bar{f} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy^2 & 2x^2yz & -3yz^2 \end{vmatrix}$

$= \bar{i}\left\{\frac{\partial}{\partial y}(-3yz^2) - \frac{\partial}{\partial z}(2x^2yz)\right\} - j\left\{\frac{\partial}{\partial x}(-3yz^2) - \frac{\partial}{\partial z}(xy^2)\right\} + \bar{k}\left\{\frac{\partial}{\partial x}(2x^2yz) - \frac{\partial}{\partial y}(xy^2)\right\}$

$= \hat{i}(-3z^2 - 2x^2y) - \hat{j}(0-0) + \bar{k}(4xyz - 2xy)$

$(\text{curl}\,\bar{f})_{(1,-1,1)} = i(-3 + 2(-2)) + \bar{k}(-4 - 2(-1))$

$= -i - 2\bar{k}$

2. S.T $\bar{f} = (y+z)\bar{i} + (z+x)\bar{j} + (x+y)\bar{k}$ is Irrotational

$\text{curl}\,\bar{f} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y+z & z+x & x+y \end{vmatrix}$

$= i\left\{\frac{\partial}{\partial y}(x+y) - \frac{\partial}{\partial z}(z+x)\right\} - j\left\{\frac{\partial}{\partial x}(x+y) - \frac{\partial}{\partial z}(y+z)\right\} + \bar{k}\left\{\frac{\partial}{\partial x}(z+x) - \frac{\partial}{\partial y}(y+z)\right\}$

$= \bar{i}(1-1) - j(1-1) + \bar{k}(1-1)$

$= 0\bar{i} + 0\bar{j} + 0\bar{k} = \bar{0}$

$\therefore \bar{f}$ is Irrotational.

3) S.T $\text{curl}\,\bar{r} = \bar{0}$

$\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

$\text{curl}\,\bar{r} = \Sigma\, i \times \frac{\partial \bar{r}}{\partial x}$

$\frac{\partial \bar{r}}{\partial x} = \bar{i}$

$= \Sigma(\bar{i} \times \bar{i})$

$= \bar{i}\times\bar{i} + \bar{j}\times\bar{j} + \bar{k}\times\bar{k}$

$= \bar{0} + \bar{0} + \bar{0} = \bar{0}$

8) find curl $\bar{f}$, when $\bar{f} = grad(x^3+y^3+z^3-3xyz)$.

Sol $\phi = x^3+y^3+z^3-3xyz$

Now,

$\frac{\partial\phi}{\partial x} = 3x^2-3yz$ $\frac{\partial\phi}{\partial y} = 3y^2-3xz$ $\frac{\partial\phi}{\partial z} = 3z^2-3xy$

$\therefore grad\,\phi = \Sigma i \frac{\partial\phi}{\partial x}$

$\bar{f} = i(3x^2-3yz)+j(3y^2-3xz)+\bar{k}(3z^2-3xy)$

$$curl\,\bar{f} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3x^2-3yz & 3y^2-3xz & 3z^2-3xy \end{vmatrix}$$

$= i\left[\frac{\partial}{\partial y}(3z^2-3xy) - \frac{\partial}{\partial z}(3y^2-3xz)\right\} - j\left\{\frac{\partial}{\partial x}(3z^2-3xy) - \frac{\partial}{\partial z}(3x^2-3yz)\right\} + \bar{k}\left\{\frac{\partial}{\partial x}(3y^2-3xz) - \frac{\partial}{\partial y}(3x^2-3yz)\right\}$

$= i(0-3x-0+3x) - \bar{j}(0-3y+0+3y) + \bar{k}(-3z+3z)$

$= i(0) - j(0) + k(0) = \bar{0}$

9) find a, b, c so that the vector $\bar{A} = (x+2y+az)i + (bx-3y-z)j + (4x+cy+2z)\bar{k}$ is irrotational. Also find ϕ such that $\bar{A} = \nabla\phi$

Sol

$$curl\,\bar{A} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x+2y+az & bx-3y-z & 4x+cy+2z \end{vmatrix}$$

$i\left\{\frac{\partial}{\partial y}(4x+cy+2z) - \frac{\partial}{\partial z}(bx-3y-z)\right\} - j\left\{\frac{\partial}{\partial x}(4x+cy+2z) - \frac{\partial}{\partial z}(x+2y+az)\right\} + k\left\{\frac{\partial}{\partial x}(bx-3y-z) - \frac{\partial}{\partial y}(x+2y+az)\right\}$

$= i(c+1) - j(-a+4) + k(b-2)$

$\therefore$ It is an irrotational, then $\text{curl}\,\bar{A} = 0$

$c+1=0$, $c=-1$ | $-(4-a)=0$, $a=+4$ | $b-2=0$, $b=2$

$\therefore \bar{A} = (x+2y+4z)\bar{i} + (2x-3y-z)\bar{j}\ (4x-y+2z)k$

$\frac{\partial\phi}{\partial x} = 1 \quad \frac{\partial\phi}{\partial y} = -3 \quad \frac{\partial\phi}{\partial z} = 2$

$\text{grad}\,\phi = \Sigma\, \bar{i}\, \frac{\partial\phi}{\partial x}$

$= \bar{i} + (-3)\bar{j} + 2\bar{k}$

$= i - 3\bar{j} + 2\bar{k}$] X

6) If $\bar{F} = x^2 y\bar{i} - 2xz\bar{j} + 2yz\bar{k}$, then find (i) curl $\bar{F}$ (ii) curl (curl $\bar{F}$).

Sol (i) curl $\bar{f}$ $= \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2y & -2xz & 2yz \end{vmatrix}$

$= i\left\{\frac{\partial}{\partial y}(2yz) - \frac{\partial}{\partial z}(-2xz)\right\} - j\left\{\frac{\partial}{\partial x}(2yz) - \frac{\partial}{\partial z}(x^2y)\right\}$
$+ k\left\{\left(\frac{\partial}{\partial x}(-2xz) - \frac{\partial}{\partial y}(x^2y)\right\}\right.$

$= i(2z+2x) - j(0) + k(-2z-x^2)$

(ii) curl (curl $\bar{f}$) $= \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2z+2x & 0 & -(2z+x^2) \end{vmatrix}$

$= i\left(\frac{\partial}{\partial y}(-2z-x^2)\right) - j\left(\frac{\partial}{\partial x}(-2z-x^2) - \frac{\partial}{\partial z}(2z+2x)\right) +$

$k\left(\frac{\partial}{\partial y}(2z+2x)\right)$

$= i(0-0) - j(-2x-2) + k(0-0)$

$= j(2x+2)$

$= 2(x+1)\bar{j}$

—— o ——

→ we have to find $\phi \rightarrow \bar{A} = \nabla\phi$

$(x+2y+4z)\bar{i} + (2x-3y-z)\bar{j} + (4x-y+2z)\bar{k}$

$= i\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z}$

$\frac{\partial\phi}{\partial x} = (x+2y+4z)$, $\frac{\partial\phi}{\partial y} = (2x-3y-z)$, $\frac{\partial\phi}{\partial z} = 4x-y+2z$

now, $\frac{\partial\phi}{\partial x} = x+2y+4z \Rightarrow \phi = \int(x+2y+4z)\,dx$

$= \frac{x^2}{2} + 2xy + 4zx + f_1(y,z)$

$\frac{\partial\phi}{\partial y} = 2x-3y-z \Rightarrow \phi = \int(2x-3y-z)\,dy$

$= 2xy - \frac{3y^2}{2} - zy + f_2(z,x)$

$\frac{\partial\phi}{\partial z} = 4x-y+2z \Rightarrow \phi = \int(4x-y+2z)\,dz$

$= 4xz - yz + z^2 + f_3(x,y)$

$\phi = \frac{x^2}{2} - \frac{3}{2}y^2 + z^2 + 2xy - yz + 4xz + c$

7) If $\bar{a}$ is a constant vector, then s. T

(i) $\text{div}(\bar{r}\times\bar{a}) = 0$, (ii) $\text{curl}(\bar{r}\times\bar{a}) = -2\bar{a}$,

(iii) $\text{grad}(\bar{r}\cdot\bar{a}) = \bar{a}$

Sol (i) By def, $\operatorname{div}(\bar{r}\times\bar{a}) = \sum i\cdot\frac{\partial}{\partial x}(\bar{r}\times\bar{a})$

$= \sum i\cdot\left(\frac{\partial \bar{r}}{\partial x}\times\bar{a} + \bar{r}\times\frac{\partial \bar{a}}{\partial x}\right)$

$\therefore \frac{\partial \bar{a}}{\partial x} = \bar{0}$ $\quad = \sum i\cdot(i\times\bar{a})$

$\bar{i}\times\bar{i} = \bar{0}$

$= 0$

(ii) By def, $\operatorname{curl}(\bar{r}\times\bar{a}) = \sum \bar{i}\times\frac{\partial}{\partial x}(\bar{r}\times\bar{a})$

$= \sum \bar{i}\times\left(\frac{\partial \bar{r}}{\partial x}\times\bar{a} + \bar{r}\,\frac{\partial \bar{a}}{\partial x}\right)$

Let $\quad = \sum \bar{i}\times(i\times\bar{a})$

$\bar{a} = a_1\bar{i} + a_2\bar{j} + a_3\bar{k}$

$i\cdot\bar{a} = a_1$

$= \sum\left[(\bar{i}\cdot\bar{a})i - (i\cdot i)\bar{a}\right]$

$= \sum(a_1 i - \bar{a})$

$= \sum a_1 i - \sum\bar{a}$

$= \bar{a} - 3\bar{a} = -2\bar{a}$

(iii) By def, $\operatorname{grad}(\bar{r}\cdot\bar{a}) = \bar{r}\cdot\operatorname{grad}\bar{a} + \bar{a}\operatorname{grad}\bar{r}$

$= \bar{r}\cdot(0) + \bar{a}(1)$

$= \bar{a}$

(or)

Here, $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

$\bar{a} = a_1\bar{i} + a_2\bar{j} + a_3\bar{k}$

$\bar{r}\cdot\bar{a} = xa_1 + ya_2 + za_3$

$\frac{\partial}{\partial x}(\bar{r}\cdot\bar{a}) = a_1(1) + 0 + 1 = a$

$\frac{\partial}{\partial y}(\bar{r}\cdot\bar{a}) = a_2, \quad \frac{\partial}{\partial z}(\bar{r}\cdot\bar{a}) = a_3$

$\operatorname{grad}(\bar{r}\cdot\bar{a}) = a_1\bar{i} + a_2\bar{j} + a_3\bar{k} = \bar{a}$

Vector Integration

Def: Let $\bar{f}(t)$ be a differentiable vector function of a scalar variable 't' and

Let $\frac{d}{dt}(\bar{F}(t)) = \bar{f}(t)$

Then $\int \bar{f}(t)\, dt = \bar{F}(t)$

$\bar{F}(t)$ is called the primitive of $\bar{f}(t)$

The set of all primitives of $\bar{f}(t)$,

$\int \bar{f}(t)\, dt = \bar{F}(t) + \bar{c}$, where $\bar{c}$ is any arbitrary constant vector, is called Indefinite Integral of $\bar{f}(t)$

Properties

(i) $\int k\bar{f}(t)\, dt = k\int \bar{f}(t)\, dt$, where k is a real constant

(ii) $\int \{\bar{f}(t) \pm \bar{g}(t)\}\, dt = \int \bar{f}(t)\, dt \pm \int \bar{g}(t)\, dt$

(iii) If $\bar{f}(t) = f_1(t)\bar{i} + f_2(t)\bar{j} + f_3(t)\bar{k}$, then

$\int \bar{f}(t)\, dt = \bar{i}\int f_1(t)\, dt + \bar{j}\int f_2(t)\, dt + \bar{k}\int f_3(t)\, dt$

Definite Integral If $\int \bar{f}(t)\, dt = \bar{F}(t) + \bar{c}$

Then $\int_a^b \bar{f}(t)\, dt = \bar{F}(b) - \bar{F}(a)$

This is called the definite Integral of $\bar{f}(t)$ b/w the limits $t = a$ and $t = b$

If $\bar{f}(t) = f_1(t)\bar{i} + f_2(t)\bar{j} + f_3(t)\bar{k}$

then $\int_a^b \bar{f}(t)\, dt = \bar{i}\int_a^b f_1(t)\, dt + \bar{j}\int_a^b f_2(t)\, dt + \bar{k}\int_a^b f_3(t)\, dt$

Results: (i) since $\frac{d}{dt}(\bar{r}\cdot\bar{s}) = \frac{d\bar{r}}{dt}\cdot\bar{s} + \bar{r}\cdot\frac{d\bar{s}}{dt}$

then $\int\left(\frac{d\bar{r}}{dt}\cdot\bar{s} + \bar{r}\cdot\frac{d\bar{s}}{dt}\right)dt = \bar{r}\cdot\bar{s} + c$

(ii) since $\frac{d}{dt}(\bar{r}^2) = \frac{d}{dt}(\bar{r}\cdot\bar{r}) = 2\bar{r}\cdot\frac{d\bar{r}}{dt}$

then $\int 2\bar{r}\,\frac{d\bar{r}}{dt} = \bar{r}^2 + \bar{c}$

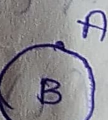
similarly, $\int 2\left(\frac{d\bar{r}}{dt}\cdot\frac{d^2\bar{r}}{dt^2}\right)dt = \left(\frac{d\bar{r}}{dt}\right)^2 + c$

(iii) $\int\left(\frac{d\bar{r}}{dt}\times\bar{s} + \bar{r}\times\frac{d\bar{s}}{dt}\right)dt = \bar{r}\times\bar{s} + \bar{c}$

(iv) $\int\left(\bar{a}\times\frac{d\bar{r}}{dt}\right)dt = \bar{a}\times\bar{r} + \bar{c}$ [where $\bar{a}$ is a constant vector]

(v) $\int\left(\bar{r}\times\frac{d^2\bar{r}}{dt^2}\right)dt = \bar{r}\times\frac{d\bar{r}}{dt} + \bar{c}$

Closed curve: Let 'c' be a curve in space whose Initial point is 'A' and terminal point is 'B'. If the two points A and B coincides then the curve 'c' is called closed curve

$\int_A^B c$ 

A curve $\bar{r} = \bar{f}(t)$ is called a smooth curve if $\bar{f}(t)$ is continuously differentiable

Line Integral

Let $\bar{r} = \bar{f}(t)$ be define a smooth curve 'c'. Joining the points A and B. Let ds be the differential of arc length at P, then 'c'. Now,

$\boxed{\frac{d\bar{r}}{ds} = \bar{T}}$, is the unit vector along the tangent to the curve 'c' at 'p'

Let $\bar{f}(r)$ be a vector point function defined and continuous along 'γ'.

The component of $\bar{f}(r)$, along the tangent at 'p' is $f(\gamma)\cdot\bar{T}$

Now the Integral $\int \bar{f}\cdot\bar{T}\cdot ds$ along the curve 'c' is called as Line Integral of .

ie $\int_A^B \bar{f}\cdot\bar{T}\,ds = \int_C \left(\bar{f}\cdot\frac{d\bar{\gamma}}{ds}\right)ds = \int_C \bar{f}\,d\bar{\gamma}$

Circulation :- If $\bar{v}$ represents the velocity of a fluid particle and 'c' is a closed curve, then $\oint_C \bar{v}\,d\bar{\gamma}$ is called circulation of $\bar{v}$ round the curve $\bar{c}$.

If $\int \bar{v}\cdot d\bar{\gamma} = 0$, then the field $\bar{v}$ is called conservative ie No work is done and the energy is conserved

Work done by a force

If $\bar{f}$ represents the force vector acting on a particle moving along an arc AB, then the total work done by $\bar{f}$ during displacement from A to B is given by the line Integral $\int_A^B \bar{f}\cdot d\bar{\gamma}$

If the force $\bar{f}$ is conservative ie $\bar{f} = \nabla\phi$, then the work done is independent of path. In this case, $\text{curl}\,\bar{f} = \text{curl}(\nabla\phi) = \bar{0}$ and ϕ is called scalar potential or potential function.

If $\text{curl}\,\bar{f} = 0$, then $\bar{f}$ is conservative force field.

1. find the work done by the force $\bar{f} = (3x^2-6zy)\bar{i} + (2y+3xz)\bar{j} + (1-4xyz^2)\bar{k}$ is moving particle from the point $(0,0,0)$ to the point $(1,1,1)$ along the curve

c': $x=t$, $y=t^2$, $z=t^3$

Sol Given curve is, $x=t$, $y=t^2$, $z=t^3$

we have, $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

$= t\bar{i} + t^2\bar{j} + t^3\bar{k}$

Now, $\frac{d\bar{r}}{dt} = \bar{i}(1) + \bar{j}(2t) + \bar{k}(3t^2)$

Given that,

$\bar{f} = (3x^2-6zy)\bar{i} + (2y+3xz)\bar{j} + (1-4xyz^2)\bar{k}$

$= (3t^2-6t^5)\bar{i} + (2t^2+3t^4) + (1-4t^9)\bar{k}$

$\because x=t, y=t^2, z=t^3$

Now,

$\bar{f}\cdot\frac{d\bar{r}}{dt} = (3t^2-6t^5)(1) + (2t^2+3t^4)(2t) + (1-4t^9)(3t^2)$

$= 3t^2 - 6t^5 + 4t^3 + 6t^5 + 3t^2 - 12t^{11}$

$= 6t^2 + 4t^3 - 12t^{11}$

$\therefore$ workdone $= \int_c \bar{f}\cdot d\bar{r} = \int_c \left(\bar{f}\cdot\frac{d\bar{r}}{dt}\right)dt$

$(0,0,0)$
$(1,1,1)$

$= \int_{t=0}^{1} (6t^2+4t^3-12t^{11})dt$

$x=t$
$\therefore t \to 0$ to

$= \left[2t^3 + t^4 - t^{12}\right]_0^1$

$= (2+1-1) - 0 = 2$

2. If $\bar{f} = 3xy\bar{i} - 5z\bar{j} + 10x\bar{k}$, find $\int_c \bar{f}\cdot d\bar{r}$ along the curve $x=t^2+1$, $y=2t^2$, $z=t^3$ from $t=1$ to $t=2$.

Sol: $\bar{f} = 2xy\bar{i} - 5z\bar{j} + 10x\bar{k}$

$x = t^2+1,\ y = 2t^2,\ z = t^3$

$\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

$= (t^2+1)\bar{i} + (2t^2)\bar{j} + t^3\bar{k}$

Now, $\frac{d\bar{r}}{dt} = (2t)\bar{i} + (4t)\bar{j} + (3t^2)\bar{k}$

Given that

$\bar{f} = 3xy\bar{i} - 5z\bar{j} + 10x\bar{k}$

$= 3(t^2+1)(2t^2)\bar{i} - 5(t^3)\bar{j} + 10(t^2+1)\bar{k}$ $\quad x = t^2+1,\ y = 2t^2,\ z = t^3$

$= 3(2t^4+2t^2)\bar{i} - 5t^3\bar{j} + 10(t^2+1)\bar{k}$

$= 6(t^2+t^4)\bar{i} - 5t^3\bar{j} + 10(t^2+1)\bar{k}$

Now, $\bar{f}\cdot\frac{d\bar{r}}{dt} = 6(t^2+t^4)\times 2t + (4t)(-5t^3) + 3t^2(10t^2+10)$

$= 12t^5 + 12t^3 - 20t^4 + 30t^4 + 30t^2$

$= 12t^5 + 10t^4 + 12t^3 + 30t^2$

$\therefore$ workdone $= \int_c \bar{f}\, d\bar{r} = \int_c \left(\bar{f}\cdot\frac{d\bar{r}}{dt}\right) dt \quad t = 1, 2.$

$= \int_1^2 (12t^5 + 10t^4 + 12t^3 + 30t^2)$

$= \left[12\left(\frac{t^6}{6}\right) + 10\left(\frac{t^5}{5}\right) + 12\left(\frac{t^4}{4}\right) + 30\left(\frac{t^3}{3}\right)\right]_1^2$

$= \left[2t^6 + 2t^5 + 3t^4 + 10t^3\right]_1^2$

$= \{2\times 64 + 2\times 32 + 3\times 16 + 10\times 8\} - \{2+2+3+10\}$

$= 320 - 17 = 303$

3) find the work done in moving a particle in the force field $\bar{f} = 3x^2\bar{i} + (2xz-y)\bar{j} + z\bar{k}$ along the St-line from $(0,0,0)$ to $(2,1,3)$

Sol: Eq to the st-line Joining $O(0,0,0)$ & $A(2,1,3)$ are

$$\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1}$$

$$\frac{x-0}{2-0} = \frac{y-0}{1-0} = \frac{z-0}{3-0}$$

$$\frac{x}{2} = \frac{y}{1} = \frac{z}{3} = t, \text{ say}$$

$x = 2t,\ y = t,\ z = 3t$

we have, $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

$= 2t\bar{i} + t\bar{j} + 3t\bar{k}$

then $\frac{d\bar{r}}{dt} = \bar{i}(2) + \bar{j}(1) + \bar{k}(3)$

Giventhat, $\bar{f} = 3x^2 i + (2xz - y)\bar{j} + z\bar{k}$

$= 3(2t)^2\bar{i} + [2(2t)(3t) - t]\bar{j} + (3t)\bar{k}$

$= 12t^2\bar{i} + (12t^2 - t)\bar{j} + 3t\bar{k}$

Now $\bar{f} \cdot \frac{d\bar{r}}{dt} = (12t^2)(2) + (12t^2 - t)(1) + (3t)(3)$

$= 36t^2 + 8t$

$O(0,0,0)$ & $A(2,1,3)$

Take $x = 2t$ as $x \to 0$ $t \to 0$

$x \to 2$ $t \to 1$

$\therefore$ 't' varies from 0 to 1

workdone $= \int_{OA} \bar{f} \cdot d\bar{r} = \int_{t=0}^{1} \left(\bar{f} \cdot \frac{d\bar{r}}{dt}\right) dt$

$= \int_0^1 (36t^2 + 8t)\, dt$

$= (12t^3 + 4t^2)_0^1$

$= 12 + 4 = 16$

H) If $\bar{f} = (-x^2+27)\bar{i} - 6yz\bar{j} + 8xz^2\bar{k}$, Evaluate

$\int_C \bar{f}\, d\bar{r}$ from the point $(0,0,0)$ to $(1,0,0)$ and to

$(1,1,0)$ and then to $(1,1,1)$

We have $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

then $d\bar{r} = \bar{i}\,dx + \bar{j}\,dy + \bar{k}\,dz$

Given that, $\bar{f} = (-x^2+27)\bar{i} - 6yz\bar{j} + 8xz^2\bar{k}$

Now, $\bar{f}\cdot d\bar{r} = (-x^2+27)dx - 6yz\,dy + 8xz^2\,dz$

(i) Along the st-line from $O(0,0,0)$ to $A(1,0,0)$

$y=0,\ z=0 \Rightarrow dy=0,\ dz=0$

x varies from 0 to 1

$\therefore \int_{OA} \bar{f}\cdot d\bar{r} = \int_0^1 (-x^2+27)\,dx = \left[-\frac{x^3}{3} + 27x\right]_0^1$

$= -\frac{1}{3} + 27 = \frac{80}{3}$

(ii) Along $\overrightarrow{AB}$, where $A=(1,0,0)$ and $B=(1,1,0)$

$x=1,\ z=0$ then $dx=0$ and $dz=0$

$y \to 0$ to 1

$\int_{AB} \bar{f}\cdot d\bar{r} = \int_{y=0}^1 -6yz\,dy = 0,\ z=0$

(iii) Along $\overrightarrow{BC}$, where $B=(1,1,0)$ and $C=(1,1,1)$

$x=1,\ y=1 \Rightarrow dx=0,\ dy=0$

$z \to 0$ to 1

$\int_{BC} \bar{f}\cdot d\bar{r} = \int_{z=0}^1 8xz^2\,dz = \int_0^1 8(1)z^2\,dz = \left(\frac{8}{3}z^3\right)_0^1$

$= 8/3$

$\therefore \int_C \bar{f}\cdot d\bar{r} = \frac{80}{3} + 0 + \frac{8}{3} = \frac{88}{3}$

5) Find $\int_c \bar{f}\cdot d\bar{r}$ where $\bar{f} = x^2y^2\bar{i} + y\bar{j}$ and the curve $y^2 = 4x$ in xy-plane from $(0,0)$ to $(4,4)$

2-dimension

Given that, $\bar{f} = x^2y^2\bar{i} + y\bar{j}$

we have, $\bar{r} = x\bar{i} + y\bar{j}$

then $d\bar{r} = \bar{i}dx + \bar{j}dy$

Now, $\bar{f}\cdot d\bar{r} = x^2y^2dx + ydy$

Given curve is $y^2 = 4x$ in xy-plane

$2ydy = 4dx \Rightarrow ydy = 2dx$

$\int_c \bar{f}\cdot d\bar{r} = \int_c x^2y^2dx + ydy$

$= \int_c x^2(4x)dx + 2dx$ ∴ curve varies from $(0,0)$ to $(4,4)$

$= \int_{x=0}^{4} (4x^3+2)dx$

$= [x^4 + 2x]_0^4 = 256 + 8 = 264$

6) If $\bar{F} = y\bar{i} + z\bar{j} + x\bar{k}$, find the circulation of $\bar{f}$ round the curve 'c' where 'c' is the circle $x^2+y^2=1, z=0$

we have, $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

then $d\bar{r} = \bar{i}dx + \bar{j}dy + \bar{k}dz$

Given that, $\bar{f} = y\bar{i} + z\bar{j} + x\bar{k}$

Now, $\bar{f}\cdot d\bar{r} = ydx + zdy + xdz$

Given curve is $x^2+y^2=1, z=0$ in xy-plane

In xy-plane, $z=0 \Rightarrow dz=0$

$\bar{F}\cdot d\bar{r} = ydx$ $\quad (y = \sqrt{1-x^2})$

$\oint_c \bar{f}\cdot d\bar{r} = \oint y\cdot dx$

$= \int_{\theta=0}^{2\pi} \sin\theta(-\sin\theta)d\theta$

$= \int_0^{2\pi} -\sin^2\theta \cdot d\theta$

$= -\int_0^{2\pi} \frac{1-\cos 2\theta}{2} \, d\theta$

$= -\frac{1}{2}\left(\theta - \frac{\sin 2\theta}{2}\right)_0^{2\pi} = -\frac{1}{2}(2\pi) = -\pi$

The parametric eq^n of the circle $x^2+y^2=1$ are

$x = \cos\theta$, $y = \sin\theta$

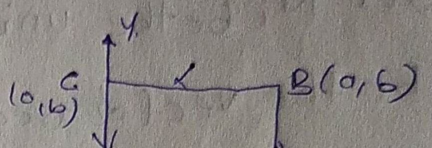
$0 \le \theta < 2\pi$

Then, $dx = -\sin\theta \, d\theta$

7) If $\bar{f} = (x^2+y^2)\bar{i} - 2xy\bar{j}$, find $\oint_C \bar{f}\cdot d\bar{r}$ where 'c' is the rectangle in xy-plane bounded by $y=0, x=0$, $y=b, x=a$

We have, $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

then $d\bar{r} = \bar{i}dx + \bar{j}dy + \bar{k}dz$



Given that, $\bar{f} = (x^2+y^2)\bar{i} - 2xy\bar{j}$

Now, $\bar{f}\cdot d\bar{r} = (x^2+y^2)dx - (2xy)dy$

(i) Line integral along $\overline{OA}$

$y=0$, x varies from 0 to a

$\therefore \int_{OA} \bar{f}\cdot d\bar{r} = \int_0^a x^2 dx = \left[\frac{x^3}{3}\right]_0^a = \frac{a^3}{3}$

(ii) Line integral along $\overline{AB}$

$dx = 0$, $x = a$, y varies from 0 to b

$\therefore \int_{AB} \bar{f}\cdot d\bar{r} = -\int_0^b 2(a)(y)dy = -2a\left[\frac{y^2}{2}\right]_0^b$

$= \left[-ay^2\right]_0^b = -ab^2$

(iii) line integral along $\overline{BC}$

$dy = 0$, $y = b$, x varies from a to 0

$\int_{BC} \bar{f}\cdot d\bar{r} = \int_a^0 (x^2+b^2)dx = \int_a^0 x^2 dx + \int_a^0 b^2 dx = \left[\frac{x^3}{3}\right]_a^0 + b^2[x]_a^0$

$= -\frac{a^3}{3} - ab^2$

(iv) line integral along $\overline{CO}$

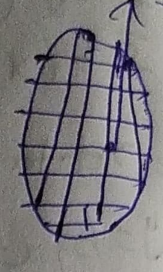
$x = 0$, y varies from b to 0, $\int_{CO} \bar{f}\cdot d\bar{r} = \int_b^0 (0) = 0$

$\therefore \oint \bar{f}\cdot d\bar{r} = \cancel{\frac{a^3}{3}} + (-ab^2) - \cancel{\frac{a^3}{3}} - ab^2 = -2ab^2$

UNIT-2

Surface Integrals

Let 's' be the region of the surface and $\bar{f}(\bar{r})$ be a continuous vector point function defined over the smooth surface $\bar{r} = \bar{f}(u,v)$



Divide the region into 'm' subregions whose areas are $\delta s_1, \delta s_2, \ldots \delta s_m$.

