Unit - I

Definition of Signal :- A Signal is defined as a Single valued function of one or more independent Variables which Contains some information.

x-axis

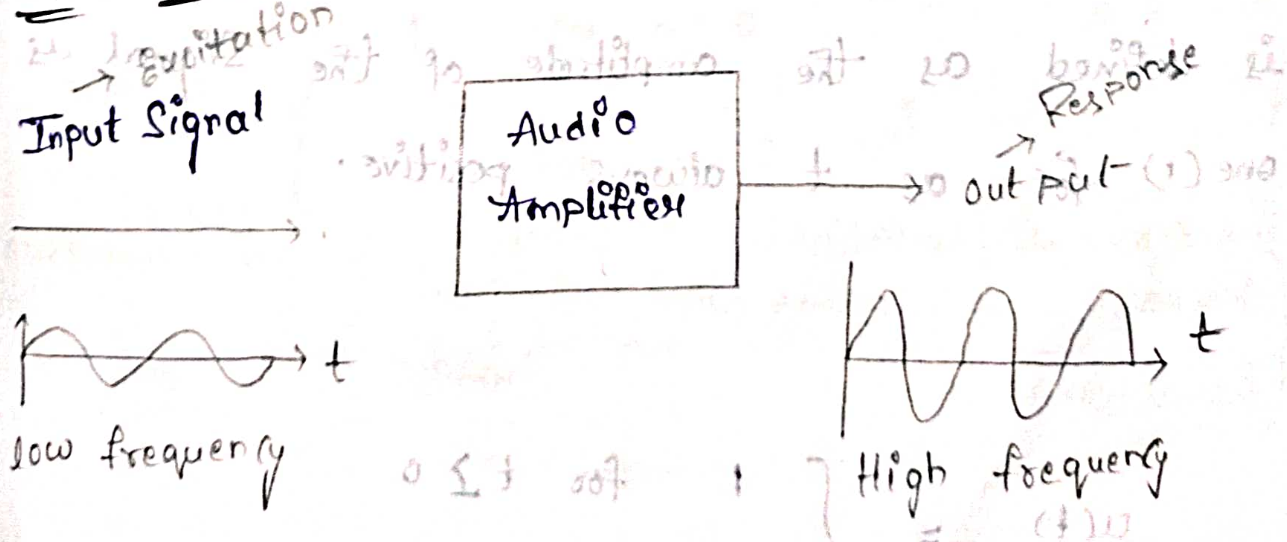
Time, frequency independent Variables

Examples :- human Speech, voltage, Current, .. etc.

Definition of System :- A System is a set of Elements of functional blocks that are Connected together and produce an output in response to an input Signal.

Ex :- Amplifiers, oscillator, attenuator etc.

Simple block diagram of Signal and System :-



Classification of Signals (Elementary Signals) :-

These Elementary are also Called as Standard Signals. These are i) unit step function.

(or)

Step Signal.

Amplitude should be 1

that should be unit

ii) unit Ramp function.

iii) unit parabolic function.

iv) unit impulse function.

v) Sinusoidal function.

vi) Real Exponential function. ($e^{\alpha t}$)

vii) Complex Exponential function. ($e^{(\alpha + j\omega)}$)

unit step function :- The Continuous time unit Step function is Expressed as $u(t)$ and it is defined as the amplitude of the Signal is one (1) for as t always positive.

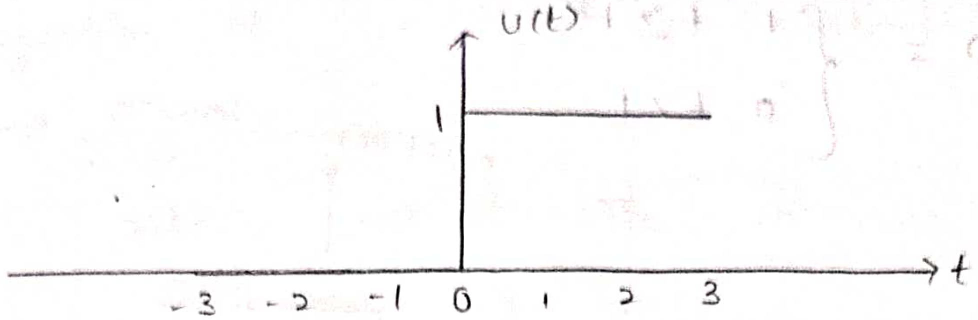
step signal

means $u(t) \rightarrow$ amplitude

Zero is undefine in U.S.F

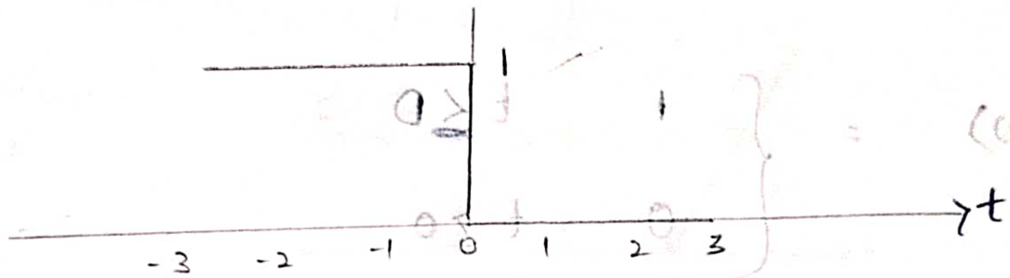
Response is nothing but output.
Excitation " " " input.

$$u(t) = \begin{cases} 1 & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases}$$



positive values of time then amplitude is 1.

u(-t)



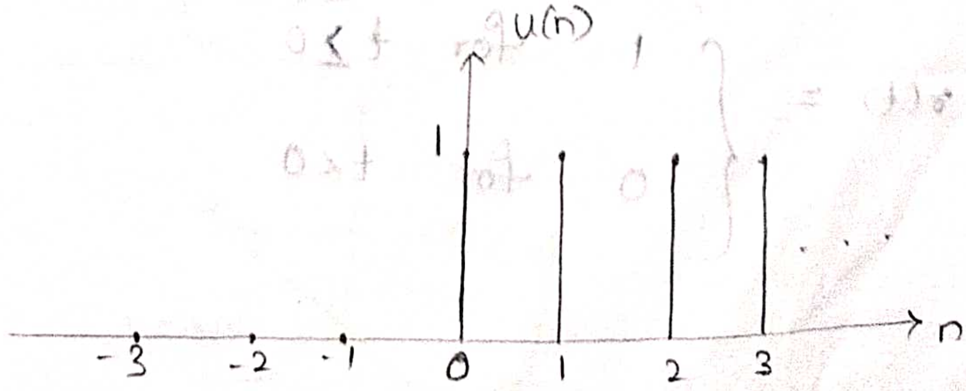
positive values of time then amplitude is 0.

negative values of time then amplitude is 1.

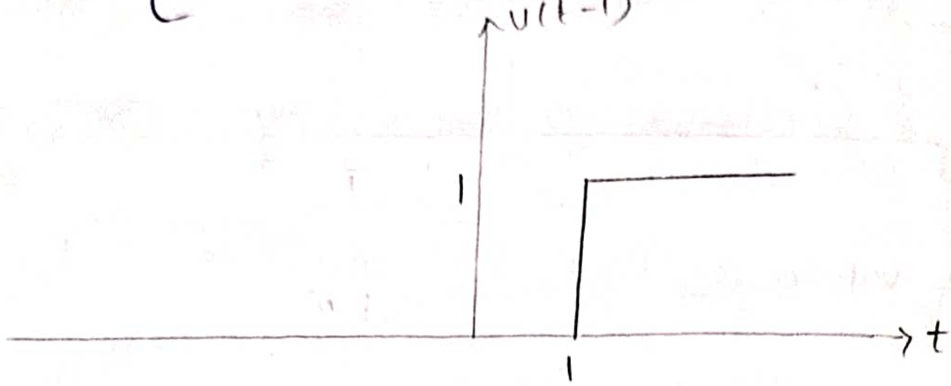
$$u(-t) = \begin{cases} 0 & \text{for } t > 0 \\ 1 & \text{for } t \leq 0 \end{cases}$$

Discret unit step signal :- It is defined as

$$u(n) = \begin{cases} 1 & \text{for } n \geq 0 \\ 0 & \text{for } n < 0 \end{cases}$$

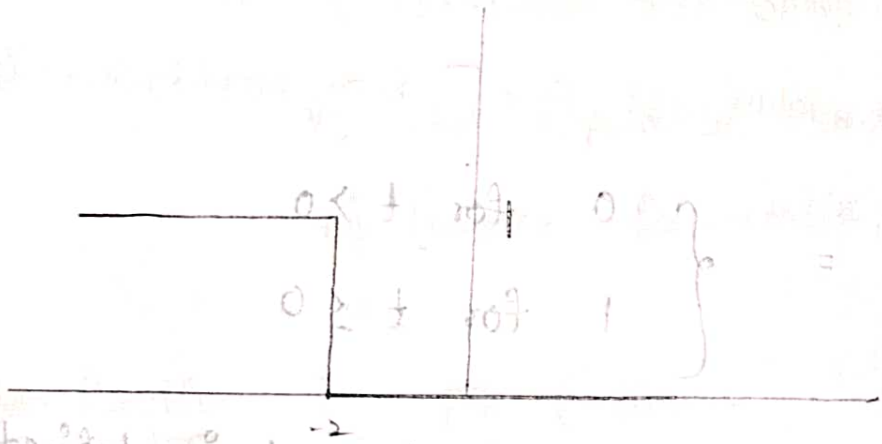


$$u(t-1) = \begin{cases} 1 & t \geq 1 \\ 0 & t < 1 \end{cases}$$



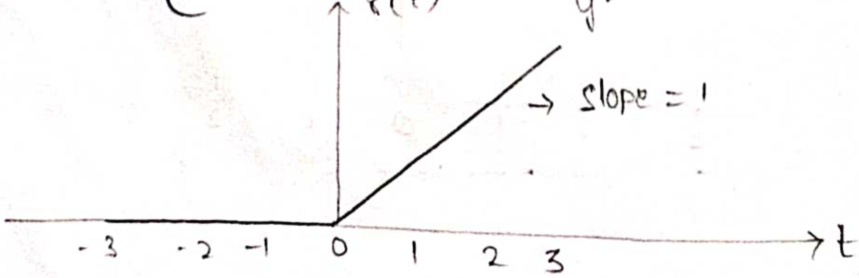
$$u(t-2) = \begin{cases} 1 & t \leq 0 \; -2 \\ 0 & t \geq 0 \end{cases}$$

internal quantity
$-t-2=0$
$t=-2$



Unit Ramp function :- the Continuous time unit Ramp function r(t) Starts at $t=0$ and increase linearly with respect to time.

