Basic Definitions of probability :-

(a) probability :- probability is the measure of chance of occurance of events.

(Or)

probability is a measuring instrument, for measuring certainities and uncertainities in the occurance of events.

(b) Experiment :- In our day to day life any closely observing phenomena is called as experiment.

→ Experiments are divided into two types.

1. Deterministic Experiment.
2. non Deterministic Experiment / Random / stochastic Experiment / Trail Experiment.

→ An Experiment is said to be Deterministic, " whose out comes can be predicted without conducting the experiment ".

Ex:- from where the sun rises.

→ In an Experiment is Said to be Random Experiment " whose out Comes Cannot be predicted with out the Conducting the Experiment ".

Ex:- Tosing a Coin

Rolling a die

picking up of a Card in a deck of 52 Cards.

Sample Space :- A set of all possible out Comes of an Experiment it is Called as a Sample Space.

Eg :- When tosing two Coins

$$S = \{ (H,H)\ (H,T)\ (T,H)\ (T,T) \}$$

When Rolling 2 dies.

$$S = \left\{ \begin{array}{cccc} (1,1) & (1,2) & \cdots & (1,6) \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ (6,1) & (6,2) & \cdots & (6,6) \end{array} \right\}$$

→ Every point in a Sample Space is Called Sample point or Event.

→ we Can define more than one Sample Space for a given Random Experiment.

Eg :- Rolling a die

$$S_1 = \{ 1, 2, 3, 4, 5, 6 \}$$

$S_2 = \{odd, even\}$

→ Sample space is divided into two types.

i) finite Sample Space.

ii) Countably ~~x~~ Infinite Sample Space.

finite Sample Space :- A Sample Space is said to be finite Sample Space when it Contains finite number of Sample points.

Eg:- when tosing the two coin.

$S = \{(H,H)\ (H,T)\ (T,H)\ (T,T)\}$

→ In finite Sample Space the Counting process will terminate after Some time.

Countably Infinite Sample Space :- A Sample space is said to be Countably Infinite ~~x~~ Sample Space when it Contains Infinite number of Countable ~~x~~ infinite Sample points

Eg:- If a bulb is tested for its life time every second then the Sample Space

$S = \{1sec, 2sec, 3sec \ldots\}$

→ In Countably Infinite Sample Space the Counting process will never terminate.

Event :- A set of possible outcomes in an experiment is called as an event.

→ An event is subset of Sample Space

[$E \in S$]

Eg :- Rolling a die

$$S = \{1, 2, 3, 4, 5, 6\}$$

$E_1 = \{1, 3, 5\}$ $E_2 = \{2, 4, 6\}$ $E_3 = \{2, 5, 8\}$ $E_4 = \{1, 3, 6\}$

Classification of Events :-

(a) Mutually Exclusive Events :- If two events are said to be mutually exclusive events, "when they cannot occur together".

Eg :- In tossing a coin, getting head and tail simultaneously is a mutually exclusive event.

(b) Independent Events :- If two events A and B are said to be independent events, "when the occurance (or) failure doesn't depend upon the other.

Eg :- In tossing a coin, Getting head & tail doesn't depend upon each other.

(c) Equally likely Events :- Two events are said to be equally likely, "when the probability of occurance of 1st event is equal to the second event and so on".

Eg:- In rolling a die, the probability of occurance of each number is equal i.e., event of getting 1, 2, 3, 4, 5, 6 are equally likely

$p(1) = p(2) = p(3) = p(4) = p(5) = p(6) = 1/6$

(d) Exhaustive Events :- Exhaustive events are defined as if all the sample points of this event are equal to the sample space.

i.e., Event = Sample Space

Eg:- In Rolling a die $S = \{1, 2, 3, 4, 5, 6\}$

$E_1 = \{1\}$, $E_2 = \{2\}$, $E_3 = \{1, 2, 3\}$, $E_4 = \{4, 5, 6\}$

$E_5 = \{1, 2, 3, 4, 5, 6\}$

(e) Favourable events :- The no of cases which are favourable to a particular event is called as favourable event.

Eg:- picking up of an ace from a pack of 52 cards.

No of favourable outcomes are 4.

(f) Certain event :- An event is said to be a certain event when it always happens.

Eg:- The sun rises in the east.

→ for a certain event, the probability = 1

(g) Impossible event :- An event is said to be an impossible event, when it never happens.

Eg:- The sun rises in the west.

→ For an impossible event, the probability = 0

Definitions of probability :-

There are several definitions available for probability out of which two of them are most frequently used. They are :

(i) relative frequency definition :- or (Apostary frequency definition) :

(ii) classical (or) axiomatic definition (Apriory definition)

(i) RFD :- let us consider a random Experiment (E) which is repeated for 'N' number of times under Essentially identical Conditions such that (N) is Very large ($N \simeq \infty$). then,

let 'm' be the number of getting Success out of N trails. then the probability of occurance of Event 'E' is given as

$$P(E) = \lim_{N \to \infty} (m/N)$$

where, m/N is called as relative frequency of getting Success.

→ It is also called as Apostary because the probability can be calculated after Conducting the Experiment.

(ii) CAD :- let us consider a random Experiment (E) which is repeated for 'n' number of times under essentially identical Conditions.

Then, the probability of occurance of Event (E) is

$$P(E) = \frac{\text{no of Cases favourable to the Event}(E)}{\text{total no of Cases.}}$$

→ This is also called as apriory probability.

Because, the probability can be calculated without conducting experiment.

Axioms of probability :-

→ this axiom says that the probability has maximum and minimum values of '0' and '1'. The probability is always a non-negative $0 \leq P(E) \leq 1$

→ $P(S) = 1$ $P(\phi) = 0$

this axiom says that probability of some space is always 1 and this probability of null set is always zero.

→ If A & B are mutually exclusive $P(A \cup B) = P(A) + P(B)$ this axiom says that if A & B are mutually exclusive events. then there are no common elements between i.e., $P(A \cap B) = 0$

Joint probability :-

Random Variable

* Definition of Random Variable
* classification of Random Variable
* Conditions for the function to become a RV.
* Distributive function ($F_X(x)$)
* properties of Distributive function $F_X(x)$
* Density function $f_X(x)$
* properties of $f_X(x)$
* Special Distribution and density functions:
  - → Binomial — Descret
  - → poissions
  - → Exponential
  - → Uniform
  - → Gaussion
  - → Rayleigh

  } Continuous
* problems

Definition of Random Variable :-

→ A Real valued function defined over a Sample Space is Called as Random Variable.

(or)

→ the Random Variable acts like a mediator or intermediate between Sample Space points and the Set of Real Numbers.

→ In all the Examples the Random Variable assigns all the Sample Space points to the Real numbers based on the function or rule.

Eg:-

i) Tossing a two Coins

$$S = \{(H,H), (T,H), (H,T), (T,T)\}$$

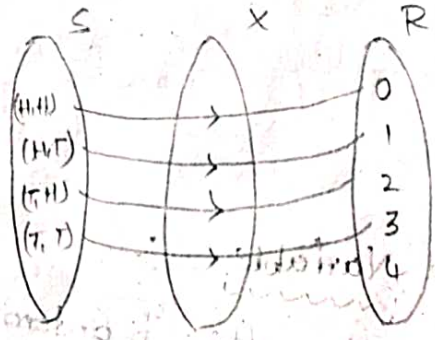
$$X = \{\text{out Come of the Experiment}\}$$

If the out Come is (H,H) then ⇒ x takes values as 0

If the out Come is (H,T) ⇒ x takes value as 1

If the outCome is (T,H) ⇒ x takes value as 2

If the outCome is (T,T) ⇒ x take value as 3



ii) Rolling a die

$$S = \{1, 2, 3, 4, 5, 6\}$$

$$X = \{\text{Squares of out Comes}\}$$

If the outcome is 1 => x-take value as 1

If the outcome is 2 => x take value as 4

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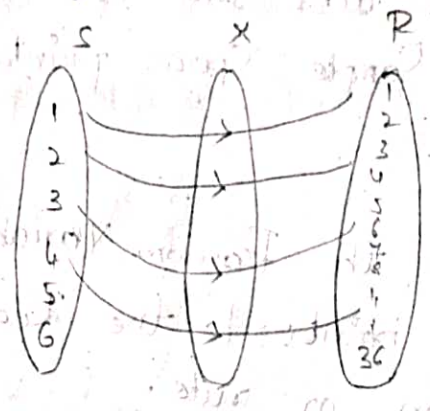
If the outcome is 6 => x-take value as 36



iii) Set of vowels

$S = \{a, e, i, o, u\}$

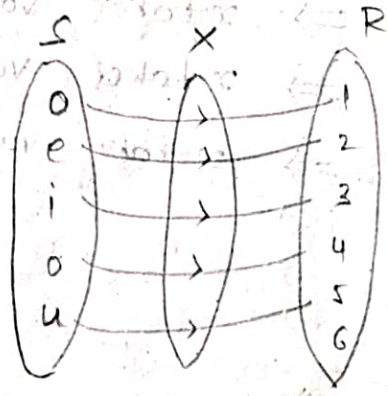
$X = \{$ positions of all alphabets $\}$

If the outcome is a => x take value as 1

- -

- -

If the outcome is u => x take value as 21



Classification of Random Variable :-

Based on the values taken by the Random Variable (x), all the Random Variables are classified into three types.

i) Continuous RV

ii) Discreate RV

iii) Mixed RV

Discrete Random Variable :- If the Random Variable assumes or takes discrete and distinct values in any finite interval then it is called as Discrete Random Variable.

Eg:- Tossing of two Coins.

$S = \{ (H,H), (H,T), (T,H), (T,T) \}$

$X = \{ \text{No of heads} \}$

If the out Come is $(H,H) \Rightarrow x$ take value as '2'

If the out Come is $(H,T) \Rightarrow x$ take value as '1'

If the out Come is $(T,H) \Rightarrow x$ take value as '1'

If the out Come is $(T,T) \Rightarrow x$ take value as '0'

$(x = 0,1,2)$

Here the Random Variable 'X' takes as value.

$x = 0,1,2$

$P(X=0) = 1/4$

$P(X=1) = 2/4 = 1/2$

$P(X=2) = 1/4$

$X=x$	0	1	2
$P(X=x)$	1/4	1/2	1/4

Probability Distribution table

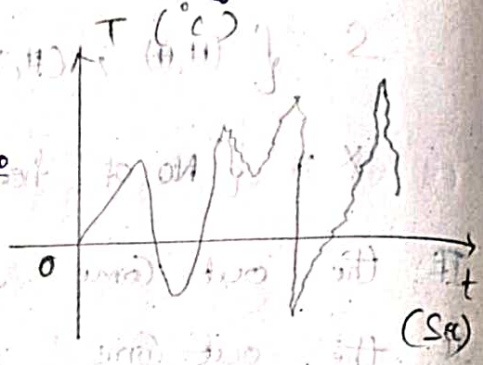
→ A probability distribution table Contains values taken by the RV Corresponding probability.

Continuous Random Variable :- If the RV assumes/takes continuous values in any interval (finite) then it is called as Continuous Random Variable.

Eg: Measurement of Temperature in a Room

$X = \{$ Temperature at any instant $\}$

In the above Example the RV x takes Continuous values of Temperature at every instant of time.

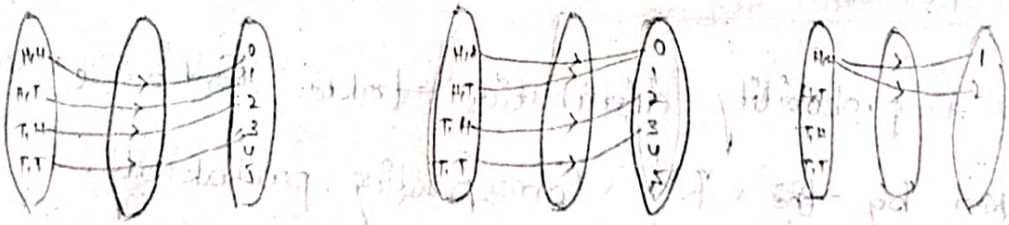


Mixed Random Variable :- If the RV assumes or takes Both Discrete and Continuous values in any finite interval then it is called as Mixed Random Variable.

**

Conditions for the function to become a Random Variable :-

→ It is not necessary that all the Sample Space points may map uniquely. Also more than one point i.e., of the Sample Space may map to one real number. But the Random Variable (random variable) can't be multi valued.



→ let $\{X \leq x\}$ be any event then the probability of occurance of this event can be obtained by adding all the individual probabilities which are less or equal to x.

i.e., $P\{X \leq x\} = P(X=0) + P(X=1) + P(X=2) + \cdots P(X=x)$

→ $P\{X = \infty\} = 0$

$P\{X = -\infty\} = 0$

Distributive function $F_X(x)$:-

→ It is also called as <u>CDF</u> or <u>PDF</u>

Cummulative Distribution Function

Probability Distribution Function.

→ It can be represented with $F_X(x)$.

→ It can be given as $F_X(x) = P(X \leq x)$

$$= P(X=0) + P(X=1) + \cdots + P(X=x)$$

$$= \sum_{x=0}^{x} P(X=x_i)$$

i.e., The probability distribution function of any Random Variable (X) can be obtained by, "By adding all the individual probabilities which are less than or equal to x."

properties of Distributive function $F_X(x)$:-

→ $F_X(x) \geq 0$ and $0 \leq F_X(x) \leq 1$. This property Says that the distributive function of a Random Variable (X) always a non-negative number. and its value lies between 0 and 1.

→ $F_X(-\infty) = 0$, $P[F_X$

$$F_X(x) = P(X \leq x) = P(X=0) + P(X=1) \cdots P(X=x)$$

$$F_X(-\infty) = P(X \leq -\infty) = 0 \text{ - min}$$

Because there are no numbers lies below $-\infty$.

→ $F_X(\infty) = 1$ - max

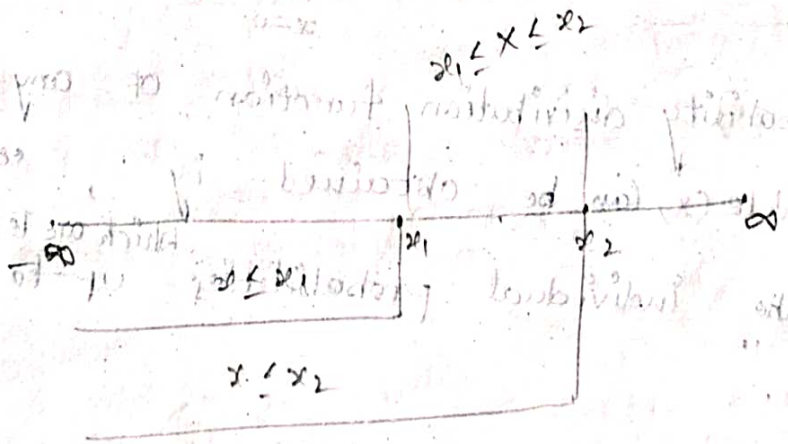
$$F_X(x) = P(X \leq x) = P(X=0) + P(X=1) \cdots P(X=x)$$

$$F_X(\infty) = P(X \leq \infty) = 1$$

→ If $x_1 < x_2$ then $F_X(x_1) \leq F_X(x_2)$

5*

→ $P(x_1 < X \leq x_2) = F_X(x_2) - F_X(x_1)$



$$(X \leq x_2) = (x \leq x_1) + (x_1 \leq X \leq x_2)$$

$$P(X \leq x_2) = P(x \leq x_1) + P(x_1 \leq X \leq x_2)$$

$$P(x_1 < X < x_2) = P(X \le x_2) - P(X \le x_1)$$
$$= F_X(x_2) - F_X(x_1)$$

Density function $f_X(x)$:- (PDF)

→ It is Represented with $f_X(x)$.

→ It is also Called as Pdf.

Probability density function.

→ It Can be given as $f_X(x) = \frac{d}{dx}(F_X(x))$

i.e., The density function of a random Variable (X) Can be obtained by "differentiating the probability distributive function".

properties of Density function $f_X(x)$:-

→ $f_X(x) \geq 0$. This probability say the density function of a Random Variable (X) always a non-negative number.

5*

→ $\int_{-\infty}^{\infty} f_X(x)\, dx = 1$

this property Says that Area under the density Curve is always Equal to 1.

proof :-

We know that $f_X(x) = \frac{d}{dx} F_X(x)$

Integrate both Sides

$$\int_{-\infty}^{\infty} f_X(x)\, dx = \int_{-\infty}^{\infty} \frac{d}{dx} F_X(x)\, dx$$

$$= \left(F_X(x)\right)_{-\infty}^{\infty}$$

$$= F_X(\infty) - F_X(-\infty)$$

$$= 1 - 0$$

$$= 1$$

3* $\rightarrow$ $\int_{x_1}^{x_2} f_X(x)\, dx = P(x_1 \le X \le x_2)$

<u>proof:-</u> We know that $f_X(x) = \frac{d}{dx} F_X(x)$

Integrate on both sides

$$\int_{x_1}^{x_2} f_X(x)\, dx = \int_{x_1}^{x_2} \frac{d}{dx} F_X(x)\, dx$$

$$= \left(F_X(x)\right)_{x_1}^{x_2}$$

$$= F_X(x_2) - F_X(x_1)$$

$$= P(x_1 \le X \le x_2)$$

5* $\rightarrow$ $F_X(x) = \int_{-\infty}^{x} f_X(x)\, dx$

This property says that the distributive function of a Random Variable (X) can be obtained by "Integrating density function".

<u>Note</u> :-

While Calculating Distributive function from the density function the upper limit must be 'x'.

$$= F_X(\infty) - F_X(-\infty)$$

$$= 1 - 0$$

$$= 1$$

3* $\rightarrow$ $$\int_{x_1}^{x_2} f_X(x)\, dx = P(x_1 \le X \le x_2)$$

<u>proof:-</u> We know that $f_X(x) = \frac{d}{dx} F_X(x)$

Integrate on both sides

$$\int_{x_1}^{x_2} f_X(x)\, dx = \int_{x_1}^{x_2} \frac{d}{dx} F_X(x)\, dx$$

$$= \left(F_X(x)\right)_{x_1}^{x_2}$$

$$= F_X(x_2) - F_X(x_1)$$

$$= P(x_1 \le X \le x_2)$$

5* $\rightarrow$ $$F_X(x) = \int_{-\infty}^{x} f_X(x)\, dx$$

This property says that the distributive function of a Random Variable (X) can be obtained by "Integrating density function".

<u>Note</u> :-

While Calculating Distributive function from the density function the upper limit must be 'x'.

problems :-

1. In a given Experiment If a Random Variable (X) has a Elements $\{1,2,3,4,5,6\}$ with probabilities $\{\frac{2}{36}, \frac{8}{36}, \frac{12}{36}, \frac{7}{36}, \frac{5}{36}, \frac{2}{36}\}$. Draw the Distributive funtion Random Variable (X). and also find the density function.

Given that $X = x = \{1, 2, 3, 4, 5, 6\}$

$$P[X=x] = \left\{\frac{2}{36}, \frac{8}{36}, \frac{12}{36}, \frac{7}{36}, \frac{5}{36}, \frac{2}{36}\right\}$$

$X = x$	1	2	3	4	5	6
$P[X=x]$	$\frac{2}{36}$	$\frac{8}{36}$	$\frac{12}{36}$	$\frac{7}{36}$	$\frac{5}{36}$	$\frac{2}{36}$

probability Distribution Table

$\therefore$ $F_X(x) = P(X \le x)$ $P(X=1) =$

$F_X(1) = P(X \le 1) = \frac{2}{36}$

$F_X(2) = P(X \le 2) = P(X=1) + P(X=2)$

$= \frac{2}{36} + \frac{8}{36} = \frac{10}{36}$

$F_X(3) = P(X \le 3) = P(X=1) + P(X=2) + P(X=3)$

$= \frac{2}{36} + \frac{8}{36} + \frac{12}{36}$

$= \frac{22}{36}$

$F_X(4) = P(X \le 4) = P(X=1) + P(X=2) + P(X=3) + P(X=4)$

$= \frac{12}{36} + \frac{8}{36} + \frac{12}{36} + \frac{7}{36} = \frac{29}{36}$

$$F_X(5) = P(X \le 5) = P(X=1) + P(X=2) + P(X=3) + P(X=4) + P(X=5)$$

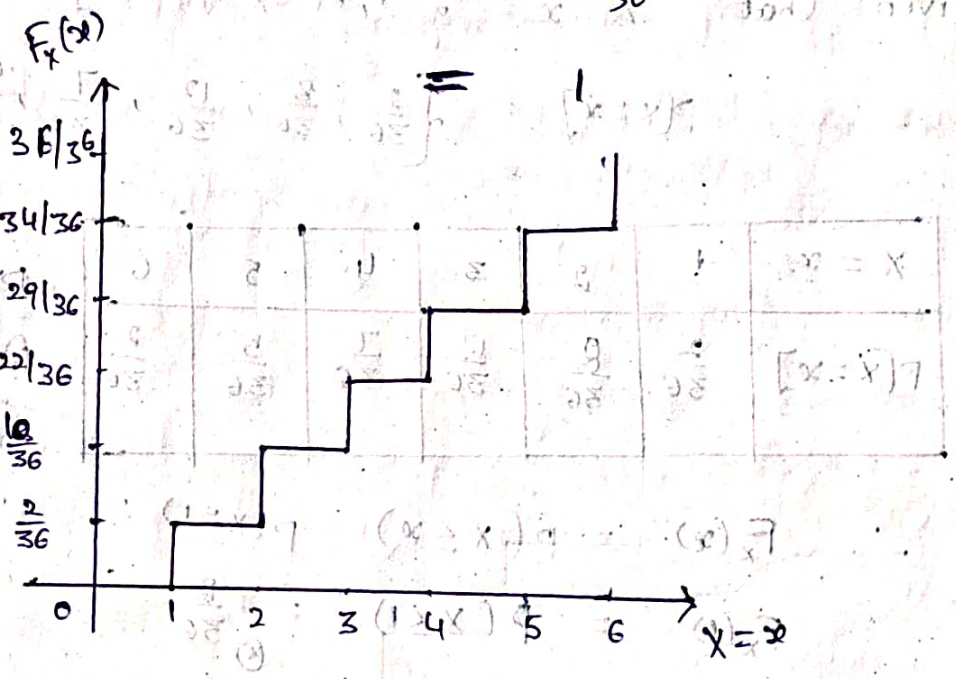
$$= \frac{2}{36} + \frac{8}{36} + \frac{12}{36} + \frac{7}{36} + \frac{5}{36}$$

$$= \frac{34}{36}$$

$$F_X(6) = P(X \le 6) = P(X=1) + P(X=2) + P(X=3) + P(X=4) + P(X=5) + P(X=6)$$

$$= \frac{2}{36} + \frac{8}{36} + \frac{12}{36} + \frac{7}{36} + \frac{5}{36} + \frac{2}{36}$$

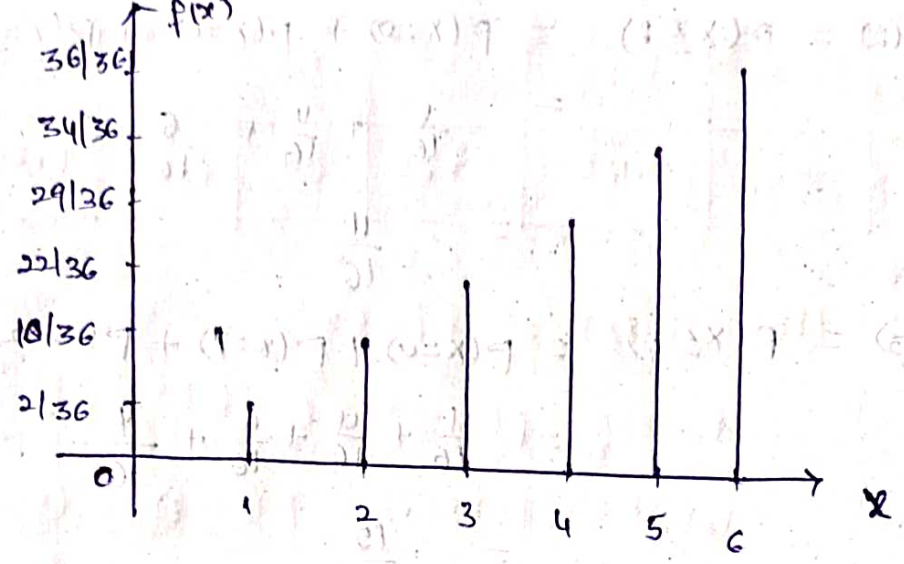
$$= 1$$



$$\therefore F_X(x) = \frac{2}{36} u(x-1) + \frac{10}{36} u(x-2) + \frac{22}{36} u(x-3) + \frac{29}{36} u(x-4) + \frac{34}{36} u(x-5) + \frac{36}{36} u(x-5)$$

Density function :- the density function of the Random Variable X $= f_X(x) = \frac{d}{dx}(F_X(x))$

$$= \frac{2}{36}\delta(x-1) + \frac{10}{36}\delta(x-2) + \frac{22}{36}\delta(x-3) + \frac{29}{36}\delta(x-4) + \frac{34}{36}\delta(x-5) + \frac{36}{36}\delta(x-5)$$



2) find the Distribution and the density function for the Random Experiment of Tossing four Coins.

Sol:- When four Coins are tossed the Sample Space

$S = \{(H,H,H,H), (H,H,H,T), (H,H,T,H), (H,T,H,H),$
$(T,H,H,H), (H,H,T,T), (H,T,T,H), (T,T,H,H),$
$(T,H,H,T), (H,T,H,T), (T,H,T,H),$
$(H,T,T,T), (T,H,T,T), (T,T,H,T)$
$(T,T,T,H), (T,T,T,T)\}$

$X = \{$no. of heads$\}$

$x = 0, 1, 2, 3, 4$

$P(X=x) = \frac{1}{16}, \frac{4}{16}, \frac{6}{16}, \frac{4}{16}, \frac{1}{16}$

$X = x$	0	1	2	3	4
$P(X=x)$	$\frac{1}{16}$	$\frac{4}{16}$	$\frac{6}{16}$	$\frac{4}{16}$	$\frac{1}{16}$

$F_X(x) = P(X \leq x)$

$F_X(0) = P(X \leq 0) = P(X=0) = \frac{1}{16}$

$F_X(1) = P(X \leq 1) = P(X=0) + P(X=1) = \frac{1}{16} + \frac{4}{16}$
$= \frac{5}{16}$

$$F_X(2) = P(X \le 2) = P(X=0) + P(X=1) + P(X=2)$$

$$= \frac{1}{16} + \frac{4}{16} + \frac{6}{16}$$

$$= \frac{11}{16}$$

$$F_X(3) = P(X \le 3) = P(X=0) + P(X=1) + P(X=2) + P(X=3)$$

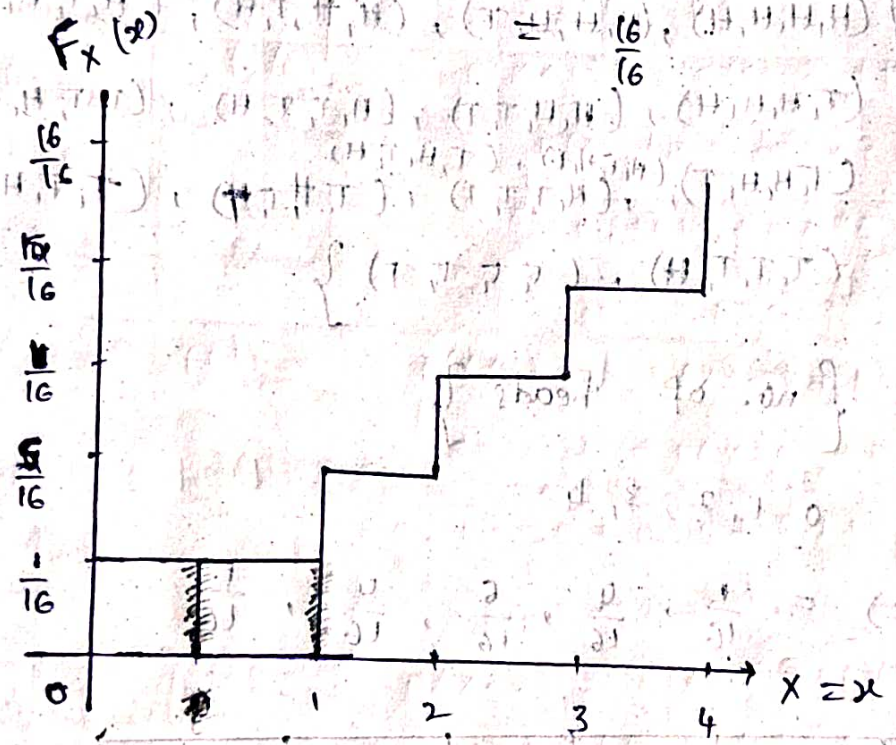
$$= \frac{1}{16} + \frac{4}{16} + \frac{6}{16} + \frac{4}{16}$$

$$= \frac{15}{16}$$

$$F_X(4) = P(X \le 4) = P(X=0) + P(X=1) + P(X=2) + P(X=3) + P(X=4) + P(X=5)$$

$$= \frac{1}{16} + \frac{4}{16} + \frac{6}{16} + \frac{4}{16} + \frac{1}{16}$$

$$= \frac{16}{16}$$

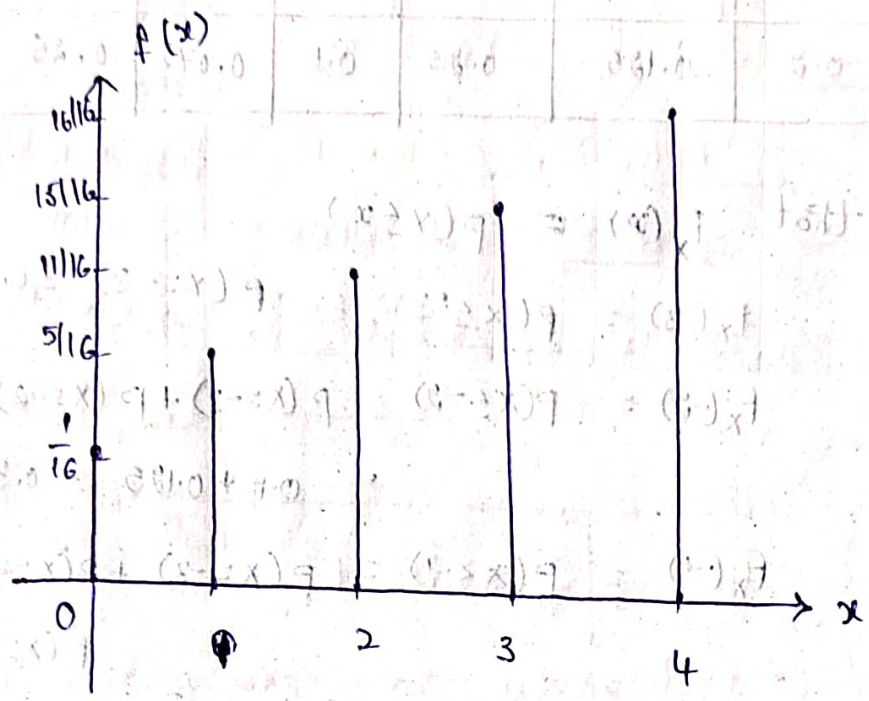


$$F_X(x) = \frac{1}{16} u(x-0) + \frac{5}{16} u(x-1) + \frac{11}{16} u(x-2) + \frac{15}{16} u(x-3) + \frac{16}{16} u(x-4)$$

Density function :-

$$f_X(x) = \frac{d}{dx}\left(F_X(x)\right)$$

$$= \frac{1}{16}\delta(x-0) + \frac{5}{16}\delta(x-1) + \frac{11}{16}\delta(x-2) + \frac{15}{16}\delta(x-3) + \frac{16}{16}\delta(x-4)$$



3.

→ The Random Variable x has the probability as shown in table.

$X = x$	-3	-2	-1	0	1	2
$P(X=x)$	0.2	0.5k	k	0.1	0.3k	k

Then i) find the value of k.

ii) find PDF and pdf

We know that Sum of all the probabilities = 1

i.e., $\sum_i P(X = x_i) = 1$

$\Rightarrow P(X=-3) + P(X=-2) + P(X=-1) + P(X=0) + P(X=1) + P(X=2) = 1$

$= 0.2 + 0.5k + k + 0.1 + 0.3k + k = 1$

$[2.8k + 0.3 = 1$ $\quad 2.8k = 0.7$

$2.8k = 0.7$ $\quad k = \frac{0.7}{2.8}$

$k = \frac{0.7}{2.8}]$ $\quad = 0.25$

$X = x$	-3	-2	-1	0	1	2
$P(X = x)$	0.2	0.125	0.25	0.1	0.075	0.25

ii) We know that $F_X(x) = P(X \le x)$

$F_X(-3) = P(X \le -3) = P(X = -3) = 0.2$

$F_X(-2) = P(X \le -2) = P(X = -3) + P(X = -2)$

$= 0.2 + 0.125 = 0.325$

$F_X(-1) = P(X \le -1) = P(X = -3) + P(X = -2) + P(X = -1)$

$= 0.2 + 0.125 + 0.25$

$= 0.575$

$F_X(0) = P(X \le 0) = P(X = -3) + P(X = -2) + P(X = -1) + P(X = 0)$

$= 0.2 + 0.125 + 0.25 + 0.1$

$= 0.675$

$F_X(1) = P(X \le 1) = P(X = -3) + P(X = -2) + P(X = -1) + P(X = 0) + P(X = 1)$

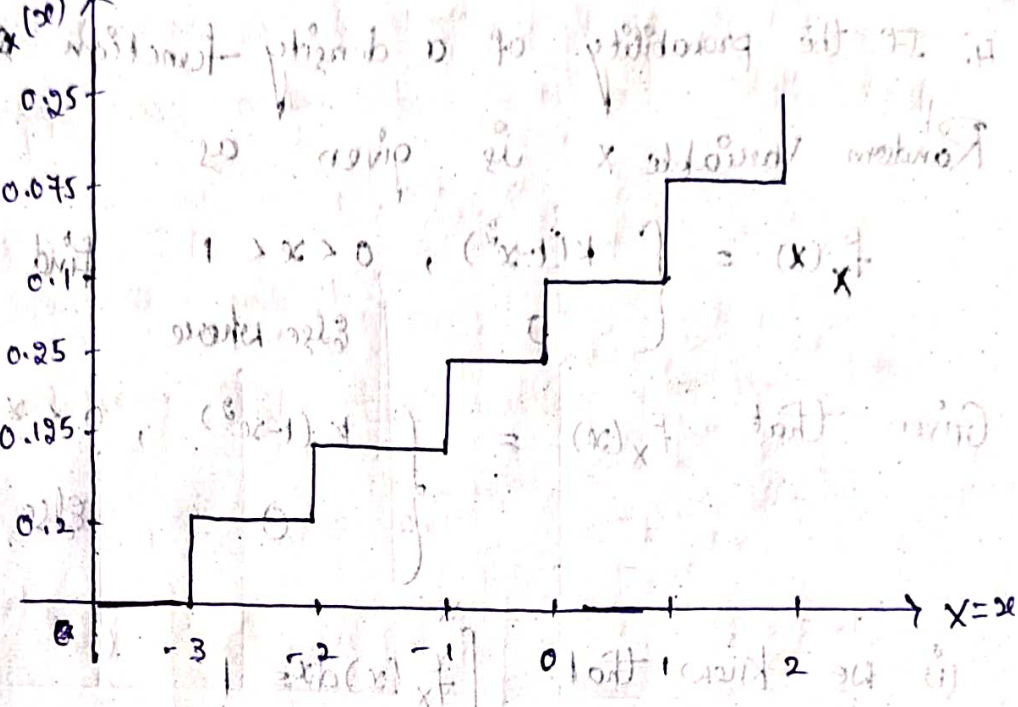
$= 0.2 + 0.125 + 0.25 + 0.1 + 0.075$

$= 0.75$

$F_X(2) = P(X \le 2) = P(X = -3) + P(X = -2) + P(X = -1) + P(X = 0) + P(X = 1) + P(X = 2)$

$= 0.2 + 0.125 + 0.25 + 0.1 + 0.075 + 0.25$

$= 1$

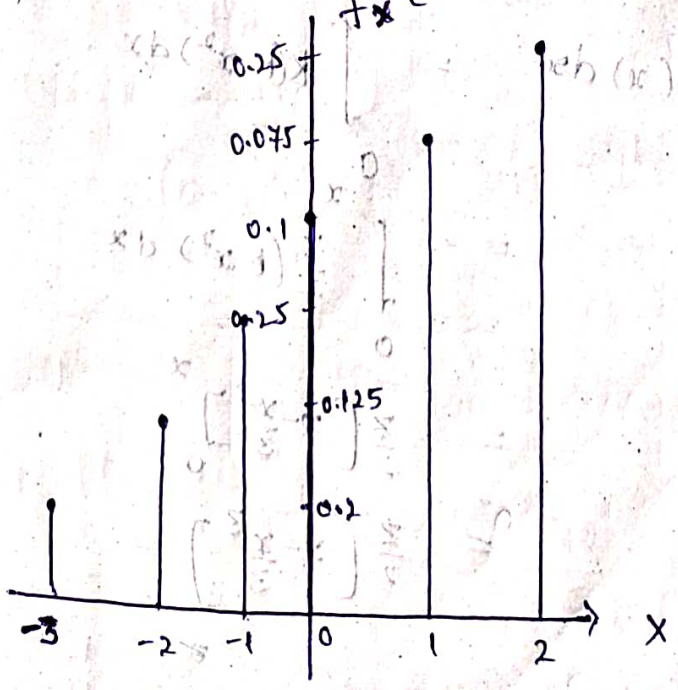


$$F_X(x) = 0.2\,u(x+3) + 0.325\,u(x+2) + 0.575\,u(x+1) + 0.675\,u(x) + 0.75\,u(x-1) + u(x-2)$$

We know that $f_X(x) = \frac{d}{dx}\left(F_X(x)\right)$

$$f_X(x) = 0.2\,\delta(x+3) + 0.325\,\delta(x+2) + 0.575\,u(x+1) + 0.675\,\delta(x) + 0.75\,\delta(x-1) + \delta(x-2)$$

~~4. If the probab~~



4. If the probability of a density function of a Random Variable x is given as

$$f_X(x) = \begin{cases} k(1-x^2), & 0 < x < 1 \\ 0 & \text{Else where} \end{cases}$$ find k of $F_X(x)$

Given that $f_X(x) = \begin{cases} k(1-x^2), & 0 < x < 1 \\ 0 & \text{Else where} \end{cases}$

(i) We know that $\int_{-\infty}^{\infty} f_X(x)\,dx = 1$

$$= \int_0^1 k(1-x^2)\,dx = 1$$

$$k\left[x - \frac{x^3}{3}\right]_0^1 = 1$$

$$k\left[1 - \frac{1}{3}\right] = 1$$

$$k\left[\frac{2}{3}\right] = 1 \Rightarrow k = \frac{3}{2}$$

ii) $F_X(x) = \int_{-\infty}^{x} f_X(x)\,dx = \int_0^x k(1-x^2)\,dx$

$$= \int_0^x \frac{3}{2}(1-x^2)\,dx$$

$$= \frac{3}{2}\left[x - \frac{x^3}{3}\right]_0^x$$

$$= \frac{3}{2}\left[x - \frac{x^3}{3}\right]$$

5. Consider $f_X(x) = a \cdot e^{-b|x|}$ where x is a Random Variable. takes the values b/w $-\infty$ to ∞ then find.

i) $F_X(x)$.

ii) Relation b/w a and b.

iii) the probability that the outcome lies b/w 1 and 2.