Let $\bar{N}_i$ be the unit normal to δs_i at P_i and $\delta \bar{A}_i$ be the vector area of δs_i then

$$\delta \bar{A}_i = \delta s_i \bar{N}_i$$

Now, $\bar{f}_i \cdot \delta \bar{A}_i = \bar{f}_r \cdot \delta s_i \bar{N}_i$

$$\underset{m\to\infty}{Lt} \sum_{r=1}^{m} \bar{f}_i \cdot \delta \bar{A}_i = \underset{m\to\infty}{Lt} \sum_{r=1}^{m} \bar{f}_r \cdot \delta s_i \bar{N}_i,$$ if exists, is called the normal surface Integral of $\bar{f}$ over region 's' of the given surface and is denoted by

$$\int_S \bar{f} \cdot \bar{N}\, ds$$

Physical Interpretation If $\bar{f}$ represents the velocity vector of a fluid, then the surface Integral $\int_S \bar{f} \cdot \bar{N}\, ds$ over a closed surface 's' represents the rate of flow of fluid through the surface.

Cartesian form: Let $\bar{f} = f_1 \bar{i} + f_2 \bar{j} + f_3 \bar{k}$

$\bar{N} = \cos\alpha \bar{i} + \cos\beta \bar{j} + \cos\gamma \bar{k}$

then $\bar{f} \cdot \bar{N} = f_1 \cos\alpha + f_2 \cos\beta + f_3 \cos\gamma$

$$\int_S \bar{f} \cdot \bar{N}\, ds = \iint_S (f_1 \cos\alpha + f_2 \cos\beta + f_3 \cos\gamma)$$

$= \iint_S f_1\, dy\,dz + f_2\, dz\,dx + f_3\, dx\,dy$

where $dy\,dz = ds\cos\alpha$, $ds\cos\beta = dz\,dx$ and $ds\cos\gamma = dx\,dy$

where $ds\cos\alpha$, $ds\cos\beta$, $ds\cos\gamma$ are the projections of 's' on yz, zx and xy- planes respectively

Note: Let R be the projection of 's' on yz-plane then $ds\cos\alpha = dy\,dz$.

$$\Rightarrow ds = \frac{dy\,dz}{\cos\alpha} = \frac{dy\,dz}{(\bar{N}\cdot\bar{i})}$$

$$\therefore \int_S \bar{f}\cdot\bar{N}\cdot ds = \iint_R \bar{f}\cdot\bar{N}\,\frac{dy\,dz}{(\bar{N}\cdot\bar{i})}$$

Similarly, by projection 's' on zx and xy-planes, we get

$$\int_S \bar{f}\bar{N}\,ds = \iint_R \bar{f}\cdot\bar{N}\,\frac{dz\,dx}{(\bar{N}\cdot\bar{j})} \quad \therefore ds\cos\beta = dz\,dx$$

and $$\int_S \bar{f}\cdot\bar{N}\,ds = \iint_R \bar{f}\cdot\bar{N}\,\frac{dx\,dy}{\bar{N}\cdot\bar{k}} \quad \therefore ds\cos\gamma = dx\,dy$$

1) Evaluate $\int_S \bar{f}\cdot\bar{N}\,ds$ where $\bar{f} = 18z\bar{i} - 12\bar{j} + 3y\bar{k}$ and 's' is the part of the plane $2x+3y+6z=12$ located in the first octant

Let ϕ $2x+3y+6z=12$

$$\frac{\partial\phi}{\partial x} = 2 \quad \frac{\partial\phi}{\partial y} = 3 \quad \frac{\partial\phi}{\partial z} = 6.$$

$$\text{grad}\,\phi = \Sigma\bar{i}\,\frac{\partial\phi}{\partial x} = 2\bar{i} + 3\bar{j} + 6\bar{k}$$

Since grad ϕ is normal to the surface, then

unit normal, $$\bar{N} = \frac{\text{grad}\,\phi}{|\text{grad}\,\phi|} = \frac{2\bar{i}+3\bar{j}+6\bar{k}}{\sqrt{2^2+3^2+6^2}}$$

$= \frac{1}{7}(2i+3j+6k)$

Given that,

$\bar{F} = 18z\bar{i} - 12\bar{j} + 3y\bar{k}$

Now. $\bar{F}\cdot\bar{N} = (18z - 12\bar{j} + 3y\bar{k})\cdot\frac{1}{7}(2\bar{i}+3\bar{j}+6\bar{k})$

$= \frac{1}{7}(36z - 36 + 18y)$

$= \frac{6}{7}(6z + 3y - 6)$

Let 'R' be the projection of 's' on xy-plane then

$\int_S \bar{F}\cdot\bar{N}\, ds = \iint_R (\bar{F}\cdot\bar{N}) \frac{dx\,dy}{(\bar{N}\cdot\bar{k})}$

Also, $\bar{N}\bar{k} = \frac{1}{7}(2\bar{i}+3\bar{j}+6\bar{k})\cdot\bar{k} = \frac{6}{7}$

Given surface is, $2x+3y+6z=12$

$z=0$, then $2x+3y=12$

$y = \frac{12-2x}{3}$

y varies from 0 to $\frac{1}{3}(12-2x)$

Now, $y=0 \Rightarrow 2x=12 \Rightarrow x=6$

x varies from 0 to 6

$\int_S \bar{F}\cdot\bar{N}\, dS = \iint_R (\bar{F}\cdot\bar{N}) \frac{dx\,dy}{\bar{N}\cdot\bar{k}}$

$= \iint_R \frac{6}{7}(6z+3y-6)\frac{dx\,dy}{6/7}$ $\quad 2x+3y+6z=12$

$= \iint_R (12-2x-3y+3y-6)\,dx\,dy$

$= \int_{x=0}^{6}\int_{y=0}^{\frac{1}{3}(12-2x)} (6-2x)\,dx\,dy$

$= \int_{x=0}^{6}$

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Sol

$$= \int_{x=0}^{6} \left\{ \int_{y=0}^{\frac{1}{3}(12-2x)} (6-2x)\,dy \right\} dx$$

$$= \int_{x=0}^{6} (6-2x) \left[y \right]_0^{\frac{12-2x}{3}} . dx$$

$$= \int_{x=0}^{6} (6-2x) \left(\frac{12-2x}{3} \right) dx$$

$$= \frac{4}{3} \int_0^6 (3-x)(6-x)\,dx.$$

$$= \frac{4}{3} \int_0^6 (18 - 9x + x^2)\,dx$$

$$= \frac{4}{3} \left\{ 18x - \frac{9}{2}x^2 + \frac{x^3}{3} \right]_0^6$$

$$= \frac{4}{3} \left\{ (18)(6) - \frac{9}{2}(6)^2 + \frac{6^3}{3} \right]$$

$$= \frac{4}{3} \left\{ (18)(6) - 9(18) + 2(36) \right]$$

$$= \frac{4}{\cancel{3}} \times \cancel{18}^{6} \left\{ 6 - 9 + 4 \right\} = 24$$

2) Evaluate $\int_S \bar{F} \cdot \bar{N}\, dS$, where $\bar{F} = z\bar{i} + x\bar{j} - 3y^2 z\bar{k}$ and 'S' is the surface of the cylinder $x^2+y^2=1$ in the first octant b/w $z=0$ and $z=2$.

Sol. Let $\phi = x^2+y^2-1$

Now, $\frac{\partial \phi}{\partial x} = 2x \quad \frac{\partial \phi}{\partial y} = 2y \quad \frac{\partial \phi}{\partial z} = 0$

$\therefore \text{grad}\,\phi = \Sigma \bar{i} \frac{\partial \phi}{\partial x} = \bar{i}(2x) + \bar{j}(2y) + \bar{k}(0)$

Since grad ϕ is normal to the surface, then unit normal, $\bar{N} = \frac{\text{grad}\,\phi}{|\text{grad}\,\phi|} = \frac{(2x)\bar{i} + \bar{j}(2y)}{\sqrt{4x^2+4y^2}}$

$$= \frac{2(x\bar{i}+y\bar{j})}{2\sqrt{1}} = (x\bar{i}+y\bar{j}) \quad \{x^2+y^2=1\}$$

Given that,

$\bar{f} = z\bar{i} + x\bar{j} - 3y^2 z\bar{k}$

Now, $\bar{f}\cdot\bar{N} = (z\bar{i} + x\bar{j} - 3y^2 z\bar{k})\cdot(x\bar{i} + y\bar{j})$

$= zx + xy - 0 = x(y+z)$

Let R be the projection of 's' on yz-plane then

$\iint_S \bar{f}\cdot\bar{N}\, ds = \iint_R \bar{f}\cdot\bar{N} \frac{dy\,dz}{(\bar{N}\cdot\bar{i})}$, $ds\cos\alpha = dy\,dz$

$\begin{cases} yz\text{-plane } (x,y,0)(1,0,0) \\ zy\text{-plane}(x,y,0)(0,1,0) \end{cases}$

Now, $\bar{N}\cdot\bar{i} = (x\bar{i} + y\bar{j})\cdot\bar{i} = x.$

Given surface is, $x^2+y^2=1$ in yz-plane,

$x=0 \Rightarrow y^2 = 1 \Rightarrow y = 1$

$\therefore$ y varies from 0 to 1

Also, given that z varies from 0 to 2.

$$\iint_S \bar{f}\cdot\bar{N}\, ds = \iint_R (\bar{f}\cdot\bar{N}) \frac{dy\,dz}{(\bar{N}\cdot\bar{i})}$$

$$= \iint_R x(y+z)\frac{dy\,dz}{x}$$

$$= \int_{y=0}^{1}\int_{z=0}^{2} (y+z)\,dy\,dz$$

$$= \int_{y=0}^{1} \left\{yz + \frac{z^2}{2}\right\}_0^2 dy$$

$$= \int_{y=0}^{1} (2y+2)\,dy = \left[y^2+2y\right]_0^1$$

$$= 3$$

3) Evaluate $\iint_S \bar{f}\cdot\bar{N}\, ds$, where $\bar{f} = yz\bar{i} + 2y^2\bar{j} + xz^2\bar{k}$ and 'S' is the surface of the cylinder $x^2+y^2=9$ included in the I octant b/w the planes $z=0$ & $z=2$

Let $\phi = x^2+y^2-9$

Now, $\frac{\partial\phi}{\partial x} = 2x$ $\quad \frac{\partial\phi}{\partial y} = 2y$ $\quad \frac{\partial\phi}{\partial z} = 0$

$\therefore \text{grad}\,\phi = \Sigma \bar{i} \frac{\partial\phi}{\partial x} = \bar{i}(2x) + \bar{j}(2y) + \bar{k}(0)$

Since grad ϕ is normal to the surface, then unit normal, $\bar{N} = \frac{\text{grad}\,\phi}{|\text{grad}\,\phi|} = \frac{\bar{i}(2x)+\bar{j}(2y)}{\sqrt{4x^2+4y^2}}$

$= \frac{\cancel{2}(x\bar{i}+y\bar{j})}{\cancel{2}\sqrt{9}}$ $\quad \{x^2+y^2=9\}$

$= \frac{x\bar{i}+y\bar{j}}{3}$

$\bar{f} = yz\bar{i} + 2y^2\bar{j} + xz^2\bar{k}$

Now, $(\bar{f}\cdot\bar{N}) = (yz\bar{i} + 2y^2\bar{j} + xz^2\bar{k})\frac{1}{3}(x\bar{i}+y\bar{j})$

$= \frac{1}{3}(xyz\cancel{\bar{i}} + 2y^3\cancel{\bar{j}})$

$= \frac{1}{3}(xyz) + 2y^3$

Let 'R' be the projection of 's' on z-x-plane then,

$\iint_S \bar{f}\cdot\bar{N}\,ds = \iint_R \bar{f}\cdot\bar{N}\,\frac{dz\,dx}{\bar{N}\cdot\bar{j}}$

Now, $\bar{N}\cdot\bar{j} = \frac{1}{3}(x\bar{i}+y\bar{j})\bar{j} = \frac{y}{3}$.

Given surface, is $x^2+y^2=9$.

In zx plane; $y=0$ $x=3$

$\therefore x$ varies from 0 to 3

$\iint \bar{f}\cdot\bar{N}\,ds = \iint_R \frac{1}{\cancel{3}}(xyz)+2y^3)\frac{dx\,dz}{y/\cancel{3}}$

$= \iint_R (xz+2y^2)\,dx\,dz$

$= \int_0^3 \left[\int_0^2 (xz+2y^2)\,dz\right]dx$

$$= \int_0^3 \left\{ x\frac{z^2}{2} + 2zy^2 \right\}_0^2 dx$$

$$= \int_0^3 2x + 4y^2\, dx$$

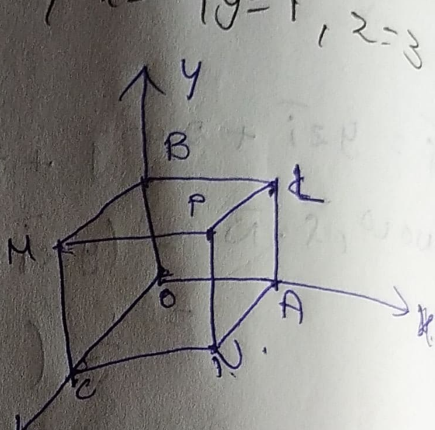
$$= 2\left(\frac{x^2}{2}\right)_0^3 + 2\int_0^3 (9-x^2)\,dx$$

$$= 2\left(\frac{9}{2}\right) + 2\left(9x - \frac{x^3}{3}\right)_0^3 = 9 + 2(27-9)$$

$$= 9 + 2(18) = 9+36 = $$

4) If $\bar{f} = 2xy\bar{i} + yz^2\bar{j} + xz\bar{k}$, find $\int_S \bar{f}\cdot\bar{N}\,ds$ over the parallelopiped $x=0, y=0, z=0, x=2, y=1, z=3$

consider the rectangular parallelopiped (OALB, CNPM)



Let,

$\bar{f} = 2xy\bar{i} + yz^2\bar{j} + xz\bar{k}$

(i) Along the region $\overleftrightarrow{ALPN}$ (R_1)

$\bar{N} = \bar{i}$, $x=2$, $ds = dy\,dz$, $x = 2$. $y \to 0$ to 1, $z \to 0$ to 3

$\bar{f}\cdot\bar{N} = 2xy = 2(2)y = 4y$

$$\iint_{R_1} \bar{f}\cdot\bar{N}\,ds = \int_{y=0}^{1}\int_{z=0}^{3} 4y\,dy\,dz$$

$$= 4\int_0^1 y\,dy \int_0^3 dz$$

$$= 4\left(\frac{y^2}{2}\right)_0^1 (z)_0^3$$

$$= 4\left(\frac{1}{2}\right)(3) = 6$$

(ii) Along the region $\overleftrightarrow{OBMC}$ (R_2)

$\bar{N} = -\bar{i}$, $x=0$, $ds = dy\,dz$. $y \to 0$ to 1, $z \to 0$ to 3

$\bar{f}\cdot\bar{N} = -2xy = 0 \quad \because x = 0$

$\therefore \iint_{R_2} \bar{f}\cdot\bar{N}\,ds = 0$

(iii) Along the Region $\overleftrightarrow{BMPL}$ (R_3)

$\bar{N} = \bar{j}$, $y = 1$, $ds = dz\,dx$ $\quad z \to 0$ to 3, $x \to 0$ to 2

$\bar{f}\cdot\bar{N} = yz^2 = (1)(z^2) = z^2$

$\therefore \iint_{R_3} \bar{f}\cdot\bar{N}\,ds = \int_{x=0}^{2}\int_{z=0}^{3} z^2\,dz\,dx$

$= \int_{x=0}^{2} dx \int_{z=0}^{3} z^2\,dz = [x]_0^2 \left[\frac{z^3}{3}\right]_0^3$

$= 2 \times 9 = 18$

(iv) Along the region $\overleftrightarrow{OCNA}$ (R_4)

$\bar{N} = -\bar{j}$, $y = 0$, $ds = dz\,dx$. $\quad z \to 0$ to 3, $x \to 0$ to 2

$\bar{f}\cdot\bar{N} = -yz^2 = 0 \quad \because y = 0$

$\therefore \iint_{R_4} \bar{f}\cdot\bar{N}\,ds = 0$

(v) Along the region $\overleftrightarrow{CNPM}$ (R_5)

$\bar{N} = \bar{k}$, $z = 3$, $ds = dx\,dy$ $\quad x \to 0$ to 2, $y \to 0$ to 1

$\bar{f}\cdot\bar{N} = xz = (x)(3) = 3x$.

$\therefore \iint_{R_5} \bar{f}\cdot\bar{N}\,ds = \int_{x=0}^{1}\int_{y=0}^{1} 3x\,dx\,dy$

$= 3\int_{x=0}^{2} x\,dx \int_{y=0}^{1} dy$

$= 3\left[\frac{x^2}{2}\right]_0^2 [y]_0^1 = 3 \times 2 = 6$

(vi) Along the region $\overleftrightarrow{OALB}$ (R_6)

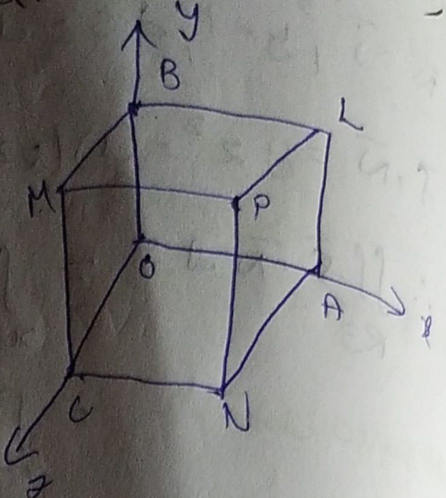
$\bar{N} = -\bar{k}$, $z = 0$, $ds = dx\,dy$ $\quad x \to 0$ to 2, $y \to 0$ to 1

$\bar{f}\cdot\bar{N} = xz = 0 \quad \therefore \iint_{R_6} \bar{f}\cdot\bar{N}\,ds = 0$

$\int_S \bar{f}\cdot\bar{N}\,ds = 6 + 0 + 18 + 0 + 6 + 0 = 30$

5) If $\bar{f} = 4xz\bar{i} - y^2z\bar{j} + yz\bar{k}$, find $\int_S \bar{F}\cdot\bar{N}\,ds$ where S is the surface of the cube bounded by $x=0$, $x=a$, $y=0$, $y=a$, $z=0$, $z=a$.

Consider a regular parallelopiped (OALB, CNPM)



Let

$\bar{f} = 4xz\bar{i} - y^2z\bar{j} + yz\bar{k}$

(i) Along the region $\overleftrightarrow{ALPN}$ (R_1)

$\bar{N} = \bar{i}$, $x = a$, $ds = dy\,dz$.

$\bar{F}\cdot\bar{N} = 4xz = 4az$.

$y \to 0$ to a

$z \to 0$ to a

$$\iint_{R_1} \bar{f}\,\bar{N}\,ds = \int_{y=0}^{a}\int_{z=0}^{a} 4az\,dz\,dy$$

$$= 4a\int_{y=0}^{a} dy \int_{z=0}^{a} z\,dz$$

$$= \overset{2}{\cancel{4}}a[a]\left[\frac{a^2}{\cancel{2}}\right] = 2a^4$$

(ii) Along the region $\overleftrightarrow{OBMC}$ (R_2).

$\bar{N} = -\bar{i}$, $x = 0$, $ds = dy\,dz$.

$z \to 0$ to a

$y \to 0$ to a

$\bar{f}\cdot\bar{N} = 4xz = 0$

$$\therefore \iint_{R2} \bar{f}\cdot\bar{N}\,ds = 0$$

(iii) Along the region $\overleftrightarrow{BMPL}$ (R_3)

$z \to 0$ to a

$x \to 0$ to a

$\bar{N} = \bar{j}$, $y = a$, $ds = dz\,dx$

$\therefore \bar{f}\cdot\bar{N} = -y^2z = -a^2z$

$$\therefore \iint_{R3} \bar{f}\cdot\bar{N}\,ds = \int_{x=0}^{a}\int_{z=0}^{a} -a^2z\,dz\,dx$$

$$= -a^2\int_{x=0}^{a} dx \int_{z=0}^{a} z\,dz.$$

$= -a^2 \left[x\right]_0^a \left[\frac{z^2}{2}\right]_0^a = -a^2 \{a\}\left\{\frac{a^2}{2}\right\} = \frac{-a^5}{2}$

(iv) Along the region $\overleftrightarrow{OCNA}$ (R_4)

$\bar{N} = -\bar{j}$, $y = 0$, $ds = dz\,dx$

$\bar{f}\cdot\bar{N} = -y^2 z = 0 \quad \because y = 0$ $\quad z \to 0$ to a, $x \to 0$ to a

$\therefore \iint_{R_4} \bar{f}\cdot\bar{N}\, ds = 0$

(v) Along the region $\overleftrightarrow{CNPM}$ (R_5)

$\bar{N} = \bar{k}$, $z = a$, $ds = dx\,dy$

$\bar{f}\cdot\bar{N} = yz = ay$ $\quad x \to 0$ to a, $y \to 0$ to a

$\therefore \iint_{R_5} \bar{f}\cdot\bar{N}\, ds = \int_{x=0}^{a}\int_{y=0}^{a} ay\, dy\, dx$

$= a\int_{x=0}^{a} dx \int_{y=0}^{a} y\, dy$

$= a\left[x\right]_0^a \left\{\frac{y^2}{2}\right\}_0^a = a \times a \times \frac{a^2}{2} = \frac{a^4}{2}$

(vi) Along the region $\overleftrightarrow{OALB}$ (R_6)

$\bar{N} = -\bar{k}$, $z = 0$, $ds = dx\,dy$

$\bar{f}\cdot\bar{N} = yz = 0 \quad \because z = 0$ $\quad x \to 0$ to a, $y \to 0$ to a

$\therefore \iint_{R_6} \bar{f}\cdot\bar{N}\, ds = 0$

$\int_S \bar{f}\cdot\bar{N}\, ds = 2a^4 + 0 - \frac{a^5}{2} + 0 + \frac{a^4}{2}$

$= a^4\left(2 + \frac{1}{2}\right) - \frac{a^5}{2}$

$= \frac{5}{2}a^4 - \frac{1}{2}a^5 = \frac{5a^4 - a^5}{2}$

(or)

$\bar{f} = 4xz\bar{i} - y^2\bar{j} + yz\bar{k}$

$\int_S \bar{f}\cdot\bar{N}\, ds = \frac{5}{2}a^4 - a^4 = \frac{3}{2}a^4$

Volume Integral

Let V be the volume bounded by a surface $\bar{r} = \bar{f}(u, v)$ and $\bar{f}(r)$ be a vector point function defined over V. Devide V into m sub-regions of volumes $\delta v_1, \delta v_2 \ldots, \delta v_m$

consider the sum $\sum_{r=1}^{m} \bar{f}(r_i)\,\delta v_i$

If $\lim_{m\to\infty} \sum_{r=1}^{m} \bar{f}(r_i)\,\delta v_i$ exists, then it is called as volume integral of $\bar{f}$ in the region 'V' and is denoted by $\int_V \bar{f}\cdot dv$.

Cartesian form

If $\bar{f} = f_1\bar{i} + f_2\bar{j} + f_3\bar{k}$

where f_1, f_2 and f_3 are functions of x, y, z then

$$\int_V \bar{f}\cdot dv = \iiint_{xyz} (f_1\bar{i} + f_2\bar{j} + f_3\bar{k})\, dx\,dy\,dz$$

$$= \bar{i}\iiint_{xyz} (f_1\,dx\,dy\,dz + \bar{j}\iiint_{xyz} f_2\,dx\,dy\,dz$$

$$+ \bar{k}\iiint_{xyz} f_3\,dx\,dy\,dz$$

1) find $\int_V \bar{f}\cdot dv$ where $\bar{f} = x\bar{i} + y\bar{j} + z\bar{k}$ and V is the region bounded by $x = 0, y = 0, y = 6, z = 4,$ $z = x^2$.