$$r(t) = \begin{cases} 1 & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases}$$



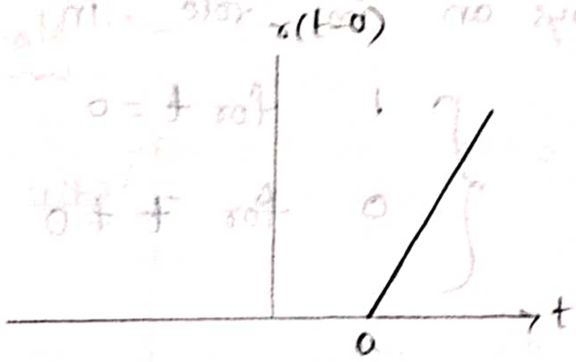
Integration of Step Signal is Equal to Ramp Signal.

$$\boxed{r(t) = \int_0^\infty u(t)\, dt}$$

$r(t-a) \rightarrow$

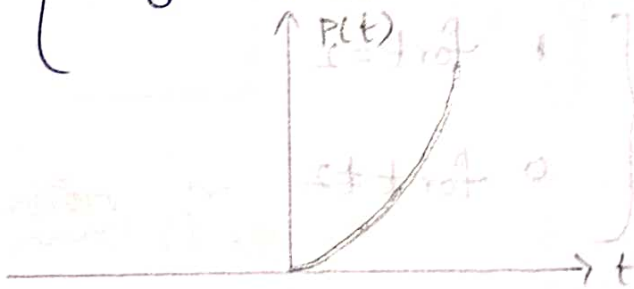
$t-a=0$

$t=a$

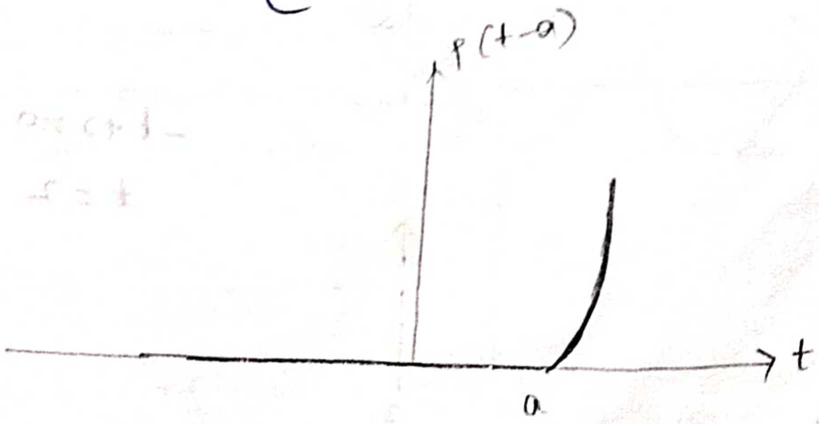


unit parabolic function :-

$$p(t) = \begin{cases} \frac{t^2}{2} & t \geq 0 \\ 0 & t < 0 \end{cases}$$



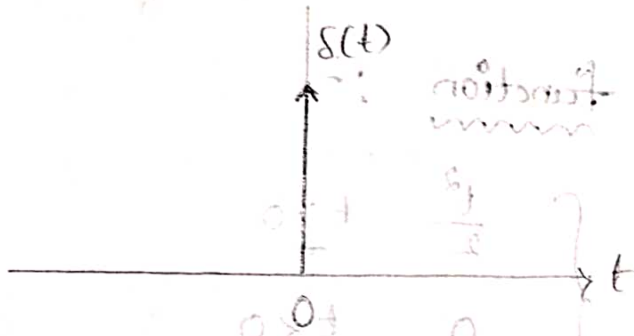
$$p(t-a) = \begin{cases} \frac{(t-a)^2}{2} & t \geq a \\ 0 & t < a \end{cases}$$



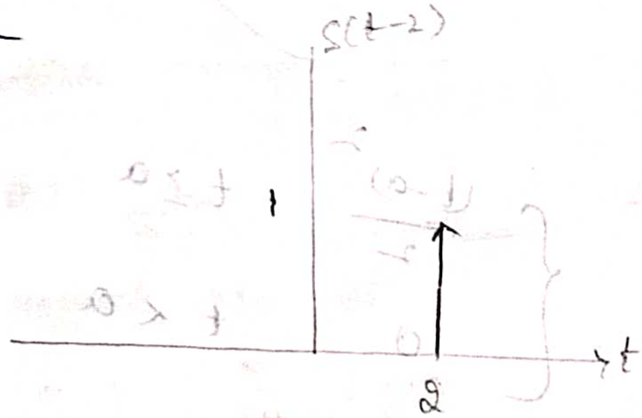
unit impulse function :-

This function is widely in the analysis of signal and system. The continuous time unit impulse function $\delta(T)$ (Delta) is also called as dirac Delta function and it plays an imp role in Signal analysis.

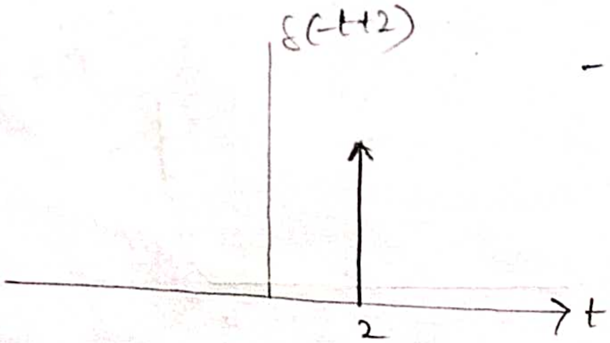
$$\delta(T) = \begin{cases} 1 & \text{for } t = 0 \\ 0 & \text{for } t \neq 0 \end{cases}$$



$$\delta(t-2) = \begin{cases} 1 & \text{for } t = 2 \\ 0 & \text{for } t \neq 2 \end{cases}$$



$\delta(-t+2)$



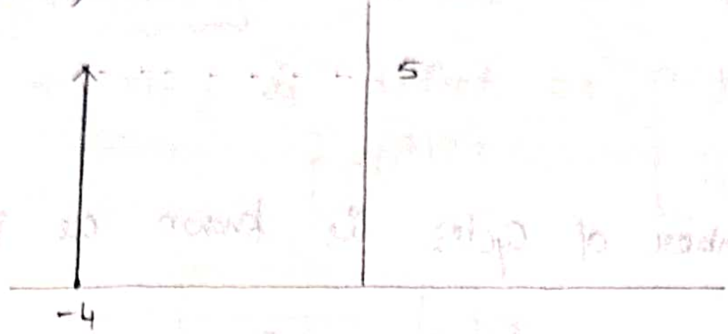
$-t+2=0$

$t=2$

$5(\delta(-t-4))$ $\quad 5\,\delta(-t-4)$

$-t-4=0$

$t=-4$

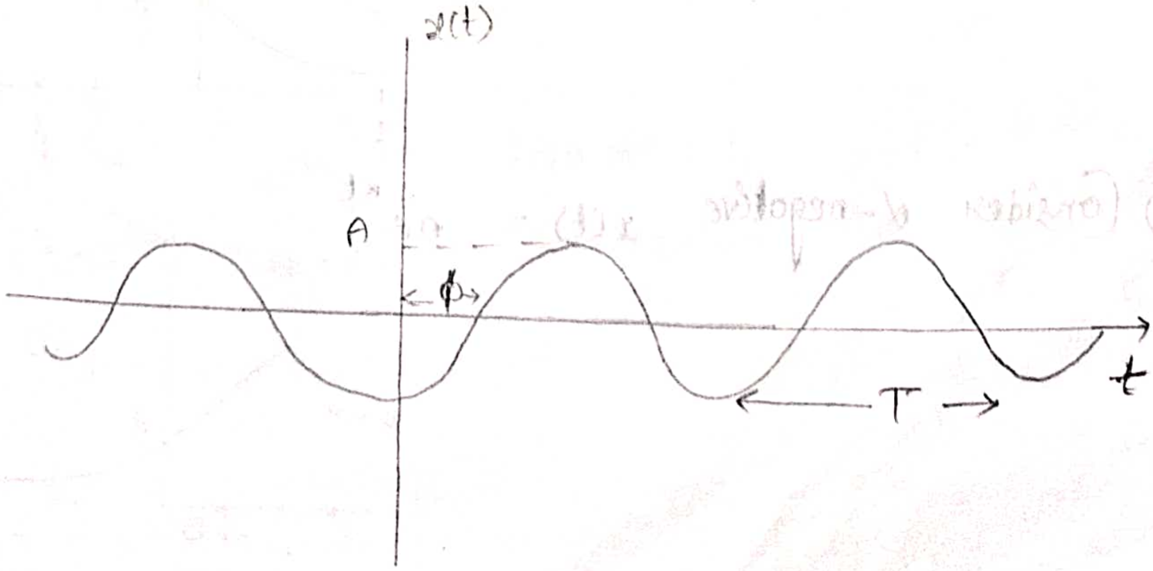


Relation b/w Signals :-

$$\boxed{\begin{aligned} \int_0^{\infty} \delta(t)\,dt &= u(t) \\ \int_0^{\infty} u(t)\,dt &= r(t) \\ \int_0^{\infty} r(t)\,dt &= p(t) \end{aligned}}$$

Sinusoidal function :-

$$x(t) = A \sin(\omega t + \phi)$$



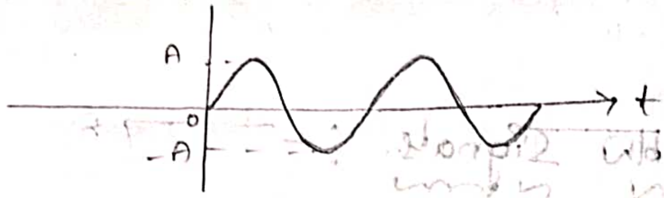
$W = 2\pi f$

$W = 2\pi \frac{1}{T}$

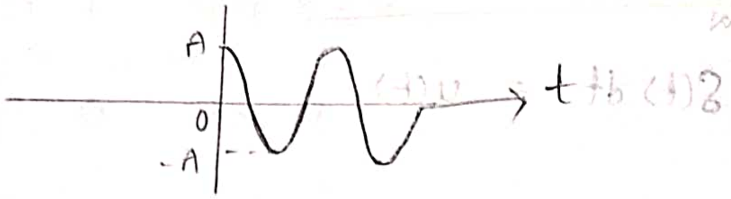
$T = \frac{2\pi}{W}$

The Repeated number of cycles is known as period.