Sol:- Given that $f_X(x) = a \cdot e^{-b|x|}$, $-\infty \le x \le \infty$

(i) $F_X(x) = \int_{-\infty}^{x} f_X(x)\,dx$

$$= \int_{-\infty}^{x} a \cdot e^{-b|x|}\,dx$$

$$= \int_{-\infty}^{0} a e^{bx}\,dx + \int_{0}^{x} a e^{-bx}\,dx$$

$$= a\left[\frac{e^{bx}}{b}\right]_{-\infty}^{0} + a\left[\frac{e^{-bx}}{-b}\right]_{0}^{x}$$

$$= \frac{a}{b}\left[e^{0} - e^{-\infty}\right] + -\frac{a}{b}\left[e^{-bx} - e^{-0}\right]$$

$$= \frac{a}{b}[1-0] - \frac{a}{b}\left[e^{-bx} - 1\right]$$

$$= \frac{a}{b}\left(1 - e^{-bx} + 1\right) = \frac{a}{b}\left(2 - e^{-bx}\right)$$

(ii) We know that $\int_{-\infty}^{\infty} f_X(x)\,dx = 1$

$$\int_{-\infty}^{\infty} a \cdot e^{-b|x|}\,dx = 1$$

$$\int_{-\infty}^{0} a e^{bx}\,dx + \int_{0}^{\infty} a e^{-bx}\,dx = 1$$

$$a\left(\frac{e^{bx}}{b}\right)_{-\infty}^{0} + a\left(\frac{e^{-bx}}{-b}\right)_{0}^{\infty} = 1$$

$$\frac{a}{b}\left[e^{0} - e^{-\infty}\right] - \frac{a}{b}\left[e^{-\infty} - e^{0}\right] = 1 \qquad (e^{\infty} = \infty,\ e^{-\infty} = 0)$$

$$\frac{a}{b}\left[1 - 0\right] - \frac{a}{b}\left[0 - 1\right] = 1$$

$$\frac{a}{b} + \frac{a}{b} = 1$$

$$2a = b$$

(iii) $P(1 < x \le 2) = \int_1^2 f_X(x)\,dx = \int_1^2 a e^{-b|x|}\,dx$

$$= \int_1^2 a e^{-bx}\,dx$$

$$= a\left(\frac{e^{-bx}}{-b}\right)_1^2$$

$$= -\frac{a}{b}\left[e^{-2b} - e^{-b}\right]$$

$$= +\frac{a}{b}\left[e^{-b} - e^{-2b}\right]$$

6. find the value of a. such that $f_X(x) = \begin{cases} 0, & x < -1 \\ A(1-x^2)\cos\frac{\pi x}{2}, & -1 \le x \le 1 \\ 0, & x > 1 \end{cases}$ is a valid density function.

We know that $\int_{-\infty}^{\infty} f_X(x) = 1$

$$\int_{-1}^{1} A(1-x^2)\cos\left(\frac{\pi x}{2}\right)dx = 1$$

$$A\left[(1-x^2)\frac{\sin\left(\frac{\pi x}{2}\right)}{\frac{\pi}{2}} - (-2x)\left(-\frac{\cos\frac{\pi x}{2}}{\left(\frac{\pi}{2}\right)^2}\right) + (-2)\left(-\frac{\sin\left(\frac{\pi}{2}x\right)}{\left(\frac{\pi}{2}\right)^3}\right)\right]_{-1}^{1} = 1$$

$$= A\left[-2\left(-\frac{\sin\frac{\pi}{2}}{\left(\frac{\pi}{2}\right)^3}\right) - (-2)\cdot\left(+\frac{\sin\frac{\pi}{2}}{\left(\frac{\pi}{2}\right)^3}\right)\right] = 1$$

$$= A\left[\frac{2}{\left(\frac{\pi}{2}\right)^3} + \frac{2}{\left(\frac{\pi}{2}\right)^3}\right]$$

$$= A\left[\frac{4}{\left(\frac{\pi}{2}\right)^3}\right] = 1$$

$$= A\left[\frac{4}{\frac{\pi^3}{8}}\right] = 1 \Rightarrow 8A\left(\frac{4}{\pi^3}\right) = 1 \Rightarrow A = \frac{\pi^3}{32}$$

$$= 0.96$$

7. The Random Variable X has CDF $F_X(x) = \sum_{n=1}^{12} \frac{n^2}{650} u(x-n)$ then find

(i) $P(-\infty < X \leq 6.5)$

(ii) $P(6 < x \leq 9)$

We know that $P(x_1 < X < x_2) = F_X(x_2) - F_X(x_1)$

(i) $P(-\infty < x \leq 6.5) = F_X(6.5) - F_X(-\infty)$ $\quad (F_X(-\infty) = 0)$

$$= F_X(6.5)$$

$$= \sum_{n=1}^{6} \frac{n^2}{650} u(x-n)$$

$$= \frac{1}{650}\left(1^2 u(x-1) + 2^2 u(x-2) + 3^2 u(x-3) + 4^2 u(x-4) + 5^2 u(x-5) + 6^2 u(x-6)\right)$$

$$= \frac{1}{650}\left[1 + 4 + 9 + 16 + 25 + 36\right]$$

$$= \frac{1}{650}\left[91\right] = \frac{91}{650} = 0.14$$

(ii) $P(6 < x \leq 9) = F_X(9) - F_X(6)$

$$\sum_{n=1}^{9} \frac{n^2}{650} u(x-n) - \sum_{n=1}^{6} \frac{n^2}{650} u(x-n)$$

$$\frac{1}{650}\left[1^2 u(x-1) + 4 u(x-2) + 9 u(x-3) + 16 u(x-4) + 25 u(x-5) + 36 u(x-6) + 49 u(x-7) + 64 u(x-8) + 81 u(x-9)\right] - \frac{91}{650}$$

$$= \frac{1}{650}\left[1+4+9+16+25+36+49+64+81\right] - \frac{91}{650}$$

$$= \frac{285}{650} - \frac{91}{650}$$

$$= \frac{194}{650} = 0.298$$

8. The probability density function Random Variable X is given by $f_X(x) = \begin{cases} c\exp\left(-\frac{x}{4}\right), & 0 < x \le 1 \\ 0, & \text{elsewhere} \end{cases}$

i) find c value?

(ii) $F_X(0.5)$

We know that $\int_{-\infty}^{\infty} f_X(x)\,dx = 1$

$$= \int_0^1 c\, e^{-\frac{x}{4}}\,dx = 1$$

$$c\left[\frac{e^{-\frac{x}{4}}}{-\frac{1}{4}}\right]_0^1 = 1 \Rightarrow -4c\left[e^{-\frac{1}{4}} - e^{0}\right] = 1$$

$$-4c\left[0.77 - 1\right] = 1$$

$$0.88c = 1$$

$$c = 1.13$$

(ii) $F_X(0.5) = \int_{-\infty}^{0.5} f_X(x)\,dx$

$$= \int_{-\infty}^{0.5} (1.13)\, e^{-\frac{x}{4}}\,dx$$

$$= 1.13\left[\frac{e^{-\frac{x}{4}}}{-\frac{1}{4}}\right]_{-\infty}^{0.5} = -1.13\times 4\left[e^{-\frac{0.5}{4}} - e^{-\frac{\infty}{4}}\right]$$

$$= -1.13\times 4\left[0.85 - 0.67\right]$$

$= (-1.33 \times 4[-0.4])$

$= (0.533)$

$= -1.33 \times -0.52$

$= 0.5$

[9. A Continuous Random Variable x has density function $f_X(x) = 3x^2, \; 0 < x < 1$ find the values of a and b.

i) $P\{x = a\} = P\{x > a\}$

ii) $P\{x > b\} = 0.05$]

Sol:-

Special Distribution and density function :-

Binomial Distribution :- Discreat

→ This was introduced by Scientist name james Bernoulli.

→ The (Random) Binomial involved is a Discrete RV in the RV.

→ The Binomial distribution is applicable to the Random Experiments having only two outcomes and it was Repeated for n number of times.

→ That means the Experiment having two outcomes i.e., Success and failure.

p is the probability of getting Success

q is the getting failure

$\therefore p + q = 1$

→ In Order to derive the expressions for Binomial distribution and density function

→ let us Consider a Random Experiment of Tossing a Coin and it is repeated for 'n' number of times.

→ Here 'p' is the probability of getting head or Success.

→ Here 'q' is the probability of getting Tail or failure.

→ Now the probability of getting 'm' number of Success is $p \cdot p \cdot p \ldots$ m times $= p^m$.

→ lly the probability of getting 'n-m' number of failures is $q \cdot q \ldots$ (n-m) times $= q^{n-m}$.

→ Also the different no of ways to get 'm' no of Success and 'n-m' no of failures is nC_m, nC_{n-m}

→ Now the probability of getting 'm' no of Sucess and 'n-m' no of failures in any Order is

$$P(X=m) = nC_m \, p^m q^{n-m}, \quad m = 0, 1, 2, 3, \ldots n$$

put $m=0 \Rightarrow P(X=0) = nC_0 \, p^0 q^{n-0}$

put $m=1 \Rightarrow P(X=1) = nC_1 \, p^1 q^{n-1}$

put $m=2 \Rightarrow P(X=2) = nC_2 \, p^2 q^{n-2}$

- - - - - -

- - - - - -

put $m=n \Rightarrow P(X=n) = nC_n \, p^n q^{n-n}$

→ the probability distribution table can be given as:-

$X = x$	0	1	...	...	...	n
$P(X=x)$	$nC_0 p^0 q^n$	$nC_1 p^1 q^{n-1}$	...	...	...	$nC_n p^n q^{n-n}$

Now the distributive function is given as

$$F_X(x) = P(X \le x)$$

$$= P(X=0) + P(X=1) + \dots P(X=n)$$

$$= nC_0 p^0 q^{n-0} + nC_1 p^1 q^{n-1} + \dots nC_n p^n q^{n-n}$$

$$= \boxed{\sum_{m=0}^{n} nC_m p^m q^{n-m} u(x-x_m)}$$

Now, the Bionomial Density function can be given as

$$f_X(x) = \frac{d}{dx} F_X(x)\, dx$$

$$= \frac{d}{dx}\left(\sum_{n=0}^{n} nC_m p^m q^{n-m} u(x-x_m) \right)$$

$$\boxed{f_X(x) = \sum_{m=0}^{n} nC_m p^m q^{n-m} \delta(x-x_m)}$$

<u>poissions Distributive function</u> :-

→ The limiting Approximation to the Bionomial Distribution is called as poissions Distribution.

i.e., By substituting two approximations in the Binomial distribution then it is converted into poissions Distribution.

→ The two approximations are

i) The probability of getting Success (p) is approximate to 0.

ii) The number of Trails (n) tends to infinity (∞).

Such that $n \cdot p =$ Constant

$$n \times p = \lambda$$

$$p = \frac{\lambda}{n}$$

$$\& \quad q = \left(1 - \frac{\lambda}{n}\right)$$

→ In Order to derive poisson Distribution and Density function.

Let us Consider the Binomial distributive function

$$P_X(x) \neq$$

$$P(X=m) = {}^nC_m \, p^m q^{n-m}, \quad m = 0, 1, 2 \ldots n$$

Now put $p = \frac{\lambda}{n}$ and $q = 1 - \frac{\lambda}{n}$

$$P(X=m) = {}^nC_m \left(\frac{\lambda}{n}\right)^m \left(1 - \frac{\lambda}{n}\right)^{n-m}$$

$$P(X=m) = \frac{n!}{(n-m)!} \cdot \frac{\lambda^m}{n^m} \cdot \frac{\left(1 - \frac{\lambda}{n}\right)^n}{\left(1 - \frac{\lambda}{n}\right)^m}$$

$$= \frac{n(n-1)(n-2)\ldots(n-m+1)\cancel{(n-m)}\cancel{(n-m-1)}\ldots}{\cancel{(n-m)}\cancel{(n-m-1)}\cancel{(n-m-2)}\ldots m!} \times \frac{\lambda^m}{n^m} \cdot \frac{\left(1 - \frac{\lambda}{n}\right)^n}{\left(1 - \frac{\lambda}{n}\right)^m}$$

$$= \frac{n(n-1)(n-2)\cdots(n-m+1)}{m!} \frac{\lambda^m}{n^m} \frac{(1-\lambda/n)^n}{(1-\frac{\lambda}{n})^m}$$

$$= \frac{n^m\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{m-1}{n}\right)}{m!} \frac{\lambda^m}{n^m} \frac{(1-\frac{\lambda}{n})^n}{(1-\frac{\lambda}{n})^m}$$

Taking limits on both sides as $n \to \infty$ ⊗

$$\Rightarrow \lim_{n\to\infty} P(X=m) = \lim_{n\to\infty} \frac{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{(m-1)}{n}\right)\cdot\lambda^m\left(1-\frac{\lambda}{n}\right)^n}{m!\left(1-\frac{\lambda}{n}\right)^m}$$

$$= \frac{\lambda^m}{m!} \lim_{n\to\infty}\left(1-\frac{1}{n}\right) \lim_{n\to\infty}\left(1-\frac{2}{n}\right) \lim_{n\to\infty}\left(1-\frac{3}{n}\right)\cdots\lim_{n\to\infty}\left(1-\frac{(m-1)}{n}\right) \cdot \frac{\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^n}{\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^m}$$

$$= \frac{\lambda^m}{m!} 1.1\cdots\cdots \frac{e^{-\lambda}}{1}$$

$$= \frac{e^{-\lambda}\lambda^m}{m!}, \quad m = 0,1,2,\ldots n$$

$$\therefore \lim_{n\to\infty} P(X=m) = \frac{e^{-\lambda}\cdot\lambda^m}{m!}$$

$$m = 0,1,2,\ldots n$$

$$\boxed{\begin{aligned} &\therefore \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = 1 \\ &\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^n = e^{-\lambda} \\ &\lim_{n\to\infty}\left(1-\frac{\lambda}{m}\right)^m = 1 \end{aligned}}$$

The poissions Distributive function is

$$\boxed{F_X(x) = \sum_{m=0}^{n} \frac{e^{-\lambda}\cdot\lambda^m}{m!} u(x-x_m)}$$

Discret

Now

(IIy) the poissions density function can be given as

$$f_X(x) = \frac{d}{dx} F_X(x)$$

$$\boxed{f_X(x) = \sum_{m=0}^{m} \frac{e^{-\lambda} \lambda^m}{m!} \delta(x - x_m)}$$

3*. Uniform Distribution and density functions :-

→ This is also called as Rectangular Distribution function.

→ The Random Variable involved in the Uniform Distribution is a Continuous Random Variable.

→ The density function of the Uniformly Distributed Random Variable in the interval (a, b) can be given as

$$f_X(x) = \begin{cases} k, & a \le x \le b \\ 0, & \text{else where} \end{cases}$$

We know that $\int_{-\infty}^{\infty} f_X(x)\,dx = 1$

$$\int_a^b k\,dx = 1$$

$$k(1)_a^b = 1$$

$$k(b-a) = 1$$

$$k = \frac{1}{b-a}$$

$$\therefore \boxed{f_X(x) = \begin{cases} \frac{1}{b-a}, & a \le x \le b \\ 0, & \text{else where} \end{cases}}$$

Now, the distributive function can be obtained as

$$F_X(x) = \int_{-\infty}^{x} f_X(x)\, dx$$

$$= \int_{+a}^{x} \frac{1}{b-a}\, dx = \frac{1}{b-a}\,[x-a] = \frac{x-a}{b-a}$$

$$\boxed{\therefore F_X(x) = \begin{cases} \dfrac{x-a}{b-a} & ; a \le x \le b \\ 0 & , \text{ else where} \end{cases}}$$

Exponential Distribution and Density function :-

→ The Random Variable in the Exponential Distributive function is a Continuous Random Variable.

→ The density function of the Exponentially Distributive Random Variable in the Interval $(0, \infty)$ can be given as

$$\boxed{f_X(x) = \begin{cases} \alpha e^{-\alpha x} & , 0 \le x \le \infty \\ 0 & , \text{ else where} \end{cases}}$$

where α is a constant whose value is always greater than or Equal to 0.

→ Now, the Exponential Distributive function can be obtained as

$$F_X(x) = \int_{-\infty}^{x} f_X(x)\, dx$$

$$= \int_0^x \alpha e^{-\alpha x} = \alpha \left(\frac{e^{-\alpha x}}{-\alpha} \right)_0^x$$

$$= -\left(e^{-\alpha x} - e^{-0x} \right)$$

$$= 1 - e^{-\alpha x}$$

$$\therefore \boxed{F_X(x) = \begin{cases} (1 - e^{-\alpha x}) & , 0 \le x \le \infty \\ 0 & , \text{else where} \end{cases}}$$

5*

Gaussion Distribution and Density function :-

→ This is also Called as Normal Distribution function

→ It is Represented with $N(m, \sigma^2)$ where m is Called as Mean of the Random Variable (X) and σ^2 is the Variance of the Random Variable (X).

→ The density function of the Gaussionly Distributive Random Variable (X) Can be given as

$$\boxed{f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \cdot e^{-\frac{(x-m)^2}{2\sigma^2}}} \quad N(m, \sigma^2)$$

→ The Gaussion density function of the Random Variable (X) with '0' mean and Variance $= \sigma^2$ is

$$\boxed{f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \cdot e^{-\frac{x^2}{2\sigma^2}} \quad N(0, \sigma^2)}$$

→ The Gaussion density of the Random Variance (x) 0 mean and Variance = 1 is given by

$$\boxed{f_X(x) = \frac{1}{\sqrt{2\pi}} \cdot e^{-x^2/2}}$$

N (0,1)

→ Here N(0,1) is also Called as Normalized or Standadized Gaussion density function.

→ In Order to draw the Gaussion density Curve

$$\frac{d}{dx}\left(f_X(x)\right) = 0$$

$$\frac{d}{dx}\left(\frac{1}{\sqrt{2\pi a^2}} \cdot e^{-\frac{(x-m)^2}{2a^2}}\right) = 0$$

$$\frac{1}{\sqrt{2\pi a^2}} \cdot e^{-\frac{(x-m)^2}{2a^2}} \cdot \frac{-2(x-m)}{2a^2} = 0$$

$$x - m = 0$$

$$x = m$$

→ That Maximum Value can be obtained by putting $x = m$ in $f_X(x)$.

i.e., $$f_X(x)\Big|_{x=m} = \frac{1}{\sqrt{2\pi a^2}} \cdot e^{-\frac{(x-m)^2}{2a^2}}\Big|_{x=m}$$

$$= \frac{1}{\sqrt{2\pi a^2}} \cdot e^{-\frac{(m-m)^2}{2a^2}}$$

$$= \frac{1}{\sqrt{2\pi a^2}}$$

→ Now find the value of $f_X(x)$ at $x = m \pm a$

$$f_X(x)\Big|_{x=m\pm a} = \frac{1}{\sqrt{2\pi a^2}} \cdot e^{-\frac{(x-m)^2}{2a^2}}\Big|_{x=m\pm a}$$

$$= \frac{1}{\sqrt{2\pi a^2}} \cdot e^{-\frac{(\cancel{m}+a-\cancel{m})^2}{2a^2}}$$

$$= \frac{1}{\sqrt{2\pi a^2}} \cdot e^{-\frac{\cancel{a^2}}{2\cancel{a^2}}}$$

$$= \frac{1}{\sqrt{2\pi a^2}} \cdot e^{-\frac{1}{2}} = \frac{0.606}{\sqrt{2\pi a^2}}$$

Now find the value of $f_X(x)$ at $x = m \pm 2a$

$$f_X(x)\Big|_{x=m\pm 2a} = \frac{1}{\sqrt{2\pi a^2}} \cdot e^{-\frac{(x-m)^2}{2a^2}}\Big|_{x=m\pm a}$$

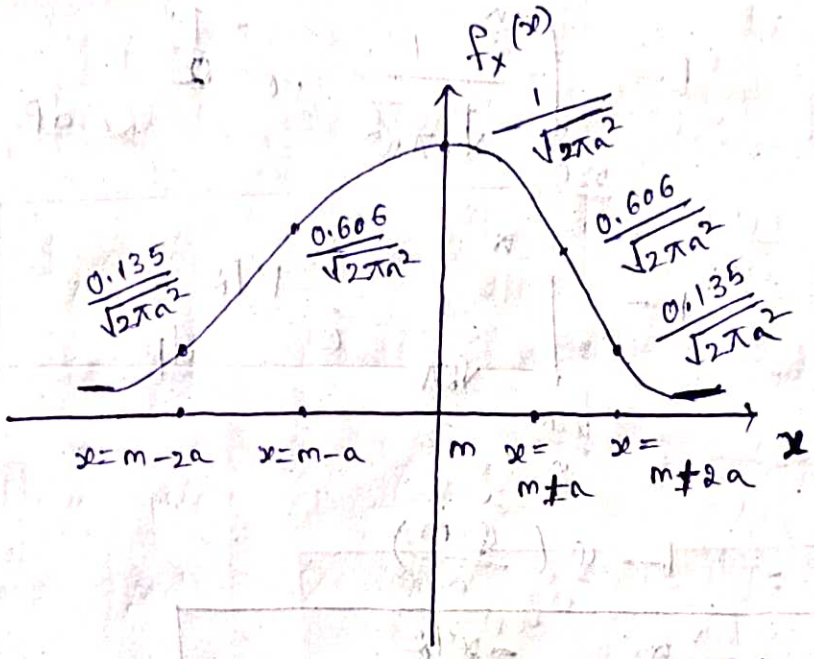
$$= \frac{1}{\sqrt{2\pi a^2}} \cdot e^{-\frac{(m\pm 2a-m)^2}{2a^2}}$$

$$= \frac{1}{\sqrt{2\pi a^2}} \cdot e^{-\frac{4a^2}{2a^2}}$$

$$= \frac{1}{\sqrt{2\pi a^2}} \cdot e^{-2}$$

$$= \frac{0.135}{\sqrt{2\pi a^2}}$$

The Gaussian density curve is a bell shaped curve.



for different values of m the graph moves horizontally

for different values of a the graph moves vertically

Gaussion Distributive function :-

$$F_X(x) = \int_{-\infty}^{\infty} f_X(x)\,dx$$

$$F_X(x) = \int_{\infty}^{x} f_X(x)\,dx$$

also $F_X(x) = P(X \le x)$

$= 1 - P(X > x)$

$= 1 - P(x < X \le \infty)$

$$= 1 - \int_{x}^{\infty} \frac{1}{\sqrt{2\pi a^2}} \cdot e^{-\frac{(x-m)^2}{2a^2}}\,dx$$

put $\frac{x-m}{a} = p \Rightarrow dx = dp$

L.L $\Rightarrow x = x \Rightarrow p = \frac{x-m}{a}$

U.L $\Rightarrow x = \infty \Rightarrow p = \infty$

$$\therefore F_X(x) = 1 - \int_{\frac{x-m}{a}}^{\infty} \frac{1}{\sqrt{2\pi a^2}} \cdot e^{-\frac{p^2}{2}} \, dp$$

$$= 1 - \int_{\frac{x-m}{a}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-p^2/2} \, dp$$

$$F_X(x) = 1 - Q\left(\frac{x-m}{a}\right)$$

$$\boxed{Q(k) = \int_{k}^{\infty} \frac{1}{\sqrt{2\pi}} \cdot e^{-x^2/2} \, dx}$$

Rayleigh Distributive And Density function :-

→ The Rayleigh Density function holds a Special Relationship with Gaussian Density function.

→ In Order to derive the Rayleigh Density function.

Let us Consider two independent Gaussian Random Variables x and y with 0 mean and Varience $= a^2$ as

$$f_X(x) = \frac{1}{\sqrt{2\pi a^2}} e^{-\frac{x^2}{2a^2}}, \quad N(0, a^2)$$

$$f_Y(y) = \frac{1}{\sqrt{2\pi a^2}} \cdot e^{-\frac{y^2}{2a^2}}, \quad N(0, a^2)$$

put $x = r\cos\theta$ & $y = r\sin\theta$

Such that $x^2+y^2=r^2$ & $dx\,dy = r\,dr\,d\theta$

$0 \le r \le \infty$ & $0 \le \theta \le 2\pi$

Now the join Density function of r & θ can be given as $f(r,\theta) = f(x,y)\,|g(x,y)|$

Where $g(x,y)$ is called as jacobian function.

$$|J(x,y)| = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r(\cos^2\theta + \sin^2\theta)$$

$$= r$$

$Q(K) = \int_K^\infty \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\,dx$

(10)

$\therefore f(r,\theta) = r \cdot f(x,y)$

$= r \cdot f_X(x) \cdot f_Y(y)$

$f_X(x)$ → Exact Notation

$f(x)$ → Simple Notation

($\because$ x & y are independent)

$$= r \cdot \frac{1}{\sqrt{2\pi a^2}} \cdot e^{-x^2/2a^2} \cdot \frac{1}{\sqrt{2\pi a^2}} \cdot e^{-y^2/2a^2}$$

$$= \frac{r}{2\pi a^2} \cdot e^{-\frac{(x^2+y^2)}{2a^2}}$$

$$= \frac{r}{2\pi a^2} \cdot e^{-r^2/2a^2}$$

Now the Rayleigh Density function

$$f_r(r) = \int_{\theta=0}^{2\pi} f(r,\theta)\,d\theta$$

$$= \int_0^{2\pi} \frac{r}{2\pi a^2} \cdot e^{-\frac{r^2}{2a^2}} \, d\theta$$

$$= \frac{r}{2\pi a^2} \cdot e^{-\frac{r^2}{2a^2}} \cdot \int_0^{2\pi} 1 \, d\theta$$

$$= \frac{r}{2\pi a^2} e^{-r^2/2a^2} (\theta)_0^{2\pi}$$

$$\boxed{f_r(r) = \frac{r}{a^2} \cdot e^{-r^2/2a^2}}$$

Distributive function :-

$$F_r(r) = \int_{-\infty}^{r} f_r(r) \, dr$$

$$= \int_{-\infty}^{r}$$

problems :-

A Random Variable x takes values $\{1,2,3,4\}$ such that plot PDF and pdf :

$$2P(X=1) = 3P(X=2) = P(X=3) = 5P(X=4)$$

Sol :- Given that $x = \{1,2,3,4\}$

$$2P(X=1) = 3P(X=2) = P(X=3) = 5P(X=4) = K \text{ (say)}$$

$$K = \frac{30}{61}$$

$P(X=1) = \frac{K}{2}$

$P(X=2) = \frac{K}{3}$

$P(X=3) = K$

$P(X=4) = \frac{K}{5}$

$X=x$	1	2	3	4
$P(X=x)$	$\frac{15}{61}$	$\frac{10}{61}$	$\frac{30}{61}$	$\frac{6}{61}$

$$\sum_i P(X=x_i) = 1$$

PDF Plot :-

We know that $F_X(x) = P(X \le x)$

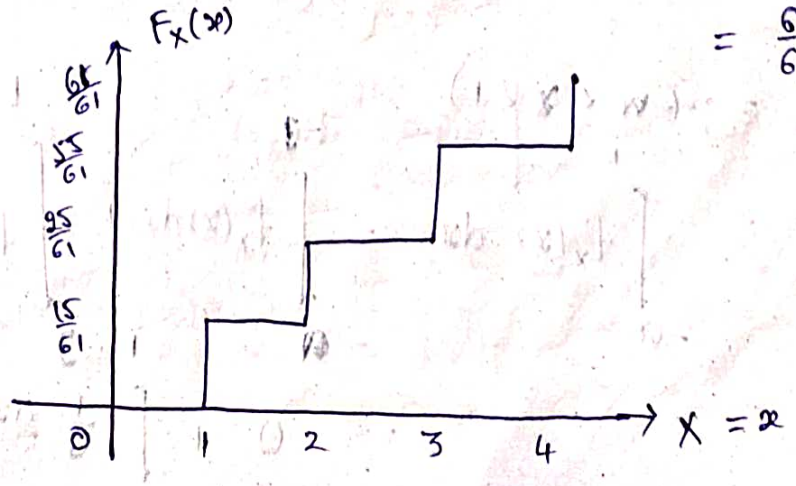
$F_X(1) = P(X \le 1) = P(X=1) = \frac{15}{61}$

$F_X(2) = P(X \le 2) = P(X=1) + P(X=2) = \frac{15}{61} + \frac{10}{61} = \frac{25}{61}$

$F_X(3) = P(X \le 3) = P(X=1) + P(X=2) + P(X=3) = \frac{55}{61}$

$F_X(4) = P(X \le 4) = P(X=1) + P(X=2) + P(X=3) + P(X=4)$

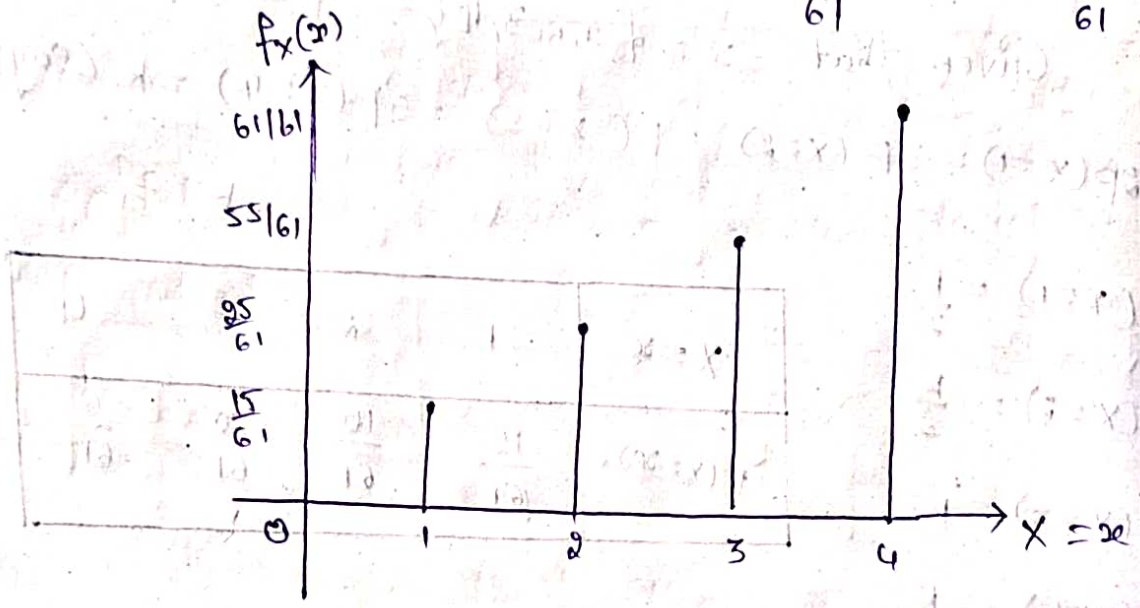
$= \frac{61}{61}$



$$F_X(x) = \frac{15}{61} u(x-1) + \frac{25}{61} u(x-2) + \frac{55}{61} u(x-3) + \frac{61}{61} u(x-4)$$

$$f_X(x) = \frac{d}{dx}(F_X x)$$

$$\frac{15}{61} \delta(x-1) + \frac{25}{61} \delta(x-2) + \frac{55}{61} \delta(x-3) + \frac{61}{61} \delta(x-4)$$



2. A Random Variable X has the following Density function $f_X(x) = \begin{cases} 1/6 & -3 \le x \le 3 \\ 0 & \text{Else where} \end{cases}$ then find

i) $P(X<1)$

ii) $P(x+3 < 4)$

iii) $P(|x| > 1)$

Sol:- Given that $f_X(x) = \begin{cases} 1/6, & -3 \le x \le 3 \\ 0, & \text{Else where} \end{cases}$

i) $P(X<1) = P(-\infty < x < 1)$

$$= \int_{-\infty}^{1} f_X(x)\,dx = \int_{-\infty}^{-3} f_X(x)\,dx + \int_{-3}^{1} f_X(x)\,dx$$

$$= 0 + \int_{-3}^{1} \frac{1}{6}\,dx$$

$= \frac{1}{6}(1+3)$

$= \frac{4}{6} = \frac{2}{3}$

ii) $P(x+3 > 4) = P(x > 1)$

$= P(1 < x < \infty)$

$= \int_1^{\infty} f_X(x)\,dx = \int_1^{+3} f_X(x)\,dx + \int_{+3}^{\infty} f_X(x)\,dx$

$= \int_1^{\infty} f_X(x)\,dx = \int_1^{\infty} \frac{1}{6}\,dx = \frac{1}{6}($

$= \frac{1}{6}(3-1)$

$= \frac{2}{6} = \frac{1}{3}$

iii) $P(|x| > 1) = P(-1 < x < 1)$

$\int_{-1}^{1} f_X(x)\,dx = \frac{1}{6}(1+1) = \frac{2}{3}$

3. find a Constant 'b' so that the function

$$\begin{cases} \frac{1}{10}e^{3x}, & 0 \le x \le b \\ 0, & \text{else where} \end{cases}$$

is a valid density function.

Given that $f_X(x) = \begin{cases} \frac{1}{10}e^{3x}, & 0 \le x \le b \\ 0, & \text{else where} \end{cases}$

$\int_{-\infty}^{\infty} f_X(x)\,dx = 1 \Rightarrow \int_0^b \frac{1}{10}e^{3x} = 1$

$\frac{1}{10}\left(\frac{e^{3x}}{3}\right)_0^b = 1$

$$\frac{1}{30}\left(e^{3x}\right)_0^b = 1$$

$$(e^{3b} - 1) = 30$$

$$e^{3b} = 31$$

$$3b = \log 31$$

$$b = 0.49$$

4. Consider the probability density function $f_X(x) = ae^{-b|x|}$ where x is a Random Variable. then find

i) $F_X(x)$

ii) Relation

4. A Rayleigh density function is given by

$$f_X(x) = \begin{cases} xe^{-\frac{x^2}{2}}, & x \geq 0 \\ 0 & x < 0 \end{cases}$$

i) prove that $\int_{-\infty}^{\infty} f_X(x)\,dx = 1$

ii) find $F_X(x)$

iii) find $P(0.5 \leq X \leq 2)$

i) Given that $f_X(x) = \begin{cases} xe^{-\frac{x^2}{2}}, & x \geq 0 \\ 0, & x < 0 \end{cases}$

$$\int_{-\infty}^{\infty} f_X(x)\,dx = \int_0^{\infty} xe^{-x^2/2}\,dx$$

put $x^2 = t$

$2x\,dx = dt$

$x\,dx = \frac{dt}{2}$

$t = 0, \infty$

$$= \frac{xe^{-\frac{x^2}{2}}}{}$$

$$= \int_0^{\infty} e^{-\frac{t}{2}} \frac{dt}{2}$$

$$= \frac{1}{2}\int_0^{\infty} e^{-\frac{t}{2}} dt = \frac{1}{2}\left[\frac{e^{-\frac{t}{2}}}{-\frac{1}{2}}\right]_0^{\infty} = -\left[e^{-\infty} - e^{0}\right]$$

$$= -[0-1]$$

$$= 1$$

Hence proved.

ii) $$F_X(x) = \int_{-\infty}^{x} f_X(x)\, dx$$

$$= \int_{+0}^{x} x\, e^{-\frac{x^2}{2}} dx$$

put $x^2 = t$

$2x\,dx = dt$

$x\,dx = \frac{dt}{2}$

$t = x^2, \infty$

$$= \int_{+0}^{x^2} e^{-\frac{t}{2}} \frac{dt}{2}$$

$$\frac{1}{2}\int_{+0}^{x^2} e^{-\frac{t}{2}} dt = \frac{1}{2}\left[\frac{e^{-\frac{t}{2}}}{-\frac{1}{2}}\right]_0^{x^2}$$

$$= -\left(e^{-\frac{x^2}{2}} - e^{0}\right)$$

$$= 1 - e^{-x^2/2}$$

iii) $$P(0.5 \le x \le 2) = F_X(2) - F_X(0.5)$$

$$= \int_{x_1}^{x_2} f_X(x)\, dx \quad \left(1 - e^{-x^2/2}\right)\Big|_{x=2} - \left(1 - e^{-x^2/2}\right)\Big|_{0.5}$$

$$= 1 - e^{-4/2} - \left(1 - e^{-(0.5)^2/2}\right)$$

$$= 1 - e^{-2} - \left(1 - e^{-0.125}\right)$$

$$= \cancel{1} - 0.135 - (\cancel{1} - 0.88)$$

$$= 0.88 - 0.135$$

$$= 0.745$$

5. If probability density function of a R.V

$$f_X(x) = \begin{cases} \frac{x}{a} + K & 0 \le x \le 6 \\ 0 & \text{else where} \end{cases}$$

i) K

ii) $P(2 \le x \le 5)$

Given that $f_X(x) = \begin{cases} \frac{x}{a} + K & 0 \le x \le 6 \\ 0 & \text{else where} \end{cases}$

i)

6. A Gaussian Random Variable with $\mu_x = 4$ and $\sigma_x = 3$ is generated find the probability of $X \le 7.75$. Write down the function and draw the graph.