$z = 4, z = x^2$

$x^2 = 4 \Rightarrow x = 2$

clearly x varies from 0 to 2

y varies from 0 to 6

z varies from x^2 to 4

$$\therefore \int_V \bar{f}\,dv = \int_V (x\bar{i} + y\bar{j} + z\bar{k})\,dv$$

$$= \bar{i}\int_{x=0}^{2}\int_{y=0}^{6}\int_{z=x^2}^{4} x\,dx\,dy\,dz + \bar{j}\int_{x=0}^{2}\int_{y=0}^{6}\int_{z=x^2}^{4} y\,dx\,dy\,dz + \bar{k}\int_{0}^{2}\int_{0}^{6}\int_{z=x^2}^{4} z\,dx\,dy\,dz$$

$$= \bar{i}\int_{x=0}^{2}\int_{y=0}^{6}\left\{x\int_{z=x^2}^{4} dz\right\}dx\,dy + \bar{j}\int_{x=0}^{2}\int_{y=0}^{6}\left\{y\int_{z=x^2}^{4} dz\right\}dx\,dy$$

$$+ \bar{k}\int_{x=0}^{2}\int_{y=0}^{6}\left\{\int_{z=x^2}^{4} z\,dz\right\}dx\,dy$$

$$= \bar{i}\int_{x=0}^{2}\int_{y=0}^{6} x\left[z\right]_{x^2}^{4} dx\,dy + \bar{j}\int_{x=0}^{2}\int_{y=0}^{6} y\left[z\right]_{x^2}^{4} dx\,dy$$

$$+ \bar{k}\int_{x=0}^{2}\int_{y=0}^{6}\left[\frac{z^2}{2}\right]_{x^2}^{4} dx\,dy$$

$$= \bar{i}\int_{x=0}^{2}\int_{y=0}^{6} x(4-x^2)\,dx\,dy + \bar{j}\int_{x=0}^{2}\int_{y=0}^{6} y(4-x^2)\,dx\,dy$$

$$+ \bar{k}\int_{x=0}^{2}\int_{y=0}^{6}\left(8-\frac{x^4}{2}\right)dx\,dy$$

$$= \bar{i}\int_{x=0}^{2} x(4-x^2)\,dx \cdot \int_{y=0}^{6} dy + \bar{j}\int_{x=0}^{2}(4-x^2)\,dx\int_{y=0}^{6} y\,dy$$

$$+ \bar{k}\int_{x=0}^{2}\left(8-\frac{x^4}{2}\right)dx\int_{y=0}^{6} dy$$

$$= \bar{i}\left[2x^2-\frac{x^4}{4}\right]_0^2\left[y\right]_0^6 + \bar{j}\left[4x-\frac{x^3}{3}\right]_0^2\left[\frac{y^2}{2}\right]_0^6$$

$$+ \bar{k}\left[8x-\frac{x^5}{10}\right]_0^2\left[y\right]_0^6$$

$$= \bar{i}(8-4)(6-0) + \bar{j}\left(8-\frac{8}{3}\right)(18-0) + \bar{k}\left(16-\frac{32}{10}\right)(6-0)$$

$$= \bar{i}(24) + \bar{j}\left(\frac{16}{3}\right)(18) + \bar{k}(16)\left(\frac{4}{5}\right)(6)$$

$$= 24\bar{i} + 96\bar{j} + \frac{384}{5}\bar{k}$$

Integral Transformations

Gauss Divergence Theorem: Let 's' be a closed surface enclosing a volume 'V'. If $\bar{f}$ is a continuously differentiable vector point function, then

$$\int_S \bar{f} \cdot \bar{N} \, dS = \int_V \operatorname{div} \bar{f} \, dV$$

where $\bar{N}$ is the outward drawn unit normal vector to 's' at any point on it.

1) Evaluate $\iint_S x\,dy\,dz + y\,dz\,dx + z\,dx\,dy$ over $x^2+y^2+z^2=1$

Given, $\iint_S x\,dy\,dz + y\,dz\,dx + z\,dx\,dy$

i.e, $\bar{f} = x\,dx + y\,dy + z\,dz$ $\quad f_1 = x$

$\text{div}\,\bar{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$ $\quad f_2 = y$

Now, $\text{div}\,\bar{f} = 1+1+1 = 3$ $\quad f_3 = z.$

By G.D theorem

$$\int_S \bar{f}\cdot\bar{N}\,dS = \int_V \text{div}\,\bar{f}\,dv$$

$$\iint_S x\,dy\,dz + y\,dz\,dx + z\,dx\,dy = \int_V \text{div}\,\bar{f}\,dv$$

$$= \int_V 3\,dv$$

volume of sphere

$v = \frac{4}{3}\pi r^3$

$= \frac{4}{3}\pi$

$r = x^2+y^2+z^2 = 1.$

$$= 3(v)$$

$$= 3 \times \frac{4}{3}\pi$$

$$= 4\pi$$

2) S.T $\int_S (ax\bar{i} + by\bar{j} + cz\bar{k})\,\bar{N}\cdot dS = \frac{4\pi}{3}(a+b+c)$

where S is the surface of the sphere $x^2+y^2+z^2$

By G.D Theorem

$$\int_S \bar{f}\cdot\bar{N}\,dS = \int_V \text{div}\,\bar{f}\,dv$$

$\bar{f} = ax\bar{i} + by\bar{j} + cz\bar{k}$

$\text{div}\,\bar{f} = a+b+c.$

$$\int_S (ax\bar{i} + by\bar{j} + cz\bar{k}) \cdot \bar{N}\, dS = \int_V (a+b+c)\, dV$$

$$= (a+b+c) \int_V dV$$

$\therefore V = \frac{4}{3}\pi r^3$

$= \frac{4}{3} \pi$

$$= (a+b+c)\, V$$

$$= (a+b+c)\frac{4\pi}{3}$$

3) Prove Gauss-divergence theorem for cube & rectangle

for cube:

(3 pages back)

part - II

$$\int_V \operatorname{div} \bar{f}\, dV$$

$$\int_0^a\int_0^a\int_0^a (4z - y)\, dx\, dy\, dz$$

$$\int_0^a\int_0^a \left[4\frac{z^2}{2} - zy\right]_0^a dx\, dy$$

$$\int_0^a\int_0^a \left[2a^2 - ay\right] dx\, dy$$

$$\int_0^a \left[2a^2 y - a\frac{y^2}{2}\right]_0^a dx$$

$$\int_0^a \left[2a^3 - \frac{a^3}{2}\right] dx$$

$$\left[2a^4 - \frac{a^4}{2}\right]$$

$$= \frac{4a^4 - a^4}{2} = \frac{3a^4}{2}$$

$$\therefore \int_S \bar{f} \cdot \bar{N}\, dS = \int_V \operatorname{div} \bar{f}\, dV$$

$\therefore$ Hence Gauss-divergence is verified

for rectangle

Part - II

$$\int_V \operatorname{div} \bar{f}\, dV$$

$$\int_0^2\int_0^1\int_0^3 \left[2y + z^2 + x\right] dx\, dy\, dz$$

$$\int_0^2\int_0^1 \left[2zy + \frac{z^3}{3} + xz\right] dx\, dy$$

$$\int_0^2\int_0^1 \left[6y + \frac{9x^3}{8} + 3x\right] dx\, dy$$

$$\int_0^2\int_0^1 \left[6y + 9 + 3x\right] dx\, dy$$

$$\int_0^2 \left[6\frac{y^2}{2} + 9y + 3xy\right]_0^1 dx$$

$$\int_0^2 \left[3 + 9 + 3x\right] dx$$

$$\left[12x + 3\frac{x^2}{2}\right]_0^2$$

$$= \left[12 + 2 + 6\right]$$

$$= 24 + 6 = 30.$$

$$\therefore \int_S \bar{F}\cdot\bar{N}\, ds = \int_V \operatorname{div}\bar{F}\, dV$$

$\therefore$ Hence Gauss theorem is verified

4) Use divergence theorem to evaluate $\iint_S \bar{f}\cdot\bar{N} ds$ where $\bar{f} = 4x\bar{i} - 2y^2\bar{j} + z^2\bar{k}$ and 's' is the surface bounded by the region $x^2+y^2=4$, $z=0$ and $z=3$.

Sol By using divergence theorem
cylinder

$$\therefore \int_S \bar{f}\cdot\bar{N}\, ds = \int_V \operatorname{div}\bar{f}\, dV$$

Given that $\bar{f} = 4x\bar{i} - 2y^2\bar{j} + z^2\bar{k}$

$$\therefore \operatorname{div}\bar{f} = \frac{\partial}{\partial x}(4x) + \frac{\partial}{\partial y}(-2y^2) + \frac{\partial}{\partial z}(z^2)$$

$$= 4 - 4y + 2z.$$

Given surface; $x^2+y^2=4$, $z=0$ & $z=3$

Now, $x^2+y^2=4 \Rightarrow y^2 = 4-x^2$, $y = \sqrt{4-x^2}$

$\therefore$ y varies from $-\sqrt{4-x^2}$ to $\sqrt{4-x^2}$

put $y=0$, then $x^2=4 \Rightarrow x=\pm 2$

Also, given that z varies from 0 to 3

$$\therefore \int_S \bar{F}\cdot\bar{N}\, ds = \int_V \operatorname{div}\bar{F}\, dV$$

$$= \int_{x=-2}^{2}\int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{z=0}^{3} (4-4y+2z)\, dx\, dy\, dz$$

$$= \int_{x=-2}^{2}\int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \left[(4-4y)z+z^2\right]_0^3 dx\, dy$$

$$= \int_{x=-2}^{2}\int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \left[3(4-4y)+9\right] dx\, dy$$

$$= \int_{x=-2}^{2}\left\{\int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (21-12y)\, dy\right\} dx$$

$$= \int_{x=-2}^{2}\left\{21\int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dy - 12\int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}} y\, dy\right\} dx$$

$$= \int_{x=-2}^{2}\left\{(21)\left[y\right]_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} - (12)\left[\frac{y^2}{2}\right]_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\right\} dx$$

$$= \int_{x=-2}^{2}\left[(21)(2)\sqrt{4-x^2} - 12(0)\right] dx \qquad \left[\sqrt{a^2-x^2}\, dx\right]$$

$$= (42)\left[\frac{x}{2}\sqrt{4-x^2} + \frac{4}{2}\sin^{-1}\left(\frac{x}{2}\right)\right]_{-2}^{2}$$

$$= (42)\left[\{0+2\sin^{-1}(1)\} - \{0+2\sin^{-1}(-1)\}\right]$$

$$= (42)\left\{2\cdot\left(\frac{\pi}{2}\right) - 2\left[-\frac{\pi}{2}\right]\right\} = (42)(2\pi) = 84\pi$$

4) Verify Gauss divergence theorem for $\bar{f} = 4xy\bar{i} - y^2\bar{j} + yz\bar{k}$, where 's' is the surface of the cube bounded by $x=0, x=1, y=0, y=1, z=0, z=1$

Sol:

Gauss divergence theorem states that

$$\int_S \bar{f}\cdot\bar{n}\,dS = \int_V \operatorname{div}\bar{f}\,dV$$

Part-①

Given that, $\bar{f} = 4xy\bar{i} - y^2\bar{j} + yz\bar{k}$

$$\operatorname{div}\bar{f} = \frac{\partial}{\partial x}(4xy) + \frac{\partial}{\partial y}(-y^2) + \frac{\partial}{\partial z}(yz)$$

$$= 4y + (-2y) + y$$

$$= 3y$$

Also, $x \to 0$ to 1, $y = 0$ to 1, $z = 0$ to 1

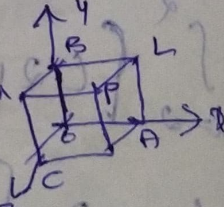
$$\therefore \int_V \operatorname{div}\bar{f}\,dV = \int_{x=0}^{1}\int_{y=0}^{1}\int_{z=0}^{1} 3y\,dx\,dy\,dz$$

$$= 3\int_{x=0}^{1} dx \int_{y=0}^{1} y\,dy \int_{z=0}^{1} dz$$

$$= 3\{x\}_0^1 \left\{\frac{y^2}{2}\right\}_0^1 \{z\}_0^1$$

$$= \frac{3}{2}$$

Part-②



Let $\bar{f} = 4xy\bar{i} - y^2\bar{j} + yz\bar{k}$

(i) Along the region ALPN (R_1) $y \to 0$ to 1, $z \to 0$ to 1

$\bar{n} = \bar{i}$, $x = 1$, $ds = dy\,dz$

$\bar{f}\cdot\bar{n} = (4y)$

$\iint_{R_1} \bar{f}\cdot\bar{n}\,dS = \int$

(ii) Along the re...

$\bar{n} = -\bar{i}$, $x =$

$\bar{f}\cdot\bar{n} = 4$

$\therefore \iint_{R_2} \bar{f}\cdot$

(iii) Along the re...

$\bar{n} = \bar{j}$, $y =$

$\therefore \bar{f}\cdot\bar{n} =$

$\therefore \iint_{R_3} \bar{f}\cdot\bar{n}$

(iv) Along the

$\bar{n} = -\bar{j}$

$\bar{f}\cdot\bar{n} =$

(v) Along th...

$\bar{n} = \bar{k}$

$\bar{f}\cdot\bar{n}$

$\therefore \iint_{R_5}$

$$\iint_{R_1} \bar{f}\cdot\bar{n}\,ds = \int_{y=0}^{1}\int_{z=0}^{1} 4y\,dz\,dy$$

$$= 4\int_{y=0}^{1} y\,dy \int_{z=0}^{1} dz$$

$$= \frac{4}{2} = 2$$

(ii) Along the region $\overleftrightarrow{OBMC}$ (R_2)

$\bar{n} = -\bar{i}$, $x = 0$, $ds = dy\,dz$

$\bar{f}\cdot\bar{n} = 4xy = 0$ $\quad z \to 0$ to 1

$\therefore \iint_{R_2} \bar{f}\cdot\bar{n}\,ds = 0$ $\quad y \to 0$ to 1

(iii) Along the region $\overleftrightarrow{BMPL}$ (R_3)

$\bar{n} = \bar{j}$, $y = 1$, $ds = dz\,dx$ $\quad z \to 0$ to 1, $x \to 0$ to 1

$\therefore \bar{f}\cdot\bar{n} = -y^2 = -1$

$$\therefore \iint_{R_3} \bar{f}\cdot\bar{n}\,ds = \int_{x=0}^{1}\int_{z=0}^{1} -1\,dz\,dx$$

$$= -\int_{x=0}^{1} dx \int_{z=0}^{1} dz$$

$$= -1$$

(iv) Along the region $\overleftrightarrow{OCNA}$ (R_4)

$\bar{n} = -\bar{j}$, $y = 0$, $ds = dz\,dx$ $\quad z \to 0$ to 1

$\bar{f}\cdot\bar{n} = -(-y^2) = 0$ $\quad x \to 0$ to 1

$\therefore \iint_{R_4} \bar{f}\cdot\bar{n}\,ds = 0$

(v) Along the region $\overleftrightarrow{CNPM}$ (R_5) $\quad x \to 0$ to 1

$\bar{n} = \bar{k}$, $z = 1$, $ds = dx\,dy$ $\quad y \to 0$ to 1

$\bar{f}\cdot\bar{n} = yz = y$

$$\iint \bar{f}\cdot\bar{n}\,ds = \int^{1}\int_{y=0}^{1} y\,dy\,dx$$

$$= \int_{x=0} dx \int_{y=0} y\, dy$$

$$= \{x\}_0^1 \left[\frac{y^2}{2}\right]_0^1$$

$$= \frac{1}{2}.$$

(vi) Along the region $\overleftrightarrow{OACB}$ (R_6)

$\bar{N} = -\bar{k}$, $z = 0$ $ds = dx\,dy$

$\bar{F} \cdot \bar{N} = yz = 0$ $\because z = 0$

$x \to 0$ to 1
$y \to 0$ to 1

$$\therefore \iint_{R_6} \bar{F} \cdot \bar{N}\, ds = 0$$

$$\int_S \bar{F} \cdot \bar{N}\, ds = 2 - 1 + \frac{1}{2} = 1 + \frac{1}{2} = \frac{3}{2}.$$

$$\therefore \int_S \bar{F} \cdot \bar{N}\, ds = \int_V \operatorname{div} \bar{F}\, dv.$$

$\therefore$ Hence Gauss divergence theorem is verified

Green's Theorem

Let 'R' be a closed region bounded by a simple closed curve 'c'. If M and N are continuous differentiable functions of x & y in R then

$$\oint_C M\,dx + N\,dy = \iint_S \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

where 'c' is traversed in positive direction

① S.T the area bounded by a simple closed curve 'c' is given by $\frac{1}{2}\oint_C x\,dy - y\,dx$. Hence find the area of the circle $x^2+y^2=a^2$

consider $\oint_C x\,dy - y\,dx$.

By Green's theorem

$$\oint_C M\,dx + N\,dy = \iint_S \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

Here, $M = -y$, $N = x$.

$$\frac{\partial M}{\partial y} = -1 \text{ and } \frac{\partial N}{\partial x} = 1$$

$$\therefore \oint_C x\,dy - y\,dx = \iint_S \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

~~Here M~~

$$= \iint_S \{1-(-1)\}\,dx\,dy$$

$= 2A$, wher A is area of the surface

$$A = \frac{1}{2}\oint_C x\,dy - y\,dx.$$

Given surface is, $x^2+y^2=a^2$ a circle

The parametric eqs of the given circle are

$x = a\cos\theta$, $y = a\sin\theta$ $\quad 0 \le \theta \le 2\pi$

$dx = -a\sin\theta\,d\theta$, $dy = a\cos\theta\,d\theta$

Area of the given circle, $A = \frac{1}{2}\oint_C x\,dy - y\,dx$

$$A = \frac{1}{2}\oint_C (a\cos\theta)(a\cos\theta\,d\theta) - (a\sin\theta)(-a\sin\theta\,d\theta)$$

$$= \frac{1}{2}\int_{\theta=0}^{2\pi} (a^2\cos^2\theta + a^2\sin^2\theta)\,d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} a^2(1)\,d\theta$$

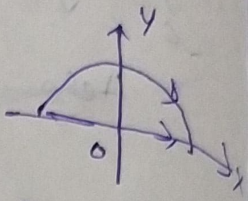
$$= \frac{a^2}{2}\{\theta\}_0^{2\pi} = \frac{a^2}{2}(2\pi) = \pi a^2$$

2) Find $\oint (2x^2-y^2)\,dx + (x^2+y^2)\,dy$ by using greens theorem where 'c' is the boundary of the area enclosed by the x-axis and upper half of the circle $x^2+y^2=a^2$.

Sol:- Here, $M = 2x^2-y^2$, $N = x^2+y^2$

$\frac{\partial M}{\partial y} = 0 - 2y$, $\frac{\partial N}{\partial x} = 2x$

By green's theorem,



$$\oint_c M\,dx + N\,dy = \iint_S \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

$$\oint_c (2x^2-y^2)\,dx + (x^2+y^2)\,dy$$

$$= \iint_S (2x - (-2y))\,dx\,dy$$

[changing cartesian co-ordinates in to polar co-ordinates by $x = r\cos\theta$, $y = r\sin\theta$, $dx\,dy = r\,dr\,d\theta$, $r \to 0$ to a, $\theta \to 0$ to π].

$$= 2\iint_S (x+y)\,dx\,dy$$

$$= 2\int_{r=0}^{a}\int_{\theta=0}^{\pi} r(\cos\theta+\sin\theta)\cdot r\,dr\,d\theta$$

$$= 2\int_0^a r^2\,dr \int_0^{\pi} (\cos\theta+\sin\theta)\,d\theta$$

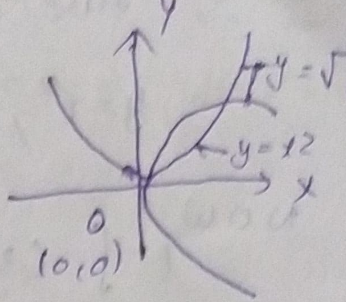
$$= 2\left[\frac{r^3}{3}\right]_0^a \left\{\sin\theta - \cos\theta\right\}_0^{\pi}$$

$$\Rightarrow 2\left(\frac{a^3}{3}\right)(1+1) = \frac{4}{3}a^3$$

8) verify green's theorem in plane for $\oint_c (3x^2-8y^2)\,dx + (4y-6xy)\,dy$ where c is the region bounded by $y=\sqrt{x}$, and $y=x$

Given curves are,

$y = \sqrt{x}$ and $y = x^2$



$x^2 = \sqrt{x}$.

$x^4 = x \Rightarrow x^3(x^3 - 1) = 0$

$x = 0, 1$

$x = 0 \Rightarrow y = 0$ and $x = 1 \Rightarrow y = 1$

$\therefore$ The two curves intersects at $O(0,0)$ and $A(1,1)$ Green's theorem states that

$$\oint_C M\,dx + N\,dy = \iint_S \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

part-① where $M = 3x^2 - 8y^2$ $\quad N = 4y - 6xy$

$$\frac{\partial M}{\partial y} = -16y \qquad \frac{\partial N}{\partial x} = -6y$$

Now,

$$\iint_S \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \iint_S (-6y + 16y)\,dx\,dy$$

$$= \int_{x=0}^{1} \int_{y=x^2}^{\sqrt{x}} 10y\,dx\,dy$$

$$= 10 \int_{x=0}^{1} \left[\int_{y=x^2}^{\sqrt{x}} y\,dy\right] dx$$

$$= 10 \int_{x=0}^{1} \left[\frac{y^2}{2}\right]_{x^2}^{\sqrt{x}} dx$$

$$= 5 \int_0^1 (x - x^4)\,dx$$

$$= 5 \left[\frac{x^2}{2} - \frac{x^5}{5}\right]_0^1$$

$$= 5\left[\frac{1}{2} - \frac{1}{5}\right] = \frac{3}{2}$$

Part-2 Line Integral along $\overrightarrow{OA}$

$y = x^2$, $dy = 2x\,dx$, $x \to 0$ to 1

Now

(i) $\int_{C_1} (3x^2 - 8y^2)\,dx + (4y - 6xy)\,dy$

$= \int_{x=0}^{1} (3x^2 - 8x^4)\,dx + (4x^2 - 6x^3)(2x\,dx)$

$= \int_0^1 (3x^2 - 8x^4 + 8x^3 - 12x^4)\,dx$

$= \left[x^3 - 4x^5 + 2x^4 \right]_0^1 = 1 - 4 + 2 = -1$

(ii) Line Integral along $\overrightarrow{AO}$ (C_2)

$y = \sqrt{x}$, then $y^2 = x$ and $2y\,dy = dx$

Also $x \to 1$ to 0

$\int_{C_2} (3x^2 - 8y^2)\,dx + (4y - 6xy)\,dy$

$= \int_{x=1}^{0} (3x^2 - 8x)\,dx + (2 - 3x)\,dx$

$= \int_1^0 (3x^2 - 11x + 2)\,dx$

$= \left[x^3 - \frac{11}{2}x^2 + 2x \right]_1^0 = 0 - \left(1 - \frac{11}{2} + 2\right)$

$= \frac{5}{2}$

$\oint_C (3x^2 - 8y^2)\,dx + (4y - 6xy)\,dy = -1 + \frac{5}{2} = \frac{3}{2}$

Thus, $\oint_C M\,dx + N\,dy = \iint_S \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx\,dy$

$\therefore$ Hence Green's theorem is verified

4) Evaluate $\int_C (2xy - x^2)\,dx + (x^2 + y^2)\,dy$, where C is the closed curve of region bounded by $y = x^2$ and $y^2 = x$.

Sol. Here, $M = 2xy - x^2$ $\quad N = x^2 + y^2$

$\frac{\partial M}{\partial y} = 2x$ $\quad \frac{\partial N}{\partial x} = 2x$

By Green's theorem

$$\oint_C M\,dx + N\,dy = \iint_S \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

$$= \int_{x=0}^{1} \int_{y=x^2}^{\sqrt{x}} (2x - 2x)\,dx\,dy$$

$$\therefore \oint_C M\,dx + N\,dy = 0\,dx\,dy = 0$$

5) Evaluate $\oint_C (x^2 + y^2)\,dx + 3xy^2\,dy$, where C is the circle $x^2 + y^2 = 4$ in xy-plane

Sol. Here, $M = x^2 + y^2$ $\quad N = 3xy^2$

$\frac{\partial M}{\partial y} = 2y$ $\quad \frac{\partial N}{\partial x} = 3y^2$

By Green's theorem

$$\oint_C M\,dx + N\,dy = \iint_S \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

$\left[\begin{array}{l} x = r\cos\theta, \\ y = r\sin\theta \\ dx\,dy = r\,dr\,d\theta \\ r \to 0 \text{ to } 2 \\ \theta \to 0 \text{ to } 2\pi \end{array} \right]$

$$= \int_{r=0}^{2} \int_{\theta=0}^{2\pi} (3r^2\sin^2\theta - 2r\sin\theta)\,r\,dr\,d\theta$$

$$= \int_{r=0}^{2} \int_{\theta=0}^{2\pi} 3r^2\sin^2\theta\,.\,dr\,d\theta - \int_{r=0}^{2} \int_{\theta=0}^{2\pi} 2r\sin\theta\,dr\,d\theta$$

$$= 3\int_{r=0}^{2} r^3\,dr \int_{\theta=0}^{2\pi} \sin^2\theta\,d\theta - 2\int_{r=0}^{2} r\,dr \int_{\theta=0}^{2\pi} \sin\theta\,d\theta$$

$$= 3\left[\frac{r^4}{4}\right]_0^2 \left[\frac{1}{2}\{\theta\}\right]_0^{2\pi} - \left[\frac{1}{4}\sin\theta\right]_0^{2\pi}$$

$$2\left[\frac{r^3}{3}\right]_0^2 \left\{-\cos\theta\right]_0^{2\pi} \qquad \left[\sin^2\theta = \frac{1-\cos 2\theta}{2}\right]$$

$$= 3[4]\left[\frac{1}{2}\{\theta\right]_0^{2\pi} - \left[\frac{1}{4}\sin\theta\right]_0^{2\pi} - 2\left[\frac{8}{3}\right]\left[-\cos 2\pi\right]$$

$$= 12\pi$$

6) Evaluate $\oint_C (y-\sin x)\,dx + \cos x\,dy$, where C is the triangle enclosed by the lines $x = \frac{\pi}{2}$, $y = 0$, $\pi y = 2x$

Sol

Here, $M = y - \sin x$ $\qquad N = \cos x$

$$\frac{\partial M}{\partial y} = 1 \qquad \frac{\partial N}{\partial x} = -\sin x$$

By green's theorem

$$\oint_C M\,dx + N\,dy = \iint_S \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

Limits

$x \to 0$ to $\frac{\pi}{2}$

$\pi y = 2x$

$y = \frac{2x}{\pi}$

$y \to 0$ to $\frac{2x}{\pi}$

$$= \int_{x=0}^{\pi/2} \int_{y=0}^{\frac{2x}{\pi}} (-\sin x - 1)\,dx\,dy$$

$$= (-1)\int_{x=0}^{\pi/2} \int_{y=0}^{\frac{2x}{\pi}} (1+\sin x)\,dx\,dy$$

$$= (-1)\int_0^{\pi/2} (1+\sin x)\,(y)_0^{2x/\pi}\,dx$$

$$= (-1)\int_0^{\pi/2} (1+\sin)\left(\frac{2x}{\pi}\right)dx$$

$$\Rightarrow \frac{-2}{\pi}\int_0^{\pi/2} x(1+\sin x)\,dx. \Rightarrow \left(\frac{-2}{\pi}\right)\left[x(x - \cos x)\right. \quad \left.\frac{x^2}{2} + \sin x\right]_0^{\pi/2}$$

$$\Rightarrow \frac{-2}{\pi}\left[-x\cos x + \frac{x^2}{2} + \sin x\right]_0^{\pi/2} \Rightarrow \frac{-2}{\pi}\left[\left(\frac{\pi^2}{8}+1\right) - (0+0\right.$$

$$= \left(\frac{-2}{\pi}\right)\left[\frac{\pi^2}{8}+1\right] = \frac{-\pi}{4} - \frac{2}{\pi} = -\left(\frac{\pi}{4} + \frac{2}{\pi}\right)$$

7) Verify Green's theorem for $\oint_C (xy+y^2)\,dx + x^2 dy$ where 'c' is bounded by $y = x$ & $y = x^2$

Sol: Given curves are

$y = x$ & $y = x^2$

$x = x^2$

$x(x-1) = 0$

$x = 0, 1$

$x = 0 \Rightarrow y = 0$

$x = 1 \Rightarrow y = 1$

$\therefore$ Two curves intersects at O (0,0) and A(1,1)

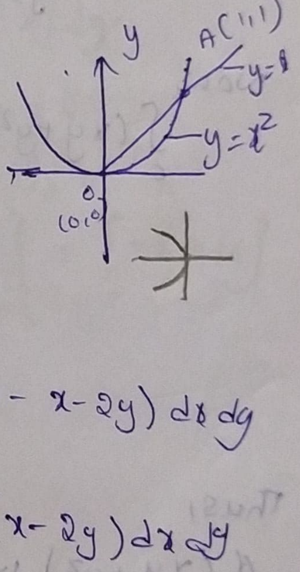
Green's theorem states that

$$\oint_C M\,dx + N\,dy = \iint_S \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

Part-1

Here, $M = xy + y^2$, $N = x^2$

$\frac{\partial M}{\partial y} = x + 2y$ $\qquad \frac{\partial N}{\partial x} = 2x.$



Now, $\iint_S \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \iint_S (2x - x - 2y)\,dx\,dy$

$$= \int_{x=0}^{1} \int_{y=x^2}^{x} (x - 2y)\,dx\,dy$$

$$= \int_{x=0}^{1} \left\{xy - 2\left(\frac{y^2}{2}\right)\right\}_{x^2}^{x} dx$$

$$= \int_{x=0}^{1} \left\{\cancel{x^2} - \cancel{x^2} - (x^3 - x^4)\right\} dx$$

$$= \int_{x=0}^{1} (x^4 - x^3)\,dx$$

$$= \left[\frac{x^5}{5} - \frac{x^4}{4}\right]_0^1$$

$$= \frac{1}{5} - \frac{1}{4} = \frac{-1}{20}$$

Part (b): (i) Line integral along $\overrightarrow{OA}$ (C_1)

$y = x^2$, $dy = 2x\,dx$ $\quad x \to 0$ to 1

Now

$$\int_{C_1} (xy+y^2)\,dx + x^2\,dy$$

$$= \int_{x=0}^{1} (x^3+x^4)\,dx + x^2(2x)\,dx$$

$$= \int_{x=0}^{1} (3x^3+x^4)\,dx$$

$$= \left\{3\left(\frac{x^4}{4}\right) + \frac{x^5}{5}\right\}_0^1 = \frac{3}{4}+\frac{1}{5} = \frac{19}{20}$$

(ii) Line integral along $\overrightarrow{AO}$ (C_2)

$y = x$ then $y^2 = x^2$ & $dy = dx$

Now,

$$\int_{C} (xy+y^2)\,dx + x^2\,dy = \int_{C_2} (x^2+x^2)\,dx + x^2\,dx$$

$$= \int_{x=1}^{0} 3x^2\,dx = 3\int_{x=1}^{0} x^2\,dx$$

$$= 3\left\{\frac{x^3}{3}\right\} = 0 - 1 = -1$$

Thus,

$$\oint_C (xy+y^2)\,dx + x^2\,dy = \frac{19}{20} - \frac{20}{20} = \frac{-1}{20}$$

$$\therefore \oint_C M\,dx + N\,dy = \iint_S \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

Hence, Green's theorem verified.