$A \sin wt \rightarrow$



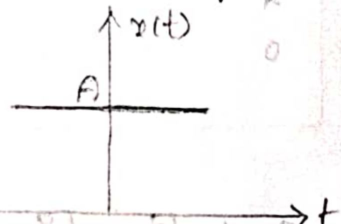
$A \cos wt \rightarrow$



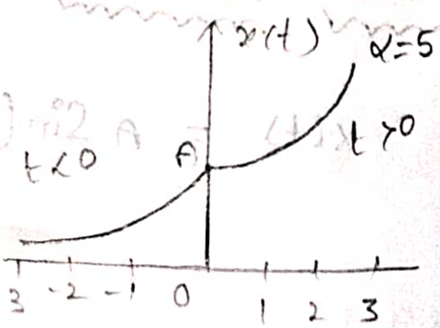
Real Exponential Signal :- A Continuous time Real Exponential Signal is defined as $x(t) = Ae^{\alpha t}$

A, α Real quantities

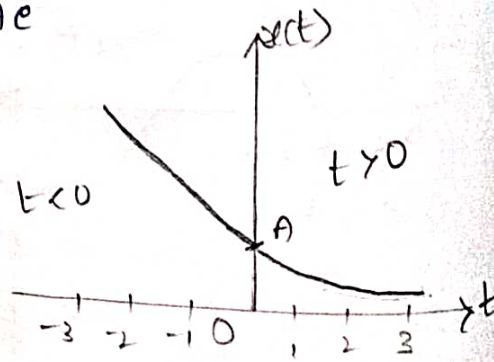
i) Consider $\alpha = 0$ $\therefore x(t) = A$



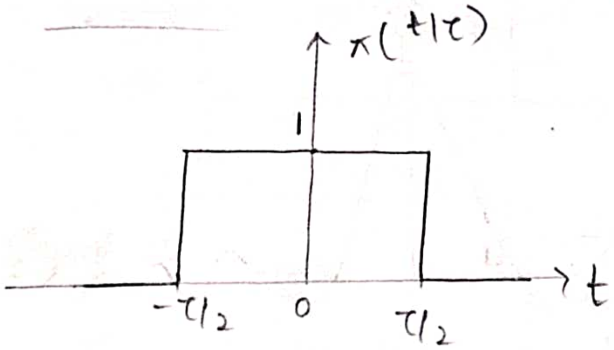
ii) Consider α-positive $x(t) = Ae^{\alpha t}$



iii) Consider α-negative $x(t) = Ae^{-\alpha t}$



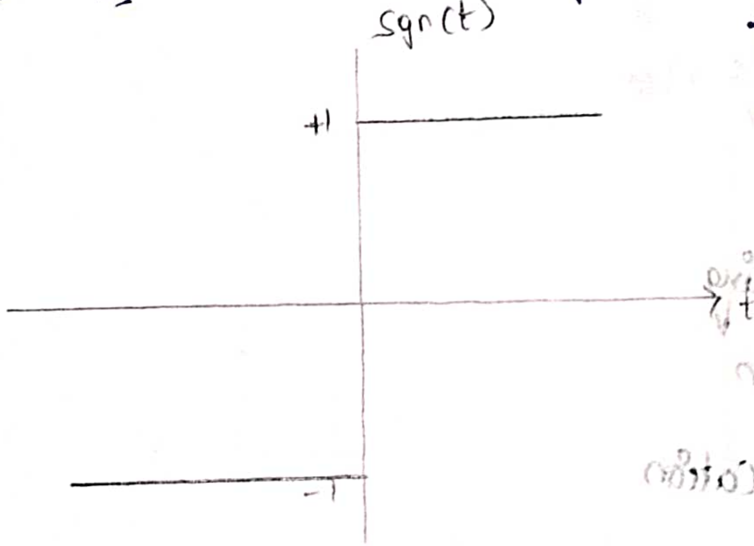
Rectangular function :- The unit Rectangular function $\pi(t/\tau)$ is defined as $\begin{cases} 1 & \text{for } |t| \le \tau/2 \\ 0 & \text{otherwise} \end{cases}$



$t \le \tau/2$

$-t \ge \tau/2$

Signum function :- The unit Signam function $Sgn(t)$ is defined as $Sgn(t) = \begin{cases} 1 & \text{for } t > 0 \\ -1 & \text{for } t < 0 \end{cases}$



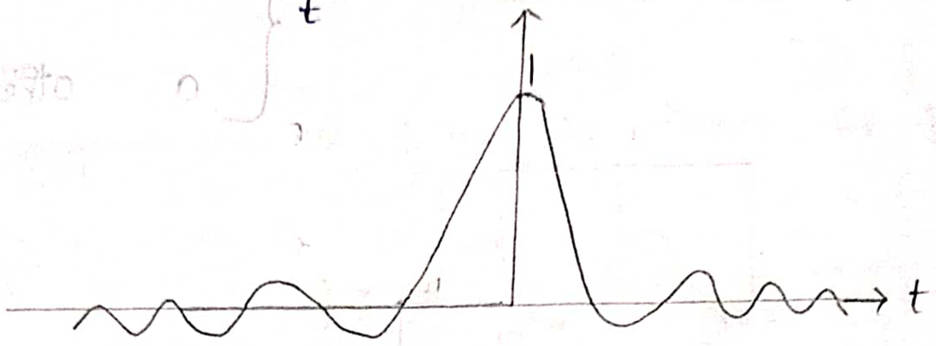
Relation b/w step Signal and Signum function :-

$$Sign(t) = -1 + 2u(t)$$

$t > 0 \quad u(t) = 1 \quad Sign(t) = 1$

$t < 0 \quad u(t) = 0 \quad Sign(t) = -1$

Sinc function :- The Sinc function is defined as $Sinc(t) = \frac{\sin t}{t}$ $-\infty < t < \infty$



Basic Operations on Signals :-

Time shifting

Time reversal

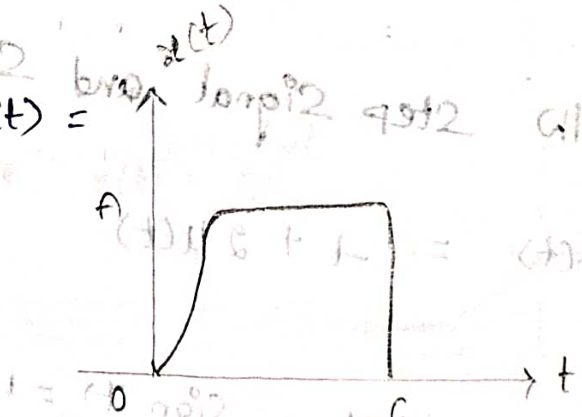
Time scaling

Amplitude scaling

Signal Addition

Signal multiplication

Time shifting :- $x(t) =$

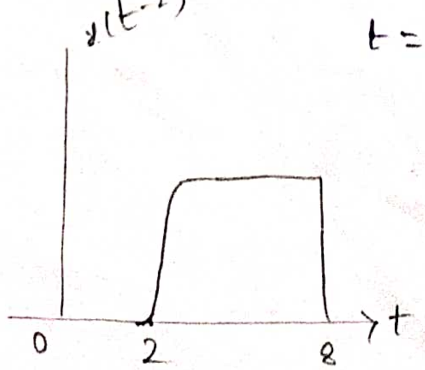


$x(t-2)$

$t-2=0$ $t=2$

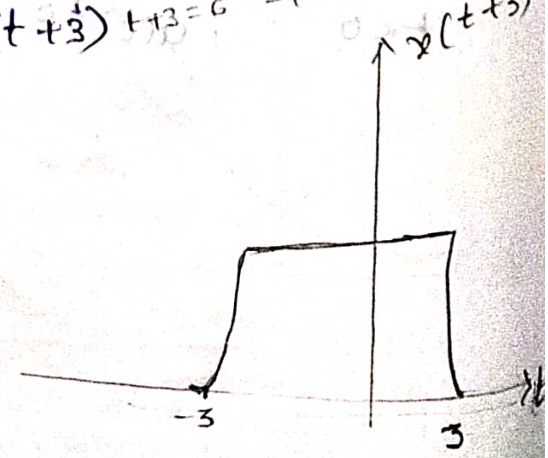
$t-2=6$

$t=8$



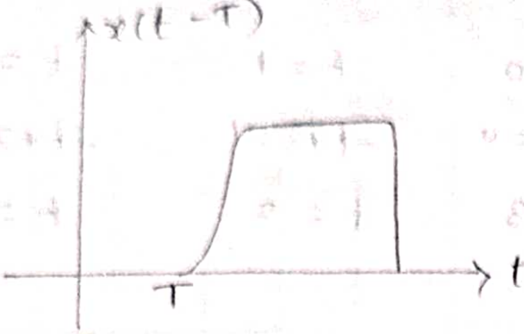
$x(t+3)$ $t+3=0 \Rightarrow t=-3$

$t+3=6 \Rightarrow t=3$



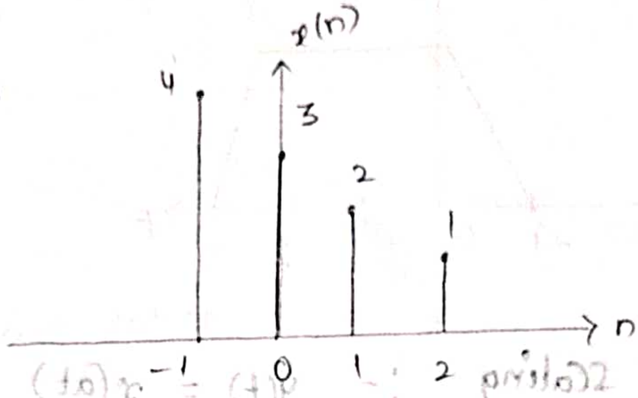
$x(t-T)$

(T.S)