Sol:- Given that Mean $= \mu_x = 4$

$\sigma_x = 3$

$P(X \le 7.75) = ?$

Now $P(X \le 7.75) = F_X(7.75)$

$= 1 - Q\left(\frac{x-m}{\sigma}\right)$

$= 1 - Q\left(\frac{7.75-4}{3}\right)$

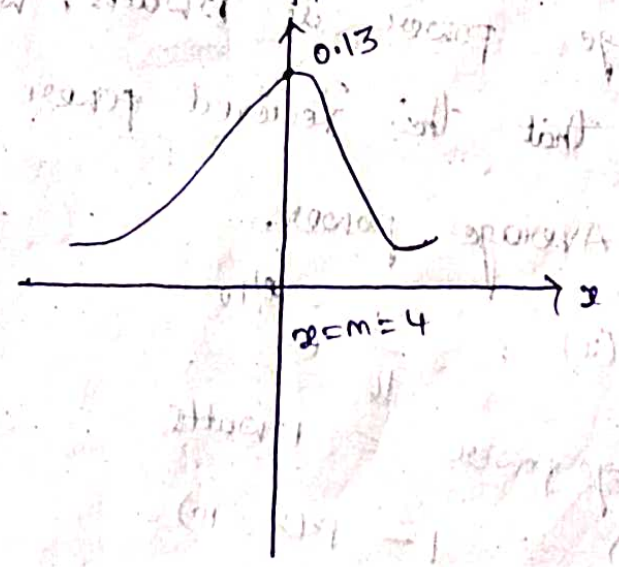
$= 1 - Q(1.25)$

$* \; Q(x) = \frac{1}{\sqrt{0.661x + 0.339 + \sqrt{x^2 + 5.51}}} \cdot \frac{e^{-x^2/2}}{\sqrt{2\pi}}$

We know that $f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \cdot e^{-\frac{(x-m)^2}{2\sigma^2}}$

$= \frac{1}{\sqrt{2\pi 3^2}} \cdot e^{-\frac{(x-4)^2}{2 \cdot 3^2}}$

$f_X(x) = 0.13 \; e^{-\frac{(x-4)^2}{18}}$



7. Assume that the height of the clouds above the ground at some location in the gaussian Random Variable (X) is a with mean value of 2km and $\sigma_x = 0.25$. find the probability of clouds higher than 2.5 km.

Given that Mean = 2km

$\sigma_x = 0.25$

$P(X \geq 2.5) = ?$ $= 1 - P(X \leq 2.5km)$

Now $P(X > 2.5) = 1 - F_X(2.5)$

$= 1 - \left(1 - Q\left(\frac{x-m}{\sigma}\right)\right)$

$= 1 - \left(1 - Q\left(\frac{2.5-2}{0.25}\right)\right)$

$= 1 - (1 - Q(2))$

$= Q(2)$

8. The power Reflected ~~from~~ from an aircraft received by radar described by an Exponential distribution. The pdf is given as $f_X(x) = \frac{1}{10} e^{-x/10}$, $x > 0$. The average power is 10 watts. What is the probability that the recieved power is greater than Average power.

Given that $f_X(x) = \frac{1}{10} e^{-x/10}$

Average power = 10 watts

$P(x > 10) = 1 - P(X \leq 10)$

$= 1 - F_X(10)$

Now $F_X(x) = \int_{-\infty}^{x} f_X(x)\,dx = \int_{0}^{x} \frac{1}{10} e^{-x/10}\,dx$

$= \left.\frac{1}{\cancel{10}} \frac{e^{-\frac{x}{10}}}{-\frac{1}{\cancel{10}}}\right|_{0}^{x} \Rightarrow -\left(e^{-\frac{x}{10}} - 1\right) = 1 - e^{-x/10}$

$\therefore P(x > 10) = 1 - F_X(10)$

$= 1 - \left(1 - e^{-10/10}\right)$

$= e^{-1}$

$= 0.368$

9. The life time of a System Expressed in weeks is a Rayleigh Random Variable (X) for which $f_X(x) = \begin{cases} \frac{x}{200} \cdot \exp\left(-x^2/400\right), & x \geq 0 \\ 0, & \text{Else where} \end{cases}$

then find

i) what is the probability that the System will not lost for a ~~few~~ one week.

ii) what is the probability the System life time will Exceed one year. $P(X > 52)$

Given that $f_X(x) = \begin{cases} \frac{x}{200} \cdot \exp\left(-\frac{x^2}{400}\right), & x \geq 0 \\ 0, & \text{Else where} \end{cases}$

i) $P(X \leq 1) = F_X(1)$

Now $F_X(x) = \int_{-\infty}^{x} f_X(x)\,dx$

$$= \int_0^{x} \frac{x}{200} \cdot e^{-\frac{x^2}{400}} \, dx$$

put $\frac{x^2}{400} = t$

$\frac{2x}{400} dx = dt$

$x dx = 200 dt$

$t = 0,$

$$\frac{1}{200} \int_0^{t^2/400} e^{-t} \, dt = \frac{e^{-t}}{-1} \Big|_0^{t^2/400} = -\left[e^{-t}\right]_0^{t^2/400}$$

$$= 1 - e^{-0.0025}$$

$$P(x \le 1) = f_x(1) = 1 - e^{-0.0025}$$

$$= 1 - 0.9975$$

$$= 0.0025$$

unit - 2

Operations On Random Variables :-

* Introduction :-

* Expected Value of the Random Variable $(E[x])$.
* Expected Value of the function of the Random Variable $E[g(x)]$.
* properties of Expectation.
* Moments.
  → Moments about the Origin. $(m_0, m_1, \ldots m_n)$.
  → Moments about the mean. / Central moments. $(\mu_0, \mu_1, \mu_2 \ldots \mu_n)$.
* Variance (μ_2), skew (μ_3).
* properties of Variance.
* characteristic function $(\phi_x(\omega))$. → higher Order moments about the origin only. generate
* properties of characteristic function.
* Transformations of a Random Variable.

Introduction :- In this chapter we will discuss Some statistical operations performed on the Random Variable like Expectation, Variance, Skew and moments etc.

5* Expected Value of the Random Variable ($E[X]$) or $\bar{x}$ or m or m_x

→ Expectation is a name given to the process of averaging when a Random Variable (X) is involved.

→ It is Represented with $E[X]$ or $\bar{x}$ or m or m_x

→ If x is a discret Random Variable then the Expected value of the Random Variable can be given as

$$\boxed{E[X] = \sum_i x_i \, P(X = x_i)}$$

→ If x is a Continuous Random Variable then the Expected value of the Random Variable can be given as

$$\boxed{E[X] = \int_{-\infty}^{\infty} x f_X(x)\, dx}$$

Expected value of the function of the Random Variable $E(g(x))$:-

→ let us Consider a Random Variable (Y) which is defined as $Y = g(x)$ where $g(x)$ is the function in terms of Random Variable (X).

→ Now the Expected value of the Random Variable Y can be given as $E[Y] = E[g(x)]$

Here $E[g(x)]$ is called as the Expected value of the function of the R.V (X).

→ It can be given as

$$\boxed{E[g(x)] = \sum_i g(x_i)\, p(X=x_i)}$$, if x is a discreat R.V

$$= \boxed{\int_{-\infty}^{\infty} g(x)\, f_X(x)\, dx}$$, if x is a Continuous RV

properties of Expectation :-

Note :- Here all the properties of Expectation are proved by Assuming the RV (X) has a Continuous Random Variable.

1. If X is a Random Variable and 'a' is a Constant then $E[a] = a$.

proof :-

We know that $E[X] = \int_{-\infty}^{\infty} x f_X(x)\, dx$

Now $E[a] = \int_{-\infty}^{\infty} a f_X(x)\, dx$

$= a \int_{-\infty}^{\infty} f_X(x)\, dx$

$= a \times 1$

$= a$

2. If 'a' is any Constant then $E[ax] = \underline{a E[x]}$

<u>proof</u> :-

We know that $E[x] = \int_{-\infty}^{\infty} x f_X(x)\, dx$

$$E[ax] = \int_{-\infty}^{\infty} a x f_X(x)\, dx$$

$$= a E[x]$$

3. If 'a' and 'b' are Constants then $E[ax+b] = aE[x] + b$

<u>proof</u> :-

We know that $E[x] = \int_{-\infty}^{\infty} x f_X(x)\, dx$

$$E[ax+b] = \int_{-\infty}^{\infty} (ax+b) f_X(x)\, dx$$

$$= aE[x] + b$$

4. If $g_1(x)$ and $g_2(x)$ are two functions of the RV (X) then Expectation of $E[g_1(x) + g_2(x)]$ is $\underline{E[g_1(x)] + E[g_2(x)]}$.

<u>proof</u> :-

We know that $E[g(x)] = \int_{-\infty}^{\infty} g(x_i) f_X(x)\, dx$

$$E[g_1(x) + g_2(x)] = \int_{-\infty}^{\infty} (g_1(x) + g_2(x)) f_X(x)\, dx$$

$$= \int_{-\infty}^{\infty} g_1(x) f_X(x)\, dx + \int_{-\infty}^{\infty} g_2(x) f_X$$

$$= E[g_1(x)] + E[g_2(x)]$$

5. If $x >= 0$, then $E[X]$ is also greater than or Equal to zero.

$\therefore$ $x >= 0$ then $E[X] > 0$

We know that

$$E[X] = \int_{-\infty}^{\infty} x f_X(x)\, dx$$

here $x > 0$

$$E[X] = \int_{0}^{\infty} x f_X(x)\, dx \geq 0$$

6. If the probability density function of the Random Variable (X) is Symmetric about a point 'a'.

i.e., $f_X(x-a) = f_X(x+a)$

In the Case that mean of the RV $E[X] = a$

7. If the probability density function of a RV is an Even function then i.e., $f_X(-x) = f_X(x)$ then in that Case the Expectation of RV $E[X] = 0$

Moments :-

→ Moments are nothing but Statistical Average.

→ The Moments of a RV (X) Can be obtained by Calculating Expected value of the function of the RV.

→ There are two types of Moments.

a) Moments about the Origin

b) Moments about the mean / Central Moments.

→ Moments about the Origin are represented with $m_0, m_1, m_2 \ldots m_n$

→ Moments about the mean are represented with $\mu_0, \mu_1, \mu_2 \ldots \mu_n$

Moments about the Origin :-

i) Moments about the Origin can be obtained by finding the Expected value of the function the RV $E(g(x))$.

→ let $g(x) = x^n$ when $n = 0, 1, 2 \ldots$ Etc

→ Now, n^{th} moment about the Origin can be given as $m_n = E(g(x)) = E[x^n]$

$$= \sum_i x_i^n \, p(X = x_i) \rightarrow x \text{ is a Discrete RV.}$$

$$= \int_{-\infty}^{\infty} x^n f_X(x)\, dx \rightarrow x \text{ is a Continuous RV}$$

Moment about the mean / Central Moments :-

i) Moments about the mean can be obtained by finding the Expected value of the function of

the RV $E(g(x))$.

→ let $g(x) = (x-m)^n$ where $n = 0, 1, 2 \ldots$

→ Now, n^{th} moment about the mean can be given

as $\mu_n = E[(x-m)^n]$

$$= \sum_i (x-m)^n \, p(X = x_i) \rightarrow x \text{ is a Discrete RV}$$

$$= \int_{-\infty}^{\infty} (x-m)^n f_X(x)\, dx \rightarrow x \text{ is a Continuous RV}$$

→ The Various moments about the Origin are

$$m_n = E[X^n] = \int_{-\infty}^{\infty} x^n f_X(x)\, dx$$

put $n = 0 \Rightarrow m_0 = E[X^0] = \int_{-\infty}^{\infty} x^0 f_X(x)\, dx = \int_{-\infty}^{\infty} f_X(x)\, dx = 1$

$$n = 1 \Rightarrow m_1 = E[X] = \int_{-\infty}^{\infty} x f_X(x)\, dx$$

→ mean of the RV.

$$n = 2 \Rightarrow m_2 = E[X^2] = \int_{-\infty}^{\infty} x^2 f_X(x)\, dx$$

→ mean Square Value of the RV (X).

→ The Various moments about the mean are

$$\mu_n = E[(x-m)^n] = \int_{-\infty}^{\infty} (x-m)^n f_X(x)\, dx$$

put $n=0 \Rightarrow \mu_0 = E[(x-m)^0] = E[1] = 1$

put $n=1 \Rightarrow \mu_1 = E[(x-m)] = E[X] - E(m]$

$= m - m = 0$

put $n=2 \Rightarrow \mu_2 = E[(x-m)^2] = E[x^2 + m^2 - 2mx]$

$= E[x^2] + E(m^2) - E[2mx]$

$= m_2 + m^2 - 2mE[X]$

$\mu_2 = m_2 - m_1^2$

↳ Variance of the RV (X)

put $n=3 \Rightarrow \mu_3 = E[(x-m)^3] = E[x^3 - m^3 - 3x^2m + 3xm^2]$

$= E[x^3] - 3mE[x^2] + 3m^2E[X] - E[m^3]$

$= m_3 - 3mm_2 + 3m^3 - m^3$

Skew of the RV

$= m_3 - 3mm_2 + 2m^3$

Variance (μ_2 (or) σ_x^2) :-

→ The Second moment about the mean (or) Second Central moment is Called as Variance.

→ It is represented with μ_2 (or) σ_x^2.

It Can be given as μ_2 (or) $\sigma_x^2 = E[(x-m)^2]$

$= \sum_i (x_i - m)^2 P(X = x_i)$; if x is a Degret R.V

$= \int_{-\infty}^{\infty} (x-m)^2 f_X(x)\, dx$ → x is a Continues RV

Relation b/w m_1, m_2 and μ_2 :-

We know that $\mu_2 = E[(x-m)^2]$

$$= E[x^2 + m^2 - 2xm]$$

$$= E[x^2] + E[m^2] - 2m\,E[x]$$

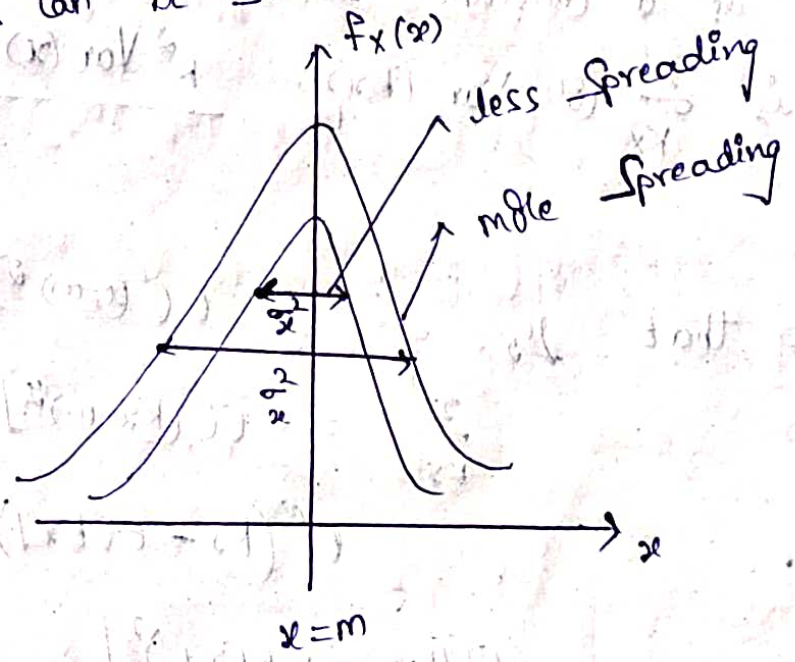
$$= m_2 + m_1^2 - 2m_1^2$$

$$\mu_2 = m_2 - m_1^2$$

Physical Significance of Variance :-

The Variance can be used to find the amount of dispersion or spreading of Density function about the mean values.

→ this can be shown in the figure as :-



Properties of Variance :-

→ The Variance of a Constant is σ_k^2 (or) $Var(k)$

$= 0$

Proof :-

We know that $\sigma_x^2 = E[(x-m)^2]$

Now $\sigma_k^2 = E[(k-m)^2]$

$= E[k^2 + m^2 - 2km]$

$= E[k^2] + E[m^2] - 2k[m]$

$= k^2 E[1] + m_1^2 E[1] - 2km E[1]$

$= k^2 + m_1^2 - 2km_1$

$= k^2 + k^2 - 2k \cdot k$

$= 0$

[m_1 is mean of Constant k.

$m_1 = E[k]$

$= k$]

→ If 'k' is a Constant then the Variance of a R.V kx is σ_{kx}^2 (or) $Var(kx) = \underline{k^2 Var(x)}$

proof :-

We know that $\mu_2 = \sigma_x^2 = E[(x-m)^2]$

$\sigma_{kx}^2 = E[(kx-m)^2]$

$= E[(kx - E[kx])^2]$

$= E[(kx - k E[x])^2]$

$= E[k^2 (x-m)^2]$

$= k^2 E[(x-m)^2]$

$= k^2 Var(x)$

→ If k is a Constant then the Variance of the Random Variable $(x+k)$ is σ^2_{x+k} & Var $(x+k)$

$= \sigma^2_x$

Proof :-

We know that $\mu_2 = \sigma^2_x = E[(x-m)^2]$

$$\sigma^2_{x+k} = E[((x+k)-m)^2]$$

$$= E[((x+k) - E[x+k])^2]$$

$$= E[((x+k) - (E[x] + E(k)))^2]$$

$$= E[((x+k) - (m+k))^2]$$

$$= E[(x+\cancel{k}-m-\cancel{k})^2]$$

$$= E[(x-m)^2]$$

$$= \sigma^2_x$$

→ If a and b are Constants then the RV $(ax+b)$ is σ^2_{ax+b} & Var $(ax+b)$ is $a^2\sigma^2_x$

Proof :-

We know that $\mu_2 = \sigma^2_x = E[(x-m)^2]$

$$\sigma^2_{ax+b} = E[((ax+b)-m)^2]$$

$$= E[[(ax+b) - E(ax+b)]^2]$$

$$= E[(ax+b) - (aE[x] + E(b))]^2]$$

$$= E\left[\left((ax+b) - (am+b)\right)^2\right]$$

$$= E\left[(ax+\cancel{b} - am - \cancel{b})^2\right]$$

$$= E\left[a^2(x-m)^2\right] = a^2 E\left[(x-m)^2\right]$$

$$= a^2 \cdot \sigma_x^2$$

→ If two RV x_1 and x_2 then the Variance of RV x_1 and x_2 is $\sigma_{x_1}^2 + \sigma_{x_2}^2$

We know that $\mu_2 = \sigma_x^2 = E\left[(x-m)^2\right]$

$$\sigma_{x_1+x_2}^2 = E\left[\left((x_1+x_2) - E[x_1+x_2]\right)^2\right]$$

$$= E\left[\left((x_1+x_2) - (E[x_1] + E[x_2])\right)^2\right]$$

$$= E\left[\left(x_1+x_2 - (m_{x_1} + m_{x_2})\right)^2\right]$$

$$= E\left[\left((x_1 - m_{x_1}) + (x_2 - m_{x_2})\right)^2\right]$$

$$= E\left[(x_1 - m_{x_1})^2 + (x_2 - m_{x_2})^2 + 2\left[(x_1 - m_{x_1})(x_2 - m_{x_2})\right]\right]$$

$$= E\left[(x_1 - m_{x_1})^2\right] + E\left[(x_2 - m_{x_2})^2\right] + 2E\left[(m_1 - m_{x_1})(m_2 - m_{x_2})\right]$$

$$= \sigma_{x_1}^2 + \sigma_{x_2}^2 + 2E\left[(x_1 - m_{x_1})(x_2 - m_{x_2})\right]$$

$$= \sigma_{x_1}^2 + \sigma_{x_2}^2 + 2E\left[(x_1 - m_{x_1})\right] E\left[(x_2 - m_{x_2})\right]$$

$$= \sigma_{x_1}^2 + \sigma_{x_2}^2 + 2\left(E[x_1] - E(m_{x_1})\right)\left(E[x_2] - E(m_{x_2})\right)$$

$$= \sigma_{x_1}^2 + \sigma_{x_2}^2 + 2\left[(m_{x_1} - m_{x_1})(m_{x_2} - m_{x_2})\right]$$

$$= \sigma_{x_1}^2 + \sigma_{x_2}^2$$

Skew (μ_3) :-

→ The third moment about the mean or the third Central moment is Called as Skew.

→ It is Represented with μ_3.

→ It Can be given as $\mu_3 = E\left[(x-m)^3\right]$

$$= \int_{-\infty}^{\infty} (x-m)^3 f_X(x)\,dx \text{ , if } X \text{ is a Continuous RV.}$$

$$= \sum_i (x_i - m)^3 P(X = x_i) \rightarrow \text{if } X \text{ is a Discreat RV.}$$

→ Relationship b/w m_1, m_2 and m_3, μ_3.

$$\mu_3 = E\left[(x-m)^3\right]$$

$$= E\left[x^3 - m^3 - 3x^2 m + 3xm^2\right]$$

$$= E[x^3] - E[m^3] - E[3x^2 m] + E[3xm^2]$$

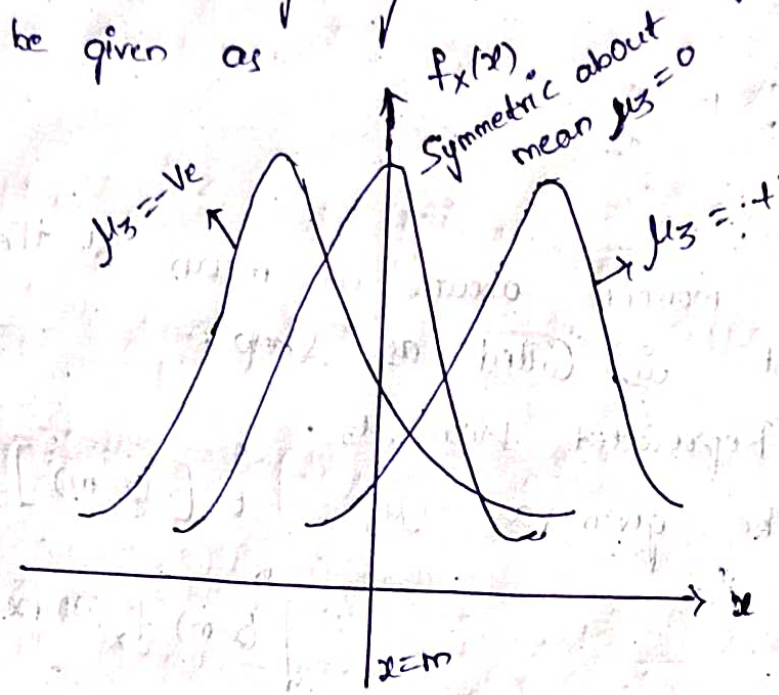
$$= m_3 - m^3 E[1] - 3mE[x^2] + 3m^2 E[x]$$

$$= m_3 - m^3 - 3m_1 m_2 + 3m_1^3$$

$$\mu_3 = m_3 - 3m_1 m_2 + 2m_1^3$$

Physical Significance of μ_3 :-

→ The skew of a RV (X) can be used to find out the amount of symmetry of the density function. This can be given as



characteristic function ($\phi_X(\omega)$) :-

→ The characteristic function is a device or tool. It can be used to generate higher order moments about the origin.

i.e., by using characteristic function we can generate $m_0, m_1, m_2, m_3, \ldots m_n$ and we unable to generate moments about the mean $\mu_0, \mu_1 \ldots \mu_n$ ✗

→ It is Represented with ($\phi_X(\omega)$).

→ It can be given as $(\phi_X(\omega)) = E[e^{j\omega x}]$

$$= \int_{-\infty}^{\infty} e^{j\omega x} f_X(x)\,dx \rightarrow \text{Continuous RV}$$

$$= \sum_i e^{j\omega x_i} \cdot P(X = x_i) \rightarrow \text{Discreat RV}$$

Advantages of $(\phi_x(\omega))$:-

1. The characteristic function always exists.

i.e., $|e^{j\omega x}| = 1$

$$|e^{j\omega x}| = |\cos\omega x + j\sin\omega x|$$

$$= \sqrt{\cos^2\omega x + \sin^2\omega x} = 1$$

2. Always $|\phi_x(\omega)| \leq 1$

We know that $(\phi_x(\omega)) = E[e^{j\omega x}]$

$$= \int_{-\infty}^{\infty} e^{j\omega x} f_X(x)\, dx$$

$$|\phi_x(\omega)| = |E[e^{j\omega x}]|$$

$$= \left|\int_{-\infty}^{\infty} e^{j\omega x} f_X(x)\, dx\right|$$

$$= \int_{-\infty}^{\infty} |e^{j\omega x}|\, |f_X(x)|\, dx$$

$$= \int_{-\infty}^{\infty} f_X(x)\, dx$$

$$= \leq 1$$

3. The characteristic function of a Random Variable (X) and its density function forms fourier transform pairs.

i.e., $\phi_x(\omega) = \int_{-\infty}^{\infty} f_X(x)\, e^{j\omega x}\, dx$

Now $f_X(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \phi_x(\omega)\, e^{-j\omega x}\, dx$

Generation of moments using $\phi_x(\omega)$:-

→ In order to generate higher order moments about the origin using characteristic function.

→ Let us Consider $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

$$\text{Now } e^{j\omega x} = 1 + \frac{(j\omega x)}{1!} + \frac{(j\omega x)^2}{2!} + \cdots$$

Now take Expectation of both sides

$$E\left[e^{j\omega x}\right] = E\left[1 + \frac{j\omega x}{1!} + \frac{(j\omega x)^2}{2!} + \cdots\right]$$

$$\phi_x(\omega) = E\left[1 + \frac{j\omega x}{1!} + \frac{(j\omega x)^2}{2!} + \cdots\right]$$

$$\phi_x(\omega) = 1 + \frac{j\omega}{1!} E[x] + \frac{(j\omega)^2}{2!} E[x^2] + \cdots \rightarrow (1)$$

Generation of m_1 about the origin :-

Now Differentiate Eq (1) w.r.t ω and substitute $\omega = 0$

$$\therefore \frac{d}{d\omega}(\phi_x(\omega)) = \frac{d}{d\omega}\left[1 + \frac{j\omega}{1!} E[x] + \frac{(j\omega)^2}{2!} E[x^2] + \cdots\right]$$

$$= 0 + \frac{j}{1!} E[x] + \frac{j^2 2\omega}{2!} E[x^2] + \cdots$$

$$\frac{d}{d\omega}(\phi_x(\omega))\Big|_{\omega=0} = 0 + \frac{j}{1!} E[x] + 0 + 0 + \cdots$$

$$E[x] = \frac{1}{j} \frac{d}{d\omega}(\phi_x(\omega))\Big|_{\omega=0}$$

$$\boxed{m_1 = \frac{1}{j} \frac{d}{d\omega}(\phi_x(\omega))\Big|_{\omega=0}}$$

Generation of m_2 :-

Now Differentiate Eq ① w.r.t ω two times and put $\omega = 0$.

$$\text{Eq ①} \Rightarrow \frac{d^2}{d\omega^2}(\phi_x(\omega)) = \frac{d}{d\omega}\left[\frac{d}{d\omega}\left(1 + \frac{j\omega}{1!}E[x] + \frac{(j\omega)^2}{2!}E[x^2] \ldots\right)\right]$$

$$\frac{d^2}{d\omega^2}(\phi_x(\omega)) = \frac{d}{d\omega}\left(0 + \frac{j}{1!}E[x] + \frac{j^2 2\omega}{2!}E[x^2] \ldots\right)$$

$$= 0 + 0 + \frac{j^2 \cdot 2}{2!}E[x^2] + \ldots$$

$$= j^2 E[x^2] + \ldots$$

$$E[x^2] = \frac{1}{j^2}\frac{d^2}{d\omega^2}(\phi_x(\omega))\Big|_{\omega=0}$$

$$\boxed{m_2 = \frac{1}{j^2}\frac{d^2}{d\omega^2}(\phi_x(\omega))\Big|_{\omega=0}}$$

Generation of nth moment :

$$\boxed{m_n = \frac{1}{j^n}\frac{d^n}{d\omega^n}(\phi_x(\omega))\Big|_{\omega=0}}$$

Properties of characteristic function :-

→ The characteristic function has unity magnitude at $\omega = 0$

i.e., $\boxed{\phi_x(\omega)\Big|_{\omega=0} = 1}$

Proof :-

We know that $\phi_x(\omega) = E[e^{j\omega x}]$

$$\phi_x(\omega)\Big|_{\omega=0} = E[e^{j\omega x}]\Big|_{\omega=0}$$

$$= E[e^0]$$

$$= E[1]$$

$$= 1$$

→ The magnitudes of $\phi_x(\omega)$ and $\phi_x(-\omega)$ are Complex Conjugates Each Other.

We know that $\phi_x(\omega) = E[e^{j\omega x}]$

$$= \int_{-\infty}^{\infty} e^{j\omega x} f_x(x)\, dx$$

Now $\phi_x(-\omega) = E[e^{-j\omega x}]$

$$= \int_{-\infty}^{\infty} e^{-j\omega x} f_x(x)\, dx$$

$$= E[e^{-j\omega x}]$$

$$= \overline{\phi_x(\omega)}$$

→ $\phi_x(\omega)$ is the characteristic function of the Random variable (X). Then the characteristic function of the RV (Y) is equal to $ax+b$ = $\dfrac{e^{j\omega b}\phi_x(a\omega)}{e^{j\omega b}\phi_{ax}(\omega)}$ (or) =

proof :-

We know that $\phi_x(w) = E[e^{jwx}]$

$$\phi_y(w) = E[e^{jwy}]$$

$$= E[e^{jw(ax+b)}] \qquad y = ax+b$$

$$= E[e^{jwax} \cdot e^{jwb}]$$

$$= e^{jwb} E[e^{jwax}]$$

$$= \boxed{e^{jwb} \phi_x(aw) \text{ (or) } e^{jwb} \phi_{ax}(w)}$$

→ $\phi_x(w)$ is the characteristic function of the Random Variable (X) then the characteristic function of the RV $(Y) = \frac{X+a}{b} = e^{jwa/b} \phi_x\left(\frac{w}{b}\right)$ or $e^{\frac{jwa}{b}} \phi_{\frac{X}{b}}(w)$

proof :-

We know that $\phi_x(w) = E[e^{jwx}]$

$$\phi_y(w) = E[e^{jwy}]$$

$$= E\left[e^{jw\left(\frac{x+a}{b}\right)}\right]$$

$$= E\left[e^{\frac{jwx}{b} + \frac{jwa}{b}}\right]$$

$$= E\left[e^{\frac{jwx}{b}} \cdot e^{\frac{jwa}{b}}\right]$$

$$= e^{\frac{jwa}{b}} E\left[e^{\frac{jwx}{b}}\right]$$

$$= \boxed{e^{\frac{jwa}{b}} \phi_x\left(\frac{w}{b}\right) \text{ (or) } e^{\frac{jwa}{b}} \phi_{\frac{X}{b}}(w)}$$

→ If A is any Real Constant then $\phi_x(aw) = \phi_{ax}(w)$

Proof :-

We know that $\phi_x(w) = E[e^{jwx}]$

$$\phi_x(aw) = E[e^{jawx}]$$

$$= E[e^{j(ax)w}]$$

$$= \phi_{ax}(w)$$

→ The characteristic function and Density function of a Random Variable X forms fourier transform pairs each other.

i.e., $\phi_x(w) = \int_{-\infty}^{\infty} f_x(x)\, e^{jwx}\, dx$

, $f_x(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi_x(w)\, dw$

$$\phi_x(w) \underset{\text{pair}}{\overset{F.T.}{\rightleftharpoons}} f_x(x)$$

→ If two RV X and Y are having Same characteristic functions $(\phi_x(w) = \phi_y(w))$ then they are having identical Distribution and density functions

Transformations of a Random Variable :-

→ Transformations are used to ~~Convert~~ Create a new Random Variables from the Existing Random Variables X

let x is a given RV then the new RV y can be given as $y = T(x)$

where $T \rightarrow$ transformation applied on x produce new RV y.

→ Generally the transformation may be linear, non linear, stair case etc) monotonic, non monotonic

→ $y = T(x)$

→ There are two types of transformations.

i) linear monotonic

ii) non linear non monotic

monotonic (linear) transformation :-

→ In monotonic transformation there is an one to one correspondence b/w the existing RV and the new RV.

i.e., when we apply transformation on existing RV then if we get one output. Then it is known as monotonic.

→ It is divided into two types.

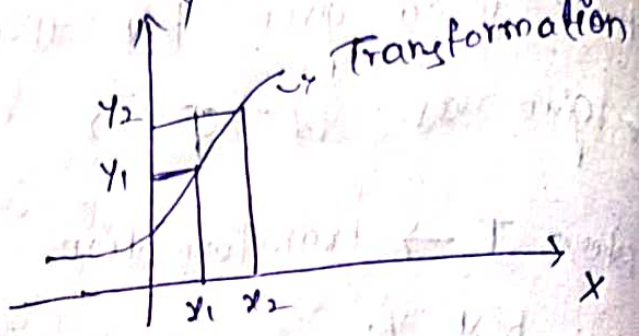
i) monotonically increasing transformations

ii) monotonically Decreasing transformations

→ if $x_1 < x_2$

then $T(x_1) < T(x_2)$

then $y_1 < y_2$

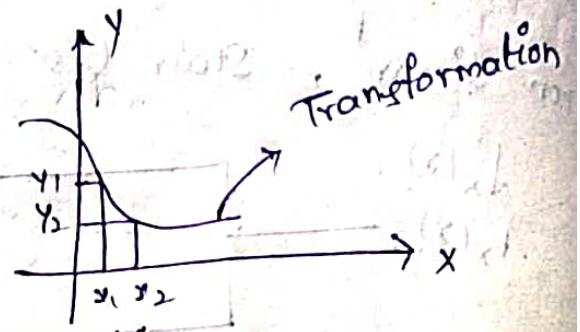


monotonically increased transformation.

→ If $x_1 < x_2$

then $T(x_1) > T(x_2)$

then $y_1 > y_2$



monotonically decreased transformation.

In order to derive monotonic transformations. let us consider $y = T(x)$

$$y_0 = T(x_0)$$

$$\Rightarrow x_0 = T^{-1}(y_0)$$

where T^{-1} is the inverse transformation

let us consider the event

$$\{Y \le y_0\} = \{X \le x_0\}$$

$$P\{Y \le y_0\} = P\{X \le x_0\}$$

$$F_Y(y_0) = F_X(x_0)$$

$$\int_{-\infty}^{y_0} f_Y(y)\,dy = \int_{-\infty}^{x_0} f_X(x)\,dx$$

Differentiating Both sides w.r.t Y.

$$\frac{d}{dy}\left(\int_{-\infty}^{y_0} f_Y(y)\,dy\right) = \frac{d}{dy}\left(\int_{-\infty}^{x_0} f_X(x)\,dx\right)$$

$$f_Y(y_0) = f_X(x_0) \cdot \frac{d}{dy}\left(\int_{-\infty}^{T^{-1}(y_0)} f_X(x)\,dx\right)$$

$$= f_X\left(T^{-1}(y_0)\right) \frac{d}{dy}\left(T^{-1}(y_0)\right)$$

$$= f_X(x_0) \cdot \frac{d}{dy}(x_0)$$

5*

$$\boxed{\therefore\ f_Y(y) = f_X(x) \cdot \left|\frac{dx}{dy}\right|}$$

non monotonic transformations :-

$$\boxed{f_Y(y) = f_X(x_1)\left|\frac{dx_1}{dy}\right| + f_X(x_2)\left|\frac{dx_2}{dy_2}\right| + \cdots}$$

→ find the Mean, Mean Square Value, Variance characteristic function of the uniformly distributed Random Variable.

We know that

for uniformly Distributed Random Variable density function

$$f_X(x) = \begin{cases} \frac{1}{b-a}, & a \le x \le b \\ 0, & \text{elsewhere} \end{cases}$$

(i) mean $m_1 = E[x]$

$$= \int_{-\infty}^{\infty} x f_X(x)\, dx$$

$$= \int_a^b x\left(\frac{1}{b-a}\right) dx = \frac{1}{b-a}\left[\frac{x^2}{2}\right]_a^b$$

$$= \frac{1}{2(b-a)}\left[b^2 - a^2\right] = \frac{b+a}{2}$$

(ii) Mean Square Value $m_2 = E[x^2]$

$$\int_a^b x^2 f_X(x)\, dx = \int_a^b x^2 \cdot \frac{1}{b-a}\, dx = \frac{1}{b-a}\left[\frac{x^3}{3}\right]$$

$$= \frac{1}{3(b-a)}\left[b^3 - a^3\right] = \frac{(b-a)(b^2+a^2+ab)}{3(b-a)} = \frac{b^2+a^2+ab}{3}$$

Variance $= m_2 - m_1^2$

μ_2

$$= \frac{a^2+b^2+ab}{3} - \left(\frac{a+b}{2}\right)^2$$

$$= \frac{a^2+b^2+ab}{3} - \left(\frac{a^2+b^2+2ab}{4}\right)$$

$$= \frac{4a^2+4b^2+4ab-3a^2-3b^2-6ab}{12}$$

$$= \frac{a^2+b^2-2ab}{12} = \frac{(a-b)^2}{12}$$

$$\phi_x(w) = E\left[e^{jwx}\right]$$

$$= \int_{-\infty}^{\infty} e^{jwx} f_x(x)\,dx$$

$$= \int_a^b e^{jwx}\left(\frac{1}{b-a}\right)dx = \frac{1}{b-a}\left.\frac{e^{jwx}}{jw}\right|_a^b = \frac{1}{jw(b-a)}\left[e^{jwb}-e^{jwa}\right]$$

$$= \frac{1}{jw(b-a)}\left[e^{jwb} - e^{jwa}\right]$$

→ find mean, mean square value, variance characteristic function of the Exponential distributed Random Variable.