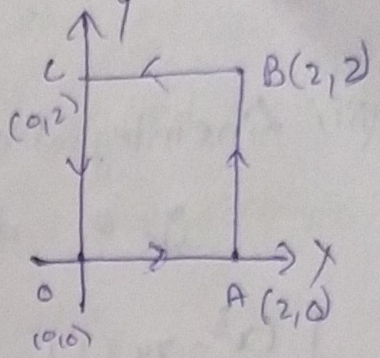
2) Verify Green's theorem for $\oint_C (x^2 - xy^3)\,dx + (y^2 - 2xy)\,dy$, where C is a square with vertices $(0,0)$, $(2,0)$, $(2,2)$ and $(0,2)$

part-I

$x \to 0$ to 2 (along OA)
$y \to 0$ to 2 (along OC)

Here,

$M = x^2 - xy^3 \qquad N = y^2 - 2xy$



$\frac{\partial M}{\partial y} = -3xy^2 \qquad \frac{\partial N}{\partial x} = -2y$

$$\iint_S \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \iint_S (-2y + 3xy^2)\,dx\,dy$$

$$= \int_{x=0}^{2}\left[\int_{y=0}^{2} \left(-\cancel{2}\frac{y^2}{\cancel{2}} + \cancel{3}x\frac{y^3}{\cancel{3}}\right)\right] dx$$

$$= \int_{x=0}^{2} \left[-y^2 + xy^3\right]_0^2 dx$$

$$= \int_{x=0}^{2} \left[-4 + 8x\right] dx$$

$$= \left[-4x + 8\frac{x^2}{2}\right]_0^2$$

$$= \left[-4(2) + 8\,\frac{4}{2}\right] = \left[-8 + 16\right]$$

$$= 8.$$

part-II

(i) Line integral along $\overrightarrow{OA}$ (C_1)

$y = 0, \; dy = 0. \qquad x \to 0$ to 2.

$$\therefore \int_{C_1} (x^2 - xy^3)\,dx + (y^2 - 2xy)\,dy = \int_{x=0}^{2} x^2 \cdot dx$$

$$= \left[\frac{x^3}{3}\right]_0^2 = \frac{8}{3}.$$

(ii) Line integral along $\overrightarrow{AB}$ (C_2)

$x = 2, \; dx = 0 \qquad y \to 0$ to 2.

$$\therefore \int_{C_2} (x^2 - xy^3)\,dx + (y^2 - 2xy)\,dy = \int_{y=0}^{2} y^2 - 4y \cdot dy$$

$$= \left[\frac{y^3}{3}\right]_0^2 - 4\left[\frac{y^2}{2}\right]_0^2$$

$$= \frac{8}{3} - 8 = \frac{8-24}{3} = \frac{-16}{3}$$

(iii) Line integral along $\overrightarrow{BC}$ (C_3)

$y = 2$, $dy = 0$, $x \to 2$ to 0.

$$\int_{C_3} (x^2 - xy^3)\,dx + (y^2 - 2xy)\,dy = \int_{C_3} x^2 - 8x \cdot dx$$

$$= \int_2^0 x^2 - 8x \cdot dx.$$

$$= \left[\frac{x^3}{3}\right]_2^0 - 8\left[\frac{x^2}{2}\right]_2^0$$

$$= \left[-\frac{8}{3}\right] - 8\left[-\frac{4}{2}\right]$$

$$= -\frac{8}{3} + 16 = \frac{-8+48}{3} = \frac{40}{3}$$

(iv) Line integral along $\overrightarrow{CO}$ (C_4)

$x = 0$, $dx = 0$, $y \to 2$ to 0.

$$\int_{C_4} (x^2 - xy^3)\,dx + (y^2 - 2xy)\,dy = \int_{C_4} y^2\,dy.$$

$$= \int_2^0 y^2\,dy$$

$$= \left[\frac{y^3}{3}\right]_2^0 = -\frac{8}{3}$$

$\therefore$ Thus,

$$\oint_C (x^2 - xy^3)\,dx + (y^2 - 2xy)\,dy = +\cancel{\frac{8}{3}} - \frac{16}{3} + \frac{40}{3} \cancel{- \frac{8}{3}}$$

$$= \frac{-16+40}{3}$$

$$= 8.$$

$$\therefore \oint_C M\,dx + N\,dy = \iint_S \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

$\therefore$ Hence Green's theorem is verified.

Stoke's Theorem

Let 's' be a open surface bounded by a simple closed curve 'c'. If $\bar{F}$ is any differentiable vector point function, then

$$\oint_C \bar{F}\cdot d\bar{r} = \int_S \text{curl}\,\bar{F}\cdot\bar{N}\,ds$$

when 'c' is traversed in the positive direction and $\bar{N}$ is outward drawn unit normal at any point of the surface

1) Verify stoke's theorem for $\bar{F} = -y^3\bar{i} + x^3\bar{j}$, where 's' is the circular disc $x^2+y^2 \le 1$, $z=0$

Sol: Given that $\bar{F} = -y^3\bar{i} + x^3\bar{j}$

The boundary 'c' of 's' is a circle in xy-plane

$$x^2+y^2=1, \quad z=0$$

Part-1

we have $\bar{r} = x\bar{i} + y\bar{j}$ $\quad z=0$

then $d\bar{r} = dx\bar{i} + dy\bar{j}$

Now, $\bar{F}\cdot d\bar{r} = -y^3dx + x^3dy$

$$\oint_C \bar{F}\cdot d\bar{r} = \oint_C -y^3dx + x^3dy$$

The parameter of the circle are $x = \cos\theta$

$y = \sin\theta$

$0 \le \theta \le 2\pi$

Also

$dx = -\sin\theta\,d\theta$

$dy = \cos\theta\,d\theta$

$$= \int_{\theta=0}^{2\pi}\{-(\sin^3\theta)(-\sin\theta\,d\theta) + (\cos^3\theta)(\cos\theta\,d\theta)\}$$

$$= \int_{\theta=0}^{2\pi}(\sin^4\theta + \cos^4\theta)\,d\theta$$

$$= \int_0^{2\pi}[(\sin^2\theta+\cos^2\theta)^2 - 2\sin^2\theta\cos^2\theta]\,d\theta$$

$$= \int^{2\pi}(1 - 2\sin^2\theta\cos^2\theta)\,d\theta$$

$$= \int_0^{2\pi} d\theta - \frac{1}{2}\int_0^{2\pi}(2\sin\theta\cos\theta)^2 d\theta$$

$$= [\theta]_0^{2\pi} - \frac{1}{2}\int_0^{2\pi}\sin^2 2\theta\, d\theta$$

$$= (2\pi - 0) - \frac{1}{2}\int_0^{2\pi}\frac{1-\cos 4\theta}{2}\, d\theta$$

$$= 2\pi - \frac{1}{4}\left(\theta - \frac{\sin 4\theta}{4}\right]_0^{2\pi}$$

$$= 2\pi - \frac{1}{4}[(2\pi - 0) - 0] = 2\pi - \frac{\pi}{2} = \frac{3\pi}{2}$$

Part-2

$$\text{curl}\,\bar{f} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y^3 & x^3 & 0 \end{vmatrix}$$

$$= \bar{i}(0-0) - j(0-0) + \bar{k}(3x^2 + 3y^2)$$

$$= 3(x^2+y^2)\bar{k}$$

$$\text{curl}\,\bar{f}\cdot\bar{N} = 3(x^2+y^2)(\bar{k}\cdot\bar{N})$$

$$\iint_S \text{curl}\,\bar{f}\cdot\bar{N}\,ds = \iint_S 3(x^2+y^2)(\bar{k}\cdot\bar{N})\,ds$$

Put (polar co-ordinats) $= \iint_R 3(x^2+y^2)\cos\gamma\, ds$

$x = r\cos\theta$

$y = r\sin\theta$

$dx\,dy = r\,dr\,d\theta$

$r \to 0$ to 1

$\theta \to 0$ to 2π

$$= 3\iint_R (x^2+y^2)\,dx\,dy$$

$$= 3\int_{r=0}^{1}\int_{\theta=0}^{2\pi} r^2 \cdot r\, dr\, d\theta$$

$$= 3\int_0^1 r^3\, dr \int_0^{2\pi} d\theta = 3\left[\frac{r^4}{4}\right]_0^1 [\theta]_0^{2\pi}$$

$$= 3 \times \frac{1}{4} \times 2\pi = \frac{3\pi}{2}$$

$$\oint_C \bar{f}\cdot d\bar{r} = \int_S \text{curl}\,\bar{f}\cdot\bar{N}\,ds$$ Stoke's theorem verified

Q) Verify Stoke's theorem for $\bar{F} = z\bar{i} + x\bar{j} + y\bar{k}$ where 'c' is the unit circle in xy-plane bounded by the hemisphere $z = \sqrt{1-x^2-y^2}$

Sol: Given that 'c' is unit circle in xy-plane

$z = 0,\ z = \sqrt{1-x^2-y^2} \Rightarrow 1-x^2-y^2 = 0$

$x^2+y^2 = 1,\ z = 0$

Part-1: $\bar{F} = z\bar{i} + x\bar{j} + y\bar{k}$

we have, $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

$d\bar{r} = dx\,\bar{i} + dy\,\bar{j} + dz\,\bar{k}$

Now, $\bar{F}\cdot d\bar{r} = z\,dx + x\,dy + y\,dz$

$= x\,dy \quad \because z=0 \Rightarrow dz = 0$

$\therefore \oint_c \bar{F}\cdot d\bar{r} = \oint_c x\,dy$

Parametric eqn of circle $x^2+y^2=1$ are $x = \cos\theta,\ y = \sin\theta$, $0 \le \theta \le 2\pi$. Also, $dy = \cos\theta\,d\theta$

$= \int_0^{2\pi} \cos\theta \cdot \cos\theta\,d\theta$

$= \int_0^{2\pi} \cos^2\theta\,d\theta$

$= \int_0^{2\pi} \frac{1+\cos 2\theta}{2}\,d\theta$

$= \frac{1}{2}\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{2\pi} = \frac{1}{2}(2\pi + 0 - 0) = \pi$

Part-2:

$$\text{curl}\,\bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & x & y \end{vmatrix}$$

$= \bar{i}(1-0) - \bar{j}(0-1) + \bar{k}(1-0)$

$= \bar{i} + \bar{j} + \bar{k}$

Here, $\bar{n} = \bar{k}$, $ds = dx\,dy$

$\text{curl}\,\bar{f}\cdot\bar{N} = (i+j+k)\cdot\bar{k} = 1$

$\therefore \int_S \text{curl}\,\bar{f}\cdot\bar{N}\,dS = \iint_S 1\,dx\,dy$

where A is area of the circle $x^2+y^2=1$

$= A$

$= \pi(1)^2 = \pi$

$\therefore \oint_C \bar{f}\cdot\bar{dr} = \int_S \text{curl}\,\bar{f}\cdot\bar{N}\,dS$

$\therefore$ Hence stoke theorem is verified

3) verify stokes theorem for $\bar{f} = (2x-y)\bar{i} - yz^2\bar{j} - yz$ over the upper half surface of sphere $x^2+y^2+z^2=1$ bound by projection of xy-plane.

$\oint_C \bar{f}\cdot\bar{dr} = \int_S \text{curl}\,\bar{f}\cdot\bar{N}\,dS$

Part-1

$C \Rightarrow x^2+y^2=1,\ z=0$

on xy-plane $\bar{dr} = dx\bar{i} + dy\bar{j}$

$\oint_C \bar{f}\cdot\bar{dr} = \oint_C (2x-y)\,dx - yz^2\,dy - yz^2\,dz.$

~~dt~~ - The parametrix equations of $x^2+y^2=1$ are

$x=\cos\theta,\ y=\sin\theta,\ dx=-\sin\theta\,d\theta,\ dy=\cos\theta\,d\theta$

θ varies from $\theta=0$ to $\theta=2\pi$

$\therefore \oint_C \bar{f}\cdot\bar{dr} = \int_{\theta=0}^{2\pi} (2\cos\theta - \sin\theta)(-\sin\theta\,d\theta)$

$= \int_{\theta=0}^{2\pi} \left[-\sin 2\theta + \frac{1-\cos 2\theta}{2}\right] d\theta$

$= \left[\frac{\cos 2\theta}{2} + \frac{\theta}{2} - \frac{\sin 2\theta}{2}\right]_0^{2\pi}$

$= \cos\frac{4\pi}{2} + \pi - \sin\frac{4}{2}\pi = (\pi - 0) = \pi$

part - II

$$\text{curl}\,\bar{f} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x-y & -yz^2 & -yz^2 \end{vmatrix} = \bar{i}\,(-2yz+2yz) - \bar{j}\,(0) + \bar{k}\,(1) = \bar{k}$$

Here $\bar{N} = \bar{k}$, $dS = dx\,dy$

$\text{curl}\,\bar{f}\cdot\bar{N} = \bar{k}\cdot(x\bar{i}+y\bar{j}+z\bar{k}) = z$

$\therefore \int_S \text{curl}\,\bar{f}\cdot\bar{N}\,dS = \iint_R \cancel{z}\,\frac{dx\,dy}{z} = \iint_R dx\,dy$ where A is area of circle $x^2+y^2=1$

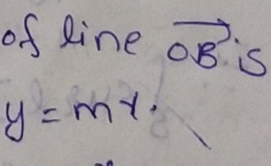
$= A$

$= \pi(1)^2 = \pi$

$\therefore \oint_C \bar{f}\cdot d\bar{r} = \int_C \text{curl}\,\bar{f}\cdot\bar{N}\,dS$

$\therefore$ Hence stoke theorem is verified

4) Evaluate $\int_C \bar{f}\cdot d\bar{r}$ by stoke's theorem, where $\bar{f} = y^2\bar{i} + x^2\bar{j} + (x+z)\bar{k}$ and 'c' is the boundary of the region with vertices (0,0,0), (1,0,0) and (1,1,0).



sol: $x \to 0$ to 1

$y \to 0$ to x.

Eq of line $\overrightarrow{OB}$ is $y = mx$, $y = x$.

$\bar{f} = y^2\bar{i} + x^2\bar{j} + (x+z)\bar{k}$

$$\text{curl}\,\bar{f} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & x^2 & x+z \end{vmatrix}$$

$= \bar{i}\,(0) - \bar{j}\,(1-0) + \bar{k}\,(2x-2y)$

$= -\bar{j} + \bar{k}\,(2x-2y)$

Here $\bar{N} = \bar{k}$, $dS = dx\,dy$.

$\text{curl}\,\bar{f}\cdot\bar{N} = \bar{k}\cdot(-\bar{j} + 2(x-y)\bar{k}) = 2(x-y)$

$$\int_S \text{curl}\,\bar{f}\cdot\bar{N}\,dS = \iint_R 2x - 2y\,dx\,dy$$

$$= \int_{x=0}^{1}\int_{y=0}^{x} 2x - 2y\,dx\,dy$$

$$= 2\int_{x=0}^{1}\left[xy - \frac{y^2}{2}\right]_0^x dx$$

$$= 2\int_{x=0}^{1}\left\{x^2 - \frac{x^2}{2}\right\} dx$$

$$= \int_{x=0}^{1} x^2\,dx = \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3}$$

5) verify stokes theorem for $\bar{f} = (x^2-y^2)\bar{i} + 2xy\bar{j}$ over the box bounded by the planes $x=0, y=0, x=a, y=b$

Sol: $\bar{f} = (x^2-y^2)\bar{i} + 2xy\bar{j}$

Part-1

$$\text{curl}\,\bar{f} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (x^2-y^2) & 2xy & 0 \end{vmatrix}$$

$$= i(0-0) - j(0-0) + k(2y+2y)$$

$$= 4y\,\bar{k}$$

Here $N = \bar{k}$ $\quad dS = dx\,dy$

$$\text{curl}\,\bar{f}\cdot\bar{N} = (4y\bar{k})\cdot\bar{k} = 4y$$

$$\therefore \int_S \text{curl}\,\bar{f}\cdot\bar{N}\,dS = \iint_S 4y\,dx\,dy$$

$$= \int_{x=0}^{a}\int_{y=0}^{b} 4y\,dx\,dy$$

$$= 4\int_0^a dx \int_0^b y\,dy$$

$$= 4\left[x\right]_0^a \left\{\frac{y^2}{2}\right\}_0^b$$

① a) $f(t) = e^{2t} + 4t^3 - 2\sin 3t + 3\cos 3t$

$$f(s) = L\{f(t)\} = L\{e^{2t}\} + 4L\{t^3\} - 2L\{\sin 3t\}$$
$$+ 3L\{\cos 3t\}$$

$$f(s) = \frac{1}{s-2} + 4\frac{(3!)}{s^{3+1}} - 2\frac{(3)}{s^2+9} + \frac{3(s)}{s^2+9}$$

$$f(s) = \frac{1}{s-2} + \frac{24}{s^4} - \frac{6}{s^2+9} + \frac{3s}{s^2+9}$$

b) $e^{4t}\{\sin 2t \cos t\}$

$$f(t) = \sin 2t \cos t$$

$$f(t) = \frac{1}{2}(2\sin 2t \cos t)$$

$$= \frac{1}{2}[\sin(2t+t) + \sin(2t-t)]$$

$$f(t) = \frac{1}{2}[\sin 3t + \sin t]$$

$$f(s) = L\{f(t)\} = \frac{1}{2}L\{\sin 3t\} + \frac{1}{2}L\{\sin t\}$$

$$= \frac{1}{2}\left[\frac{3}{s^2+9}\right] + \frac{1}{2}\left[\frac{1}{s^2+1}\right]$$

$$f(s) = \frac{3}{2(s^2+9)} + \frac{1}{2(s^2+1)}$$

now

$$L\{e^{4t}\{\sin 2t \cos t\}\} = f(s-4)$$

$$= \frac{3}{2((s-4)^2+9)} + \frac{1}{2((s-4)^2+1)}$$

$$= \frac{3}{2(s^2-8s+25)} + \frac{1}{2(s^2-8s+17)}$$

(2) (a) $e^{-3t}(2\cos 5t - 3\sin 5t)$

$$f(t) = 2\cos 5t - 3\sin 5t$$

$$f(s) = L\{f(t)\} = 2L\{\cos 5t\} - 3L\{\sin 5t\}$$

$$= 2\frac{s}{s^2+25} - \frac{3(5)}{s^2+25}$$

$$f(s) = \frac{2s-15}{s^2+25}$$

NOW

$$L\{e^{-3t}(2\cos 5t - 3\sin 5t)\}$$

$$= f(s+3)$$

$$= \frac{2(s+3)-15}{(s+3)^2+25}$$

$$= \frac{2s+6-15}{s^2+9+6s+25}$$

$$= \frac{2s-9}{s^2+6s+34}$$

2) (b) $\cos 3t \sin 5t$

$$f(t) = \cos 3t \sin 5t$$

$$f(t) = \frac{1}{2}(2\sin 5t \cos 3t)$$

$$= \frac{1}{2}[\sin(8t) + \sin(2t)]$$

$$f(s) = L\{f(t)\} = \frac{1}{2}L\{\sin(8t)\} + \frac{1}{2}L\{\sin 2t\}$$

$$= \frac{1}{\cancel{2}}\left(\frac{\cancel{8}^{4}}{s^2+64}\right) + \frac{1}{\cancel{2}}\left(\frac{\cancel{2}}{s^2+4}\right)$$

$= \frac{4}{s^2+64} + \frac{1}{s^2+4}$

3)(a) $\int_0^t e^{-t}\cos t \, dt$

$f(s) = \int_0^\infty e^{-st} f(t) dt$

comparing formula with quest

$f(t) = \cos t$ & $s = 1$

$F(s) = L\{f(t)\} = L\{\cos t\} = \frac{s}{s^2+1}$

so now

$s = 1$

$f(1) = \int_0^t e^{-t}\cos t \, dt$

$= \frac{(1)}{(1)^2+(1)} = \frac{1}{2}$

3)(b) $\frac{\sin 3t \cos t}{t}$

$= \frac{\frac{1}{2}(\sin 4t) + \frac{1}{2}\sin 2t}{t}$

$f(t) = \sin 3t \cos t = \frac{1}{2}(\sin(4t) + \sin(2t))$

$L\{f(t)\} = \frac{1}{2} L\{\sin 4t\} + \frac{1}{2} L\{\sin 2t\}$

$= \frac{1}{\cancel{2}}\left(\frac{\cancel{4}^{2}}{s^2+16}\right) + \frac{1}{\cancel{2}}\left(\frac{\cancel{2}}{s^2+4}\right)$

$f(s) = \frac{2}{s^2+16} + \frac{1}{s^2+4}$

$L\left\{\frac{f(t)}{t}\right\} = \frac{1}{2} L\left\{\frac{\sin 4t + \sin 2t}{t}\right\}$

$= \frac{1}{2}\int_s^\infty L\{\sin 4t + \sin 2t\} ds$

$= \int_s^\infty \left(\frac{2}{s^2+16} + \frac{1}{s^2+4}\right) ds$

$$= \int_s^\infty \left(\frac{\frac{2}{16}}{\left(\frac{s}{4}\right)^2+1}\right) ds + \int_s^\infty \frac{\frac{1}{4}}{\left(\frac{s}{2}\right)^2+1} ds$$

$$= \frac{1}{8}\left(\tan^{-1}\left(\frac{s}{4}\right)\right]_s^\infty + \frac{1}{4}\left(\tan^{-1}\left(\frac{s}{2}\right)\right]_s^\infty$$

$$= \frac{1}{8}\left[\tan^{-1}(\infty) - \tan^{-1}\left(\frac{s}{4}\right)\right] + \frac{1}{4}\left(\tan^{-1}(\infty) - \tan^{-1}\left(\frac{s}{2}\right)\right)$$

$$= \frac{1}{8}\left(\cancel{\tan^{-1}}(\cancel{\tan}\tfrac{\pi}{2}) - \tan^{-1}\left(\frac{s}{4}\right)\right] + \frac{1}{4}\left(\cancel{\tan^{-1}}(\cancel{\tan}\tfrac{\pi}{2}) - \tan^{-1}\left(\frac{s}{2}\right)\right)$$

$$= \frac{1}{8}\cot^{-1}\left(\frac{s}{4}\right) + \frac{1}{4}\cot^{-1}\left(\frac{s}{2}\right)$$