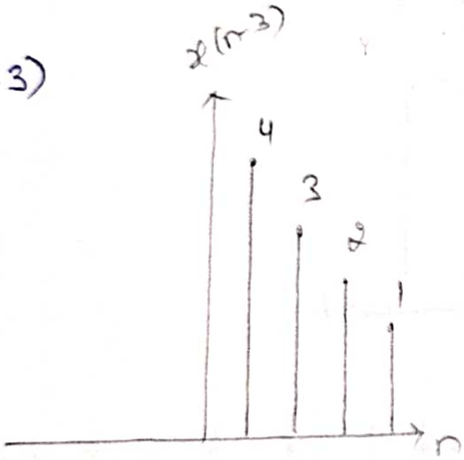


Discret Signal :-

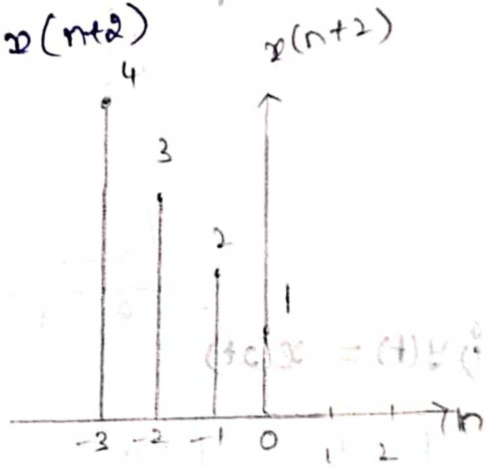
$x(n)$



$x(n-3)$



$x(n+2)$



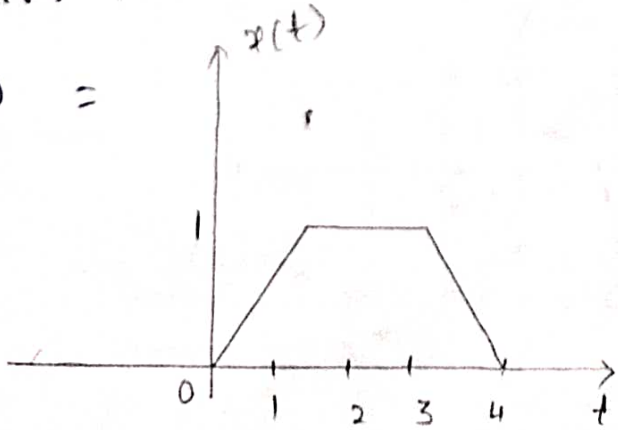
$n=0$	$n=1$	$n=-1$
$n-3=0$	$n-3=1$	$n-3=-1$
$n=3$	$n=4$	$n=2$

$n=0$	$n=1$
$n+2=0$	$n+2=1$
$n=-2$	$n=-1$

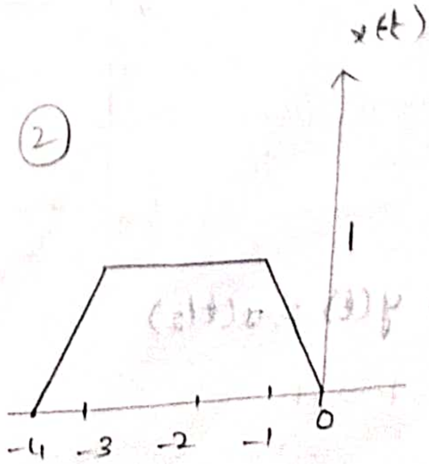
Time Reversal :- $(x(-t))$

$x(t)$ =

①



②



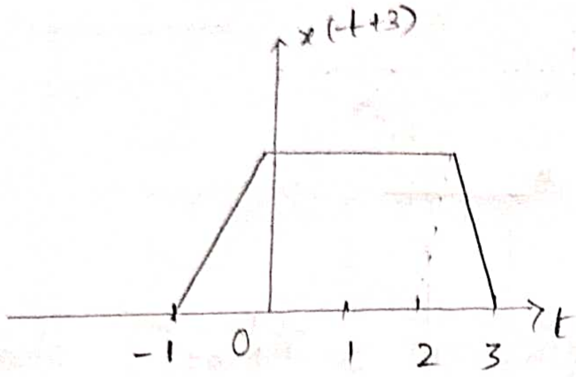
$x(-t+3)$

According to ①

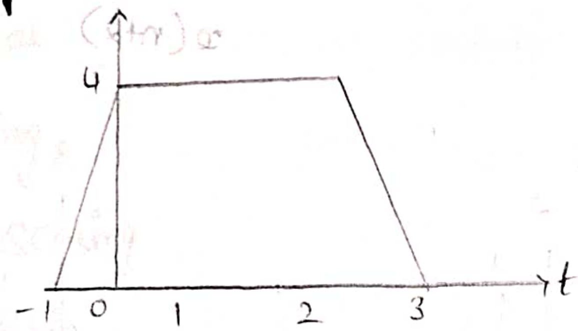
$t=0$ $\quad$ $t=1$ $\quad$ $t=2$

$-t+3=0$ $\quad$ $-t+3=1$ $\quad$ $-t+3=2$

$t=3$ $\quad$ $t=2$ $\quad$ $t=1$



Time scaling :- $y(t) = x(at)$

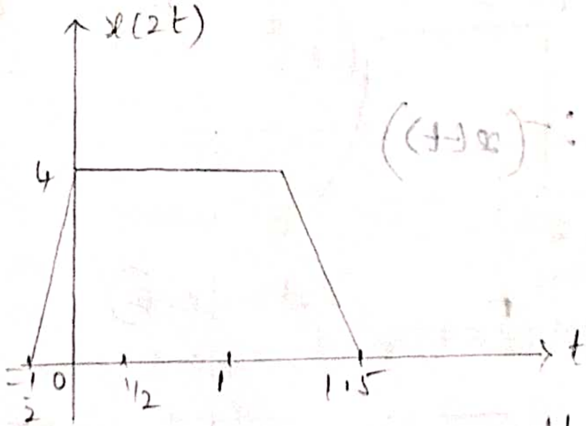


i) $y(t) = x(2t)$

$t=-1$ $\quad$ $t=0$ $\quad$ $t=1$ $\quad$ $t=2$ $\quad$ $t=3$

$2t=-1$ $\quad$ $2t=0$ $\quad$ $2t=1$ $\quad$ $2t=2$ $\quad$ $2t=3$

$t=-\frac{1}{2}$ $\quad$ $t=0$ $\quad$ $t=\frac{1}{2}$ $\quad$ $t=1$ $\quad$ $t=\frac{3}{2}$