We know that

for Exponentialy

$$f_x(x) = \begin{cases} \alpha e^{-\alpha x} & 0 \le x \le \infty \\ 0 & \text{else where} \end{cases}$$

i) mean $m_1 = E[x]$

$$= \int_0^\infty x f_X(x)\, dx = \int_0^\infty x \cdot \alpha e^{-\alpha x}\, dx$$

$$x \frac{e^{-\alpha x}}{-\alpha} - \int 1 \cdot \frac{e^{-\alpha x}}{-\alpha}$$

$$x \frac{e^{-\alpha x}}{-\alpha} - \frac{e^{-\alpha x}}{\alpha^2}$$

$$= \alpha \left[x \frac{e^{-\alpha x}}{-\alpha} - \frac{e^{-\alpha x}}{\alpha^2} \right]_0^\infty$$

$$= \alpha \left[\infty - \frac{e^{-\infty}}{\alpha^2} - 0 + \frac{e^0}{\alpha^2} \right]$$

$$= \alpha \cdot \frac{1}{\alpha^2} = \frac{1}{\alpha}$$

ii) mean Square Value $m_2 = E[x^2]$

$$\int_0^\infty x^2 f_X(x)\, dx = \int_0^\infty x^2 \alpha e^{-\alpha x}\, dx$$

$$= \alpha \left[x^2 \cdot \frac{e^{-\alpha x}}{-\alpha} \right]_0^\infty - \int_0^\infty 2x \frac{e^{-\alpha x}}{-\alpha} \Bigg]$$

$$= \frac{2\alpha}{-\alpha} \int_0^\infty x e^{-\alpha x} dx$$

$$= \frac{2}{\alpha^2}$$

iii) Variance $\mu_2 = m_2 - m_1^2$

$$= \frac{2}{\alpha^2} - \frac{1}{\alpha^2} = \frac{1}{\alpha^2}$$

iv) $\phi_X(x) = E[e^{j\omega x}]$

$$= \int_0^\infty e^{j\omega x} f_X(x)\, dx$$

$$= \int_0^\infty e^{j\omega x} \alpha e^{-\alpha x}\, dx$$

$$= \alpha \left[\cancel{x^2} \frac{e^{-\alpha x}}{\cancel{-\alpha}} \right]_0^\infty$$

$$= \alpha \int_0^\infty e^{x(j\omega-\alpha)} dx$$

$$= \alpha \left[\frac{e^{(j\omega-\alpha)x}}{j\omega-\alpha} \right]_0^\infty$$

$$= \frac{-\alpha}{j\omega-\alpha} = \frac{\alpha}{\alpha-j\omega}$$

→ find Mean, Mean Square Value, Variance, characteristic function of the poission's Distributed Discreat Random Variable.

We know that

for a poissions RV (X)

$$P(X=m) = \frac{e^{-\lambda} \cdot \lambda^m}{m!} \; ; \; m = 0, 1, 2, \ldots n$$

i) mean $m_1 = E[x]$

$$= \sum_i x_i \, P(X = x_i)$$

$$= \sum_{x=0}^{n} x \frac{e^{-\lambda} \cdot \lambda^x}{x!} \; ; \; x = 0, 1, 2 \ldots$$

$$= e^{-\lambda} \cdot \lambda + \frac{2e^{-\lambda} \cdot \lambda^2}{2!} + \frac{3e^{-\lambda} \lambda^3}{3!} + \ldots$$

$$= e^{-\lambda} \cdot \lambda + e^{-\lambda} \cdot \lambda^2 + e^{-\lambda} \lambda^3]$$

$$e^{-\lambda} \sum_{x=1}^{n} \frac{\lambda^x \cdot \cancel{x}}{\cancel{x}(x-1)!}$$

$$e^{-\lambda} \sum_{x=1}^{n} \frac{\lambda^x}{(x-1)!}$$

$$= e^{-\lambda}\left[\frac{\lambda^0}{(0-1)!} + \frac{\lambda^1}{(1-1)!} + \frac{\lambda^2}{(2-1)!} + \cdots\right]$$

$$= e^{-\lambda}\left[\frac{\lambda}{1} + \frac{\lambda^2}{1!} + \cdots\right]$$

$$= \lambda e^{-\lambda}\left[1 + \frac{\lambda}{1!} + \frac{\lambda^2}{2!} + \cdots\right]$$

$$= \lambda \cancel{e^{-\lambda}} \cdot \cancel{e^{\lambda}}$$

$$= \lambda$$

ii) Mean Square Value $m_2 = E[x^2]$

$$\sum_i x_i^2 \, P(X = x_i)$$

$$= \sum_{x=0}^{n} \frac{x^2 e^{-\lambda} \cdot \lambda^x}{x!}$$

$$= e^{-\lambda} \sum_{x=0}^{n} \frac{\lambda^x [x(x-1) + x]}{x(x-1)!}$$

$$= e^{-\lambda}\left[\sum_{x=0}^{n} \frac{\lambda^x x(x-1)}{x!} + \sum_{x=0}^{n} \frac{\lambda^x x}{x!}\right]$$

$$= e^{-\lambda}\left[\sum_{x=0}^{n} \frac{\lambda^x \cancel{x(x-1)}}{\cancel{x(x-1)}(x-2)!} + \lambda\right]$$

$$= e^{-\lambda}\left[\sum_{x=2}^{n} \frac{\lambda^x}{(x-2)!}\right] + \lambda$$

$$= e^{-\lambda}\left(\frac{\lambda^2}{0!} + \frac{\lambda^3}{1!} + \frac{\lambda^4}{2!} + \cdots\right) + \lambda$$

$$= e^{-\lambda}\left[\lambda^2\left(1 + \frac{\lambda}{1!} + \frac{\lambda^2}{2!} + \cdots\right)\right] + \lambda$$

$$= e^{-\lambda} \cdot \lambda^2 \cdot e^{\lambda} + \lambda = \lambda^2 + \lambda$$

iii) variance $\sigma_x^2 = m_2 - m_1^2$

$$= \lambda^2 + \lambda - \lambda^2$$

$$= \lambda$$

iv) characteristic function :-

$$\phi_X(\omega) = E[e^{j\omega X}]$$

$$= \sum_x e^{j\omega x} p(X = x)$$

$$= \sum_{x=0}^{n} \frac{e^{j\omega x} \cdot e^{-\lambda} \cdot \lambda^x}{x!}$$

$$= e^{-\lambda} \sum_{x=0}^{n} \frac{e^{j\omega x} \cdot \lambda^x}{x!} \qquad 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots = e^x$$

$$= e^{-\lambda} \sum_{x=0}^{n} \frac{(e^{j\omega} \cdot \lambda)^x}{x!}$$

$$= e^{-\lambda} \left(1 + \frac{\lambda e^{j\omega}}{1!} + \frac{(\lambda e^{j\omega})^2}{2!} + \cdots \right)$$

$$= e^{-\lambda} e^{\lambda e^{j\omega}}$$

→ find Mean, Mean Square Value, Variance, characteristic function of a Binomial Distribution
↓
Discreat

for Binomial RV (x)

$$P(X = m) = \sum_i {}^nC_m p^m q^{n-m} u(x - x_m)$$

↓
unit

$$= \sum_{x=0}^{n} x_i \, {}^nC_m p^m q^{n-m}$$

i) Mean $m_1 = E[x]$

$$= \cancel{\sum_{n=0}^{m}} x \, {}^nC_x \, p^x q^{n-x}$$

$$= {}^nC_m p^m q^{n-m} \sum_{x=0}^{n} x$$

$$= \cancel{\sum_{n=0}}$$

$$= \sum_{x=0}^{n} x \, {}^nC_x \, p^x q^{n-x}$$

$$= \sum_{x=0}^{n} x \frac{n!}{(n-x)!\,x!} p^x q^{n-x}$$

$$= \sum_{x=0}^{n} \cancel{x} \frac{n!}{(n-x)!\,\cancel{x}(x-1)!} p^x q^{n-x}$$

$$= \sum_{x=0}^{n} \frac{n!}{(n-x)!\,(x-1)!} p^x q^{n-x}$$

$$= \sum_{x=0}^{n} \frac{n(n-1)!}{((n-1)-(x-1))!\,(x-1)!} \cdot p \cdot p^{x-1} q^{((n-1)-(x-1))}$$

$$= np \sum_{x=0}^{n} \frac{(n-1)!}{((n-1)-(x-1))!\,(x-1)!} p^{x-1} q^{((n-1)-(x-1))}$$

$$= np \sum_{x=0}^{n} {}^{n-1}C_{x-1} \, p^{x-1} q^{((n-1)-(x-1))}$$

$$= np\,(p+q)^{n-1} \qquad \downarrow (p+q)^{n-1}$$

$$p+q = 1$$

$$= np \qquad 1^{n-1} = (1)$$

(ii) Mean Squared Value $m_2 = E[x^2]$

$$= \sum_{x=0}^{n} x^2 \, {}^nC_x \, p^x q^{n-x}$$

$$= \sum_{x=0}^{n} x^2 \frac{n(n-1)!}{(n-x)!\,x!} p^x q^{n-x}$$

$$\Rightarrow \sum_{x=0}^{n} x \frac{n(n-1)!}{(n-x)!\,(x-1)!} p^x q^{n-x}$$

$$= \sum_{x=0}^{n} \left\{ \frac{(x-1)\, n(n-1)!}{(n-x)!\,(x-1)(x-2)!} p^x q^{n-x} + \frac{n(n-1)}{(n-x)!\,(x-1)!} \cdot p^x q^{n-x} \right\}$$

$$= \sum_{x=0}^{n} \left\{ \frac{n(n-1)(n-2)!}{((n-2)-(x-2))!\,(x-2)!} p^{x-2} \cdot p^2 \cdot q^{(n-2)-(x-2)} + \frac{n(n-1)!}{((n-1)-(x-1))!\,(x-1)!} \cdot p^{x-1} p \cdot q^{(n-1)-(x-1)} \right.$$

$$= n(n-1)p^2 \sum_{x=0}^{n} \frac{(n-2)!}{((n-2)-(x-2))!\,(x-2)!} p^{x-2} q^{(n-2)-(x-2)} + np \frac{(n-1)!}{((n-1)-(x-1))!\,(x-1)!} p^{x-1} q^{(n-1)-(x-1)}$$

$$= n(n-1)p^2 (p+q)^{n-2} + np (p+q)^{n-1}$$

$$= n(n-1)p^2 + np$$

Variance :- $\mu_2 = m_2 - m_1^2$

$$= n(n-1)p^2 + np - (np)^2$$

$$= \cancel{n^2p^2} - np^2 + np - \cancel{(np)^2}$$

$$= np(1-p)$$

→ If x is a Discreat Random Variable. with Probability Distribution table as Shown in figure

$x = x_i$	-2	-1	0	1	2
$P(x = x_i)$	$1/5$	$2/5$	$1/10$	$1/10$	$1/15$

Then find i) mean ii) mean Square Value iii) Variance

iv) $E[(2x+3)]$ v) $E[(2x+1)^2]$

i) mean $= E[x] = \sum_i x_i \, p(x = x_i)$

$$= \cancel{\Sigma} \; -2 \cdot \frac{1}{5} + (-1) \cdot \frac{2}{5} + 0 \cdot \frac{1}{10} + 1 \cdot \frac{1}{10} + 2 \cdot \frac{1}{15}$$

$$= -\frac{2}{5} - \frac{2}{5} + \frac{1}{10} + \frac{2}{15}$$

$$= -\frac{4}{5} + \frac{1}{10} + \frac{2}{15} = \frac{-24+7}{30}$$

$$= \frac{-17}{30}$$

ii) mean Square Value $E[x^2] = \sum_i x_i^2 \, p(x = x_i)$

$$= 4 \cdot \frac{1}{5} + 1 \cdot \frac{2}{5} + 0 \cdot \frac{1}{10} + 1 \cdot \frac{1}{10} + 4 \cdot \frac{1}{15}$$

$$= \frac{6}{5} + \frac{1}{10} + \frac{4}{15} = \frac{36+3+8}{30} = \frac{47}{30}$$

iii) Variance $= m_2 - m_1^2$

$$= \frac{47}{30} - \frac{289}{900} = \frac{1410-289}{900}$$

$$= \frac{1121}{900}$$

iv) $E[2x+3] = 2E[x] + E[3]$

$= 2 \cdot \frac{-17}{30} + 3$

$= \frac{-17}{15} + 3 = \frac{-17+45}{15} = \frac{28}{15}$

v) $E[(2x+1)^2] = E[4x^2 + 4x + 1]$

$= 4E[x^2] + 4[x] + E(1)$

$= 4 \cdot \frac{47}{30} + 4 \cdot \frac{-17}{5} + 1$

$= \frac{188}{30} - \frac{68}{5} + 1$

$\frac{188}{30}$ $\frac{218}{}$ $\quad$ $\frac{408}{218}$ $\frac{-190}{}$

$= \frac{188 - 408 + 30}{30} = \frac{189 - 408}{30} = \frac{-219}{30}$

→ If x is a uniform RV with density

$f_X(x) = \frac{1}{10}, \quad -5 \le x \le 5$ then find i) $E[x]$

$0, \quad$ Else where

ii) $E[x^2]$ iii) variance iv) $E[2x+3]$ v) $E[(2x+1)^2]$

for uniform Distribution with Density function

$$f_X(x) = \begin{cases} \frac{1}{b-a} & a \le x \le b \\ 0, & \text{Else where} \end{cases}$$

i) mean $E[x] = \int_a^b x f_X(x)\, dx$

$= \int_{-5}^{5} x \cdot \frac{1}{10}$

$= \frac{1}{10}\left[\frac{x^2}{2}\right]_{-5}^{5} \Rightarrow \frac{1}{20}\left[25 - 25\right] = 0$

Expected value $E[x^2]$

$$= \int_a^b x^2 f_X(x)\,dx = \int_{-5}^{5} x^2 \frac{1}{10}\,dx = \frac{1}{10}\left[\frac{x^3}{3}\right]_{-5}^{5}$$

$$= \frac{1}{30}\left[125 + 125\right]$$

$$= \frac{250}{30} = \frac{25}{3}$$

iii) Variance $= m_2 - m_1^2 = \frac{25}{3} - 0 = \frac{25}{3}$

iv) $E[2x+3] = 2E[x] + 3$

$= 2 \cdot 0 + 3 = 3$

v) $E[(2x+1)^2] = E[4x^2 + 1 + 4x]$

$= 4E[x^2] + 4E[x] + E[1]$

$= 4 \cdot \frac{25}{3} + 0 + 1 = \frac{100}{3} + 1 = \frac{103}{3}$

→ The probability density function of a R.V. is given as $f_X(x) = b e^{-|ax|}$ where a, b are constants. then find characteristic function, mean and variance.

→ characteristic function $\phi_X(x) = E[e^{jwx}]$

$$= \int_{-\infty}^{\infty} e^{jwx} f_X(x)\,dx$$

$$= \int_{-\infty}^{0} e^{jwx} \cdot b e^{-(-ax)} + \int_{0}^{\infty} e^{jwx}\, b e^{-(ax)}\,dx$$

$$= \int_{-\infty}^{0} e^{j\omega x} \cdot b e^{ax} + \int_{0}^{\infty} e^{j\omega x} \cdot b e^{-ax} \, dx$$

$$= b\left[\int_{-\infty}^{0} e^{j\omega x} \cdot e^{ax} + \int_{0}^{\infty} e^{j\omega x} \cdot e^{-ax} \, dx \right]$$

$$= b\left[\int_{-\infty}^{0} e^{j\omega x + ax} + \int_{0}^{\infty} e^{+j\omega x - ax} \, dx \right]$$

$$= b\left[\int_{-\infty}^{0} e^{x(a+j\omega)} \, dx + \int_{0}^{\infty} e^{-x(a-j\omega)} \, dx \right]$$

$$= b\left[\frac{e^{(a+j\omega)x}}{a+j\omega} \right]_{-\infty}^{0} + b\left[\frac{e^{-x(a-j\omega)}}{-(a-j\omega)} \, dx \right]_{0}^{\infty}$$

$$= b\left[\frac{1-0}{a+j\omega} + 0 + \frac{1}{-(a-j\omega)} \right]$$

$$= \left[\frac{2ab}{a^2-(j\omega)^2} \right] = \frac{-2ab}{a^2+\omega^2}$$

mean $m_1 = \frac{1}{j} \cdot \frac{d}{d\omega} \left(\phi_x(\omega) \right) \Big|_{\omega=0}$

$$= \frac{1}{j} \cdot \frac{d}{d\omega} \left(\frac{2ab}{a^2+\omega^2} \right) \Big|_{\omega=0}$$

$$= \frac{1}{j} \, 2ab \left(\frac{-1}{(a^2+\omega^2)^2} \cdot 2\omega \right) \Big|_{\omega=0}$$

$$= \frac{1}{j} \, 2ab \left(\frac{-1}{a^2+0^2} \right) 2(0)$$

$$= 0$$

$$m_2 = \frac{1}{j^2}\frac{d^2}{d\omega^2}\phi_X(\omega)\Big|_{\omega=0}$$

$$= \frac{1}{j^2}\frac{d^2}{d\omega^2}\left(\frac{2ab}{a^2+\omega^2}\right)\Big|_{\omega=0}$$

$$= \frac{1}{j^2}\,2ab\,\frac{d}{d\omega}\left(-\frac{1}{(a^2+\omega^2)^2}\cdot 2\omega\right)\Big|_{\omega=0}$$

$$= \frac{2ab}{j^2}\left(\frac{2}{(a^2+\omega^2)^3}\cdot 4\omega^2\right)\Big|_{\omega=0}$$

$$= \frac{-4ab}{j^2}\frac{d}{d\omega}\left(\frac{\omega}{(a^2+\omega^2)^2}\right)\Big|_{\omega=0}$$

$$= -\frac{4ab}{j^2}\left[\frac{(a^2+\omega^2)^2 + \omega\cdot(2(a^2+\omega^2)\cdot 2\omega}{((a^2+\omega^2)^2)^2}\right]\Big|_{\omega=0}$$

$$= \frac{-4ab}{j^2}\left[\frac{a^4+0}{a^8}\right] \Rightarrow \frac{-4ab}{j^2}\left[\frac{1}{a^4}\right]$$

$$= \frac{-4b}{a^3(-1)} = \frac{4b}{a^3}$$

Variance $= m_2 - m_1^2 = \frac{4b}{a^3}$

→ A RV X has $f_X(x) = \begin{cases} \frac{1}{2}\cdot\cos x, & -\frac{\pi}{2} < x < \frac{\pi}{2} \\ 0, & \text{Else where} \end{cases}$

then find the mean of the function $g[x] = 4x^2$.

$$f_X(x) = \begin{cases} \frac{1}{2}\cos x, & -\frac{\pi}{2} < x < \frac{\pi}{2} \\ 0, & \text{Else where} \end{cases}$$

We know that $E[g(x)] = \int_{-\infty}^{\infty} g(x)\, f_X(x)\,dx$

$$= \int_{-\pi/2}^{\pi/2} 4x^2 \cdot \frac{1}{2} \cos x \, dx$$

$$= 2 \left[\int_{-\pi/2}^{\pi/2} \underset{u}{x^2} \underset{v}{\cos x} \, dx \right]$$

$$2 \left[(x^2 \cdot \sin x)_{-\pi/2}^{\pi/2} - \right.$$

$$= 2 \left[x^2 \cdot \sin x - 2x \cdot -\cos x + 2(-\sin x) \right]_{-\pi/2}^{\pi/2}$$

$$= 2 \left[x^2 \sin x + 2x \cos x - 2 \sin x \right]_{-\pi/2}^{\pi/2}$$

$$= 2 \left[\frac{\pi^2}{4} - 2 + \frac{\pi^2}{4} - 2 - \frac{\pi^2}{4} + 2 \right] \qquad \pi^2 - 8$$

$$= 2 \left[\frac{\pi^2}{2} - 4 \right] = \pi^2 - 8$$

→ Consider RV X. $E[x] = 5$ $\sigma_{x^2} = 3$.

$Y = -8x + 10$

i) $E[x^2]$

ii) $E(xY)$

iii) $E[Y^2]$

iv) σ_Y

Given that $E[x] = 5$; $\overline{x^2} = 3$, $Y = 8x + 10$

$\therefore \boxed{m_1 = 5}$, $\boxed{\mu_2 = 3}$

$\mu_2 = m_2 - m_1^2 \Rightarrow 3 = m_2 - 25$

$\boxed{m_2 = 28}$

i) $E[x^2] = m_2 = 28$

ii) $E[xy] = E[x(-8x+10)]$

$= E[-8x^2 + 10x]$

$= -8\,E[x^2] + 10\,E[x]$

$= -8(28) + 10(5) = -224 + 50 = -174$

3) $E[y^2] = E[(-8x+10)^2]$

$= E[64x^2 + 100 - 160x]$

$= 64 m_2 + 100 - 160(5) = 64(28) + 100 - 160 \times 5$

$= 1792 + 100 - 800 = 1092$

4) $\overline{y^2} = E[y^2] - (E[y])^2$

We find $E[y] = E(-8x+10)$

$= -8 \times 5 + 10$

$= -40 + 10 = 30$

$\therefore \overline{y^2} = 1092 - (-30)^2$

$= 1092 - 900 = 192$

→ The density function of a RV X. $f_X(x)$

$$= \begin{cases} 0.3507\sqrt{x} & 0 < x < 3 \\ 0 & \text{Elsewhere} \end{cases}$$

then find mean, mean Square value and variance.

$$f_X(x) = \begin{cases} 0.3507\sqrt{x} & 0 < x < 3 \\ 0 & \text{Else where} \end{cases}$$

mean $m_1 = \int_{-\infty}^{\infty} x f_X(x)\, dx = \int_0^3 x\, 0.3507\sqrt{x}\, dx$

$$= 0.3507 \int_0^3 x^{3/2}\, dx \Rightarrow 0.30507 \left[\frac{x^{3/2+1}}{\frac{3}{2}+1} \right]_0^3$$

$$= 2.19$$

mean square value $m_2 = E[x^2]$

$$= \int_{-\infty}^{\infty} x^2 f_X(x)\, dx$$

$$= \int_0^3 x^2 \cdot 0.3507\sqrt{x}\, dx \Rightarrow 0.3507 \int_0^3 x^{\frac{5}{2}}\, dx$$

$$= 0.3507 \times \left. \frac{x^{\frac{7}{2}}}{\frac{7}{2}} \right|_0^3 \Rightarrow \frac{0.6114}{7} \left[3^{\frac{7}{2}} - 0 \right]$$

$$\Rightarrow 0.102\,(46.77)$$

$$= 4.685$$

Variance $= m_2 - m_1^2$

$$= 4.685 - 4.796$$

$$= \cancel{-0.0} \; -0.06$$

→ Find the characteristic function of a RV X, $f_X(x) = ae^{-bx}$, $b > 0$, find m_1, m_2.

$$\phi_X(\omega) = E[e^{j\omega x}]$$

$$= \int_{-\infty}^{\infty} e^{j\omega x} f_X(x)\,dx = \int_{0}^{\infty} e^{j\omega x} \cdot e^{-bx} \cdot a\,dx$$

$$= a\int_{0}^{\infty} e^{-x(b-j\omega)}\,dx \Rightarrow a \left.\frac{e^{-x(b-j\omega)}}{-(b-j\omega)}\right|_{0}^{\infty}$$

$$\Rightarrow \frac{-a}{b-j\omega}\left(e^{-x(b-j\omega)}\right)\Big|_{0}^{\infty}$$

$$\Rightarrow \frac{-a}{b-j\omega}(0-1) \Rightarrow \frac{a}{b-j\omega}$$

$\therefore$ We have $m_n = \left(\frac{1}{j}\right)^n \cdot \frac{d^n}{d\omega^n}\left(\phi_X(\omega)\right)\Big|_{\omega=0}$

$$m_1 = \frac{1}{j} \cdot \frac{d}{d\omega}\left(\phi_X(\omega)\right)\Big|_{\omega=0}$$

$$= \frac{1}{j}\frac{d}{d\omega}\left(\frac{a}{b-j\omega}\right)\Big|_{\omega=0}$$

$$\Rightarrow \frac{a}{j}\left[\frac{-1}{(b-j\omega)^2} \cdot -j\right]\Big|_{\omega=0}$$

$$\Rightarrow a\left[\frac{1}{b^2-0)}\right] = \frac{a}{b^2}$$

$$m_2 = \frac{1}{j^2} \cdot \frac{d^2}{d\omega^2}\left(\phi_X(\omega)\right)\Big|_{\omega=0}$$

$$= \frac{1}{j^2}\frac{d^2}{d\omega^2}\left(\frac{a}{b-j\omega}\right)\Big|_{\omega=0}$$

$$= \frac{a}{j^2} \frac{d}{d\omega}\left(\frac{j}{(b-j\omega)^2}\right)\Big|_{\omega=0}$$

x^{-2}
$-2x^{-3}$
$\frac{-2}{x^3}$

$$= \frac{a}{j} \frac{d}{d\omega}\left(\frac{1}{(b-j\omega)^2}\right)\Big|_{\omega=0}$$

$$= \frac{a}{j} \cdot \left(\frac{-2}{(b-j\omega)^3} \cdot -j\right)\Big|_{\omega=0}$$

$$\Rightarrow \quad 2\left[\frac{1}{(b-0)^3}\right] = \frac{2a}{b^3}$$

→ find the density function of a RV 'x' if its characteristic function is $\phi_x(\omega) = \begin{cases} 1-|\omega|, & |\omega| \le 1 \\ 0, & \text{elsewhere} \end{cases}$

$$\phi_x(\omega) = \begin{cases} 1-|\omega|, & |\omega| \le 1 \\ 0 & \text{Else where} \end{cases}$$

$$f_x(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \phi_x(\omega) \cdot e^{-j\omega x}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-1}^{1} 1-|\omega| \cdot e^{-j\omega x}\, d\omega$$

$$= \frac{1}{2\pi}\left\{\int_{-1}^{0} (1+\omega) e^{-j\omega x}\, d\omega + \int_{0}^{1} (1-\omega)\, e^{-j\omega x}\, d\omega\right\}$$

$$= \frac{1}{2\pi}\left\{\left((1+\omega)\cdot\frac{e^{-j\omega x}}{-jx} - 1\cdot\frac{e^{-j\omega x}}{(-jx)^2}\right) + \left((1-\omega)\cdot\frac{e^{-j\omega x}}{-jx} - (-1)\frac{e^{-j\omega x}}{(-jx)^2}\right)\right\}$$

$$= \frac{1}{2\pi}\left\{1\cdot\frac{1}{-jx} - \frac{1}{(-jx)^2} + \frac{e^{jx}}{(-jx)^2} + \frac{e^{-jx}}{(-jx)^2} - 1\cdot\frac{1}{-jx} - \frac{1}{(-jx)^2}\right\}$$

$$= \frac{1}{2\pi}\left\{\cancel{\frac{1}{-jx}} - \frac{1}{(jx)^2} + \frac{e^{jx}}{(jx)^2} + \frac{e^{-jx}}{(jx)^2} + \cancel{\frac{1}{jx}} - \frac{1}{(jx)^2}\right\}$$

$$= \frac{1}{2\pi}\left\{ \frac{1}{x^2} + \frac{e^{jx}}{x^2} + \frac{e^{-jx}}{x^2} + \frac{1}{x^2} \right\}$$

$$\frac{1}{\pi x^2}(1-\cos x)$$

$$= \frac{1}{2\pi}\left\{ \frac{2}{x^2} + \frac{e^{jx}}{x^2} + \frac{e^{-jx}}{x^2} \right\}$$

$$\frac{\cos\theta + i\sin\theta}{2x^2} + \frac{\cos\theta - i\sin\theta}{2x^2}$$

$$= \frac{1}{2\pi}\left\{ \frac{2(1-\cos x)}{x^2} \right\}$$

$$\frac{2 + 2\cos\theta}{2x^2}$$

$$= \frac{1}{\pi x^2}(1-\cos x)$$

$$\frac{(1-\cos\theta)}{x^2}$$

→ S.T the density function for which the characteristic function is $e^{-|\omega|}$ has the density function $f_X(x) = \frac{1}{\pi(1+x^2)}$.

Sol: $\phi_X(\omega) = e^{-|\omega|}$

$$f_X(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \phi_X(\omega)\cdot e^{-j\omega x}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-|\omega|}\cdot e^{-j\omega x}\, d\omega$$

$$= \frac{1}{2\pi}\left\{ \int_{-\infty}^{0} e^{\omega}\cdot e^{-j\omega x}\, d\omega + \int_{0}^{\infty} e^{-\omega}\cdot e^{-j\omega x}\, d\omega \right\}$$

$$= \frac{1}{2\pi}\left\{ \int_{-\infty}^{0} e^{-\omega(jx-1)}\, d\omega + \int_{0}^{\infty} e^{-\omega(jx+1)}\, d\omega \right\}$$

$$= \frac{1}{2\pi}\left\{ \frac{e^{-(jx-1)\omega}}{-(jx-1)}\Bigg|_{-\infty}^{0} + \frac{e^{-(jx+1)\omega}}{-(jx+1)}\Bigg|_{0}^{\infty} \right\}$$

$$= \frac{1}{2\pi}\left\{ \frac{1}{+(jx-1)} - 0 + 0 + \frac{1}{-(jx+1)} \right\}$$

$$= \frac{1}{2\pi}\left\{ \frac{1}{-x+1} + \frac{1}{-x-1} \right\}$$

$$= \frac{1}{2\pi}\left\{ \frac{x-1+x+1}{x^2-1} \right\} = \frac{1}{2\pi}\left\{ \frac{-x-1-x-1}{(-x+1)(-x-1)} \right\}$$

$$= \frac{1}{2\pi}\left\{ \qquad = \frac{1}{2\pi}\left\{ \frac{}{1-x^2} \right] \right.$$

$$= \frac{1}{2\pi}\left\{ \frac{e^{-(jx-1)\omega}}{1-jx}\Bigg|_{-\infty}^{0} + \frac{e^{-(1+jx)\omega}}{-(1+jx)}\Bigg|_{0}^{\infty} \right\}$$

$$= \frac{1}{2\pi}\left\{ \frac{1}{1-jx} - 0 + 0 + \frac{1}{1+jx} \right\}$$

$$= \frac{1}{2\pi}\left\{ \frac{1+jx+1-jx}{1-j^2x^2} \right\} \Rightarrow \frac{\cancel{2}}{\cancel{2}\pi(1+x^2)} = \frac{1}{\pi(1+x^2)}$$

Hence proved.

→ If $f_X(x) = \frac{1}{\sqrt{2\pi}} \cdot e^{-x^2/2}$ then find the density function of a Random variable

$Y = \frac{x^2}{9}$

Given that $f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$ and

$$Y = \frac{x^2}{9}$$

$f_Y(y) = ?$

$$f_Y(y) = f_X(x) \cdot \left| \frac{dx}{dy} \right|$$

According to transformations of a Random Variable

Now Consider $y = \frac{x^2}{9} \Rightarrow x = 3y^{1/2}$

$$\frac{dx}{dy} = 3\,\frac{1}{y^{\frac{1}{2}}} = \frac{3/2}{\sqrt{y}} \Rightarrow \frac{3}{2\sqrt{y}}$$

$$\therefore f_Y(y) = \frac{1}{\sqrt{2\pi}} \cdot e^{-x^2/2} \cdot \left|\frac{3}{2\sqrt{y}}\right|$$

$$= \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{9y}{2}} \left|\frac{3}{2\sqrt{y}}\right|$$

$$= \frac{3}{2\sqrt{2\pi}} \frac{e^{-9y/2}}{\sqrt{y}}$$

→ The density of a RV x is given by $f_X(x) = \frac{x}{20}$ $2 \le x \le 5$ then find the density function of a RV $y = 3x - 5$

According to transformation of a Random Variable

$$f_Y(y) = f_X(x) \cdot \left|\frac{dx}{dy}\right|$$

$$y = 3x - 5 \Rightarrow x = \frac{y+5}{3} \Rightarrow \frac{dx}{dy} = \frac{1}{3}$$

$$f_Y(y) = \frac{x}{20} \times \frac{1}{3}$$

$$2 \le \frac{y+5}{\cancel{180}\ 5} \le 5$$

$$6 \le y + 5 \le 15$$

$$1 \le y \le 10$$

→ A guassian RV x $\sigma_x^2 = 10$. and $m = 5$ is transformed to another random variable 'y' $= e^x$ then find density function of RV 'y'?

We have

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \cdot e^{-\frac{(x-m)^2}{2\sigma^2}}$$

$$= \frac{1}{\sqrt{20\pi}} \cdot e^{-\frac{(x-5)^2}{20}}$$

$$\boxed{f_Y(y) = f_X(x) \left| \frac{dx}{dy} \right|}$$

Now $y = e^x$, $x = \log y$

$$\left| \frac{dx}{dy} \right| = \left| \frac{1}{y} \right|$$

$$\therefore f_Y(y) = \frac{1}{y\sqrt{20\pi}} \cdot e^{-\frac{(\log y - 5)^2}{20}}, \quad y \geq 0$$

→ A RV θ is uniformly Distributed in the Interval (θ_1, θ_2). Such that $\theta_1 < \theta < \theta_2$ then find density function of the RV $Y = \cos\theta$

$$f_X(x) = \begin{cases} \frac{1}{b-a} & a \leq x \leq b \\ 0 & \text{Else where} \end{cases}$$

$$f_X(x) = \begin{cases} \frac{1}{\theta_2 - \theta_1} & \theta_1 \leq \theta \leq \theta_2 \\ 0 & \text{Else where} \end{cases}$$

Now we know that $f_Y(y) = f_X(x) + \left| \frac{dx}{dy} \right|$

$y = \cos\theta \qquad \theta = \cos^{-1} y$

$$\left| \frac{d\theta}{dy} \right| = \left| \frac{-1}{\sqrt{1-y^2}} \right|$$

$f_Y(y) = \frac{1}{\sqrt{1-y^2}}$

→ If $f_X(x) = \frac{1}{\sqrt{2\pi}} \cdot e^{-x^2/2}$ then find the RV 'X'

→ If X is a uniformly distributed RV in the interval (0,6) and it is transformed with $Y = 2(x-3)^2 - 4$ then find the density function of Y.

$$f_X(x) = \begin{cases} \frac{1}{6} & 0 \le x \le 6 \\ 0 & \text{else where} \end{cases}$$

According to transformation

$$f_Y(y) = f_X(x) \cdot \left| \frac{dx}{dy} \right|$$

Here $y = 2(x-3)^2 - 4 \Rightarrow \frac{y+4}{2} = (x-3)^2$

$$x = \pm \sqrt{\frac{y+4}{2}} + 3$$

$$\frac{dx}{dy} = \pm \frac{1}{\sqrt{2}} \cdot (y+4)^{\frac{1}{2}-1} \cdot \frac{1}{2} + 0$$

$$= \left| \pm \frac{1}{2\sqrt{2}} \frac{1}{\sqrt{y+4}} \right|$$

$$f_Y(y) = f_X(x) \left| \frac{dx_1}{dy} \right| + f_X(x) \left| + \frac{dx_2}{dy} \right|$$

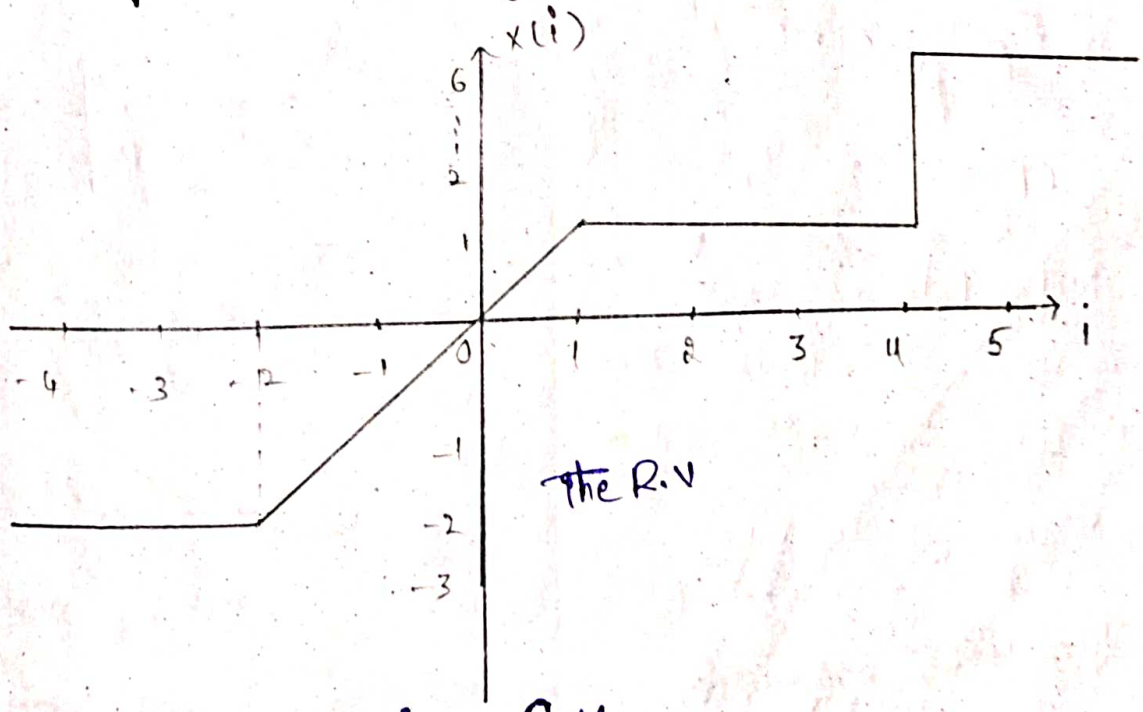
$$f_Y(y) = \frac{1}{6}\left|\frac{1}{2\sqrt{2}}\frac{1}{\sqrt{y+4}}\right| + \frac{1}{6}\left|\frac{-1}{2\sqrt{2}}\frac{1}{\sqrt{y+4}}\right|$$

$$= \frac{1}{12\sqrt{2}}\frac{1}{\sqrt{y+4}} + \frac{1}{12\sqrt{2}}\frac{1}{\sqrt{y+4}}$$

$$= \frac{1}{6\sqrt{2}}\frac{1}{\sqrt{y+4}}$$

* The RV X is defined by $X(i) = \begin{cases} -2, & i \le -2 \\ i, & -2 \le i < 1 \\ 6, & 1 \le i \le 4 \\ 6 & i > 4 \end{cases}$

Show by a sketch the values of X in which the values of i are mapped by X. what type of RV is X? (march 2021)



The RV involved is Continuous.