(4) (a) $L\{t(\underbrace{\sin 3t \cos 2t}_{f(t)})\}$

$$f(t) = \sin 3t \cos 2t$$

$$= \frac{1}{2}(\sin 5t + \sin t)$$

~~$L\{t f(t)\}$~~

$$F(s) = L\{f(t)\} = \frac{1}{2}L\{\sin 5t\} + \frac{1}{2}L\{\sin t\}$$

$$F(s) = \frac{1}{2}\left(\frac{5}{s^2+25}\right) + \frac{1}{2}\left(\frac{1}{s^2+1}\right)$$

$$L\{t(\sin 3t \cos 2t)\} = \frac{1}{2}(-1)^1\left[\frac{d}{ds}\left(\frac{5}{s^2+25}\right) + \frac{d}{ds}\left(\frac{1}{s^2+1}\right)\right]$$

$$= \frac{-1}{2}\left[\frac{(s^2+25)(0) - 5(2s)}{(s^2+25)^2} + \frac{(s^2+1)0 - 1(2s)}{(s^2+1)^2}\right]$$

$$= \frac{-1}{2}\left[\frac{-2s(5)}{(s^2+25)^2}\right] + \left(\frac{-1}{2}\right)\left[\frac{-2s}{(s^2+1)^2}\right]$$

$$= \frac{\cancel{10}^{5} s}{\cancel{2}(s^2+25)^2} + \frac{\cancel{2}s}{\cancel{2}(s^2+1)^2}$$

$$= \frac{5s}{(s^2+25)^2} + \frac{s}{(s^2+1)^2}$$

4) (b) $f(s) = \int_0^t t e^{-t} \sin 2t \, dt$

$f(t) = t\sin 2t \quad , s = +1$

$$f(s) = L\{f(t)\} = (-1)^1 \frac{d}{ds}\left[\frac{2}{s^2+4}\right]$$

$$= (-1)\left[\frac{(s^2+4)(0) - (2s)(2)}{(s^2+4)^2}\right]$$

$$f(s) = \frac{(-1)(-4s)}{(s^2+4)^2} = \frac{4s}{(s^2+4)^2}$$

$$f(1) = \int_0^t e^{-t}\, t\sin 2t \, dt$$

$$f(1) = \frac{4(1)}{(1+4)^2} = \frac{4}{25}$$

5) (b) $L\{t e^{3t} \sin 2t\}$

$$f(s) = L\{f(t)\} = L\{t\sin 2t\} = -\frac{d}{ds}\frac{2}{s^2+4}$$

$$= -\left(\frac{0 - 2(2s)}{(s^2+4)^2}\right)$$

$$= \frac{4s}{(s^2+4)^2}$$

$$L\{e^{3t} t\sin 2t\} = f(s-3)$$

$$= \frac{4(s-3)}{((s-3)^2+4)^2}$$

$$= \frac{4s-12}{(s^2+9-6s+4)^2}$$

$$= \frac{4s-12}{(s^2-6s+13)^2}$$

5) a) $\frac{e^{-at}-e^{-bt}}{t}$ $\boxed{f(t) = e^{-at} - e^{-bt}}$

$$f(s) = L\{f(t)\} = \frac{1}{s+a} - \frac{1}{s+b}$$

$$L\left\{\frac{e^{-at}-e^{-bt}}{t}\right\} = \int_s^{\infty} \left(\frac{1}{s+a} - \frac{1}{s+b}\right) ds$$

$$= \left(\log(s+a) - \log(s+b)\right)_s^{\infty}$$

$$= \left(\log\left(\frac{s+a}{s+b}\right)\right)_s^{\infty}$$

$$= \log(1) - \log\left(\frac{s+a}{s+b}\right)$$

$$= -\log\left|\frac{s+a}{s+b}\right|$$

⑥ a) $\dfrac{3(s^4+4-4s^2)}{2s^5} = \dfrac{3s^4+12-12s^2}{2s^5} = \dfrac{3s^4}{2s^5}+\dfrac{12}{2s^5}-\dfrac{12s^2}{2s^5}$

$$f(s) = \frac{3}{2}\left(\frac{1}{s}\right) + 6\left(\frac{1}{s^5}\right) - 6\left(\frac{1}{s^3}\right)$$

$$f(t) = L^{-1}(f(s)) = \frac{3}{2}L^{-1}\left(\frac{1}{s}\right) + 6L^{-1}\left(\frac{1}{s^5}\right) - 6L^{-1}\left(\frac{1}{s^3}\right)$$

$$= \frac{3}{2}(1) + 6\frac{t^4}{4!} - 6\frac{t^2}{2!}$$

$$f(t) = \frac{3}{2} + \frac{6t^4}{24} - 6\frac{t^2}{4}$$

⑦ b) $\dfrac{4}{(s+1)(s+2)} = \dfrac{A}{s+1} + \dfrac{B}{s+2}$

$$4 = A(s+2) + B(s+1)$$

Put $s=-1$ | put $s=-2$

$A = 4$ | $B = -4$

$$\frac{4}{(s+1)(s+2)} = \frac{4}{s+1} - \frac{4}{s+2}$$

$$\frac{4}{(s+1)(s+2)} = 4\left[\frac{1}{s+1} - \frac{1}{s+2}\right]$$

$$f(t) = L^{-1}\left\{\frac{4}{(s+1)(s+2)}\right\} = 4\left[L^{-1}\left\{\frac{1}{s+1}\right\} - L^{-1}\left\{\frac{1}{s+2}\right\}\right]$$

$$f(t) = 4\left[e^{-t} - e^{-2t}\right]$$

6) (b) $L^{-1}\left\{\frac{s^2}{(s^2+4)(s^2+25)}\right\}$

Let $F(s) = \frac{s}{s^2+4}$ & $G(s) = \frac{s}{s^2+25}$

$L^{-1}\{F(s)\} = L^{-1}\left\{\frac{s}{s^2+(2)^2}\right\} = \cos 2t = f(t)$ say

$L^{-1}\{G(s)\} = L^{-1}\left\{\frac{s}{(s^2+(5)^2}\right\} = \cos 5t = g(t)$ say

$L^{-1}\left\{\frac{s^2}{(s^2+4)(s^2+25)}\right\} = L^{-1}\left\{\frac{s}{(s^2+4)} \cdot \frac{s}{(s^2+25)}\right\}$

$= L^{-1}\{F(s) \cdot G(s)\}$

$= f(t) * g(t)$ by using C.T

$= \int_0^t f(u)\, g(t-u)\, du$

$= \int_0^t \cos 2u \cdot \cos 5(t-u)\, du$

$= \frac{1}{2}\int_0^t 2\cos 2u \cdot \cos(5t-5u)\, du$

$= \frac{1}{2}\int_0^t (\cos(2u-5t+5u) +$
$\cos(2u+5t-5u))$

$= \frac{1}{2}\int_0^t (\cos(7u-5t) + \cos(5t-3u))\, du$

$= \frac{1}{2}\left[\int_0^t \cos(7u-5t)\, du + \int_0^t \cos(5t-3u)\, du\right]$

$= \frac{1}{2}\left[\frac{\sin(7u-5t)}{7}\right]_0^t + \frac{1}{2}\left[\frac{\sin(5t-3u)}{-3}\right]_0^t$

$$\int_0^t \cos 2u \cdot \cos(5t-5u)$$

$$\frac{1}{2}\int_0^t \cos(2u+5t-5u)+\cos(2u+5u-5t)$$

$$\frac{1}{2}\int_0^t \cos(5t-3u)\,du + \int_0^t \cos(7u-5t)\,du$$

$$\frac{1}{2}\left[\left[\frac{\sin(5t-3u)}{-3}\right]_0^t + \left[\frac{\sin(7u-5t)}{7}\right]_0^t\right.$$

$$\left[\frac{\sin(2t)}{-3}+\frac{\sin(5t)}{3}\right]+\left[\sin\left(\frac{2t}{7}\right)-\frac{\sin(-5t)}{7}\right]$$

$$\frac{1}{2}\left[\frac{\sin 2t}{-3}+\frac{\sin 2t}{7}+\frac{\sin 5t}{3}+\frac{\sin 5t}{7}\right.$$

$$-\frac{2}{21}\sin 2t+\frac{5}{21}\sin 5t \quad \frac{4\sin 2t}{-21}+\frac{10\sin 5t}{21}$$

7 (a) $L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\}$

Let $F(s) = \frac{s}{s^2+a^2}$ and $G(s) = \frac{1}{s^2+a^2}$

$L^{-1}\{F(s)\} = L^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \cos at = f(t)$ say

$L^{-1}\{G(s)\} = L^{-1}\left\{\frac{1}{s^2+a^2}\right\} = \frac{1}{a}\sin at = g(t)$ say

$L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\} = L^{-1}\left\{\frac{s}{(s^2+a^2)}\cdot\frac{1}{(s^2+a^2)}\right\}$

$= L^{-1}\{F(s)\cdot G(s)\}$

$= f(t) * g(t)$ C.T

$= \int_0^t f(u)\cdot g(t-u)\,du$

$= \frac{1}{a}\int_0^t \cos au \sin a(t-u)\,du$

$= \frac{1}{a(2)}\int_0^t 2\cos au \sin(at-au)\,du$

$= \frac{1}{2a}\int_0^t \sin(at-\cancel{au}+\cancel{au})$

$+\sin(at-au-au)\,du$

$= \frac{1}{2a}\int_0^t \sin at + \sin(at-2au)$

$= \frac{1}{2a}\left[\left[\int_0^t \sin at\,du\right]\right.$

$\left.+\int_0^t \sin(at-2au)\,du\right]$

$= \frac{1}{2a}\left[(u\sin at)_0^t + \frac{\cos(at-2au)}{2a}\Big|_0^t\right]$

$= \frac{1}{2a}[t\sin at] + \frac{1}{4a^2}\left(\cancel{\cos(-at) - \cos at}\right)$

$= \frac{1}{2a}(t\sin at)$

(8) $(D^2+4D+5)y = 5$ $\quad y(0)=0,\ y'(0)=0$

$y''+4y'+5y=5$

Apply Laplace on BS

$$L\{y''\}+4L\{y'\}+5L\{y\} = L\{5\}$$

$$s^2L\{y\} - \boxed{s\,y(0)} - \boxed{y'(0)} + 4\left[sL\{y\} - \boxed{y(0)}\right] + 5L\{y\} = 5/s$$

$$L\{y\}\left[s^2+4s+5\right] - 0 = \frac{5}{s}$$

$$L\{y\} = \frac{5}{s(s^2+4s+5)}$$

$$\frac{5}{s(s^2+4s+5)} = \frac{A}{s} + \frac{(Bs+C)}{s^2+4s+5} \longrightarrow (1)$$

$$5 = A(s^2+4s+5) + (Bs+C)s$$

$$5 = As^2+Bs^2+4As+Cs+5A$$

s^2	s	c
$0 = A+B$	$0 = 4A+C$	$\cancel{5} = 5A$
		$A = 1$

$A+B=0$	$4(1)+C=0$
$1+B=0$	$C=-4$
$B=-1$	

sub [A, B, C] values in (1)

$$L\{y\} = \frac{5}{s(s^2+4s+5)} = \frac{1}{s} - \frac{(s+4)}{(s^2+4s+5)}$$

$$L\{y\} = \frac{1}{s} - \frac{(s+2)+2}{(s+2)^2+(1)^2}$$

$$L\{y\} = \frac{1}{s} - \frac{(s+2)}{(s+2)^2+(1)^2} - \frac{2}{(s+2)^2+(1)^2}$$

Apply L^{-1} on BS

$$y = L^{-1}\left\{\frac{1}{s}\right\} - L^{-1}\left\{\frac{(s+2)}{(s+2)^2+(1)^2}\right\} - 2\,L^{-1}\left\{\frac{1}{(s+2)^2+(1)^2}\right\}$$

$$= 1 - e^{-2t}\cos t - 2e^{-2t}\sin t$$

(9) $(D^2 - 2D + 2)x = 0 \qquad x(0) = 0,\ x'(0) = 1$

$$x'' - 2x' + 2x = 0$$

Apply 'L' on B.S

$$L\{x''\} - 2L\{x'\} + 2L\{x\} = L\{0\}$$

$$s^2L\{x\} - \boxed{s\,x(0)} - x'(0) - 2[sL\{x\} - \boxed{x(0)}] + 2L\{x\} = 0$$

$$L\{x\}[s^2 - 2s + 2] - 1 = 0$$

$$L\{x\} = \frac{1}{s^2 - 2s + 1 + 1}$$

$$L\{x\} = \frac{1}{(s-1)^2 + 1}$$

Apply L^{-1} on BS

$$x = L^{-1}\left\{\frac{1}{(s-1)^2+1}\right\}$$

$$= e^{t}\sin(1)t$$

$$= e^{t}\sin t$$

⑩ $y'' + 7y' + 10y = 4e^{-3t}$ $y(0)$, $y'(0) = -1$

Apply L on BS

$L\{y''\} + 7L\{y'\} + 10L\{y\} = 4L\{e^{-3t}\}$

$s^2L\{y\} - s\boxed{y(0)} - y'(0) + 7[sL\{y\} - \boxed{y(0)}] + 10L\{y\} = 4L\{e^{-3t}\}$

$L\{y\}[s^2 + 7s + 10] + 1 = 4\left(\frac{1}{s+3}\right)$

$$L\{y\} = \frac{4 - (s+3)}{(s+3)(s+5)(s+2)}$$

$$L\{y\} = \frac{1-s}{(s+3)(s+5)(s+2)}$$

$$L\{y\} = \frac{1-s}{(s+3)(s+5)(s+2)} = \frac{A}{s+3} + \frac{B}{s+5} + \frac{C}{s+2}$$

$1 - s = (s+5)(s+2)(A) + (s+3)(s+2)B + C(s+3)(s+5)$

put $s = -3$

$1 + 3 = (2)(-1)A$

$A = -2$

put $s = 5$

$1 + 5 = (2)(-3)B$

$B = 1$

put $s = -2$

$1 + 2 = (1)(3)C$

$C = 1$

$$L\{y\} = \frac{-2}{s+3} + \frac{1}{s+5} + \frac{1}{s+2}$$

Apply L^{-1} on B.S

$y = -2L^{-1}\left\{\frac{1}{s+3}\right\} + L^{-1}\left\{\frac{1}{s+5}\right\} + L^{-1}\left\{\frac{1}{s+2}\right\}$

$$= -2e^{-3t} + e^{-5t} + e^{-2t}$$

Inverse Laplace Transforms

$\mathcal{L}^{-1}\{f(s)\} = f(t)$

$\mathcal{L}^{-1}\{k/s\} = k$

$\mathcal{L}^{-1}\left\{\frac{1}{s^n}\right\} = \frac{t^{n-1}}{(n-1)!}$

$\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} = 1$

$\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = t$

$\mathcal{L}^{-1}\left\{\frac{1}{s^{n+1}}\right\} = \frac{t^n}{n!}$

$\mathcal{L}^{-1}\left\{\frac{n!}{s^{n+1}}\right\} = t^n$

$\mathcal{L}^{-1}\left\{\frac{1}{s+a}\right\} = e^{-at}$

$\mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$

$\mathcal{L}^{-1}\left\{\frac{a}{s^2-a^2}\right\} = \sinh at \Rightarrow \mathcal{L}^{-1}\left\{\frac{1}{s^2-a^2}\right\} = \frac{1}{a}\sinh at$

$\mathcal{L}^{-1}\left\{\frac{s}{s^2-a^2}\right\} = \cosh at$

$\mathcal{L}^{-1}\left\{\frac{1}{s^2+a^2}\right\} = \frac{1}{a}\sin at$

$\mathcal{L}^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \cos at$

1) find $\mathcal{L}^{-1}\left\{\frac{5}{s+2}\right\} = 5\left[\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\}\right] = 5e^{-2t}$

2) find $\mathcal{L}^{-1}\left\{\frac{4s}{s^2+9}\right\} \Rightarrow 4\left[\mathcal{L}^{-1}\left\{\frac{s}{s^2+3^2}\right\}\right] = 4\cos 3t$

3) $\mathcal{L}^{-1}\left\{\frac{3s-4}{16-s^2}\right\}$

$\mathcal{L}^{-1}\left\{\frac{4-3s}{s^2-16}\right\} = \mathcal{L}^{-1}\left\{\frac{4}{s^2-16} - \frac{3s}{s^2-16}\right\}$

$= \mathcal{L}^{-1}\left\{\frac{4}{s^2-16}\right\} - 3\,\mathcal{L}^{-1}\left\{\frac{s}{s^2-16}\right\}$

$= \cancel{4}\left(\frac{1}{\cancel{4}}\sinh 4t\right) - 3\cosh 4t$

$= \sinh 4t - 3\cosh 4t$

4) $\mathcal{L}^{-1}\left\{\frac{8s^3+5}{s^7}\right\}$

$\mathcal{L}^{-1}\left\{\frac{8s^3}{s^7} + \frac{5}{s^7}\right\}$

$\mathcal{L}^{-1}\left\{\frac{8}{s^4} + \frac{5}{s^7}\right\}$

$= 8\left\{\mathcal{L}^{-1}\left\{\frac{1}{s^4}\right\}\right\} + 5\left\{\frac{1}{s^7}\right\}$

$= 8\left[\mathcal{L}^{-1}\frac{1}{s^{3+1}}\right] + 5\left[\mathcal{L}^{-1}\frac{1}{s^{6+1}}\right]$

$= 8\left(\frac{t^3}{3!}\right) + 5\left(\frac{t^6}{6!}\right)$

5) $\mathcal{L}^{-1}\left\{\frac{6s-5}{s^2+7}\right\}$

$\mathcal{L}^{-1}\left\{\frac{6s}{s^2+7} - \frac{5}{s^2+7}\right\}$

$6\mathcal{L}^{-1}\left\{\frac{s}{s^2+7}\right\} - 5\mathcal{L}^{-1}\left\{\frac{1}{s^2+7}\right\}$

$= 6(\cos\sqrt{7}t) - 5\left[\frac{\sin\sqrt{7}t}{\sqrt{7}}\right]$

6) $\mathcal{L}^{-1}\left[\frac{s^2+4s-4}{s^3-4s}\right]$

$\mathcal{L}^{-1}\left\{\frac{s^2+4s-4}{s(s^2-4)}\right\} = \mathcal{L}^{-1}\left\{\frac{s^2-4}{s(s^2-4)} + \frac{4s}{s(s^2-4)}\right\}$

$= \mathcal{L}^{-1}\left\{\frac{1}{s} + \frac{4}{s^2-4}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + 4\mathcal{L}^{-1}\left\{\frac{1}{s^2-4}\right\}$

$= 1 + 4 \times \frac{1}{2}\sinh 2t$

$= 1 + 2\sinh 2t$

Linearity property

If $\mathcal{L}^{-1}\{f(s)\} = f(t)$ and $\mathcal{L}^{-1}\{g(s)\} = g(t)$,

then $\mathcal{L}^{-1}\{c_1 f(s) + c_2 g(s)\} = c_1 \mathcal{L}^{-1}\{f(s)\} + c_2 \mathcal{L}^{-1}\{g(s)\}$

where c_1 & c_2 are constants

7) find $\mathcal{L}^{-1}\left\{\frac{4}{(s+1)(s+2)}\right\}$

Sol: Consider. $\frac{4}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2}$

$4 = A(s+2) + B(s+1)$

put $s=-1$, $A=4$ and $s=-2$, $B=-4$.

$\therefore \mathcal{L}^{-1}\left\{\frac{4}{(s+1)(s+2)}\right\} = \mathcal{L}^{-1}\left\{\frac{4}{s+1} - \frac{4}{s+2}\right\}$

$= 4 \cdot \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} - 4 \cdot \mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\}$

$= 4 \cdot e^{-t} - 4 \cdot e^{-2t} = 4\left(e^{-t} - e^{-2t}\right)$

8) find $\mathcal{L}^{-1}\left\{\frac{s-2}{s^2+5s+6}\right\}$

Sol: Consider, $\frac{s-2}{s^2+5s+6} = \frac{s-2}{(s+2)(s+3)} = \frac{A}{s+2} + \frac{B}{s+3}$

$s-2 = A(s+3) + B(s+2)$

$A = -4$ and $B = 5$

$\mathcal{L}^{-1}\left\{\frac{s-2}{s^2+5s+6}\right\} = \mathcal{L}^{-1}\left\{\frac{-4}{s+2} + \frac{5}{s+3}\right\}$

$= -4\,\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} + 5\,\mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\}$

$= -4e^{-2t} + 5e^{-3t}$

a) find $\mathcal{L}^{-1}\left\{\dfrac{1}{(s+1)(s^2+1)}\right\}$

i) Consider,

$$\frac{1}{(s+1)(s^2+1)} = \frac{A}{s+1} + \frac{Bs+C}{s^2+1} \quad -(1)$$

$1 = A(s^2+1) + (Bs+C)(s+1) =$

$(A+B)s^2 + (B+C)s + (A+C)$

Comparing the coefficients of s^2, s & constant o.B.s

$A+B=0$, $B+C=0$ and $A+C=1$

solving these, we get

$$A = \frac{1}{2},\ B = \frac{-1}{2},\ C = \frac{1}{2}$$

$\therefore$ from (1), we have

$$\frac{1}{(s+1)(s^2+1)} = \frac{1}{2}\left[\frac{1}{s+1} - \frac{s}{s^2+1} + \frac{1}{s^2+1}\right]$$

Hence $\mathcal{L}^{-1}\left\{\dfrac{1}{(s+1)(s^2+1)}\right\} = \dfrac{1}{2}\left[\mathcal{L}^{-1}\left\{\dfrac{1}{s+1}\right\} - \mathcal{L}^{-1}\left\{\dfrac{s}{s^2+1}\right\} + \mathcal{L}^{-1}\left\{\dfrac{1}{s^2+1}\right\}\right]$

$$= \frac{1}{2}\left\{e^{-t} - \cos t + \sin t\right\}$$

iii) find $\mathcal{L}^{-1}\left\{\dfrac{s^2+s-2}{s(s+3)(s-2)}\right\}$

Sol Consider,

$$\frac{s^2+s-2}{s(s+3)(s-2)} = \frac{A}{s} + \frac{B}{s+3} + \frac{C}{s-2} \quad -(1)$$

$s^2+s-2 = A(s+3)(s-2) + Bs(s-2) + Cs(s+3) \quad -(2)$

To find A, put $s=0$ in (2)

$\therefore -2 = -6A \Rightarrow A = \dfrac{1}{3}$

To find B, put $s=-3$ in Eq (2)

$\therefore 4 = 15B \Rightarrow B = \dfrac{4}{15}$

To find C, put $s=2$ in Eq (2)

$\therefore 4 = 10C$

$C = \frac{2}{5}$

Sub A, B, C in eq ①

$$\frac{s^2+s-2}{s(s+3)(s-2)} = \frac{1}{3s} + \frac{4}{15(s+3)} + \frac{2}{5(s-2)}$$

Hence,

$$L^{-1}\left\{\frac{s^2+s-2}{s(s+3)(s-2)}\right\} = \frac{1}{3}L^{-1}\left\{\frac{1}{s}\right\} + \frac{4}{15}L^{-1}\left\{\frac{1}{s+3}\right\} + \frac{2}{5}L^{-1}\left\{\frac{1}{s-2}\right\}$$

$$= \frac{1}{3}\cdot 1 + \frac{4}{15}e^{-3t} + \frac{2}{5}e^{2t}$$

$$= \frac{1}{3} + \frac{4}{15}e^{-3t} + \frac{2}{5}e^{2t}$$

10) Find $L^{-1}\left\{\frac{s^2}{(s^2+4)(s^2+25)}\right\}$.

Sol: Take

$$\frac{1}{(s^2+4)(s^2+25)} = \frac{1}{21}\left[\frac{1}{s^2+4} - \frac{1}{s^2+25}\right]$$

$$\therefore L^{-1}\left\{\frac{s^2}{(s^2+4)(s^2+25)}\right\} = L^{-1}\left\{\frac{1}{21}\frac{s^2}{s^2+4} - \frac{s^2}{s^2+25}\right\}$$

$$= \frac{1}{21}\cdot L^{-1}\left\{\frac{s^2+4-4}{s^2+4} - \frac{s^2+25-25}{s^2+25}\right\}$$

$$= \frac{1}{21}\cdot L^{-1}\left\{\left[1-\frac{4}{s^2+4}\right] - \left[1-\frac{25}{s^2+25}\right]\right\}$$

$$= \frac{1}{21}L^{-1}\left\{\frac{25}{s^2+5^2} - \frac{4}{s^2+2^2}\right\}$$

$$= \frac{1}{21}\left[5\cdot L^{-1}\left\{\frac{5}{s^2+5^2}\right\} - 2\cdot L^{-1}\left\{\frac{2}{s^2+2^2}\right\}\right]$$

$$= \frac{1}{21}\left[5\cdot \sin 5t - 2\sin 2t\right]$$

11) $\mathcal{L}^{-1}\left\{\dfrac{s^2}{(s^2+4)(s^2+9)}\right\}$

Sol Take,

$$\frac{1}{(s^2+4)(s^2+9)} = \frac{1}{5}\left[\frac{1}{s^2+4} - \frac{1}{s^2+9}\right]$$

$$\therefore \mathcal{L}^{-1}\left\{\frac{s^2}{(s^2+4)(s^2+9)}\right\} = \frac{1}{5}\mathcal{L}^{-1}\left\{\frac{(s^2+4)-4}{s^2+4} - \frac{(s^2+9)-9}{s^2+9}\right\}$$

$$= \frac{1}{5}\mathcal{L}^{-1}\left\{\left[1-\frac{4}{s^2+4}\right] - \left[1 - \frac{9}{s^2+9}\right]\right\}$$

$$= \frac{1}{5}\mathcal{L}^{-1}\left\{\frac{9}{s^2+9} - \frac{4}{s^2+4}\right\}$$

$$= \frac{1}{5}\left[3\cdot\mathcal{L}^{-1}\left\{\frac{3}{s^2+3^2}\right\} - 2\cdot\mathcal{L}^{-1}\left\{\frac{2}{s^2+2^2}\right\}\right]$$

$$= \frac{1}{5}(3\sin 3t - 2\sin 2t)$$

(12) $\mathcal{L}^{-1}\left\{\dfrac{2s+3}{s^3-6s^2+11s-6}\right\}$

Sol Take,

$$\frac{2s+3}{s^3-6s^2+11s-6} = \frac{2s+3}{(s-1)(s-2)(s-3)} = \frac{A}{s-1} + \frac{B}{s-2} + \frac{C}{s-3} \quad \text{—(1)}$$

$$2s+3 = A(s-2)(s-3) + B(s-1)(s-3) + C(s-1)(s-2) \quad \text{—(2)}$$

Put $s=1$ in (2) $5 = A(1-2)(1-3)$ $\therefore A = 5/2$

Put $s=2$ in (2) $7 = B(2-1)(2-3)$ $\therefore B = -7$

Put $s=3$ in (2) $9 = C(3-1)(3-2)$ $\therefore C = 9/2$

Sub A, B, C in Eq (1)

$$\frac{2s+3}{s^3-6s^2+11s-6} = \frac{5}{2(s-1)} - \frac{7}{(s-2)} + \frac{9}{2(s-3)}$$

$$\therefore \mathcal{L}^{-1}\left\{\frac{2s+3}{s^3-6s^2+11s-6}\right\} = \frac{5}{2}\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} - 7\cdot\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} + \frac{9}{2}\cdot\mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\}$$

$$= \frac{5}{2}e^{t} - 7e^{2t} + \frac{9}{2}e^{3t}$$

First Shifting Theorem

If $\mathcal{L}^{-1}\{f(s)\} = f(t)$, then $\mathcal{L}^{-1}\{f(s-a)\} = e^{at} \cdot f(t)$ 3)

w.k.t. $\mathcal{L}\{e^{at} f(t)\} = f(s-a)$

$$\mathcal{L}^{-1}\{f(s-a)\} = e^{at} \cdot f(t) = e^{at} \cdot \mathcal{L}^{-1}\{f(s)\}$$

similarly, $\mathcal{L}^{-1}\{f(s+a)\} = e^{-at} f(t) = e^{-at} \mathcal{L}^{-1}\{f(s)\}$

1) $\mathcal{L}^{-1}\left\{\dfrac{1}{(s+2)^2+16}\right\}$

Sol

Here, $f(s+2) = \dfrac{1}{(s+2)^2+16} \Rightarrow f(s) = \dfrac{1}{s^2+16}$

Now,

$$\mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2+16}\right\} = \mathcal{L}^{-1}\{f(s+2)\}$$

$= e^{-2t} \cdot \mathcal{L}^{-1}\{f(s)\}$, by f.s.T

$$= e^{-2t} \cdot \mathcal{L}^{-1}\left\{\frac{1}{s^2+16}\right\}$$

$$= e^{-2t} \cdot \frac{1}{4} \sin 4t = \frac{1}{4} e^{-2t} \sin 4t$$

2) $\mathcal{L}^{-1}\left\{\dfrac{2s+3}{s^2+2s+2}\right\}$

Sol

$\mathcal{L}^{-1}\left\{\dfrac{2(s+1)+1}{(s+1)^2+1}\right\} = \mathcal{L}^{-1}\{f(s+1)\} = e^{-t} \cdot \mathcal{L}^{-1}\{f(s)\}$, by f.s.T

$$= e^{-t} \mathcal{L}^{-1}\left\{\frac{2s+1}{s^2+1}\right\}$$

$$= e^{-t} \mathcal{L}^{-1}\left\{\frac{2s}{s^2+1} + \frac{1}{s^2+1}\right\}$$

$$= e^{-t}\left\{2\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}\right\}$$

$$= e^{-t}\{2\cos t + \sin t\}$$

3) $L^{-1}\left\{\frac{s}{(s+3)^2}\right\} = L^{-1}\left\{\frac{(s+3)-3}{(s+3)^2}\right\}$

$= L^{-1}\{f(s+3)\}$

$= e^{-3t} L^{-1}\{f(s)\}$

$= e^{-3t} L^{-1}\left\{\frac{s-3}{s^2}\right\}$

$= e^{-3t} L^{-1}\left\{\frac{1}{s} - \frac{3}{s^2}\right\}$

$= e^{-3t}\left[L^{-1}\left\{\frac{1}{s}\right\} - 3L^{-1}\left\{\frac{1}{s^2}\right\}\right]$

$= e^{-3t}[1 - 3t]$.

