UNIT-1

Mathematical Logic : Propositional Calculus: Statements and Notations, Connectives, Truth Tables, Tautologies, Equivalence of Formulas, Duality law, Tautological Implications, Normal Forms, Theory of Inference for Statement Calculus, Consistency of Premises, Indirect Method of Proof.
Predicate calculus: Predicative Logic, Statement Functions, Variables and Quantifiers, Free & Bound Variables, Inference theory for predicate calculus.

Introduction: ***Logic*** is the study of reasoning. One of the main aims of logic is to provide rules by which one can determine whether any particular argument or reasoning is valid. Any collection of rules needs a language in which these rules are stated. Natural languages are not always precise enough. So a formal language called "***Object language***" is used. A formal language is one in which the syntax is well defined. Inorder to avoid ambiguity, we use symbols in object languages. An additional reason to use symbols is that they are easy to write and manipulate. The study of object language requires the use of another language. For this purpose we can choose any of the natural languages. This natural language (preferably english) will be called as "***metalanguage***".

Propositional Calculus:

Statements and Notations:-

- Statements are the basic units of object language.
- Statements are also called as "*propositions".*
- Statements are of mainly 2 types
  - Primary statements
  - Compound statements

Primary statements:- A set of declarative sentences which cannot be further broken down into simpler sentences are called primary statements. These are also called as **atomic or primitive statements**. Each primary statement can have one and only one of the two possible values called "***truth values***". The truth values are "*true*" and "*false*" denoted by the symbols T and F respectively (sometimes with 1 or 0). Since we admit only 2 possible truth values, our logic is sometimes called a "***two-valued logic***".
Primitive statements are denoted by distinct symbols from the capital letters A,B,C......,P,Q,.....

> *"All declarative sentences to which it is possible to assign one and only one truth value are called statements".*

Examples:

1. India is a country.
2. 1 + 101 = 110.
3. Close the door.
4. This statement is false.
5. Delhi is the capital of Nepal.

Statements (1) and (5) have truth values *true* and *false* respectively. Statement (2) has truth value depending on the context. If both the numbers are treated as ***decimal*** the truth value is '*false'* but if they are considered as ***binary*** the truth value is '*true'*. So it has only one truth value depending on the context. Statement (3) is not a statement but an order. Statement (4) is not a statement as we can't properly assign a truth value to it.

So we can represent the statements as below:

P : India is a country

Q : 1 + 101 = 110

R : Delhi is the capital of Nepal.

Connectives:

These are the connecting words or expressions used to construct complicated statements from simpler statements. The statements constructed using simper statements (primary) are called ***compound statements***. Connectives in natural language are "and", "but", "or" etc.

Connectives in object language:

1) **Negation:** The negation of a statement is generally formed by introducing the word "not" at a proper place in the statement or by prefixing the statement with the phrase "It is not the case that". If "P" denotes a statement, then the negation of "P" is written as "$\neg$P" and read as "not P". If the truth value of "P" is "T", then the truth value of "$\neg P$" is "F" and vice-versa.

P	**$\neg$P**
T	F
F	T

Truth table for negation

Examples:

- P : London is a city.
  $\neg$P : London is not a city (or)
  $\neg$P : It is not the case that London is a city.

- P : I went to my class yesterday.
  $\neg$P : I did not go to my class yesterday.
  $\neg$P : I was absent from my class yesterday.
  $\neg$P : It is not the case that I went to my class yesterday.

Negation is a unary operation and the alternate symbols used are '~', bar ,NOT, (~p, $\overline{p}$, NOT p)

2) **Conjunction:** The conjunction of two statements P and Q is the statement $P \wedge Q$ which is read as "P and Q". The statement $P \wedge Q$ has the truth value T whenever both P and Q have truth value T, otherwise it has the truth value F.

P	**Q**	**P $\wedge$ Q**
T	T	T
T	F	F
F	T	F
F	F	F

Truth table

Examples:

- P : It is raining today.
  Q : There are 20 tables in this room.
  $P \wedge Q$: It is raining today and there are 20 tables in this room.
- Translate the following into symbolic statements : *Jack and Jill went up the hill*
  Step1:-
  Jack went up the hill.
  Jill went up the hill. (paraphrasing)
  Step2:-
  P : Jack went up the hill.
  Q : Jill went up the hill.
  Step3:-
  $P \wedge Q$
  Note:- $P \wedge Q$ and $Q \wedge P$ should have the same truth values.

- Roses are red and violets are blue.
  P : Roses are red.
  Q : Violets are blue.
  $P \wedge Q$
- He opened the book and started to read
  P: He opened the book
  Q: He started to read
  In this example P∧Q is not the same as Q∧P. So conjunction should not be used.(Q∧P means he started to read and opened the book which is not meaningful)

3) Disjunction: The disjunction of two statements P and Q is the statement $P \vee Q$ which is read as "P or Q". The statement $P \vee Q$ has the truth value F only when both P and Q have truth value F, otherwise it has the truth value T.

P	Q	PVQ
T	T	T
T	F	T
F	T	T
F	F	F

Truth table

Examples:

- I shall watch the game on television or go to the game.
  Step1:-
  P : I shall watch the game on television.
  Q : I go to the game.
  Step2:-
  In this case "or" is used in the ***exclusive*** sense i.e, one or other possibility exists but not both. So "$\vee$" is not used here.

- There is something wrong with the bulb or with the wiring.
  Here the "or" was ***inclusive*** i.e, the intended meaning is clearly one or other or both. So "$\vee$" can be used here.

Step1:-

P : There is something wrong with the bulb.

Q : There is something wrong with the wiring.

Step2:-

$P \vee Q$

Statement Formulas and Truth tables:-

- ✓ The statements which do not contain any connectives are called ***atomic*** or *primary* or *simple* statements.
- ✓ The statements which contain one or more primary statements and some connectives are called ***molecular*** or *composite* or *compound* statements.
- ✓ Examples: Let P and Q are two statements, then some of the compound statements formed by using P and Q are
  - $\neg P$
  - $P \vee Q$
  - $(P \wedge Q) \vee (\neg P)$
  - $\neg P \vee (\neg Q)$ ...

Examples: Construct the truth table for the statement formulas

1) $((P \wedge Q) \vee \neg Q)$

P	Q	$P \wedge Q$	$\neg Q$	$((P \wedge Q) \vee \neg Q)$
T	T	T	F	T
T	F	F	T	T
F	T	F	F	F
F	F	F	T	T

2) $\neg (P \wedge Q) \vee (\neg R)$

P	Q	R	$P \wedge Q$	$\neg(P \wedge Q)$	$\neg R$	$\neg(P \wedge Q) \vee \neg R$
T	T	T	T	F	F	F
T	T	F	F	T	T	T
T	F	T	F	T	F	T
T	F	F	F	T	T	T
F	T	T	F	T	F	T
F	T	F	F	T	T	T
F	F	T	F	T	F	T
F	F	F	F	T	T	T

Conditional statements:- If P and Q are any two statements , then the statement $P \rightarrow Q$ which is read as "If P then Q" is called a conditional statement. The statement $P \rightarrow Q$ has a truth value F when Q has truth value F and P the truth value T, otherwise it has the truth value T.

P	Q	$P \rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

The statement P is called the antecedent and Q the consequent in $P \to Q$

Examples:

- Express in english the statement $P \to Q$ where

  P : The sun is shining today.

  Q : 2 + 7 > 4

  Sol:- If the sun is shining today, then 2 + 7 > 4

- Write the following statement in symbolic form

  If either Jerry takes calculus or Ken takes sociology, then Larry will take English

  Sol:- P : Jerry takes calculus

  Q : Ken takes sociology.

  R : Larry takes English.

  $$(P \vee Q) \to R$$

- Write the following statement in symbolic form

  The crop will be destroyed if there is a flood.

  Sol:- A : The crop will be destroyed.

  B : There is a flood.

  $$B \to A$$

Example: Construct the truth table $(P \vee \neg Q) \to (P \wedge Q)$

P	**Q**	**¬Q**	**P∨ ¬Q**	**P∧Q**	**(P∨ ¬Q) → (P∧Q)**
T	T	F	T	T	T
T	F	T	T	F	F
F	T	F	F	F	T
F	F	T	T	F	F

Biconditional statements: If P and Q are any two statements , then the statement $P \leftrightarrow Q$ which is read as "P If and only if Q" and abbreviated as "P iff Q" is called a biconditional statement. The statement $P \leftrightarrow Q$ has a truth value T whenever both P and Q have identical values. $P \leftrightarrow Q$ is also translated as "P is necessary and sufficient for Q".

P	**Q**	**P↔ Q**
T	T	T
T	F	F
F	T	F
F	F	T

Example: Construct the truth table $(\neg P \wedge \neg Q) \leftrightarrow (P \wedge \neg Q)$

P	**Q**	$\neg P$	$\neg Q$	$\neg P \wedge \neg Q$	$P \wedge \neg Q$	$(\neg P \wedge \neg Q) \leftrightarrow (P \wedge \neg Q)$
T	T	F	F	F	F	T
T	F	F	T	F	T	F
F	T	T	F	F	F	T
F	F	T	T	T	F	F

Operator Precedence:-

Operator	Precedence
$\neg$	1
$\wedge$ $\vee$	2 3
$\rightarrow$ $\leftrightarrow$	4 5

Logical NAND:- The logical NAND is an operation on two logical values, typically the values of two propositions, that produces a value of *false* if both of its operands are true. In other words, it produces a value of *true* if at least one of its operands is false.The truth table for P NAND Q (also written as P ↑ Q or P | Q) is as follows:

Logical NAND		
P	*Q*	*P* ↑ *Q*
T	T	F
T	F	T
F	T	T
F	F	T

The negation of a conjunction: ¬(*P* ∧ *Q*), and the disjunction of negations: (¬*P*) ∨ (¬*Q*) can be tabulated as follows:

P	*Q*	*P* ∧ *Q*	¬(*P* ∧ *Q*)	¬*P*	¬*Q*	(¬*P*) ∨ (¬*Q*)
T	T	T	F	F	F	F
T	F	F	T	F	T	T
F	T	F	T	T	F	T
F	F	F	T	T	T	T

Logical NOR :The logical NOR is an operation on two logical values, typically the values of two propositions, that produces a value of *true* if both of its operands are false. In other words, it produces a value of *false* if at least one of its operands is true. ↓ is also known as the Peirce arrow after its inventor, Charles Sanders Peirce, and is a Sole sufficient operator.

The truth table for **P NOR Q** (also written as **P ↓ Q** or **P ⊥ Q**) is as follows:

Logical NOR		
P	***Q***	***P*↓ *Q***
T	T	F
T	F	F
F	T	F
F	F	T

The negation of a disjunction ¬(*P* ∨ *Q*), and the conjunction of negations (¬*P*) ∧ (¬*Q*) can be tabulated as follows:

P	Q	$P \vee Q$	$\neg(P \vee Q)$	$\neg P$	$\neg Q$	$(\neg P) \wedge (\neg Q)$
T	T	T	F	F	F	F
T	F	T	F	F	T	F
F	T	T	F	T	F	F
F	F	F	T	T	T	T

Well – Formed Formulas:-

A statement formula is an expression which is a string consisting of variables, parenthesis and connective symbols. A well-formed formula (wff) which is the recursive definition of a statement formula can be generated by the following rules:

1. A statement variable standing alone is a wff.
2. If A is a wff, then $\neg A$ is a wff.
3. If A and B are wff's, then $(A \wedge B), (A \vee B), (A \rightarrow B)$ and $(A \leftrightarrow B)$ are wffs.
4. A string of symbols containing the statement variables, connectives and parantheses is a wff, iff it can be obtained by finitely many applications of the rule 1, 2 and 3.

   Examples:

   $P \wedge Q$

   $(P \vee Q) \rightarrow R$

   $P \vee Q \rightarrow P$

   $(P \rightarrow Q) \rightarrow (\neg Q \rightarrow \neg P)$

   $\neg \neg P$

   $P \neg \neg$ this is not a wff.

Tautologies :-

- ✓ A statement formula which is true regardless of the truth values of the statements which replace the variables in it is called as a "*universally valid formula*" or a ***tautology*** or a *logical truth.*

  Examples:

  - A round circle.
  - A good- looking beautiful women.
  - A big giant.
- ✓ A statement formula which is false regardless of the truth values of the statements which replace the variables in it is called as a "***contradiction***".
- ✓ A statement formula which is neither a tautology nor a contradiction is called a "**contingency**".

A straight forward method to determine whether a given formula is a tautology is to construct its truth table. But it becomes tedious when the number of distinct variables is large or when the formula is complicated.

The number of rows in a truth table is 2^n,where n is the number of distinct variables in the formula

Eg 1: Show that $P \vee \neg P$ is a tautology and $P \wedge \neg P$ is a contradiction.

P	¬P	P∨¬P	P∧¬P
T	F	T	F
F	T	T	F

Eg 2: Show that $(P\to Q) \leftrightarrow (\neg Q\to\neg P)$ is a tautology.

P	Q	P→Q	¬Q	¬P	¬Q→¬P	(P→Q) ↔ (¬Q→¬P)
T	T	T	F	F	T	T
T	F	F	T	F	F	T
F	T	T	F	T	T	T
F	F	T	T	T	T	T

Equivalence of Formulas:- Let A and B be two statement formulas and let $P_1,P_2,....P_n$ denote all the varibles occuring in both A and B. Consider an assignment of truth values to $P_1,P_2,....P_n$ and the resulting truth values of A and B. If the truth value of A is equal to B for every one of the 2^n possible sets of truth values assigned to $P_1,P_2,....P_n$, then A and B are said to be equivalent.

Examples:

- $\neg\neg P$ is equivalent to P.
- $P \vee P$ is equivalent to P.
- $(P \wedge \neg P) \vee Q$ is equivalent to Q.
- $P \vee \neg P$ is equivalent to $Q \vee \neg Q$.

Symbol for equivalence is $\Leftrightarrow$

Eg 1: Prove $\neg P \vee Q$ *and* $P \to Q$ are logically equivalent

P	Q	¬P	¬P ∨Q	P → Q
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

Equivalent Formulas:

Equivalence	Name
$P\wedge T \Leftrightarrow P$ $P\vee F \Leftrightarrow P$	Identity Laws
$P\vee T \Leftrightarrow T$ $P\wedge F \Leftrightarrow F$	Domination Laws
$P\vee P \Leftrightarrow P$ $P\wedge P \Leftrightarrow P$	Idempotent Laws
$\neg(\neg P) \Leftrightarrow P$	Double Negation Law
$P\vee Q \Leftrightarrow Q\vee P$ $P\wedge Q \Leftrightarrow Q\wedge P$	Commutative Laws

Equivalence	Name
$P\vee (Q \vee R) \Leftrightarrow (P \vee Q) \vee R$ $P\wedge (Q \wedge R) \Leftrightarrow (P \wedge Q) \wedge R$	Associative Laws
$P\vee (Q \wedge R) \Leftrightarrow (P \vee Q) \wedge (P$ $P\wedge (Q \vee R) \Leftrightarrow (P \wedge Q) \vee (P$	Distributive Laws
$\neg(P\vee Q) \Leftrightarrow \neg P \wedge \neg Q$ $\neg(P\wedge Q) \Leftrightarrow \neg P \vee \neg Q$	De Morgan Laws
$P\vee (P \wedge Q) \Leftrightarrow P$ $P\wedge (P \vee Q) \Leftrightarrow P$	Absorption Laws
$P\vee \neg P \Leftrightarrow T$ $P\wedge \neg P \Leftrightarrow F$	Negation Laws

Example: Show that $(P \wedge Q) \rightarrow (P \vee Q)$ is a tautology

$(P \wedge Q) \rightarrow (P \vee Q)$	$\Leftrightarrow$	$\neg(P \wedge Q) \vee (P \vee Q)$	
	$\Leftrightarrow$	$(\neg P \vee \neg Q) \vee (P \vee Q)$	(De Morgan's Law)
	$\Leftrightarrow$	$((\neg P \vee \neg Q \vee P) \vee Q$	(Associative Law)
	$\Leftrightarrow$	$((\neg P \vee P \vee \neg Q) \vee Q$	(Commutative Law)
	$\Leftrightarrow$	$(\neg P \vee P) \vee (\neg Q \vee Q)$	(Associative Law)
	$\Leftrightarrow$	**T** $\vee$ **T**	(Negation Law)
	$\Leftrightarrow$	**T**	

Hence proved.

Example: Show that $\neg(P \vee (\neg P \wedge Q))$ and $(\neg P \wedge \neg Q)$ are equivalent.

$\neg(P \vee (\neg P \wedge Q))$	$\Leftrightarrow$	$\neg P \wedge \neg(\neg P \wedge Q)$	[De Morgan's Law]
	$\Leftrightarrow$	$\neg P \wedge (\neg(\neg P) \vee \neg Q)$	[De Morgan's Law]
	$\Leftrightarrow$	$\neg P \wedge (P \vee \neg Q)$	[Double negation law]
	$\Leftrightarrow$	$(\neg P \wedge P) \vee (\neg P \wedge \neg Q)$	[Distributive Law]
	$\Leftrightarrow$	**F** $\vee (\neg P \wedge \neg Q)$	[Negation Law]
	$\Leftrightarrow$	$(\neg P \wedge \neg Q) \vee$ **F**	[Commutative Law]
	$\Leftrightarrow$	$(\neg P \wedge \neg Q)$	[Identity Law]

Equivalence Proved

Duality Law:-

Two formulas A and A^* are said to be duals to each other if either one can be obtained from the other by replacing $\wedge$ by $\vee$ and $\vee$ by $\wedge$. These connectives $\wedge$ and $\vee$ are called duals of each other. If the formula A contains T or F, then A^* is obtained by replacing T by F and F by T in addition to the above mentioned interchanges.

Eg :- Write the duals of

a) $(P \vee Q) \wedge R$ $\Leftrightarrow$ The dual is $(P \wedge Q) \vee R$

b) $(P \wedge Q) \vee T$ $\Leftrightarrow$ The dual is $(P \vee Q) \wedge F$

c) $\neg(P \vee Q) \wedge (P \vee \neg(Q \wedge \neg S))$ $\Leftrightarrow$ The dual is $\neg(P \wedge Q) \vee (P \wedge \neg(Q \vee \neg S))$

Tautological Implications:- The connectives $\wedge, \vee$ and $\leftrightarrow$ are symmetric i.e, $P \wedge Q \Leftrightarrow Q \wedge P$, $P \vee Q \Leftrightarrow Q \vee P$, $P \leftrightarrow Q \Leftrightarrow Q \leftrightarrow P$. But $P \rightarrow Q \neq Q \rightarrow P$.

For any statement formula $P \rightarrow Q$, the statement formula

- ✓ "$Q \rightarrow P$" is called converse,
- ✓ "$\neg P \rightarrow \neg Q$" is called inverse and
- ✓ "$\neg Q \rightarrow \neg P$" is called its contrapositive.

				Conditional	Converse	Inverse	Contrapositive
P	**Q**	$\neg P$	$\neg Q$	$P \rightarrow Q$	$Q \rightarrow P$	$\neg P \rightarrow \neg Q$	$\neg Q \rightarrow \neg P$
T	T	F	F	T	T	T	T
T	F	F	T	F	T	T	F
F	T	T	F	T	F	F	T
F	F	T	T	T	T	T	T

A statement A is said to be ***tautologically imply*** a statement B iff $A \to B$ is a tautology. We shall denote this as $A \Rightarrow B$ and read as 'A implies B'.

Implications:

$$P \wedge Q \Rightarrow P \qquad (1)$$
$$P \wedge Q \Rightarrow Q \qquad (2)$$
$$P \Rightarrow P \vee Q \qquad (3)$$
$$\neg P \Rightarrow P \to Q \qquad (4)$$
$$Q \Rightarrow P \to Q \qquad (5)$$
$$\neg(P \to Q) \Rightarrow \neg Q \qquad (6)$$
$$P \wedge (P \to Q) \Rightarrow Q \qquad (7)$$
$$\neg P \wedge (P \vee Q) \Rightarrow Q \qquad (8)$$
$$(P \to Q) \wedge (Q \to R) \Rightarrow (P \to R) \qquad (9)$$
$$(P \vee Q) \wedge (P \to R) \wedge (Q \to R) \Rightarrow R \qquad (10)$$
$$\neg(P \to Q) \Rightarrow P \qquad (11)$$
$$\neg Q \wedge (P \to Q) \Rightarrow \neg P \qquad (12)$$

Both implication and equivalence are transitive. To say that equivalence is transitive means if $A \Leftrightarrow B$ and $B \Leftrightarrow C$ then $A \Leftrightarrow C$. To say that implication is transitive means if $A \Rightarrow B$ and $B \Rightarrow C$ then $A \Rightarrow C$.

Eg:- *Show that implication is transitive.*

sol:- Let $A \Rightarrow B$ and $B \Rightarrow C$ i.e, $A \to B$ and $B \to C$ are tautologies. Hence ($A \to B$) $\wedge$ ($B \to C$) is also a tautology. But from (9) ($A \to B$) $\wedge$ ($B \to C$) $\Rightarrow (A \to C)$. Hence $A \to C$ is also a tautology. That means $A \Rightarrow C$.

Note:- $P \leftrightarrow Q \Leftrightarrow (P \to Q) \wedge (Q \to P)$

$P \to Q \Leftrightarrow \neg P \vee Q$

Eg 1: Write an equivalent formula for $P \wedge (Q \leftrightarrow R) \vee (R \leftrightarrow P)$ *which does not contain the biconditional.*

sol:- $P \wedge (Q \leftrightarrow R) \vee (R \leftrightarrow P)$

$\Leftrightarrow P \wedge ((Q \to R) \wedge (R \to Q)) \vee ((R \to P) \wedge (P \to R))$

Eg 2: Write an equivalent formula for $P \wedge (Q \leftrightarrow R)$ which contains neither the biconditional nor the conditional.

sol:-

$P \wedge (Q \leftrightarrow R)$

$\Leftrightarrow P \wedge ((Q \to R) \wedge (R \to Q))$

$\Leftrightarrow P \wedge ((\neg Q \vee R) \wedge (\neg R \vee Q))$

$\Leftrightarrow P \wedge (\neg Q \vee R) \wedge (\neg R \vee Q)$

Normal Forms:

Let A ($P_1, P_2, \ldots P_n$) be a statement formula where $P_1 P_2, \ldots P_n$ are the atomic variables. If A has the truth value T for atleast one combination of truth values assigned to $P_1, P_2, \ldots P_n$, the A is said to be satisfiable.

The problem of determining in a finite number of steps, whether a given statement formula is a tautology or a contradiction or atleast satisfiable is known as ***decision problem***.

Disjunctive Normal Forms (DNF's):-

We use the word "**product**" in place of **"conjunction**" and "**sum**" in place of "**disjunction**".

A product of the variables and their negations in a formula is called an ***elementary product***. Similarly, a sum of the variables and their negations is called an ***elementary sum.***

Let P and Q be any two atomic variables. Then P, $P \wedge \neg P \wedge Q$, $\neg Q \wedge \neg P \wedge P$, $P \wedge \neg P$ are some examples of elementary products. On the other hand, P, $\neg P \vee \neg Q$, $P \vee Q \vee \neg P$ etc are examples of elementary sums.

A formula which is equivalent to a given formula and which consists of **sum of elementary products** is called a **distinjunctive normal form** of the given formula.

Some examples:

- $(P\wedge Q\wedge\neg R\wedge S) \vee (\neg Q\wedge S) \vee (P\wedge S)$ is in disjunctive normal form.
- $(P\vee Q\vee\neg R\vee S) \wedge (\neg Q\vee S) \wedge\neg S$ is in conjunctive normal form.
- $(P\vee R) \wedge (Q \wedge (P\vee\neg Q))$ is not in a normal form.
- $\neg P\vee Q\vee R$ and $\neg P\wedge Q\wedge R$ are in both normal forms.

Eg 1: Obtain DNF for $(P \to Q) \to (\neg R \wedge Q)$

sol:-$(P \to Q) \to (\neg R \wedge Q)$

$$\Leftrightarrow \neg (P \to Q) \vee (\neg R \wedge Q) \qquad [P \to Q \Leftrightarrow \neg P \vee Q]$$
$$\Leftrightarrow \neg (\neg P \vee Q) \vee (\neg R \wedge Q) \qquad [P \to Q \Leftrightarrow \neg P \vee Q]$$
$$\Leftrightarrow (P\wedge\neg Q) \vee (\neg R \wedge Q) \qquad \text{[De Morgan's Law]}$$

Eg 2: Obtain DNF for $P \to ((P \to Q) \wedge \neg(\neg Q \vee \neg P))$

sol:- $P\to ((P\to Q) \wedge\neg (\neg Q\vee\neg P))$

$$\Leftrightarrow \neg P \vee [(P\to Q) \wedge\neg (\neg Q\vee\neg P)] \qquad [P \to Q \Leftrightarrow \neg P \vee Q]$$
$$\Leftrightarrow \neg P \vee [(\neg P \vee Q) \wedge \neg(\neg Q \vee \neg P)] \qquad [P \to Q \Leftrightarrow \neg P \vee Q]$$
$$\Leftrightarrow \neg P \vee [(\neg P \vee Q) \wedge \neg(\neg(Q \wedge P))] \qquad \text{[De Morgan's Law]}$$
$$\Leftrightarrow \neg P \vee [(\neg P \vee Q) \wedge (Q \wedge P)] \qquad \text{[Double Negation Law]}$$
$$\Leftrightarrow \neg P \vee [(\neg P \wedge (Q \wedge P)) \vee (Q \wedge (Q \wedge P))] \qquad \text{[Distributive Law]}$$
$$\Leftrightarrow \neg P \vee [(\neg P \wedge P) \wedge Q)) \vee (Q \wedge Q) \wedge P)] \qquad \text{[Associative Law]}$$
$$\Leftrightarrow \neg P \vee [(F \wedge Q) \vee (Q \wedge Q) \wedge P)] \qquad \text{[Negation Law]}$$
$$\Leftrightarrow \neg P \vee [(F \wedge Q)) \vee (Q \wedge P)] \qquad \text{[Idempotent Law]}$$
$$\Leftrightarrow \neg P \vee (F \vee (Q \wedge P)) \qquad \text{[Domination Law]}$$
$$\Leftrightarrow \neg P \vee (Q \wedge P) \qquad \text{[Identity Law]}$$

Conjunctive Normal Forms:- A formula which is equivalent to a given formula and which consists of **product of elementary sums** is called a **conjunctive normal** form of the given formula.