→ Given that the RV X has the following possible values State if X is a discreat, Continuous and mixed

i) $\{-20 < x < -5\}$

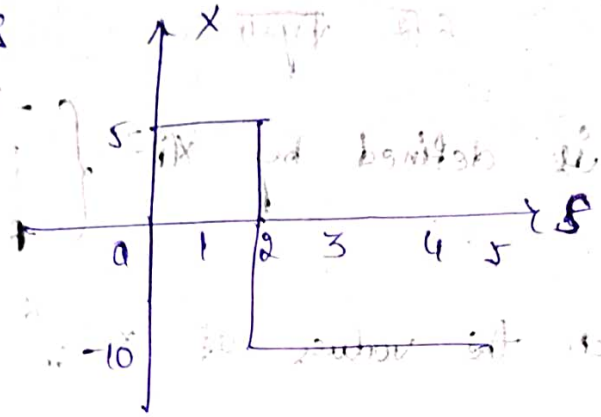
ii) $(10, 12 < x < 5 \quad 14, 15, 17)$

iii) $\{-10$ for $s > 2$, 5 for $s < 2$ where $1 \le s \le 6$

iv) $\{4, 3+1, -2\}$

i) Continuous ii) Discreat ii) mixed.

iii) Continuous



→ find the mean and variance of a Gaussian RV

→ March, 2021 Set-2 (7m)

For a quassian RV $a_x = 0$ (mean) $\sigma_x = 1$ then ST

$$\int_{-\infty}^{\infty} x f_x(x)\, dx = a_x$$

We know that for a quassian RV

$$f_x(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \cdot e^{-\frac{(x-m)^2}{2\sigma^2}}$$

Here $a_x = 0$ $\sigma_x = 1$

$$f_x(x) = \frac{1}{\sqrt{2\pi}} \cdot e^{-x^2/2}$$

Consider L.H.S $\int_{-\infty}^{\infty} x f_x(x)\, dx = \int_{-\infty}^{\infty} x \cdot \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \underset{u}{x} \cdot \underset{v}{e^{-x^2/2}}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \left(x \cdot \frac{e^{-x^2/2}}{-\frac{2x}{2}} - 1. \right) \quad \left(\frac{1}{\sqrt{2\pi}} \left[x \cdot \frac{e^{-x^2/2}}{-\frac{2x}{2}} \Big|_{-\infty}^{\infty} - \int 1 \cdot \frac{e^{-x^2/2}}{-\frac{2x}{2}} \right] \right)$$

$$= 0 = a_x$$

Hence proved.

→ The RV X is uniformly Distributed Interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and X is transformed to a new RV Y $= T(x) = a \tan x$. where $a > 0$. then find the probability density function of Y?

We know that for uniformly Distributed RV

$$f_X(x) = \begin{cases} \frac{1}{b-a} & a \le x \le b \quad -\frac{\pi}{2} \le x \le \frac{\pi}{2} \\ 0 & \text{Else where} \end{cases}$$

Given that $y = a \tan x$

$$\frac{y}{a} = \tan x$$

$$x = \tan^{-1}\left(\frac{y}{a}\right)$$

$$\frac{dx}{dy} = \frac{1}{1+\left(\frac{y}{a}\right)^2} \cdot \left(\frac{1}{a}\right)$$

$$= \frac{a}{a^2+y^2}$$

$$f_X(y) = f_X(x)\left|\frac{dx}{dy}\right| = \frac{1}{b-a}\left|\frac{a}{a^2+y^2}\right|$$

$$= \frac{1}{\pi}\left|\frac{a}{a^2+y^2}\right|$$

→ A RV X Can have values $-4, -1, 2, 3, 4$ Each probability of 1/5 then find the Density function mean Variance of the RV $Y = 3x^2$.

$x = \{-4, -1, 2, 3, 4\}$

x_i	-4	-1	2	3	4
$P(X=x_i)$	1/5	1/5	1/5	1/5	1/5

$$f_X(x) = \sum_i P(X=x_i)\,\delta(x-x_i)$$

$$= \frac{1}{5}\delta(x+4) + \frac{1}{5}\delta(x+1) + \frac{1}{5}\delta(x-2) + \frac{1}{5}\delta(x$$

$$+ \frac{1}{5}\delta(x-4)$$

$m_1 = \sum_i x_i p(x - x_i)$

$m_2 = \sum_i x_i^2 \, (p(x = x))$

$\sigma_x^2 = \mu_x = m_2 - m_1$

→ A RV x has $\bar{x} = 3$, $\overline{x^2} = 11$, $\sigma_x^2 = 2$. for a new RV $Y = 2x - 3$. then find i) $\bar{Y}$ ii) $\overline{Y^2}$ iii) σ_y^2

Given that $\bar{x} = -3$, $\overline{x^2} = 11$, $\sigma_x^2 = 2$ $Y = 2x - 3$

$m_1 = 3$ $m_2 = 11$

i) $E[x] = -3$ $E[x^2] = 11$

$E[Y] = 2E[x] - 3 = 2(-3) - 3 = -9$

ii) $\overline{Y^2} = (2x-3)^2 = 4x^2 + 9 - 12x$

$= 4 \times 11 + 9 - 12 \times 3 = 44 + 9 + 36$

$= 53 + 36 = 89$

iii) $\sigma_y^2 = m_2 - m_1^2 = 89 - 81 = 8$

→ find the Expected value of the function $g[x] = x^3$ where x is a rv in a Density function

$f_x(x) = \frac{1}{2} u(x) \exp(-x/2)$

$g[x] = x^3$

$f_x(x) = \frac{1}{2} u(x) e^{-x/2}$

$E[g(x)] = \int_{-\infty}^{\infty} x^3 \cdot \frac{1}{2} e^{-x/2}$

$$= \frac{1}{2}\int_{-\infty}^{\infty} x^3 \cdot e^{-x/2} = \frac{1}{2}\int_{0}^{\infty} x^3 \cdot e^{-x/2}$$

$$\frac{1}{2}\left[x^3 \cdot \frac{e^{-x/2}}{-\frac{1}{2}} - 3x^2 \cdot \frac{e^{-x/2}}{\frac{1}{4}} + 6x \cdot \frac{e^{-x/2}}{-\frac{1}{8}} - 6 \cdot \frac{e^{-x/2}}{\frac{1}{16}} \right]_0^{\infty}$$

$$\frac{1}{2}\left[0 - 0 - 0 - 0 + \frac{6}{\frac{1}{16}} \right]$$

$$= \frac{1}{\cancel{2}} \times 16 \times \overset{3}{\cancel{6}} = 48$$

→ 2018 may set-1

2m

10 unbaised throw simultaneously. find the probability of getting atleast 7 heads.

$$P(X \geq 7) = 1 - P(X \leq 7)$$

$$P(X = m) = {}^nC_m p^m q^{n-m}$$

$n = 10$, $p = \frac{1}{2}$, $q = \frac{1}{2}$, $m = 7$

$$= 1 - {}^{10}C_7 \left(\frac{1}{2}\right)^7 \cdot \left(\frac{1}{2}\right)^3$$

$$= 1 - {}^{10}C_3 \left(\frac{1}{2}\right)^{10}$$

$$= 1 - \frac{10!}{7!\,3!} \cdot \frac{1}{2^{10}} \Rightarrow 1 - \frac{10 \times 9 \times 8 \times \cancel{7!}}{\cancel{7!} \times 3 \times 2 \times 1}$$

→ A RV has the probability function Density

$$f_X(x) = \begin{cases} \frac{5}{4}(1 - x^4), & 0 \leq x \leq 1 \\ 0, & \text{Else where} \end{cases}$$

i) $\in [ux + 12]$

ii) $\in [x^2]$

$$f_X(x) = \frac{5}{4}(1 - x^4)$$

$$\int_0^1 x f(x)\,dx$$

$$\int_0^1 x \cdot \frac{5}{4}(1-x^4)$$

$$\frac{5}{4}\int_0^1 x(1-x^4)\,dx$$

$$= \frac{5}{4}\int_0^1 (x-x^5)\,dx$$

$$= \frac{5}{4}\left[\frac{x^2}{2} - \frac{x^6}{6}\right]_0^1$$

$$\frac{5}{4}\left[\frac{1}{2} - \frac{1}{6}\right]$$

$$= \frac{5}{4}\left[\frac{3-1}{6}\right]$$

$$= \frac{5}{4}\left[\frac{2}{6}\right] = \frac{5}{12}$$

i) $E[ax+b]$

$$4 \cdot \frac{5}{12} + 12 = \frac{5}{3} + 12 = \frac{5+36}{3}$$

$$= \frac{11}{3} = \frac{41}{3}$$

ii) $E[x^2] = \int_0^1 x^2 f_X(x)\,dx$

$$= \frac{5}{4}\int_0^1 (x^2 - x^6)\,dx$$

$$= \frac{5}{4}\left[\frac{x^3}{3} - \frac{x^7}{7}\right]$$

$$= \frac{5}{21}$$

→ Given that If y is a uniformly Distributed RV with $(0,1)$ then $f_Y(y)$ if $y = x^3$

x^n

nx^{n-1}

$$f_X(x) = \begin{cases} \frac{1}{b-a} & a \le x \le b \\ 0 & \text{Else where} \end{cases}$$

$$y = x^3 \Rightarrow x = (y)^{1/3}$$

$$\frac{dx}{dy} = \frac{1}{3} y^{\frac{1}{3}-1} = \frac{1}{3} y^{\frac{1-3}{3}} = \frac{1}{3} y^{-\frac{2}{3}}$$

$$= \frac{1}{3(y)^{2/3}}$$

$$f_Y(y) = f_X(x) \left| \frac{dx}{dy} \right|$$

$$= \frac{1}{1-0} \left| \frac{1}{3(y)^{2/3}} \right| = \frac{1}{3y^{2/3}} \quad 0 < y < 1$$

$$\therefore f_Y(y) = \begin{cases} \frac{1}{3y^{2/3}} & 0 \le y \le 1 \\ & \text{Else where} \end{cases}$$

→ A number of Cars arriving at a Certain Bank During any 10 min period is a poissions RV with $b = 2$. Find the probability that more than three Cars will arrive during any 10 min period.

Given that X is a possions RV

$$F_X(x = m) = \frac{e^{-\lambda} \cdot \lambda^m}{m!}$$

$\lambda = n \times p$

$2 = np$

$1 - P(X \leq 3)$

$$P(X > 3) = 1 - \frac{e^{-2} \cdot (2)^3}{3!} = \frac{e^{-2} \cdot 8^4}{6\,3} = [0.13 \times 1.3]$$

$$= 0.18$$

$$P(X > 3) = 1 - P(X \leq 3)$$

$$= 1 - F_X(3)$$

$$= 1 - \sum_{m=0}^{3} \frac{e^{-\lambda} \cdot \lambda^m}{m!}$$

$$= 1 - \left(\frac{e^{-2} \cdot 2^0}{0!} + \frac{e^{-2} \cdot 2^1}{1!} + \frac{e^{-2} \cdot 2^2}{2!} + \frac{e^{-2} \cdot 2^3}{3!} \right)$$

$$= 1 - \left(\frac{1}{e^2} + \frac{2}{e^2} + \frac{4}{2e^2} + \frac{8}{3e^2} \right)$$

→ find the mean of the Rayleigh RV.

→ Gaussian Random voltages 'x' for which $\sigma_x = 0$ $\bar{x} = 4.2V$ appears across a 100Ω resistor with power rating of 0.25W. what is the probability that the instantaneous voltage will cause an power that exceeds resistor Rating.

mean = 0

Variance $= (4.2)^2 = 17.64$

$$P(x > 0.25) = 1 - P(x \leq 0.25)$$

$$= 1 - F_X(0.25)$$

$$= 1 - \left(1 - \varphi\left(\frac{x-m}{\sigma}\right)\right)$$

$$= 1 - \left(1 - \varphi\left(\frac{0.25 - 0}{4.2}\right)\right)$$

$$= \varphi\left(\frac{0.25}{4.2}\right) = \varphi(0.059)$$

→ A Guassian RV X has $\sigma_x = 2$ $\bar{x} = 2$ then find (i) $P(X > 1.0)$ (b) $P(X \leq -1.0)$

(i) $P(X > 1.0) = 1 - P(X \leq 1.0) = 1 - F_X(1.0)$

$$= 1 - 1 + \varphi\left(\frac{x-m}{\sigma}\right) = 1 - 1 + \varphi\left(\frac{1.0 - 2}{2}\right)$$

$= \varphi\left(\frac{-1}{2}\right)$

$= \varphi(-0.5)$

ii) $P(X \leq -1.0) = \cancel{F(X \leq}\ F_X(-1.0)$

$= 1 - \varphi\left(\frac{-1.0 - 2}{2}\right)$

$= 1 - \varphi\left(\frac{-3}{2}\right)$

$= 1 - \varphi(-1.5)$

Multiple Random Variables

PART A :-

→ Joint Distribution function. ($F_{XY}(x,y)$)

→ properties of joint Distribution function.

→ joint Density function ($f_{XY}(x,y)$)

→ properties of joint Density function

→ Marginal Distribution functions ($F_X(x)$, $F_Y(y)$)

→ Marginal Density functions ($f_X(x)$, $f_Y(y)$)

→ Statistically independent Random Variables.

→ Sum of two Random Variables.

Joint Distribution function ($F_{XY}(x,y)$) :-

→ It is represented with $F_{XY}(x,y)$

→ It can be given as $F_{XY}(x,y) = P(X \le x, Y \le y)$

i.e., joint Distribution function can be obtained by "adding all the joint probabilities which ~~is~~ are less than or equal to x, y".

$$\therefore F_{XY}(x,y) = \sum_i \sum_j P(x_i, y_j)\, u(x - x_i)\, u(y - y_j)$$

properties of joint Distribution function :-

→ $F_{XY}(x,y) \ge 0$, $0 \le F_{XY}(x,y) \le 1$

→ $F_{XY}(-\infty,-\infty) = 0$, $F_{XY}(x,-\infty) = 0$

$F_{XY}(-\infty, y) = 0$

→ $F_{XY}(+\infty,+\infty) = 1$, $F_{XY}(x,\infty) = F_X(x)$ — marginal D.F of x

$F_{XY}(\infty, y) = F_Y(y)$ — marginal D.F of y

→ marginal D.f is calculated from the joint D.f.

→ marginal D.f means sub $x = \infty$ in j.D.f we get m.D.f of y

→ ~~$P(x_1 < X \le x_2)$~~ ~~by~~ $y = \infty$ " " " " " x

$$P(x_1 < X \le x_2, y_1 < Y \le y_2) = F_{XY}(x_1, y_1) + F_{XY}(x_2, y_2)$$

* first property says that always joint Distribution function is a non-negative number. and it values lies between $0 \le F_{XY}(x,y) \le 1$.

* $F_{XY}(-\infty,-\infty) = P(x \le -\infty,\ y \le -\infty)$

$= P(\phi)$

$= 0$

$F_{XY}(x,-\infty) = P(X \le x,\ y \le -\infty)$

$= P((X \le x) \cap \phi)$

$= P(\phi)$

$= 0$

$F_{XY}(-\infty,y) = P(X \le -\infty,\ Y \le y)$

$= P((\phi) \cap (Y \le y))$

$= P(\phi)$

$= 0$

* $F_{XY}(\infty,\infty) = P(X \le \infty,\ Y \le \infty)$

$= P(S \cap S)$

$= P(S)$

$= 1$

$F_{XY}(x,\infty) = P(X \le x,\ y \le \infty)$

$= P((X \le x) \cap S)$

$= P(X \le x) = F_X(x)$

$F_{XY}(\infty,y) = P(X \le \infty,\ Y \le y)$

$= P(S \cap (Y \le y))$

$= P(Y \le y)$

$= F_Y(y)$

Joint Density function ($f_{XY}(x,y)$) :-

→ It is Represented with $f_{XY}(x,y)$.

→ It Can be given as

$$\boxed{\begin{aligned} f_{XY}(x,y) &= \frac{\partial^2}{\partial x \partial y}\left(F_{XY}(x,y)\right) \\ &= \sum_i \sum_j p(x_i, y_j)\, \delta(x-x_i)\, \delta(y-y_j) \end{aligned}}$$

i.e.,

the joint ~~Distribu~~ Density function of a Random Variable Can be obtained by double differentiation of joint Distribution function with Respect to x and y.

properties of joint Density function :-

→ $f_{XY}(x,y) \geq 0$, this property says that the joint density function is always a non-negative number.

→ The joint ~~Distr~~ Distribution function Can be obtained by $F_{XY}(x,y) = \int_{x=-\infty}^{x} \int_{y=-\infty}^{y} f_{XY}(x,y)\, dx\, dy$

→ $P(x_1 < X \le x_2, y_1 < Y \le y_2) = \int_{x_1}^{x_2}\int_{y_1}^{y_2} f_{XY}(x,y)\,dx\,dy$

→ The marginal density function of x and y is given as

$$f_X(x) = \int_{y=-\infty}^{\infty} f_{XY}(x,y)\,dy$$

$$f_Y(y) = \int_{x=-\infty}^{\infty} f_{XY}(x,y)\,dx$$

→ $$F_X(x) = \int_{x=-\infty}^{x}\int_{y=-\infty}^{\infty} f_{XY}(x,y)\,dx\,dy$$

$$F_Y(y) = \int_{x=-\infty}^{\infty}\int_{y=-\infty}^{y} f_{XY}(x,y)\,dx\,dy$$

Marginal Distribution functions $(F_X(x), F_Y(y))$:-

→ The Distribution functions which are Calculated from joint distribution functions are Called as marginal Distribution functions.

→ To Calculate marginal Distribution function of x in $F_{XY}(x,y)$.

put $y = \infty$

i.e., $F_{XY}(x,y) = P(X \le x, Y \le y)$

$$F_{XY}(x,\infty) = P(X \le x, Y \le \infty)$$

$$= P(X \le x \cap S)$$

$$= P(X \le x)$$

$$= F_X(x)$$

lly $F_{XY}(\infty, y) = P(X \le x, Y \le y)$

$$= P(X \le \infty, Y \le y)$$

$$= P(S \cap Y \le y)$$

$$= P(Y \le y)$$

$$= F_Y(y)$$

Marginal Density functions $(f_X(x), f_Y(y))$:-

→ The density functions which are Calculated from joint density functions are called as marginal density functions.

→ To Calculate marginal Density function of x integrate $f(x,y)$ with respect y with limits $-\infty$ to $+\infty$

$$f_X(x) = \int_{y=-\infty}^{\infty} f_{XY}(x,y)\, dy$$

Similarly,

$$f_Y(y) = \int_{x=-\infty}^{\infty} f_{XY}(x,y)\, dx$$

Statistically independent Random Variables :-

→ let us Consider two Random Variables X and Y when x and y are independent RV the occurance of failure of ~~first one~~ RV of X does not depends upon the occurance of failure of RV of Y.

then

$$\boxed{f_{XY}(x,y) = f_X(x) \cdot f_Y(y)}$$

$$\boxed{F_{XY}(x,y) = F_X(x) \cdot F_Y(y)}$$

$$F_{X|Y}(x|y) = \frac{F_{XY}(x,y)}{F_Y(y)} = \frac{F_X(x) \cdot F_Y(y)}{F_Y(y)} = F_X(x)$$

$$F_{Y|X}(y|x) = \frac{F_{XY}(x,y)}{F_X(x)} = \frac{F_X(x) \cdot F_Y(y)}{F_X(x)} = F_Y(y)$$

Sum of two Random Variables : $(f_W(w))$:-

It can be obtained by Convolution of their corresponding density functions

Let us Conside $F_W(w) = P(W \le w)$

$$= P(X+Y \le w)$$

$$= \int_{x=-\infty}^{\infty} \int_{y=-\infty}^{w-x} f_{XY}(x,y)\, dx\, dy$$

$$= \int_{x=-\infty}^{\infty} \int_{y=-\infty}^{w-x} f_X(x) \cdot f_Y(y)\, dx\, dy$$

$$= \int_{x=-\infty}^{\infty} f_X(x) \int_{y=-\infty}^{w-x} f_Y(y)\, dy\, dx$$

Now Diff w.r.t w

$$\frac{d}{dw}\left(F_W(w)\right) = \frac{d}{dw}\left(\int_{x=-\infty}^{\infty} f_X(x) \int_{y=-\infty}^{w-x} f_Y(y)\, dy\, dx\right)$$

$$f_W(w) = \int_{x=-\infty}^{\infty} f_X(x)\left(f_Y(y)\right)_{-\infty}^{w-x} dx$$

$$= \int_{x=-\infty}^{\infty} f_X(x)\left(f_Y(w-x) - f_Y(-\infty)\right) dx$$

$$= \int_{x=-\infty}^{\infty} f_X(x)\, f_Y(w-x)\, dx$$

$$\boxed{f_W(w) = f_X(x) * f_Y(y)}$$

problems :-

→ The probabilities of X and Y are shown in table then find

i) find the value of K.

ii) joint distribution function and joint functions

iii) ~~joint~~ marginal Distribution and Density functions.

x\y	-1	0	1
0	3/18	2/18	3/18
1	1/18	K/18	1/18
2	2/18	1/18	2/18

i) we know that Sum of all joint probabities = 1

$$\sum_i \sum_j P(x_i, y_j) = 1$$

$$\frac{3}{18} + \frac{2}{18} + \frac{3}{18} + \frac{1}{18} + \frac{K}{18} + \frac{1}{18} + \frac{2}{18} + \frac{1}{18} + \frac{2}{18} = 1$$

$$\frac{15+K}{18} = 1 \Rightarrow K = 3$$

ii) The joint Distribution function x and y

Can given as

$$\boxed{F_{XY}(x,y) = \sum_i \sum_j P(x_i, y_j)\, u(x-x_i)\, u(y-y_j)}$$

$$= \frac{3}{18} u(x+1) u(y) + \frac{2}{18} u(x) u(y) + \frac{2}{18} u(x-1) u(y)$$

$$+ \frac{1}{18} u(x+1) u(y-1) + \frac{3}{18} u(x)(y-1) u + \frac{1}{18} u(x-1)(y-1) u$$

$$+ \frac{2}{18} u(x+1) u(y-2) + \frac{1}{18} u(x) u(y-2) + \frac{2}{18} u(x-1)$$

$$u(y-2)$$

$$\boxed{f_{XY}(x,y) = \sum_i \sum_j P(x_i, y_j)\, \delta(x-x_i)\, \delta(y-y_j)}$$

$$= \frac{3}{18} \delta(x+1) \delta(y) + \frac{2}{18} \delta(x) \delta(y) + \frac{3}{18} \delta(x-1) \delta(y)$$

$$+ \frac{1}{18} \delta(x+1) u(y-1) + \frac{3}{18} \delta(x) \delta(y-1) + \frac{1}{18} \delta(x-1) \delta(y-1) +$$

$$\frac{2}{18} \delta(x+1) \delta(y-2) + \frac{1}{18} \delta(x) \delta(y-2) + \frac{2}{18} \delta(x-1) \delta(y-2)$$

iii) Marginal Distribution function :-

$$F_{XY}(x,y) = F_Y(y)$$

$$F_{XY}(x,y) = F_X(x)$$

$$F_X(x) = \frac{3}{18} u(x+1) + \frac{2}{18} u(x) + \frac{3}{18} u(x-1) + \frac{1}{18} u(x+1)$$

$$+ \frac{3}{18} u(x) + \frac{1}{18} u(x+1) + \frac{2}{18} u(x+1) + \frac{1}{18} u(x) + \frac{2}{18} u(x-1)$$

$$F_Y(y) = \frac{3}{18} u(y) + \frac{2}{18} u(y) + \frac{3}{18} u(y) + \frac{1}{18} u(y-1) +$$

$$\frac{3}{18} u(y-1) + \frac{1}{18} u(y-1) + \frac{2}{18} u(y-2) + \frac{1}{18} u(y-2) + \frac{2}{18} u(y-2)$$

→ The joint probability density function is given $f_{XY}(x,y) = A.e^{-(2x+y)}$, $x \geq 0$, $y \geq 0$ then find the value of (i) A

(ii) check whether the Random Variables x, y are independent or not.

i) we know that, Area under the joint density Curve is Equal to 1.

i.e.,
$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)\,dx\,dy = 1$$

$$= \int_{0}^{\infty}\int_{0}^{\infty} A.e^{-(2x+y)}\,dx\,dy = 1$$

$$A\left[\int_{0}^{\infty} e^{-2x}\,dx + \int_{0}^{\infty} e^{-y}\,dy\right] = 1$$

$$A\left[\frac{e^{-2x}}{-2}\Big|_{0}^{\infty} + \frac{e^{-y}}{-1}\Big|_{0}^{\infty}\right] = 1 \qquad A\left[0 - \frac{1}{2} - 0 + 1\right] = 1$$

$$\left[\frac{A}{-2}\left[[0-1]\right] + (0+1)\right] = 1 \qquad A = 2$$

$$\frac{A}{-2}(-2) = 1 \qquad \frac{A}{2}\left[\frac{1}{2} - 1\right] = 1$$

$$(A = 1) \qquad A\left[-\frac{1}{2}\right] = 1$$

$$A = -2$$

→ If x and y are independent then,

$$\boxed{f_{XY}(x,y) = f_X(x) \cdot f_Y(y)}$$

$$f_X(x) = \int_{y=-\infty}^{\infty} f_{XY}(x,y)\,dy = \int_0^{\infty} Ae^{-2x-y}\,dy$$

$$Ae^{-2x}\int_0^{\infty} e^{-y}\,dy = Ae^{-2x}\cdot\left[\frac{e^{-y}}{-1}\right]_0^{\infty}$$

$$= Ae^{-2x}\left[0+1\right]$$

$$= 2e^{-2x}$$

$$f_Y(y) = \int_{x=-\infty}^{\infty} f_{XY}(x,y)\,dx = \int_0^{\infty} Ae^{-2x-y}\,dx$$

$$Ae^{-y}\left[\frac{e^{-2x}}{-2}\right]_0^{\infty} = \frac{Ae^{-y}}{2}\left[0+\frac{1}{2}\right]$$

$$= e^{-y}$$

$$f_{XY}(x,y) = f_X(x)\cdot f_Y(y)$$

$$= 2e^{-2x}\cdot e^{-y} = 2e^{-2x-y} = 2e^{-(2x+y)}$$

∴ Hence proved.

∴ X and Y are independent RV.

→ The joint density function of two RV is given by $f_{XY}(x,y) = \frac{1}{18}e^{-\left(\frac{x}{16}+\frac{y}{3}\right)}u(x)u(y)$. then S.T X and Y are independent RV and also find the Conditional density functions.

Given that $f_{XY}(x,y) = \frac{1}{18}e^{-\left(\frac{x}{16}+\frac{y}{3}\right)}$.

$$f_X(x) = \int_{y=-\infty}^{\infty} f_{XY}(x,y)\,dy =$$

If x and y are independent then

$$\boxed{f_{XY}(x,y) = f_X(x) \cdot f_Y(y)}$$

$$f_X(x) = \int_{y=-\infty}^{\infty} f_{XY}(x,y)\,dy = \int_{y=-\infty}^{\infty} \frac{1}{18} e^{-\left(\frac{x}{16}+\frac{y}{3}\right)} u(x)u(y)\,dy$$

$$= \frac{1}{18} e^{-\frac{x}{16}} u(x) \int_{y=-\infty}^{\infty} e^{-\frac{y}{3}} u(y)\,dy$$

$$= \frac{1}{18} e^{-\frac{x}{16}} u(x) \int_{y=0}^{\infty} e^{-\frac{y}{3}}\,dy$$

$$= \frac{1}{18} e^{-\frac{x}{16}} u(x) \left[\frac{e^{-\frac{y}{3}}}{-\frac{1}{3}}\right]_0^{\infty}$$

$$= \frac{1}{18} e^{-\frac{x}{16}} u(x) \cdot -3\left[0-1\right]$$

$$= \frac{1}{\cancel{18}_{6}} e^{-\frac{x}{16}} u(x) \cdot \cancel{3} = \boxed{\frac{e^{-\frac{x}{16}}}{6} u(x)}$$

$$f_Y(y) = \int_{x=-\infty}^{\infty} f_{XY}(x,y)\,dy = \int_{x=-\infty}^{\infty} \frac{1}{18} e^{-\left(\frac{x}{16}+\frac{y}{3}\right)} u(x)u(y)\,dy$$

$$= \frac{1}{18} e^{-\frac{y}{3}} u(y) \int_{x=-\infty}^{\infty} e^{-\frac{x}{16}} u(x)\,dx$$

$$= \frac{1}{18} e^{-\frac{y}{3}} u(y) \int_{x=0}^{\infty} e^{-\frac{x}{16}}\,dx$$

$$= \frac{1}{18} e^{-\frac{y}{3}} u(y) \cdot -16\left[e^{-\frac{x}{16}}\right]_0^{\infty}$$

$$= \frac{1}{18} e^{-\frac{y}{3}} u(y) \cdot 16 = \boxed{\frac{8}{9} e^{-\frac{y}{3}} u(y)}$$

$$f_{XY}(x,y) = \frac{e^{-\frac{x}{16}}}{6} u(x) \cdot \frac{8}{9} e^{-\frac{y}{3}} u(y)$$

$$f_X(x) \cdot f_Y(y) = \frac{8}{54} e^{-\left(\frac{x}{16}+\frac{y}{3}\right)} u(x)\, u(y)$$

$$= \frac{4}{27} e^{-\left(\frac{x}{16}+\frac{y}{3}\right)} u(x)\, u(y)$$

$$\neq f_{XY}(x,y)$$

$\therefore$ the RV X and Y are not independent

ii) the Conditional density function of 'X' given by Y

$$f_X(x|y) = \frac{f_{XY}(x,y)}{f_Y(y)}$$

The Conditional density function of 'y' given by X

$$f_Y(y|x) = \frac{f_{XY}(y,x)}{f_X(x)}$$

→ find the Constant b interms of a Such that the function $f_{XY}(x,y) = \begin{cases} b e^{-(x+y)}, & 0 \le x \le a \\ & 0 \le y \le \infty \\ 0, & \text{Else where} \end{cases}$ is a valid density function.

$$f_{XY}(x,y) = \begin{cases} b e^{-(x+y)}, & 0 \le x \le a \\ & 0 \le y \le \infty \\ 0, & \text{Else where} \end{cases}$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y) = 1$$

$$\int_0^a\int_0^\infty b e^{-(x+y)}\,dx\,dy = 1 \qquad b\left[\int_0^a e^{-x}\,dx + \int_0^\infty e^{-y}\,dy\right] = 1$$

$$b\int_0^a \frac{e^{-(x+y)}}{-1}\Big|_0^\infty dx = 1 \qquad b\left[\frac{e^{-x}}{-1}\Big|_0^a + \frac{e^{-y}}{-1}\Big|_0^\infty\right] = 1$$

$$-b\int_0^a (0-1)\,dx = 1 \qquad -b\left(\left[e^{-0}-1\right] - \left[0-1\right]\right) = 1$$

$$b\int_0^a dx = 1 \qquad \cancel{-}b\left[\cancel{e^{-0}-1-1}\right] = 1$$

$$\cancel{-}b\left[\cancel{e^{-a}-2}\right] = 1$$

$$b\left[a-0\right] = 1 \qquad -b\left[e^{-a}+1\right] = 1$$

$$ab = 1 \qquad be^{-a} + b = 1$$

$$be^{-a} = 1-b$$

$$e^{-a} = \frac{b-1}{b}$$

$$b = \frac{1}{1-e^{-a}}$$

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→ find the value of ~~function~~ Constant 'b' such that the function shown below is

$$f_{XY}(x,y) = b\,x^2 y \exp(-2xy)\,u(x-2)\,u(y-1)$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y) = 1$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} b\,x^2 y\, e^{-2xy}\,u(x-2)\,u(y-1)\,dx\,dy$$

$$\int_2^\infty b x^2 y e^{-2xy} \, dx \int_1^\infty b x y^2 e^{-2xy} \, dy$$

$$= b\left[y \int_2^\infty x^2 e^{-2xy} \, dx \int_1^\infty x^2 e^{-2xy} \, y \, dy \right]$$

$$= b\left[y\left[x^2 \frac{e^{-2xy}}{-2y} - 2x \cdot \frac{e^{-2xy}}{4y^2} + 2 \cdot \frac{e^{-2xy}}{-8y^3} \right]_2^\infty \cdot x^2 e^{-2xy} \cdot \left(\frac{y^2}{2}\right)_1^\infty \right]$$

$$= b\left[y\left[-4 \cdot \frac{e^{-4y}}{-2y} + 4 \cdot \frac{e^{-4y}}{4y^2} + \frac{1}{4} \cdot \frac{e^{-4y}}{y^3} \right] \cdot \frac{x^2 e^{-2xy}}{2} \right]$$

$$= \int_{x=2}^\infty b x^2 \int_{y=1}^\infty y e^{-2xy} \, dy = 1$$

$$= \int_{x=2}^\infty b x^2 \left(y \cdot \frac{e^{-2xy}}{-2x} - 1 \cdot \frac{e^{-2xy}}{4x^2} \right) dx = 1$$

$$= \int_{x=2}^\infty b x^2 \left[\frac{e^{-2x}}{2x} + \frac{e^{-2x}}{4x^2} \right] = 1$$

$$b \int_2^\infty x^2 e^{-2x} \left[\frac{1}{2x} + \frac{1}{4x^2} \right] = 1$$

$$b \int_2^\infty x^2 e^{-2x} \left[\frac{x^2}{2x} + \frac{x^2}{4x^2} \right] = 1$$

$$b \int_2^\infty x^2 e^{-2x} \left(\frac{2x+1}{4x^2} \right) dx = 1$$

$$= \frac{b}{4} \int_{x=2}^{\infty} (2x+1) e^{-2x} dx$$

$$= \frac{b}{4} \left[(2x+1) \frac{e^{-2x}}{-2} - 2 \cdot \frac{e^{-2x}}{4} \Big|_{2}^{\infty} \right] = 1$$

$$= \frac{b}{4} \left[\frac{5e^{-4}}{-2} + \frac{2e^{-4}}{4} \right] = 1$$

$$= \frac{b}{4} \left[e^{-4} \left(\frac{5}{2} + \frac{2}{4} \right) \right] = 1$$

$$\Rightarrow \frac{b e^{-4}}{4} \left[\frac{24}{8} \right] = 1$$

$$b = 4 e^{4}$$

→ find the density function of $w = x+y$ where the densities of x and y are assumed to be $4\, u(x) e^{-4x} = f_X(x)$ and $f_Y(y) = 5\, u(y) \cdot e^{-5y}$

Given that $f_X(x) = 4\, u(x) e^{-4x}$

$f_Y(y) = 5\, u(y) e^{-5y}$

$w = x+y$

We know that the density function of the sum of two RV is $f_W(w) = f_X(x) * f_Y(y)$

$$\int_{-\infty}^{\infty} f_X(x) \cdot f_Y(w-x)\, dx$$

$$\int_{-\infty}^{\infty} 4\, u(x) e^{-4x} \cdot 5\, u(w-x) \cdot e^{-5(w-x)}\, dx$$

$$\int_0^{\omega} 4e^{-4x} \cdot 5e^{-5(\omega-x)} \, dx$$

$$20 \int_0^{\omega} e^{-4x} \cdot e^{-5\omega} \cdot e^{5x} \, dx$$

$$20e^{-5\omega} \left[\int_0^{\omega} e^{x} \, dx \right]$$

$$= 20e^{-5\omega} \cdot (e^{\omega} - e^{0})$$

$$= 20e^{-5\omega} (e^{\omega} - 1) = 20\left[e^{-4\omega} - e^{-5\omega}\right]$$

$$= 20 \left\{ \right.$$

→ In a binary Communication System the probability of bit error is 0.01. If a block of 8 bits are transmitted find the probability that.

i) Exactly two bit errors will occur.

ii) Atleast two bit errors will occur.

iii) more than two bit errors will occur.

iv) all the bits will be errorances.