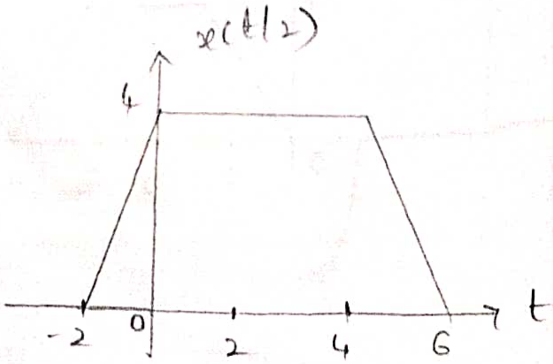


ii) $y(t) = x(t/2)$

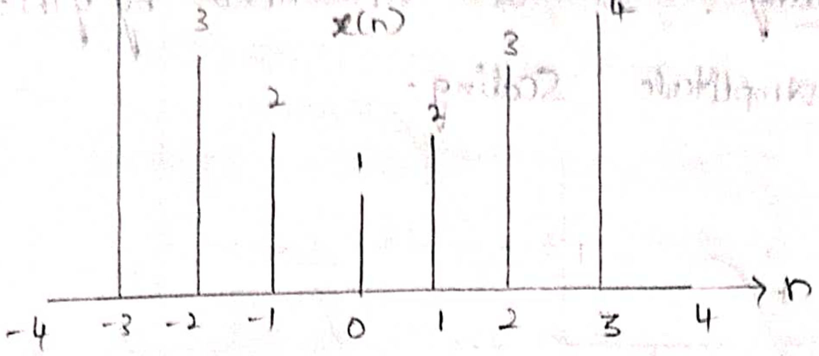
$t=-1$

$\frac{t}{2}=-1$

$t=-2$



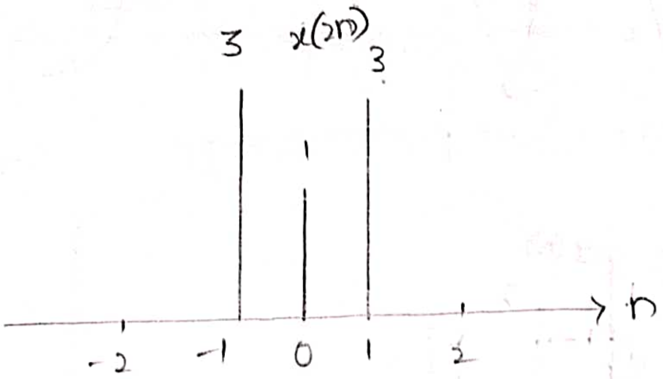
$x(n)$



$x(2n)$

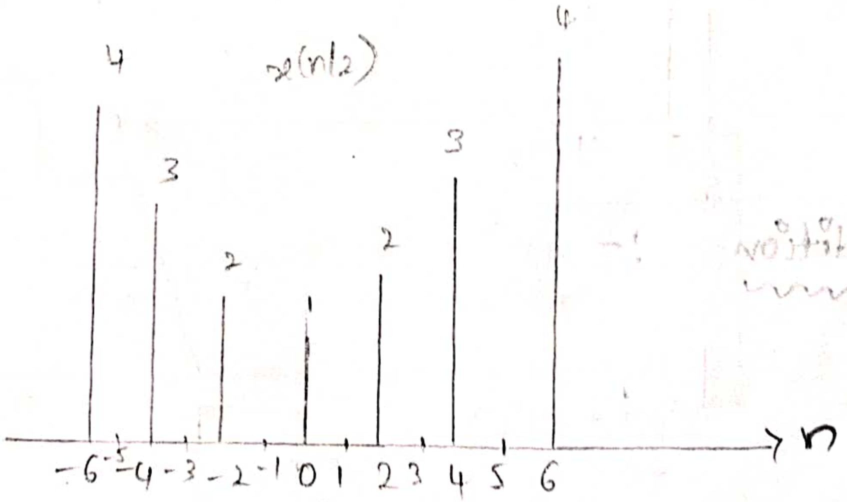
$n=0$	$n=2$
$2n=0$	$2n=2$
$n=0$	$n=1$

It is defined in Discret values

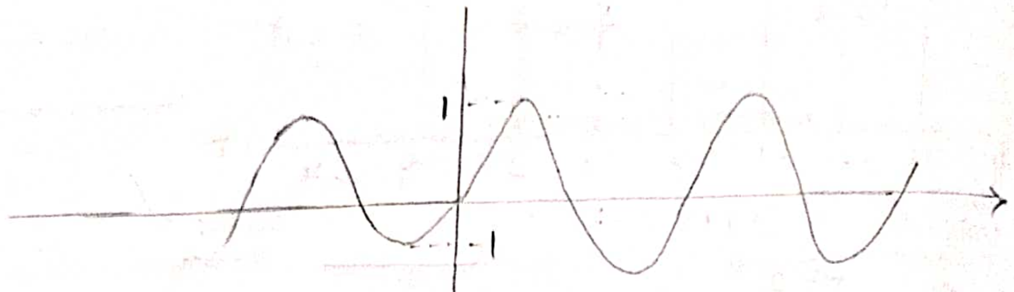


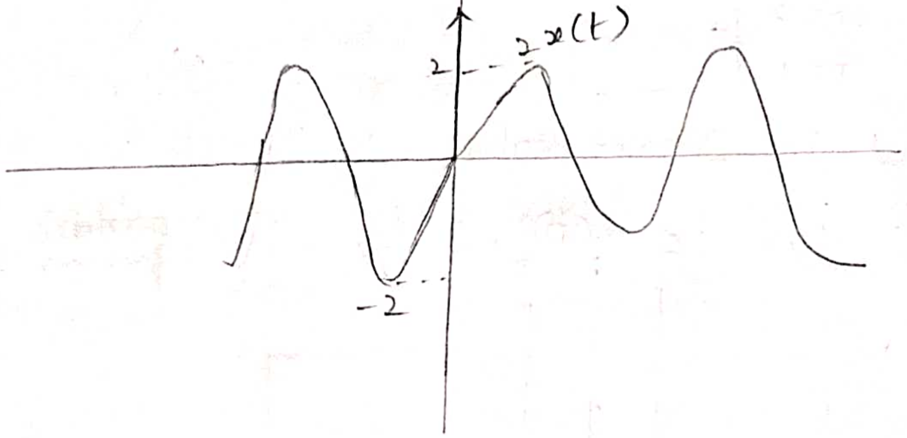
$x(n/2)$

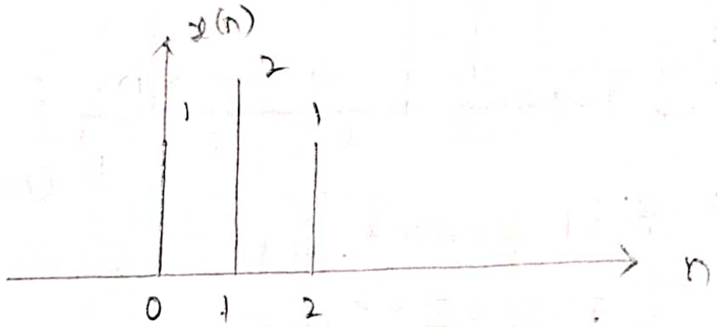
$n=0$	$n=1$	$n=2$
$\frac{n}{2}=0$	$\frac{n}{2}=1$	$\frac{n}{2}=2$
$n=0$	$n=2$	$n=4$

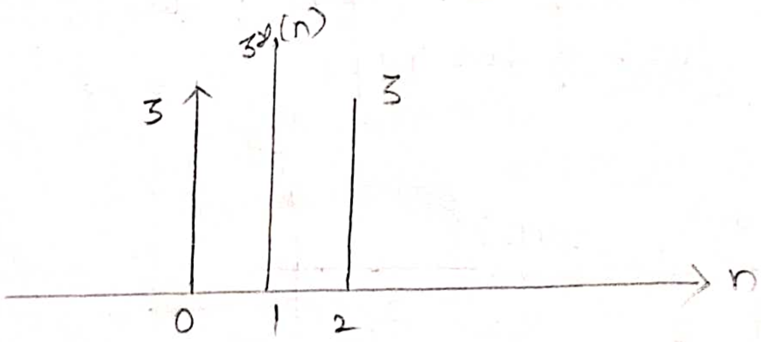


Amplitude Scaling :- It is represented by $y(t) = Ax(t)$ is called Amplitude Scaling.