Eg 1: Obtain CNF for $\neg (\neg P \wedge (Q \vee \neg(R \wedge S)))$

sol:-

$\neg (\neg P \wedge (Q \vee \neg(R \wedge S)))$

$$\Leftrightarrow \neg (\neg P) \vee \neg (Q \vee \neg (R \wedge S)) \qquad \text{[DeMorgan's Law]}$$
$$\Leftrightarrow P \vee \neg (Q \vee \neg(R \wedge S)) \qquad \text{[Double Negation Law]}$$
$$\Leftrightarrow P \vee (\neg Q\wedge \neg\neg (R \wedge S)) \qquad \text{[DeMorgan's Law]}$$
$$\Leftrightarrow P \vee (\neg Q \wedge (R \wedge S)) \qquad \text{[Double Negation Law]}$$
$$\Leftrightarrow (P\vee \neg Q) \wedge (P\vee R) \wedge (P\vee S) \qquad \text{[Distributive Law]}$$

Eg 2: Obtain CNF for $\neg((\neg P \to \neg Q) \wedge \neg R)$

sol:-

$$
\begin{aligned}
\neg((\neg P \to \neg Q) \wedge \neg R) &\Leftrightarrow \neg((\neg\neg P \vee \neg Q) \wedge \neg R) && [P \to Q \Leftrightarrow \neg P \vee Q]\\
&\Leftrightarrow \neg((P \vee \neg Q) \wedge \neg R) && [\text{Double negation Law}]\\
&\Leftrightarrow \neg(P \vee \neg Q) \vee \neg\neg R && [\text{DeMorgan's Law}]\\
&\Leftrightarrow \neg(P \vee \neg Q) \vee R && [\text{Double negation Law}]\\
&\Leftrightarrow (\neg P \wedge \neg\neg Q) \vee R && [\text{DeMorgan's Law}]\\
&\Leftrightarrow (\neg P \wedge Q) \vee R && [\text{Double negation Law}]\\
&\Leftrightarrow (\neg P \vee R) \wedge (Q \vee R) && [\text{Distributive Law}]
\end{aligned}
$$

Eg 3: Obtain CNF for $P \to (P \to Q) \wedge \neg(\neg Q \vee \neg P)$

sol:-

$$
\begin{aligned}
& P \to [(P \to Q) \wedge \neg(\neg Q \vee \neg P)] \\
&\Leftrightarrow \neg P \vee [(P \to Q) \wedge \neg(\neg Q \vee \neg P)] && [P \to Q \Leftrightarrow \neg P \vee Q]\\
&\Leftrightarrow \neg P \vee [(\neg P \vee Q) \wedge \neg(\neg(Q \wedge P))] && [P \to Q \Leftrightarrow \neg P \vee Q]\\
&\Leftrightarrow \neg P \vee [(\neg P \vee Q) \wedge (Q \wedge P)] && [\text{Double negation Law}]\\
&\Leftrightarrow \neg P \vee [(\neg P \wedge Q \wedge P) \vee (Q \wedge Q \wedge P)] && [\text{Distributive Law}]\\
&\Leftrightarrow \neg P \vee [(\neg P \wedge Q \wedge P) \vee (Q \wedge P)] && [\text{Idempotent Law}]\\
&\Leftrightarrow \neg P \vee [(\neg P \wedge P \wedge Q) \vee (Q \wedge P)] && [\text{Commutative Law}]\\
&\Leftrightarrow \neg P \vee [(F \wedge Q) \vee (Q \wedge P)] && [\text{Negation Law}]\\
&\Leftrightarrow \neg P \vee (F \vee (Q \wedge P)) && [\text{Domination Law}]\\
&\Leftrightarrow \neg P \vee (F \vee Q) \wedge (F \vee P) && [\text{Distributive Law}]\\
&\Leftrightarrow \neg P \vee (Q \wedge P) && [\text{Identity Law}]\\
&\Leftrightarrow (\neg P \vee Q) \wedge (\neg P \vee P) && [\text{Distributive Law}]\\
&\Leftrightarrow (\neg P \vee Q) \wedge T && [\text{Negation Law}]\\
&\Leftrightarrow (\neg P \vee Q) && [\text{Identity Law}]
\end{aligned}
$$

Principal Disjunctive Normal Forms:- Let P and Q be two statement variables. Let us construct all possible formulas which consists of conjunctions of P or its negations and conjunctions of Q or its negations. None of the formulas should contain both a variable and its negation.

- → For any two variables P and Q, $P \vee Q, P \vee \neg Q, \neg P \vee Q and \neg P \vee \neg Q$ are called ***minterms*** or **boolean conjunctions**.
- → For three variables P , Q and R, the min terms are $P \vee Q \vee R, P \vee Q \vee \neg R, P \vee \neg Q \vee R, \neg P \vee Q \vee R, \neg P \vee \neg Q \vee \neg R, \neg P \vee Q \vee \neg R,$ $P \vee \neg Q \vee \neg R and \neg P \vee \neg Q \vee \neg R$

P	Q	$P \wedge Q$	$P \wedge \neg Q$	$\neg P \wedge Q$	$\neg P \wedge \neg Q$
T	T	T	F	F	F
T	F	F	T	F	F
F	T	F	F	T	F
F	F	F	F	F	T

From the truth table of minterms it is clear that no two minterms are equivalent. Each minterm has the truth value T for exactly one combination of the truth values of the variables P and Q.

For every truth value T in the truth table of the given formula, select the minterm which also has the value T for the same combination of the truth values of P and Q. The disjunction of these minterms will be equivalent to the given formula.

For a given formula, an equivalent formula consisting of ***"disjunctions of minterms only"*** is known as its **Principal Disjunctive Normal Form.** Such a form is also called the ***"sum of products cononical form".***

Constructing PDNF's using truth tables

Example:- The truth tables for $P \to Q, P \vee Q$ and $\neg(P \wedge Q)$ is given below. Obtain PDNF of these formulas.

P	Q	$P \to Q$	$P \vee Q$	$\neg(P \wedge Q)$
T	T	T	T	F
T	F	F	T	T
F	T	T	T	T
F	F	T	F	T

Sol:-

$$P \to Q \Leftrightarrow (P \wedge Q) \vee (\neg P \wedge Q) \vee (\neg P \wedge \neg Q)$$
$$P \vee Q \Leftrightarrow (P \wedge Q) \vee (P \wedge \neg Q) \vee (\neg P \wedge Q)$$
$$\neg(P \wedge Q) \Leftrightarrow (P \wedge \neg Q) \vee (\neg P \wedge Q) \vee (\neg P \wedge \neg Q)$$

Example: Obtain PDNF of $(P \wedge Q) \vee (\neg P \wedge R) \vee (Q \wedge R)$

			A		B	C		
P	Q	R	$P \wedge Q$	$\neg P$	$\neg P \wedge R$	$Q \wedge R$	$A \vee B$	$A \vee B \vee C$
T	T	T	T	F	F	T	T	T
T	T	F	T	F	F	F	T	T
T	F	T	F	F	F	F	F	F
T	F	F	F	F	F	F	F	F
F	T	T	F	T	T	T	T	T
F	T	F	F	T	F	F	F	F
F	F	T	F	T	T	F	T	T
F	F	F	F	T	F	F	F	F

$$\Leftrightarrow (P \wedge Q \wedge R) \vee (P \wedge Q \wedge \neg R) \vee (\neg P \wedge Q \wedge R) \vee (\neg P \wedge \neg Q \wedge R)$$

Constructing PDNF's without using truth tables:-

- *Step 1:* Replace conditionals and biconditionals by their equivalent formulas containing only $\vee, \wedge$ and $\neg$.
- *Step 2*: Negations are applied to the variables using De Morgan's laws.
- *Step 3:* Apply distributive laws.
- *Step 4:* Drop contradictory elementary products.
- *Step 5:* Minterms are obtained in disjunctions by introducing the missing factors.
- *Step 6:* Finally delete identical minterms.

Eg 1: Obtain PDNF of $P \to Q$

$$P \to Q \Leftrightarrow \neg P \vee Q$$

$$\Leftrightarrow (\neg P \wedge T) \vee (Q \wedge T)$$
$$\Leftrightarrow (\neg P \wedge (Q \vee \neg Q)) \vee (Q \wedge (P \vee \neg P))$$
$$\Leftrightarrow (\neg P \wedge Q) \vee (\neg P \wedge \neg Q) \vee (Q \wedge P) \vee (Q \wedge \neg P)$$

Eg 2: Obtain PDNF of $P \vee Q$

$$P \vee Q \Leftrightarrow (P \wedge T) \vee (Q \wedge T)$$
$$\Leftrightarrow (P \wedge (Q \vee \neg Q)) \vee (Q \wedge (P \vee \neg P))$$
$$\Leftrightarrow (P \wedge Q) \vee (P \wedge \neg Q) \vee (Q \wedge P) \vee (Q \wedge \neg P)$$

Principal Conjunctive Normal Forms:- For a given formula, an equivalent formula consisting of ***"conjunctions of maxterms only"*** is known as its **Principal Conjunctive Normal Form**. Such a form is also called the ***"products-of-sums cononical form".*** **Maxterms** are called "**boolean disjunctions**".

- → For any two variables P and Q, $P \vee Q, P \vee \neg Q, \neg P \vee Q and \neg P \vee \neg Q$ are called min terms.
- → For three variables P , Q and R, the min terms are $P \vee Q \vee R, P \vee Q \vee \neg R, P \vee \neg Q \vee R, \neg P \vee Q \vee R, \neg P \vee \neg Q \vee \neg R, \neg P \vee Q \vee \neg R,$ $P \vee \neg Q \vee \neg R and \neg P \vee \neg Q \vee \neg R$

Eg 1: Obtain PCNF of

I. $\neg(P \vee Q)$
II. $\neg(P \rightarrow Q)$
III. $\neg(P \leftrightarrow Q)$

P	Q	$P \vee Q$	$\neg(P \vee Q)$	$P \rightarrow Q$	$\neg(P \rightarrow Q)$	$P \leftrightarrow Q$	$\neg(P \leftrightarrow Q)$
T	T	T	F	T	F	T	F
T	F	T	F	F	T	F	T
F	T	T	F	T	F	F	T
F	F	T	T	T	F	T	F

I. $\neg(P \vee Q) \Leftrightarrow (\neg P \vee \neg Q) \wedge (\neg P \vee Q) \wedge (P \vee \neg Q)$
II. $\neg(P \rightarrow Q) \Leftrightarrow (\neg P \vee \neg Q) \wedge (P \vee \neg Q)$
III. $\neg(P \leftrightarrow Q) \Leftrightarrow (\neg P \vee \neg Q) \wedge (P \vee Q)$

PCNF without using truth tables:-

Eg 1:- Obtain PCNF of $\neg(P \vee Q)$

$$\neg(P \vee Q) \Leftrightarrow \neg P \wedge \neg Q$$
$$\Leftrightarrow (\neg P \vee F) \wedge (\neg Q \vee F)$$
$$\Leftrightarrow [\neg P \vee (Q \wedge \neg Q)] \wedge [\neg Q \vee (P \wedge \neg P)]$$
$$\Leftrightarrow (\neg P \vee Q) \wedge (\neg P \vee \neg Q) \wedge (P \vee \neg Q) \wedge (\neg P \vee \neg Q)$$
$$\Leftrightarrow (\neg P \vee Q) \wedge (\neg P \vee \neg Q) \wedge (P \vee \neg Q)$$

Eg 2:- Obtain PCNF of $(\neg P \rightarrow R) \wedge (Q \leftrightarrow P)$

$$(\neg P \rightarrow R) \wedge (Q \leftrightarrow P) \Leftrightarrow (P \vee R) \wedge ((Q \rightarrow P) \wedge (P \rightarrow Q))$$

$$\Leftrightarrow (P \vee R) \wedge ((\neg Q \vee P) \wedge (\neg P \vee Q))$$
$$\Leftrightarrow (P \vee R \vee F) \wedge (\neg Q \vee P \vee F) \wedge (\neg P \vee Q \vee F)$$
$$\Leftrightarrow (P \vee R \vee (Q \wedge \neg Q)) \wedge (\neg Q \vee P \vee (R \wedge \neg R)) \wedge (\neg P \vee Q \vee (R \wedge \neg R))$$
$$\Leftrightarrow (P \vee R \vee Q) \wedge (P \vee R \vee \neg Q) \wedge (P \vee \neg Q \vee R) \wedge (P \vee \neg Q \vee \neg R) \wedge (\neg P \vee Q \vee R) \wedge (\neg P \vee Q \vee \neg R)$$
$$\Leftrightarrow (P \vee R \vee Q) \wedge (P \vee \neg Q \vee R) \wedge (P \vee \neg Q \vee \neg R) \wedge (\neg P \vee Q \vee R) \wedge (\neg P \vee Q \vee \neg R)$$

<u>Converting one P-normal form to other:-</u>

If we are given a PCNF, we can obtain its equivalent PDNF and viceversa.

Eg 1: Obtain PCNF of $(\neg P \wedge Q) \vee (\neg P \wedge \neg Q) \vee (P \wedge Q)$

Given one is a PDNF. Let it be S

$$S \Leftrightarrow (\neg P \wedge Q) \vee (\neg P \wedge \neg Q) \vee (P \wedge Q)$$
$$\neg S \Leftrightarrow (P \wedge \neg Q) \qquad (\because \text{Remaining minterms of PDNF})$$
$$\neg(\neg S) \Leftrightarrow \neg(P \wedge \neg Q)$$
$$\Leftrightarrow \neg P \vee Q \qquad \text{which is the required PCNF.}$$

Ordering and Uniqueness of Normal Forms:- Given any 'n' statement variables, we first arrange them in fixed order.

1. If capital letters are used to denote variables, then they may be arranged in alphabetical order.
   Eg:- P, A, Q, S
   sol: A, P, Q, S
2. If subscripted letters are also used, then the following is an illustration of the order that may be used: A,B,....., x, A_1,B_1,.......Z_2,.....C_3.....
   Eg:- P_1,Q,R_3,S_1,T_2,Q_3
   sol:- Q,P_1,S_1,T_2,Q_3,R_3

Minterms:- If two variables P, Q are given then we can have 2^2=4 minterms (0 to 3).

0 – 0 0	$\neg P \wedge \neg Q$ - m_0	If there are 3 variables then we have 2^3 = 8 minterms (0 to 7)
1 – 0 1	$\neg P \wedge Q$ - m1	
2 – 1 0	$P \wedge \neg Q$ - m_2	
3 – 1 1	$P \wedge Q$ - m_3	

0 – 0 0 0	5 – 1 0 1	$(\neg P \wedge \neg Q \wedge \neg R)$ -- m_0
1 – 0 0 1	6 – 1 1 0	..
2 – 0 1 0	7 – 1 1 1	..
3 – 0 1 1		$(P \wedge Q \wedge R)$ -- m_7
4 – 1 0 0		

Maxterms:- If two variables P, Q are given then we can have 2^2=4 maxterms (0 to 3).

0 – 0 0	3 – 1 1	$\neg P \vee Q$ - m_2
1 – 0 1	$P \vee Q$ - m_0	$\neg P \vee \neg Q$ - m_3
2 – 1 0	$P \vee \neg Q$ - m_1	

If there are 3 variables then we have $2^3 = 8$ minterms (0 to 7)

- Sum of products or sum of minterms can be designated by the notation Σ and product of sums or product of maxterms can be designated by the notation Π.

  Eg:- $(P \wedge Q) \vee (\neg P \wedge Q) \vee (\neg P \wedge \neg Q)$

  $\Rightarrow \Sigma(m_3, m_1, m_0)$

  $\Rightarrow \Sigma(m_0, m_1, m_3)$

  Eg:- $(\neg P \vee Q) \wedge (\neg P \vee \neg Q) \wedge (P \vee \neg Q)$

  $\Rightarrow \Pi(M_2, M_3, M_1)$

  $\Rightarrow \Pi(M_1, M_2, M_3)$

The Theory of inference for statement calculus: The main function of logic is to provide rules of inference or principles of reasoning. The theory associated with such rules is known as "*Inference theory*" because it is concerned with the inference of a conclusion from certain premises. When a conclusion is derived from certain premises by using the accepted rules of reasoning, the process of derivation is called a *deduction or formal proof*.

In any argument, a conclusion is admitted to be true provided the premises are accepted as true and the reasoning used in deriving the conclusion from the premises follows certain accepted rules of logic inference. Such an argument is called a valid argument. Any conclusion that is arrived at by following these rules is called a *valid conclusion* and the argument is called a *valid argument.*

Validity using truth table: Let A and B be two formulas. We say that "B logically follows from A" or "B is a valid conclusion of the premise A" iff $A \rightarrow B$ is a tautology i.e, $A \Rightarrow B$.

Eg 1: Determine whether the conclusion C follows logically from the premises H_1 and H_2.

a) $H_1 : P \rightarrow Q,\ H_2 : P, \quad C : Q$
b) $H_1 : P \rightarrow Q,\ H_2 : \neg P, \quad C : Q$
c) $H_1 : P \rightarrow Q,\ H_2 : \neg(P \wedge Q), \quad C : Q$
d) $H_1 : \neg P,\ H_2 : P \leftrightarrow Q, \quad C : \neg(P \wedge Q)$
e) $H_1 : P \rightarrow Q,\ H_2 : Q, \quad C : P$

Sol:-

a)

P (H_2)	Q (C)	$P \rightarrow Q$ (H_1)
T	T	T
T	F	F
F	T	T
F	F	T

C is a valid conclusion of H_1 and H_2.

b)

P	$\neg$P (H_2)	Q (C)	$P \rightarrow Q$ (H_1)
T	F	T	T
T	F	F	F
F	T	T	T
F	T	F	T

C is not a valid conclusion from H_1 and H_2.
C is true only in 3rd row but not 4th.

c)

P	$\neg$P (C)	Q	$\neg(P\wedge Q)$ (H$_2$)	$P\to Q$ (H$_1$)
T	F	T	F	T
T	F	F	T	F
F	T	T	T	T
F	T	F	T	T

C is a valid conclusion of H_1 and H_2.

d)

P	$\neg$P (H$_1$)	Q	$P\leftrightarrow Q$ (H$_2$)	$\neg(P\wedge Q)$ (C)
T	F	T	T	F
T	F	F	F	T
F	T	T	F	T
F	T	F	T	T

C is a valid conclusion of H_1 and H_2.

e)

P (C)	Q (H$_2$)	$P\to Q$ (H$_1$)
T	T	T
T	F	F
F	T	T
F	F	T

C is not a valid conclusion of H_1 and H_2.

Rules of inference:-

There are two rules of inference, ‘P’ and ‘T’ through which we demonstrate that a particular formula is a valid consequence of a given set of premises.

Rule P : A premise may be introduced at any point in the derivation.

Rule T: A formula S may be introduced in a derivation if S is tatutologically implied by any one or more of the preceding formulas in the derivation.

Before we proceed, we list some important implications and equivalences that will be referred to frequently.

Implications:-

1) $P, P\to Q \Rightarrow Q$ (Modus ponens)
2) $P, Q \Rightarrow P\wedge Q$
3) $\neg P, P\vee Q \Rightarrow Q$
4) $\neg Q, P\to Q \Rightarrow \neg P$ (Modus tollens)
5) $P\to Q, Q\to R \Rightarrow P\to R$ (Hypothetical Syllogism)
6) $P\vee Q, P\to R, Q\to R \Rightarrow R$ (Dilemma)

Equivalences:-

1) $P\to Q \Leftrightarrow \neg Q\to\neg P$
2) $P\to(Q\to R) \Leftrightarrow (P\wedge Q)\to R$
3) $\neg(P\leftrightarrow Q) \Leftrightarrow P\leftrightarrow\neg Q$
4) $P\leftrightarrow Q \Leftrightarrow (P\to Q)\vee(Q\to P)$
5) $P\leftrightarrow Q \Leftrightarrow (P\wedge Q)\vee(\neg P\wedge\neg Q)$

Eg 1: Show that R is a valid inference from the premises $P \to Q, Q \to R$ and P.

sol:	{1}	(1)	$P \to Q$	Rule P
	{2}	(2)	P	Rule P
	{1,2}	(3)	Q	Rule T, (1), (2), P , $P \to Q \Rightarrow Q$
	{4}	(4)	$Q \to R$	Rule P
	{1,2,4}	(5)	R	Rule T, (3), (4), P , $P \to Q \Rightarrow Q$

Eg 2: Show that $R \vee S$ *follows logically from the premises* $C \vee D$, $(C \vee D) \to \neg H$, $\neg H \to (A \wedge \neg B)$, *and* $(A \wedge \neg B) \to (R \vee S)$.

sol:	{1}	(1)	$(C \vee D) \to \neg H$	Rule P
	{2}	(2)	$\neg H \to (A \wedge \neg B)$	Rule P
	{1,2}	(3)	$(C \vee D) \to (A \wedge \neg B)$	Rule T, (1), (2), $(P \to Q) \wedge (Q \to R) \Rightarrow (P \to R)$
	{4}	(4)	$(A \wedge \neg B) \to (R \vee S)$	Rule P
	{1,2,4}	(5)	$(C \vee D) \to (R \vee S)$	Rule T, (3), (4), $(P \to Q) \wedge (Q \to R) \Rightarrow (P \to R)$
	{6}	(6)	$C \vee D$	Rule P
	{1,2,4,6}	(7)	$R \vee S$	Rule T, (5), (6), $P, P \to Q \Rightarrow Q$

Eg 3: Show that $S \vee R$ *is a tautologically implied by* $(P \vee Q) \wedge (P \to R) \wedge (Q \to S)$

sol:	{1}	(1)	$P \vee Q$	Rule P
	{1}	(2)	$\neg P \to Q$	Rule T, (1), $P \vee Q \Leftrightarrow \neg P \to Q$
	{3}	(3)	$Q \to S$	Rule P
	{1,3}	(4)	$\neg P \to S$	Rule T, (2), (3), $P \to Q$ & $Q \to R \Rightarrow P \to R$
	{1,3}	(5)	$\neg S \to P$	Rule T, (4), $P \to Q \Leftrightarrow \neg Q \to \neg P$
	{6}	(6)	$P \to R$	Rule P
	{1,3,6}	(7)	$\neg S \to R$	Rule T, (5), (6), $P \to Q$ & $Q \to R \Rightarrow P \to R$
	{1,3,6}	(8)	$S \vee R$	Rule T, (7), $P \to Q \Leftrightarrow \neg P \vee Q$

Eg 4: Show that $R \wedge (P \vee Q)$ *is a valid inference from the premises* $P \vee Q, Q \to R$ and $P \to M, \neg M$.

sol:	{1}	(1)	$P \to M$	Rule P
	{2}	(2)	$\neg M$	Rule P
	{1,2}	(3)	$\neg P$	Rule T, (1), (2), $\neg Q, P \to Q \Rightarrow \neg P$
	{4}	(4)	$P \vee Q$	Rule P
	{1,2,4}	(5)	Q	Rule T, (3),(4), $\neg P, P \vee Q \Rightarrow Q$
	{6}	(6)	$Q \to R$	Rule P
	{1,2,4,6}	(7)	R	Rule T, (5), (6), $P, P \to Q \Rightarrow Q$
	{1,2,4,6}	(8)	$R \wedge (P \vee Q)$	Rule T, (4), (7), $P, Q \Rightarrow P \wedge Q$

Eg 5: Show that $\neg Q,\ P \to Q \Rightarrow \neg P$.

sol:	{1}	(1)	$P \to Q$	Rule P
	{1}	(2)	$\neg Q \to \neg P$	Rule T, (1), $P \to Q \Leftrightarrow \neg Q \to \neg P$
	{3}	(3)	$\neg Q$	Rule P
	{1,3}	(4)	$\neg P$	Rule T, (2), (3), $P, P \to Q \Rightarrow Q$

Rule CP (or) Rule of conditional proof:-If we can derive S from R and a set of premises, then we can derive $R \to S$ from the set of premises alone.

- Rule CP is also called as deduction theorem and is generally used if the conclusion is of form $R \to S$. In such cases, R is taken as an additional premise and S is derived from the given premises and R.

Eg 1: Show that $R \to S$ can be derived from the premise $P \to (Q \to S), \neg R \vee P$, and Q.

sol:	{1}	(1)	$\neg R \vee P$	Rule P
	{2}	(2)	R	Rule P(Assumed premise)
	{1,2}	(3)	P	Rule T, (1), (2), $\neg P, P \vee Q \Rightarrow Q$
	{4}	(4)	$P \to (Q \to S)$	Rule P
	{1,2,4}	(5)	$Q \to S$	Rule T, (3),(4), $P, P \to Q \Rightarrow Q$
	{6}	(6)	Q	Rule P
	{1,2,4,6}	(7)	S	Rule T, (5), (6), $P, P \to Q \Rightarrow Q$
	{1,2,4,6}	(8)	$R \to S$	Rule CP

Consistency of premises and Indirect Method of Proof:-

Consistency of premises:- A set of formulas $H_1, H_2, ... H_m$ is said to be consistent if their conjunction has the truth value T for some assignment of the truth values to the atomic variables appearing in $H_1, H_2, ... H_m$.

A set of formulas $H_1, H_2, ... H_m$ is inconsistent if their conjunction implies a contradication i.e, $[H_1 \wedge H_2 \wedge ... H_m \Rightarrow R \wedge \neg R]$ where R is any formula.