$$P(x=m) = {}^{n}C_{m} \, p^{m} \cdot q^{n-m}$$

$n = 8$ $\quad p = 0.01$ $\quad q = 1 - 0.01 = 0.99$

(i) $P(x=2)$ $\quad$ ii) $P(x \geq 2) = 1 - P(x<2)$

iii) $P(x>2)$ $\quad$ iv) $P(x=8)$

i) $P(x=2) = {}^8C_2 (0.01)^2 (0.99)^{8-2}$

$= {}^8C_2 (0.01)(0.01)(0.99)^6$

$= (\cancel{8\times7\times6\times5\times4\times3\times2\times1})$

$= \frac{8!}{6!\cdot2!} \times 1\times10^{-04} \times (0.99)^6$

$= \frac{8\times7}{2!} \times 10^{-04} \times (0.99)^6$

$= 28\times10^{-04} \times (0.99)^6 = 2.63\times10^{-03}$

ii) $P(x \geq 2) = 1 - P(x<2)$

$= 1 - [P(x=0) + P(x=1)]$

$= 1 - {}^8C_0 (0.01)^0 (0.99)^8 = 1 - (0.99)^8$

$= 1 - 0.92 = 0.08$

→ ST the characteristic function of a RV having Bionomial density function is $\phi_x(\omega) = [1-P+Pe^{j\omega}]$

for a Binomial distribution we know that

$P[x=m] = {}^nC_m p^m q^{n-m}$

$\phi_x(\omega) = \sum_i e^{j\omega x_i} P(x=x_i)$

$\phi_x(\omega) = E[e^{j\omega x}]$

$= \sum_{m=0}^{n} e^{j\omega x} \cdot {}^nC_m p^m q^{n-m}$

$= [e^{j\omega} {}^nC_0 p^0 q^{n-0} + e^{j\omega} {}^nC_1 p^1 q^{n-1} + \dots]$

$= [pe^{j\omega}+q] = [pe^{j\omega}+1-p]$

A RV X has a density function $f_X(x) = \begin{cases} \frac{\pi}{16} \cos\left(\frac{\pi x}{8}\right), & -4 \le x \le 4 \\ 0, & \text{elsewhere} \end{cases}$

$$f_X(x) = \frac{\pi}{16} \cos\left(\frac{\pi x}{8}\right)$$

mean, $m_1 = E[x]$

$$= \int_{-\infty}^{\infty} x f_X(x)\, dx$$

$$= \int_{-4}^{4} x \cdot \frac{\pi}{16} \cos\left(\frac{\pi x}{8}\right) dx$$

$$\frac{\pi}{16} \int_{-4}^{4} x \cdot \cos\left(\frac{\pi x}{8}\right) dx$$

$$= \frac{\pi}{16} \left[x \cdot \frac{\sin \frac{\pi x}{8}}{\frac{\pi}{8}} - 1 \cdot \frac{-\cos \frac{\pi x}{8}}{\left(\frac{\pi}{8}\right)^2} \Bigg|_{-4}^{4} \right]$$

$$= \frac{\pi}{16} \left[4 \cdot \frac{8}{\pi} \cdot \sin\frac{4\pi}{8} + \cos\left(\frac{4\pi}{8}\right) \cdot \left(\frac{8}{\pi}\right)^2 + 4 \cdot \frac{\sin\left(-\frac{4\pi}{8}\right)}{\frac{\pi}{8}} - \frac{\cos\left(\frac{4\pi}{8}\right)}{\left(\frac{\pi}{8}\right)^2} \right]$$

$$= \frac{\pi}{16} \left[\frac{32}{\pi} \cdot 0 \right] = 0$$

$m_2 = E[x^2]$

$$\int_{-\infty}^{\infty} x^2 f_X(x)\, dx = \int_{-4}^{4} x^2 \cdot \frac{\pi}{16} \cos\left(\frac{\pi x}{8}\right) dx$$

$$\frac{\pi}{16} \int_{-4}^{4} x^2 \cdot \cos\left(\frac{\pi x}{8}\right) dx \Rightarrow$$

$$\frac{\pi}{16}\left[x^2 \cdot \frac{\sin\frac{\pi x}{8}}{\frac{\pi}{8}} - 2x \cdot \frac{-\cos\frac{\pi x}{8}}{\left(\frac{\pi}{8}\right)^2} + 2 \cdot \frac{-\sin\frac{\pi}{8}x}{\left(\frac{\pi}{8}\right)^3}\right]_{-4}^{4}$$

$$= \frac{\pi}{16}\left[\frac{8x^2}{\pi}\sin\left(\frac{\pi x}{8}\right) + 2x\left(\frac{8}{\pi}\right)^2\cos\frac{\pi x}{8} - 2\left(\frac{8}{\pi}\right)^3\sin\frac{\pi}{8}x\right]_{-4}^{4}$$

$$= \frac{\pi}{16}\left[\frac{16\times 8}{\pi}\sin\frac{4\pi}{8} + 4\cdot\left(\frac{8}{\pi}\right)^2\cos\frac{4\pi}{8} - 2\left(\frac{8}{\pi}\right)^3\sin\left(\frac{4\pi}{8}\right) - \right.$$

$$\left.\frac{8\times 16}{\pi}\sin\left(\frac{-4\pi}{8}\right) - 8\left(\frac{8}{\pi}\right)^2\cos\left(\frac{4\cdot x}{8}\right) + 2\left(\frac{8}{\pi}\right)^3\sin\left(\frac{-4\pi}{8}\right)\right]$$

$$= \frac{\pi}{16}\left[\frac{16\times 8}{\pi} + 0 - 2\cdot\left(\frac{8}{\pi}\right)^3 + \frac{8\times 16}{\pi} - 0 - 2\left(\frac{8}{\pi}\right)^3\right]$$

$$= \frac{\pi}{16}\left[\frac{16\times 8}{\pi} + \frac{8\times 16}{\pi} - 2\cdot\left(\frac{8}{\pi}\right)^3 - 2\left(\frac{8}{\pi}\right)^3\right]$$

$$\begin{array}{r} 64 \\ 64 \\ \hline 128 \end{array}$$

$$= \frac{\pi}{\cancel{16}}\left[\cancel{16}\left(\frac{8}{\pi} + \frac{8}{\pi} - \frac{8^2}{\pi^3}\right) - \frac{8^2}{\pi^3}\right]$$

$$= \pi\left[\frac{16}{\pi} - \frac{128}{\pi^3}\right]$$

$$= \cancel{\pi}\left[\frac{16\pi^2 - 128}{\pi^{\cancel{3}\,2}}\right] = 16 - \frac{128}{\pi^2}$$

$$= \frac{16\pi^2 - 128}{\pi^2}$$

Part - B

1) Operations on Multiple Random Variables

(i) Expected Value of the function of the random variable [$E\{g(x,y)\}$]

(ii) Joint Moments about origin (m_{nk})

(iii) Joint Moments about mean (or) Joint central moments. (μ_{nk})

(iv) Joint Characteristic function [$\phi_{xy}(\omega_1, \omega_2)$]

Expected Value of the function of the random variable [$E[g(x,y)]$]

Let x and y be two random variables and $g(x,y)$ be the function in terms of random variable x & y then, the expected value of the function of the random variable given as

$$\boxed{E\{g(x,y)\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)\, f_{xy}(x,y)\, dx\, dy}$$

'x' & 'y' are continuous R.V

$$\boxed{E\{g(x,y)\} = \sum_{i=-\infty}^{\infty}\sum_{j=-\infty}^{\infty} g(x,y)\, p(x_i, y_j)}$$

'x' & 'y' are discrete R.V.

Joint Moments about origin (m_{nk})

The Joint moments about the origin is represented with 'm_{nk}'.

x y.

Here $(n+k)$ is called order of the moments

It can be obtained by calculating expected value of the function of the random variable.

$$g(x,y) = x^n \cdot y^k$$

It can be given as.

$$m_{nk} = E[g(x,y)]$$
$$= E[x^n \cdot y^k]$$

$$\boxed{m_{nk} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^n y^k \cdot f_{xy}(x,y)\,dx\,dy}$$

↳ x & y are contin

$$\boxed{m_{nk} = \sum_{i=0}^{n}\sum_{j=0}^{k} x^n y^k p(x_i, y_j)}$$

If x & y are discrete.

If $n=0, k=0$

$$m_{00} = E[x^0 \cdot y^0] = E[1] = 1$$

$n=1, k=0$

$$m_{10} = E[x^1 \cdot y^0] = E[x] = m_x$$

$n=0, k=1$

$$m_{01} = E[x^0 y^1] = E[y] = m_y$$ (mean of the R.V 'y')

$n=2, k=0$

$$m_{20} = E[x^2 y^0] = E[x^2]$$ = mean square value of 'x'.

→ second order moments

$n=0, k=2$

$$m_{02} = E[x^0 y^2] = E[y^2]$$ = mean square value of 'y'

$n=1, k=1$

$$m_{11} = E[x^1 y^1] = E[xy]$$ = correlation of random variable of x & y

R_{xy}.

* Correlation coefficient

$$\rho = \frac{C_{xy}}{\sigma_x \sigma_y}$$

Note:

(i) Correlation gives us amount of similarity and co-variance gives us amount of difference

(ii) If $R_{xy} = 0$ then Random variable x & y are called as orthogonal

(iii) If $R_{xy} = E[xy] = E[x]E[y]$ then two random variables are uncorrelated

Joint Moments about Mean (μ_{nk})

It is represented with μ_{nk}

$\therefore$ Here $n+k$ is called as order of the moments

* It can be calculated by finding the expected value of the function of the random variable.

~~It can~~

$$g(x,y) = (x-m_x)^n (y-m_y)^k$$

It can be given as

$$\mu_{nk} = E[g(x,y)]$$

$$= E[(x-m_x)^n (y-m_y)^k]$$

$$\boxed{\mu_{nk} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x-m_x)^n (y-m_y)^k f_{xy}(x,y)\,dx\,dy}$$

$\therefore$ x & y are continuous

$$\boxed{\mu_{nk} = \sum_{i=0}^{n}\sum_{j=0}^{k} (x_i-m_x)^n (y_j-m_y)^k p(x_i,y_j)}$$

$\therefore$ x & y are discrete

If $n=0, k=0$

$$\mu_{00} = E[(x-m_x)^0 (y-m_y)^0] = E[1] = 1$$

If $n=1, k=0$

$\mu_{10} = E\left[(x-m_x)^1 (y-m_y)^0\right] = E\left[(x-m_x)\right]$

$= E[x] - E[m_x]$

$= m_x - m_x = 0.$

If $n=0, k=1$

$\mu_{01} = E\left[(x-m_x)^0 (y-m_y)^1\right] = E\left[(y-m_y)\right]$

$= E[y] - E[m_y]$

$= m_y - m_y = 0$

If $n=2, k=0.$

$\mu_{20} = E\left[(x-m_x)^2 (y-m_y)^0\right] = E\left[(x-m_x)^2\right]$

$= \mu_{2x}$ (or) σ^2_x

If $n=0, k=2$

$\mu_{02} = E\left[(x-m_x)^0 (y-m_y)^2\right] = E\left[(y-m_y)^2\right]$

$= \mu_{2y}$ (or) σ^2_y.

If $n=1, k=1$

$\mu_{11} = E\left[(x-m_x)^1 (y-m_y)^1\right] = C_{xy}$

= covariance of 'x' & 'y'

Note:

(i) If $C_{xy} = 0$ then the two random variable x & y are uncorrelated.

Relationship between R_{xy} and C_{xy}

Now, consider

$C_{xy} = \mu_{11}$

$= E\left[(x-m_x)^1 (y-m_y)^1\right]$

$= E\left[xy - xm_y - ym_x + m_x m_y\right]$

$= E\{xy\} - E\{x m_y\} - E\{y m_x\} + E\{m_x m_y\}$

$= R_{xy} - m_y E\{x\} - m_x E\{y\} + m_x m_y E\{1\}$

$= R_{xy} - m_y m_x - m_x m_y + m_x m_y$

$$\boxed{C_{xy} = R_{xy} - E\{x\}E\{y\}}$$

Joint Characteristic Function $[\phi_{xy}(\omega_1, \omega_2)]$

* It is represented with $\phi_{xy}(\omega_1, \omega_2)$.
* The Joint characteristic function used to generate higher order joint moments about the origin only. It can be given as.

$$m_{nk} = E\{x^n y^k\}$$

$$= \left(\frac{1}{j}\right)^{n+k} \frac{\partial^{n+k}}{\partial \omega_1^n \partial \omega_2^k} \left(\phi_{xy}(\omega_1, \omega_2)\right)\Big|_{\omega_1 = 0, \omega_2 = 0}$$

$$\boxed{\therefore m_{nk} = \left(\frac{1}{j}\right)^{n+k} \frac{\partial^{n+k}}{\partial \omega_1^n \partial \omega_2^k} \left(\phi_{xy}(\omega_1, \omega_2)\right)\Big|_{\omega_1 = 0, \omega_2 = 0}}$$

Problems

1) If $f_{xy}(x,y) = \begin{cases} \frac{1}{100} & 0 < x < 5,\ 0 < y < 20 \\ 0 & \text{elsewhere.} \end{cases}$ then find the expected value of the function (i) xy (ii) $x^2 y$.

Sol (i) $E[xy] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^n y^k \cdot f_{xy}(x, y)\, dx\, dy$

$$= \int_0^5 \int_0^{20} xy \cdot \frac{1}{100}\, dx\, dy = \frac{1}{100}\left[\frac{x^2}{2}\right]_0^5 \left[\frac{y^2}{2}\right]_0^{20}$$

$$= \frac{1}{100}\left[\frac{25}{2}\right]\left[\frac{400}{2}\right] = 25.$$

$$E[x^2y] = \int_0^5 \int_0^{20} x^2 y \frac{1}{100}\, dx\, dy$$

$n=2, k=1$

$$= \frac{1}{100}\left[\frac{x^3}{3}\right]_0^5 \left[\frac{y^2}{2}\right]_0^{20}$$

$$= \frac{1}{100}\left[\frac{125}{3}\right]\left[\frac{400}{2}\right]$$

$$= \frac{250}{3}$$

1) The density function of the random variables is given by $f_{xy}(x,y) = e^{-2(x+y)} u(x)\, u(y)$ then find $E\, e^{-(x+y)}$

the mean value of

Sol:- $$E\left[e^{-(x+y)}\right] = \int_0^\infty \int_0^\infty e^{-2(x+y)} \cdot e^{-(x+y)}\, dx\, dy$$

$$= \int_0^\infty \int_0^\infty e^{-3(x+y)}\, dx\, dy$$

$$= \int_0^\infty \left.\frac{e^{-3(x+y)}}{-3x}\right|_0^\infty dy$$

$$= \int_0^\infty \frac{e^{-3x}}{-3x}\, dx$$

$$= \left[\frac{e^{-3x}}{+9}\right]_0^\infty$$

$$= \frac{1}{9}$$

4) If $f_{XY}(x,y) = \begin{cases} \frac{(x+y)^2}{40}, & -1 \le x \le 1, -3 \le y \le 3 \\ 0, & \text{elsewhere} \end{cases}$

(i) Variances of x & y

(ii) Also find correlation coefficient

Sol) (i) $f_X(x) = \int_{y=-\infty}^{\infty} f_{XY}(x,y)\,dy$

$$= \int_{y=-3}^{3} \frac{(x^2+y^2+2xy)}{40}\,dy$$

$$\cancel{= \frac{x^2}{40}\left[\frac{y^3}{3}\right]_{-3}^{3} + 2x\left[\frac{y^2}{2}\right]_{-3}^{3}}$$

$$= \frac{1}{40}\int_{y=-3}^{3} (x^2+y^2+2xy)\,dy$$

$$= \frac{1}{40}\left[x^2\{y\}_{-3}^{3} + \left[\frac{y^3}{3}\right]_{-3}^{3} + 2x\left[\frac{y^2}{2}\right]_{-3}^{3}\right]$$

$$= \frac{1}{40}\left\{x^2[6] + \left[\frac{27}{3} + \frac{27}{3}\right] + 2x\left[\frac{9}{2} - \frac{9}{2}\right]\right\}$$

$$= \frac{1}{40}\{6x^2 + 18\}$$

$$= \frac{3x^2+9}{20}$$

$$f_Y(y) = \int_{x=-\infty}^{\infty} f_{XY}(x,y)\,dx$$

$$= \int_{-1}^{1} \frac{(x+y)^2}{40}\,dx$$

$$= \frac{1}{40}\int_{-1}^{1} (x^2+y^2+2xy)\,dx$$

$$= \frac{1}{40}\left[\left(\frac{x^3}{3}\right)_{-1}^{1} + y^2[x]_{-1}^{1} + 2y\left[\frac{x^2}{2}\right]_{-1}^{1}\right]$$

$$= \frac{1}{40}\left[\left(\frac{1}{3}+\frac{1}{3}\right) + y^2 2\right]$$

$$= \frac{1}{40}\left[\frac{2}{3} + 2y^2\right] = \frac{6y^2+2}{120} = \frac{3y^2+1}{60}$$

we know that

$$\sigma_X^2 = E[x^2] + \{E[x]\}^2,$$

$$\sigma_Y^2 = E[y^2] + [E(y)]^2.$$

~~variance~~

$$E[x] = \int_{-\infty}^{\infty} x f_X(x)\,dx.$$

$$= \int_{-1}^{1} x\,\frac{(3x^2+9)}{20}\,dx = \frac{1}{20}\int_{-1}^{1} 3x^3+9x\,dx$$

$$= \frac{1}{20}\left[3\left(\frac{x^4}{4}\right)_{-1}^{1} + 9\left(\frac{x^2}{2}\right)_{-1}^{1}\right]$$

$$= \frac{1}{20}\left[\frac{3}{4}(1-1) + \frac{9}{2}(1-1)\right]$$

$$= \frac{1}{20}[0+0] = 0.$$

$$E[x^2] = \int_{-\infty}^{\infty} x^2 f_X(x)\,dx.$$

$$= \int_{-1}^{1} x^2\,\frac{(3x^2+9)}{20}\,dx = \frac{1}{20}\int_{-1}^{1} 3x^4+9x^2\,dx$$

$$= \frac{1}{20}\left[3\left(\frac{x^5}{5}\right)_{-1}^{1} + 9\left(\frac{x^3}{3}\right)_{-1}^{1}\right]$$

$$= \frac{1}{20}\left[\frac{3}{5}(1+1) + \frac{9}{3}(1+1)\right]$$

$$= \frac{1}{20}\left[\frac{6}{5} + \frac{18}{3}\right] = \frac{3}{150} + \frac{9}{30} = \frac{9}{25}$$

$$E[Y] = \int_{-\infty}^{\infty} y\, f_Y(y)\, dy$$

$$= \int_{-3}^{3} y \cdot \left(\frac{3y^2+1}{60}\right) dy$$

$$= \frac{1}{60}\int_{-3}^{3} 3y^3 + y\, dy$$

$$= \frac{1}{60}\left[3\left(\frac{y^4}{4}\right)_{-3}^{3} + \left(\frac{y^2}{2}\right)_{-3}^{3}\right]$$

$$= \frac{1}{60}\left[\frac{3}{4}(81-81) + \frac{1}{2}(9-9)\right]$$

$$= \frac{1}{60}[0+0] = 0$$

$$E[Y^2] = \int_{-\infty}^{\infty} y^2 f_Y(y)\, dy$$

$$= \int_{-3}^{3} y^2\left(\frac{3y^2+1}{60}\right) dy$$

$$= \frac{1}{60}\int_{-3}^{3} 3y^4 + y^2\, dy$$

$$= \frac{1}{60}\left[3\left(\frac{y^5}{5}\right)_{-3}^{3} + \left(\frac{y^3}{3}\right)_{-3}^{3}\right]$$

$$= \frac{1}{60}\left[\frac{3}{5}(243+243) + \frac{1}{3}(27+27)\right]$$

$$= \frac{1}{60}\left[\frac{3}{5}(486) + \frac{1}{3}(54)\right]$$

$$= \frac{1458}{300} + \frac{54}{180} = \frac{149}{25}$$

$\sigma_x^2 = E[x^2] - (E(x))^2$

$= \frac{9}{25} - 0$

$= \frac{9}{25}$

$\sigma_y^2 = E[y^2] - (E(y))^2$

$= \frac{129}{125} - 0$

$= \frac{129}{25}$

(ii) $C_{xy} = R_{xy} - E[x]E[y]$

$= R_{xy} - 0 \times 0$

$= R_{xy}$

$R_{xy} = E[xy]$

$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f_{xy}(x,y)\,dx\,dy$

$= \int_{x=-1}^{1}\int_{y=-3}^{3} xy \frac{(x+y)^2}{40}\,dx\,dy$

$= \frac{1}{40}\int_{-1}^{1}\int_{-3}^{3} xy\,[x^2+y^2+2xy]\,dx\,dy$

$= \frac{1}{40}\int_{-1}^{1}\int_{-3}^{3} x^3y + xy^3 + 2x^2y^2\,dx\,dy$

$= \frac{1}{40}\int_{-1}^{1}\left[x^3\left(\frac{y^2}{2}\right) + x\left(\frac{y^4}{4}\right) + 2x^2\left(\frac{y^3}{3}\right)\right]_{-3}^{3} dx$

$= \frac{1}{40}\int_{-1}^{1} \frac{x^3}{2}[9-9] + \frac{x}{4}[27-27] + \frac{2x^2}{3}[27+27]\,dx$

$= \frac{1}{40}\int_{-1}^{1} \frac{2x^2}{3}[54]\,dx$

$= \cancel{\frac{1}{40}\int_{-1}^{1} 54x^2\,dx} = \frac{90}{120}\left[\frac{x^3}{3}\right]_{-1}^{1}$

$= \cancel{\frac{24}{120}}\,[\;] = \frac{108}{360}\left(1^3 - (-1)^3\right)$

$= \frac{108 \times 2}{360} = 3/5$

$$\therefore \rho = \frac{C_{xy}}{\sigma_x \sigma_y} = \frac{3/5}{\sqrt{\frac{9}{25}}\sqrt{\frac{129}{25}}}$$

$$= \frac{3/5}{\frac{3}{5} \times \frac{\sqrt{129}}{5}} = 0.44$$

5) 'x' & 'y' are two independent random variables such that $E[X] = \lambda_1$, $E[Y] = \lambda_2$ & $\sigma^2_x = \sigma_1^2$ and $\sigma^2_y = \sigma_2^2$ then s.t $var(x,y) = \sigma_1^2\sigma_2^2 + \lambda_1^2\sigma_2^2 + \lambda_2^2\sigma_1^2$

Sol

$E[X] = \lambda_1$ and $E[Y] = \lambda_2$

we know that two independent random variables

$$F_{xy}(x,y) = F_x(x)F_y(y)$$

Variance of $\sigma^2_x = m_2 - m_1^2$.

$$\sigma^2_x = E[x^2] - [E(x)]^2 \quad \text{—} (1)$$

$$\sigma^2_{xy} = E[(xy)^2] - (E[xy])^2$$

$$= E[x^2]E[y^2] - (E[x])^2(E[y])^2$$

$$= (\sigma^2_x + (Ex)^2)(\sigma^2_y + (Ey)^2) - [E(x)]^2[E(y)]^2$$

(from (1))

$$= (\sigma_1^2 + \lambda_1^2)(\sigma_2^2 + \lambda_2^2) - \lambda_1^2\lambda_2^2$$

$$= \sigma_1^2\sigma_2^2 + \sigma_1^2\lambda_2^2 + \lambda_1^2\sigma_2^2 + \cancel{\lambda_1^2\lambda_2^2} - \cancel{\lambda_1^2\lambda_2^2}$$

$$\sigma^2_{xy} = \sigma_1^2\sigma_2^2 + \lambda_1^2\sigma_2^2 + \lambda_2^2\sigma_1^2$$

$\therefore$ Hence proved

6) Two statically independent random variable 'x' & 'y' have mean values $\bar{x} = E[x] = 2$, $\bar{y} = E[y] = 4$. They have second moments $\overline{x^2} = E[x^2] = 8$, $\overline{y^2} = E[y^2] = 25$ then find (i) Mean value. (ii) second moment (iii) The variance of the random variable $w = 3x - y$ [March 2021]

Sol

(i) $\bar{x} = E[x] = 2$, $\bar{y} = E[y] = 4$

$w = 3x - y$

$E[w] = 3E[x] - E[y]$

$= 3 \times 2 - 4$

$= \cancel{+8-4} = 6 - 4 = 2$

(ii) $E[w^2] = \cancel{3E[x^2] - E[y^2]}$

$\cancel{= 3 \times 8 - 25} = 24$

$= E[(3x-y)^2]$

$= E[9x^2 + y^2 - 6xy]$

$= 9E[x^2] + E[y^2] - 6E[x]E[y]$

$= 9 \times 8 + 25 - 6 \times 2 \times 4$

$= 72 + 25 - 48 = 97 - 48 = 49$

(iii) $\sigma_w^2 = E[w^2] = E[w^2] - [E(w)]^2$

$= 49 - 2^2 = 49 - 4 = 45$

Define marginal density function and find the marginal density function of x & y such that $f_{xy} = \frac{1}{12} u(x) u(y) e^{-x/3} e^{-y/4}$. [set 3]

Ans: $f_{xy} = \frac{1}{12} u(x) u(y) e^{-x/3} e^{-y/4}$

$$f_x(x) = \int_{y=-\infty}^{\infty} f_{xy}(x,y)\,dy$$

The density functions of 'x' & 'y' which are separated from Joint density are called as marginal density function.

Now,

$$f_x(x) = \int_{y=-\infty}^{\infty} \frac{1}{12} u(x) u(y) e^{-x/3} e^{-y/4}\,dy$$

$$= \frac{1}{12} u(x) e^{-x/3} \int_{y=0}^{\infty} e^{-y/4}\,dy$$

$$= \frac{1}{12} u(x) e^{-x/3} \left[\frac{e^{-y/4}}{-1/4} \right]_0^{\infty}$$

$$= \frac{-4}{12} u(x) e^{-x/3} \left[0 - 1 \right]$$

$$f_x(x) = \frac{1}{3} u(x) e^{-x/3}$$

$$f_y(y) = \int_{x=-\infty}^{\infty} \frac{1}{12} u(x) u(y) e^{-x/3} e^{-y/4}\,dx$$

$$= \frac{1}{12} u(y) e^{-y/4} \int_{x=0}^{\infty} e^{-x/3}\,dx$$

$$= \frac{1}{12} u(y) e^{-y/4} \left[\frac{e^{-x/3}}{-1/3} \right]_0^{\infty}$$

$$= -\frac{3}{12} u(y) e^{-y/4} \left[0 - 1 \right]$$

$$= +1/4\, u(y) e^{-y/4}$$

8) Two random variables having the Joint characteristic function $\phi_{xy}(\omega_1, \omega_2) = \exp(-2\omega_1^2, -8\omega_2^2)$ then find m_{10}, m_{01}, m_{11}

Sol

$$m_{nk} = \left(\frac{1}{j}\right)^{n+k} \frac{\partial^{n+k}}{\partial\omega_1^n \, \partial\omega_2^k} \left(\phi_{xy}(\omega_1, \omega_2)\right) \Big|_{\substack{\omega_1=0 \\ \omega_2=0}}$$

(i) $$m_{10} = \left(\frac{1}{j}\right)^{1+0} \frac{\partial^{1+0}}{\partial\omega_1^1 \, \partial\omega_2^0} \exp(-2\omega_1^2 - 8\omega_2^2)$$

$$= \left(\frac{1}{j}\right) \frac{\partial}{\partial\omega_1} e^{-2\omega_1^2} - e^{-8\omega_2^2}$$

$$= \left(\frac{1}{j}\right) e^{-8\omega_2^2} (-4)\omega_1 e^{-2\omega_1^2} \Big|_{\omega_1 = 0}$$

$$= 0$$

(ii) $$m_{01} = \left(\frac{1}{j}\right)^{0+1} \frac{\partial^{0+1}}{\partial\omega_1^0 \, \partial\omega_2^1} e^{(-2\omega_1^2 - 8\omega_2^2)}$$

$$= \left(\frac{1}{j}\right) \frac{\partial}{\partial\omega_2^1} e^{-2\omega_1^2} \times e^{-8\omega_2^2}$$

$$= \left(\frac{1}{j}\right) e^{-2\omega_1^2} \frac{\partial}{\partial\omega_2^1} \left(e^{-8\omega_2^2}\right)$$

$$= \left(\frac{1}{j}\right) e^{-2\omega_1^2} (16\omega_2) \Big|_{\omega = 0}$$

$$= 0$$

(iii) $$m_{11} = \left(\frac{1}{j}\right)^{1+1} \frac{\partial^2}{\partial\omega_1 \partial\omega_2} e^{-2\omega_1^2} \times e^{-8\omega_2^2} \Big|_{\omega = 0}$$

$$= \left(\frac{1}{j}\right)^2 \frac{\partial}{\partial\omega_1} \left(e^{-2\omega_1^2}\right) \frac{\partial}{\partial\omega_1} \left(e^{-8\omega_2^2}\right)_{\omega=0}$$

$$= \left(\frac{1}{j}\right)^2 (4\omega_1)(16\omega_2) \Big|_{\omega = 0} = 0$$

Define random variables v & w by $v = x + ay$ and $w = x - ay$ where 'a' is a real number and 'x' & 'y' are random variables. Determine 'a' in terms of 'x' and 'y' such that v & w are orthogonal

$v = x + ay \qquad w = x - ay$

v and w are orthogonal.

$R_{xy} = 0 \qquad R_{vw} = 0$

$E[xy] = 0 \qquad E[vw] = 0$

$$E[(x+ay)(x-ay)] = 0$$

$$E[x^2 - \cancel{axy} + \cancel{axy} - a^2y^2] = 0$$

$$E[x^2] - a^2 E[y^2] = 0$$

$$E[x^2] = a^2 E[y^2]$$

$$a^2 = \frac{E[x^2]}{E[y^2]}$$

$$a = \sqrt{\frac{E[x^2]}{E[y^2]}}$$

10) Consider R.V y_1 & y_2 related to a arbitary random variables x & y has $y_1 = x\cos\theta + y\sin\theta$, $y_2 = -x\sin\theta + y\cos\theta$ then find (i) the co-variance of y_1 & y_2 (ii) for what values 'θ' the R.V y_1 & y_2 are uncorrelated.

Sol: (i) $C_{xy} = R_{xy} - E[x]E[y]$

$$C_{y_1y_2} = R_{y_1y_2} - E[y_1]E[y_2]$$

$$E[y_1] = E[x\cos\theta + y\sin\theta] = \cos E[x] + \sin\theta E[y]$$

$$E[y_2] = E[-x\sin\theta + y\cos\theta]$$
$$= -\sin\theta\, E[x] + \cos\theta\, E[y]$$

$$R[y_1 y_2] = E[y_1 y_2]$$
$$= E[(x\cos\theta + y\sin\theta).(-x.\sin\theta + y\cos\theta)]$$
$$= E[-x^2\sin\theta\cos\theta + xy\cos^2\theta - xy\sin^2\theta + y^2 \sin\theta\cos\theta]$$
$$= E[\sin\theta\cos\theta(y^2 - x^2) + xy(\cos^2\theta - \sin^2\theta)]$$
$$= \sin\theta\cos\theta\, E[y^2 - x^2] + (\cos^2\theta - \sin^2\theta)E[xy]$$
$$R_{y_1 y_2} = \frac{1}{2}\sin 2\theta\, E[y^2 - x^2] + \cos 2\theta\, E[xy]$$

$$E[y_1]E[y_2] = [\cos\theta\, E(x) + \sin\theta\, E(y)][-\sin\theta\, E[x] + \cos\theta\, E[y]]$$
$$= -\sin\theta\cos\theta[E(x)^2] + \cos^2\theta\, E[x]E[y] - \sin^2\theta\, E[x]E[y] + \sin\theta\cos\theta[E(y)^2]$$

$$C_{y_1 y_2} = R_{y_1 y_2} - E[y_1]E[y_2]$$
$$= \sin\theta\cos\theta\, E[y^2 - x^2] + (\cos^2\theta - \sin^2\theta)E[xy] \cancel{- \sin\theta\cos\theta\, E[x^2]}$$
$$+ \sin\theta\cos\theta[E(x)]^2 - \cos^2\theta\, E[x]E[y] + \sin^2\theta\, E[x]E[y] - \sin\theta\cos\theta[E(y)]^2$$

$$= \sin\theta\cos\theta\left[E\{y^2\} - E\{x^2\} + E\{x\}^2 - (E(y))^2\right]$$
$$+ (\cos^2\theta - \sin^2\theta)\left[E\{xy\} - E\{x\}E\{y\}\right]$$

$$= \sin\theta\cos\theta(\sigma_y^2 - \sigma_x^2) + (\cos^2\theta - \sin^2\theta)E_{xy}$$

$$\boxed{C_{y_1y_2} = \frac{1}{2}\sin 2\theta(\sigma_y^2 - \sigma_x^2) + \cos 2\theta\, E_{xy}.}$$

(ii) Equate the above equation to zero

$$C_{y_1y_2} = 0.$$

$$\frac{1}{2}\sin 2\theta(\sigma_y^2 - \sigma_x^2) + \cos 2\theta\, E_{xy} = 0.$$

4. RANDOM PROCESS - Temporial characteristics.

Random Process:-

" The concept of Random Process is an extension of the concept of Random Variable by taking time into consideration."

* The Random Process is represented with $X(\lambda, t)$.
  where 'λ' is values taken by Random Process.
  't' is time function.

i.e., Thus the Random Process is a Random Variable i.e., function of time.

Interpretations on Random Process:-

let us consider a Random Process $X(\lambda, t)$, where 'λ' is value taken by Random Process and 't' is time. Then

(i) In $X(\lambda, t)$, if 't' is fixed/ constant, then random Process $X(\lambda, t)$ is converted into time function/ signal $(X(t))$.

(ii) If 't' is fixed, the Random Process $(X(\lambda, t))$ is converted into Random variable $(X(\lambda))$.

(iii) If both 't' and 'λ' are fixed, then random Process $(X(\lambda, t))$ becomes a constant value/ Number.

****

classification of Random Process:-

For the given Random Process $X(\lambda, t)$, 'λ' takes values b/w $-\infty$ to ∞. Similarly 't' takes limits b/w $-\infty$ to ∞.

i.e $-\infty \le \lambda \le \infty$, $-\infty \le t \le \infty$

Depending upon the values taken by (λ, t) all the Random Process are classified into 4-types. They are.

1) continous Random Process
2) Discrete Random Process

3) Continous Random Sequence

4) Discrete Random Sequence

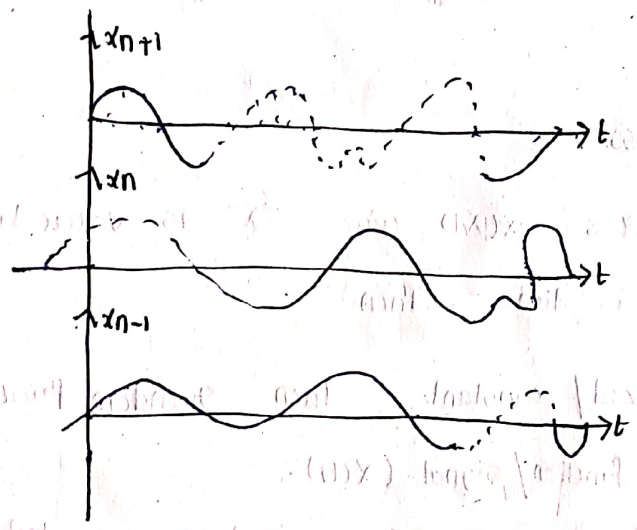
1) Continous Random Process:-

* "In this Random Process 'λ' and 't' are continous i.e., they takes each and every value along the axis."

Ex:-

(i) Desolving of sugar crystals in coffee.

(ii) Thermal Noise Generated in a circuit.



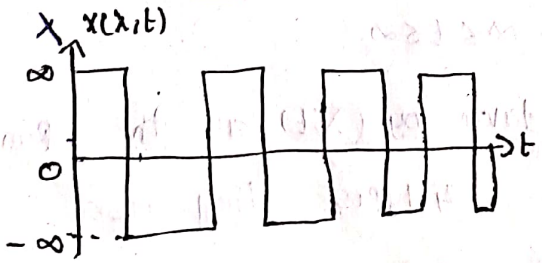
(Fig.01)

* A continous Random Process is one in which the values taken by Random Process 'λ' and time 't' both are continous."

2) Discrete Random Process:-

* "A Discrete Random Process is one in which the values taken by Random Process 'λ' is discrete and time 't' is continous."

Ex:-



This fig. can be obtained by sampling above fig (Ex.C.RP)

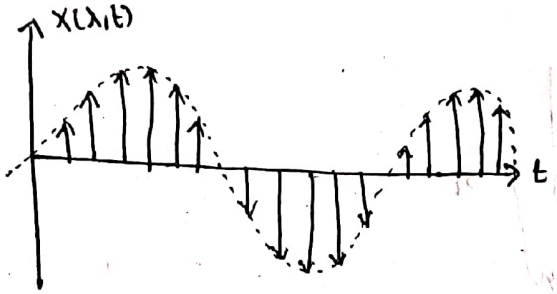
1) Here this fig., has two amplitudes (-∞ & ∞) at each.

every time instant.

3) Continous Random sequence:-

* " A continous Random Sequence is one, in which the values taken by Random Process (λ) is continous and time 't' is Discrete".

Ex:-

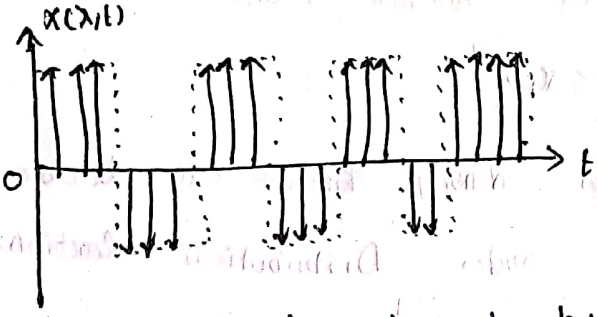


The above signal can be obtained by Periodically sampling Fig.1 w.r.t time.

4) Discrete Random Sequence:-

* " The Discrete Random sequence is one, in which the value taken by Random Process (λ) and time (t) both are Discrete".

Ex:

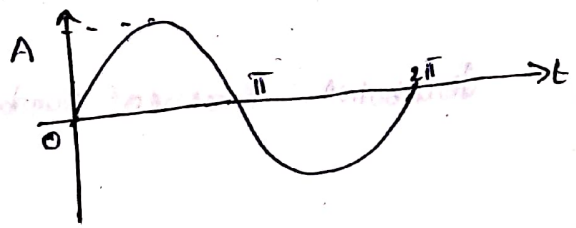


The above signal can be obtained by Periodically sampling Fig.1 with respect to both amplitude and time.