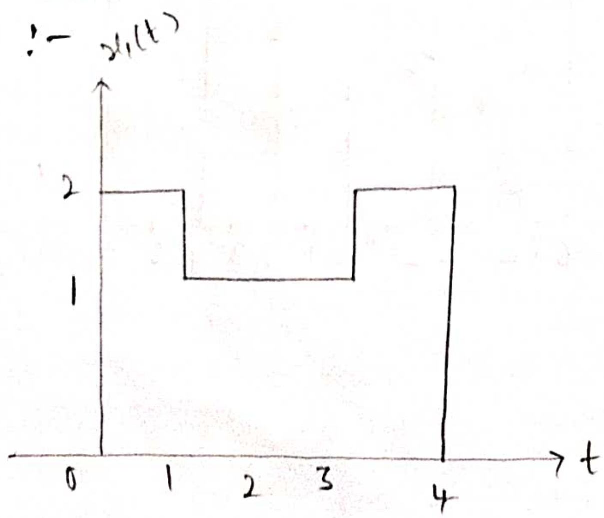


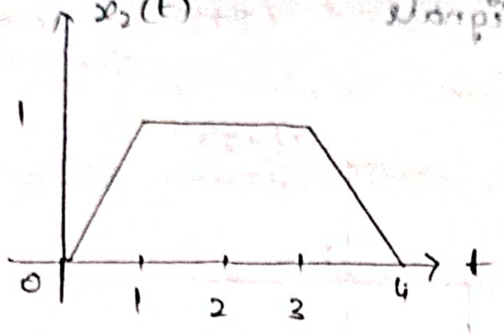






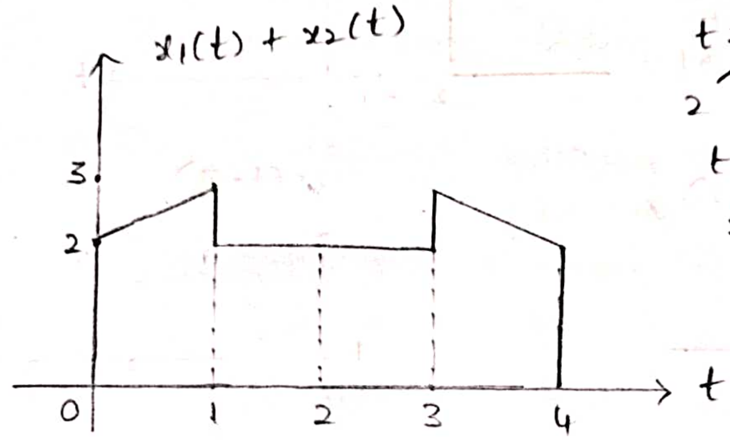
Signal Addition :-





$x_1(t) + x_2(t)$

Addition of Signal



$t = 0$

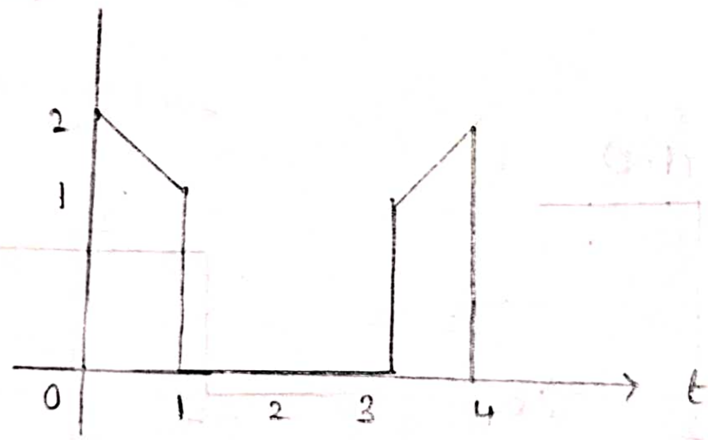
$2 \quad 0 \Rightarrow 2 + 0 = 2$

$t = 1$

$2 \quad 1 \Rightarrow 2 + 1 = 3$

$x_1(t) - x_2(t)$

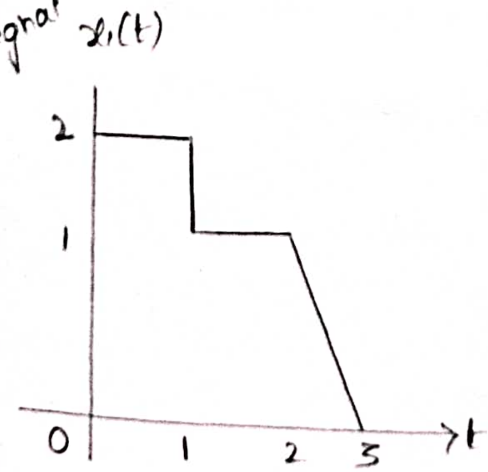
Subtraction of Signal

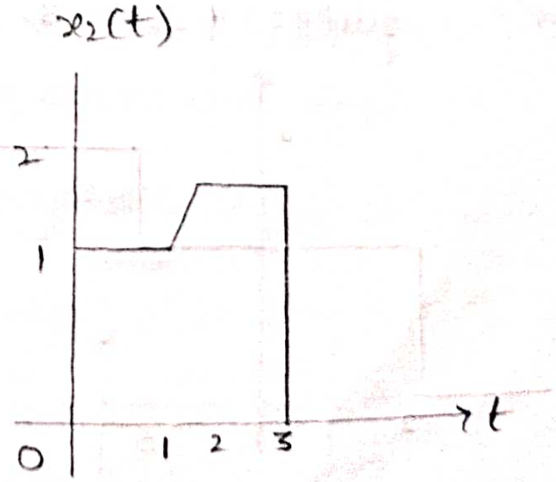


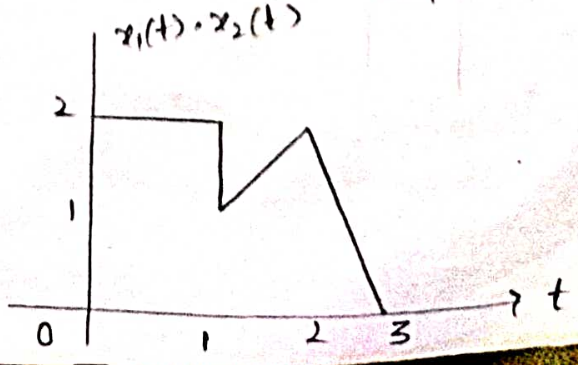
Con – incr = decr

Con – dec = incre

Multiplication of Signal





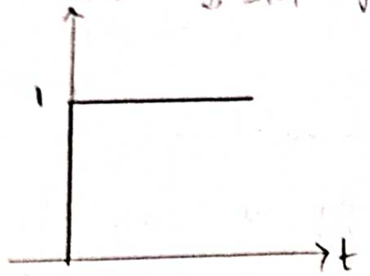


Sketch the following Signals

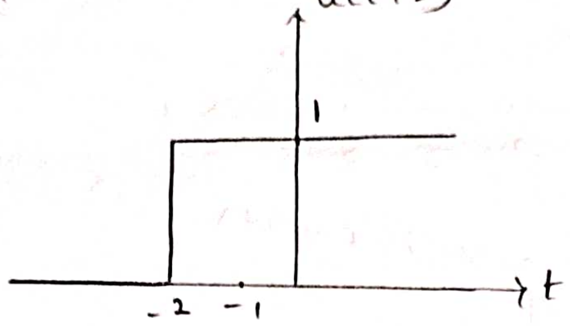
Time shifting

$2u(t+2) - 2u(t-3)$

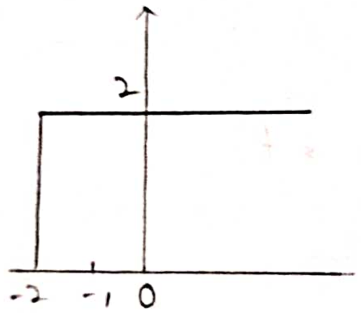
$u(t)$ ↓ Step signal



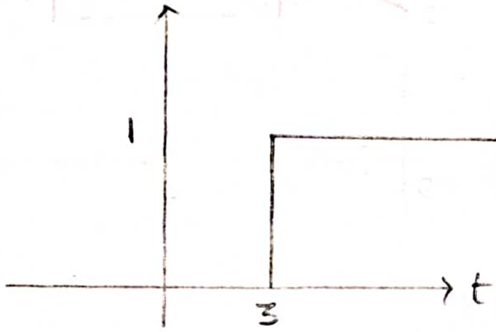
$u(t+2)$



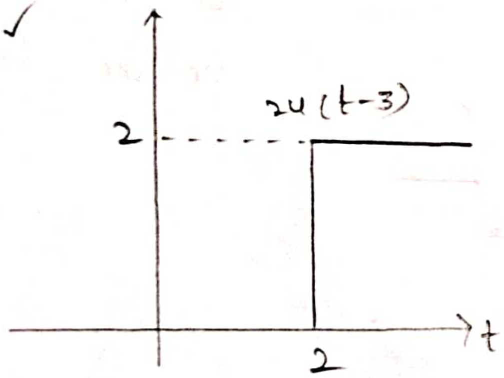
$2u(t+2)$ ↓ Amplitude placing



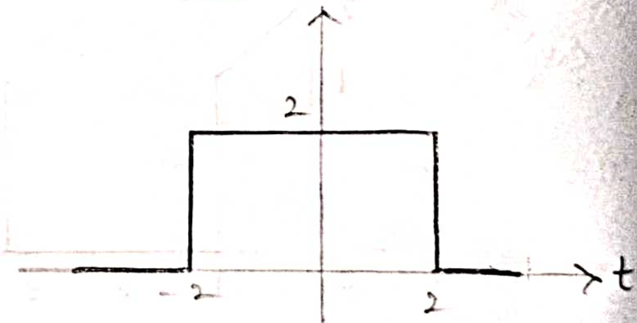
$u(t-3)$



$2u(t-3)$

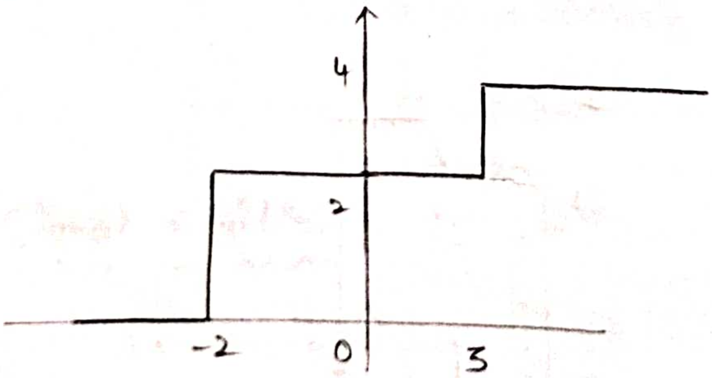


$2u(t+2) - 2u(t-3)$



Addition

$2u(t+2) + 2u(t-3)$



ii) $u(t+4) \times u(-t+4)$

Classification of Signals based on nature and characteristics in time domine Signals are classified as i) Continuous and Discrete Signals

ii) Deterministic and Random Signals

iii) periodic and non periodic Signals

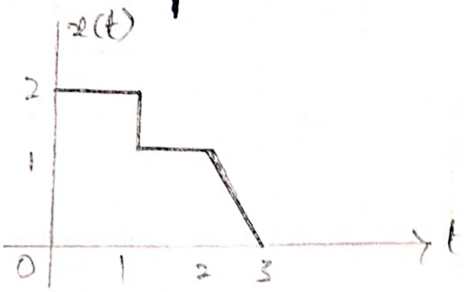
iv) Energy and power Signals.

v) Even & odd signals.

ia)

Continuous Signals :- The Signals are defined for Every instant of time are known as Continuous time Signal.

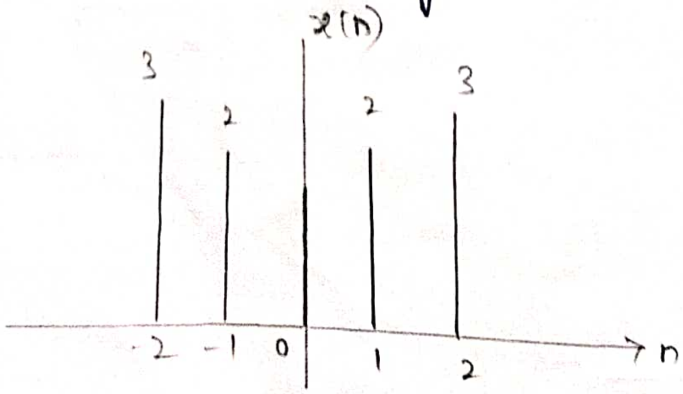
Ex:- It is denoted by $x(t)$



ib)

Descret signals :- The Signals are defined at only descret instant of time. It is denoted by $x(n)$ where 'n' is an integer.

Ex:-

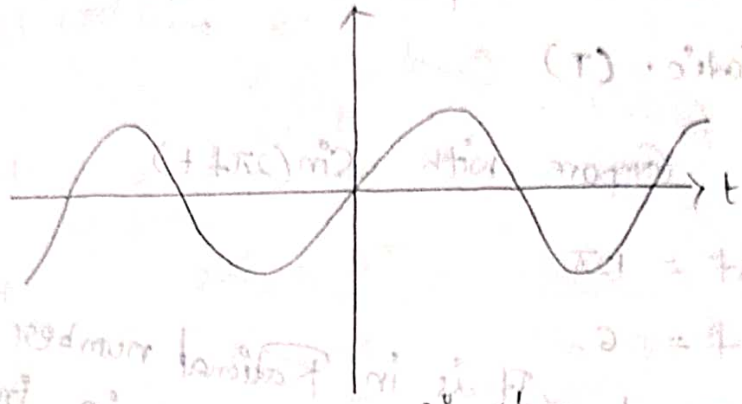


bits

2^{20} → mega

2^{10} → kilo

(ii) Deterministic Signal and Random Signals:



$x(t) = \sin wt$

(iii) periodic and non-periodic Signals:

A Signal which repeats itself at regular interval of time is Called periodic Signal.

A Signal which doesn't repeat at regular intervals of time Called non-periodic or a periodic Signals.

Mathematically, a Continuous time Signal x(t) is Called periodic iff

$$x(t+T) = x(t) \text{ for all } 't' \ (-\infty < t < \infty)$$

where 'T' is Called periodicity (period of x(t)).

reciprocal of 'T' is Called frequency $f = 1/T$.

Angular frequency $(w) = 2\pi f = 2\pi/T \Rightarrow f = \frac{2\pi}{w}$

find whether the following Signals periodic Signals or not. if periodic Determine fundamental periodic. (T)

→ i) $\sin(12\pi t)$ Continuous Signal Compare with $\sin(2\pi f t)$

Here $2\pi f = 12\pi$

$f = 6$

$T = \frac{1}{6}$ It is in Rational number (N & D are in Integers)

So, it is periodic signal.

fundamental period $T = \frac{1}{6}$ Sec

ii) $e^{j4\pi t}$ Compare $e^{j2\pi f t}$

$\Rightarrow 2\pi f t = 4\pi t$

$f = 2$

$T = \frac{1}{2}$ Rational

Periodicity = fundamental periodicity

It is periodic

fundamental period $= \frac{1}{2}$ Sec

iii) $3\sin 200\pi t + 4\cos 100\pi t$

$\sin 2\pi f_1 t + \cos 2\pi f_2 t$

$2\pi f_1 t = 200\pi$ $\quad$ $2\pi f_2 = 100\pi$

$f_1 = 100$ $\quad$ $f_2 = 50$

$T_1 = \frac{1}{100}$ $\quad$ $T_2 = \frac{1}{50}$

find $\frac{T_1}{T_2} = \frac{50}{100} = \frac{1}{2}$ Rational number

periodic Signal.

fundamental period $T_1, T_2 = \frac{1}{100}, \frac{1}{50} \Rightarrow \frac{\text{lcm of N.m}}{\text{GCF of D.m}}$

$\frac{\ }{50}$

iv) $6e^{j(4t+\pi/3)} + 8e^{j(3\pi t+\pi/4)}$

$\Downarrow$

$e^{j2\pi f_1 t} + e^{j2\pi f_2 t}$

$2\pi f_1 = 4$ $\qquad 2\pi f_2 = 3\pi$

$f_1 = \frac{2}{\pi}$ $\qquad f_2 = \frac{3}{2}$

$T_1 = \frac{\pi}{2}$ $\qquad T_2 = \frac{2}{3}$

find $\frac{T_1}{T_2} = \frac{\frac{\pi}{2}}{\frac{2}{3}}$

$= \frac{3\pi}{4}$ (It is not Rational number)

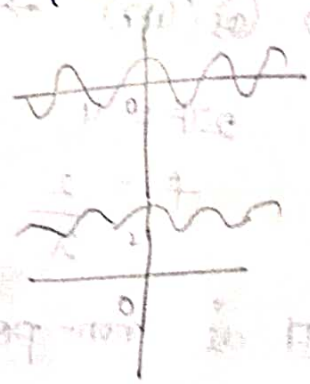
It is non periodic.

v) $2 + \cos(2\pi t)$ DC component (it doesn't influence frequency)

$\downarrow$

Constant so $f=0$, $2\pi f = 2\pi \Rightarrow f = 1 \Rightarrow T = 1$ Rational Num.