Eg 1:- Show that the premises are $a \to (b \to c), d \to (b \wedge \neg c), a \wedge d$ *inconsistent*

Sol:-	{1}	(1)	$A \wedge D$	Rule P
	{1}	(2)	A	Rule T, (1) $P \wedge Q \Rightarrow P$
	{1}	(3)	D	Rule T, (1) $P \wedge Q \Rightarrow Q$
	{4}	(4)	$A \to (B \to C)$	Rule P
	{1,4}	(5)	$B \to C$	Rule T, (2),(4) $P, P \to Q \Rightarrow Q$
	{1,4}	(6)	$\neg B \vee C$	Rule T, (5) $P \to Q \Leftrightarrow \neg P \vee Q$
	{7}	(7)	$D \to (B \wedge \neg C)$	Rule P
	{7}	(8)	$\neg(B \wedge \neg C) \to \neg D$	Rule T, (7) $P \to Q \Leftrightarrow \neg Q \to \neg P$
	{7}	(9)	$\neg B \vee C \to \neg D$	Rule T, (8) Demorgan's law
	{1,4,7}	(10)	$\neg D$	Rule T, (6),(9) Modus ponens
	{1,4,7}	(11)	$D \wedge \neg D$	Rule T, (3),(10) $P, Q \Rightarrow P \wedge Q$

Eg 2:- Show that the following premises are inconsistent

1. *If Jack misses many classes because of illness, then he fails high school*
2. *If Jack fails high school, then he is uneducated.*
3. *If Jack reads a lot of books, then he is not uneducated.*
4. *Jack misses many classes because of illness and reads a lot of books.*

Sol:- Let, P: jack misses many classes because of illness

Q: Jack fails high school.

R: Jack is uneducated

S: Jack reads a lot of books

Therefore the premises are:

$$P \to Q, Q \to R, S \to \neg R, P \wedge S$$

{1}	(1)	$P \to Q$	Rule P
{2}	(2)	$Q \to R$	Rule P

{1,2}	(3)	$P \to R$	Rule T, (1) (2) $P \to Q$ & $Q \to R \Rightarrow P \to R$
{4}	(4)	$S \to \neg R$	Rule P
{4}	(5)	$R \to \neg S$	Rule T, (4) $P \to Q \Leftrightarrow \neg Q \to \neg P$
{1,2,4}	(6)	$P \to \neg S$	Rule T,(3),(5)

$(P \to Q) \wedge (Q \to R) \Leftrightarrow P \to R$

{1,2,4}	(7)	$\neg P \vee \neg S$	Rule T, (6) $P \to Q \Leftrightarrow \neg P \vee Q$
{1,2,4}	(8)	$\neg(P \wedge S)$	Rule T, (6) Demorgan's law
{9}	(9)	$P \wedge S$	Rule P
{1,2,4,9}	(10)	$(P \wedge S) \wedge \neg(P \wedge S)$	Rule T, (8),(9) $P, Q \Rightarrow P \wedge Q$

$(P \wedge S) \wedge \neg(P \wedge S) \Leftrightarrow F$

$\therefore$ The given premises are inconsistent

<u>Indirect method of Proof :</u>

To show that a conclusion 'C' follows logically from the premises H_1, H_2, - - - - -H_m, we assume an additional premise $\neg C$.

If the new set of premises is inconsistent then they imply a contradiction. That shows $\neg C$ is false i.e C is true. Therefore C follows logically from the premises H_1,H_2, - - - - H_m.

Eg 1:- Show that $\neg(P \wedge Q)$ follows from $\neg P \wedge \neg Q$.

Sol:-

{1}	(1)	$\neg\neg(P \wedge Q)$	Rule P (Assumed premise)
{1}	(2)	$(P \wedge Q)$	Rule T, (1) $\neg(\neg P) \Rightarrow P$
{1}	(3)	P	Rule T, (2), $P \wedge Q \Rightarrow P$
{4}	(4)	$\neg P \wedge \neg Q$	Rule P
{4}	(5)	$\neg P$	Rule T, (4) $P \wedge Q \Rightarrow P$
{1,4}	(6)	$P \wedge \neg P$	Rule T,(3) (5) $P, Q \Leftrightarrow P \wedge Q$

$P \wedge \neg P \Leftrightarrow F$ i.e, our assumption is wrong.

$\therefore$ *$\neg(P \wedge Q)$ follows from $\neg P \wedge \neg Q$.*

Eg 2:- Using indirect method, Show that $P \to Q, Q \to R, \neg(P \wedge R), P \vee R \Leftrightarrow R$

Sol:-

{1}	(1)	$\neg R$	Rule P (Assumed premise)
{2}	(2)	$Q \to R$	Rule P
{1,2}	(3)	$\neg Q$	Rule T,(1),(2), $P \to Q, \neg Q \Rightarrow \neg P$
{4}	(4)	$P \to Q$	Rule P
{1,2,4}	(5)	$\neg P$	Rule T,(3),(4) $P \to Q, \neg Q \Rightarrow \neg P$
{6}	(6)	$P \vee R$	Rule P
{6}	(7)	$\neg P \to R$	Rule T,(6) $\neg P \to Q \Leftrightarrow P \vee Q$
{1,2,4,6}	(8)	R	Rule T,(5) (7) Modus ponens
{1,2,4,6}	(9)	$R \wedge \neg R$	Rule T,(1) (8) $P, Q \Leftrightarrow P \wedge Q$

$R \wedge \neg R \Leftrightarrow F$ i.e, our assumption is wrong.

$\therefore$ The given conclusion follows from the premises.

PREDICATE CALCULUS

Predicate calculus:

By using propositional logic, it was not possible to express the fact that any atomic statements have some features in common. In order to investigate questions of this nature, we introduce the concept of a "predicate" in an atomic statement. The logic based on the analysis of predicates in any statement is called predicate logic.

Predicate: A part of declarative sentence describing the properties of a subject or relation among subjects is called predicate.

Ex: Ram is a bachelor -> predicate
Shyam is a bachelor

$\left.\begin{array}{l}\textit{Radha}\\ \textit{Shyam}\end{array}\right\}$ Subject of the statement

" is a bachelor " -> refers to a property that the subject can have, is called the predicate.

- Predicate is denoted by capital letters.
- Subject is denoted by small letters.

From the example:

Radha is a girl
Seeta is a girl

" is a girl" is a predicate, let denote by "G".
" Radha " is denoted by 'r'.
" Seeta " is denoted by 's'.

$\therefore$ Radha is a girl $\Rightarrow$ G(r)
$\therefore$ Seeta is a girl $\Rightarrow$ G(s)

Types of predicates:

- 1 place predicate
- 2 place predicate
- 3 place predicate
- m place predicate

1 place predicate: If there is one name associated with a predicate then it is called a 1 place predicate.
Ex: Ravi is a teacher $\Rightarrow$T(s)

2 place predicate: If two names are associated with a predicate then it is called a 2 place predicate.
Ex: Ravi is senior than ramesh $\Rightarrow$ S(r,m)

3 place predicate: If three names are associated with a predicate then it is called a 3 place predicate.
Ex: Ravi sits between ram and ramesh $\Rightarrow$ S(r,a,h)
m place predicate: If 'm' names are associated with a predicate then it is called m place predicate.
Ex:Ram sits among ravi, raju, rahul.. $\Rightarrow$ S(a,b,c,d…………….)

Statement functions:

Consider the following statements:

Somu is mortal

India is mortal
A table is mortal

Let H be the predicate '*is mortal*', s be the name *somu*, i be the name '*india*' and t be the *table*. Then, H(s), H(i), and H(t) denote the above statements. If we write H(x) for "x is mortal", then H(s), H(i) and H(t) can be obtained form H(x) by replacing x by an appropriate name.

Note:

- → H(x) is not a statement, but when x is replaced by the name of a subject, it becomes a statement.
- → A simple statement function of one variable is defined as an expression consisting of a predicate symbol and an individual variable.
- → A statement function becomes a statement when the variable is replaced by the name of any subject.

Example 1: *Let P(x) denote the statement 'x > 3'. What are the truth values of P(4) and P(2)?*

- The statement P(4) can be obtained by setting x = 4 in the statement x > 3. Hence, P(4) is the statement '4 > 3', which is true.
- The statement P(2) can be obtained by setting x = 2 in the statement x > 3. Hence, P(2) is the statement '2 > 3', which is false.

Example 2: *Let Q(x) denote the statement 'x = y+3'. What are the truth values of Q(1,2) and Q(3,0) ?*

- The statement *Q(1,2)* can be obtained by setting x = 1 and y = 2 in the statement '*x = y+3*'. Hence, *Q(1,2)* is the statement '1 = 2 + 3', which is false.
- The statement *Q(3,0)* can be obtained by setting x = 3 and y = 0 in the statement '*x = y+3*'. Hence, *Q(3,0)* is the statement '3 = 0 + 3', which is true.

Quantifiers:

There are two types of quantifiers

1. Universal quantifier
2. Existential quantifier

1. Universal quantifier:

Consider the following statements

- ✓ All men are mortal
- ✓ Every apple is red

The statements can be written as

- ➢ For all x, if x is a man, then x is mortal
- ➢ For all x, if x is an apple, then x is red.

These statements can be symbolized as

$$(x)\quad (M(x) \rightarrow H(x));$$
$$(x)\quad (A(x) \rightarrow R(x));$$

$M(x)$: x is a man; $A(x)$: x is an apple;
$H(x)$: x is mortal; $R(x)$: x is red;

✦ We symbolize "for all x" by the symbol $(\forall x)$ or (x).It is called universal quantifier.

- The notation $(\forall x)P(x)$ denotes the universal quantification of $P(x)$.

2.Existential quantifier:

Consider the following statements.

- There exists a man
- Some men are clever
- Some real numbers are rational

The statements can be written as

- ✓ There exists an x such that x is a man
- ✓ There exists an x such that x is a man and x is clever
- ✓ There exists an x such that x is a real number and x is a rational number

These statements can be symbolized as

$(\exists x)\quad M(x)$;

$(\exists x)\quad (M(x) \wedge C(x))$;

$(\exists x)\quad (R_1(x)) \vee R_2(x))$;

$M(x)$: x is a man;

$C(x)$: x is clever;

$R_1(x)$: x is a real number;

$R_2(x)$: x is a rational number;

- We symbolize "there is at least one x such that" or "there exists an x such that" or " for some x" by the symbol $(\exists x)$.It is called universal quantifier.
- The notation $(\exists x)P(x)$ denotes the existential quantification of $P(x)$.

Example:- ***What is the truth value of the following quantifications?***

1. *$(\forall x)P(x)$, where $P(x)$ is the statement 'x < 2' and the universe discourse consists of all real numbers?*

   sol:- $P(x)$ *= x < 2*

   $P(x)$ is not true for every real number x, since P(3) is false.

   $\therefore (\forall x)P(x)$ is false.

2. *$(\forall x)P(x)$, where $P(x)$ is the statement '$x^2 < 10$' and the universe discourse consists of the positive integers not exceeding 4?*

   sol:- The statement $(\forall x)P(x)$ is $P(1) \wedge P(2) \wedge P(3) \wedge P(4)$, since the universe of discourse is {1,2,3,4}

   P(4) is the statement $4^2 < 10$, which is not true.

   $\therefore (\forall x)P(x)$ is false.

3. *$(\exists x)P(x)$ where $P(x)$ is the statement 'x > 3' and the universe discourse consists of all real numbers?*

   Sol:- $P(x)$ *= x > 3*

   when x =4 , 4 > 3 which is true

   $\therefore (\exists x)P(x)$ is true.

4. $(\exists x)P(x)$ *where* $P(x)$ *is the statement 'x = x+1' and the universe discourse is the set of all real numbers?*
Sol:- $P(x)$ *:* x = x + 1
Since $P(x)$ is false for every real number x, the existential quantification of $P(x)$ *is false.* $(\exists x)P(x)$ is false.

Example:

S(x): x is a student; I(x): x is intelligent; M(x): x likes music
Write the following statements in symbolic form.

1. All students are intelligent.
   For all x if x is a student then x is intelligent
   $(\forall x)$ [S(x) $\rightarrow$ I(x)]
2. Some intelligent students like music
   There exists an x such that x is a student and x is intelligent and x likes music
   (∃x)[S(x) $\wedge$ I(x) $\wedge$ M(x)]
3. Everyone who likes music is a stupid student
   For all x, if x likes music then x is a student and x is not intelligent(stupid)
   $(\forall x)$ [M(x) $\rightarrow$ S(x) $\wedge$ I'(x)]

Universe of discourse:-

The process of symbolizing a statement in predicate calculus, which is quite complicated, can be simplified by limiting the class of individuals or objects under consideration. This restricted class is called universe of discourse or simply the universe. If the discussion refers to human beings only, then the universe of discourse is the class of human beings.
Definition: The collection of values that a variable x can take is called x's universe of discourse.
Example: Symbolize the statement " All men are giants".
Solution : Using *G(x): x* is a giant.
M(x): x is a man.
The given statement can be symbolized as *(x) (M (x)* $\rightarrow$ *G(x)).* However, if we restrict the variable *x* to the universe which is the class of men, then the statement is *(x) G(x)*
The universe of discourse, if any, must be explicitly stated, because the truth value of a statement depends upon it.
For instance, consider the predicate *Q(x): x* is less than *5* and the statements *(x) Q(x)* and *(∃ x) Q(x).* If the universe of discourse is given by the set

1. {-1, 0, 1, 2, 4}
2. {3, –2, 7, 8, -2}
3. { 15, 20, 24} then *(x)Q(x)* is true for the universe of discourse *(1)* and false for *(2)* and *(3).* The statement *(∃ x)Q(x)* is true for both *(1)* and *(2),* but false for *(3).*

Free and Bound variables

Definition: – An expression like P (x) is said to have a free variable x (meaning, x is undefined). A quantifier (either ∀ or ∃) operates on an expression having one or more free variables, and binds one or more of those variables, to produce an expression having one or more bound variables.
Examples:
Eg1. ∃ x [x + y = z], x is bound but y and z are free variables.
Eg2: P (x,y) has 2 free variables, x and y.

Eg3: ∀x P (x , y) has 1 free variable y, and one bound variable x.
" P (x), where x=3 " is another way to bind x.
Given a formula containing a part of the form *(x)P(x)* or *(∃x) P(x),* such a part is called *x* bound part of that formula. Any occurrence of *x* in an *X*-bound part of a formula is called a bound occurrence of *x*, while any occurrence of *x* or any variable that is not a bound occurrence is called a free occurrence. The formula *P(x)* either in *(x) P (x)* or in *(∃x) P(x)* is described as the scope of the quantifier.

Theory of Inference for the Predicate Calculus:

Rule US: universal specification or instantiation

$$(x)A(x) \Rightarrow A(y)$$

From $(x)A(x)$, one can conclude $A(y)$

Rule ES: Existential specification

$$(\exists x)A(x) \Rightarrow A(y)$$

Rule EG: Existential generalization

$$A(x) \Rightarrow (\exists y)A(y)$$

From $A(x)$, one can conclude $(\exists y)A(y)$

Rule UG: universal generalization

$$A(x) \Rightarrow (y)A(y)$$

Ex1: Prove that $(\exists x)(P(x) \wedge Q(x)) \Rightarrow (\exists x)P(x) \wedge (\exists x)Q(x)$

{1}	(1)	$(\exists x)(P(x) \wedge Q(x)$	rule P
{1}	(2)	$(P(y) \wedge Q(y)$	rule ES
{1}	(3)	$P(y)$	rule T $P \wedge Q \Rightarrow P$
{1}	(4)	$(\exists x)(P(x)$	rule EG
{1}	(5)	$Q(y)$	rule T, $P \wedge Q \Rightarrow Q$
{1}	(6)	$(\exists x)P(x) \wedge (\exists x)Q(x)$	rule T, P,Q$\Rightarrow P \wedge Q$

Ex 2: Show that *(x)(H(x) → M(x)) ∧ H(s) ⇔ M(s).*This problem is a symbolic translation of a well-known argument known as the **"Socrates argument**" which is given by:

All men are mortal. Socrates is a man. Therefore Socrates is a mortal.

If we denote *H (x): x* is a man, *M (x) : x* is a mortal, and s: Socrates, we can put the argument in the above form.

Solution:

{1}	(1)	*(x) (H(x) → M(x))*	***rule P***
{1}	(2)	*H(s) → M(s)*	***rule US**, (1)*
{3}	(3)	*H(s)*	***rule P***
{1,3}	(4)	*M(s)*	***rule T**, (2), (3)*

Note that in step 2 first we remove the universal quantifier.

Ex 3: Show that *(x)(P(x) →Q(x)) ∧ (x) (Q(x) →R(x)) ⇔ (x) (P(x) →R(x))*

Solution:

{1}	(1)	*(x)(P(x) → Q(x))*	***rule P***
{1}	(2)	*P(y) → Q(y)*	***rule US**, (l)*
{3}	(3)	*(x) (Q(x) → R(x))*	***rule P***

{3}	(4)	$Q(y) \to R(y)$	***rule* US,** (3)
{1,3}	(5)	$P(y) \to R(y)$	***rule T**, (2), (4)*
{1,3}	(6)	$(x)(P(x) \to R(x))$	***rule UG**, (5)*

Ex 4: Show that $(\exists x)M(x)$ follows logically from the premises $(x)(H(x) \to M(x))$ and $(\exists x)H(x)$
Solution:

{1}	(1)	$(\exists x)H(x)$	***rule P***
{1}	(2)	$H(y)$	***rule ES**, (1)*
{3}	(3)	$(x)(H(x) \to M(x))$	***rule P***
{3}	(4)	$H(y) \to M(y)$	***rule US, (**3)*
{1,3}	(5)	$M(y)$	***rule T**, (2), (4)*
{1,3}	(6)	$(\exists x)M(x)$	***rule EG**, (5)*

UNIT-VI

Graph Theory: *Basic Concepts of Graphs, Sub graphs, Matrix Representation of Graphs: Adjacency Matrices, Incidence Matrices, Isomorphic Graphs, Paths and Circuits, Eulerian and Hamiltonian Graphs, Multigraphs, (Problems and Theorems without proofs), Planar Graphs, Euler's Formula, Graph Colouring, Chromatic Number,(Problems and Theorems without proofs) Trees, Binary Trees, Decision Trees, Spanning Trees: Properties, Algorithms for Spanning trees and Minimum Spanning Trees*

GRAPH THEORY

Basic concepts of graphs:

Graph:- A graph **G** consists of a pair **(V,E)** where **V** is a non-empty finite set whose elements are called ***vertices*** (or nodes or points) and **E** is another set whose elements are called ***edges*** (or lines) such that each edge e∊E is associated with ordered or unordered pair of elements of V, that is there is a mapping from the set of edge E to the set of ordered or unordered pair of elements of V.

The graph G with vertices V and edge E is written as G = (V,E) or G(V,E).

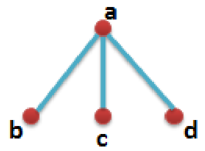
(or)

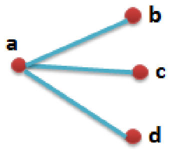
A graph G = (V,E) consists of two objects V and E such that V = {v_1,v_2,........} called vertices and E = {e_1,e_2,......} called edges and each edge e_k is associated with an unordered pair (v_i,v_j) of vertices.

- Edge e is said to connect u and v then u,v are called as *endpoints* of e.
- The edge e that joins the nodes u and v is said to be *incident* on each of its end points u & v.
- If two distinct edges e_1 and e_2 are incident with a common point then they are called *adjacent edges*.
- Any two vertices connected by an edge in a graph are called as *adjacent vertices*.
- Vertex that is not adjacent to any other vertex is called as *isolated vertex*.

Eg-1:- V = {a,b,c,d} and E = {(a,b) (a,c) (a,d)} draw a graph for this information

Sol: *G = (V,E) = (4,3)*

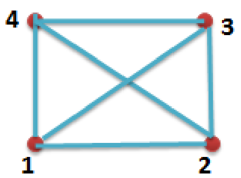




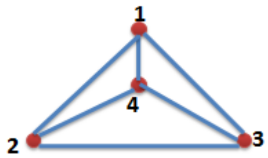
In the graph a and b are adjacent, b and c are non-adjacent.

Eg-2:- let V = {1,2,3,4} and E = {(1,2) (1,3) (1,4) (2,3) (2,4) (3,4)} draw a graph.

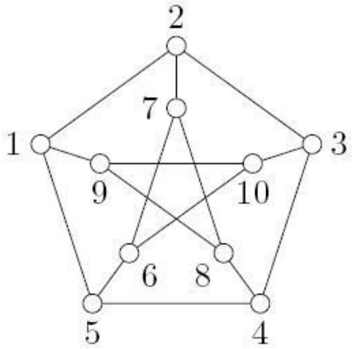
Sol: G = (V,E) = (4,6)



In the graph edge (1,3) & (2,4) intersect, however their intersection is not a vertex of the graph.



Eg-3:- Consider (10, 15)



This graph is called as Petersen graph.

V = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}
E = {(1,2) (2,3) (3,4) (4,5) (5,1) (1,9) (2,7) (3,10) (4,8) (5,6) (6,7) (6,10) (7,8) (8,9) (9,10)}

Types of Graphs:

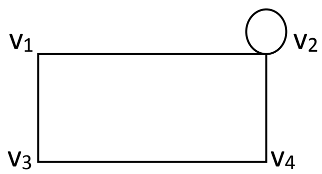
Null graph:- A graph in which number of edge is zero is called as null graph

Eg: $.v_1$ $.v_2$

$.v_3$ $.v_4$

Self loop:- An edge joining a vertex to itself is called as self loop. A loop may be defined as an edge (v_l,v_m) , where $v_l = v_m$.

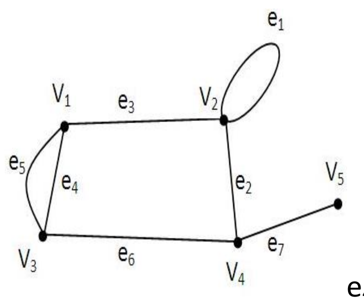
Eg:



v_2 is a self loop.

Parallel edges:- If there is more than one edge between two vertices they are called parallel edges.

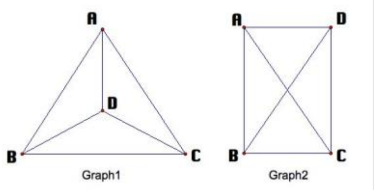
Eg:



e_4 and e_5 are parallel edges

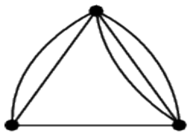
Simple Graph:- A graph which contains neither self-loops nor parallel edges is called a simple graph.

Eg:



Multi graph:- A graph which contains parallel edges is called a multi graph.

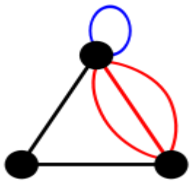
Eg:



multigraph

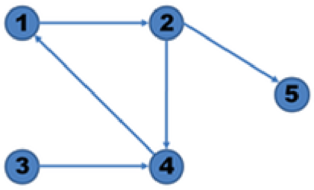
Pseudograph:- A graph in which loops and multiple edges are allowed is called a pseudo graph.

Eg:



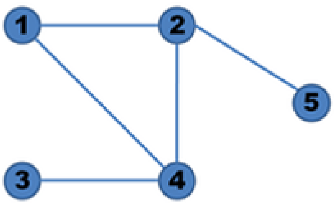
Directed graph:- A directed graph or digraph G consists of a set V of vertices and a set E of edges such that each edge e ϵ E is associated with an ordered pair of vertices. In other words, if each edge of the graph G has direction then the graph is called directed graph.

Eg:



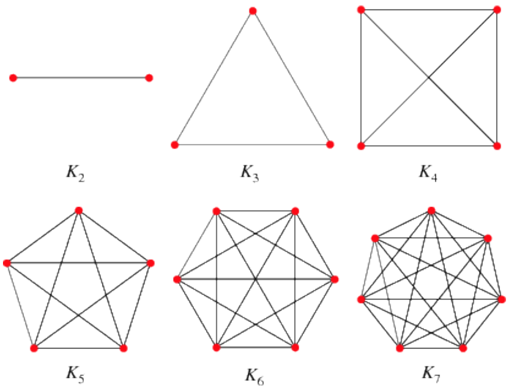
Undirected graph:- An undirected graph G consists of a set V of vertices and the set E of edges such that each edge e ϵ E is associated with an unordered pair of vertices.

Eg:



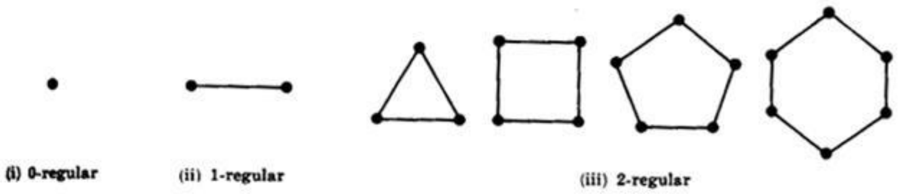
Complete graph:- A simple graph G is said to be complete if every vertex in G is connected with every other vertex if G. A complete graph is usually denoted by K_n .

Eg:



Regular graph:- A graph in which all vertices are of equal degree is called a regular graph. If the degree of each vertex is r, then the graph is called a regular graph of degree r.

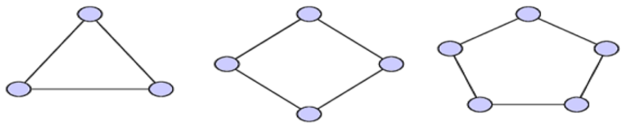
Eg:



Note:- Every null graph is a regular graph of degree zero.

Cycle Graph:- A cycle C_n , $n \geq 3$ consists of n vertice $v_1, v_2, \ldots v_n$ and edges $\{v_1,v_2\},\{v_2,v_3\}\ldots\ldots\{v_{n-1},v_n\}$ and $\{v_n,v_1\}$.