Determinstic & Non-Determinstic Random Process:-

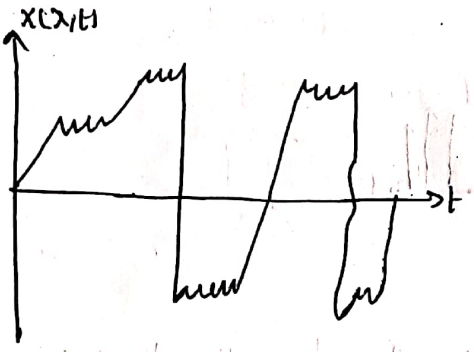
* "If future value of any sample function can be predicted from knowledge of past values, then the Random Process is called as Determinstic Random Process".

Ex: $X(\lambda,t) = A\sin(\omega t + \phi)$.



* "If future values of a sampled function can't be predicted from the values of past, then the Random Process is called as Non-Deterministic Random Process."

Ex:- (i) Dissolving sugar crystals in coffee.



Distribution & Density functions of Random Process:-

1(a) as consider a Random Process $X(x,t)$ at $t=t_1$ then the Random Process $X(x,t)$ become $X(t_1)$ or x_1 where $X(t_1)$ (or) x_1 is called as Random variable at $t=t_1$. Now the distribution Function of Random Process can be given as

$$\boxed{F_X(x_1,t_1) = P(X(t_1) \le x_1)}$$

1) also the first order density function can be obtained by differentiation first order Distribution function.

$$\boxed{f_X(x_1,t_1) = \frac{\partial}{\partial x_1}(F_X(x_1,t_1))}$$

Second order Distribution & Density functions:-

let us consider a Random Process $X(x,t)$ at $t=t_1$, $X(x,t)$ become $X(t_1)$ or x_1. Similarly at $t=t_2$ the Random Process $X(x,t)$ become $X(t_2)$ or x_2 where x_1 or x_2 are called as Random variables at $t=t_1$ and t_2.

Now, the second order distribution Function of Random Process can be given as

$$F_X(x_1, x_2 : t_1, t_2) = P(X(t_1) \le x_1, \; X(t_2) \le x_2)$$

Similarly the second order density function can be given as

$$f_X(x_1, x_2 : t_1, t_2) = \frac{\partial^2}{\partial x_1 \partial x_2} \left(F_X(x_1, x_2 : t_1, t_2) \right)$$

nth order Distribution & density Functions:-

nth order Distribution Function can be given as.

$$F_X(x_1, x_2 \ldots x_n : t_1, t_2 \ldots t_n) = P(X(t_1) \le x_1, \; X(t_2) \le x_2 \ldots X(t_n) \le x_n)$$

nth order density function is

$$f_X(x_1, x_2 \ldots x_n : t_1, t_2 \ldots t_n) = \frac{\partial^n}{\partial x_1 \partial x_2 \ldots \partial x_n} \left(F_X(x_1, x_2 \ldots x_n : t_1, t_2 \ldots t_n) \right)$$

Statistical independance:-

Case:-I

The Random $X(x,t)$ is said to be independent when, "The nth order Joint density function is equal to the Product of their marginal density functions".

$$f_X(x_1, x_2, x_3, \ldots x_n : t_1, t_2, t_3 \ldots t_n) = f_X(x_1, t_1) \, f_X(x_2, t_2) \ldots f_X(x_n, t_n)$$

Case:-II

$X(x,t)$ & $Y(y,t')$, these two random Process are said to be statistically independent when, "Their nth order Joint density function is equal to Product of their Joint density functions".

$$f_X(x_1, x_2, x_3 \ldots x_n, y_1 y_2 y_3 \ldots y_n : t_1 t_2 t_3 \ldots t_n, t_1' t_2' \ldots t_n') = f_X(x_1, x_2 \ldots x_n : t_1 t_2 t_3 \ldots t_n) \cdot f_Y(y_1, y_2 \ldots y_n : t_1' t_2' \ldots t_n')$$

Statistical Properties of Random Process:-

1) Mean:- The mean of the Random Process $X(x,t)$ (or) $X(t)$ can be given as $m_X(t) = E[X(t)]$

$$= \int_{-\infty}^{\infty} x(t) \, f_X(x) \, dx$$

2) auto correlation:-

"The auto correlation of Random Process $x(t)$ (or) $x(x,t)$ can be given as

$$R_{x(t)x(t)} = R_{x_1x_2}$$
$$= E[x_1x_2]$$
$$= E[x(t_1), x(t_2)]$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(t_1)\, x(t_2)\, f_X(x_1, x_2 : t_1, t_2)\, dx\, dy$$ "

3) Cross correlation:-

" The Cross correlation of Random Process $x(t)$ and $y(t)$ can be given as

$$R_{x(t)y(t)} = R_{XY}$$
$$= E[XY] = E[x(t) \cdot y(t)]$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(t)\, y(t)\, f_{XY}(x, y : t_1, t_2)\, dx\, dy$$ "

4) Co-variance:-

" The co-variance of Random Process $x(t)$ is given as

$$C_{x(t)x(t)} = C_{x_1x_2} = R_{x_1x_2} - E[x_1]E[x_2]$$
$$= E[R_{x_1(t)x_2(t)} - E[x_1(t)]E[x_2(t)]]$$

5) cross variance:-

" The cross variance of Random Process $x(t)$ and $y(t)$ is given as

$$C_{x(t)y(t)} = C_{XY} = R_{XY} - E[X]E[Y]$$
$$= R_{x(t)y(t)} - E[x(t)]E[y(t)]$$

****

Stationarity:-

" A Random Process is said to be stationary when if all the statistical properties like mean, variance, skew etc., are Not changing with respect time".

* Stationarity is of different types which depends upon

the density function of the Random variables.

* If the Random Process that are not Stationary, then they are called as non-Stationary Random Process.

* Stationarity is of different types. They are

a) First order Stationary Random Process
b) Secon order " " "
c) wide sense " " " (weak sense)
d) Strict sense " " " (N^{th} order)
e) Jointly wide sense Stationary Random Process

FSWSJ

a) First order Stationary Random Process:-

"A Random Process is said to be first order stationary." If the first order density is not changing with respect to time".

This can be given as

$$\boxed{f_X(x_1, t) = f_X(x_1, t + \Delta t)}$$

where Δt = change in time

* The condition for first order stationary is the mean of random process $X(t)$ is constant

$$\boxed{E[X(t)] = \text{constant}}$$

b) Second order Stationary Random Process:-

"A Random Process is said to be second order stationary," If the second order density is not changing with respect to time.

This can be given as

$$\boxed{f_X(x_1, x_2; t_1, t_2) = f_X(x_1, x_2; t_1 + \Delta t, \Delta t + t_2)}$$

* The condition for second order stationary is it's auto correlation function depends upon the time difference and independent of absolute time.

This can be given as $R_{XX}(t_1, t_2) = E[X(t_1)X(t_2)]$

Now $\tau = t_1 - t_2$

$\Rightarrow R_{XX}(t_1, t_1+\tau) = E[X(t_1) \cdot X(t_1+\tau)]$

$= R_{XX}(\tau)$.

Note:-

A second order stationary process is also a first order stationary random process.

3) Weak Sense Stationary (Wide sense stationary) Random Process:-

" A Random Process $X(t)$ is said to be weak sense stationary, then it has to satisfy two constant

i) Mean of Random Process is constant i.e

$E[X(t)] =$ constant

ii) Auto correlation function is a function of Time difference (τ) and independent of absolute time.

$R_{XX}(t_1, t_1+\tau) = E[X(t_1) \cdot X(t_1+\tau)] = R_{XX}(\tau)$

Note:-

1) If a Random Process $X(t)$ is second order stationary Random Process then it is called as weak sense stationary Random Process but converse is not true.

4) Jointly weak sense stationary Random Process:-

" If two Random Processes $X(t)$ and $Y(t)$ are jointly weak sense stationary Random Random Processes then it has to satisfy three conditions.

1) Mean of Random Process $X(t)$ is constant

$E[X(t)] =$ constant

2) Mean of Random Process $Y(t)$ is constant

$E[Y(t)] =$ constant

9) It's cross correlation is independent on time and function of time difference (τ).

$$R_{XY}(t, t+\tau) = E[X(t) \cdot Y(t+\tau)]$$
$$= R_{XY}(\tau)$$

3) Strict Sense (Nth order) Stationary Process:-

" A Random Process is said to be Strict Sense Stationary," when its nth order density function is not changing w.r.t time".

$$\boxed{f_X(x_1, x_2 \ldots x_n : t_1, t_2 \ldots t_n) = f_X(x_1, x_2 \ldots, x_n : t_1+\Delta T, t_2+\Delta T, \ldots t_n+\Delta T)}$$

(*****)

Auto Correlation:-

" Auto correlation is used to find the amount of similarity (or) coherance between the given Random Process and replica of the same Random Process delayed by some time".

The auto correlation is represented with $R_{XX}(t_1, t_2)$, which can be given as

$$R_{XX}(t_1, t_2) = E[X(t_1) \cdot X(t_2)]$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(t_1)\, x(t_2)\, f_X(x_1, x_2 : t_1, t_2)\, dx_1 dx_2.$$

* For a stationary Random Process $X(t)$, the auto correlation can be given as $R_{XX}(\tau)$

$$R_{XX}(\tau) = E[X(t) \cdot X(t+\tau)]$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(t)\, x(t+\tau)\, f_X(x_1, x_2 : t, t+\tau)\, dx_1 dx_2.$$

Properties:-

Here the properties of auto correlation are proved by assuming the given Random Process $X(t)$ as a wide sense stationary Random Process.

(i) $E[x(t)] =$ constant

(ii) $R_{xx}(\tau) = E[x(t)\, x(t+\tau)]$

1) The mean square value of Random process can substituting $\tau = 0$ in auto correlation.

Proof:-

we know that

$$R_{xx}(\tau) = E[x(t)\, x(t+\tau)]$$

$\boxed{\text{Put } \tau = 0}$

$$\Rightarrow R_{xx}(0) = E[x(t)\, x(t)]$$

$$= E[x^2(t)] = \text{m.s.v of RP}$$

2) The auto correlation $R_{xx}(\tau)$ is even function of 'τ' i.e.

$$R_{xx}(-\tau) = R_{xx}(\tau)$$

Proof:- we know that

$$R_{xx}(\tau) = E[x(t)\, x(t+\tau)]$$

Put $\tau = -\tau$

$$R_{xx}(-\tau) = E[x(t)\, x(t-\tau)]$$

Put $t - \tau = u$

$t = u + \tau$

$$= E[x(u+\tau)\, x(u)]$$

$$= R_{xx}(\tau)$$

3) If $x(t)$ is Periodic then auto correlation $R_{xx}(\tau)$ is also Periodic

Proof:- Given that $x(t)$ is Periodic i.e., $x(t) = x(t+T_0) = x(t+2T_0) = \ldots$

$x(t+\tau) = x(t+\tau+T_0) = x(t+\tau+2T_0) = \ldots$

we know that $R_{xx}(\tau) = E[x(t)\, x(t+\tau)]$

now $R_{xx}(\tau+T_0) = E[x(t)\, x(t+\tau+T_0)]$

$= E[x(t)\, x(t+\tau)]$ [$\because$ since $x(t)$ is Periodic]

$= R_{xx}(\tau)$

4) A Random Process $Z(t)$ is a sum of two random Process $x(t)$ and $y(t)$, then $R_{ZZ}(\tau) =$

Proof:-

Given $Z(t) = X(t) + Y(t)$

Now $R_{ZZ}(\tau) = E[Z(t) \cdot Z(t+\tau)] = E[(X(t)+Y(t)) \cdot Z(t+\tau)]$

$= E[[X(t)+Y(t)] [X(t+\tau) + Y(t+\tau)]]$

$= E[[X(t) X(t+\tau)] + X(t) Y(t+\tau) + Y(t) X(t+\tau) + Y(t) Y(t+\tau)]$

$= E[X(t) X(t+\tau)] + E[X(t) Y(t+\tau)] + E[Y(t) X(t+\tau)] + E[Y(t) Y(t+\tau)]$

$R_{ZZ}(\tau) = R_{XX}(\tau) + R_{XY}(\tau) + R_{YX}(\tau) + R_{YY}(\tau).$

5) If Random Process $x(t)$ with zero mean has a DC component 'A' as $Y(t) = A + x(t)$, then $R_{YY}(\tau) =$

Proof:-

we know that $Y(t) = A + x(t)$ and $E[x(t)] = 0$.

Now $R_{YY}(\tau) = E[Y(t) Y(t+\tau)]$

$= E[(A + x(t)) (A + x(t+\tau))]$

$= E[A^2 + A x(t+\tau) + A x(t) + x(t) x(t+\tau)]$

$= E[A^2] + A E[x(t+\tau)] + A E[x(t)] + E[x(t) x(t+\tau)]$

$= A^2 + 0 + 0 + R_{XX}(\tau)$

$R_{YY}(\tau) = A^2 + R_{XX}(\tau)$

By The Auto correlation Function of Random Process $R_{XX}(\tau)$ cannot have any arbitary ~~Shift~~ Shape.

Cross Correlation:-

" Cross correlation is used to find the amount of coherance (or) similarity between two Random Process $x(t)$ and $Y(t)$"

The Cross correlation is Represented by $R_{XY}(t_1, t_2)$ this can given as

$R_{XY}(t_1,t_2) = E[X(t_1)\, Y(t_2)]$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(t_1)\, Y(t_2)\, f_{XY}(x,y;t_1,t_2)\, dx\,dy \quad \text{--(1)}$$

* For Stationary Random Process $x(t)$ the ~~auto~~ cross Correlation can be given as $R_{XY}(\tau)$

$$R_{XY}(\tau) = E[x(t)\, y(t+\tau)]$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(t)\, Y(t+\tau)\, f_{XY}(x,y;\, t, t+\tau)\, dx\,dy.$$

Properties:-

i) $R_{XY}(\tau) = R_{YX}(-\tau)$ is a Symmentric Property.

Proof:-

We know that $R_{XY}(\tau) = E[x(t)\, Y(t+\tau)]$

and $R_{YX}(\tau) = E[Y(t)\, x(t+\tau)]$

Now Put $\tau = -\tau$

$R_{YX}(-\tau) = E[Y(t)\, x(t-\tau)]$ Put $t-\tau = u$

$t = u+\tau$

$= E[Y(u+\tau)\, x(u)]$

$$\boxed{R_{YX}(-\tau) = R_{XY}(\tau)}$$

(ii) If two Random Process $x(t)$ and $Y(t)$ are Statistically independent and are Jwss Process then $R_{XY}(\tau) =$

Proof:-

Given that

$x(t)$ and $Y(t)$ are Statistically independent

$E[X(t)\, Y(t)] = E[X(t)] \cdot E[Y(t)]$

$E[x(t+\tau)\, Y(t+\tau)] = E[X(t+\tau)]\, E[Y(t+\tau)]$

$x(t)$ & $Y(t)$ are wss,

(i) $E[X(t)] =$ constant

(ii) $E[Y(t)] =$ constant

(iii) $R_{XY}(t_1,t_2) = R_{XY}(\tau)$

Now, $R_{XY}(\tau) = E[X(t)\,Y(t+\tau)]$

$= E[X(t)]\,E[Y(t+\tau)]$ (∵ independent to each other).

$= E[X(t)]\,E[Y(t)]$ (∵ mean constant)

$= \bar{X}\bar{Y}$

3) If two Random Process $X(t)$ and $Y(t)$ are Statistically independent with zero mean and are Jointly WSS. then

$$\lim_{\tau \to 0} R_{XY}(\tau) = \cancel{\lim_{\tau \to 0} E[x(t)\,y(t+\tau)]}$$

$E[x(t)] = E[x(t+\tau)]$

Sol Given that $X(t)$ and $Y(t)$ are Independent

$E[X(t)\,Y(t)] = E[X(t)]\,E[Y(t)]$

$E[X(t+\tau)\,Y(t+\tau)] = E[X(t+\tau)]\,E[Y(t+\tau)]$

and $X(t)$ & $Y(t)$ are JWSS

(i) $E[X(t)] = E[Y(t)] =$ constant

(ii) $R_{XY}(t_1, t_2) = R_{XY}(\tau)$.

W.K.T $R_{XY}(\tau) = E[X(t)\,Y(t+\tau)]$

$R_{XY}(\tau) = E[X(t)]\,E[Y(t+\tau)]$

$\Rightarrow \lim_{\tau \to 0} R_{XY}(\tau) = E[X(t)]\,E[Y(t)]$

$= 0 \times 0 = 0$.

4) If $R_{XX}(\tau)$ and $R_{YY}(\tau)$ are the auto correlation functions of Random Process $X(t)$ and $Y(t)$ Respectively. Then the cross correlation Function satisfies the inequality $|R_{XY}(\tau)| \le \sqrt{R_{XX}(0)\,R_{YY}(0)}$

Proof:

In order to prove this inequality, let us consider

$$E\left[\frac{X(t)}{\sqrt{R_{XX}(0)}} + \frac{Y(t)}{\sqrt{R_{YY}(0)}}\right]^2 \ge 0$$

$$E\left[\frac{(X(t))^2}{R_{XX}(0)}\right] + E\left[\frac{(Y(t))^2}{R_{YY}(0)}\right] + \frac{2}{\sqrt{R_{XX}(0)\,R_{YY}(0)}}\,E[X(t)\,Y(t)] \ge 0$$

$$\Rightarrow \frac{R_{XX}(0)}{R_{XX}(0)} + \frac{R_{YY}(0)}{R_{YY}(0)} + \frac{2R_{XY}(\tau)}{\sqrt{R_{XX}(0)\,R_{YY}(0)}} \ge 0 \Rightarrow |R_{XY}(\tau)| \le \sqrt{R_{XX}(0)\,R_{YY}(0)}$$

Time averages of R.P:-

1) Time averaged Mean ($\overline{x}$):-

* It is represented with $\overline{x}$, it is defined as the average value of $x(t)$ taken over all the times is called as Time averaged mean. It can be given as

$$\overline{x} = A\{x(t)\}$$

$$= \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} x(t)\,dt$$

2) Time Auto Correlation:- ($R_{xx}(\tau)$)

* It is Represented with $R_{xx}(\tau)$. It is defined as the time averaged Product $x(t)$ & $x(t+\tau)$ is called Time averaged auto correlation function. It can be given as.

$$R_{xx}(\tau) = A\{x(t)\,x(t+\tau)\}$$

$$= \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} x(t)\,x(t+\tau)\,dt$$

3) Time averaged Cross correlation ($R_{xy}(\tau)$)

* It is Represented with $R_{xy}(\tau)$. It is defined as the time averaged of Product $x(t)$ & $y(t+\tau)$ is called Time averaged Cross correlation function. It is given as

$$R_{xy}(\tau) = A\{x(t)\,y(t+\tau)\}$$

$$= \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} x(t)\,y(t+\tau)\,dt$$

Ergodic theorm / Ergodicity:-

" If the all the statiscal properties of the Random process are equal to the time averaged properties of that Random process then it is called Ergodicity / Ergodic RP"

* Ergodicity is of four types.

① Mean Ergodicity Random Process:-

" If the statistical mean of the Random process is equal time averaged mean of Random process."

$$\boxed{\begin{aligned} \bar{X} &= \overline{X} \\ E[X] &= A\{x(t)\} \\ \int_{-\infty}^{\infty} x f_X(x)\,dx &= \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} x(t)\,dt. \end{aligned}}$$

② Auto correlation Ergodicity Random Process:-

" If the statiscal A-C of R.P is equal to time averaged A-C of RP".

$$\boxed{\begin{aligned} R_{XX}(\tau) &= \overline{R}_{XX}(\tau) \\ \int_{-\infty}^{\infty} x(t)\,x(t+\tau)\,f_X(x)\,dx &= \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} x(t)\,x(t+\tau)\,dt \end{aligned}}$$

③ Cross correlation Ergodicity Random Process:-

" If statiscal C-C of R.P is equal to time averaged C-C of R.P"

$$\boxed{R_{XY}(\tau) = \overline{R}_{XY}(\tau)}$$

④ Jointly ergodic R.P:-

(i) $\bar{X} = \overline{X}$

(ii) $\bar{Y} = \overline{Y}$

(iii) $R_{XY}(\tau) = \overline{R}_{XY}(\tau)$.

Q) A Random Process 'X(t)' is $x(t) = At$ where 'A' is Random Variable Uniformly distributed in interval (0,2). Find whether $x(t)$ is WSS (or) Not.

Sol Given that $x(t) = At$

and 'A' is uniformly distributed R.V in inter (0,2)

i.e.

$$f_A(A) = \begin{cases} \frac{1}{b-a}, & a \le A \le b \\ 0, & \text{elsewhere} \end{cases}$$

$$\Rightarrow \quad f_A(A) = \begin{cases} \frac{1}{2}, & 0 < A < 2 \\ 0, & \text{otherwise} \end{cases}$$

For WSS Process, $E[X(t)] =$ constant

$$R_{XX}(t_1, t_2) = R_X(\tau)$$

mean:- $E[X(t)] = E[At] = \int_{-\infty}^{\infty} At \, f_A(A) \, dA$

$$= \int_0^2 At \, \frac{1}{2} dA = \frac{t}{2}\left(\frac{A^2}{2}\right)_0^2 = \frac{t}{4}(4) = t.$$

hence mean is Not constant. So, it is not WSS Process.

Q) A Random Process $Y(t)$ is given as $Y(t) = X(t)\cos(\omega t + \theta)$, where $X(t)$ is WSS Process, ω is constant and θ is Random Variable independent of $X(t)$, uniformly distributed in interval $(-\pi, \pi)$ then find

① mean of $Y(t)$.

② auto correlation of $Y(t)$

Sol Given $Y(t) = X(t)\cos(\omega t + \theta)$

and $f_\theta(\theta) = \begin{cases} \frac{1}{2\pi}, & -\pi < \theta < \pi \\ 0, & \text{otherwise.} \end{cases}$

(i) mean of $Y(t)$

$$E[Y(t)] = E[X(t)\cos(\omega t + \theta)] = E[X(t)]\,E[\cos(\omega t + \theta)]$$

$$= K\,E[\cos\omega t \cos\theta - \sin\omega t \sin\theta]$$

$= K\,E[\cos\omega t \cos\theta] - K\,E[\sin\omega t \sin\theta]$

$= K\int_{-\infty}^{\infty} \cos\omega t \cos\theta\, f(\theta)\,d\theta - K\int_{-\infty}^{\infty} \sin\omega t \sin\theta\, f(\theta)\,d\theta$

$= K\int_{-\pi}^{\pi} \cos\omega t \cos\theta \frac{1}{2\pi}\,d\theta - K\int_{-\pi}^{\pi} \sin\omega t \sin\theta \cdot \frac{1}{2\pi}\,d\theta$

$= \frac{K}{2\pi}\cos\omega t\,(\sin\theta)_{-\pi}^{\pi} - \frac{K}{2\pi}\sin\omega t\,(-\cos\theta)_{-\pi}^{\pi}$

$= \frac{K}{2\pi}\sin\omega t\,(-1+1) = 0.$

② auto correlation of $Y(t)$ is.

$R_{YY}(\tau) = E[Y(t)Y(t+\tau)] = E[X(t)\cos(\omega t+\theta)\; X(t+\tau)\cos(\omega t+\omega\tau+\theta)]$

$= E\left[\frac{X(t)X(t+\tau)}{2}\left(\cos(2\omega t+\omega\tau+2\theta)+\cos\omega\tau\right)\right]$

$= \frac{1}{2}E[X(t)\cos\cos(2\omega t+2\theta)X(t+\tau) + X(t)X(t+\tau)\cos\omega\tau]$

$= E[X(t)X(t+\tau)]\frac{E}{2}[\cos(2\omega t+2\theta)+\omega\tau) + \cos\omega\tau]$

$= \frac{R_{XX}(\tau)}{2}\left[\int_{-\infty}^{\infty}\cos(2\omega t+2\theta+\omega\tau)\,f(\theta)\,d\theta + \int_{-\infty}^{\infty}\cos\omega\tau\, f(\theta)\,d\theta\right]$

$= \frac{R_{XX}(\tau)}{2}\cos\omega\tau$

) If a R.P $X(t) = a\cos\omega t + b\sin\omega t$ where a and b are un correlated zero mean R.V and variance is σ^2 then find

① Auto correlation.

② S.T $X(t)$ is WSS Process.

g Given that, $X(t) = a\cos\omega t + b\sin\omega t$

and a & b are uncorrelated R.V with mean=0 & variance=σ^2.

① Auto correlation $R_{XX}(\tau) = E[X(t)X(t+\tau)]$

$\Rightarrow E[A] = E[B] = 0$

$\Rightarrow E[AB] = E[A]E[B]$

$\Rightarrow \sigma_A^2 = \sigma_B^2 = \sigma^2$

① $\Rightarrow R_{xx}(\tau) =$ $\qquad \mu_2 = m_2 - m_1^2$

$E[(a\cos\omega t + b\sin\omega t)(a\cos\omega(t+\tau) + b\sin\omega(t+\tau))]$

$\Rightarrow E[a^2\cos\omega t\cos\omega(t+\tau) + ab\cos\omega t\sin\omega(t+\tau) + ab\cos\omega(t+\tau)\sin\omega t$

$+ b^2\sin\omega t\sin\omega(t+\tau)]$

$\Rightarrow E\left[\frac{a^2}{2}\right]E[\cos(\omega t+\omega t+\omega\tau) + \cos(\omega\tau)] + E\left[\frac{ab}{2}\right]E[\sin(\omega t+\omega t+\omega\tau) + \sin(\omega\tau)]$

$+ E\left[\frac{ab}{2}\right][\sin(\omega t+\omega t+\omega\tau) + \sin(\omega\tau)] - \left[\frac{b^2}{2}\right][\cos(\omega t+\omega t+\omega\tau) - \cos(\omega\tau)]$

$\Rightarrow \frac{a^2}{2}\int$

$= a^2 E[\cos\omega t\cos(\omega t+\omega\tau)] + E[A]E[B]\sin(2\omega t+\omega\tau) + a^2 E[\sin\omega t\sin\omega(t+\tau)]$

$= a^2 E[\cos\omega\tau] = a^2\cos\omega\tau.$

② mean

$E[x(t)] = 0$

The give R.P $x(t)$ is w.s.s. process.

Q A stationary R.P $x(t)$ has auto-correlation function $R_{xx}(\tau)$

$R_{xx}(\tau) = \frac{25\tau^2 + 36}{6.25\tau^2 + 4}$, then mean, mean square value, variance.

Sol (i) Mean $= \bar{x} = m_1 = E[x(t)] = \lim_{\tau\to\infty} R_{xx}(\tau).$

$= \lim_{\tau\to\infty} \frac{\tau^2\left(25 + \frac{36}{\tau^2}\right)}{\tau^2\left(6.25 + \frac{4}{\tau^2}\right)} = 4$

(ii) Mean square value $= m_2 = E(x^2(t))$

$= R_{xx}(\tau)|_{\tau=0}$

$= \frac{36}{4} = 9.$

(iii) Variance $\mu_2 = E[(x-m_1)^2] = m_2 - m_1^2$

$= 9 - 16 = -7$

Q) let $x(t)$ be the WSS Process with auto correlation function $R_{xx}(\tau) = e^{-a|\tau|}$ where a is constant. Assume that $x(t)$ amplitude modulates carrier $\cos(\omega_0 t + \theta)$, where ω_0 is constant, 'θ' is uniformly distributed R.v in $(-\pi, \pi)$ & Produces $y(t)$. If $x(t)$ and θ are statiscally indepent, determine auto correlation function of $y(t)$.

Sy Given that

$x(t)$ is WSS Process,

$$R_{xx}(\tau) = e^{-a|\tau|}$$

$x(t)$ amplitude Modulates carrier signal $\cos(\omega_0 t + \theta)$ then

$$Y(t) = x(t) \cos(\omega_0 t + \theta).$$

and $$f_\theta(\theta) = \begin{cases} \frac{1}{2\pi}, & -\pi < \theta < \pi. \\ 0, & \text{otherwise,} \end{cases}$$

and $x(t)$ and θ are statiscally independent then....

$$E[x(t) \cos(\omega_0 t + \theta)] = E[x(t)]\, E[\cos(\omega_0 t + \theta)]$$

$$\Rightarrow R_{YY}(\tau) = E[Y(t)\, Y(t+\tau)]$$

$$= E[x(t) \cos(\omega_0 t + \theta) \cdot x(t+\tau) \cos(\omega_0 (t+\tau) + \theta)]$$

$$= E[x(t)\, x(t+\tau) \cos(\omega_0 t + \theta) \cos(\omega_0 t + \omega_0 \tau + \theta)]$$

$$= \frac{1}{2} E[x(t)\, x(t+\tau) (\cos(2(\omega_0 t + \theta) + \omega_0 \tau) + \cos(\omega_0 \tau))]$$

$$= \frac{1}{2} E[x(t)\, x(t+\tau) \cos(2(\omega_0 t + \theta) + \omega_0 \tau)] + \frac{1}{2} E[$$

$$= \frac{1}{2} E[x(t)\, x(t+\tau)] \cdot E[\cos(2(\omega_0 t + \theta) + \omega_0 \tau) + \cos(\omega_0 \tau)]$$

$$= \frac{1}{2} R_{xx}(\tau) \left(\int_{-\pi}^{\pi} \cos(2(\omega_0 t + \theta) + \omega_0 \tau) \frac{1}{2\pi} d\theta \right) + \int_{-\pi}^{\pi} \cos(\omega_0 \tau) \left(\frac{1}{2\pi}\right) d\theta.$$

$$= \frac{R_{xx}(\tau)}{4\pi} \left(\cos(\omega_0 \tau)(2\pi) \right) = \frac{R_{xx}(\tau)}{2} \cos(\omega_0 \tau)$$

$$R_{YY}(\tau) = \frac{e^{-a|\tau|}}{2} \cos(\omega_0 \tau)$$

Q) A Random Process is described by $X(t) = A^2\cos^2(\omega_c t+\theta)$, where 'A' and $\omega_c t$ are constant, 'θ' is uniformly D.RV where A, ω_c are constant. 'θ' is uniformly distributed RV in interval $(-\pi, \pi)$. is X(t) is stationary (or) Not?

Sol Given $X(t) = A^2\cos^2(\omega_c t+\theta)$

and $f_\theta(\theta) = \begin{cases} \frac{1}{2\pi}, & -\pi<\theta<\pi \\ 0, & \text{otherwise.} \end{cases}$

$\cos 2\theta = \cos^2\theta - \sin^2$

$1+\cos 2\theta =$

mean $= \bar{X} = E[X(t)] = \int_{-\infty}^{\infty} X(t) f_\theta(\theta)\, d\theta$

$$= \int_{-\pi}^{\pi} A^2\cos^2(\omega_c t+\theta)\, \frac{1}{2\pi}\, d\theta$$

$$= \frac{A^2}{2\pi}\int_{-\pi}^{\pi} \frac{1+\cos 2(\omega_c t+\theta)}{2}\, d\theta$$

$$= \frac{A^2}{4\pi}\left((\theta)_{-\pi}^{\pi} + \left(\frac{\sin 2(\omega_c t+\theta)}{2}\right)_{-\pi}^{\pi}\right)$$

$$= \frac{A^2}{2}$$

Now $R_{XX}(\tau) = E[X(t)\,X(t+\tau)]$

$$= E[A^2\cos^2(\omega_c t+\theta)\, A^2\cos^2(\omega_c(t+\tau)+\theta)]$$

$$= E[A^4\cos^2(\omega_c t+\theta)\,\cos^2(\omega_c(t+\tau)+\theta)]$$

$$= A^4 E\left[\left(\frac{1+\cos 2(\omega_c t+\theta)}{2}\right)\left(\frac{1+\cos 2(\omega_c(t+\tau)+\theta)}{2}\right)\right]$$

$$= \frac{A^4}{4} E\left[1+\cos 2(\omega_c t+\omega_c\tau+\theta) + \cos 2(\omega_c t+\theta) + \cos 2(\omega_c t+\theta)\right.$$

$$= \frac{A^4}{4}\left[1+\int_{-\pi}^{\pi}\cos 2(\omega_c t+\omega_c\tau+\theta)\frac{1}{2\pi}\, d\theta + \int_{-\pi}^{\pi}\cos 2(\omega_c t+\theta)\frac{1}{2\pi}\, d\theta\right.$$

$\cos 2(\omega_c t+\omega_c\tau)+$

$$= \frac{A^4}{4}\left(\cos\omega_c\tau(2\pi)\right) = \frac{A^4}{2}\cos$$

8) Consider a Random Process $X(t) = A\cos(\omega_1 t + \theta)$ and $Y(t) = B\cos(\omega_2 t + \phi)$. where A, B, ω_1, ω_2 are constants θ, ϕ are statistically independent and uniformly distributed R.V's in interval $(0, 2\pi)$. Then S.T $X(t)$ and $Y(t)$ are jointly w.s.s.

Sol) Given that

$X(t) = A\cos(\omega_1 t + \theta)$ and $Y(t) = B\cos(\omega_2 t + \phi)$

and A, B, ω_1, ω_2 are constant.

$$f_\theta(\theta) = \begin{cases} \frac{1}{2\pi}, & 0 < \theta < 2\pi \\ 0, & \text{elsewhere} \end{cases} ; \quad f_\phi(\phi) = \begin{cases} \frac{1}{2\pi}, & 0 < \phi < 2\pi \\ 0 & \text{otherwise.} \end{cases}$$

Now,

$$\text{mean} = \bar{X} = E[X(t)] = \int_{-\infty}^{\infty} X(t)\, f_\theta(\theta)\, d\theta.$$

$$= \int_{-\infty}^{\infty} A\cos(\omega_1 t + \theta) \cdot f_\theta(\theta)\, d\theta$$

$$= \int_0^{2\pi} A\cos(\omega_1 t + \theta) \cdot \frac{1}{2\pi}\, d\theta = \frac{A}{2\pi} \left(\sin(\omega_1 t + \theta)\right)_0^{2\pi}$$

$$= \frac{A}{2\pi} \left(\sin\omega_1 t \cos\theta + \cos\omega_1 t \sin\theta\right)_0^{2\pi}$$

$$= \frac{A}{2\pi} \left((\sin\omega_1 t(1) + 0) - (\sin\omega_1 t + 0)\right) = 0.$$

Now

$$\text{mean} = \bar{Y} = E[Y(t)] = \int_{-\infty}^{\infty} Y(t)\, f_\phi(\phi)\, d\phi.$$

$$= \int_0^{2\pi} B\cos(\omega_2 t + \phi) \cdot \frac{1}{2\pi}\, d\phi$$

$\cos(A+B)$

$$= \frac{B}{2\pi} \left(\sin(\omega_2 t + \phi)\right)_0^{2\pi} = 0.$$

Now,

Auto correlation $R_{XX}(\tau) = E[X(t)\, X(t+\tau)]$

$$= E[A\cos(\omega_1 t + \theta) \cdot (B\cos(\omega_2(t+\tau) + \phi)].$$

$$= AB\, E[\cos(\omega_1 t + \theta) \cdot \cos(\omega_2 t + \omega\tau + \theta)].$$

$$= \frac{AB}{2} E[\cos(\omega_1 t + \omega_2 t + \omega\tau + 2\theta) + \cos(\omega_1 t - \omega_2 t - \omega\tau)]. = 0.$$

Q) If $Y_1(t) = X_1\cos(\omega t) + X_2\sin(\omega t)$ and $Y_2(t) = X_1\sin(\omega t) + X_2\cos(\omega t)$ where X_1 and X_2 are zero mean, independent R.V's. with unity variance. S.T the R.P $Y_1(t)$ and $Y_2(t)$ are individually W.S.S but not Jointly.

Sol. Given $Y_1(t) = X_1\cos(\omega t) + X_2\sin(\omega t)$ and $Y_2(t) = X_1\sin(\omega t) + X_2\cos(\omega t)$

$\bar{X_1} = 0$, $\bar{X_2} = 0 \Rightarrow E[X_1] = E[X_2] = 0$.

$E[X_1X_2] = E[X_1]E[X_2]$

and unity variance $\sigma_{X_1}^2 = \sigma_{X_2}^2 = 0$.

$\Rightarrow E[X_1^2] = E[X_2^2] = 1$.

Now, $\bar{Y_1} = E[Y_1] = \int_{-\infty}^{\infty} E[X_1\cos(\omega t) + X_2\sin(\omega t)]$

$= E[X_1\cos\omega t] + E[X_2\sin\omega t]$

$= \cos\omega t\, E[X_1] + \sin\omega t\, E[X_2]$

$= 0$.

$\bar{Y_2} = E[Y_2] = E[X_1\sin\omega t + X_2\cos\omega t]$

$= \sin\omega t\, E[X_1] + \cos\omega t\, E[X_2]$

$= 0$.

Auto correlation $R_{Y_1Y_1}(\tau) = E[Y_1(t)\, Y_1(t+\tau)]$

$= E[(X_1\cos(\omega t) + X_2\sin\omega t)(X_1\cos(\omega(t+\tau)) + X_2\sin(\omega(t+\tau)))]$

$= E[X_1^2\cos\omega t\cos(\omega(t+\tau)) + X_1X_2\sin(\omega t)\sin(\omega(t+\tau)) + X_1X_2\sin\omega t\cos(\omega(t+\tau))$
$+ X_2^2\sin\omega t\sin(\omega(t+\tau))]$

$= \cos\omega t\cos(\omega t+\tau) + \sin\omega t\sin(\omega(t+\tau)) + \sin\omega t\sin(\omega(t+\tau)) \cdot 0 + 0$

$= R_{Y_1Y_1}(\tau) = \cos\omega t\cos(\omega t+\tau) + \sin\omega t\sin(\omega(t+\tau))$

$= \cos(\omega\tau)$

auto correlation $R_{Y_2Y_2(\tau)}$

$$= E[(x_1 \sin\omega t + x_2 \cos\omega t)(x_1 \sin\omega(t+\tau) + x_2 \cos\omega(t+\tau))]$$

$$= E[x_1^2 \sin\omega t \sin\omega(t+\tau) + x_1x_2 \sin\omega t \cos\omega(t+\tau) + x_1x_2 \cos\omega t \sin\omega(t+\tau) + x_2^2 \cos\omega t \cos\omega(t+\tau)]$$

$$= \sin\omega t \sin\omega(t+\tau) + \cos\omega t \cos\omega(t+\tau)$$

$$= \cos(\omega\tau)$$

$\therefore$ The Random $Y_1(t)$ and $Y_2(t)$ are WSS Processes.