4) $L^{-1}\left\{\frac{s}{s^4+4a^4}\right\}$

take, $s^4+4a^4 = (s^2+2a^2)^2 - 4a^2s^2$

$= (s^2+2a^2+2as) - (2as)^2$

$= (s^2+2a^2+2as)(s^2+2a^2-2as)$.

$L^{-1}\left\{\frac{s}{s^4+4a^4}\right\} = L^{-1}\left\{\frac{s}{(s^2+2as+2a^2)(s^2-2as-2a^2)}\right\}$

$= \frac{1}{4a} L^{-1}\left\{\frac{1}{s^2-2as+2a^2} - \frac{1}{s^2+2as+2a^2}\right\}$

$= \frac{1}{4a}\left[L^{-1}\left\{\frac{1}{(s-a)^2+a^2}\right\} - L^{-1}\left\{\frac{1}{(s+a)^2+a^2}\right\}\right]$

$= \frac{1}{4a}\left[e^{at} L^{-1}\left\{\frac{1}{s^2+a^2}\right\} - e^{-at} L^{-1}\left\{\frac{1}{s^2+a^2}\right\}\right]$

By f.S.T

$= \frac{1}{4a}\left[e^{at}\cdot\frac{1}{a}\sin at - e^{-at}\frac{1}{a}\sin at\right]$

$= \frac{1}{4a^2}(e^{at}-e^{-at})\sin at$

$= \frac{1}{2a^2}\sinh at \cdot \sin at$

5) $\mathcal{L}^{-1}\left\{\frac{3s-2}{s^2-4s+20}\right\} = \frac{3(s-2)+4}{(s-2)^2+16} = \mathcal{L}^{-1}\{f(s-2)\}$

$= e^{2t}\mathcal{L}^{-1}\{f(s)\} = e^{2t}\mathcal{L}^{-1}\left\{\frac{3s+4}{s^2+16}\right\}$

$= e^{2t}\mathcal{L}^{-1}\left\{\frac{3s}{s^2+4^2} + \frac{4}{s^2+4^2}\right\}$

$= e^{2t}\left\{3\mathcal{L}^{-1}\left\{\frac{s}{s^2+4^2}\right\} + 4\mathcal{L}^{-1}\left\{\frac{1}{s^2+4^2}\right\}\right\}$

$= e^{2t}(3\cos 4t + \frac{4}{4}\sin 4t)$

$= e^{2t}(3\cos 4t + \sin 4t)$

6) $\mathcal{L}^{-1}\left\{\frac{s+3}{s^2-10s+29}\right\} = \frac{(s-5)+8}{(s-5)^2+4} = \mathcal{L}^{-1}\{f(s-5)\}$

$= e^{5t}\mathcal{L}^{-1}\{f(s)\} = e^{5t}\mathcal{L}^{-1}\left\{\frac{s+8}{s^2+4}\right\}$

$= e^{5t}\mathcal{L}^{-1}\left\{\frac{s}{s^2+4} + \frac{8}{s^2+4}\right\}$

$= e^{5t}\left[\mathcal{L}^{-1}\left(\frac{s}{s^2+2^2}\right) + 8\mathcal{L}^{-1}\left(\frac{1}{s^2+2^2}\right)\right]$

$= e^{5t}(\cos 2t + 4\sin 2t)$

7) $\mathcal{L}^{-1}\left\{\frac{2s+12}{s^2+6s+13}\right\} = \frac{2(s+3)+6}{(s+3)^2+4} = \mathcal{L}^{-1}\{f(s+3)\}$

$= e^{-3t}\mathcal{L}^{-1}\{f(s)\} = e^{-3t}\mathcal{L}^{-1}\left\{\frac{2s+6}{s^2+4}\right\}$

$= e^{-3t}\mathcal{L}^{-1}\left\{\frac{2s}{s^2+4} + \frac{6}{s^2+4}\right\}$

$= e^{-3t}\left[2\mathcal{L}^{-1}\frac{s}{s^2+2^2} + 6\mathcal{L}^{-1}\frac{1}{s^2+2^2}\right]$

$= e^{-3t}(2\cos 2t + 3\sin 2t$

$$\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^3}\right\} = \mathcal{L}^{-1}\{f(s+1)\} = e^{-t}\mathcal{L}^{-1}\{f(s)\}$$

$$= e^{-t}\mathcal{L}^{-1}\left\{\frac{1}{s^3}\right\} = e^{-t}\frac{t^2}{2!}$$

Second Shifting Theorem

If $\mathcal{L}^{-1}\{f(s)\} = f(t)$, then $\mathcal{L}^{-1}\{e^{-as}\cdot f(s)\} = f(t-a)$, $t > a$

$= 0$, $t < a$.

or $\mathcal{L}^{-1}\{e^{-as}\cdot f(s)\} = f(t-a)\cdot H(t-a)$

1) $\mathcal{L}^{-1}\left\{\frac{e^{-3s}}{(s-4)^2}\right\}$, Here $f(s) = \frac{1}{(s-4)^2}$

$\mathcal{L}^{-1}\{f(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{(s-4)^2}\right\} = e^{4t}\cdot\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\}$ by f.S.T

$= e^{4t}\cdot t \Rightarrow f(t)$, say

$\therefore \mathcal{L}^{-1}\left\{\frac{e^{-3s}}{(s-4)^2}\right\} = f(t-3)H(t-3)$, by S.S.T

$= (t-3)\,e^{4(t-3)}\cdot H(t-3)$.

2) $\mathcal{L}^{-1}\left\{\frac{e^{4-3s}}{(s+4)^{3/2}}\right\}$

$n+1 = \frac{3}{2}$

$n = \frac{3}{2} - 1 = \frac{1}{2}$

Here, $f(s) = \frac{1}{(s+4)^{3/2}}$

$\mathcal{L}^{-1}\left\{\frac{1}{s^{n+1}}\right\} = \frac{t^n}{\sqrt{n+1}}$

$\mathcal{L}^{-1}\{f(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{(s+4)^{3/2}}\right\}$

$= e^{-4t}\cdot\mathcal{L}^{-1}\left\{\frac{1}{s^{3/2}}\right\}$, by f.S.T

$= e^{-4t}\cdot\frac{t^{1/2}}{\sqrt{\frac{1}{2}+1}} = 2e^{-4t}\cdot\frac{\sqrt{t}}{\sqrt{\pi}}$ $\left[\because \sqrt{1/2} = \sqrt{\pi}\right]$

$\therefore \mathcal{L}^{-1}\left\{\frac{e^{4-3s}}{(s+4)^{3/2}}\right\} = e^4\cdot\mathcal{L}^{-1}\{e^{-3s}\cdot f(s)\}$

$= e^4\cdot f(t-3)\,H(t-3)$, by S.S.T

$\therefore e^{4} 2e^{-4(t-3)} \sqrt{t/\pi} \cdot H(t-3)$

$= \cancel{e^{4(1-t)} \sqrt{t/\pi} \cdot H(t-3)} = 2e^{4(1-t+3)} \sqrt{\frac{t-3}{\pi}} \cdot H(t-3)$

$= 2e^{-4(t+2)} \sqrt{\frac{t-3}{\pi}} \cdot H(t-3)$

3) $\mathcal{L}^{-1}\left\{\frac{e^{-\pi(s+2)}}{s+2}\right\}$

Sol: $\mathcal{L}^{-1}\left\{\frac{e^{-\pi(s+2)}}{s+2}\right\} = e^{-2t}\mathcal{L}^{-1}\left\{\frac{e^{-\pi s}}{s}\right\}$, by f.s.T.

$= e^{-2t} f(t-\pi) H(t-\pi)$, by s.s.T

$\left[\because f(t) = \mathcal{L}^{-1}\left(\frac{1}{s}\right) = 1\right]$

$= e^{-2t} H(t-\pi)$.

4) $\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s^2+4s+5}\right\}$

Sol: $\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s^2+4s+5}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2+1}\right\} = e^{-2t}\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}$

$= e^{-2t}\sin t = f(t)$

$\therefore$ By s.s.T

$\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s^2+4s+5}\right\} = e^{-2(t-2)}\sin(t-2)\, H(t-2)$

or $\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s^2+4s+5}\right\} = \begin{cases} e^{-2(t-2)}\sin(t-2), & t>2 \\ 0, & t<2. \end{cases}$

Change of Scale property

If $\mathcal{L}^{-1}\{f(s)\} = f(t)$, then $\mathcal{L}^{-1}\{f(as)\} = \frac{1}{a} f\left(\frac{t}{a}\right)$

1) If $\mathcal{L}^{-1}\left\{\frac{s}{(s^2+1)^2}\right\} = \frac{1}{2}t\sin t$, find $\mathcal{L}^{-1}\left\{\frac{8s}{(4s^2+1)^2}\right\}$

Sol: we have, $\mathcal{L}^{-1}\left\{\frac{s}{(s^2+1)^2}\right\} = \frac{1}{2}t\sin t$

By change of scale property

$\mathcal{L}^{-1}\left\{\frac{as}{(a^2s^2+1)^2}\right\} = \frac{1}{a} \times \frac{1}{2} \times \frac{t}{a}\sin\left(\frac{t}{a}\right)$

put $a=2$,

$$\mathcal{L}^{-1}\left\{\frac{2s}{(4s^2+1)^2}\right\} = \frac{1}{8}t\sin\frac{t}{2})$$

$$\mathcal{L}^{-1}\left\{\frac{8s}{(4s^2+1)^2}\right\} = \frac{t}{2}\sin\frac{t}{2}$$

2) $\mathcal{L}^{-1}\left\{\frac{e^{-1/s}}{s^{1/2}}\right\} = \frac{\cos 2\sqrt{t}}{\sqrt{\pi t}}$, find $\mathcal{L}^{-1}\left\{\frac{e^{-a/s}}{s^{1/2}}\right\}$

Sol: we have $\mathcal{L}^{-1}\left\{\frac{e^{-1/s}}{s^{1/2}}\right\} = \frac{\cos 2\sqrt{t}}{\sqrt{\pi t}}$

By change of scale property

$$\mathcal{L}^{-1}\left\{\frac{e^{-1/ks}}{\sqrt{ks}}\right\} = \frac{1}{k}\frac{\cos 2\sqrt{t/k}}{\sqrt{\pi t/k}}$$

put $k=\frac{1}{a}$, we get

$$\mathcal{L}^{-1}\left\{\frac{e^{-a/s}}{\sqrt{s}}\right\} = \frac{\cos 2\sqrt{at}}{\sqrt{\pi t}}$$

Inverse Laplace Transform of Integrals

If $\mathcal{L}^{-1}\{f(s)\} = f(t)$, then $\mathcal{L}^{-1}\left\{\int_s^\infty f(s)\,ds\right\} = \frac{f(t)}{t}$

$$= \frac{1}{t}\mathcal{L}^{-1}\{f(s)\}$$

1) find $\mathcal{L}^{-1}\left\{\frac{2s}{(s^2-4)^2}\right\}$

Sol: Let $f(s) = \frac{2s}{(s^2-4)^2}$.

W.K.T.

$$\mathcal{L}^{-1}\{f(s)\} = t\cdot\mathcal{L}^{-1}\left\{\int_s^\infty f(s)\,ds\right\}$$

$$\mathcal{L}^{-1}\left\{\frac{2s}{(s^2-4)^2}\right\} = t\cdot\mathcal{L}^{-1}\left\{\int_s^\infty \frac{2s}{(s^2-4)^2}\,ds\right\}$$

$$= t\cdot\mathcal{L}^{-1}\left\{\left[\frac{-1}{s^2-4}\right]_s^\infty\right\}$$

Put, $s^2-4=t$, $2s\,ds = dt$

$$= t\cdot\mathcal{L}^{-1}\left\{0+\frac{1}{s^2-4}\right\} = t\cdot\frac{1}{2}\sinh 2t$$

2) $L^{-1}\left\{\frac{s}{(s^2-a^2)^2}\right\}$.

Sol Let $f(s) = \frac{s}{(s^2-a^2)^2}$ then

$$L^{-1}\{f(s)\} = t\, L^{-1}\left\{\int_s^\infty \frac{s}{(s^2-a^2)^2}\, ds\right\}$$

$$= t\, L^{-1}\left\{\frac{1}{2}\int_s^\infty \frac{2s}{(s^2-a^2)^2}\, ds\right\}$$

$$= t \cdot L^{-1}\left\{\frac{1}{2}\left[-\frac{1}{s^2-a^2}\right]_s^\infty\right\}$$

$$= \frac{t}{2} L^{-1}\left\{\frac{1}{s^2-a^2}\right\} = \frac{t}{2}\frac{1}{a}\sinh at$$

3) $L^{-1}\left\{\int_s^\infty \left(\frac{1}{u}-\frac{1}{u+1}\right) du\right\}$

Sol) $L^{-1}\left\{\int_s^\infty \left(\frac{1}{u}-\frac{1}{u+1}\right) du\right\}$

Let $f(u) = \frac{1}{u} - \frac{1}{u+1}$

$$L^{-1}\{f(u)\} = L^{-1}\left\{\frac{1}{u} - \frac{1}{u+1}\right\}$$

$$= L^{-1}\left\{\frac{1}{u}\right\} - L^{-1}\left\{\frac{1}{u+1}\right\} = 1 - e^{-t}$$

$\therefore$ By Laplace transform of Integrals,

$$L^{-1}\left\{\int_s^\infty \left(\frac{1}{u}-\frac{1}{u+1}\right) du\right\} = L^{-1}\left\{\int_s^\infty f(u)\, du\right\}$$

$$= \frac{1}{t} \cdot L^{-1}\{f(s)\}$$

$$= \frac{1-e^{-t}}{t}$$

Multiplication by power of 's':

If $L^{-1}\{f(s)\} = f(t)$ and $f(0)=0$, then

$$L^{-1}\{s f(s)\} = f'(t)$$

In general,

$$L^{-1}\{s^n f(s)\} = f^{(n)}(t), \quad \text{if } f^{(n)}(0)=0$$

for $n = 1, 2, \ldots n-1$

Q. find

i) $L^{-1}\left\{\frac{s}{(s+2)^2}\right\}$.

sol: $f(s) = \frac{1}{(s+2)^2}$ then

$$L^{-1}\{f(s)\} = L^{-1}\left\{\frac{1}{(s+2)^2}\right\} = e^{-2t} L^{-1}\left\{\frac{1}{s^2}\right\}$$

clearly $f(0)=0$ $\qquad = e^{-2t} \cdot t = f(t)$

$$L^{-1}\left\{\frac{s}{(s+2)^2}\right\} = L^{-1}\left\{s \cdot \frac{1}{(s+2)^2}\right\} = L^{-1}\{s \cdot f^{-1}(s)\}$$

$$= f'(t) = \frac{d}{dt}(te^{-2t}) = t(-2e^{-2t}) + e^{-2t},$$

$$= e^{-2t}(1-2t).$$

ii) $L^{-1}\left\{\frac{s}{(s+3)^2}\right\}$

sol: $L^{-1}\left\{\frac{s}{(s+3)^2}\right\} = L^{-1}\left\{\frac{(s+3)-3}{(s+3)^2}\right\}$

$$= L^{-1}\left\{\frac{1}{s+3} - \frac{3}{(s+3)^2}\right\} = L^{-1}\left\{\frac{1}{s+3}\right\} - 3L^{-1}\left\{\frac{1}{(s+3)^2}\right\}$$

$$= e^{-3t} - 3e^{-3t} L^{-1}\left\{\frac{1}{s^2}\right\} \text{ (By f.s.T)}$$

$$= e^{-3t} - 3e^{-3t} \cdot t = e^{-3t}(1-3t)$$

2) find $\mathcal{L}^{-1}\left\{\frac{s^2}{(s-3)^2}\right\}$.

Sol Let $f(s) = \frac{1}{(s-3)^2}$

$\mathcal{L}^{-1}\{f(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{(s-3)^2}\right\} = e^{3t}\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\}$, by f.s.T

$= e^{3t}.t = f(t)$, say

also, $f(0) = 0$

$\therefore \mathcal{L}^{-1}\left\{\frac{s^2}{(s-3)^2}\right\} = \mathcal{L}^{-1}\{s^2 f(s)\}$

$= \frac{d^2}{dt^2} f(t) = \frac{d^2}{dt^2}(te^{3t})$

$= 3e^{3t}(2+3t)$

3) find $\mathcal{L}^{-1}\left\{\frac{s+1}{(s^2+2s+2)^2}\right\}$.

Sol $\mathcal{L}^{-1}\left\{\frac{s+1}{(s^2+2s+2)^2}\right\} = \mathcal{L}^{-1}\left\{\frac{s+1}{[(s+1)^2+1]^2}\right\}$

$= e^{-t}.\mathcal{L}^{-1}\left\{\frac{s}{(s^2+1)^2}\right\}$, by f.s.T.

$= e^{-t}.\frac{t}{2}\sin t = \frac{t}{2}e^{-t}\sin t$

Inverse Laplace Transforms of Derivatives

If $\mathcal{L}^{-1}\{f(s)\} = f(t)$, then $\mathcal{L}^{-1}\{f^n(s)\} = (-1)^n t^n f(t)$

In particular, if $n=1$, $\mathcal{L}^{-1}\{f'(s)\} = (-1)^1 t^1 . f(t)$

$= (-1)t\,\mathcal{L}^{-1}\{f(s)\}$

1) find $\mathcal{L}^{-1}\left\{\log\left(1+\frac{1}{s^2}\right)\right\}$.

Sol Let $f(s) = \log\left(1+\frac{1}{s^2}\right) = \log\left(\frac{s^2+1}{s^2}\right)$

$= \log(1+s^2) - 2\log s$

$$f'(s) = \frac{2s}{1+s^2} - \frac{2}{s}$$

$$\therefore \mathcal{L}^{-1} f'(s) = \mathcal{L}^{-1}\left\{\frac{2s}{1+s^2} - \frac{2}{s}\right\}$$

$$\Rightarrow (-1)(t)\mathcal{L}^{-1}\{f(s)\} = 2\left\{\mathcal{L}^{-1}\left[\frac{s}{1+s^2}\right] - \mathcal{L}^{-1}\left\{\frac{1}{s}\right\}\right\}$$

$$-t\cdot\mathcal{L}^{-1}\left\{\log\left(1+\frac{1}{s^2}\right)\right\} = 2\left[\cos t - 1\right]$$

$$\mathcal{L}^{-1}\left\{\log\left(1+\frac{1}{s^2}\right)\right\} = \frac{-2}{t}(\cos t - 1)$$

2) find $\mathcal{L}^{-1}\left\{\log\left(\frac{s+1}{s-1}\right)\right\}$

sol: Let $f(s) = \log\left(\frac{s+1}{s-1}\right)$

$$\mathcal{L}\{f(t)\} = \log\left(\frac{s+1}{s-1}\right) = \log(s+1) - \log(s-1)$$

$$\therefore \mathcal{L}\{t\cdot f(t)\} = (-1)\frac{d}{ds}\left[\log(s+1) - \log(s-1)\right]$$

$$= (-1)\left[\frac{1}{s+1} - \frac{1}{s-1}\right] = \frac{1}{s-1} - \frac{1}{s+1}$$

$\left[\because \sinh x = \frac{e^x - e^{-x}}{2}\right]$

$$= \mathcal{L}\{e^t - e^{-t}\} = \mathcal{L}\{2\sinh t\}$$

$$t\cdot f(t) = 2\sinh t$$

$$f(t) = \frac{2}{t}\sinh t$$

3) find i) $\mathcal{L}^{-1}\left\{\cot^{-1}\left(\frac{s+a}{b}\right)\right\}$

Let $f(s) = \cot^{-1}\left(\frac{s+a}{b}\right)$

$$f'(s) = \frac{-1}{1+\left(\frac{s+a}{b}\right)^2}\cdot\frac{1}{b} = \frac{-b}{b^2+(s+a)^2}$$

$$\therefore \mathcal{L}^{-1}\{f'(s)\} = \mathcal{L}^{-1}\left\{\frac{-b}{(s+a)^2+b^2}\right\}$$

$$(-1)t\,\mathcal{L}^{-1}\{f(s)\} = (-b)\cdot e^{-at}\mathcal{L}^{-1}\left\{\frac{1}{s^2+b^2}\right\},$$

by f.S.T

$= (-b)\, e^{-at} \cdot \frac{1}{b} \sin bt$

$\mathcal{L}^{-1}\{f(s)\} = \frac{e^{-at}}{t} \cdot \sin bt$

$\therefore \mathcal{L}^{-1}\left\{\cot^{-1}\left(\frac{s+a}{b}\right)\right\} = \frac{e^{-at}}{t} \sin bt$

(ii) $\mathcal{L}^{-1}\left\{\cot^{-1}\left(\frac{s+2}{3}\right)\right\}$

Sol Let, $f(s) = \cot^{-1}\left(\frac{s+2}{3}\right)$

$f'(s) = \frac{-1}{1+\left(\frac{s+2}{3}\right)^2} \cdot \frac{1}{3} = \frac{-3}{3^2+(s+2)^2}$

$\therefore \mathcal{L}^{-1}\{f'(s)\} = \mathcal{L}^{-1}\left\{\frac{-3}{(s+2)^2+3^2}\right\}$

$(-1)\, t\, \mathcal{L}^{-1}\{f(s)\} = (-3) \cdot e^{-2t}\, \mathcal{L}^{-1}\left\{\frac{1}{s^2+3^2}\right\}$, by f.s.t

$= (-3)\, e^{-2t} \cdot \frac{1}{3} \sin 3t$

$\mathcal{L}^{-1}\{f(s)\} = \frac{e^{-2t}}{t} \sin 3t$

$\mathcal{L}^{-1}\left\{\cot^{-1}\left(\frac{s+2}{3}\right)\right\} = \frac{e^{-2t}}{t} \sin 3t$

② Division by 's': If $\mathcal{L}^{-1}\{f(s)\} = F(t)$, then $\mathcal{L}^{-1}\left\{\frac{f(s)}{s}\right\}$

$= \int_0^t F(u)\,du.$

1) Find $\mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2+a^2)}\right\}$

Sol we have, $\mathcal{L}^{-1}\left\{\frac{1}{s^2+a^2}\right\} = \frac{1}{a} \sin at$

$\therefore \mathcal{L}^{-1}\left\{\frac{1}{s} \cdot \frac{1}{s^2+a^2}\right\} = \int_0^t \frac{1}{a} \sin au\,du = \frac{1}{a}\left[\frac{-\cos au}{a}\right]_0^t$

$= -\frac{1}{a^2}(\cos at - 1)$

Thus,

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2+a^2)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s}\cdot\frac{1}{s(s^2+a^2)}\right\}$$

$$= \int_0^t \frac{1}{a^2}(1-\cos at)\,dt$$

$$= \frac{1}{a^2}\left[t - \frac{\sin at}{a}\right]_0^t$$

$$= \frac{1}{a^2}\left\{t - \frac{1}{a}\sin at\right\}$$

2) find $\mathcal{L}^{-1}\left\{\dfrac{s+1}{s^2(s^2+1)}\right\}$.

Sol Let $f(s) = \dfrac{s+1}{s^2+1} = \dfrac{s}{s^2+1} + \dfrac{1}{s^2+1}$

$$\mathcal{L}^{-1}\{f(s)\} = \mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}$$

$$= \cos t + \sin t$$

Now,

$$\mathcal{L}^{-1}\left\{\frac{1}{s}\cdot f(s)\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s}\cdot\frac{s+1}{s^2+1}\right\}$$

$$= \int_0^t f(u)\,du$$

$$= \int_0^t (\cos u + \sin u)\,du$$

$$= (\sin u - \cos u)_0^t = \sin t - \cos t + 1$$

Thus,

$$\mathcal{L}^{-1}\left\{\frac{s+1}{s^2(s^2+1)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s}\cdot\frac{s+1}{s(s^2+1)}\right\}$$

$$= \int_0^t (\sin u - \cos u + 1)\,du.$$

$$= (-\cos u - \sin u + u)_0^t$$

$$= -\cos t - \sin t + t - (-1)$$

$$= 1 + t - \cos t - \sin t$$

3) find $\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+2s+2)}\right\}$

Sol Let, $f(s) = \frac{1}{s^2+2s+2} = \frac{1}{(s+1)^2+1}$. then

$$\mathcal{L}^{-1}\{f(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+1}\right\} = e^{-t}\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}$$

$$= e^{-t}\sin t = f(t)$$

thus

$$\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+2s+2)}\right\} = \mathcal{L}^{-1}\left\{\frac{f(s)}{s}\right\} = \int_0^t f(u)\,du$$

$$= \int_0^t e^{-u}\sin u\,du$$

$$= \left[\frac{e^{-u}}{1+1}(-\sin u - \cos u)\right]_0^t$$

$$= -\frac{1}{2}\left[e^{-u}(\sin u + \cos u)\right]_0^t$$

$$= -\frac{1}{2}\left\{e^{-t}(\sin t + \cos t) - (0+1)\right\}$$

$$= \frac{1}{2}\left\{1 - e^{-t}(\sin t + \cos t)\right\}.$$

4) find $\mathcal{L}^{-1}\left\{\frac{1}{s(s^2-1)(s^2+1)}\right\}$

Sol Let, $f(s) = \frac{1}{(s^2-1)(s^2+1)}$, then

$$\mathcal{L}^{-1}\left\{\frac{1}{(s^2-1)(s^2+1)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{2}\left[\frac{1}{s^2-1} - \frac{1}{s^2+1}\right]\right\}$$

$$= \frac{1}{2}\left[\mathcal{L}^{-1}\left\{\frac{1}{s^2-1}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}\right]$$

$$= \frac{1}{2}(\sinh t - \sin t) = f(t)$$

Thus,

$$\mathcal{L}^{-1}\left\{\frac{1}{s(s^2-1)(s^2+1)}\right\} = \int_0^t f(u)\,du = \frac{1}{2}\int_0^t(\sinh u - \sin u)\,du$$

$$= \frac{1}{2}(\cosh u + \cos u)_0^t$$

$$= \frac{1}{2}\left[(\cosh t + \cos t)\right] - (1+1)]$$

$$= \frac{1}{2}\left[\cosh t + \cos t - 2\right]$$

Convolution :- Let $f(t)$ and $g(t)$ be two function defined for $t>0$. The convolution product of $f(t)$ and $g(t)$ denoted by $f(t)*g(t)$ and is defined as $f(t)*g(t)=\int_0^t f(u)\,g(t-u)\,du$

note: (i) $f(t)*g(t)=g(t)*f(t)$

(ii) $f(t)*[g(t)*h(t)]=[f(t)*g(t)]*h(t)$

(iii) $f(t)*0=0*f(t)=0$

Convolution theorem

If $L\{f(t)\}=F(s)$ and $L\{g(t)\}=G(s)$, then

$L\{f(t)*g(t)\}=F(s)\cdot G(s)$ (or)

$L^{-1}\{F(s)\,G(s)\}=f(t)*g(t)$

1) using convolution theorem, find

(i) $L^{-1}\left\{\frac{1}{(s^2+a^2)^2}\right\}$

Sol: Let $F(s)=\frac{1}{s^2+a^2}$ and $G(s)=\frac{1}{s^2+a^2}$

$L^{-1}\{F(s)\}=L^{-1}\left\{\frac{1}{s^2+a^2}\right\}=\frac{1}{a}\sin at=f(t)$, say

Also, $L^{-1}\{G(s)\}=L^{-1}\left\{\frac{1}{s^2+a^2}\right\}=\frac{1}{a}\sin at=g(t)$, say

$\therefore L^{-1}\left\{\frac{1}{(s^2+a^2)^2}\right\}=L^{-1}\left\{\frac{1}{s^2+a^2}\cdot\frac{1}{s^2+a^2}\right\}$

$=L^{-1}\{F(s)\cdot G(s)\}$

$=f(t)*g(t)$. by using convolution theorem

$=\int_0^t f(u)\,g(t-u)\,du$

$=\int_0^t \frac{1}{a}\sin au\cdot\frac{1}{a}\sin a(t-u)\,du$

$$= \frac{1}{a^2}\int_0^t \frac{1}{2}\cdot 2\sin au \cdot \sin(at-au)\,du$$

$$= \frac{1}{2a^2}\int_0^t \cos(au-at+au) - \cos(\cancel{au}+at-\cancel{au})\,du$$

$$= \frac{1}{2a^2}\int_0^t \cos(2au-at) - \cos at\,)\,du$$

$$= \frac{1}{2a^2}\left\{\frac{1}{2a}\sin(2au-at) - u\cos at\right]_0^t$$

$$= \frac{1}{2a^2}\left[\frac{1}{2a}\sin at - t\cos at + \frac{1}{2a}\sin at\right]$$

$$= \frac{1}{2a^3}(\sin at - at\cos at)$$

2) (i) $\mathcal{L}^{-1}\left\{\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right\}$.