It is a periodic signal



* $T_1 \quad T_2$

$\frac{T_1}{T_2}$ and $\frac{T_1}{T_3}$ (for example we got $\frac{3}{2}, \frac{1}{1}$)

Both are Rational, then it is periodic

$T_1, T_2, T_3 = \frac{\text{lcm of Nr } (3,2,1)}{\text{GCF Dm } (3,2,1)}$

3, 2, 1

In Discreat Signals How to find periodicity

Consider Discret time Sequence $x(n)$. the given Sequence is periodic by obtain the value

$\frac{K}{N}$.

if $\frac{K}{N}$ is Rational number then the given Signal is periodic Signal and fundamental period is N

1. $\sin(0.02\pi n)$ Discret signal

Compare with $\sin(2\pi f n)$

$2\pi f = 0.02\pi$

$f = \frac{0.02}{2} = \frac{1}{100} = \frac{K}{N}$ (Rational number)

It is periodic.

fundamental period $N = 100$

ii) $\cos(4n) \rightarrow \cos 2\pi f n$

$2\pi f = 4$

$f = \frac{2}{\pi} = \frac{K}{N}$ It is not Rational

It is non-periodic Signal.

iii) $x(n) = \sin\left(\frac{2\pi n}{3}\right) + \cos\left(\frac{2\pi n}{5}\right)$

$\downarrow \qquad \qquad \downarrow$

$2\pi f_1 n \qquad 2\pi f_2 n$

$2\pi f_1 = \frac{2\pi}{3} \Rightarrow f_1 = \frac{1}{3} = \frac{K}{N_1}$

$2\pi f_2 = \frac{2\pi}{5} \Rightarrow f_2 = \frac{1}{5} = \frac{K}{N_2}$

$N_1 = 3$ $\quad \frac{N_1}{N_2} = \frac{3}{5}$ Rational number

$N_2 = 5$

It is periodic.

fundamental period $N = \underbrace{N_1, N_2}_{\text{LCM of}} = 3, 5 = 15$

(iii) $x(n) = \cos\left(\frac{\pi}{2} + 0.3n\right)$

$2\pi f n \rightarrow \quad 2\pi f = 0.3 = \frac{3}{20\pi}$

$f = \frac{0.3}{2\pi} = \frac{K}{N}$ (not a Rational)

It is non-period.

Energy & power signals :-

$$E = \lim_{T \to \infty} \int_{-T/2}^{T/2} |x(t)|^2 \, dt$$

$$\boxed{E = \int_{-\infty}^{\infty} |x(t)|^2 \, dt}$$

$$P = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 \, dt \rightarrow (1)$$

$$\boxed{= \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{+T} |x(t)|^2 \, dt} \rightarrow (2)$$

$$E = \sum_{n=-\infty}^{\infty} |x(n)|^2$$

Non-periodic

$$\boxed{P = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x(n)|^2}$$

Constant

→ As t increases then Amplitude increases - power

→ As t increases then Amplitude decreases - Energy

→ As t increases then Amplitude increase - Neither energy nor power

→ All Energy Signals are non periodic Signals.

→ All power Signals are periodic Signals.

→ Some signals are neither Energy nor power.

→ Both Energy and power Signals are not Exists

→ A Signal is Said to be Energy Signal only if the total Energy 'E' is always finite. $(0 < E < \infty)$

→ for Energy Signal the average power $= 0$.

→ Non periodics are Examples of Energy signals.

→ A Signal is said to be power Signal if its average power is finite.

→ The Energy of power Signal is always infinity.

Eg:- periodic Signals. periodic.

→ All periodic Signals are power Signals but All P.S are not

1. Determine Average power and Rms power of the given Signal $x(t) = A\sin(\omega_0 t + \theta)$

$$P = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 \, dt$$

$$= \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |A \sin(\omega_0 t + \theta)|^2 \, dt$$

$$= \lim_{T\to\infty} \frac{A^2}{T} \int_{-T/2}^{T/2} \left(\frac{1 - \cos 2(w_0 t + \theta)}{2} \right) dt$$

$$= \frac{A^2}{T} \lim_{T\to\infty} \int_{-T/2}^{T/2} \frac{1}{2} dt - \int_{-T/2}^{T/2} \frac{\cos(w_0 t + \theta)}{2} dt$$

Integration of Cos function over full cycles is always zero.

$$= \frac{A^2}{T} \lim_{T\to\infty} \frac{1}{2} \int_{-T/2}^{T/2} dt - 0$$

$$= \frac{A^2}{T} \frac{1}{2} [t]_{-T/2}^{T/2}$$

$$= \frac{A^2}{2T} \left[\frac{T}{2} + \frac{T}{2} \right] = \frac{A^2}{2}$$

RMS power : $\sqrt{\dfrac{A^2}{2}} = \dfrac{A}{\sqrt{2}}$

find out Energy

$$E = \lim_{T\to\infty} \int_{-T/2}^{T/2} |x(t)|^2 dt$$

$$= \lim_{T\to\infty} \int_{-T/2}^{T/2} |A \sin(w_0 t + \theta)|^2 dt$$

$$= \lim_{T\to\infty} \int_{-T/2}^{T/2} A^2 \left(\frac{1 - \cos 2(w_0 t + \theta)}{2} \right) dt$$

$$= \lim_{T\to\infty} \frac{A^2}{2} \int_{-T/2}^{T/2} dt - 0$$

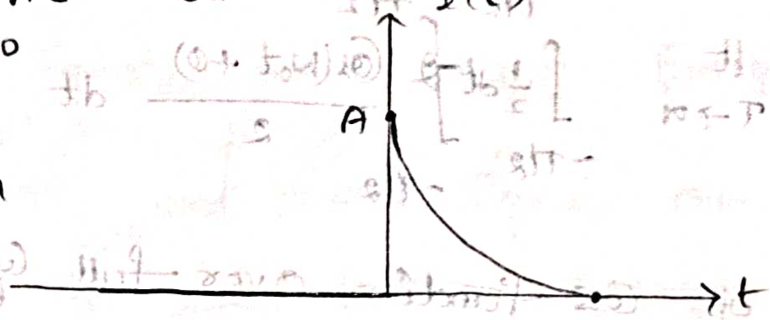
$$= \frac{A^2}{2} (t)_{-T/2}^{T/2} = \frac{A^2}{2} \left[\frac{T}{2} + \frac{T}{2} \right]$$

$$= \infty$$

Determine energy and power of the given signal

$x(t) = A e^{-at} u(t) \quad a > 0 \qquad a = 1$

u(t) - unit step
u(t) - negative - 0
then A·u(t) = 0
u(t) - positive - 1



$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{+0}^{\infty} |A e^{-at}|^2 dt$$

$$= \int_{0}^{\infty} A^2 e^{-2at} dt$$

$$= A^2 \frac{e^{-2at}}{-2a}$$

$$= -\frac{A^2}{2a} \left[e^{-\infty} - e^{-0} \right]$$

$$= -\frac{A^2}{2a} \left[0 - 1 \right] = \frac{A^2}{2a}$$

$$P = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 dt$$

$$= \lim_{T \to \infty} \frac{1}{T} \int_{0}^{T/2} \left(A e^{-2at}\right)^2 dt$$

$$= \lim_{T \to \infty} \frac{A^2}{T} \left[\frac{e^{-2at}}{-2a} \right]_0^{T/2}$$

$$= \lim_{T \to \infty} \frac{A^2}{T} \left[e^{-2aT/2} - 1 \right] \frac{1}{-2a}$$

$$\lim_{T \to \infty} \frac{A^2}{-2aT} \left[e^{-2aT/2} - 1 \right]$$

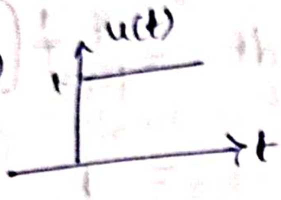
$$= \frac{A^2}{2a \cdot \infty} = 0$$

Find whether the following signals are Energy & power signals and calculate Energy or power.

1) $x(t) = u(t)$

2) $x(t) = t\,u(t)$

3) $x(t) = e^{j(2t + \pi/2)}$

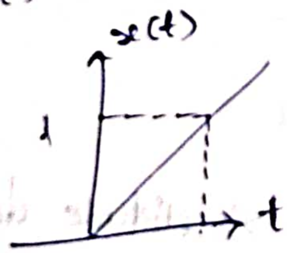
1) 

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_0^{\infty} 1^2 dt = t\Big|_0^{\infty} = \infty$$

$$P = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} |x(t)|^2 . dt = \lim_{T\to\infty} \frac{1}{T}\int_0^{T/2} 1 dt$$

$$= \lim_{T\to\infty} \frac{1}{T}\, t\Big|_0^{T/2} = \frac{T/2 - 0}{T} = \frac{1}{2}$$

2) $x(t) = t\,u(t)$



$x(t) = t$ as $t \geq 0$

$= 0$

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_0^{\infty} |t\,u(t)|^2 dt = \int_0^{\infty} t^2 dt$$

$$= \frac{t^3}{3}\Big]_0^{\infty} = \infty$$

$$P = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} |x(t)|^2 dt = \lim_{T\to\infty} \frac{1}{T}\int_0^{T/2} |t|^2 dt$$

$$= \lim_{T\to\infty} \frac{1}{T}\, \frac{t^3}{3}\Big|_0^{T/2} = \lim_{T\to\infty} \frac{1}{3T}\left[\frac{T^3}{8}\right] = \frac{T^2}{24} = \infty$$

Neither Energy nor power Signal.

5) $x(t) = e^{j(3t+\pi/2)}$

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |e^{j(3t+\pi/2)}|^2 dt = \int_{-\infty}^{\infty} 1\, dt = \infty$$

$e^{j\theta} = \cos\theta + j\sin\theta$

$|e^{j\theta}| = \sqrt{\cos^2\theta + \sin^2\theta} = 1$

$$P = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 dt = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} 1\, dt = \lim_{T\to\infty} \frac{1}{T}(T)$$

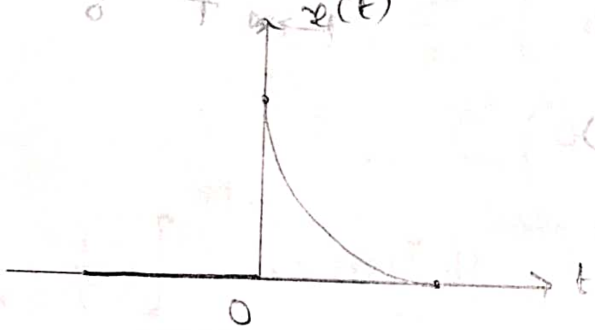
$= 1$

Sketch the following Signals and find Energy or power.

1) $e^{-5t}u(t)$ 2) $u(t) - u(t-4)$ 3) $\sin \omega t . u(t-1) \, u(9-$

4) $u(t) + u(t-2)$ 5) $t^2 u(t)$

1)

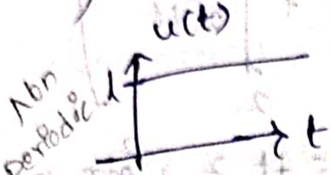


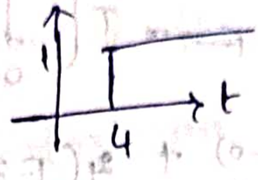
It is non periodic t is increas Amplitude decrease so it comes under Energy Signals. so there power is 0.