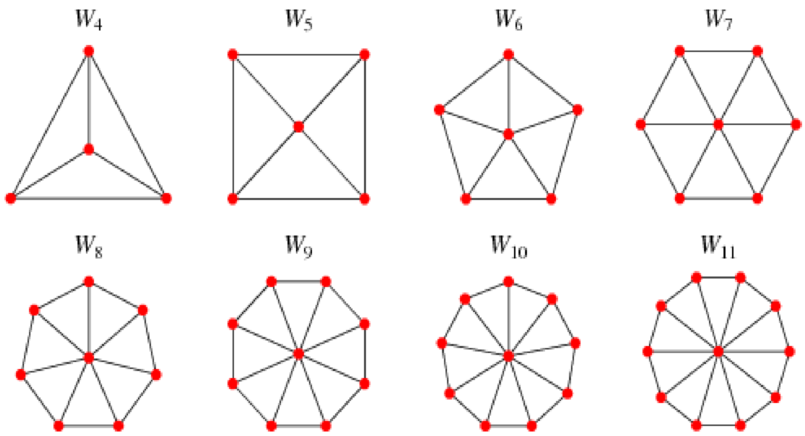
Eg:



C_3 C_4 C_5

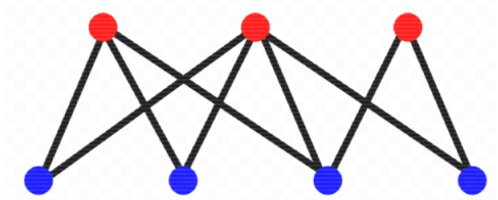
Wheels:- A wheel W_n is obtained when an additional vertex is added to the cycle C_n, for n>=3 and this new vertex is connected to each of the n vertices in C_n .

Eg:



Bipartite graphs:- A graph G = (V,E) is said to be bipartite if the vertex set V can be partitioned into two disjoint subsets V_1 and V_2 such that every edge in E connects a vertex in V_1 and a vertex V_2 so that no edge in G connects either two vertices in V_1 or two vertices in V_2. (V_1,V_2) is called a bipartition of G.

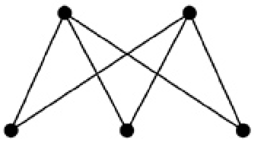
Eg:



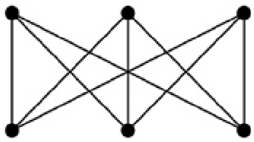
Note:- A bipartite graph can have no loop.

Complete bipartite graph:-A complete bipartite graph on m and n vertices denoted by $k_{m,n}$ is a graph whose vertex set is partitioned into sets V_1 with ***m*** vertices and V_2 with ***n*** vertices in which there is an edge between each pair of vertices v_1 and v_2 where v_1 is in V_1 and v_2 is in V_2 .

Eg:



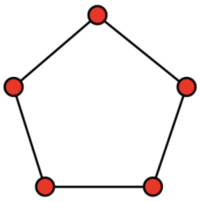
$K_{2,3}$



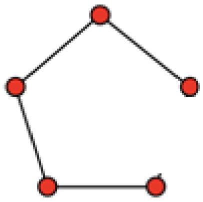
$K_{3,3}$

Path graph:- Path graph of order n which will be obtained by removing any one edge from cycle graph C_n and is denoted by P_n.

Eg:



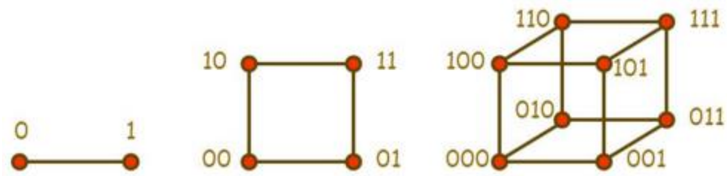
C_5



P_5

N – cube graph:- An N-cube is a graph that has vertices representing the 2^n bit string of length n. An N-cube is denoted by Q_n.

Eg:

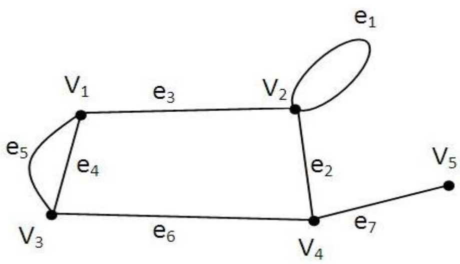


Q_1 Q_2 Q_3

Finite graph and Infinite graph:-A graph is finite if both its vertex set and the edge set are finite. Otherwise it is an infinite graph.

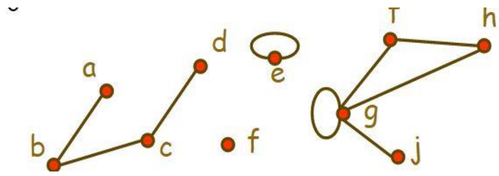
Degree of a vertex in a non-directed graph and degree sequence:-

- The degree of a vertex V of a graph G is the number of edges of G, which are incident with V.
- A vertex of degree zero is called an **isolated vertex**.
- A vertex with degree one is called a **pendant vertex**.
- A vertex of odd degree is an **odd vertex** and a vertex of even degree is an **even vertex**.



$Deg(V_1) = 3$
$Deg(V_2) = 4$
$Deg(V_3) = 3$
$Deg(V_4) = 3$
$Deg(V_5) = 1$

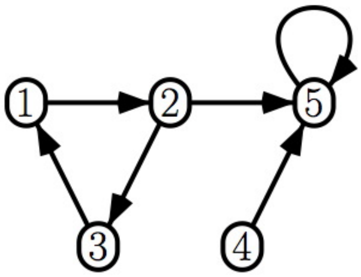
In the above graph V_5 is a pendant. The degree of V_2 is 4 because there are 2 edges incident with it and a loop which should be counted as 2.



In this example,
f is an isolated vertex.
a,d,j are pendant vertices

Degree of a vertex in a directed graph:-

- The number of edges incident to a vertex is called the ***indegree*** of the vertex and the number of edges incident from it is called its ***outdegree*** for a digraph.
- The indegree of a vertex v in a graph G is denoted by deg $G^+(v)$ and the outdegree of a vertex v in a graph G is denoted by deg $G^-(v)$.
- The degree of a vertex is determined by counting each loop incident on it twice and each other edge once.
- The minimum of all the degree of the vertices of a graph G is denoted by $\delta(G)$ and the maximum of all the degree of the vertices of G is denoted by $\Delta(G)$.

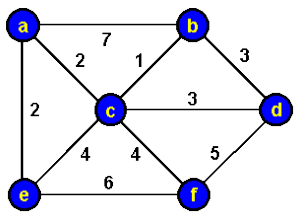


$Deg^+(1) = 1$ $\quad$ $Deg^-(1) = 1$
$Deg^+(2) = 1$ $\quad$ $Deg^-(2) = 2$
$Deg^+(3) = 1$ $\quad$ $Deg^-(3) = 1$
$Deg^+(4) = 0$ $\quad$ $Deg^-(4) = 1$
$Deg^+(5) = 3$ $\quad$ $Deg^-(5) = 1$

$Deg(1) = Deg^+(1) + Deg^-(1) = 1+1=2$
$Deg(2) = Deg^+(2) + Deg^-(2) = 1+2=3$
$Deg(3) = Deg^+(3) + Deg^-(3) = 1+1=2$
$Deg(4) = Deg^+(4) + Deg^-(4) = 0+1=1$
$Deg(5) = Deg^+(5) + Deg^-(5) = 3+1=4$

$\delta(G)=1$, $\Delta(G)=4$

Weighted graph: A graph in which weights are assigned to every edge is called a weighted graph



Theorem-1: The sum of all vertex degrees is equal to twice the number of edges. (or) The sum of the degrees of the vertices of G is even.

$$\boxed{\sum_{i=1}^{n} d_i = 2|E|}$$

- Note:- Any graph has even number of odd degree vertices.

Theorem-2: (The handshaking theorem):- If G = (V, E) be an undirected graph with e edges then

$$\boxed{\sum_{v\in V}\deg G(v)=2e}$$ where e is the number of edges.

Theorem-3:- If G = (V,E) be a directed graph with e edges then

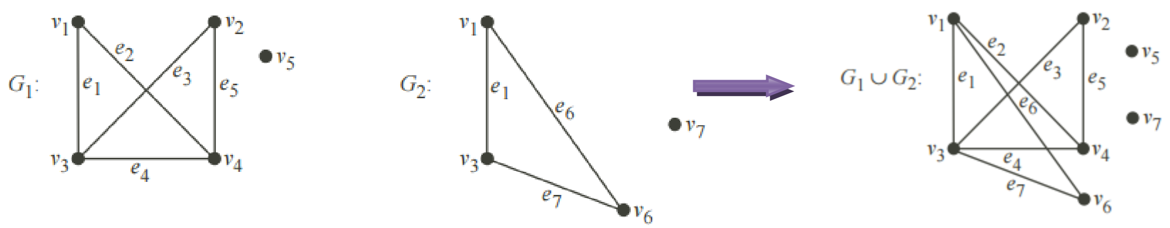
$$\boxed{\sum_{v\in V}\deg^{+}G(v)=\sum_{v\in V}\deg^{-}G(v)=e}$$

Graph Operations:-

Union of two graphs:- Let $G_1 = (V_1,E_1)$ and $G_2 = (V_2,E_2)$ be two graphs. The union of G_1 and G_2 will be a graph (V,E) such that

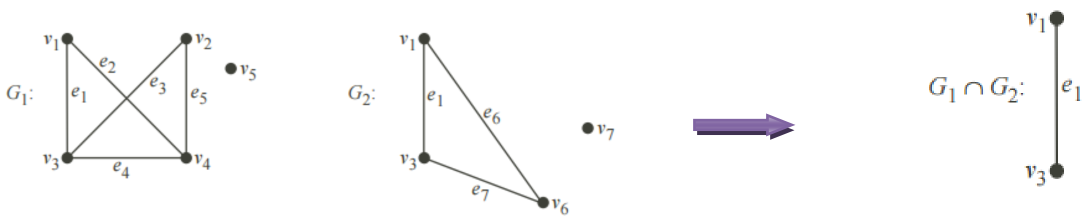
$$V = V_1 \cup V_2$$
$$E = E_1 \cup E_2$$

Eg:



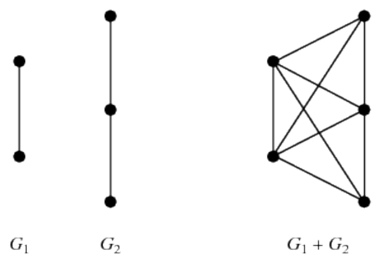
Intersection of two graphs:- Let $G_1 = (V_1,E_1)$ and $G_2 = (V_2,E_2)$ be two graphs with atleast one vertex in common. The intersection of G_1 and G_2 will be a graph (V,E) such that $V = V_1 \cap V_2$, $E = E_1 \cap E_2$.

Eg:



Sum of two graphs:- Let $G_1 = (V_1,E_1)$ and $G_2 = (V_2,E_2)$ be two graphs such that $V_1 \cap V_2 = \phi$. The sum G_1+G_2 is defined as the graph whose vertex set is V_1+V_2 and the edge set consists of those edges which are in G_1 and in G_2 and the edges obtained by joining each vertex of G_1 to that of G_2 .

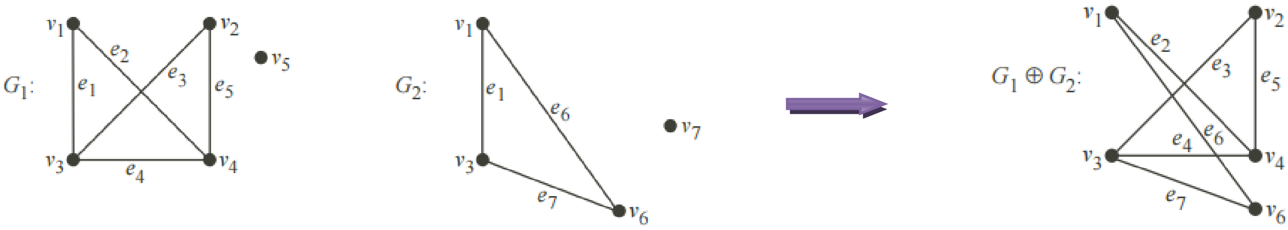
Eg:



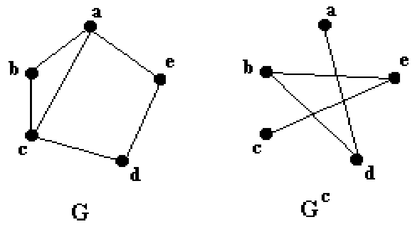
Ring Sum:- Let $G_1 = (V_1,E_1)$ and $G_2 = (V_2,E_2)$ be two graphs. The ring sum $G_1 \oplus G_2$ is a graph G = (V,E) such that

$$V = V_1 \cup V_2$$
$$E = (E_1 \cup E_2) - (E_1 \cap E_2)$$

Eg:



Complement of a graph:- The complement $\overline{G}$ or G^C of a graph G = (V,E) is the graph with the vertex set V such that two vertices are adjacent in $\overline{G}$ if an only if these vertices are non-adjacent in G i.e, G is a graph with $V(\overline{G}) = V(G)$ and $E(\overline{G}) = \{(x, y)/(x, y) \notin E(G)\}$.



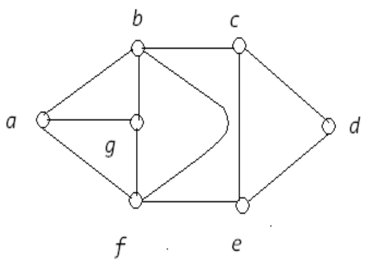
Note: Suppose G is a graph with n-vertices and m-edges. Then the number of edges in its complement is given by $\boxed{\overline{G} = \frac{n(n-1)}{2} - m}$.

For the above graph G, n=5 and m= 6. So its complement $\overline{G}$ have 5(4)/2 – 6 = 10 – 6 = 4 edges.

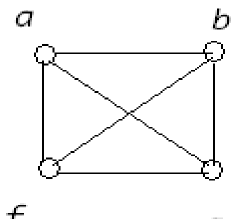
Subgraphs:

- If G and H are two graphs with vertex sets V(H) and V(G) and edge sets E(H) and E(G) respectively such that V(H) $\subseteq$ V(G) and E(H) $\subseteq$ E(G), then we call H as a sub graph of G.
- If V(H) $\subset$ V(G) and E(H) $\subset$ E(G) then H is a ***proper sub graph*** of G and
- If V(H)= V(G) then we say that H is ***spanning sub graph*** of G.
- A spanning sub graph need contain all the edges in G.

  If H is a sub graph of G, then
  - All the vertices of H are in G.
  - All the edges of H are in G and
  - Each edge of H has the same end points in H as in G.



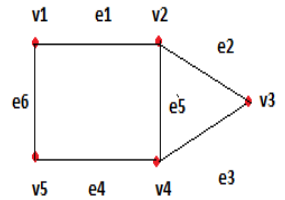
G



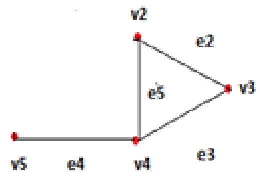
H

Vertex deleted subgraph:- Let G=(V,E) be a graph. Let $v_i \in V$. The subgraph of G obtained by removing the vertex v_i and all the edges incident with v_i is called a vertex deleted subgraph and is denoted by G- v_i.

Eg:



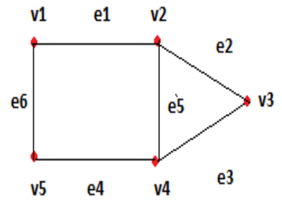
G



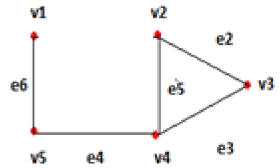
G- v_1

Edge deleted subgraph:- Let G=(V,E) be a graph. Let $e_j \in E$. Then G- e_j =(V,E-{ e_j}) is called the edge deleted subgraph of G obtained by the removal of edge e_j.

Eg:



G



G- e_1

Isomorphic Graphs:- Let G_1=(V(G_1),E(G_1)) and G_2=(V(G_2,E(G_2)) be two graphs. A function $f:G_1 \rightarrow G_2$ is called an isomorphism if

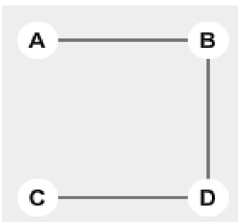
i. f is one one
ii. f is onto
iii. (x,y) $\in$ E(G_1) if and only if f(x),f(y) $\in$ E(G_2) i.e two vertices x and y are adjacent in G_1 iff f(x),f(y) are adjacent in G_2. If the graph G_1 is isomorphic to G_2 then we write $G_1 \cong G_2$.

Properties of isomorphism:

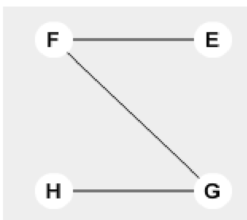
If two graphs G_1 and G_2 are isomorphic then

1. No of vertices of G_1= No of vertices of G_2
   $$|V(G_1)|=|(V(G_2)|$$
2. No of edges of G_1= No of edges of G_2
   $$|E(G_1)|=|(E(G_2)|$$
3. Degree sequences(ascending order of degrees of all the vertices) of G_1 and G_2 must be the same.
4. If (u,u) is a loop in G_1 then (f(u),f(u)) must be a loop in G_2.
5. If v_0-v_1-v_2......v_{n-1}-v_n-v_0 is cycle of length n in G_1 then f(v_0)-f(v_1)-f(v_2)......f(v_{n-1}) - f(v_n) -f(v_0) must be cycle of length n in G_2.
6. If two graphs are isomorphic then their adjacency matrices are same.

Eg-1: Determine whether the following graphs are isomorphic or not



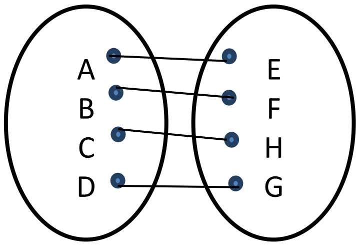
G_1



G_2

Sol:

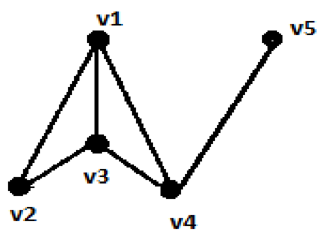
1. Number of vertices of G_1 and G_2 are equal
   $|V(G_1)| = |V(G_2)| \quad \Rightarrow \quad 4 = 4$
2. Number of edges of G_1 and G_2 are equal
   $|E(G_1)| = |E(G_2)| \quad \Rightarrow \quad 3 = 3$
3. Mapping (Function from G_1 to G_2)
   f(A) =E , f(B) =F , f(C) =H , f(D) =G
   Clearly the given function is one-to-one and onto
4. Degree sequence of G_1 = Degree sequence of G_2
   (1, 1, 2, 2) = (1, 1, 2, 2)
5. There are no loops in G_1 and G_2.
6. There are no cycles in G_1 and G_2.
7. Adjacency matrices of G_1 and G_2 are equal

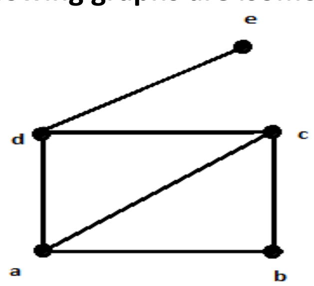


$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array}\begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \end{array} = \begin{array}{c} \\ E \\ F \\ H \\ G \end{array}\begin{array}{c} \begin{array}{cccc} E & F & H & G \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

$\therefore$ The given graphs G_1 and G_2 are isomorphic.

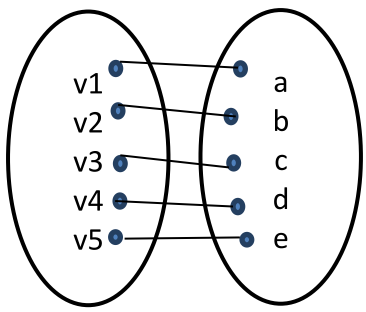
Eg-2: Determine whether the following graphs are isomorphic or not





Sol:

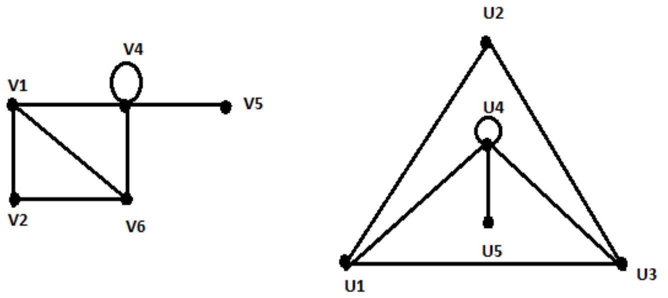
1. Number of vertices of G_1 and G_2 are equal
   $|V(G_1)| = |V(G_2)| \quad \Rightarrow \quad 5 = 5$
2. Number of edges of G_1 and G_2 are equal
   $|E(G_1)| = |E(G_2)| \quad \Rightarrow \quad 6 = 6$
3. Mapping (Function from G_1 to G_2)
   $f(v_4)$ =d , $f(v_5)$ =e , $f(v_3)$ =c , $f(v_2)$ =b , $f(v_1)$ =a
   Clearly the given function is one-to-one and onto
4. Degree sequence of G_1 = Degree sequence of G_2
   (1,2,3,3,3) = (1,2,3,3,3)
5. There are no loops in G_1 and G_2.
6. i) **v_1-v_2-v_3-v_4-v_1** is a cycle in G_1 and **$f(v_1)$-$f(v_2)$-$f(v_3)$-$f(v_4)$-$f(v_1)$** i.e **a-b-c-d-a** is a cycle in G_2 .
   ii) **v_1-v_2-v_3-v_1** is a cycle in G_1 and **$f(v_1)$-$f(v_2)$-$f(v_3)$-$f(v_1)$** i.e **a-b-c-a** is a cycle in G_2 .
   iii) **v_1-v_3-v_4-v_1** is a cycle in G_1 and **$f(v_1)$-$f(v_3)$-$f(v_4)$-$f(v_1)$** i.e **a-c-d-a** is a cycle in G_2 .
   Clearly If v_0-v_1-v_2……v_{n-1}-v_n-v_0 is cycle of length n in G_1 then $f(v_0)$-$f(v_1)$-$f(v_2)$……$f(v_{n-1})$ -$f(v_n)$ -$f(v_0)$ is a cycle of length n in G_2.
7. Adjacency matrices of G_1 and G_2 are equal



$$\begin{bmatrix} & v1 & v2 & v3 & v4 & v5 \\ v1 & 0 & 1 & 1 & 1 & 0 \\ v2 & 1 & 0 & 1 & 0 & 0 \\ v3 & 1 & 1 & 0 & 1 & 0 \\ v4 & 1 & 0 & 1 & 0 & 1 \\ v5 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} & a & b & c & d & e \\ a & 0 & 1 & 1 & 1 & 0 \\ b & 1 & 0 & 1 & 0 & 0 \\ c & 1 & 1 & 0 & 1 & 0 \\ d & 1 & 0 & 1 & 0 & 1 \\ e & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

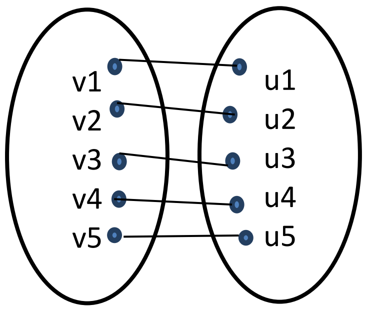
$\therefore$ The given graphs G_1 and G_2 are isomorphic.

Eg-3: Determine whether the following graphs are isomorphic or not



G_1 **G_2**

Sol:



1. Number of vertices of G_1 and G_2 are equal
   $|V(G_1)| = |V(G_2)| \quad \Rightarrow \quad 5 = 5$
2. Number of edges of G_1 and G_2 are equal
   $|E(G_1)| = |E(G_2)| \quad \Rightarrow \quad 6 = 6$
3. Mapping (Function from G_1 to G_2)
   $f(v_4) = u_4$, $f(v_5) = u_5$, $f(v_3) = u_3$, $f(v_2) = u_2$, $f(v_1) = u_1$
   Clearly the given function is one-to-one and onto
4. Degree sequence of G_1 = Degree sequence of G_2
   (1,2,3,3,5) = (1,2,3,3,5)
5. v4 is a loop in G1 and f(v4) i.e u4 is a loop in G2.Clearly if (u,u) is a loop in G1 then (f(u),f(u)) is a loop in G2.

6. i) **v_1-v_2-v_3-v_4-v_1** is a cycle in G_1 and **$f(v_1)$-$f(v_2)$-$f(v_3)$-$f(v_4)$-$f(v_1)$** i.e **u_1-u_2-u_3-u_4-u_1** is a cycle in G_2 .
   ii) **v_1-v_4-v_3-v_2-v_1** is a cycle in G_1 and **$f(v_1)$-$f(v_4)$-$f(v_3)$-$f(v_2)$-$f(v_1)$** i.e **u_1-u_4-u_3-u_2-u_1** is a cycle in G_2 .
   Clearly If v_0-v_1-v_2……v_{n-1}-v_n-v_0 is cycle of length n in G_1 then $f(v_0)$-$f(v_1)$-$f(v_2)$……$f(v_{n-1})$ -$f(v_n)$ -$f(v_0)$ is a cycle of length n in G_2.
7. Adjacency matrices of G_1 and G_2 are equal

$$\begin{bmatrix} & v1 & v2 & v3 & v4 & v5 \\ v1 & 0 & 1 & 1 & 1 & 0 \\ v2 & 1 & 0 & 1 & 0 & 0 \\ v3 & 1 & 1 & 0 & 1 & 0 \\ v4 & 1 & 0 & 1 & 1 & 1 \\ v5 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} & u1 & u2 & u3 & u4 & u5 \\ u1 & 0 & 1 & 1 & 1 & 0 \\ u2 & 1 & 0 & 1 & 0 & 0 \\ u3 & 1 & 1 & 0 & 1 & 0 \\ u4 & 1 & 0 & 1 & 1 & 1 \\ u5 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$\therefore$ The given graphs G_1 and G_2 are isomorphic.