Now, Cross correlation,

$$R_{Y_1Y_2}(\tau) = E[Y_1(t)\, Y_2(t+\tau)]$$

$$= E[(x_1 \cos\omega t + x_2 \sin\omega t)(x_1 \sin\omega(t+\tau) + x_2 \cos\omega(t+\tau))]$$

$$= E[x_1^2 \cos\omega t \sin\omega(t+\tau) + x_1x_2 \cos(\omega t) \cos\omega(t+\tau) + x_1x_2 \sin\omega t \sin\omega(t+\tau) + x_2^2 \sin\omega t \cos\omega(t+\tau)]$$

$$= \cos\omega t \sin\omega(t+\tau) + \sin\omega t \cos\omega(t+\tau).$$

$$R_{Y_1Y_2}(\tau) = \sin(\omega\tau)$$

Here $Y_1(t)$ and $Y_2(t)$ are Not Jointly W.S.S.

2) P.T a R.P $X(t) = A\cos(\omega t + \theta)$ is a WSS Process. where ω is constant 'θ' is uniformly distributed R.V $(0, 2\pi)$.

Now Given $f_\theta(\theta) = \begin{cases} \frac{1}{2\pi}, & 0 < \theta < 2\pi \\ 0, & \text{otherwise.} \end{cases}$

$$\overline{X(t)} = E[X(t)] = \int_{-\infty}^{\infty} x(t)\, f_\theta(\theta)\, d\theta$$

$$= \int_0^{2\pi} A\cos(\omega t + \theta) \frac{1}{2\pi} d\theta$$

$$= \frac{A}{2\pi} \left(\sin(\omega t + \theta)\right)_0^{2\pi} = 0.$$

and $R_{XX}(t, t+\tau) = E[X(t) X(t+\tau)]$

$$= \int_{-\infty}^{\infty} = E[A\cos(\omega t + \theta) \cdot A\cos(\omega(t+\tau) + \theta)]$$

$$= A^2 E[\cos(\omega t + \theta) \cos(\omega t + \omega\tau + \theta)]$$

$$= \frac{A^2}{2} E[\cos(\omega t + \omega t + \omega\tau + 2\theta) + \cos(\omega\tau)]$$

$$= \frac{A^2}{2}\left[\left(\int_0^{2\pi} \cos(2(\omega t+\theta) + \omega\tau) \cdot \frac{1}{2\pi}\, d\theta\right) + \int_0^{2\pi} \cos\omega\tau \, d\theta\right]$$

$$= \frac{A^2}{2}\left[\frac{1}{2\pi}\left(\sin(2\omega t + \omega\tau + \omega\tau)\right)_0^{2\pi} + \cos\omega\tau\,(2\pi)\right]$$

$$= A^2\pi \cos\omega\tau.$$

$\therefore$ Given R.P $(X(t))$ is W.S.S.

Unit - V

Random processes : spectral characteristics

Random process $x(t)$ amplitude is randomly varied with time.
so that Fourier transform is not possible.

To study the spectral characteristics of R.P auto correlation function is used.

$$R_{xx}(\tau) = E[x(t)\, x(t+\tau)]$$

Power spectrum (or) power spectral density (PSD)

Psd : ($S_{xx}(\omega)$)

Def 1 : The psd of a WSS R.P'l $x(t)$ is defined as Fourier Transform of the auto correlation function. $R_{xx}(\tau)$.

$$S_{xx}(\omega) = \int_{-\infty}^{\infty} R_{xx}(\tau)\, e^{-j\omega\tau}\, d\tau$$

$$R_{xx}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{xx}(\omega)\, e^{j\omega\tau}\, d\omega .$$

$$R_{xx}(\tau) \xleftrightarrow{F.T} S_{xx}(\omega)$$

$x(t)$

Def 2 :

$$S_{xx}(\omega) = \lim_{T\to\infty} \frac{E\left[|X_T(\omega)|^2\right]}{2T},$$

$X_T(\omega)$ is F.T of $x(t)$ in $[-T, T]$

Average Power of the Random Process: (P_{xx})

The Average Power P_{xx} of a WSS R.P $x(t)$ is defined of time avg of auto correlation at $\tau = 0$.

$$\boxed{P_{xx} = A\{E[x^2(t)]\}}$$

$$\boxed{P_{xx} = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} E[x^2(t)]\, dt.}$$

$$\boxed{P_{xx} = R_{xx}(\tau)\Big|_{\tau=0}}$$

$$R_{xx}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{xx}(\omega)\, e^{j\omega\tau}\, d\omega$$

$$\boxed{P_{xx} = R_{xx}(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{xx}(\omega)\, d\omega.}$$

Q1.pdf

Consider WSS R.P $x(t)$

Take F.T $[-T, T]$ Let $x_T(t)$

$$x_T(t) = \begin{cases} x(t) & , \quad -T < t < T \\ 0 & , \quad \text{o.w} \end{cases}$$

$$X_T(\omega) = \int_{-\infty}^{\infty} x_T(t)\, e^{-j\omega t}\, dt = \int_{-T}^{T} x(t)\, e^{-j\omega t}\, dt.$$

$$\text{Energy} = \int_{-T}^{T} |x_T(t)|^2\, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X_T(\omega)|^2\, d\omega.$$

$$\therefore \quad P_{xx} = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |x_T(t)|^2\, dt = \frac{1}{2\pi} \lim_{T \to \infty} \int_{-\infty}^{\infty} \frac{|X_T(\omega)|^2}{2T}\, d\omega.$$

R.P

$$\therefore \quad P_{xx} = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} E[x^2(t)]\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \lim_{T\to\infty} \frac{E[|X_T(\omega)|^2]}{2T}\,d\omega$$

$$P_{xx} = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{xx}(\omega)\,d\omega.$$

$$S_{xx}(\omega) = \lim_{T\to\infty} \frac{E[|X_T(\omega)|^2]}{2T}$$

PSD

Properties of Power spectral density ($S_{XX}(\omega)$)

The properties of PSD $S_{XX}(\omega)$ for WSS R.P is

(i) $S_{XX}(\omega) \geq 0$

Proof

$$S_{XX}(\omega) = \lim_{T \to \infty} \frac{E[|X_T(\omega)|^2]}{2T}$$ is always non negative

because $E[|X_T(\omega)|^2]$ is always +ve.

(ii) The PSD at zero frequency is area under the curve of auto correlation.

$$S_{xx}(0) = \int_{-\infty}^{\infty} R_{xx}(\tau)\, d\tau.$$

Proof

$$S_{xx}(\omega) = \int_{-\infty}^{\infty} R_{xx}(\tau)\, e^{-j\omega\tau}\, d\tau$$

$$\omega = 0$$

$$S_{xx}(0) = \int_{-\infty}^{\infty} R_{xx}(\tau)\, d\tau$$

(iii) $S_{xx}(\omega)$ is always a real function.

Proof

$$S_{xx}(\omega) = \lim_{T \to \infty} \frac{E[|X_T(\omega)|^2]}{2T}$$

$|X_T(\omega)|^2$ is real so that $S_{xx}(\omega)$ is real.

(iv) The PSD of real process is even function.

$$S_{xx}(-\omega) = S_{xx}(\omega).$$

Proof

$$S_{xx}(\omega) = \int_{-\infty}^{\infty} R_{xx}(\tau)\, e^{-j\omega\tau}\, d\tau.$$

$$S_{xx}(-\omega) = \int_{-\infty}^{\infty} R_{xx}(\tau)\, e^{j\omega\tau}\, d\tau$$

$$\tau = -\tau$$

$$= \int_{-\infty}^{\infty} R_{xx}(-\tau)\, e^{-j\omega\tau}\, d\tau \qquad R_{xx}(-\tau) = R_{xx}(\tau)$$

$$= \int_{-\infty}^{\infty} R_{xx}(\tau)\, e^{-j\omega\tau}\, d\tau = S_{xx}(\omega)$$

$$S_{xx}(-\omega) = S_{xx}(\omega).$$

(v) $S_{xx}(\omega)$ is psd of a WSS R.P $x(t)$

$$\text{Avg power} = R_{xx}(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{xx}(\omega)\, d\omega.$$

<u>Proof</u>

$$R_{xx}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{xx}(\omega)\, e^{j\omega\tau}\, d\omega$$

$$\tau = 0$$

$$P_{xx} = R_{xx}(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{xx}(\omega)\, d\omega$$

(vi) *** The power spectral density ($S_{xx}(\omega)$) and time average auto correlation function ($A[R_{xx}(\tau)]$) form a fourier transform pair.

$$A[R_{xx}(\tau)] \xleftrightarrow{F.T} S_{xx}(\omega)$$

Proof

$$A[R_{xx}(\tau)] = A\{E[x(t)\,x(t+\tau)]\}$$

$$S_{xx}(\omega) = \lim_{T\to\infty} \frac{E[|X_T(\omega)|^2]}{2T}$$

$$|X_T(\omega)|^2 = X_T(\omega)\; X_T^*(\omega)$$

$$X_T(\omega) = \int_{-T}^{T} x(t_1)\, e^{-j\omega t_1}\, dt_1$$

$$X_T^*(\omega) = \int_{-T}^{T} x(t)\, e^{j\omega t}\, dt$$

$$S_{xx}(\omega) = \lim_{T\to\infty} \frac{1}{2T} E\left\{ \int_{-T}^{T} x(t_1)\, e^{-j\omega t_1}\, dt_1 \int_{-T}^{T} x(t)\, e^{j\omega t}\, dt \right\}$$

$$S_{xx}(\omega) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T}\int_{-T}^{T} \underbrace{E[x(t_1)\,x(t)]}_{R_{xx}(t,t_1)} e^{-j\omega(t_1-t)}\, dt\, dt_1$$

$$S_{xx}(\omega) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T}\int_{-T}^{T} R_{xx}(t,t_1)\, \underbrace{e^{-j\omega(t_1-t)}}\, dt\, dt_1 .$$

Part: 1

Take Inverse of $S_{xx}(\omega)$. PSD

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} S_{xx}(\omega)\, e^{j\omega\tau}\, d\omega = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T}\int_{-T}^{T} R_{xx}(t,t_1) \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{j\omega\tau}\, e^{-j\omega(t_1-t)}\, d\omega .\, dt\, dt_1$$

$$= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T}\int_{-T}^{T} R_{xx}(t,t_1) \underbrace{\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-j\omega(t_1-t-\tau)}\, d\omega}. \; dt . dt_1 .$$

$$\int_{-\infty}^{\infty} (1)\, e^{-j\omega(t_1 - t - \tau)}\, d\omega = 2\pi\,\delta(t_1 - t - \tau)$$

$$\boxed{\int_{-\infty}^{\infty} (1)\, e^{-j\omega t}\, dt = 2\pi\,\delta(\omega)}$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} S_{xx}(\omega)\, e^{j\omega\tau}\, d\omega$$

$$= \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T}\int_{-T}^{T} R_{xx}(t, t_1)\, \frac{1}{\cancel{2\pi}}\, \cancel{2\pi}\,\delta(t_1 - t - \tau)\, dt\, dt_1.$$

$t_1 = t + \tau$

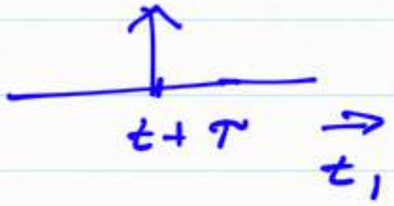
$$= \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \underbrace{\int_{-T}^{T} R_{xx}(t, t_1)\, \delta(t_1 - t - \tau)\, dt_1}\, dt.$$

$$= \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} R_{xx}(t, t+\tau)\, dt.$$

$$= A\{R_{xx}(t, t+\tau)\} = A\{R_{xx}(\tau)\}.$$

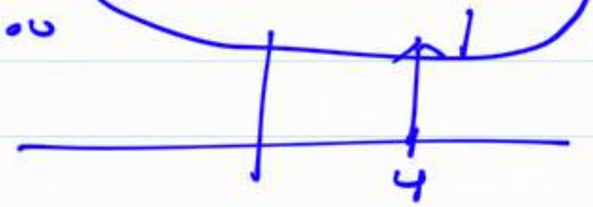
Wiener - Khintchine Relation

$\delta[t_1 - (t+\tau)]$



$$\int_{-\infty}^{\infty} \cos(t)\,\underset{t=0}{\underset{\parallel}{\delta(t)}}\,dt = \cos(t)\Big|_{t=0} = 1 \cdot 1 = 1$$

$$\int_{-\infty}^{\infty} e^{-at}\,\delta(t-4)\,dt = e^{-at}\cdot 1\Big|_{t=4} = e^{-a4}\cdot 1 = e^{-4a}.$$



(9b) Determine which of the following fn's are valid PSD

(a) $\dfrac{\cos(8\omega)}{2+\omega^4}$ (b) $e^{-(\omega-1)^2}$

Sol

$$S_{xx}(\omega) = S_{xx}(-\omega)$$

(a) $$S_{xx}(\omega) = \frac{\cos(8\omega)}{2+\omega^4}$$

$$S_{xx}(-\omega) = \frac{\cos(-8\omega)}{2+(-\omega)^4}$$

$$= \frac{\cos(8\omega)}{2+\omega^4}$$

$$= S_{xx}(\omega)$$

Real & positive valid PSD

(b) $$S_{xx}(\omega) = e^{-(\omega-1)^2}$$

$$S_{xx}(-\omega) = e^{-(-\omega-1)^2}$$

$$= e^{-(\omega+1)^2}$$

$$\neq S_{xx}(\omega)$$

Not valid PSD.

Cross Power density spectrum ($S_{xy}(\omega)$)

Def: 1

Consider two R.P's $x(t)$ & $y(t)$ they are jointly WSS then cross power density spectrum is defined as Fourier transform of cross correlation ($R_{xy}(\tau)$)

$$S_{xy}(\omega) = \int_{-\infty}^{\infty} R_{xy}(\tau)\, e^{-j\omega\tau}\, d\tau.$$

$$S_{yx}(\omega) = \int_{-\infty}^{\infty} R_{yx}(\tau)\, e^{-j\omega\tau}\, d\tau$$

$$R_{xy}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{xy}(\omega)\, e^{j\omega\tau}\, d\omega.$$

$$R_{yx}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{yx}(\omega)\, e^{j\omega\tau}\, d\omega.$$

Def: 2

$x(t)$ & $y(t)$ are two R.P's

$X_T(\omega)$ & $Y_T(\omega)$ are fourier transforms in $[-T, T]$

Then cross PSD $(S_{xy}(\omega))$ is defined as

$$S_{xy}(\omega) = \lim_{T \to \infty} \frac{E[X_T^*(\omega)\, Y_T(\omega)]}{2T}$$

$$S_{yx}(\omega) = \lim_{T \to \infty} \frac{E[Y_T^*(\omega)\, X_T(\omega)]}{2T}$$

Average cross power (P_{xy})

The avg cross power for WSS $x(t)$ & $y(t)$ is def. of the cross correlation at $\tau = 0$.

$$P_{xy} = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} R_{xy}(\tau)\, d\tau.$$

$$P_{xy} = R_{xy}(\tau) \Big|_{\tau=0} = R_{xy}(0)$$

$$P_{xy} = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{xy}(\omega)\, d\omega.$$

$$P_{yx} = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{yx}(\omega)\, d\omega.$$

Properties of cross PSD ($S_{XY}(\omega)$)

(i) $S_{XY}(\omega) = S_{YX}(-\omega) = S^*_{YX}(\omega)$

Proof

$$S_{XY}(\omega) = \int_{-\infty}^{\infty} R_{XY}(\tau)\, e^{-j\omega\tau}\, d\tau.$$

$$R_{YX}(-\tau) = R_{XY}(\tau)$$

$$S_{YX}(\omega) = \int_{-\infty}^{\infty} R_{YX}(\tau)\, e^{-j\omega\tau}\, d\tau$$

$$S_{YX}(-\omega) = \int_{-\infty}^{\infty} R_{YX}(\tau)\, e^{j\omega\tau}\, d\tau.$$

$$\tau = -\tau$$

$$= \int_{-\infty}^{\infty} R_{YX}(-\tau)\, e^{-j\omega\tau}\, d\tau$$

$$S_{YX}(-\omega) = \int_{-\infty}^{\infty} R_{XY}(\tau)\, e^{-j\omega\tau}\, d\tau$$

$$= S_{XY}(\omega).$$

(ii) The Real part of $S_{xy}(\omega)$ & $S_{yx}(\omega)$ are even functions of ω.

$$Re\{S_{xy}(\omega)\} = Re\{S_{xy}(-\omega)\}$$

Proof

$$S_{xy}(\omega) = \int_{-\infty}^{\infty} R_{xy}(\tau)\, e^{-j\omega\tau}\, d\tau.$$

$$e^{-j\omega\tau} = \cos(\omega\tau) - j\sin(\omega\tau)$$

$$Re\{S_{xy}(\omega)\} = \int_{-\infty}^{\infty} R_{xy}(\tau)\cos(\omega\tau)\, d\tau.$$

$$Re\{S_{xy}(-\omega)\} = \int_{-\infty}^{\infty} R_{xy}(\tau)\cos(-\omega\tau)\, d\tau$$

$$= \int_{-\infty}^{\infty} R_{xy}(\tau)\cos(\omega\tau)\, d\tau$$

$$= Re\{S_{xy}(\omega)\}$$

(iii) The imaginary part of $S_{XY}(\omega)$ is odd fn'l in 'ω'

$$\mathrm{Im}\{S_{XY}(\omega)\} = -\mathrm{Im}\{S_{XY}(-\omega)\}$$

(iv) $S_{xy}(\omega) = 0$ & $S_{yx}(\omega) = 0$ if $x(t)$ & $y(t)$ are orthogonal

Proof

$x(t)$ & $y(t)$ are orthogonal

$$R_{xy}(\tau) = E[x(t)\, y(t+\tau)] = 0$$

$$S_{xy}(\omega) = 0$$

(v) If $x(t)$ & $y(t)$ are uncorrelated then

$$S_{xy}(\omega) = 2\pi\, \bar{x}\, \bar{y}\, \delta(\omega)$$

Proof

$$R_{xy}(\tau) = E[x(t)\, y(t+\tau)]$$

$$= E[x(t)]\; E[y(t+\tau)]$$

$$= \bar{x}\, \bar{y}$$

$$S_{XY}(\omega) = \int_{-\infty}^{\infty} R_{XY}(\tau)\, e^{-j\omega\tau}\, d\tau$$

$$= \int_{-\infty}^{\infty} \bar{X}\,\bar{Y}\, e^{-j\omega\tau}\, d\tau$$

$$= \bar{X}\,\bar{Y} \underbrace{\int_{-\infty}^{\infty} (1)\, e^{-j\omega\tau}\, d\tau}$$

$$= \bar{X}\,\bar{Y}\; 2\pi\, \delta(\omega)$$

$$= 2\pi\, \bar{X}\,\bar{Y}\, \delta(\omega)$$

(6)** $\quad \{R_{xy}(\tau)\} \xleftrightarrow{FT} S_{xy}(\omega)$

<u>proof</u> Two RPs $x(t)$ & $y(t)$. $x_T(t)$ & $y_T(t)$ are defined in time interval $[-T, T]$

$$x_T(\omega) = \int_{-T}^{T} x(t)\, e^{-j\omega t}\, dt \quad \checkmark$$

$$y_T(\omega) = \int_{-T}^{T} y(t_1)\, e^{-j\omega t_1}\, dt_1 \quad \checkmark$$

$$S_{xy}(\omega) = \lim_{T\to\infty} \frac{1}{2T} E\left[X_T^*(\omega)\, Y_T(\omega) \right]$$

$$= \lim_{T\to\infty} \frac{1}{2T} E\left[\int_{-T}^{T} x(t)\, e^{j\omega t}\, dt \int_{-T}^{T} y(t_1)\, e^{-j\omega t_1}\, dt_1 \right]$$

$$= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T}\int_{-T}^{T} E\left[x(t)\, y(t_1) \right] e^{-j\omega(t_1 - t)}\, dt\, dt_1$$

$$E\left[x(t)\, y(t+\tau) \right]$$

Let $t_1 = t + \tau$

$$S_{xy}(\omega) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T}\int_{-T}^{T} R_{xy}(t, t+\tau)\, e^{-j\omega(t-t_1)}\, dt\, dt_1$$

Inverse fourier Transform of $S_{xy}(\omega)$.

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} S_{xy}(\omega)\, e^{j\omega\tau_1}\, d\omega$$

$$= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T}\int_{-T}^{T} R_{xy}(t, t+\tau)\, \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-j\omega(t-t_1)}\, e^{j\omega\tau_1}\, d\omega\, .\, dt\, dt_1$$

$$= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T}\int_{-T}^{T} R_{xy}(t, t+\tau)\, \underbrace{\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-j\omega(t-t_1-\tau_1)}\, d\omega}_{\delta}\, dt\, dt_1$$

$$\int_{-\infty}^{\infty} e^{-j\omega(t-t_1-\tau_1)}\, d\omega = 2\pi\,\delta(t-t_1-\tau_1)$$

$$= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T}\int_{-T}^{T} R_{xy}(t, t+\tau)\, \frac{1}{\cancel{2\pi}}\, \cancel{2\pi}\, \delta(t-t_1-\tau_1)\, .\, dt\, dt_1$$

$x' = x^{\tau_1}$

$$= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T}\int_{-T}^{T} R_{xy}(t, t+\tau)\, \delta(t-t_1-\tau_1)\, dt\, dt_1$$

$$= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} R_{xy}(t_1+\tau_1,\ t_1+\tau_1+\tau)\, dt_1$$

$t_1+\tau_1 = \eta$.

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$$= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} R_{xy}(\eta,\ \eta+\tau)\, .\, d\eta$$

$$= A\{R_{xy}(\tau)\}$$

(15) Find the PSD of WSS process $x(t)$ having auto correlation function $R_{xx}(\tau) = a\, e^{-b|\tau|}$

Proof

$$S_{xx}(\omega) = f\{R_{xx}(\tau)\}$$

$$= \int_{-\infty}^{\infty} R_{xx}(\tau)\, e^{-j\omega\tau}\, d\tau$$

$$= \int_{-\infty}^{\infty} a\, e^{-b|\tau|}\, e^{-j\omega\tau}\, d\tau$$

$$= a \int_{-\infty}^{\infty} e^{-b|\tau|}\, e^{-j\omega\tau}\, d\tau$$

$$= a \int_{-\infty}^{0} e^{+b\tau}\, e^{-j\omega\tau}\, d\tau + \int_{0}^{\infty} e^{-b\tau}\, e^{-j\omega\tau}\, d\tau$$

$$e^{-a|t|} \longleftrightarrow \frac{2a}{a^2+\omega^2}$$

$$e^{-b|\tau|} \longleftrightarrow \frac{2b}{b^2+\omega^2}$$

$$= a \left\{ \int_{-\infty}^{0} e^{(b - j\omega)\tau} \, d\tau + \int_{0}^{\infty} e^{-(b + j\omega)\tau} \, d\tau \right\}$$

$$= a \left\{ \frac{e^{(b - j\omega)\tau}}{b - j\omega} \Bigg|_{-\infty}^{0} + \frac{e^{-(b + j\omega)\tau}}{-(b + j\omega)} \Bigg|_{0}^{\infty} \right.$$

$$= a \frac{1}{b - j\omega} + \frac{+1}{+(b + j\omega)}$$

$$= \frac{2ab}{b^2 + \omega^2} .$$

(Pb) Assume ergodic R.P $x(t)$ has auto correlation

$$R_{xx}(\tau) = 18 + \frac{2}{6+\tau^2}\left[1 + 4\cos(12\tau)\right]$$

what is average power of $x(t)$.

Sol

$$P_{xx} = R_{xx}(0)$$

$$= 18 + \frac{2}{6+0}\left[1 + 4\cos(12 \cdot 0)\right]$$

$$= 18 + \frac{2}{6}\left[1 + 4\right] = 18 + 5/3$$

$$= 19.7 \text{ watt.}$$

(Q) A Random process has the PSD $S_{xx}(\omega) = \frac{6\omega^2}{1+\omega^4}$

Find the avg power.

Sol

$$P_{xx} = R_{xx}(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{xx}(\omega)\, e^{j\omega\tau}\, d\omega \Big|_{\tau=0}$$

$$P_{xx} = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{xx}(\omega)\, d\omega .$$

$$= \frac{6}{2\pi}\int_{-\infty}^{\infty} \frac{\omega^2}{1+\omega^4}\, d\omega .$$

$$= \frac{\cancel{6}^{\,3}}{\cancel{2}\pi}\,\frac{\cancel{\pi}}{2\sqrt{2}} = \frac{3}{2\sqrt{2}} .$$

$$\boxed{\int_{-\infty}^{\infty} \frac{\omega^2}{1+\omega^4}\, d\omega = \frac{\pi}{2\sqrt{2}}}$$

(Pb) Find the avg power of WSS R.P $x(t)$ which has PSD

$$S_{xx}(\omega) = \frac{\omega^2 - 17}{(\omega^2+49)(\omega^2+16)} \, d\omega .$$

Sol

$$P_{xx} = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{xx}(\omega) \, d\omega .$$

$$\frac{\omega^2 - 17}{(\omega^2+49)(\omega^2+16)} = \frac{A}{\omega^2+49} + \frac{B}{\omega^2+16}$$

$$A = \frac{-66}{-33} = 2 \qquad B = \frac{-33}{33} = -1$$

$$= \frac{2}{\omega^2+49} - \frac{1}{\omega^2+16}$$

$$P_{xx} = \frac{1}{2\pi}\left[\int_{-\infty}^{\infty} \frac{2}{\omega^2+49}\, d\omega - \int_{-\infty}^{\infty} \frac{1}{\omega^2+16}\, d\omega\right]$$

$$= \frac{1}{2\pi}\left\{ 2\,\frac{1}{7}\tan^{-1}\left(\frac{\omega}{7}\right)\Big|_{-\infty}^{\infty} - \frac{1}{4}\tan^{-1}\left(\frac{\omega}{4}\right)\Big|_{-\infty}^{\infty} \right\}$$

$$= \frac{1}{2\pi}\left\{ \frac{2}{7}\cdot\left[\tan^{-1}\infty - \tan^{-1}(-\infty)\right] - \frac{1}{4}\left(\tan^{-1}(\infty) - \tan^{-1}(-\infty)\right)\right\}$$

$$= \frac{1}{2\pi}\left[\frac{2}{7}\cdot\pi - \frac{1}{4}\pi\right]$$

$$= \frac{1}{2}\left[\frac{2}{7} - \frac{1}{4}\right] = \frac{1}{2}\,\frac{1}{28} = \frac{1}{56}.$$

(Pb) If $x(t)$ is stationary R.P find psd of

$Y(t) = A_0 + B_0 x(t)$ in terms of power spectrum of $x(t)$.

Sol

$$R_{YY}(\tau) = E[Y(t)\,Y(t+\tau)]$$

$$= E[\{A_0 + B_0 x(t)\}\{A_0 + B_0 x(t+\tau)\}]$$

$$= E[A_0^2 + A_0 B_0 x(t) + A_0 B_0 x(t+\tau) + B_0^2 x(t)\,x(t+\tau)]$$

$$= A_0^2 + A_0 B_0 E[x(t)] + A_0 B_0 E[x(t+\tau)] + B_0^2 E[x(t)\,x(t+\tau)]$$

$$f[1] \longleftrightarrow 2\pi\delta(\omega)$$

$$= A_0^2 + A_0 B_0 \bar{x} + A_0 B_0 \bar{x} + B_0^2 R_{xx}(\tau)$$

$$R_{yy}(\tau) = A_0^2 + 2 A_0 B_0 \bar{x} + B_0^2 R_{xx}(\tau)$$

$$S_{yy}(\omega) = f\{R_{yy}(\tau)\}$$

$$= f\{A_0^2 + 2 A_0 B_0 \bar{x} + B_0^2 R_{xx}(\tau)\}$$

$$= f[A_0^2] + f[2 A_0 B_0 \bar{x}] + B_0^2 f[R_{xx}(\tau)]$$

$$S_{yy}(\omega) = A_0^2 2\pi\delta(\omega) + 2 A_0 B_0 \bar{x} 2\pi\delta(\omega) + B_0^2 S_{xx}(\omega)$$

Response of Linear systems to Random signals:

Consider an LTI system

$$x(t) \longrightarrow \underset{h(t)}{\boxed{LTI}} \longrightarrow y(t)$$

$$y(t) = x(t) * h(t)$$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau.$$

For a LTI system a R.P $X(t)$ is I/P & $h(t)$ impulse response then

$$X(t) \longrightarrow \boxed{h(t)} \longrightarrow Y(t)$$

$$Y(t) = X(t) * h(t)$$

$$= \int_{-\infty}^{\infty} h(\tau)\, X(t-\tau)\, d\tau$$

Mean value of output Random process $\{E[Y(t)]\}$

$$Y(t) = X(t) * h(t) \qquad \text{WSS } X(t)$$

$$= \int_{-\infty}^{\infty} h(\tau)\, X(t-\tau)\, d\tau.$$

$$H(\omega) = \mathcal{F}\{h(\tau)\}$$

$$= \int_{-\infty}^{\infty} h(\tau)\, e^{-j\omega\tau}\, d\tau.$$

$$E[Y(t)] = E\left\{\int_{-\infty}^{\infty} h(\tau)\, X(t-\tau)\, d\tau\right\}$$

$$= \int_{-\infty}^{\infty} h(\tau)\, E\{X(t-\tau)\}\, d\tau.$$

$$H(0) = \int_{-\infty}^{\infty} h(\tau)\, d\tau.$$

$$= \int_{-\infty}^{\infty} h(\tau)\, \bar{x}\, d\tau$$

$$= \bar{x} \int_{-\infty}^{\infty} h(\tau)\, d\tau.$$

$$E[Y(t)] = \bar{x}\, H(0).$$

Auto correlation function of output Random process [$R_{YY}(\tau)$]

$X(t)$ is i/p R.P with Auto correlation $R_{XX}(\tau)$

The Auto correlation of o/p R.P

$$R_{YY}(t_1, t_2) = E[Y(t_1)\, Y(t_2)]$$

$$Y(t) = X(t) * h(t)$$

$$= E[\{X(t_1) * h(t_1)\}\{X(t_2) * h(t_2)\}]$$

$$= E\left[\int_{-\infty}^{\infty} h(\tau_1)\, X(t_1-\tau_1)\, d\tau_1 \cdot \int_{-\infty}^{\infty} h(\tau_2)\, X(t_2-\tau_2)\, d\tau_2\right]$$

$$= E\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} X(t_1-\tau_1)\, X(t_2-\tau_2)\, h(\tau_1)\, h(\tau_2)\, d\tau_1\, d\tau_2\right.$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E[x(t_1-\tau_1)\,x(t_2-\tau_2)]\,h(\tau_1)\,h(\tau_2)\,d\tau_1\,d\tau_2 .$$

$$E[x(t_1-\tau_1)\,x(t_2-\tau_2)] = R_{xx}(t_2-\tau_2-t_1+\tau_1)$$
$$= R_{xx}(t_2-t_1-\tau_2+\tau_1)$$
$$= R_{xx}(\tau-\tau_2+\tau_1)$$

$$R_{xy}(\tau) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_{xx}(\tau-\tau_2+\tau_1)\,h(\tau_1)\,h(\tau_2)\,d\tau_1\,d\tau_2 .$$

$$= \int_{-\infty}^{\infty} h(\tau_1) \underbrace{\int_{-\infty}^{\infty} R_{xx}(\tau+\tau_1-\tau_2)\,h(\tau_2)\,d\tau_2}\,d\tau_1$$

$$= \int_{-\infty}^{\infty} h(\tau_1)\left\{R_{xx}(\tau+\tau_1) * h(\tau+\tau_1)\right\} d\tau_1 .$$

$$R_{yy}(\tau) = R_{xx}(\tau) * h(\tau) * h(-\tau)$$

Mean square value of output process ($E[y^2(t)]$)

$$E[y^2(t)] = E\left[\left[h(t) * x(t)\right]^2\right]$$

$$= E\left[h(t) * x(t) \cdot h(t) * x(t)\right]$$

$$= E\left[\int_{-\infty}^{\infty} h(\tau_1)\, x(t-\tau_1)\, d\tau_1 \cdot \int_{-\infty}^{\infty} h(\tau_2)\, x(t-\tau_2)\, d\tau_2\right]$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E\left[x(t-\tau_1)\, x(t-\tau_2)\right] h(\tau_1)\, h(\tau_2)\, d\tau_1\, d\tau_2$$

$$E[y^2(t)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_{xx}(\tau_1-\tau_2)\, h(\tau_1)\, h(\tau_2)\, d\tau_1\, d\tau_2$$

Cross correlation function $\{R_{xy}(\tau)\}$

$x(t)$ i/p R.P $\quad y(t)$ o/p R.P.

$$R_{xy}(\tau) = E\{x(t)\,y(t+\tau)\}$$

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} h(\tau_1)\,x(t-\tau_1)\,d\tau_1$$

$$y(t+\tau) = \int_{-\infty}^{\infty} h(\tau_1)\,x(t+\tau-\tau_1)\,d\tau_1$$

$$R_{xy}(\tau) = E\left\{x(t)\int_{-\infty}^{\infty} h(\tau_1)\,x(t+\tau-\tau_1)\,d\tau_1\right.$$

$$= \int_{-\infty}^{\infty} h(\tau_1)\,E[x(t)\,x(t+\tau-\tau_1)]\,d\tau_1$$

$$= \int_{-\infty}^{\infty} h(\tau_1)\,R_{xx}(\tau-\tau_1)\,d\tau_1$$

$$R_{xy}(\tau) = R_{xx}(\tau) * h(\tau)$$

$$\boxed{R_{xy}(\tau) = R_{xx}(\tau) * h(\tau)}$$

similarly

$$\boxed{R_{yx}(\tau) = R_{xx}(\tau) * h(-\tau)}$$

$$R_{yy}(\tau) = R_{xx}(\tau) * h(\tau) * h(-\tau)$$

$$\boxed{R_{yy}(\tau) = R_{xy}(\tau) * h(-\tau)}$$

$$\boxed{R_{yy}(\tau) = R_{yx}(\tau) * h(\tau)}$$

** Spectral characteristic of system Response.

$X(t)$ R.P of WSS $R_{XX}(\tau)$ applied to LTI system. o/p
$Y(t)$ R.P of WSS $R_{YY}(\tau)$

PSD of Response ($S_{YY}(\omega)$)

$X(t)$ is applied to LTI system with Transfer fn. $H(\omega)$
If $S_{XX}(\omega)$ is i/p PSD then $S_{YY}(\omega)$ is o/p PSD

$$S_{YY}(\omega) = |H(\omega)|^2 S_{XX}(\omega).$$

Proof

$$S_{YY}(\omega) = F.T\{R_{YY}(\tau)\}$$
$$= \int_{-\infty}^{\infty} R_{YY}(\tau)\, e^{-j\omega\tau}\, d\tau$$

$$R_{yy}(\tau) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_{xx}(\tau+\tau_1-\tau_2)\, h(\tau_1)\, h(\tau_2)\, d\tau_1\, d\tau_2 .$$

$$S_{yy}(\omega) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_{xx}(\tau+\tau_1-\tau_2)\, h(\tau_1)\, h(\tau_2)\, d\tau_1\, d\tau_2\, e^{-j\omega\tau}\, d\tau .$$

$$= \int_{-\infty}^{\infty} h(\tau_1) \int_{-\infty}^{\infty} h(\tau_2) \int_{-\infty}^{\infty} R_{xx}(\tau+\tau_1-\tau_2)\, e^{-j\omega\tau}\, d\tau .\; d\tau_2\, d\tau_1 .$$

$$= \int_{-\infty}^{\infty} h(\tau_1)\, e^{j\omega\tau_1} \int_{-\infty}^{\infty} h(\tau_2)\, e^{-j\omega\tau_2} \int_{-\infty}^{\infty} R_{xx}(\tau+\tau_1-\tau_2)\, e^{-j\omega(\tau+\tau_1-\tau_2)}\, d\tau .\; d\tau_2\, d\tau_1 .$$

$$\tau+\tau_1-\tau_2 = \eta$$

$$= \int_{-\infty}^{\infty} h(\tau_1)\, e^{j\omega\tau_1} \int_{-\infty}^{\infty} h(\tau_2)\, e^{-j\omega\tau_2} \underbrace{\int_{-\infty}^{\infty} R_{xx}(\eta)\, e^{-j\omega\eta}\, d\eta}_{S_{xx}(\omega)} .\; d\tau_2\, d\tau_1$$

$$= \int_{-\infty}^{\infty} h(\tau_1) e^{j\omega\tau_1} d\tau_1 \int_{-\infty}^{\infty} h(\tau_2) e^{-j\omega\tau_2} d\tau_2 \cdot S_{xx}(\omega).$$

$$= H^*(\omega) \quad H(\omega) \quad S_{xx}(\omega)$$

(or)

$$= H(-\omega) \quad H(\omega) \quad S_{xx}(\omega).$$

$$\boxed{S_{yy}(\omega) = |H(\omega)|^2 S_{xx}(\omega).}$$

Cross PSD $\{S_{xy}(\omega)\}$

$$R_{xy}(\tau) = R_{xx}(\tau) * h(-\tau)$$

$$S_{xy}(\omega) = S_{xx}(\omega)\, H^*(\omega) = S_{xx}(\omega)\, H(-\omega)$$

$$R_{yx}(\tau) = R_{xx}(\tau) * h(\tau)$$

$$S_{yx}(\omega) = S_{xx}(\omega)\, H(\omega).$$

$$S_{yy}(\omega) = S_{xy}(\omega) \cdot H(\omega)$$

$$S_{yy}(\omega) = S_{yx}(\omega) \cdot H^*(\omega)$$