Sol $\mathcal{L}^{-1}\left\{\frac{s}{s^2+a^2}\cdot\frac{s}{s^2+b^2}\right\}$

Let $f(s) = \frac{s}{s^2+a^2}$, $g(s) = \frac{s}{s^2+b^2}$

Then $f(t) = \cos at$ and $g(t) = \cos bt$

$$\mathcal{L}^{-1}\left\{\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right\} = \cos at * \cos bt$$

$$= \int_0^t \cos au\cdot\cos b(t-a)\,du$$

$$= \frac{1}{2}\int_0^t 2\cos au\cdot\cos b(t-u)\,du$$

$$= \frac{1}{2}\int_0^t \{\cos\cdot(au+bt-bu) + \cos(au-bt+bu)\}du$$

$$= \frac{1}{2}\int_0^t \{\cos[(a-b)u+bt] + \cos[(a+b)u-bt]\}du$$

$$= \frac{1}{2}\left[\frac{\sin\{(a-b)u+bt\}}{a-b} + \frac{\sin\{(a+b)u-bt\}}{a+b}\right]_0^t$$

$$=\frac{1}{2}\left[\frac{1}{a-b}(\sin at-\sin bt+\frac{1}{a+b}(\sin at+\sin bt)\right]$$

$$=\frac{1}{2}\left\{\sin at\left[\frac{1}{a-b}-\frac{1}{a+b}\right]+\sin bt\left[\frac{1}{a+b}-\frac{1}{a-b}\right]\right\}$$

$$=\frac{a\sin at-b\sin bt}{a^2-b^2}.$$

(ii) $\mathcal{L}^{-1}\left\{\frac{s^2}{(s^2+4)(s^2+9)}\right\}$

Sol putting $a=2$ and $b=3$ in above problem, we get

$$\mathcal{L}^{-1}\left\{\frac{s^2}{(s^2+4)(s^2+9)}\right\}=-\frac{1}{5}(2\sin 2t-3\sin 3t)$$

(iii) $\mathcal{L}^{-1}\left\{\frac{1}{(s^2+5^2)^2}\right\}$

Sol putting $a=5$ in first problem. we get

$$\mathcal{L}^{-1}\left\{\frac{1}{(s^2+5^2)^2}\right\}=\frac{1}{2(5)^3}(\sin 5t-5t\cos 5t)$$

$$=\frac{\sin 5t-5t\cos 5t}{250}$$

3) find $\mathcal{L}^{-1}\left\{\frac{1}{s^2(s+1)^2}\right\}$

Sol:- Let $f(s)=\frac{1}{s^2}$ and $g(s)=\frac{1}{(s+1)^2}$.

$\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\}=t$
$=f(t)$, say

and $\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2}\right\}=e^{-t}\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\}$
$=te^{-t}$
$=g(t)$, say

$\therefore \mathcal{L}^{-1}\left\{\frac{1}{s^2(s+1)^2}\right\}=\mathcal{L}^{-1}\{f(s)\cdot g(s)\}$

$=f(t)*g(t)$, by using convolution

$=\int_0^t f(u)\cdot g(t-u)\,du.$

$= \int_0^t u e^{-u}(t-u)\,du$

$= t\int_0^t u e^{-t}\,du - \int_0^t u^2 e^{-u}\,du$

$= t\left[-(t+1)e^{-t}+1\right] - \left\{-e^{-t}(t^2+2t+2)+2\right\}$

$= -t^2e^{-t} - te^{-t} + t + t^2e^{-t} + 2te^{-t} + 2e^{-t} - 2$

$= t(e^{-t}+1) + 2(e^{-t}-1)$

Applications of ODE

1) Solve $(D^2+4D+5)y = 5$, given that $y(0)=0,\ y'(0)=0$

Sol Given that, $y''+4y'+5y=5.$

$\mathcal{L}\{y''+4y'+5y\} = \mathcal{L}\{5\}$

$\mathcal{L}\{y''\} + 4\mathcal{L}\{y'\} + 5\mathcal{L}\{y\} = 5\cdot\frac{1}{s}$

$[s^2Y(s) - sy(0) - y'(0)\} + 4\{sY(s) - y(0)\} + 5Y(s) = \frac{5}{s}$

$s^2Y(s) + 4sY(s) + 5Y(s) = \frac{5}{s}$

$(s^2+4s+5)\,Y(s) = \frac{5}{s}$

$Y(s) = \frac{5}{s(s^2+4s+5)}$

$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{5}{s(s^2+4s+5)}\right\}$

$y(t) = \mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{s+4}{s^2+4s+5}\right\}$, by using partial fraction

$= \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \mathcal{L}^{-1}\left\{\frac{s+4}{s^2+4s+5}\right\}$

$= 1 - \mathcal{L}^{-1}\left\{\frac{(s+2)+2}{(s+2)^2+1}\right\}$

$= 1 - e^{-2t}\,\mathcal{L}^{-1}\left\{\frac{s+2}{s^2+1}\right\}$, by F.S.T

$= 1 - e^{-2t} \cdot \mathcal{L}^{-1}\left\{\frac{s}{s^2+1} + \frac{2}{s^2+1}\right\}$

$= 1 - e^{-2t}(\cos t + 2\sin t)$

Q) Solve $(D^2+2D+1)x = 3te^{-t}$, given $x(0)=4$, $x'(0)=0$

Sol: Given that,

$$x'' + 2x' + x = 3te^{-t}$$

$$\mathcal{L}\{x'' + 2x' + x\} = \mathcal{L}\{3te^{-t}\}$$

$$\mathcal{L}\{x''(t)\} + 2\mathcal{L}\{x'(t)\} + \mathcal{L}\{x(t)\} = 3 \cdot \mathcal{L}\{e^{-t} \cdot t\}$$

$$\{s^2 X(s) - s x(0) - x'(0)\} + 2\{s X(s) - x(0)\} + X(s) = 3 \cdot \frac{1}{(s+1)^2}$$ by F.S.T

$$(s^2+2s+1) X(s) - s(4) - 8 = \frac{3}{(s+1)^2}$$

$$(s^2+2s+1) X(s) = \frac{3}{(s+1)^2} + 4s + 8$$

$$X(s) = \frac{3}{(s+1)^4} + \frac{4s}{(s+1)^2} + \frac{8}{(s+1)^2}$$

$$\mathcal{L}^{-1}\{X(s)\} = \mathcal{L}^{-1}\left\{\frac{3}{(s+1)^4} + \frac{4s}{(s+1)^2} + \frac{8}{(s+1)^2}\right\}$$

$$\therefore x(t) = 3 \cdot \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^4}\right\} + 4 \cdot \mathcal{L}^{-1}\left\{\frac{s}{(s+1)^2}\right\} + 8\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2}\right\}$$

$$= 3 \cdot e^{-t} \cdot \mathcal{L}^{-1}\left\{\frac{1}{s^4}\right\} + 4 \cdot e^{-t} \cdot \mathcal{L}^{-1}\left\{\frac{s-1}{s^2}\right\} + 8e^{-t}\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\}$$

$$= 3 \cdot e^{-t} \cdot \frac{t^3}{3!} + 4e^{-t}(1-t) + 8e^{-t}(t)$$

$$= e^{-t}\left(\frac{t^3}{2} + 4t + 4\right)$$

Fourier Series

Fourier Series is an Infinite Series representation of a periodic function in terms of sines & cosines.

Suppose that a given function $f(x)$ defined in $[-\pi, \pi]$ or $[0, 2\pi]$ or in any other Interval can be expressed as a trigonometric series as

$$f(x) = \frac{a_1}{2} + a_1 \cos x + a_2 \cos 2x + \cdots$$
$$+ b_1 \sin x + b_2 \sin 2x + \cdots$$
$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

Such Series is known as the fourier series of $f(x)$ and the constants a_0, a_n, b_n $(n = 1, 2, \ldots)$ are called fourier coefficients of $f(x)$

Euler's Formulae: The fourier series for the fn $f(x)$ in $[c, c+2\pi]$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

where,

$$a_0 = \frac{1}{\pi} \int_c^{c+2\pi} f(x)\, dx$$

$$a_n = \frac{1}{\pi} \int_c^{c+2\pi} f(x) \cos nx\, dx$$

$$b_n = \frac{1}{\pi} \int_c^{c+2\pi} f(x) \sin nx\, dx.$$

Note: If $c = 0$, then the fourier coefficients in $[0, 2\pi]$ are

$$a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x)\, dx; \quad a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx\, dx.$$

$$b_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin nx\, dx$$

If $c=-\pi$, then the fourier series coefficients in $[-\pi,\pi]$ are

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx, \quad a_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx;$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

② Dirichlet Conditions:- A function $f(x)$ has a valid fourier series expansion of the form

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)$$

Provided: (i) $f(x)$ is well defined, periodic, single-valued and finite (ii) $f(x)$ has a finite no. of finite discontinuities in any period (iii) $f(x)$ has at most a finite no. of maxima and minima in the interval.

① Determine the fourier series expansion of

$$f(x)=\frac{1}{12}(3x^2-6x\pi+2\pi^2) \text{ in } (0,2\pi)$$

Sol: let

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)$$

where,

$$a_0=\frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$$

$$=\frac{1}{\pi}\int_0^{2\pi}\left(\frac{3x^2-6x\pi+2\pi^2}{12}\right)dx$$

$$=\frac{1}{12\pi}\left[x^3-3x^2\pi+2\pi^2 x\right]_0^{2\pi}$$

$$=\frac{1}{12\pi}\left[8\pi^3-12\pi^3+4\pi^3\right]=0$$

$$a_n=\frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$$

$$=\frac{1}{\pi}\int_0^{2\pi}\frac{1}{12}(3x^2-6x\pi+2\pi^2)\cos nx\,dx$$

$$= \frac{1}{12\pi}\left[3x^2 - 6x\pi + 2\pi^2\left(\frac{\sin nx}{n}\right) - \int (6x-6\pi)\frac{\sin nx}{n}\,dx\right]_0^{2\pi}$$

$$= \frac{1}{12\pi}\left[(3x^2-6x\pi+2\pi^2)\left(\frac{\sin nx}{n}\right)\right]_0^{2\pi} - \frac{1}{12\pi}\frac{6}{n}$$

$$\left[(x-\pi)\left(-\frac{\cos nx}{n}\right) - \int 1\cdot\frac{\cos nx}{n}\,dx\right]_0^{2\pi}$$

$$= \frac{1}{12\pi}(0) - \frac{1}{2n\pi}\left[\frac{\pi - x}{n}\cos nx + \frac{1}{n^2}\sin nx\right]_0^{2\pi}$$

$$= 0 - \frac{1}{2n\pi}\left\{\left\{-\frac{\pi}{n}(1) + \frac{1}{n^2}(0)\right\} - \left\{\frac{\pi}{n}(1) + 0\right\}\right\}$$

$$= \frac{-1}{2n\pi}\left(-\frac{\pi}{n} - \frac{\pi}{n}\right) = \frac{-1}{2n\pi}\left(\frac{-2\pi}{n}\right) = \frac{1}{n^2}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_0^{2\pi}\left(\frac{3x^2-6x\pi+2\pi^2}{12}\right)\sin nx\,dx$$

$$= \frac{1}{12\pi}\left[(3x^2-6x\pi+2\pi^2)\left(-\frac{\cos nx}{n}\right) - \int (6x-6\pi)\left(-\frac{\cos nx}{n}\right)dx\right]_0^{2\pi}$$

$$= \frac{1}{12\pi}\left[(3x^2-6x\pi+2\pi^2)\left(-\frac{1}{n}\cos nx\right)\right]_0^{2\pi}$$

$$+ \frac{1}{12\pi}\frac{6}{n}\left\{(x-\pi)\frac{\sin nx}{n} - \int 1\cdot\frac{\sin nx}{n}\,dx\right]_0^{2\pi}$$

$$= \frac{1}{12\pi}\left[(12\pi^2 - 12\pi^2 + 2\pi^2)\left(-\frac{1}{n}\right) - (2\pi^2)\left(-\frac{1}{n}\right)\right]$$

$$+ \frac{1}{2n\pi}\left[\frac{x-\pi}{n}\sin nx + \frac{1}{n^2}\cos nx\right]_0^{2\pi}$$

$$= \frac{1}{12\pi}(0) + \frac{1}{2n\pi}\left[\left(0 + \frac{1}{n^2}\right) - \left(0 + \frac{1}{n^2}\right)\right] = 0$$

Hence, the fourier series expansion of $f(x)$ is

$$f(x) = \sum_{n=1}^{\infty}\frac{1}{n^2}\cos nx$$

$$\frac{1}{12}(3x^2 - 6x\pi + 2\pi^2) = \frac{1}{1^2}\cos x + \frac{1}{2^2}\cos 2x + \cdots$$

2). find the fourier series $f(x) = x,\ 0 < x < 2\pi$.

Sol Let,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

where,

$$a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x)\, dx = \frac{1}{\pi} \int_0^{2\pi} x \cdot dx = \frac{1}{\pi} \left[\frac{x^2}{2}\right]_0^{2\pi}$$

$$= 2\pi$$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx\, dx = \frac{1}{\pi} \int_0^{2\pi} x \cos nx\, dx$$

$$= \frac{1}{\pi} \left[x \left(\frac{\sin nx}{n}\right) - 1 \cdot \left(\frac{-\cos nx}{n^2}\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi} \left[\frac{1}{n} x \sin nx + \frac{1}{n^2} \cos nx\right]_0^{2\pi}$$ $[\because \cos 2n\pi = \cos 0 = 1]$.

$$= \frac{1}{\pi} \left[\frac{1}{n^2} \cos 2n\pi - \frac{1}{n^2}\right] = \frac{1}{\pi}\left[\frac{1}{n^2} - \frac{1}{n^2}\right] = 0.$$

$$b_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin nx\, dx = \frac{1}{\pi} \int_0^{2\pi} x \sin nx\, dx.$$

$$= \frac{1}{\pi} \left[x \left(\frac{-\cos nx}{n}\right) - 1 \left(-\frac{\sin nx}{n^2}\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi} \left[\frac{-1}{n} x \cos nx + \frac{1}{n^2} \sin nx\right]_0^{2\pi}$$

$$= \frac{1}{\pi} \left[\left(\frac{-1}{n} 2\pi \cos 2n\pi + 0\right) - (0+0)\right] = \frac{-2}{n}.$$

$\therefore$ Hence fourier series expansion is

$$f(x) = \pi - 2 \sum_{n=1}^{\infty} \frac{1}{n} \sin nx.$$

3) Find the fourier series $f(x) = \frac{(\pi - x)^2}{4}$ in $0 < x < 2\pi$

Sol Let,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx.)$$

where,

$$a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x)\, dx = \frac{1}{\pi} \int_0^{2\pi} \frac{(\pi - x)^2}{4}\, dx = \frac{1}{4\pi} \left[-\frac{1}{3}(\pi - x)^3\right]_0^{2\pi}$$

$$= -\frac{1}{12\pi} \left[(-\pi^3) - \pi^3\right] = \frac{\pi^2}{6}$$

$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_0^{2\pi} \frac{(\pi-x)^2}{4}\cos nx\,dx.$

$= \frac{1}{4\pi}\left[(\pi-x)^2\left[\frac{\sin nx}{n}\right] - 2(\pi-x)(-1)\left[\frac{-\cos nx}{n^2}\right] + 2\left[\frac{-\sin nx}{n^3}\right]\right]_0^{2\pi}$

$= \frac{1}{4\pi}\left[\left[0 + \frac{2\pi\cos 2n\pi}{n^2} + 0\right] - \left[0 - \frac{2\pi\cos 0}{n^2} + 0\right]\right]$

$= \frac{1}{4\pi}\left[\frac{2\pi}{n^2} + \frac{2\pi}{n^2}\right] \quad [\because \cos 2n\pi = \cos 0 = 1]$

$= \frac{1}{4\pi}\left[\frac{4\pi}{n^2}\right] = \frac{1}{n^2}.$

finally,

$b_n = \frac{1}{\pi}\int_0^{2\pi} \frac{(\pi-x)^2}{4}\sin nx\,dx.$

$= \frac{1}{4\pi}\left[(\pi-x)^2\left(\frac{-\cos nx}{n}\right) - 2(\pi-x)(-1)\left[\frac{-\sin nx}{n^2}\right] + 2\left[\frac{\cos nx}{n^3}\right]\right]_0^{2\pi}$

$= \frac{1}{4\pi}\left[\left(\frac{-\pi^2\cos 2n\pi}{n} - 0 + 2\frac{\cos 2n\pi}{n^3}\right) - \left(\frac{-\pi^2}{n} - 0 + \frac{2}{n^3}\right)\right]$

$= \frac{1}{4\pi}\left[\left(\frac{-\pi^2}{n} + \frac{2}{n^3}\right) - \left(\frac{-\pi^2}{n} + \frac{2}{n^3}\right)\right]$

$= 0$

$\therefore$ Hence fourier series is.

$\frac{(\pi-x)^2}{4} = \frac{\pi^2}{12} + \sum_{n=1}^{\infty}\frac{1}{n^2}\cos nx = \frac{\pi^2}{12} + \frac{\cos x}{1^2} + \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} + \cdots$

4) Expand $f(x) = x\sin x$, $0 < x < 2\pi$, as a fourier series.

Sol) Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx)$.

where, $a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$

$= \frac{1}{\pi}\int_0^{2\pi} x\sin x\,dx$

$= \frac{1}{\pi}\left[x(-\cos x) - \int 1\cdot(-\cos x)\,dx\right]_0^{2\pi}$

$= \frac{1}{\pi}(-x\cos x + \sin x)_0^{2\pi}$

$= \frac{1}{\pi}(-2\pi)(1) + 0 - 0 = -2.$

$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx.$

$= \frac{1}{\pi}\int_0^{2\pi} x\sin x\cos nx\,dx$

$2\sin A\cos B = \sin(A+B)+\sin(A-B)$

$= \frac{1}{2\pi}\int_0^{2\pi} x(2\sin x\cos nx)\,dx$

$= \frac{1}{2\pi}\int_0^{2\pi} x\left[\sin(x+nx)+\sin(x-nx)\right]dx$

$= \frac{1}{2\pi}\int_0^{2\pi} x\left[\sin(n+1)x - \sin(n-1)x\right]dx \quad \because \sin(-a) = -\sin a$

$= \frac{1}{2\pi}\left[x\left\{-\frac{\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\right\} - \int (1)\left\{\frac{-\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\right\}dx\right]_0^{2\pi}$

$= \frac{1}{2\pi}\left[x\left\{-\frac{\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\right\}\right]_0^{2\pi} - \frac{1}{2\pi}\left[\frac{-\sin(n+1)x}{(n+1)^2} + \frac{\sin(n-1)x}{(n-1)^2}\right]_0^{2\pi}, \quad n \neq 1$

$= \frac{1}{2\pi}\left[2\pi\left\{-\frac{\cos 2(n+1)\pi}{n+1} + \frac{\cos 2(n-1)\pi}{n-1}\right\}\right]$

$= \frac{-1}{n+1} + \frac{1}{n-1} = \frac{2}{n^2-1}.$

$\because \cos n\pi = 1$ & $\sin n\pi = 0$,

If $n=1$, $a_1 = \frac{1}{\pi}\int_0^{2\pi} x\sin x\cdot\cos x\,dx$

$= \frac{1}{2\pi}\int_0^{2\pi} x\cdot\sin 2x\,dx.$

$= \frac{1}{2\pi}\left[x\left(\frac{-\cos 2x}{2}\right) - \int 1\cdot\frac{-\cos 2x}{2}dx\right]_0^{2\pi}$

$= \frac{1}{2\pi}\left[-\frac{x}{2}\cos 2x + \frac{1}{4}\sin 2x\right]_0^{2\pi}$

$= \frac{1}{2\pi}\left[(-\pi)(1)+0-0\right] = \frac{-1}{2}.$

$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$

$$= \frac{1}{\pi}\int_0^{2\pi} x\sin x \cdot \sin nx\, dx.$$

$2\sin A \sin B$
$= \cos(A-B) - \cos(A+B)$

$$= \frac{1}{2\pi}\int_0^{2\pi} x(2\sin x \sin nx)\, dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x\{\cos(n-1)x - \cos(n+1)x\}\, dx.$$

$$= \frac{1}{2\pi}\left[x\left\{\frac{\sin(n-1)x}{n-1} - \frac{\sin(n+1)x}{n+1}\right\} - \int(1)\left\{\frac{\sin(n-1)x}{n-1} - \frac{\sin(n+1)x}{n+1}\right\}dx\right]_0^{2\pi},$$

for $n \neq 1$

$$= \frac{1}{2\pi}\left[x\left\{\frac{\sin(n-1)x}{n-1} - \frac{\sin(n+1)x}{n+1}\right\}\right]_0^{2\pi} - \frac{1}{2\pi}\left[-\frac{\cos(n-1)x}{(n-1)^2} + \frac{\cos(n+1)x}{(n+1)^2}\right]_0^{2\pi}, \; n \neq 1$$

$$= \frac{1}{2\pi}(0) - \frac{1}{2\pi}\left[\left\{\frac{-1}{(n-1)^2} + \frac{1}{(n-1)^2}\right\} - \left\{-\frac{1}{(n+1)^2} + \frac{1}{(n+1)^2}\right\}\right]$$

$$= 0 - \frac{1}{2\pi}(0), \; n \neq 1$$

$$= 0, \; \text{for } n \neq 1$$

If $n = 1$, $b_1 = \frac{1}{\pi}\int_0^{2\pi} x\sin x \cdot \sin x\, dx.$

$$= \frac{1}{\pi}\int_0^{2\pi} x \cdot \sin^2 x\, dx.$$

$$= \frac{1}{\pi}\int_0^{2\pi} x \cdot \frac{(1-\cos 2x)}{2}\, dx$$

$$= \frac{1}{2\pi}\left[x\left(x - \frac{\sin 2x}{2}\right) - \int(1)\left(x - \frac{\sin 2x}{2}\right)dx\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[x^2 - \frac{x}{2}\sin 2x - \frac{x^2}{2} - \frac{1}{4}\cos 2x\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[\frac{x^2}{2} - \frac{x}{2}\sin 2x - \frac{1}{4}\cos 2x\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[\left(2\pi^2 - 0 - \frac{1}{4}\right) - \left(-\frac{1}{4}\right)\right] = \pi$$

∴ The fourier series expansion of $f(x)$ is,

$$x\sin x = \frac{-2}{2} + \left(\frac{-1}{2}\right)\cos x + \pi\sin x + \sum_{n=2}^{\infty} \frac{2n}{n^2-1}\cos x$$

$$= -1 - \frac{1}{2}\cos x + \pi\sin x + \sum_{n=2}^{\infty} \frac{2n}{n^2-1}\cos x.$$

5) Find the fourier expansion of $f(x) = x\cos x$, $0 < x < 2\pi$

Sol Let,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n\cos nx + b_n\sin nx)$$

where,

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{2\pi} x\cos x\,dx.$$

$$= \frac{1}{\pi}\left[x\cdot\sin x - 1\cdot(-\cos x)\right]_0^{2\pi} = \frac{1}{\pi}(x\sin x + \cos x)_0^{2\pi}$$

$$= \frac{1}{\pi}\left[0 + \cos 2\pi - 0 - 1\right] = \frac{1}{\pi}(1-1) = 0$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_0^{2\pi} x\cos x\cos nx\,dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x(2\cos x\cos nx)\,dx = \frac{1}{2\pi}\int_0^{2\pi} x\{\cos(1+n)x + \cos(1-n)x\}\,dx$$

$$= \frac{1}{2\pi}\left[x\left(\frac{\sin(1+n)x}{1+n} + \frac{\sin(1-n)x}{1-n}\right) - 1\cdot\left(\frac{-\cos(1+n)x}{(1+n)^2} - \frac{\cos(1-n)x}{(1-n)^2}\right)\right]_0^{2\pi} \quad (n \neq 1)$$

$$= \frac{1}{2\pi}\left[\frac{\cos(1+n)x}{(1+n)^2} + \frac{\cos(1-n)x}{(1-n)^2}\right]_0^{2\pi}$$