Paths and Circuits:

Walk:

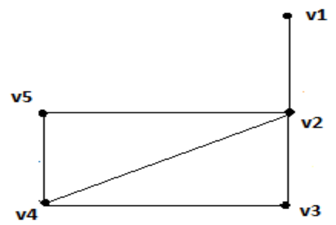
- A walk of a graph G is defined as an alternating sequence of vertices and edges $v_0,e_1,v_1,e_2,v_2,\ldots\ldots v_{n-1},e_n,v_n$ beginning and ending with vertices such that each line e_i is incident with v_{i-1} and v_i.
- A walk joining v_0 and v_n is called v_0-v_n walk. Here v_0 is called the ***initial vertex*** and v_n is called the ***terminal vertex*** of the walk
- The number of edges in the walk is known as the ***length*** of the walk.
- If the length of the walk is zero, then the walk has no edges and it contains only a single vertex. Such a walk is called a ***trivial walk***.

Trail: A walk is called a trail if its edges are distinct.
Path: A walk is called a path if all its vertices are distinct.

Note: Every path is a trail, but every trail need not be a path.

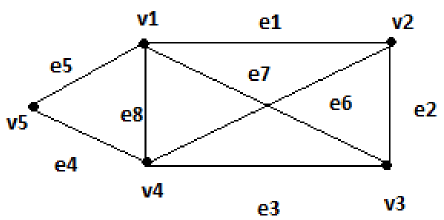
Closed path: A closed path is a path that starts and ends at the same point.
Circuits: A circuit or cycle is defined as a closed path of non zero length that does not contain a repeated edge.

Eg-1:



i) v1-v2-v3-v4-v5 is a walk of length 4(no of edges in the walk v1-v2,v2-v3,v3-v4,v4-v5)
ii) v1-v2-v4-v3-v2-v5 is a trail(edges are distinct) but not a path(because the vertex v2 is repeated)
iii) v1-v2-v4-v5 is a path and also a trail

Eg-2:

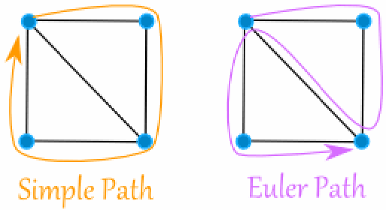


i) v1-e1-v2-e6-v4-e3-v3-e2-v2 is a walk and a trail but not a path (because v2 is repeated) and circuit.
ii) v1-e1-v2-e2-v3-e3-v4-e4-v5 is a walk, trail and a path but not a circuit
iii) v2-e2-v3-e3-v4-e4-v5-e5-v1-e1-v2 is a walk, trail, path and a circuit

Eulers (or) Eulerian Graph:

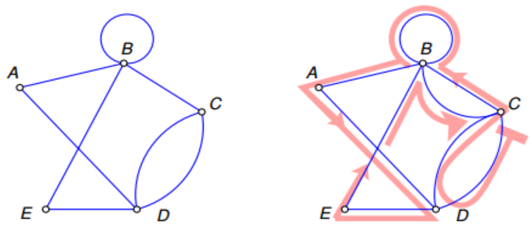
1. An ***Euler's path*** in a multigraph is a path that travels all edges exactly once and vertices can be travelled at least once.

2. In any multigraph we can get an Euler's path if there are two vertices of odd degree and remaining vertices degree is even.
3. The Euler's path will always start at odd degree vertex and end at other odd degree vertex.



Euler's circuit/cycle:

If the starting and ending vertices of Euler's path are same then such a path is called Euler's circuit.

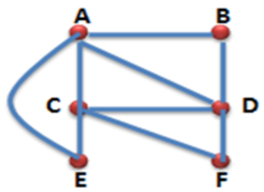


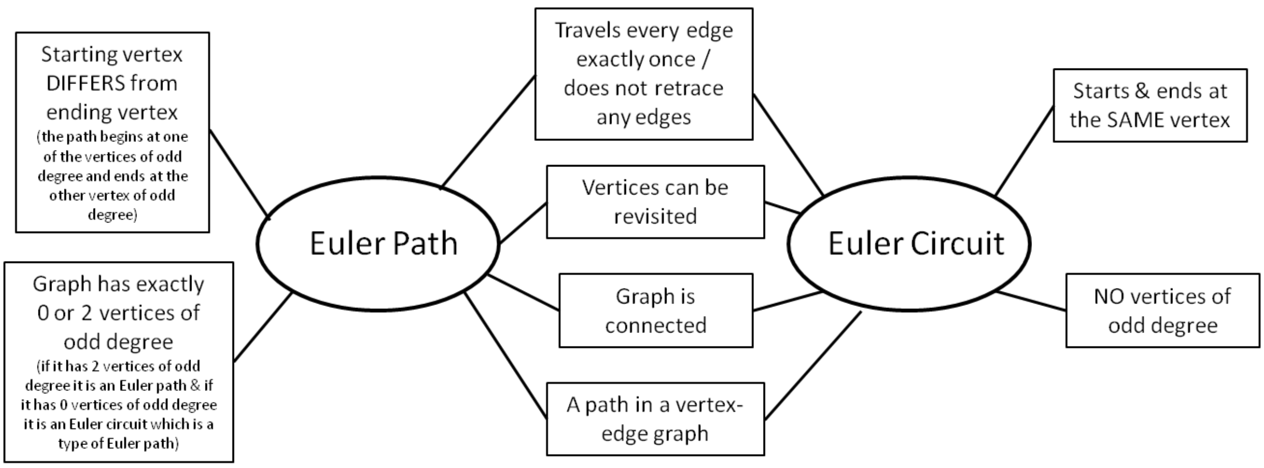
An Euler circuit: CDCBBADEBC

Euler's graph:

A graph having an Euler's circuit is called an Euler's graph.

Eg:



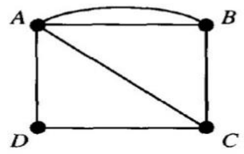


Theorem 1: If G is a graph in which the degree of every vertex is atleast two then G contains a cycle.
Theorem 2: A non-empty connected graph G is Eulerian if and only if its vertices are all of even degree.

Hamiltonian graph:

Hamiltonian circuit: A circuit in a graph G is called a Hamiltonian graph if it contains every vertex of G exactly once, except for the starting and ending vertex that appears twice.

Eg:

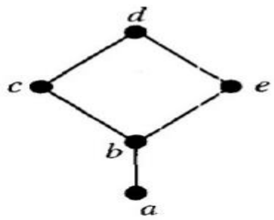


Hamiltonian circuit: A, D, C, B, A

Hamiltonian graph: A graph is said to be a Hamiltonian graph if it contains Hamiltonian cycle i.e a graph G is Hamiltonian if there exists a cycle containing every vertex of G.

Hamiltonian path:A path in a graph G is called a Hamiltonian graph if it contains every vertex of G where the end points may be distinct.

Eg:

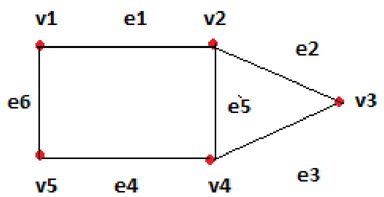


Hamiltonian path: a, b, c, d, e

Note: Hamiltonian path in G may be obtained by deleting an edge from a Hamiltonian cycle/
Note: Every graph which has a Hamiltonian cycle or circuit contains a Hamiltonian path but a graph containing Hamiltonian path may not have a Hamiltonian cycle.

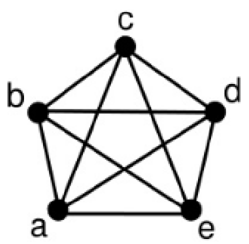
Theorem 1: (Dirac's theorem) Every graph G with $n \geq 3$ vertices.If $deg(v) \geq n/2$ for all vertices of G, then G is ***Hamiltonian.***

Eg-1:



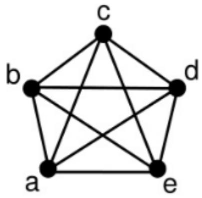
The given graph has n=5 which is ≥ 3 vertices and n/2 =5/3 =2.deg(v1) =2, deg(v2) =3, deg(v3) =2, deg(v4) =3 and deg(v5) =2.
∴ $deg(v) \geq n/2$ for all vertices of G. So the given graph is Hamiltonian.

Theorem 2: If the graph G has no loops or parallel edges, if $V|G|= n \geq 3$ and if $deg(v) \geq n/2$ for each vertex v of G, then G is Hamiltonian.



Given graph has no loops, no parallel edges,V (G) = 5 which is ≥ 3, n/2= 5/2 =2
deg (a) = 4, deg (b) = 4, deg(c) = 4,
deg (d) = 4, deg (e) = 4 and deg (v) $\geq$ n/2
i.e. $4 \geq 2$ for each vertex v of G.
∴ The given graph is Hamiltonian and circuit is a, b, c, d, e

Theorem 3: Let G be a graph with n vertices with no loops or parallel edges. Then, G has atleast ½ (n-1) (n-2)+2 edges in Hamiltonian.

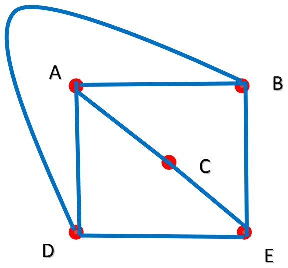


Given graph has 5 vertices & it has no loops and parallel edges.

$$=\frac{(n-1)(n-2)}{2}+2=\frac{(5-1)(5-2)}{2}+2 \quad =\frac{(4)(3)}{2}+2$$

= 6+2 =8.Hence the graph has atleast 8 edges (Minimum 8 edges and they may be exceed 8, in this case 10) .So the given graph is a Hamiltonian graph.

Theorem 4: If the graph G has no loops or parallel edges, if V|G|= n $\geq$ 3. If deg(u) + deg(v) $\geq$ n for each two vertices u and v not connected by an edge in G is Hamiltonian.



Given graph has n=5 which is $\geq$ 3 vertices. First find all the possible vertices u and v not connected by an edge in G.

S.No	u	v	deg(u)	deg(v)	deg(u)+ deg(v)
1	A	E	3	3	6
2	B	C	3	2	5
3	C	D	2	3	5

Clearly it is evident from the table that deg(u) + deg(v) $\geq$ n for each two vertices u and v not connected by an edge in G. So the given graph is Hamiltonian.

Representation of graphs:-

Matrix representation of graphs: A diagrammatic representation of a graph is very useful for visual study. Any graph can be represented by a matrix. A matrix is a very effective and convenient way of representing a graph. There are two types of matrix representation.

1. Adjacency matrix
2. Incident matrix

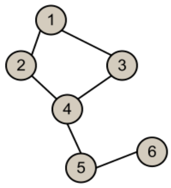
1. ***Adjacency matrix:*** Representation of different types of graphs as adjacency matrices is described as follows.

   a. *Representation of undirected graph:*

   Let G be a graph with n vertices, where n>0 and no parallel edges. The adjacency matrix of G denoted by A_G is an nxn matrix, A=[a_{ij}], whose elements are defined as follows:

   a_{ij}=1, if there is an edge between i^{th} and j^{th} vertices
   =0, if there is no edge between them.

Undirected Graph & Adjacency Matrix



Undirected Graph

	1	2	3	4	5	6
1	0	1	1	0	0	0
2	1	0	0	1	0	0
3	1	0	0	1	0	0
4	0	1	1	0	1	0
5	0	0	0	1	0	1
6	0	0	0	0	1	0

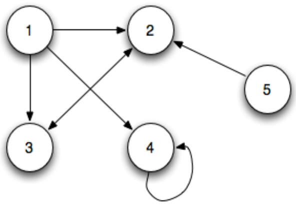
Adjacency Matrix

b. *Representation of directed graph:*

The adjacency matrix of a digraph D with n vertices is an nxn matrix $A_G=(a_{ij})$ in which

$a_{ij} = 1$, if (v_i,v_j) is in D

$= 0$, otherwise



	1	2	3	4	5
1	0	1	1	1	0
2	0	0	1	0	0
3	0	1	0	0	0
4	0	0	0	1	0
5	0	1	0	0	0

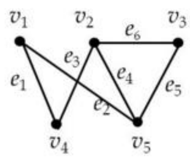
2. *Incidence matrix:*

a. *Representation of undirected graph:*

Let G=(V,E) be an undirected graph with n labeled vertices and m labeled edges. The incidence matrix $I_G=(b_{i,j})$ is the nxm matrix , where

$b_{ij} = 1$, if the j^{th} edge e_j is incident on the i^{th} vertex v_i

$= 0$, otherwise



	e_1	e_2	e_3	e_4	e_5	e_6
v_1	1	1	0	0	0	0
v_2	0	0	1	1	0	1
v_3	0	0	0	0	1	1
v_4	1	0	1	0	0	0
v_5	0	1	0	1	1	0

← edges

↑ vertices

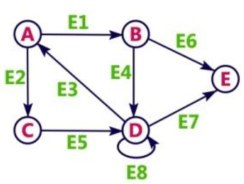
b. *Representation of directed graph:*

The incidence matrix $I_G =(b_{ij})$ of digraph D with n vertices and m edges is the nxm matrix, where

$b_{ij} = 1$, if the j^{th} edge is incident out of the i^{th} vertex.

$= -1$, if the j^{th} edge is incident into the ith vertex.

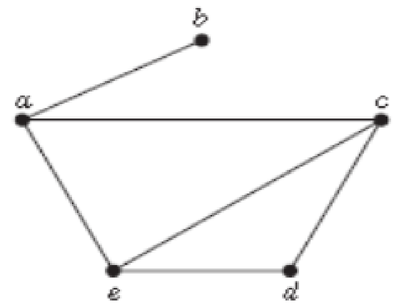
$= 0$, if j^{th} edge in not incident on i^{th} vertex.



	E1	E2	E3	E4	E5	E6	E7	E8
A	1	1	-1	0	0	0	0	0
B	-1	0	0	1	0	1	0	0
C	0	-1	0	0	1	0	0	0
D	0	0	1	-1	-1	0	1	1
E	0	0	0	0	0	-1	-1	0

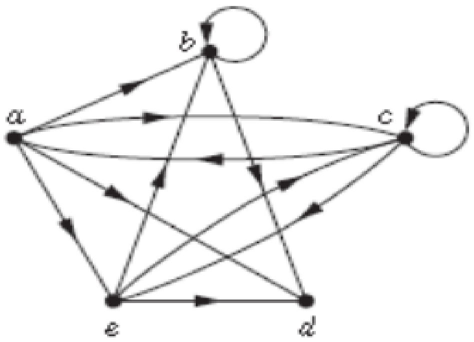
Linked Representation: In this a list of vertices adjacent to each vertex is maintained. This is also called adjacency list representation.

The adjacency list for simple undirected graph is shown below:



Vertex	Adjacent Vertices
a	b, c, e
b	a
c	a, d, e
d	c, e
e	a, c, d

The adjacency list for a simple directed graph is shown below:



Initial Vertex	Terminal Vertices
a	b, c, d, e
b	b, d
c	a, c, e
d	
e	b, c, d

Fleury's Algorithm for finding Euler's circuit:- Let G = (V,E) be an Eulerian connected graph with each vertex of even degree. The following steps may be used to construct an Eulerian circuit.

Step 1:- Choose any arbitrary vertex v_0 from V as the starting vertex.

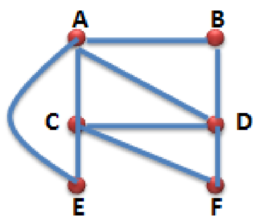
Step 2:- Select an edge e = (v_0,v). If there are many such edges, select the one that is not a bridge. If there are more than one edges of that type choose any of them. If e is a bridge, select only if there is no possibility.

Step 3:- Now delete that edge from the edge list and the graph.

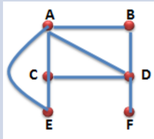
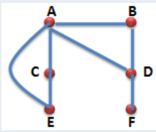
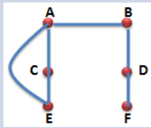
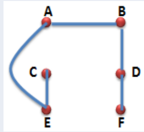
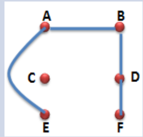
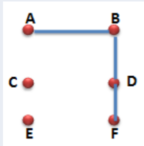
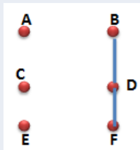
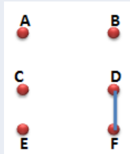
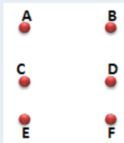
Step 4:- Repeat step-2 and step-3 until $E = \phi$.

Note:- To find an Euler's path in a graph, Fleury's algorithm can be used with a slight modification. The choice of selection of starting vertex is limited to one of the two odd – degree vertices. If all the vertices are of even degree then no modification is required.

Eg-1: Find Euler's circuit for the following graph

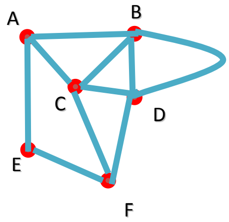


Sol:

Current Path	Next Edge	Reason | Remark	
1.F	{ F, C }	No edge from F is a bridge. So choose anyone. Let it be C i.e, {F, C}	
2. F, C	{ C, D }	No edge from C is a bridge. So choose anyone. Let it be D i.e, {C, D}	
3. F, C, D	{ D, A }	No edge from D is a bridge. So choose anyone. Let it be A i.e, {D, A}	
4.F, C, D,A	{ A,C }	From A edges are {A,B}, {A,C}, {A,E}. Since {A, B} is a bridge choose the remaining edges. Let it be {A, C}.	
5. F,C,D,A,C	{ C, E }	From C there is only one edge {C,E}. So choose it.	
6. F,C,D,A,C,E	{ E, A }	From E there is only one edge {E,A}. So choose it.	
7. F,C,D,A,C, E, A	{ A, B }	From A there is only one edge {A,B}. So choose it.	
8. F,C,D,A,C, E, A, B	{ B, D }	From B there is only one edge {B,D}. So choose it.	
9. F,C,D,A,C, E, A,B,D	{ D, F}	From D there is only one edge {D,F}. So choose it.	
10. F,C,D,A,C,E,A,B,D, F	ϕ	No more edges left	

∴ Euler's circuit is ***F, C, D, A, C, E, A, B, D, F***

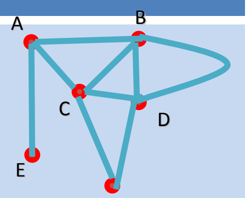
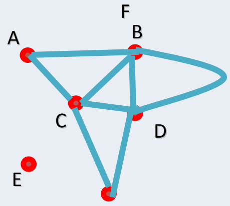
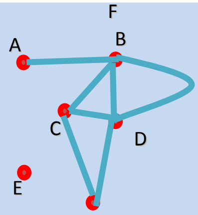
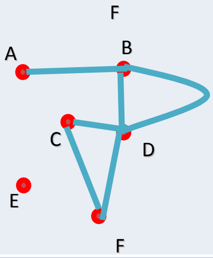
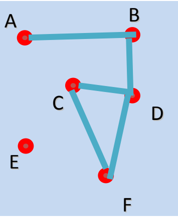
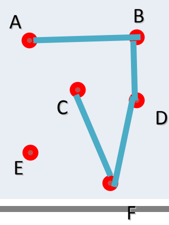
Eg-2:

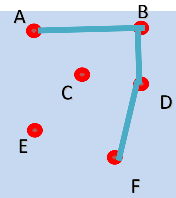
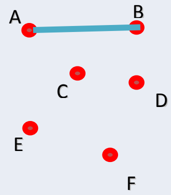
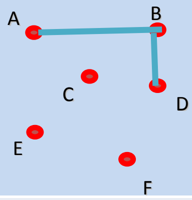
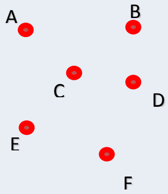


Sol:

deg(A)=3,deg(B)=deg(C)=deg(D)=4,deg(E)=2 and deg(F)=3. So the given graph has 2 odd degree vertices A and F. So only path exists. So start at any odd degree vertex. Let's start at F.

Note: Euler's path starts at one degree vertex and ends at the other odd degree vertex.

Current Path	Next Edge	Reason | Remark	
1.F	{ F, E }	No edge from F is a bridge. So choose anyone. Let it be {F, E}	
2. F, E	{ E, A }	From E there is only one edge {E,A}. So choose it.	
3. F, E,A	{ A, C }	No edge from A is a bridge. So choose anyone. Let it be {A, C}	
4.F, E,A,C	{ C,B }	From C there are 3 edges {C,B},{C,D} and {C,F} out of which {C,F} is bridge. So select one from remaining two edges. Let it be {C,B}	
5. F,E,A,C,B	{ B, D }	From B there is a multiple edge {B,D} and another edge {B,A} which are not bridges. So choose anyone. Let it be {B,D}	
6. F,E,A,C,B,D	{ D, C }	From D, {D,B} is bridge. So choose from {D,C} or {D,F}.Let it be {D,C}.	

7. F,E,A,C,B,D,C	{ C, F }	From C there is only one edge {C,F}.So choose it	
8. F,E,A,C,B,D,C,F	{ F, D }	From F there is only one edge {F,D}.So choose it	
9. F,E,A,C,B,D,C,F,D	{ D, B}	From D there is only one edge {D,B}.So choose it	
10. F,E,A,C,B,D,C,F,D,B	{ B, A}	From B there is only one edge {B,A}.So choose it	
11. F,E,A,C,B,D,C,F,D,B,A	ϕ	No more edges left	

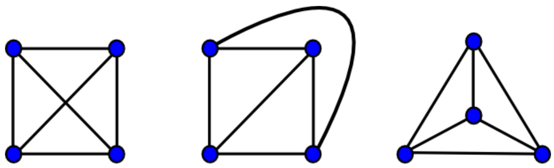
∴ Euler's path is **F,E,A,C,B,D,C,F,D,B**

Planar graphs:

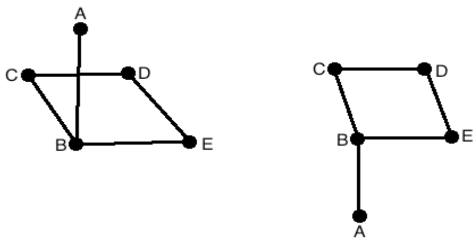
A graph G is called a planar graph if it can be drawn on a plane such that no two edges intersect except at the common vertex.

→ A graph may be planar even if it is usually drawn with crossings since it may be possible to draw it in a different way without crossings.

Eg-1:



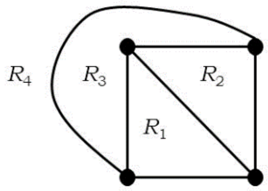
Eg-2:



Euler's theorem/ Euler's Formula:

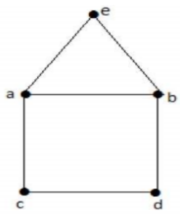
If a connected planar graph 'G' has n_v vertices, n_e edges and n_f faces or regions then $n_v - n_e + n_f = 2$.

Eg-1:



Sol: For the given graph $n_v = 4$, $n_e = 6$ and $n_f = 4$ and $n_v - n_e + n_f = 4-6+4 = 8-6 = 2$. So it is a planar graph.

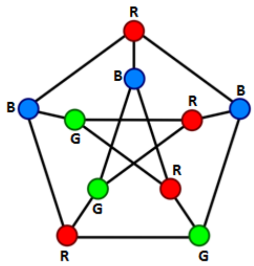
Eg-2:



Sol: : For the given graph $n_v = 5$, $n_e = 6$ and $n_f = 3$ and $n_v - n_e + n_f = 5-6+3 = 8-6 = 2$. So it is a planar graph.

Graph coloring:

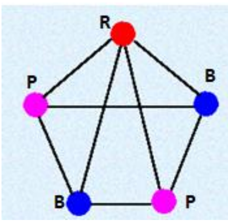
→ An assignment of colors to the vertices of a graph so that no two adjacent vertices get the same color is called coloring of the graph or simply vertex coloring.
→ The ‘n’ coloring of G refers to the coloring of G using n-colors.
→ If G has n-coloring then G is said to be n-colorable.



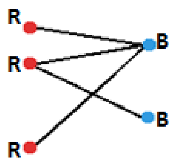
Chromatic number:

- The chromatic number of a graph G is the minimum number of colors needed to color the vertices of a graph G is denoted by $\chi(G)$.
- A graph G is n-colorable if $\chi(G) \leq$ n.

Eg’s :



Chromatic number = 3

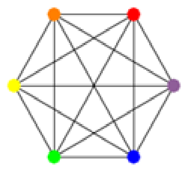


Chromatic number = 2

The chromatic number of some familiar graphs:

graph G	Chromatic number $\chi(G)$
complete graph K_n	$\chi(K_n) = n$
graph complement $\overline{K_n}$	$\chi(\overline{K_n}) = 1$
cycle C_n	$\chi(C_n) = 2$, for n – even $\chi(C_n) = 3$, for n – odd
wheel W_n	$\chi(W_n) = 4$, for n – even $\chi(W_n) = 3$, for n – odd
star S_n	$\chi(S_n) = 2$
tree T	$\chi(T) = 2$

- Theorem:- Let G be a non-trivial simple graph. Then $\chi(G)$=2 if and only if G is a bipartite graph.
- Theorem:- For any simple graph G, $\chi(G) \leq \Delta(G)+1$.
- Determine the chromatic number of complete graph K_n with n vertices.



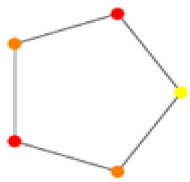
$\boxed{\chi = 6}$

$\chi(K_n) = n$

- Determine the chromatic number of C_n if n = even and n = odd.

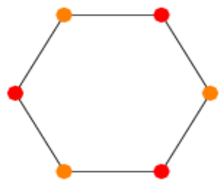
  Sol:- $\chi(C_n) = 2$, if n is even.

  $\chi(C_n) = 3$, if n is odd.



$\chi = 3$

$n = 5$



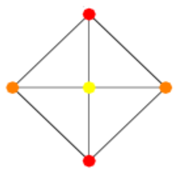
$\chi = 2$

$n = 6$

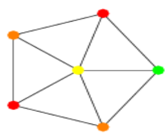
- Determine the chromatic number of wheel graph with n vertices.

  Sol:- $\chi(w_n) = 3$, if n is odd.

  $\chi(w_n) = 4$, if n is even.



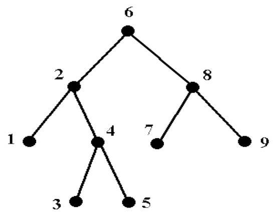
$\chi = 3$, n=5

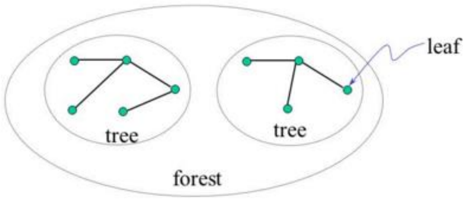


$\chi = 4$, n=6

Trees:

A graph that contains no cycles is called **acyclic graph** and a connected acyclic graph is called a **tree**. In other words, a connected graph with no cycles is known as a tree. Its edges are called branches. Connected: There is a path between any pair of nodes.

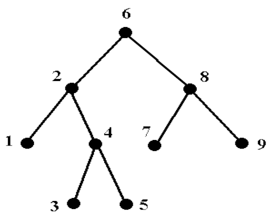




Tree properties:

i. There is only one path between every pair of vertices, in a tree T.
ii. If in a graph G, there is one and only one path between every pair of vertices then G is a tree.
iii. A tree with n vertices has (n-1) edges.
iv. For any +ve integer n, if G is a connected graph with n vertices and n-1 edges , then G is a tree.
v. In a tree with more than one vertex there are at least two vertices of degree 1.

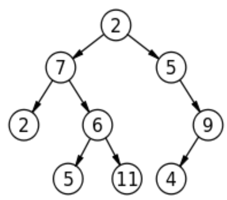
Rooted Trees:- A rooted tree is a tree in which a particular vertex is distinguished from the others and is called the **root**.
Height:- It is defined as maximum level.
Depth:- It is defined as the length of the path from the root to V.



- Root of Tree = 6
- Find all leaves vertices
  1 ,3, 5, 7, 9
- What's the level of 4 and 8 → 2 and 1
- Children of 8 are 7 and 9.
- Internal vertices are 2, 8 and 4

m-ary tree: A rooted tree is an m-ary tree if every internal vertex has **atmost** m children.

Binary Tree: A 2-ary tree is called a binary tree. A binary tree is a rooted tree in which each vertex has at most two children(means every vertex should have 0 children or l child or 2 children), designated as left child and right child. If a vertex has one child, that child is designated as either a left child or a right child, but not both. A full binary tree is a binary tree in which each vertex has exactly two children or none.



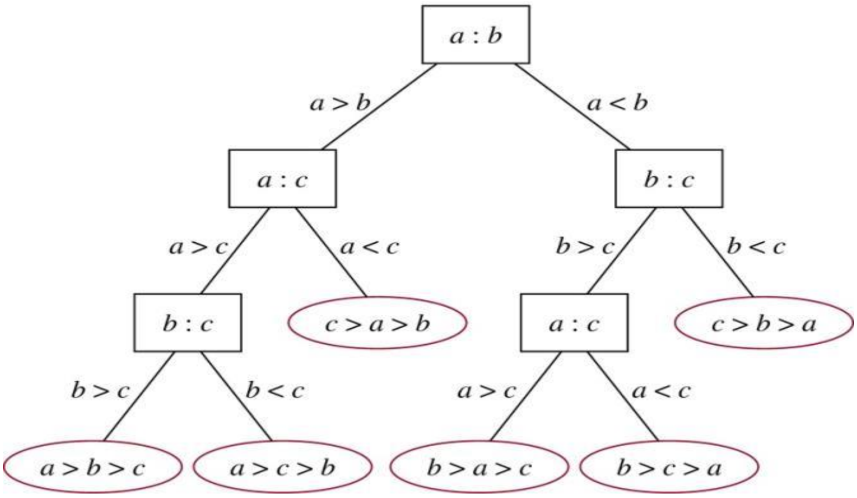
Note: If T is a full binary tree with i internal vertices, then T has i+1 terminal vertices and 2i + 1 total vertices.

Decision Trees: Decision trees are those which are used to model problems in which a series of decisions lead to a solution.

Eg-1: Sort three distinct elements a, b, and c
Sol:

Case (i) : Let a>b. Then compare a and c.
If a<c then c>a>b.
If a >c , then compare b and c.
If b<c then a>c>b, otherwise a>b>c.
Case (ii): The case a< b can be discussed similarly.

The following tree is the decision tree for this problem.



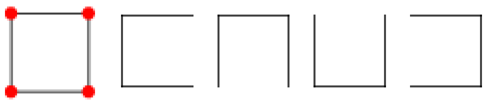
Spanning Trees:

A sub graph 'T' of a graph 'G' said to be a spanning tree if

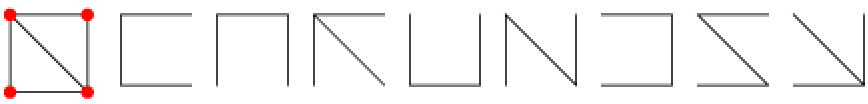
a. 'T' is a tree.
b. If 'T' includes every vertex of 'G' i.e. V(T) = V(G)

Below are the examples of some graphs and all possible spanning trees of them.

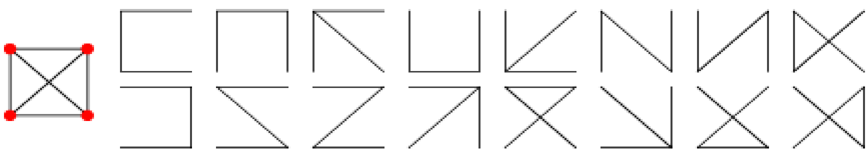
Eg-1:



Eg-2:



Eg-3:



Algorithms for spanning trees:

1. BFS
2. DFS

BFS (Breadth First Search): In this algorithm a rooted tree is constructed and the undirected graph of this rooted tree forms the spanning tree.

The idea of BFS is to visit all the vertices on a given level before going to the next level.

Procedure:

Step1: Choose a vertex arbitrarily and designated it as the root.

Step2: Add all the edges incident to this vertex such that the addition of vertex does not produce any cycle. The new vertices added at this stage become the vertices at level **1** in the spanning tree. Order them in as arbitrary manner.

Step3: For each vertex at level **1**, visited in order, add each edge incident to vertex to the tree as long as it does not produce any cycle.

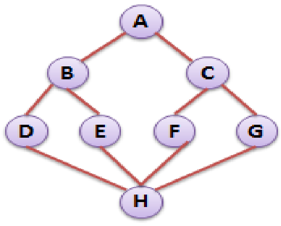
Step4: Arbitrarily order the children of each vertex at level **1**. This produces the vertices at level **2** in the tree.

Step5: Continue with the same procedure until all the vertices in the tree have been added.

This procedure ends, since there are only a finite no of edges in the graph.

Since we have produced a tree without cycle containing every vertex of the graph, the produced tree is a spanning tree.

Eg: Find Breadth First Search (BFS) order for the given graph:



Graph

Sol:

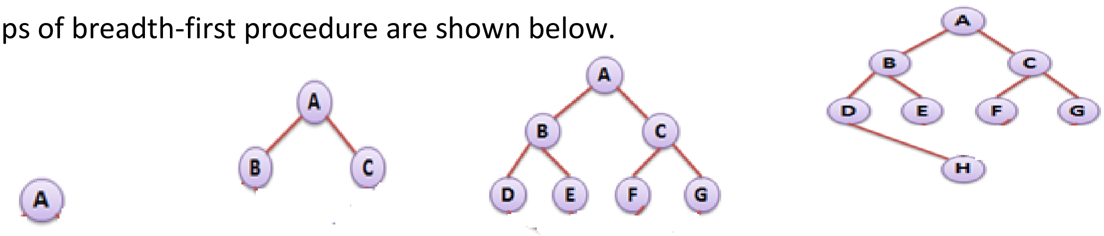
Step 1: Choose the vertex 'A' to be the root.

Step-2: Add all the edges incident to the vertex A such that the addition of edges does not produce any cycle. Hence the edges {A,B},{A,C} are added. The two vertices B and C are in level **1** in the tree.

Step-3: Add edges from these vertices at level 1 to adjacent vertices which are not already in the tree. Hence the edges {B,D}, {B,E}, {C,F} and {C,G} are added. The vertices D,E,F and G are in level **2** in the tree.

Step-4: Add edges from these vertices at level 2 to adjacent vertices which are not already in the tree. Hence the edges {D,H} is added. The vertex H is in level **3.**
in the tree.

The steps of breadth-first procedure are shown below.



(a) (b) (c) (d)

So the required path is A-B-C-D-E-F-G-H and the BFS spanning tree is fig(d) given above.

DFS (Depth First Search): DFSA is also known as backtracking.

Procedure:

Step1: Arbitrarily choose a vertex from the vertices of the graph and designate it as the root.

Step2: Form a path starting at this vertex by successively adding edges as long as possible where each new edge is incident with the last vertex in the path without producing any cycle.

Step3: If the path goes through all the vertices of the graph, then the tree consisting of this path is a spanning tree otherwise go the next step.

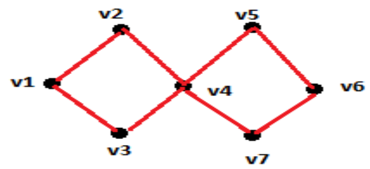
Step4: Move back to the next vertex in the graph. If possible, form a new path starting at this vertex passing through vertices that were not already visited.

Step5: If this cannot be done, move back another vertex in the path and repeat the process described in step2.

Step6: Repeat this procedure, beginning at the last vertex visited, moving back up the path one vertex at a time, and form a new path until no more edges can be added.

This process ends, since the graph has a finite number of edges and is connected. Hence a spanning tree is produced.

Eg-1: Find the Depth First Traversal order for the given graph



Sol:

Step1: Choose the vertex v1 to be the root.

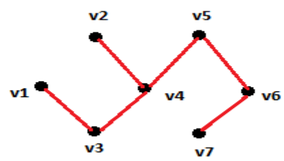
Step2: Form a path by successively adding edges incident with vertices which are not already in the path as long as possible. This procedure produces the path (v1, v3, v4, v5, v6, v7).

Step3: Now backtrack to v6. There is no path beginning at v6 containing vertices not already visited.

Step4: Similarly on backtracking to v5, we observe that there is no path.

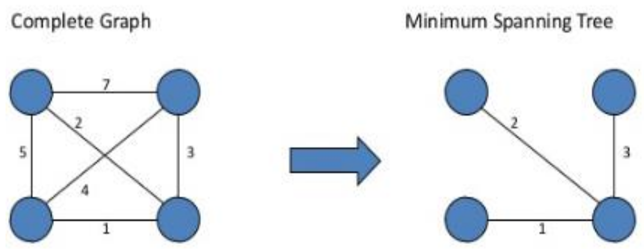
Step5: Backtrack to v4 and form the path (v4, v2).This produces the required spanning tree.

So the required path is v1, v3, v4, v5, v6, v7,v2 and the DFS spanning tree is shown below



Minimum Cost Spanning Tree (or) Minimal Spanning Tree:

Let 'G' be a weighted graph. A minimal spanning tree of G is defined as a spanning tree of 'G' with minimum weight.



Algorithms for finding minimal Spanning Tree:

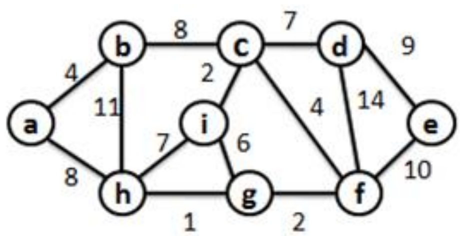
1. Kruskal's
2. Prim's

1. **Kruskal's:**

 i/p: A connected weighted graph 'G'.
 o/p: A minimum cost(minimal) spanning tree 'T' for 'G'

Procedure:
Step1: Choose an edge with minimal weight.
Step2: At each stage, choose (from the edges not yet chosen) the edge of lowest weight whose inclusion will not produce a cycle.
Step3: Stop the process of step2 when (n-1) edges are selected. These (n-1) edges contribute a minimum spanning tree for 'G'. Otherwise repeat step (2).

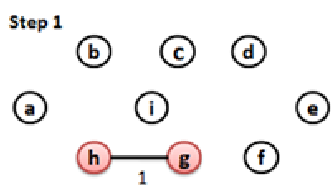
Eg: Find Minimum spanning tree for a graph in Fig using Kruskal's algorithm



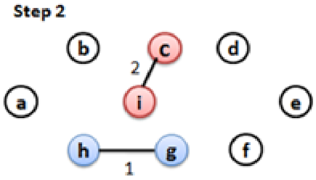
Sol: We first tabulate the edges with their weights(costs) in non-decreasing order.

Edge	(h,g)	(i,c)	(g,f)	(a,b)	(c,f)	(i,g)	(c,d)	(i,h)	(a,h)	(b,c)	(d,e)	(e,f)	(b,h)	(d,f)
Weight	1	2	2	4	4	6	7	7	8	8	9	10	11	14

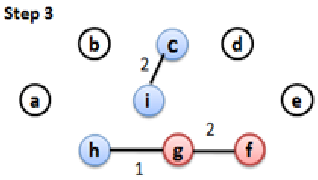
Step-1: Choose the edge (h,g) which has the minimal weight



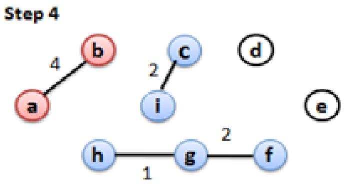
Step-2: There are two edges (i,c) and(g,f) with the next minimal cost. So choose both of them each in a step.So add the next edge (i,c).



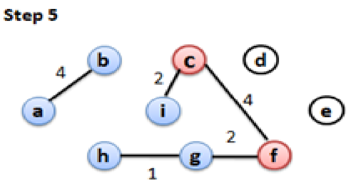
Step-3: Add the next edge (g,f).



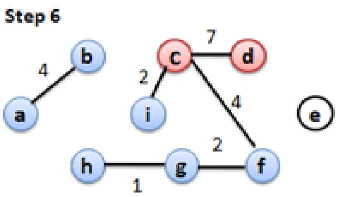
Step-4: Add the next edge (a,b).



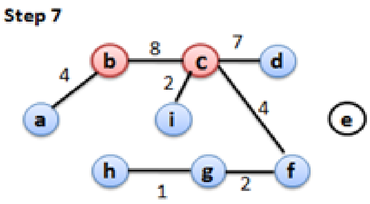
Step-5: Add the next edge (c,f).



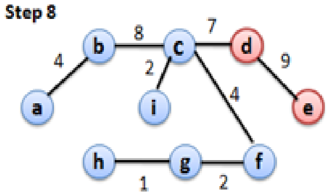
Step-6: The next edges with minimal cost are (i,g) and (i,h). But their inclusion will produce a cycle. So discard them and add the next edge (c,d).



Step-7: Add the next edge (b,c).



Step-8: The next edge with minimal cost is (a,h) but we discard it as its inclusion produces a cycle. So add the next edge (d,e).



Step-9: Discard all the remaining edges as their inclusion produces a cycle.Since the graph has 9 vertices it is enough to choose only 8 edges.Therefore we stop the procedure and the tree obtained in the Step-8 is the required Minimal Spanning tree and the Cost of the minimum spanning tree = 4+8+7+9+2+4+2+1 = 37

2. Prim's algorithm:

i/p: A connected weighted graph 'G'

o/p: A minimal spanning tree 'T'

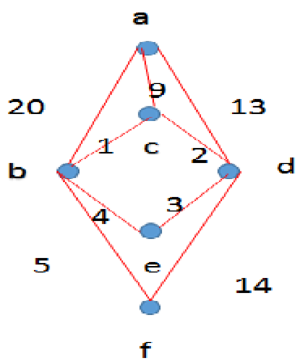
Procedure:

Step1: Select any vertex and choose the edge and minimum weight from 'G'.

Step2: At each stage choose the edge of smallest weight joining a vertex already included to vertex not yet included.

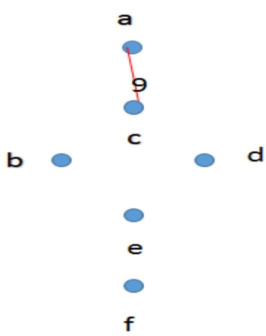
Step3: Continue until all vertices are included or If G has n vertices, then stop after (n-1) edges have been chosen otherwise repeat step(2).

Eg:

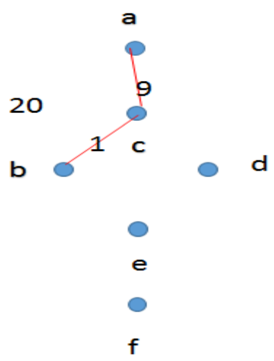


Sol:

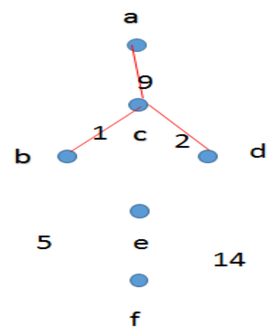
Step-1: Choose the vertex **'a'**. Now edges incident on 'a' are (a,b) , (a,c) and (a,d). Choose the edge with minimum weight i.e (a,c).



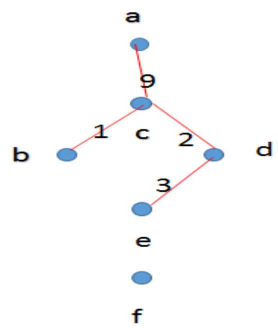
Step-2: The recent vertex is c. Now we have to choose the edge with minimum cost which are incident from both a and c which are already not included and which does not form a cycle. We have w(a,b)=20,w(a,d) =13,w(c,b) =1 and w(c,d) =2. So choose the edge (c,b)



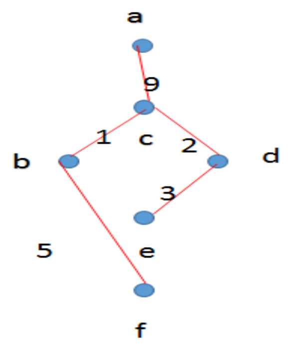
Step-3: The recent vertex is b. Now we have to choose the edge with minimum cost which are incident from a ,c and b which are already not included and which does not form a cycle. We have w(a,b)=20,w(a,d) =13,w(c,d) =2, w(be)=4 and w(b,f)=5. So choose the edge (c,d)



Step-4: The recent vertex is d. Now we have to choose the edge with minimum cost which are incident from a ,c,b and d which are already not included and which does not form a cycle. We have w(a,b)=20,w(a,d) =13, w(be)=4 ,w(b,f)=5,w(d,e)=3 and w(df)=14. So choose the edge (d,e)



Step-5: The recent vertex is e. Now we have to choose the edge with minimum cost which are incident from a ,c,b,d and e which are already not included and which does not form a cycle. We have w(a,b)=20,w(a,d) =13, w(be)=4 ,w(b,f)=5,w(df)=14. The edge with the next minimum cost is (b,e) but it is discarded as it forms a cycle.So choose the edge (b,f)



Step-6: The recent vertex is f. Now we have to choose the edge with minimum cost which are incident from a ,c,b,d,e and f which are already not included and which does not form a cycle. We have w(a,b)=20,w(a,d) =13, w(be)=4 and w(df)=14. We have to discard all these edges as they form a cycle .As all the vertices are covered we can stop this procedure and the tree obtained in the Step-5 is the required Minimal Spanning tree and the cost of the tree
= 9+1+2+3+5 =20
.

UNIT-5

Recurrence Relations : Generating Function of Sequences, Partial Fractions, Calculating Coefficient of Generating Functions, Recurrence Relations, Formulation as Recurrence Relations, Solving linear homogeneous recurrence Relations by substitution, generating functions and the Method of Characteristic Roots. Solving Inhomogeneous Recurrence Relations.

Recurrence relations

Definition: An equation that express a_n in terms of one or more of the previous terms of the sequence namely $a_0, a_1, a_2, \ldots a_{n-1}$ for all integers n with $n \geq n_0$ where n_0 is a non-negative integer is called a **recurrence relation** for the sequence **$\{a_n\}$** or a **difference equation**.

If a_n terms of sequence satisfy the recurrence relation, then the sequence is called solution of the recurrence relation.

Eg-1: Let a_n be the sequence that satisfies the recurrence relation $a_n = a_{n-1} + 3a_{n-2}$ for n = 2,3,… and a_0 = 1 and a_1 = 2. What are the values of a_2 and a_3.

Sol:- Given recurrence relation is $a_n = a_{n-1} + 3a_{n-2}$

$$a_2 = a_{2-1} + 3a_{2-2} = a_1 + 3a_0 = 2 + 3(1) = 5$$

$$a_3 = a_{3-1} + 3a_{3-2} = a_2 + 3a_1 = 5 + 3(2) = 11$$

Eg-2: Find the first five terms of the sequence if

I. $a_n = a^2_{n-1}$, $a_1 = 2$

II. $a_n = a_{n-1} + a_{n-3}$, $a_0 = 1, a_1 = 2, a_2 = 0$

III. $a_n = na_{n-1} + n^2 a_{n-2}$, $a_0 = 1, a_1 = 1$

Sol:-

I. $a_n = a^2_{n-1}$, $a_1 = 2$

If n = 2, $a_2 = a^2_{2-1} = (a_1)^2 = 2^2 = 4$

If n = 3, $a_3 = a^2_{3-1} = (a_2)^2 = 4^2 = 16$

If n = 4, $a_4 = a^2_{4-1} = (a_3)^2 = 16^2 = 256$

If n = 5, $a_5 = a^2_{5-1} = (a_4)^2 = 256^2 = 65356$.

∴ The first five terms of the sequence are 2, 4, 16, 256, 65356.

II. $a_n = a_{n-1} + a_{n-3}$, $a_0 = 1, a_1 = 2, a_2 = 0$

If n = 3,

$$a_3 = a_{3-1} + a_{3-3} = a_2 + a_0 = 0 + 1 = 1$$

If n = 4,

$$a_4 = a_{4-1} + a_{4-3} = a_3 + a_1 = 1 + 2 = 3$$

∴ The first five terms of the sequence are 1, 2, 0, 1, 3.

III. $a_n = na_{n-1} + n^2 a_{n-2}$, $a_0 = 1, a_1 = 1$

If n = 2,

$$a_2 = 2a_{2-1} + 2^2 a_{2-2} = 2a_1 + 4a_0 = 2(1) + 4(1) = 2 + 4 = 6$$

If n = 3,

$a_3 = 3a_{3-1} + 3^2 a_{3-2} = 3a_2 + 9a_1 = 3(6) + 9(1) = 18+9 = 27$

If n = 4,

$a_4 = 4a_{4-1} + 4^2 a_{4-2} = 4a_3 + 16a_2 = 4(27) + 16(6)$

$= 108+96 = 204$

∴ The first five terms of the sequence are 1, 1, 6, 27, 204.

Eg-3: Determine whether the sequence $\{a_n\}$ is a solution of recurrence relation $a_n = a_{n-1} + 2a_{n-2} + 2n - 9$

i. $a_n = -n+2$

ii. $a_n = 3(-1)^n + 2^n - n + 2$

Sol:-

i. $a_n = -n+2$

Given $a_n = -n+2$

But question contains n-1 and n-2

$a_{n-1} = -(n-1) + 2$

$= -n+1+2$

$= -n+3$ ----- (1)

$a_{n-2} = -(n-2) + 2$

$= -n+2+2$

$= -n+4$ ----- (2)

Consider,

R.H.S $= a_{n-1} + 2a_{n-2} + 2n - 9$ (substitute 1 and 2 in R.H.S)

$= (-n+3) + 2(-n+4) + 2n - 9$

$= -n - 3 \cancel{-2n} + 8 + \cancel{2n} - 9$

$= -n + 11 - 9$

$= -n+2$

$= a_n =$ L.H.S

∴ $a_n = -n+2$ is a solution of given recurrence relation.

ii. $a_n = 3(-1)^n + 2^n - n + 2$

Given $a_n = 3(-1)^n + 2^n - n + 2$

But question contains n-1 and n-2

$a_{n-1} = 3(-1)^{n-1} + 2^{n-1} - (n-1) + 2$

$= 3(-1)^{n-1} + 2^{n-1} - n + 1 + 2$

$= 3(-1)^{n-1} + 2^{n-1} - n + 3$ ----- (1)

$a_{n-2} = 3(-1)^{n-2} + 2^{n-2} - (n-2) + 2$

$= 3(-1)^{n-2} + 2^{n-2} - n + 2 + 2$

$= 3(-1)^{n-2} + 2^{n-2} - n + 4$ ----- (2)

Consider,

R.H.S $= a_{n-1} + 2a_{n-2} + 2n - 9$ (substitute 1 and 2 in R.H.S)

$= 3(-1)^{n-1} + 2^{n-1} - n + 3 + 2(3(-1)^{n-2} + 2^{n-2} - n + 4) + 2n - 9$

$= 3(-1)^{n-1} + 2^{n-1} - n + 3 + 6(-1)^{n-2} + 2.2^{n-2} \cancel{-2n} + 8 + \cancel{2n} - 9$

$= 3(-1)^{n-1} + 2^{n-1} - n + 3 + 6(-1)^{n-2} + 2.2^{n-2} + 8 - 9$

$= 6(-1)^{n-2} + 3(-1)^{n-1} + 2.2^{n-2} + 2^{n-1} - n + 2$

$$= \frac{6(-1)^n}{(-1)^2} + \frac{3(-1)^n}{(-1)^1} + 2.\frac{2^n}{2^2} + \frac{2^n}{2^1} - n + 2$$

$$= 6(-1)^n - 3(-1)^n + \frac{2^n}{2^1} + \frac{2^n}{2^1} - n + 2$$

$$= 3(-1)^n + 2.\frac{2^n}{2} - n + 2$$

$$= 3(-1)^n + 2^n - n + 2 = a_n = \text{L.H.S}$$

∴ $a_n = 3(-1)^n + 2^n - n + 2$ is a solution of given recurrence relation.