If $n=1$, $a_1 = \frac{1}{\pi}\int_0^{2\pi} x\cos^2 x\,dx. = \frac{1}{2\pi}\int_0^{2\pi} x(1+\cos 2x)\,dx.$

$$= \frac{1}{2\pi}\left[x\left(x + \frac{\sin 2x}{2}\right) - 1\cdot\left(\frac{x^2}{2} - \frac{\cos 2x}{4}\right)\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[\frac{x^2}{2} + \frac{x}{2}\sin 2x + \frac{1}{4}\cos 2x\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[\left(2\pi^2 + 0 + \frac{1}{4}\right) - \left(0 + 0 + \frac{1}{4}\right)\right]$$

$$= \frac{1}{2\pi}(2\pi^2) = \pi$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_0^{2\pi} x\cos x\sin nx\,dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x(2\cos x\sin nx)\,dx = \frac{1}{2\pi}\int_0^{2\pi} x\left[\sin(1+n)x - \sin(1-n)x\right]dx$$

$$= \frac{1}{2\pi}\left[x\left(-\frac{\cos(1+n)x}{1+n} + \frac{\cos(1-n)x}{1-n}\right) - 1\cdot\left(-\frac{\sin(1+n)x}{(1+n)^2} + \frac{\sin(1-n)x}{(1-n)^2}\right)\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[2\pi\left[-\frac{\cos 2(1+n)\pi}{(1+n)} + \frac{\cos 2(1-n)\pi}{(1-n)}\right]\right]$$

$$= \frac{-1}{1+n} + \frac{1}{1-n} = \frac{2n}{1-n^2}.$$

If $n=1$, then

$$b_1 = \frac{1}{\pi}\int_0^{2\pi} x\cos x\sin x\,dx = \frac{1}{2\pi}\int_0^{2\pi} x\sin 2x\,dx$$

$$= \frac{1}{2\pi}\left[x\left(-\frac{\cos 2x}{2}\right) - 1\cdot\left(-\frac{\sin 2x}{4}\right)\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[-\frac{x\cos 2x}{2} + \frac{\sin 2x}{4}\right]_0^{2\pi} = \frac{1}{2\pi}\left[\left(-\frac{2\pi}{2}+0\right) - (0+0)\right]$$

$$= \frac{-1}{2}.$$

$\therefore$ the fourier series expansion of $f(x)$ is,

$$x\cos x = \pi\cos x - \frac{1}{2}\sin x + 2\sum_{n=2,3,\dots}^{\infty} \frac{n}{1-n^2}\sin nx.$$

Functions having points of Discontinuity

1) Find the fourier series of $f(x)$, if

$$f(x) = x, \text{ when } 0 \le x \le \pi$$
$$= 2\pi - x, \text{ when } \pi \le x \le 2\pi$$

Sol) Hence deduce that $1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots = \frac{\pi^2}{8}$.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx)$$

where, $a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$

$$= \frac{1}{\pi}\left[\int_0^{\pi} x\,dx + \int_{\pi}^{2\pi}(2\pi - x)\,dx\right]$$

$$= \frac{1}{\pi}\left[\left(\frac{x^2}{2}\right)_0^{\pi} + \left(2\pi x - \frac{x^2}{2}\right)_{\pi}^{2\pi}\right]$$

$$= \frac{1}{\pi}\left(\frac{\pi^2}{2} + \left(4\pi^2 - \frac{4\pi^2}{2}\right) - \left(2\pi^2 - \frac{\pi^2}{2}\right)\right]$$

$$= \frac{1}{\pi}(\pi^2) = \pi$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} x\cos nx\,dx + \int_{\pi}^{2\pi}(2\pi - x)\cos nx\,dx\right]$$

$$= \frac{1}{\pi}\left[x\frac{\sin nx}{n} - \int 1\cdot\frac{\sin nx}{n}\,dx\right]_0^{\pi} + \frac{1}{\pi}\left[(2\pi - x)\frac{\sin nx}{n} - \int(-1)\frac{\sin nx}{n}\,dx\right]_{\pi}^{2\pi}$$

$$= \frac{1}{\pi}\left[\frac{x}{n}\sin nx + \frac{1}{n^2}\cos nx\right]_0^{\pi} + \frac{1}{\pi}\left[\frac{(2\pi - x)}{n}\sin nx - \frac{1}{n^2}\cos nx\right]_{\pi}^{2\pi}$$

$$= \frac{1}{\pi}\left[0 + \frac{1}{n^2}(-1)^n - \left(0 + \frac{1}{n^2}\right)\right] + \frac{1}{\pi}\left[0 - \frac{1}{n^2} - \left\{0 - \frac{1}{n}(-1)^n\right\}\right]$$

$$= \frac{1}{\pi}\left\{\frac{1}{n^2}\{(-1)^n - 1\}\right] + \frac{1}{n}\left[\frac{1}{n^2}\{(-1)^n - 1\}\right]$$

$$= \frac{2}{n^2\pi}\left[(-1)^n - 1\right\} = \begin{cases} 0, & \text{if } 'n' \text{ is even} \\ \frac{-4}{n^2\pi}, & \text{if } 'n' \text{ is odd} \end{cases}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx.$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} x\sin nx\,dx + \int_{\pi}^{2\pi}(2\pi - x)\sin nx\,dx\right]$$

$$= \frac{1}{\pi}\left[x\left(-\frac{\cos nx}{n}\right) - \int(1)\frac{-\cos nx}{n}\,dx\right]_0^{\pi} + \frac{1}{\pi}$$

$$\left[\frac{(2\pi-x)(-\cos nx)}{n} - \int(-1)\left(\frac{-\cos nx}{n}\right) dx \right]_{\pi}^{2\pi}$$

$$= \frac{1}{\pi}\left[-\frac{x}{n}\cos nx + \frac{1}{n^2}\sin nx\right]_0^{\pi} + \frac{1}{\pi}\left[\frac{x-2\pi}{n}\cos nx - \frac{\sin nx}{n^2}\right]_{\pi}^{2\pi}$$

$$= \frac{1}{\pi}\left\{-\frac{\pi}{n}(-1)^n + 0 - (0)\right] + \frac{1}{\pi}\left[0 - \left(\frac{-\pi}{n}(-1)^n - 0\right)\right]$$

$$= -\frac{1}{n}(-1)^n + \frac{1}{n}(-1)^n = 0$$

The fourier series expansion of $f(x)$ is

$$f(x) = \frac{\pi}{2} + \sum_{n=1,3,5\ldots}^{\infty}\left(\frac{-4}{n^2\pi}\right)\cos nx$$

$$= \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1,3,5\ldots}^{\infty}\frac{1}{n^2}\cos nx$$

$$= \frac{\pi}{2} - \frac{4}{\pi}\left[\frac{1}{1^2}\cos x + \frac{1}{3^2}\cos 3x + \frac{1}{5^2}\cos 5x + \ldots\right] \leftarrow (I)$$

Clearly, $f(x)$ is continuous at $x=\pi$. Thus, the fourier series is $f(\pi)$ put $x=\pi$ in (I)

$$f(\pi) = \frac{\pi}{2} - \frac{4}{\pi}\left[\frac{1}{1^2}(-1) + \frac{1}{3^2}(-1) + \ldots\right]$$

$$\pi = \frac{\pi}{2} + \frac{4}{\pi}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\right]$$

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5} + \ldots = \left(\pi - \frac{\pi}{2}\right)\frac{\pi}{4} = \frac{\pi^2}{8}$$

2) Find the fourier series of $f(x) = \begin{cases} 0, & \text{for } -\pi < x < 0 \\ x^2, & \text{for } 0 < x < \pi \end{cases}$

Sol: $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos nx + b_n \sin nx)$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} 0\,dx + \int_0^{\pi} x^2 dx\right] = \frac{1}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi} = \frac{\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} 0\cdot\cos nx\,dx + \int_0^{\pi} x^2\cos nx\,dx\right]$$

$$= \frac{1}{\pi}\int_0^{\pi} x^2 \cos nx\,dx.$$

$$= \frac{1}{\pi}\left[x^2\left[\frac{\sin nx}{n}\right] - 2x\left[\frac{-\cos nx}{n^2}\right] + 2\left[\frac{-\sin nx}{n^3}\right]\right]_0^{\pi}$$

$$= \frac{1}{\pi}\cdot\frac{2\pi}{n^2}\cos n\pi = \frac{2}{n^2}(-1)^n, \text{ for } n = 1, 2, 3 \ldots$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_{-\pi}^{0} 0 \sin nx\,dx + \frac{1}{\pi}\int_0^{\pi} x^2 \sin nx\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi} x^2 \sin nx\,dx$$

$$= \frac{1}{\pi}\left[x^2\left[\frac{-\cos nx}{n}\right] - 2x\left[\frac{-\sin nx}{n^2}\right] + 2\left[\frac{\cos nx}{n^3}\right]\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{-\pi^2}{n}\cos n\pi + \frac{2}{n^3}(\cos n\pi - 1)\right]$$

$$= \frac{-\pi}{n}(-1)^n + \frac{2}{\pi n^3}\left[(-1)^n - 1\right]$$

$\therefore$ The fourier series of expansion of $f(x)$ is

$$f(x) = \frac{\pi^2}{6} + 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx + \sum_{n=1}^{\infty}\left[\frac{\pi}{n}(-1)^{n+1} + \frac{2}{\pi n^3}\left((-1)^n - 1\right)\right]\sin nx.$$

Fourier Series of Even and Odd functions

The fourier series expansion of $f(x)$ in $(-\pi, \pi)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos nx + b_n \sin nx)$$

where, a_0, a_n and b_n are

case (i): If $f(x)$ is even

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx.$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = 0$$

then, $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx.$

Case (ii): If $f(x)$ is odd

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = 0$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$$

$$\therefore f(x) = \sum_{n=1}^{\infty} b_n \sin nx\,dx.$$

1) Expand $f(x) = x^2$ as a fourier series in $(-\pi, \pi)$.
Hence deduce that (i) $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots = \frac{\pi^2}{12}$

(ii) $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = \frac{\pi^2}{6}$ (iii) $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$

Sol: clearly, $f(x) = x^2$ is an even function

$\therefore$ the fourier series expansion of $f(x)$ in $(-\pi, \pi)$

is. $$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \quad \text{---} ①$$

where, $a_1 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$

$$= \frac{2}{\pi}\int_0^{\pi} x^2\,dx = \frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi} = \frac{2}{3}\pi^2$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi} x^2 \cos nx\,dx.$$

$$= \frac{2}{\pi}\left[x^2\left(\frac{\sin nx}{n}\right) - \int 2x\,\frac{\sin nx}{n}\,dx\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[0 + 2\pi\,\frac{\cos n\pi}{n^2} + 2\cdot 0\right] = \frac{4\cos n\pi}{n^2} = \frac{4}{n^2}(-1)^n$$

from ①

$$x^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2}(-1)^n \cos nx.$$

$$= \frac{\pi^2}{3} - 4\left(\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}\cos nx\right)$$

$$= \frac{\pi^2}{3} - 4\left(\cos x - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \frac{\cos 4x}{4^2} + \cdots\right)$$

(i) putting $x=0$ in above equation

$$0 = \frac{\pi^2}{3} - 4\left[1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots\right]$$

$$= 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots = \frac{\pi^2}{12} \quad \text{— (2)}$$

(ii) put $x=\pi$ in abov equation

$$\pi^2 = \frac{\pi^2}{3} - 4\left[\cos\pi - \frac{\cos 2\pi}{2^2} + \frac{\cos 3\pi}{3^2} - \frac{\cos 4\pi}{4^2} + \cdots\right]$$

$$= \frac{\pi^2}{3} - 4\left[-1 - \frac{1}{2^2} - \frac{1}{3^2} - \frac{1}{4^2} - \cdots\right]$$

$$= \frac{\pi^2}{3} + 4\left[1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots\right]$$

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = \frac{\pi^2}{6} \quad \text{— (3)}$$

Adding (2) & (3) we get

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} \cdots = \frac{\pi^2}{8}$$

2) fourier series of $f(x)$ is $f(x) = \sin ax$. ~~$\sum_{n=1} b_n \sin nx$~~.

Sol

$$\sin ax = \sum_{n=1}^{\infty} b_n \sin nx$$

where $b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$

$$= \frac{2}{\pi}\int_0^{\pi} \sin ax \sin nx\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi} \cos(a-n)x - \cos(a+n)x\,]\,dx$$

$$\therefore b_n = \frac{1}{\pi}\left[\frac{\sin(a-n)x}{a-n} - \frac{\sin(a+n)x}{a+n}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{\sin(a-n)\pi}{a-n} - \frac{\sin(a+n)\pi}{a+n}\right]$$

$$= \frac{1}{\pi}\left[\frac{\sin a\pi\cos n\pi - \cos\pi\sin n\pi}{a-n} - \frac{\sin a\pi\cos n\pi + \cos a\pi\sin n\pi}{a+n}\right]$$

$$= \frac{1}{\pi}\left[\frac{\sin a\pi \cos n\pi}{a-n} - \frac{\sin a\pi \cos n\pi}{a+n}\right] \quad [\because \sin n\pi = 0]$$

$$= \frac{1}{\pi} \sin a\pi \cos n\pi \left[\frac{1}{a-n} - \frac{1}{a+n}\right]$$

$$= \frac{1}{\pi} \sin a\pi (-1)^n \left[\frac{a+n-a+n}{a^2-n^2}\right]$$

$$= \frac{(-1)^n 2n}{\pi(a^2 n^2)} \sin a\pi$$

$$\sin a x = \frac{2\sin a\pi}{\pi} \sum_{n=1}^{\infty} \frac{n(-1)^n}{a^2 n^2} \sin nx = \frac{2\sin a x}{\pi}\left[\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2 - a^2} \sin nx\right]$$

$$= \frac{2\sin a\pi}{\pi}\left[\frac{\sin x}{1^2-a^2} - \frac{2\sin 2x}{2^2-a^2} + \frac{3\sin 3x}{3^2-a^2} - \right]$$

Half Range Fourier Series

The half range sine series of $f(x)$ in $(0, \pi)$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

where, $b_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx \, dx$

The half range cosine series of $f(x)$ in $(0, \pi)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

where, $a_0 = \frac{2}{\pi}\int_0^{\pi} f(x) \, dx$

and $a_n = \frac{2}{\pi}\int_0^{\pi} f(x) \cos nx \, dx$.

1. find the half range cosine series of $f(x) = x(\pi - x)$ in $0 < x < \pi$

Sol: The fourier cosine series of $f(x)$ in $(0, \pi)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

where, $a_0 = \frac{2}{\pi}\int_0^{\pi} f(x) \, dx = \frac{2}{\pi}\int_0^{\pi} x(\pi - x) \, dx$

$$= \frac{2}{\pi}\left[\pi\frac{x^2}{2} - \frac{x^3}{3}\right]_0^\pi$$

$$= \frac{2}{\pi}\left[\frac{\pi^3}{2} - \frac{\pi^3}{3}\right] = \frac{2}{\pi}\cdot\frac{\pi^3}{6} = \frac{\pi^2}{3}$$

$$a_n = \frac{2}{\pi}\int_0^\pi f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^\pi x(\pi - x)\cos nx\,dx$$

$$= \frac{2}{\pi}\left[(\pi x - x^2)\frac{\sin nx}{n} - \int(\pi - 2x)\frac{\sin nx}{n}\,dx\right]_0^\pi$$

$$= \frac{2}{\pi}\left[(\pi x - x^2)\frac{\sin nx}{n}\right]_0^\pi - \frac{2}{n\pi}\left\{(\pi - 2x)\left(-\frac{\cos nx}{n}\right) - \int(-2)\left(-\frac{\cos nx}{n}\right)dx\right\}_0^\pi$$

$$= \frac{2}{\pi}(0) - \frac{2}{n\pi}\left[\frac{(2x-\pi)}{n}\cos nx - \frac{2}{n}\cdot\frac{\sin nx}{n}\right]_0^\pi$$

$$= 0 - \frac{2}{n\pi}\left[\frac{\pi}{n}(-1)^n - \left\{-\frac{\pi}{n}(1) - \frac{2}{n}(0)\right\}\right]$$

$$= \frac{-2}{n\pi}\cdot\frac{\pi\{(-1)^n+1\}}{n} = \frac{-2}{n^2}\{(-1)^n+1\} = \begin{cases} -4/n^2, & \text{if } n \text{ is even} \\ 0, & \text{if } n \text{ is odd}\end{cases}$$

$\therefore$ The Half-range cosine series of $f(x)$ is

$$x(\pi - x) = \frac{\pi^2/3}{2} + \sum_{n=2,4,6\ldots}^{\infty}\left(\frac{-4}{n^2}\right)\cos nx \quad [n = 2,4,6\ldots]$$

$$= \frac{\pi^2}{6} - 4\sum_{n=2,4,6\ldots}^{\infty}\frac{1}{n^2}\cos nx$$

$$= \frac{\pi^2}{6} - 4\left(\cancel{\ldots}x + \frac{1}{2^2}\cos 2x + \frac{1}{4^2}\cos 4x + \ldots\right)$$

Fourier Series for functions having period '2l'

The fourier series expansion for $f(x)$ in the interval $c < x < c+2l$ is,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{l} + b_n\sin\frac{n\pi x}{l}\right)$$

where, $a_0 = \frac{1}{l}\int_c^{c+2l} f(x)\,dx$

$$a_n = \frac{1}{l} \int_c^{c+2l} f(x) \cos \frac{n\pi x}{l}\, dx$$

and $$b_n = \frac{1}{l} \int_c^{c+2l} f(x) \sin \frac{n\pi x}{l}\, dx.$$

(ii) **Fourier series of $f(x)$ in $[0, 2l]$**

Put $c = 0$, we get

$$a_0 = \frac{1}{l} \int_0^{2l} f(x)\, dx \; ; \; a_n = \frac{1}{l} \int_0^{2l} f(x) \cos \frac{n\pi x}{l}\, dx$$

$$b_n = \frac{1}{l} \int_0^{2l} f(x) \sin \frac{n\pi x}{l}\, dx$$

(iii) **Fourier series of $f(x)$ in $(-l, l)$**

Put $c = -l$, we get

$$a_0 = \frac{1}{l} \int_{-l}^{l} f(x)\, dx \; ; \; a_n = \frac{1}{l} \int_{-l}^{l} f(x) \cos \frac{n\pi x}{l}\, dx$$

$$b_n = \frac{1}{l} \int_{-l}^{l} f(x) \sin \frac{n\pi x}{l}\, dx$$

Fourier series of even and odd functions in $(-l, l)$

(i) **Case (i)** If $f(x)$ is even, then

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l}$$

where,
$$a_0 = \frac{2}{l} \int_0^{l} f(x)\, dx \; ; \; a_n = \frac{2}{l} \int_0^{l} f(x) \cos \frac{n\pi x}{l}\, dx.$$

(ii) **Case (ii)** If $f(x)$ is odd, then

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}.$$

where,
$$b_n = \frac{2}{l} \int_0^{l} f(x) \sin \frac{n\pi x}{l}\, dx$$

Half Range Series of $f(x)$ in $(0, l)$

Half-Range sine series of $f(x)$ in $(0, l)$ is.

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}, \text{ where}$$

$$b_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l} \, dx.$$

Half-range cosine series of $f(x)$ in $(0, l)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l}, \text{ when}$$

$$a_0 = \frac{2}{l} \int_0^l f(x)\, dx \; ; \; a_n = \frac{2}{l} \int_0^l f(x) \cos \frac{n\pi x}{l} \, dx.$$

(1) find the fourier series of $f(x) = x^2 - 2$, where $-2 \le x \le 2$.

Sol: Here, $l = 2$. Also $f(x)$ is even

$\therefore$ fourier series of $f(x)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{2} \quad \because l = 2.$$

where,

$$a_0 = \frac{2}{l} \int_0^l f(x)\, dx.$$

$$= \frac{2}{2} \int_0^2 (x^2 - 2)\, dx = \left[\frac{x^3}{3} - 2x \right]_0^2$$

$$= \frac{8}{3} - 4 - 0 = \frac{-4}{3}$$

$$a_n = \frac{2}{l} \int_0^l f(x) \cos \frac{n\pi x}{l} \, dx = \frac{2}{2} \int_0^2 (x^2 - 2) \cos \frac{n\pi x}{2} \, dx$$

$$= \left[(x^2 - 2) \frac{\sin \frac{n\pi x}{2}}{n\pi/2} - \int 2x \cdot \frac{\sin \frac{n\pi x}{2}}{n\pi/2} \, dx \right]_0^2$$

$$= \left[(x^2 - 2) \cdot \frac{2}{n\pi} \sin \frac{n\pi x}{2} \right]_0^2 - \frac{4}{n\pi} \left\{ x \left(- \frac{\cos \frac{n\pi x}{2}}{\frac{n\pi}{2}} \right) - \int 1 \cdot \frac{-\cos \frac{n\pi x}{2}}{n\pi/2} \, dx \right]_0^2$$

$$= (0 - 0) + \frac{4}{n\pi} \left[\frac{2}{n\pi} x \cos \frac{n\pi x}{2} - \frac{2}{n\pi} \frac{\sin \frac{n\pi x}{2}}{\frac{n\pi}{2}} \right]_0^2$$

$$= \frac{4}{n\pi} \cdot \frac{2}{n\pi}\left[x \cos\frac{n\pi x}{2} - \frac{2}{n\pi}\sin\frac{n\pi x}{2}\right]_0^2$$

$$= \frac{8}{n^2\pi^2}\left[\{2(-1)^n - 0\} - 0\right] = (-1)^n \frac{16}{n^2\pi^2}.$$

$\therefore$ fourier series of $f(x)$ in $(-2,2)$ is

$$f(x) = \frac{-4/3}{2} + \sum_{n=1}^{\infty} (-1)^n \frac{16}{n^2\pi^2} \cos\frac{n\pi x}{2}.$$

$$x^2 - 2 = \frac{-2}{3} + \frac{16}{\pi^2}\sum_{n=1}^{\infty} (-1)^n \cos\frac{n\pi x}{2}$$

$$= \frac{-2}{3} + \frac{16}{\pi^2}\left[-\cos\frac{\pi x}{2} + \cos\pi x - \cos\frac{3\pi}{2}x + \cdots\right]$$

(2) find the half-range cosine series for the function $f(x) = (x-1)^2$ in the interval $0 < x < 1$. Hence S.T

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6}.$$

Sol Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi x$

Then, $a_0 = 2\int_0^1 f(x)\,dx = 2\int_0^1 (x-1)^2\,dx.$

$$= 2\left[\frac{(x-1)^3}{3}\right]_0^1 = \frac{2}{3}(0+1) = \frac{2}{3}$$

$a_n = 2\int_0^1 f(x) \cos n\pi x\, dx$ $[\because l = 1]$.

$= 2\int_0^1 (x-1)^2 \cos n\pi x\, dx.$ [Apply Bernoulli's rule]

$$= 2\left\{(x-1)^2\left[\frac{\sin n\pi x}{n\pi}\right] - 2(x-1)\left[-\frac{\cos n\pi x}{n^2\pi^2}\right] + 2\left[-\frac{\sin n\pi x}{n^3\pi^3}\right]\right\}_0^1$$

$$= 2\left\{(0+0-0) - \left[0 - \frac{2}{n^2\pi^2} - 0\right]\right\} = \frac{4}{n^2\pi^2}.$$

The half range cosine series is

$$f(x) = \frac{1}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty} \frac{1}{n^2}\cos n\pi x.$$

$$(x-1)^2 = \frac{1}{3} + \frac{4}{\pi^2}\left[\frac{1}{1^2}\cos \pi x + \frac{1}{2^2}\cos 2\pi x + \cdots\right]$$

Put $x=0$ in above equation

$$1-\frac{1}{3} = \frac{4}{\pi^2}\left[\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots\right]$$

$$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots = \frac{\pi^2}{6}.$$

(3) Obtain the half-range sine series for $f(x)=e^x$ in $0<x<1$.

Sol The half-range sine series of e^x is

$$e^x + \sum_{n=1}^{\infty} b_n \sin n\pi x, \text{ since } l=1$$

Here, $b_n = 2\int_0^1 e^x \sin n\pi x\,dx$. $\left[\int e^{ax}\sin bx\,dx\right]$

$$= 2\left[\frac{e^x}{n^2\pi^2+1}(\sin n\pi x - n\pi\cos n\pi x)\right]_0^1$$

$$= 2\left[\frac{e}{n^2\pi^2+1}(0-n\pi\cos n\pi) - \frac{1}{n^2\pi^2+1}(0-n\pi)\right]$$

$$= 2\left[\frac{1}{n^2\pi^2+1}(n\pi - n\pi e\cos n\pi)\right] = \frac{2n\pi}{n^2\pi^2+1}\left[1-e(-1)^n\right]$$

Hence $e^x = \frac{2n\pi}{n^2\pi^2+1}\left[1+(-1)^{n+1}e\right]\sin n\pi x$

$$e^x = \pi\left[\frac{2(1+e)}{\pi^2+1}\sin \pi x + \frac{2(1-e)}{4\pi^2+1}\sin 2\pi x + \cdots\right]$$

(4) Find the half-range cosine series for $f(x)=x(2-x)$, in $0\le x\le 2$ and hence find sum of the series

$$\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\cdots$$

Sol $x(2-x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{2}$ $(\because l = 2)$ — (1)

where, $a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx$

$= \frac{2}{2}\int_0^2 x(2-x)\cos\frac{n\pi x}{2}\,dx.$

$= \int_0^2 (2x-x^2)\cos\frac{n\pi x}{2}\,dx$

$= \left[(2x-x^2)\frac{2}{n\pi}\left(\sin\frac{n\pi x}{2}\right) + (2-2x)\frac{4}{n^2\pi^2}\cos\frac{n\pi x}{2} + \right.$

$\left.(2)\frac{8}{n^3\pi^3}\sin\frac{n\pi x}{2}\right]_0^2$ using Bernoulli's Rule

$= \frac{-8}{n^2\pi^2}\cos n\pi - \frac{8}{n^2\pi^2} = \frac{-8}{n^2\pi^2}\left[1+(-1)^n\right]$

$\therefore a_n = \begin{cases} \frac{-16}{n^2\pi^2}, & \text{when } n \text{ is even} \\ 0, & \text{when } n \text{ is odd} \end{cases}$

and $a_0 = \frac{2}{2}\int_0^2 (2x-x^2)\,dx = \frac{4}{3}.$

from (1)

$2x - x^2 = \frac{2}{3} - \frac{16}{\pi^2}\sum_{n=2,4,6-}^{\infty}\left[\frac{1}{n^2}\cos\frac{n\pi x}{2}\right]$

$= \frac{2}{3} - \frac{16}{\pi^2}\left[\frac{1}{2^2}\cos\pi x + \frac{1}{4^2}\cos 2\pi x + \frac{1}{6^2}\cos 3\pi x + \cdots\right]$

$= \frac{2}{3} - \frac{16}{\pi^2}\cdot\frac{1}{2^2}\left[\cos\pi x + \frac{1}{2^2}\cos 2\pi x + \frac{1}{3^2}\cos 3\pi x - \cdots\right]$

Put $x = 1$ in above equation:

$2 - 1 = \frac{2}{3} - \frac{4}{\pi^2}\left[\cos\pi + \frac{1}{2}\cos 2\pi + \frac{1}{3^2}\cos 3\pi + \cdots\right]$

$1 - \frac{2}{3} = \frac{-4}{\pi^2}\left[-1 + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} + \cdots\right]$

$\frac{1}{3} = \frac{4}{\pi^2}\left[1 - \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} + \cdots\right]$

$\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots = \frac{\pi^2}{12}.$