Eg-4:- Let $a_n=2^n+5(3^n)$ for n= 0, 1, 2....
i) Find a_0, a_1, a_2, a_3 and a_4
ii) Show that $a_2=5a_1-6a_0$, $a_3=5a_2 - 6a_1$ and $a_4= 5a_3-6a_2$
iii) Show that $a_n=5a_{n-1} - 6a_{n-2}$ for all n ≥2

Sol:- *Given $a_n=2n+5(3n)$, n≥0*

i) $a_0 = 2^0+5.3^0 = 1+5(1) = 1+5 =6$
$a_1 = 2^1+5(3^1) = 2+5(3) = 2+15 =17$
$a_2 = 2^2+5(3^2) = 4+5(9) = 4+45 =49$
$a_3 = 2^3+5(3^3) = 8+5(27) = 8+135 =143$
$a_4 =2^4+5(3^4) = 16+5(81) = 16+405 =421$

ii) L.H.S = $5a_1 - 6a_0 = 5(17)-6(6) = 49 = a_2$ = R.H.S
L.H.S = $5a_2 - 6a_1 =5(49) -6(17) =143 = a_3$ = R.H.S
L.H.S = $5a_3 - 6a_2=5(143)-6(49) =421 = a_4$ = R.H.S
∴ Hence proved.

iii) L.H.S $= 5a_{n-1} - 6a_{n-2}$
$= 5 [2^{n-1}+5(3^{n-1})] - 6[2^{n-2}+5(3^{n-2})]$
$= 2^{n-2}[-6+10] + 3^{n-2}[-30+75]$
$= 2^{n-2}x4 +3^{n-2}x 45 \quad = 2^{n-2}x2^2 + 3^{n-2}x3^2x5$
$=2^n+5(3^n) \quad = a_n$ =R.H.S
∴ Hence proved.

Eg-5: A person deposits Rs.1000 in an account that yields 9% interest compounded yearly.

I. Set up recurrence relation for amount in account at the end of n years.
II. Find an explicit formula for amount in account at end of n years.
III. How much money will be in account after 100 years?

Sol:-

I) Let S_n denote the amount in the account after n years.

But the amount in the account after n years = amount in the account after (n-1) years + interest for the n^{th} year. (Interest = 9/100 = 0.09)

$S_n = S_{n-1} + (0.09) S_{n-1}$
$= S_{n-1} (1 + 0.09)$
∴ $S_n = S_{n-1} (1.09)$

This is the required recurrence relation.

II) Explicit formula for Sn

$S_1 = S_{1-1}(1.09) = (1.09)\, S_0$

$S_2 = S_{2-1}(1.09) = (1.09)\, S_1 = (1.09)^2\, S_0$

$S_3 = S_{3-1}(1.09) = (1.09)\, S_2 = (1.09)^3\, S_0$

....

$S_n = S_{n-1}(1.09) = (1.09)\, S_{n-1} = (1.09)^n\, S_0$

$\therefore S_n = (1.09)^n\, S_0 = (1.09)^n \times 1000\ [S_0=1000]$ ---- (1)

Using mathematical induction we can prove the validity of equation (1)

When n = 0

$S_0 = (1.09)^0 \times 1000 = 1000$

$\therefore S_n$ is true for n = 0

We assume that $S_k = (1.09)\ k \times 1000$ is true ---- (2)

We need to prove that S_{k+1} is true i.e

$S_{k+1} = (1.09)\ k+1 \times 1000$ is true

From recurrence relation we have $S_{k+1} = S_k\,(1.09)$

$= (1.09)\,(1.09)\ k \times 1000$

$= (1.09)\ k+1 \times 1000$

$\therefore$ S_{k+1} is true.

Thus by the principle of mathematical induction S_n is true for all n.

$\therefore$ Explicit formula $S_n = (1.09)\ n \times 1000$

III) When n=100, we have

$S_{100} = (1.09)100 \times 1000$

= Rs. 1000 x (1.09)100

$\therefore$ Money in the account after 100 years = Rs. 1000 x (1.09)100

Eg-6: Suppose the number of bacteria in a colony triples every hour.

1) Set up a recurrence relation for the number of bacteria after n hours have elapsed.

2) If 100 bacteria are used to begin a new colony, how many bacteria will be there in the colony in 10 hours.

Sol:- 1) Let a_n be the number of bacteria after n hours and a_{n-1} be the number of bacteria after n-1 hours.

$\therefore a_n = 3\, a_{n-1}$ is the required recurrence relation.

2) Explicit formula

Let $a_0 = 100$ then

$a_1 = 3\, a_0 = 3 \times 100$

$a_2 = 3\, a_1 = 3 \times 3 \times 100 = 3^2 \times 100$

$a_3 = 3\, a_2 = 3 \times 3 \times 3 \times 100 = 3^3 \times 100$

$a_n = 3\, a_{n-1} = 3^n \times 100$ -----(1)

By mathematical induction let's validate equation (1)

If n = 0 then

$a_0 = 3^0 \times 100$

$= 100$ which is true

$\therefore$ a_0 is true.

We assume that $a_k = 3^k \times 100$ is true

From (1) $a_{k+1} = 3a_k$

$= 3.3^k \times 100$

$= 3^{k+1} \times 100$

$\therefore a_{k+1}$ is true

Explicit formula is $a_n = 3^n \times 100$

If n = 10

$a_{10} = 3^{10} \times 100$

$= 59,04,900$

$\therefore$ No. of bacteria in the colony in 10 hours = 59,04,900

Linear homogeneous recurrence relations with constant coefficients:-

A recurrence relation of the form $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}$ where $c_1, c_2, \ldots c_k$ are constants and $c_k \neq 0$ is called a linear homogeneous recurrence relation of **degree k** with constant coefficients.

Examples:

1. $a_n = 2a_{n-1}$ // homogeneous with constant coefficient.
2. $a_n - a_{n-1} = 3$ // Non- homogeneous
3. $a_n = 2a_{n-1} + a^2_{n-2}$ // It is not linear
4. $a_n = 2a_{n-1} + a_{n-2}$ // Linear homogeneous with constant coefficients.

Solving linear homogenous recurrence relations:-

A linear homogeneous recurrence relation can be solved by using 3 methods. They are:

1. Substitution (also called as Iteration).
2. Method of characteristic roots.
3. Generating functions.

1) Substitution:- In this method the recurrence relation for a_n is used repeatedly to solve for a general expression of a_n in terms of n.

Eg: Solve the recurrence relation $a_n = a_{n-1} + f(n)$, $n \geq 1$ by ***substitution.***

$a_1 = a_{1-1} + f(1) = a_0 + f(1)$

$a_2 = a_{2-1} + f(2) = a_1 + f(2) = a_0 + f(1) + f(2)$

$a_3 = a_{3-1} + f(3) = a_2 + f(3) = a_0 + f(1) + f(2) + f(3)$

$$a_n = a_{n-1} + f(n) = a_0 + \sum_{k=1}^{n} f(k)$$

More generally c is a constant

we can solve $a_n = C\, a_{n-1} + f(n)$

$a_1 = C\, a_0 + f(1)$

$a_2 = C\, a_1 + f(2)$

$= C(Ca_0 + f(1)) + f(2)$

$= C^2 a_0 + C\, f(1) + f(2)$

$a_3 = C\, a_2 + f(3)$

$= C(C^2 a_0 + C\, f(1) + f(2)) + f(3)$

$= C^3 a_0 + C^2 f(1) + C\, f(2) + f(3)$

$a_n = C\, a_{n-1} + f(n)$

$= C^n a_0 + C^{n-1} f(1) + C^{n-2} f(2) + \text{---} + C\, f(n-1) + f(n)$

$= C^n a_0 + \sum_{k=1}^{n} C^{n-k} f(k)$

2) The method of characteristic roots:- Consider the recurrence relation $a_n = c_1 a^{n-1} + c_2 a^{n-2} + \ldots . c_k a^{n-k}$ where $c_1, c_2 \ldots . c_k$ are constants and $c_k \neq 0$.
The **characteristic equation** is given by $r^k - c_1 r^{k-1} - c_2 r^{k-2} \ldots . - c_{k-1} r - c_k = 0$
The solutions to this equation are called **characteristic roots**.

Distinct Roots:- If the characteristic equation has distinct roots $r_1, r_2, r_3, \ldots . r_k$ then the formula is $a_n = c_1 r_1^n + c_2 r_2^n + c_3 r_3^n + \ldots .. c_k r_k^n$ where $c_1, c_2, c_3, \ldots c_k$ are constants.

Equal Roots:- If the characteristic equation has equal roots then the formula is $\mathbf{a_n = c_1 r_1^n + c_2 n r_2^n}$ where c_1, c_2 are constants.

Imaginary Roots:- If the characteristic equation has equal roots then the formula is $a_n = r^n(c_1 \cos n\Theta + c_2 \sin n\Theta)$ where $r = \sqrt{(a^2+b^2)}$ & $\Theta = \tan^{-1}(b/a)$

S.No	Type of roots	Formula
1	Real and equal	$a_n = c_1 r_1^n + c_2 n r_2^n$
2	Real and distinct	$a_n = c_1 r_1^n + c_2 r_2^n$
3	Imaginary	$r^n(c_1 \cos n\Theta + c_2 \sin n\Theta)$ where $r = \sqrt{(a^2+b^2)}$ & $\Theta = \tan^{-1}(b/a)$

Eg-1: Solve $a_n = a_{n-1} + 2a_{n-2}$, initial condition $a_0 = 0$, $a_1 = 1$.

Sol:- *Given recurrence relation is* $a_n = a_{n-1} + 2a_{n-2}$

$a_n - a_{n-1} - 2a_{n-2} = 0$ is second order linear homogeneous recurrence relation.

Characteristic equation is

$$r^2 - r - 2 = 0$$
$$r^2 - 2r + r - 2 = 0$$
$$r(r-2) + 1(r-2) = 0$$
$$(r-2)(r+1) = 0$$
$$r = 2, -1 \qquad \text{Distinct real roots}$$

The roots r_1 and r_2 are real and distinct, therefore solution is $a_n = c_1 r_1^n + c_2 r_2^n$ where c_1 and c_2 are constants.

∴ Solution is $a_n = c_1 2^n + c_2 (-1)^n$.

Now the values of c1 and c2 must be calculated using the initial conditions.

If $\mathbf{a_0 = 0} \Rightarrow n=0$

$$a_n = c_1 2^0 + c_2(-1)^0$$
$$0 = c_1 + c_2$$
$$c_1 = -c_2 \qquad \text{----- (1)}$$

If $\mathbf{a_1 = 1} \Rightarrow n=1$

$$a_n = c_1 2^1 + c_2(-1)^1$$
$$1 = 2c_1 - c_2$$
$$2c_1 - c_2 = 1 \qquad \text{----- (2)}$$

By substituting (1) in (2)

$$2c_1 - c_2 = 1$$

$$2(-c_2)-c_2=1$$
$$-3c_2=1$$
$$c_2=-\tfrac{1}{3} \qquad \text{----- (3)}$$

Solving (3) and (1)

$$c_1=-c_2$$
$$c_1=-(-\tfrac{1}{3})$$
$$c_1=\tfrac{1}{3}$$

∴ Solution is $a_n=\tfrac{1}{3}2^n+(-\tfrac{1}{3})(-1)^n$.

Eg-2: Solve $a_n = 8a_{n-1} - 16a_{n-2}$, initial conditions $a_0 = 16$, $a_1 = 80$.

Sol:- *Given recurrence relation is* $a_n = 8a_{n-1} - 16a_{n-2}$

$a_n - 8a_{n-1} + 16a_{n-2}= 0$ is second order linear homogeneous recurrence relation.

Characteristic equation is

$$r^2-8r+16=0$$
$$r^2-4r-4r+16=0$$
$$r(r-4)-4(r-4)=0$$
$$(r-4)(r-4)=0$$
$$r=4,4 \qquad \text{real roots and equal}$$

The roots r_1 and r_2 are real and equal, therefore solution is $a_n=c_1r_1^n+c_2n\,r_2^n$ where c_1 and c_2 are constants.

∴ Solution is $a_n=c_14^n+c_2n\,4^n$.

Now the values of c1 and c2 must be calculated using the initial conditions.

If **a_0 = 16** ⇒ n=0

$$a_n=c_14^0+c_2 0.4^0$$
$$16=c_1+0$$
$$c_1=16 \qquad \text{----- (1)}$$

If **a_1 = 80** ⇒ n=1

$$a_n=c_14^1+c_2 1.4^1$$
$$80=4c_1+4c_2$$
$$c_1+c_2=20 \qquad \text{----- (2)}$$

By substituting (1) in (2)

$$c_1+c_2=20$$
$$16+c_2=20$$
$$c_2=4.$$

∴ Solution is $a_n=16.4^n+4n.4^n$

$$=4^n(16+4n)$$

Eg-3:- Solve the recurrence relation $a_n=2a_{n-1}-2a_{n-2}$, $a_0=1$, $a_1=2$.

Sol: Given recurrence relation is $a_n=2a_{n-1}-2a_{n-2}$

$a_n- 2a_{n-1}+2a_{n-2}=0$ is second order linear homogeneous recurrence relation.

Characteristic equation is $r^2-2r+2=0$

$a=1, b=-2, c=2$. Substitute these values in the formula $-b\pm\sqrt{b^2-4ac}/2a$

$=2\pm\sqrt{4-4(1)(2)}/2$ $=2\pm\sqrt{4-8}/2$ $=2\pm\sqrt{-4}/2$

$=2\pm\sqrt{4i^2}/2$ $=2\pm2i/2$ $=2(1\pm i)/2$ $=i\pm1$

Roots are imaginary.

∴Solution is $a_n=r^n(c_1\cos n\Theta+c_2\sin n\Theta)$ where $r=\sqrt{(a^2+b^2)}$ & $\Theta=\tan^{-1}(b/a)$

Here a=1 and b=1.

$r=\sqrt{1^2+1^2}$ $=\sqrt{1+1}$ $=\sqrt{2}$

$\Theta=\tan^{-1}(1/1)$ $=\tan^{-1}(1)$ $=45°=\pi/4$

∴ $i\pm1=\sqrt{2}(c_1\cos n\pi/4+c_2\sin n\pi/4)$

Now the values of c1 and c2 must be calculated using the initial conditions.

a_0=1 ⇒ n=0

$=\sqrt{2}^0(C_1\cos 0\pi/4+C_2\sin 0\pi/4)$

$=1(C_1\cos 0+C_2\sin 0)$ = $C1=1$---- (1)

a_1=2 ⇒ n=1

$\sqrt{2}^1(C_1\cos\pi/4+C_2\sin\pi/4)$

$\sqrt{2}(C_1\cos\pi/4+C_2\sin\pi/4)$

$=\sqrt{2}(C1\ I/\sqrt{2}+C_2\ I/\sqrt{2})$

$=\sqrt{2}(C_1+C_2)/\sqrt{2}$ $=C_1+C_2=2$ -----(2)

From (1) & (2) $C_1+C_2=2$ $=1+C_2=2$

$=C_2=2-1$ $=C_2=1$

∴ The required solution is $\sqrt{2}^n(\cos n\pi/4+\sin n\pi/4)$

Partial Fractions (Some Important Formulae)

Form of the Rational Function	Form of the Partial Fraction
$\dfrac{px+q}{(x-a)(x-b)}, a\neq b$	$\dfrac{A}{x-a}+\dfrac{B}{x-b}$
$\dfrac{px+q}{(x-a)^2}$	$\dfrac{A}{x-a}+\dfrac{B}{(x-a)^2}$
$\dfrac{px^2+qx+r}{(x-a)(x-b)(x-c)}$	$\dfrac{A}{x-a}+\dfrac{B}{x-b}+\dfrac{C}{x-c}$
$\dfrac{px^2+qx+r}{(x-a)^2(x-b)}$	$\dfrac{A}{x-a}+\dfrac{B}{(x-a)^2}+\dfrac{C}{x-b}$
$\dfrac{px^2+qx+r}{(x-a)(x^2+bx+c)}$	$\dfrac{A}{x-a}+\dfrac{Bx+C}{x^2+bx+c}$
Where x^2+bx+c cannot be factorised further	

Some Important formulae:

Series	Expansion
$(1+x)^{-1}$ (or) $1/(1+x)$	$1-x+x^2-x^3+x^4-.....$
$(1-x)^{-1}$ (or) $1/(1-x)$	$1+x+x^2+x^3+x^4+....$
$(1+x)^{-2}$ (or) $1/(1+x)^2$	$1-2x+3x^2-4x^3+....$
$(1-x)^{-2}$ (or) $1/(1-x)^2$	$1+2x+3x^2+4x^3+....$

Generating Functions: - Many counting problems are solved by using generating functions. The generating function of a sequence $a_0, a_1, a_2 \dots a_n$ is given as

$$G(z) = a_0 + a_1 z^1 + a_2 z^2 + ... + a_n z^n$$

$$= \sum_{n=0}^{\infty} a_n z^n$$ is called generating function of numeric function a.

Eg-1: Find the generating function for the sequence 2, 2, 2, 2, 2, 2

Sol:- $G(z) = 2+2z+2z^2+2z^3+2z^4+2z^5$

Eg-2: Find the generating function for the sequence 1,2,3,4… (Or) for the sequence $\{a_n\}$, $a_n=n+1$

Sol:- $G(z) = 1+2z+3z^2+4z^3+\dots \quad \Rightarrow 1/(1-z)^2 \quad \Rightarrow (1-z)^{-2}$

Note:- *The generating function of a sequence* $a_0, a_1, a_2, \dots a_n$ is denoted by $< a_0, a_1, a_2, \dots a_n >$

Eg-3: Find the generating function of the sequence <5, 3,-4,-2, 0, 1>

Sol:- $G(z)= 5+ 3z-4z^2-2z^3+0z^4+1z^5 \quad \Rightarrow G(z)= 5+ 3z-4z^2-2z^3+z^5$

Calculating Coefficient of Generating Functions:-

The coefficient of generating function of the form $(1+x+x^2+x^3+\dots)^n$ *i.e* $\frac{1}{(1-x)^n}$ *is given by*

$$1+\binom{1+n-1}{1}x+\binom{2+n-1}{2}x^2+\dots+\binom{r+n-1}{r}x^r$$

Eg-1: Find the coefficient of x^{16} in the expansion of $(x^2+x^3+x^4+....)^5$

Sol:- $(x^2+x^3+x^4+....)^5 = [x^2(1+x+x^2+x^3+\dots)]^5$

$$\Rightarrow (x^2)^5 (1+x+x^2+x^3+\dots)^5 \quad \Rightarrow x^{10}.(1+x+x^2+x^3+\dots)^5$$

$$\Rightarrow x^{10}.\frac{1}{(1-x)^5}$$

$$\frac{1}{(1-x)^5} \Rightarrow n=5$$

$$\Rightarrow 1+\binom{1+5-1}{1}x+\binom{2+5-1}{2}x^2+\dots+\binom{r+5-1}{r}x^r$$

$$\therefore x^{10}.\frac{1}{(1-x)^5} = x^{10}\binom{r+5-1}{r}x^r \quad \Rightarrow x^{10+r}\binom{r+5-1}{r}x^r$$

But we want the coefficient of x^{16} $\quad \Rightarrow x^{10+r} = x^{16} \quad \Rightarrow 10+r = 16$

$$\Rightarrow r=16-10 \quad \Rightarrow r=6$$

$\therefore$ Coefficient of f x^{16} = $\binom{6+5-1}{6}$ $\Rightarrow \binom{10}{6}$ $\Rightarrow {}^{10}C_6$

$\Rightarrow {}^{10}C_4$ $[\because {}^nc_r = {}^nc_{n-r}]$ $\Rightarrow \dfrac{10\times 9\times 8\times 7}{4\times 3\times 2\times 1}$ $\Rightarrow 210$

Eg-2: Find the coefficient of x^{20} in the expansion of $(x^3+x^4+x^5+....)^5$

Sol:- $(x^3+x^4+x^5+....)^5 = [x^3(1+x+x^2+x^3+...)]^5$

$\Rightarrow (x^3)^5(1+x+x^2+x^3+...)^5$ $\Rightarrow x^{15}.(1+x+x^2+x^3+...)^5$

$\Rightarrow x^{15}.\dfrac{1}{(1-x)^5}$

$\dfrac{1}{(1-x)^5} \Rightarrow n=5$

$\Rightarrow 1+\binom{1+5-1}{1}x+\binom{2+5-1}{2}x^2+...+\binom{r+5-1}{r}x^r$

$\therefore x^{15}.\dfrac{1}{(1-x)^5} = x^{15}\binom{r+5-1}{r}x^r$ $\Rightarrow x^{15+r}\binom{r+5-1}{r}x^r$

But we want the coefficient of x^{20} $\Rightarrow x^{15+r} = x^{20}$ $\Rightarrow 15+r = 20$

$\Rightarrow r=20-15$ $\Rightarrow r=5$

$\therefore$ Coefficient of f x^{20} = $\binom{5+5-1}{5}$ $\Rightarrow \binom{9}{5}$ $\Rightarrow {}^9C_5$

$\Rightarrow {}^9C_4$ $[\because {}^nc_r = {}^nc_{n-r}]$ $\Rightarrow \dfrac{9\times 8\times 7\times 6}{4\times 3\times 2\times 1}$ $\Rightarrow 126$

3) Solution of recurrence relations using generating functions

Eg-1: Solve $a_n = 3a_{n-1} + 2$, $n\geq 1$ with initial condition $a_0 = 1$

Sol:- *Given $a_n = 3a_{n-1} + 2$, $n\geq 1$ ------ (1)*

Let the generating function of the sequence $\{a_n\}$ be G(z) $= \sum_{n=0}^{\infty} a_n z^n$

Multiplying both sides of equation (1) with z^n and applying summation,we get

$$\sum_{n\geq 1}^{\infty} a_n z^n = 3\sum_{n\geq 1}^{\infty} a_{n-1} z^n + 2\sum_{n\geq 1}^{\infty} z^n$$

$$\Rightarrow \sum_{n\geq 1}^{\infty} a_n z^n = 3z\sum_{n\geq 1}^{\infty} a_{n-1}\frac{z^n}{z} + 2z\sum_{n\geq 1}^{\infty}\frac{z^n}{z}$$

$$\Rightarrow G(z) - a_0 = 3z\sum_{n\geq 1}^{\infty} a_{n-1} z^{n-1} + 2z\sum_{n\geq 1}^{\infty} z^{n-1}$$

$$\Rightarrow G(z) - a_0 = 3z\,G(z) + 2z\,(1+z+z^2+...)$$

$$\Rightarrow G(z) - 1 = 3z\,G(z) + (\frac{2z}{1-z})$$

$$\Rightarrow G(z) - 1 = 3z\,G(z) + 2z(1-z)^{-1}$$

$$\Rightarrow G(z) - 3z\,G(z) = 2z(1-z)^{-1} + 1$$

$$\Rightarrow G(z)[1-3z] = \frac{2z}{1-z} + 1$$

$$\Rightarrow G(z) = \frac{2z + 1(1-z)}{(1-z)(1-3z)} = \frac{z+1}{(1-z)(1-3z)}$$

Let

$$\frac{1+z}{(1-z)(1-3z)} = \frac{A}{(1-z)} + \frac{B}{(1-3z)}$$ [By using partial fractions]

Equating numerators on both sides ----- (2)

$\Rightarrow (1+z) = (1-3z)A + B(1-z)$

Put 1-z=0 $\Rightarrow$ z= 1 in B $(1-z)$ to calculate A in equation (2)

$\Rightarrow$ (1 +1) = (1 – 3) A $\Rightarrow 2=-2A$

$\Rightarrow$ A = -2/2 $\Rightarrow A=-1$

Put (1-3z) =0 $\Rightarrow$ 1=3z $\Rightarrow$ z = 1/3 in $(1-3z)A$ to calculate B in equation (2)

$$(1+\frac{1}{3}) = B(1-\frac{1}{3})$$

$$\Rightarrow \frac{4}{3} = B(\frac{2}{3})$$

$$B = 2$$

$$\therefore G(z) = \frac{-1}{(1-z)} + \frac{2}{(1-3z)}$$

$\therefore a_n = 2(3)^n - 1(1)^n \quad \Rightarrow a_n = 2(3)^n - 1$

Eg-2: Solve $a_n = a_{n-1} + 2(n-1)$, initial condition $a_0 = 3$, n≥1

Sol:- *Given $a_n = a_{n-1} + 2(n-1)$, n≥1 ------ (1)*

Let the generating function of the sequence $\{a_n\}$ be G(z) $= \sum_{n=0}^{\infty} a_n z^n$

Multiplying both sides of equation (1) with z^n and applying summation,we get

$$\sum_{n\geq 1}^{\infty} a_n z^n = \sum_{n\geq 1}^{\infty} a_{n-1} z^n + 2\sum_{n\geq 1}^{\infty} n z^n - 2\sum_{n\geq 1}^{\infty} z^n$$

$$\Rightarrow \sum_{n\geq 1}^{\infty} a_n z^n = z\sum_{n\geq 1}^{\infty} a_{n-1}\frac{z^n}{z} + 2\sum_{n\geq 1}^{\infty} n.z^n - 2\sum_{n\geq 1}^{\infty} z^n$$

$$\Rightarrow G(z) - a_0 = z[G(z)] + 2[z + 2z^2 + 3z^3 + ...] - 2[z + z^2 + z^3 + ...]$$

$$\Rightarrow G(z) - a_0 = z[G(z)] + 2z[1 + 2z + 3z^2 + ... + (n+1)z^n] - 2[(1-z)^{-1} - 1]$$

$$\Rightarrow G(z) - zG(z) = 3 + 2z(1-z)^{-2} - 2(1-z)^{-1} + 2$$

$$\Rightarrow G(z)[1-z] = 3 + 2z(1-z)^{-2} - 2(1-z)^{-1} + 2$$

$$\Rightarrow G(z)[1-z] = 3 + \frac{2z}{(1-z)^2} - \frac{2}{(1-z)^1} + 2$$

$$\Rightarrow G(z) = \frac{5}{(1-z)} + \frac{2z}{(1-z)^3} - \frac{2}{(1-z)^2}$$

$$\therefore a_n = 5.1^n + 2n(n+1) - 2(n+1)$$

$$= 5.1^n + 2n^2 + 2n - 2n - 2$$

$$= 2n^2 + 3$$

Eg-3: Solve a_n -2 a_{n-1} – 3a_{n-2}=0, with a_0 = 3, a_1 = 1, n≥2

Sol:- *Given $a_n = 2a_{n-1} + 3a_{n-2}$=0, n≥2* ------ (1)

Let the generating function of the sequence {a_n} be G(z) $= \sum_{n=0}^{\infty} a_n z^n$

Multiplying both sides of equation (1) with z^n and applying summation,we get

$$\sum_{n\geq 2}^{\infty} a_n z^n = 2\sum_{n\geq 2}^{\infty} a_{n-1} z^n + 3\sum_{n\geq 2}^{\infty} a_{n-2} z^n z^n$$

$$\Rightarrow\ 2z\sum_{n\geq 2}^{\infty} a_{n-1}\frac{z^n}{z} + 3z^2\sum_{n\geq 2}^{\infty} a_{n-2} z^{n-2}$$

$$\Rightarrow\ G(z) - a_0 - a_1 z = 2z\,[\mathrm{G(z)} - \mathrm{a}_0] + 3\mathrm{z}^2\, G(z)$$

$$\Rightarrow\ G(z) - a_0 - a_1 z = 2z\mathrm{G(z)} - 2\mathrm{z\,a}_0 + 3\mathrm{z}^2\, G(z)$$

$$\Rightarrow\ G(z) - 2z\mathrm{G(z)} - 3\mathrm{z}^2\, G(z) = -2\mathrm{z\,a}_0 + a_0 + a_1 z$$

$$\Rightarrow\ G(z) - 2z\mathrm{G(z)} - 3\mathrm{z}^2\, G(z) = -2\mathrm{z}\,(3) + 3 + (1)z$$

$$\Rightarrow\ G(z) - 2z\mathrm{G(z)} - 3\mathrm{z}^2\, G(z) = -6\mathrm{z} + 3 + z$$

$$\Rightarrow\ G(z) - 2z\mathrm{G(z)} - 3\mathrm{z}^2\, G(z) = -5\mathrm{z} + 3$$

$$\Rightarrow\ G(z)(1 - 2z - 3\mathrm{z}^2) = -5\mathrm{z} + 3$$

$$\Rightarrow\ G(z) = -5\mathrm{z} + 3/(1 - 2z - 3\mathrm{z}^2)$$

$$\Rightarrow\ G(z) = \frac{3 - 5\mathrm{z}}{(1 - 2z - 3z^2)} = \frac{3 - 5\mathrm{z}}{(1 - 3z)(1 + z)}$$

Let

$$\frac{3 - 5\mathrm{z}}{(1 - 3z)(1 + z)} = \frac{A}{(1 - 3z)} + \frac{B}{(1 + z)}$$

Equating numerators on both sides

⇒ 3-5z=A (1+z) +B (1-3z) ----- (2)

Put 1+z=0 ⇒ z=-1 in A (1+z) to calculate B in equation (2)

⇒ 3-5(-1) =B(1-3(-1)) ⇒ 3+5=B(1+3)

⇒ 8=B(4) ⇒ B=2

Equating coefficients of z on both sides in equation (2)

⇒ A-3B=-5 ⇒ A-3(2) =-5 ⇒ A-6—5

⇒ A=-5+6 ⇒ A=1

$$\therefore \mathrm{G(z)} = \frac{3 - 5\mathrm{z}}{(1 - 3z)(1 + z)} = \frac{1}{(1 - 3z)} + \frac{2}{(1 + z)}$$

$\therefore a_n = 1(3)^n + 2(-1)^n$

Eg-4: Solve a_n =4 a_{n-1} + 3n.2^n, with a_0 = 4, n≥1

Sol:- *Given a_n = 4 a_{n-1} + 3n.2^n, n≥1* ------ (1)

Let the generating function of the sequence {a_n} be G(z) $= \sum_{n=0}^{\infty} a_n z^n$

Multiplying both sides of equation (1) with z^n and applying summation,we get

$$\sum_{n\geq 1}^{\infty} a_n z^n = 4\sum_{n\geq 1}^{\infty} a_{n-1} z^n + 3\sum_{n\geq 1}^{\infty} n.2^n z^n$$

$$\Rightarrow \quad 4z\sum_{n\geq 1}^{\infty} a_{n-1}\frac{z^n}{z} + 3\sum_{n\geq 1}^{\infty} n.2^n z^n$$

$$\Rightarrow \quad 4z\sum_{n\geq 1}^{\infty} a_{n-1}\frac{z^n}{z} + 3\sum_{n\geq 1}^{\infty} n.(2z)^n$$

$$\Rightarrow G(z) - a_0 = 4zG(z) + 3(2z)\sum_{n\geq 1}^{\infty} n(2z)^{n-1}$$

$$\Rightarrow G(z) - 4zG(z) = a_0 + 3(2z)\sum_{n\geq 1}^{\infty} n(2z)^{n-1}$$

$$\Rightarrow G(z)(1-4z) = 4 + 6z[1 + 2.2z + 3.2z^2 + ..]$$

$$\Rightarrow G(z)(1-4z) = 4 + 6z(1-2z)^{-2}$$

$$\Rightarrow G(z) = \frac{4}{(1-4z)} + \frac{6z}{(1-4z)\ (1-2z)^2}$$

Let $\frac{6z}{(1-4z)\ (1-2z)^2} = \frac{A}{(1-4z)} + \frac{B}{(1-2z)} + \frac{C}{(1-2z)^2}$ [By using partial fractions]

Equating numerators on both sides

$\Rightarrow$ 6z = A(1-2z)^2+B(1-4z)(1-2z)+C(1-4z) ----- (2)

Put 1-4z=0 $\Rightarrow$ -4z= -1 $\Rightarrow$ z=1/4 in C$(1-4z)$ to calculate A in equation (2)

$\Rightarrow 6(1/4) = A(1-2(1/4))^2 + B(1-4(1/4))(1-2(1/4)) + C(1-4(1/4))$

$\Rightarrow$ 3/2= A(1-2/4)^2+B(1-4/4)*(1-2/4)+C(1-4/4)

$\Rightarrow$ 3/2 = A(1-1/2)^2+B(1-1)(1-1/2)+C(1-1)

$\Rightarrow$ 3/2 = A(1/2)^2+B(0)+C(0)

$\Rightarrow$ 3/2=A (1/4) $\Rightarrow$ A= 3/2(4) $\Rightarrow$ A= 3(2) $\Rightarrow$ A=6

Put (1-2z)^2=0 $\Rightarrow$ 1-2z =0 $\Rightarrow$ 2z=1 $\Rightarrow$ z=1/2 in A ((1-2z)^2 to calculate C in equation (2)

$\Rightarrow$ 6(1/2) = A (1-2(1/2))^2+B (1-4(1/2)) (1-2(1/2)) +C (1-4(1/2))

$\Rightarrow$ 3 = A (1-1)^2+B (1-2) (1-1) +C (1-2)

$\Rightarrow$ 3=A (0) +B (0) +C (-1) $\Rightarrow$ 3= C (-1) $\Rightarrow$ C=-3

Equating the coefficients of z on both sides in equation (2) to calculate B

$\Rightarrow$ 6z=-4zA-6zB-4zC $\Rightarrow$ 6$\cancel{z}$ =$\cancel{z}$ (-4A-6B-4C) $\Rightarrow$ 6=-4A-6B-4C

$\Rightarrow$ 6=-4(6)-6B-4(-3) $\Rightarrow$ 6=-24+12-6B $\Rightarrow$ 6= -12-6B

$\Rightarrow$ 6+12 = -6B $\Rightarrow$ 18 = -6B $\Rightarrow$ 6B = -18

$\Rightarrow$ B=-18/6 $\Rightarrow$ B=-3

$$\therefore G(z) = \frac{4}{(1-4z)} + \frac{6}{(1-4z)} - \frac{3}{(1-2z)} - \frac{3}{(1-2z)^2}$$

$$\Rightarrow G(z) = \frac{10}{(1-4z)} - \frac{3}{(1-2z)} - \frac{3}{(1-2z)^2}$$

$\therefore a_n$= 10(4)^n – 3(2)^n – 3(n+1)2^n

$\Rightarrow$ 10(4)^n – 3(2)^n – 3n (2)^n – 3(2)^n

$\Rightarrow$ 10(4)^n –2. 3(2)^n– 3n (2)^n

$\Rightarrow$ 10(4)^n –6(2)^n– 3n (2)^n

$\therefore a_n$ = $\Rightarrow$ 10(4)^n -(2)^n(3n+6)

Solving Non-homogeneous or Inhomogeneous Recurrence Relation:- A linear inhomogeneous recurrence relation with constant coefficients of degree k is a recurrence relation of the form $a_n=C_1a^{n-1}+C_2a^{n-2}+....C_ka^{n-k}+G(n)$. G (n) is a function not identically zero. The general solution of this is given by $a_n = a_n^{(h)} + a_n^{(p)}$ where

$a_n^{(h)}$ = Associated linear homogeneous recurrence relation and

$a_n^{(p)}$ = particular solution.

S.No	G(n)	Particular Solution
1	x^n and root = x	$d.n.x^n$
2	x^n and root ≠ x	$d.x^n$
3	constant(eg:2,3,100 etc)	d or nd or n^2d
4	c_0+c_1n where c_0 & c_1 are constants	d_0+d_1n

Eg-1: Solve $a_n = 2a_{n-1} + 2^n$, initial condition $a_0 = 2$.

Sol:- *The given recurrence relation $a_n - 2a_{n-1} = 2^n$ is a first order Inhomogeneous recurrence relation.*

1) The associated homogeneous recurrence relation is $a_n - 2a_{n-1} =0$

Characteristic equation is $r-2=0 \Rightarrow r=2$

∴ Homogeneous *solution is* $a_n^{(h)} = c_1 r_1^n$ *where c_1 is a constant.*

$\Rightarrow a_n^{(h)} = c_1 2^n$

2) Since R.H.S of the given is 2^n and 2 is a characteristic root, the particular solution be $a_n^{(p)}=d.n.2^n$

Substituting in the given relation we get

$dn2^n-2d(n-1)2^{n-1}=2^n \quad \Rightarrow dn2^n- d(n-1)2^n =2^n$

$\Rightarrow 2^n[dn-d(n-1)] = 2^n \quad \Rightarrow dn-d(n-1) =1$

$\Rightarrow dn-dn+d =1 \quad \Rightarrow d=1$

$\therefore a_n^{(p)}=n.2^n$

∴*General solution is* $a_n = a_n^{(h)} + a_n^{(p)}$

$a_n = c_1 2^n + n.2^n$

The value of c_1 must be calculated using initial condition.

If $\boldsymbol{a_0 = 2 \Rightarrow n=0}$

$a_n = c_1 2^n + n.2^n \quad \Rightarrow 2 = c_1 2^0 + 0.2^n \quad \Rightarrow c_1 = 2$

∴ *The required solution is* $a_n = 2.2^n + n.2^n$

Eg-2: Solve $a_n = 3a_{n-1} + 2n$, initial condition a0 = 1.

Sol: *The given recurrence relation $a_n - 3a_{n-1} = 2^n$ is a first order Inhomogeneous recurrence relation.*

1) The associated homogeneous recurrence relation is $a_n - 3a_{n-1} =0$

Characteristic equation is $r-3=0 \Rightarrow r=3$

∴*Homogeneous solution is* $a_n^{(h)} = c_1 r_1^n$ *where c_1 is a constant.*

$\Rightarrow a_n^{(h)} = c_1 3^n$

2) Since R.H.S of the given equation is 2^n and 2 is not the characteristic root, the particular solution be $a_n^{(p)}=d2^n$

Substituting in the given relation we get

$d2^n-3d2^{n-1}=2^n \quad \Rightarrow \quad 2^n[d-3/2d] = 2^n$

$\Rightarrow d-3/2d=1 \quad \Rightarrow 2d-3d=2$

$\Rightarrow -d=2 \quad \Rightarrow d=-2$

$\therefore a_n^{(p)}=-2.2^n$

$\therefore$ *General solution is* $a_n = a_n^{(h)} + a_n^{(p)}$

$a_n=c_1.3^n-2.2^n$

The value of c_1 must be calculated using initial condition.

If $\mathbf{a_0 = 1 \Rightarrow n=0}$

$\Rightarrow a_0=c_1.3^0-2.2^0 \quad \Rightarrow c_1-2 =1 \quad \Rightarrow c_1=1+2 \quad \Rightarrow c_1=3$

$\therefore$ *Required solution is $a_n=3.3^n-2.2^n$*

Eg-3: Solve $a_n - 2a_{n-1} + a_{n-2} = 2$, initial condition $a_0 = 25$ and $a_1 = 16$.

Sol:- *The given recurrence relation $a_n - 2a_{n-1} + a_{n-2} = 2$ is a second order Inhomogeneous recurrence relation.*

1) Associated homogeneous recurrence relation is $a_n - 2a_{n-1} + a_{n-2} = 0$

Characteristic equation is $r^2 - 2r + 1 = 0$

$\Rightarrow (r-1)^2 = 0 \quad \Rightarrow r = 1,1$

$\therefore$ Homogeneous solution is $a_n^{(h)} = c_1 1^n + c_2.n.1^n$

$= (c_1 + c_2.n).1^n$

2) Since R.H.S of the given(2) is a constant, the particular solution be $a_n = d$

Substituting in the given relation we get d-2d+d = 2

$\Rightarrow$ 0 = 2 which is impossible

So take nd

$\Rightarrow$ nd – 2 (n-1)d + (n-1)d = 2 $\quad \Rightarrow$ nd - 2nd + 2d + nd – 2d = 2

$\Rightarrow$ ~~2nd~~ - ~~2nd~~ + ~~2d~~ - ~~2d~~ = 2

$\Rightarrow$ 0 = 2 which is impossible

So take n^2 d

$\Rightarrow n^2 d - 2(n-1)^2 d + (n-2)^2 d = 2$

$\Rightarrow n^2 d - 2d(n^2 + 1 - 2n) + (n^2 + 4 - 4n) \ d = 2$

$\Rightarrow n^2 d - 2dn^2 - 2d + 4dn + dn^2 + 4d - 4nd = 2$

$\Rightarrow d(n^2 - 2n^2 - 2 + 4n + n^2 + 4 - 4n) = 2$

$\Rightarrow d(2) = 2$

$\Rightarrow d = 1$

$a_n^{(p)} = n^2.1$ is the particular solution.

$\therefore$ General solution is $a_n = a_n^{(h)} + a_n^{(p)}$

$a_n = (c_1 + nc_2)1^n + n^2$

The values of c1 and c2 must be calculated using the initial conditions.

If $\mathbf{a_0 = 25} \Rightarrow n=0$

$$a_n = (c_1 + nc_2)1^n + n^2$$
$$\Rightarrow a_0 = (c_1 + 0c_2)1^0 + 0^2$$
$$\Rightarrow 25 = c_1 + 0$$
$$\Rightarrow c_1 = 25 \quad \text{-----(1)}$$

If $a_1 = 16 \Rightarrow n=1$

$$a_n = (c_1 + nc_2)1^n + n^2$$
$$\Rightarrow a_1 = (c_1 + 1c_2)1^1 + 1^2$$
$$\Rightarrow c_1 + c_2 + 1 = 16$$
$$\Rightarrow c_1 + c_2 = 15 \quad \text{-----(2)}$$

From (1) and (2)

$$c_1 + c_2 = 15$$
$$\Rightarrow c_2 = 15 - 25$$
$$\Rightarrow c_2 = -10$$

∴ Required solution is $a_n = (25 - 10n)1^n + n^2$

Eg-4: Solve $a_n - 7a_{n-1} + 10a_{n-2} = 6+8n$, initial conditions $a_0 = 1$ and $a_1 = 2$.

Sol:- *The given recurrence relation $a_n - 7a_{n-1} + 10a_{n-2} = 6+8n$ is a second order Inhomogeneous recurrence relation*

1) Associated homogeneous recurrence relation is $a_n - 7a_{n-1} + 10a_{n-2} = 0$

Characteristic equation is $r^2-7r+10=0$

$\Rightarrow r^2-5r-2r+10 \quad \Rightarrow \quad r(r-5) -2(r-5)=0$

$\Rightarrow (r-5)(r-2) = 0 \quad \Rightarrow \quad r=5, r=2$

∴ Homogeneous Solution is $a_n^{(h)} = c_1 2^n + c_2 5^n$ where c_1 and c_2 are constants.

2) Let $a_n^{(p)} = d_0+d_1n$ be the particular solution as the R.H.S is of the form c_0+c_1n

$\Rightarrow (d_0+d_1n)-7(d_0+d_1(n-1)) +10(d_0+d_1(n-2)) = 6+8n$

$\Rightarrow (d_0+d_1n)-7(d_0+d_1n-d_1) +10(d_0+d_1n-2d_1) = 6+8n$

$\Rightarrow d_0+d_1n-7d_0-7d_1n+7d_1+10d_0+10d_1n-20d_1 = 6+8n$

$\Rightarrow (d_0-7d_0+10d_0) + (d_1n-7d_1n+7d_1+10d_1n-20d_1) = 6+8n$

$\Rightarrow (d_0-7d_0+10d_0) +d_1(n-7n+10n+7-20) = 6+8n$

$\Rightarrow 4d_0+ d_1(4n-13) = 6+8n$

$\Rightarrow (4d_0-13d_1) + d_1 4n = 6+8n \quad \text{------ (1)}$

Equating coefficients on both sides of equation (1)

$(4d_0-13d_1) = 6 \quad \text{------ (2)}$

$d_1 4n = 8n \quad \text{------ (3)}$

Calculating the value of d_1 from equation (3)

$d_1 4\cancel{n} = 8\cancel{n} \quad \Rightarrow \quad 4d_1=8 \quad \Rightarrow \quad d_1=8/4 \quad \Rightarrow \quad d_1=2$

Substituting the value of d_1 in equation (3)

$4d_0-13d_1=6 \quad \Rightarrow \quad 4d_0-13(2) = 6 \quad \Rightarrow \quad 4d_0 = 6+26$

$\Rightarrow 4d0 = 32 \quad \Rightarrow \quad d_0=32/4 \quad \Rightarrow \quad d_0=8$

∴$a_n^{(p)} = 8+2n$

General solution is $a_n = a_n^{(h)} + a_n^{(p)}$

$\Rightarrow c_1 2^n + c_2 5^n + 8+2n$

If $a_0 = 1 \Rightarrow n=0$

$c_1 2^0 + c_2 5^0 + 8+2(0) = 1 \Rightarrow \quad c_1+c_2+8=1 \quad \Rightarrow \quad c_1+c_2 = 1-8 \quad \Rightarrow \quad c_1+c_2 = -7 \text{ ----- (4)}$

If $a_1 = 2 \Rightarrow n=1$

$c_1 2^1 + c_2 5^1 + 8+2(1) = 1 \Rightarrow \quad c_1 2+c_2 5+8+2=2 \quad \Rightarrow 2c_1+5c_2 = 2-10 \quad \Rightarrow \quad 2c_1+5c_2 = -8 \text{ ----- (5)}$

By solving (4) and (5) we will get the values of c_1 and c_2

$c_2 = 2$, $c1=-9$

$\therefore$ The required solution is $(-9)2^n+2.5^n+8+2n$

Eg-5: Solve $a_n = 4a_{n-1} - 4a_{n-2} + 3n + 2^n$, initial conditions $a_0 = 1$ and $a_1 = 1$.

Sol:- *The given recurrence relation $a_n - 4a_{n-1} + 4a_{n-2} - 3n - 2^n$ is a second order Inhomogeneous recurrence relation*

1) Associated homogeneous recurrence relation is $a_n - 4a_{n-1} + 4a_{n-2} = 0$

Characteristic equation is $r^2-4r+4=0$

$\Rightarrow (r-2)^2=0 \quad \Rightarrow \quad r=2, 2$

$\therefore$ Homogeneous solution is $a_n^{(h)} = c_1 2^n + c_2.n.2^n$ where c_1 and c_2 are constants.

2) R.H.S = $3n+2^n$

$\therefore$ Particular solution $a_n^{(p)} = a_n^{(p1)} + a_n^{(p2)}$ where p1 for 3n and p2 for 2^n

$a_n^{(p1)} = d_0+d_1n$ [3n can be written as 3n+0 which is of the form c_0+c_1n]

$\Rightarrow (d_0+d_1n) -4[d_0+d_1(n-1)] + 4[d_0+d_1(n-2)] = 3n$

$\Rightarrow (d_0+d_1n) -4(d_0+d_1n-d_1) +4(d_0+d_1n-2d_1) = 3n$

$\Rightarrow (d_0+d_1n) - (4d_0+4d_1n-4d_1) + (4d_0+4d_1n-8d_1) = 3n$

$\Rightarrow (d_0\cancel{-4d_0+4d_0}) + d_1(n\cancel{-4n}+4+\cancel{4n}-8) = 3n$

$\Rightarrow d_0 + d_1(n-4) = 3n \qquad \Rightarrow d_0 + d_1.n - 4d_1 = 3n$

$\Rightarrow (d_0 - 4d_1) + d_1.n = 3n$ ----- (1)

Equating coefficients of equation (1) on both sides

$d_1.n = 3n$ ----- (2)

$(d_0 - 4d_1) = 0$ ----- (3)

From equation (2) $d_1.\cancel{n} = 3\cancel{n} \qquad \Rightarrow d_1 = 3$

Substituting the value of d_1 in equation (3)

$\Rightarrow (d_0 - 4d_1) = 0 \qquad \Rightarrow (d_0 - 4(3)) = 0 \qquad \Rightarrow d_0 - 12 = 0 \quad \Rightarrow d_0 = 12$

$\therefore a_n^{(p1)} = 12+3n$

$a_n^{(p2)} = d.n^2.2^n$[since R.H.S is 2^n and characteristic roots are (2,2)]

$\Rightarrow dn^2 2^n - 4d(n-1)^2 2^{n-1} + 4d(n-2)^2 2^{n-2} = 2^n$

$\Rightarrow dn^2 2^n - 4d(n^2+1-2n) 2^{n-1} + 4d(n^2+4-4n) 2^{n-2} = 2^n$

$\Rightarrow 2^n [dn^2 - 4/2d(n^2+1-2n) + 4/4d(n^2+4-4n)] = 2^n$

$\Rightarrow \cancel{2^n}[dn^2 - 2d(n^2+1-2n) + d(n^2+4-4n)] = \cancel{2^n}$

$\Rightarrow [dn^2 - 2d(n^2+1-2n) + d(n^2+4-4n)] = 1$

$\Rightarrow dn^2 - 2dn^2 - 2d + 4nd + dn^2 + 4d - 4nd = 1$

$\Rightarrow \cancel{2dn^2} - \cancel{2dn^2} + \cancel{4nd} \cancel{-4nd} - 2d + 4d = 1$

$\Rightarrow 2d = 1 \qquad \Rightarrow d = 1/2$

$\therefore a_n^{(p2)} = ½ n^2.2^n \qquad = n^2.2^{n-1}$

$\therefore a_n^{(p)} = a_n^{(p1)} + a_n^{(p2)} \Rightarrow 12+3n+n^2.2^{n-1}$

General solution is $a_n = a_n^{(h)} + a_n^{(p)}$

$\Rightarrow a_n = a_n^{(h)} + a_n^{(p)} \qquad \Rightarrow (c_1+c_2n)2^n+12+3n+n^2.2^{n-1}$ ----- (4)

If $a_0 = 1 \Rightarrow$ n=0 [substitute the values in equation (4)]

$\Rightarrow (c_1+c_2.0)2^0+12+3(0) + 0^2.2^{0-1} = 1 \qquad \Rightarrow (c_1).1+12 = 1$

$\Rightarrow c_1 = 12-1 \qquad \Rightarrow c_1 = -11$

If $a_1 = 1 \Rightarrow$ n=1 [substitute the values of n, a_1 and c_1 in equation (4) to get the value of c_2]

$\Rightarrow (c_1+c_2.1)2^1+12+3(1) + 1^2.2^{1-1} = 1$

$\Rightarrow (c_1+c_2)2+12+3+1.2^0 = 1 \qquad \Rightarrow 2(c_1+c_2) + 15 + 1 = 1$

$\Rightarrow 2(c_1+c_2) + 16 = 1 \qquad \Rightarrow 2c_1 + 2c_2 = 1-16$

$\Rightarrow 2c_1 + 2c_2 = -15 \qquad \Rightarrow 2(-11) + 2c_2 = -15 \qquad \Rightarrow -22 + 2c_2 = -15$

$\Rightarrow 2\, c_2 = -15+22$ $\Rightarrow 2\, c_2 = 7$ $\Rightarrow c2= 7/2$

$\therefore$ The required solution is $(-11 + (7/2)\, n)\, 2^{n+}12+3n+ n^2.2^{n-1}$