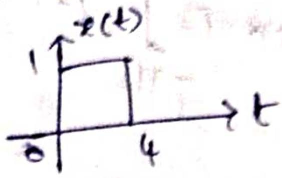
$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_0^{\infty} |e^{-5t}|^2 u(t) = \int_0^{\infty} e^{-10t} dt$$

$$= \left.\frac{e^{-10t}}{-10}\right|_0^{\infty} = -\frac{1}{10}(0-1) = \frac{1}{10}$$

2) $u(t) - u(t-4)$

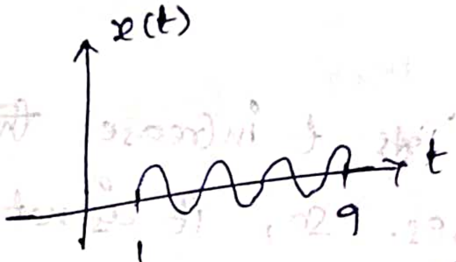






$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_0^4 1^2 dt = 4 \text{ joul.}$$ power = 0

3) $\sin wt \cdot u(t-1)\, u(9-t)$



It is non-periodic. Energy signal

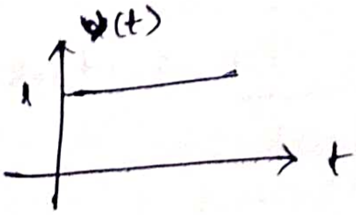
$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_1^9 \sin^2 wt\, dt = \int \frac{1-\cos 2wt}{2} dt$$

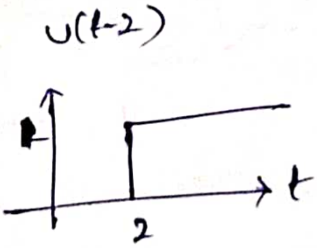
$$= \int_1^9 \frac{1}{2} dt - \frac{1}{2}\int_1^9 \cos 2wt\, dt$$

$$= \frac{1}{2}[t]_1^9 - 0 = \frac{1}{2} \times 8 = 4 \text{ joules.}$$

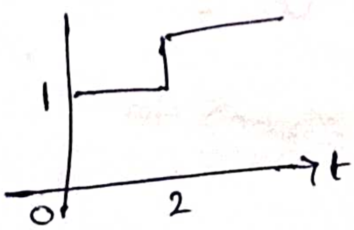
∴ power is 0.

4) $u(t) + u(t-2)$





$u(t) + u(t-2)$



non periodic

It is power signal

$$P = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 dt = \lim_{T\to\infty} \frac{1}{T} \left[\int_0^2 1 dt + \int_2^{T/2} 2^2 dt \right]$$

$$= \frac{1}{T}\left[(2-0) + 4\left(\frac{T}{2}-2\right)\right] = \lim_{T\to\infty} \frac{2}{T} + \frac{2T}{T} - \frac{8}{T}$$

$$= \lim_{T\to\infty} \frac{2}{T} - \frac{8}{T} + 2$$

$$= 2W$$

5) $t^2 u(t)$

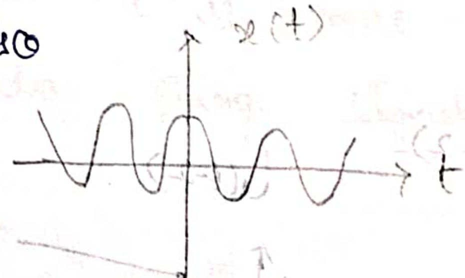
It is non periodic signal. As t increases then here amplitude also increases. So, it is not energy signal and power signal. So energy and power signals are infinity.

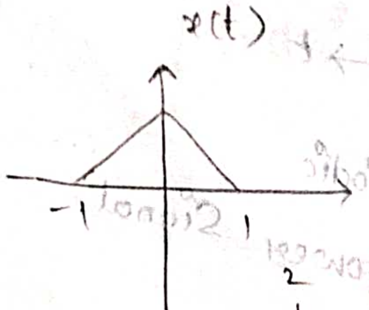
<u>Even & odd Signals</u> :- A continous time signal $x(t)$ is said to be Even (Symmetric) signal If it satisfies the condition $x(t) = x(-t)$ for all the values of 't'

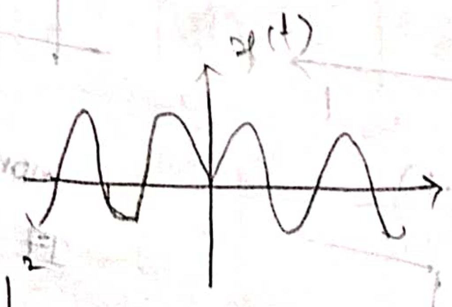
Ex:- $\cos wt$ or $\cos\theta$

$x(t) = x(-t)$

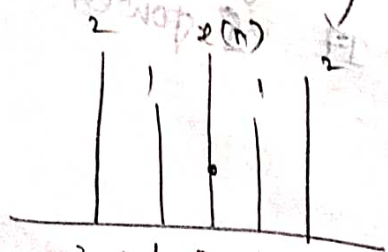
$\cos\theta = \cos(-\theta)$
$= \cos\theta$







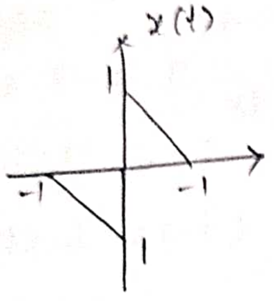
$x(n) = x(-n)$ for 'n'

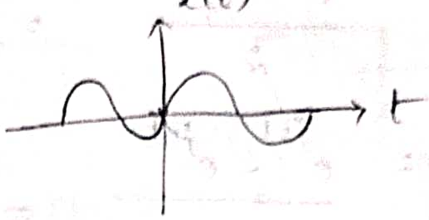


→ The signal $x(t)$ is said to be odd (Anti Symmetric) signal if it satisfies the condition for all values of 't'

$x(t) = -x(-t)$ for all values of t

$\sin\theta = -\sin(-\theta)$





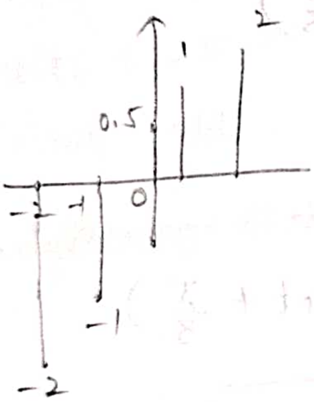
$x(n) = -x(-n)$

n = -1

A = -1

n = -2

A = -2



Evaluation of Even and odd parts of the signals

To find the Even part of the signal

$$x_e(t) = \frac{x(t) + x(-t)}{2}$$

To find the odd part of the signal

$$x_o(t) = \frac{x(t) - x(-t)}{2}$$

$x(t) = x_e(t) + x_o(t)$

even $= \frac{\text{sym}}{2}$

odd $= \frac{\text{Anti sym}}{2}$

find even and odd components of the following Signals.

$\cos\theta + i\sin\theta = e^{i\theta}$

$\cos\theta - i\sin\theta = e^{-i\theta}$

1) $x(t) = e^{j2t}$

$$x_e(t) = \frac{e^{j2t} + e^{j-2t}}{2} = \cos 2t$$

$$x_o(t) = \frac{e^{i2t} - e^{j2t}}{2} = j\sin 2t$$

2) $x(t) = \cos\left(w_0 t + \frac{\pi}{3}\right)$

$$x_e(t) = \frac{\cos\left(w_0 t + \frac{\pi}{3}\right) + \cos\left(-w_0 t + \frac{\pi}{3}\right)}{2}$$

$$= \cos\frac{\pi}{3}\cos w_0 t = \frac{1}{2}\cos w_0 t$$

$$x_o(t) = \frac{\cos\left(w_0 t + \frac{\pi}{3}\right) - \cos\left(-w_0 t + \frac{\pi}{3}\right)}{2}$$

$$= \frac{\cancel{2}}{\cancel{2}} - \sin\frac{\pi}{3}\sin w_0 t = -\frac{\sqrt{3}}{2}\sin w_0 t$$

3) $x(t) = 1 + 2t + 3t^2 + 4t^3$

$$x_e(t) = \frac{1 + \cancel{2t} + 3t^2 + \cancel{4t^3} + 1 - \cancel{2t} + 3t^2 - \cancel{4t^3}}{2}$$

$$= \frac{2 + 6t^2}{2} = 1 + 3t^2$$

$$x_o(t) = \frac{1 + 2t + 3t^2 + 4t^3 - (1 - 2t + 3t^2 - 4t^3)}{2}$$

$$= \frac{\cancel{1} + 2t + \cancel{3t^2} + 4t^3 - \cancel{1} + 2t - \cancel{3t^2} + 4t^3}{2}$$

$$= \frac{4t + 8t^3}{2} = 2t + 4t^3$$

find whether the following signals are even or odd.

1) $x(t) = e^{-3t}$

$x(t) = e^{-3t}$

$x(-t) = e^{3t}$

$x(t) \neq x(-t)$

not Even

$x(t) = e^{-3t}$

$-x(-t) = -e^{-3t}$

$x(t) \neq -x(-t)$

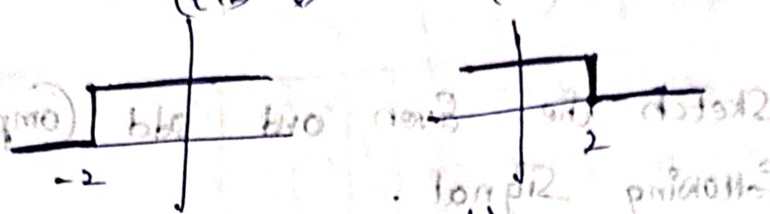
not odd

neither even nor odd

2) $x(t) = u(t+2)$

$x(t) = u(t+2)$, $x(-t) = u(-t+2)$

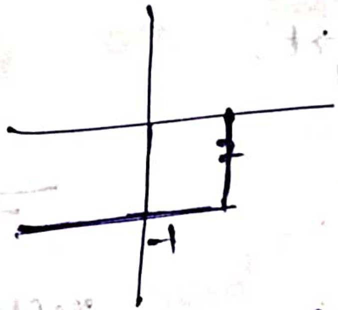
$(t+2)$ ✓ $(-t+2)$ ↓



$x(t) \neq x(-t)$

not Even.

$-x(-t) = -u(-t+2)$



$x(t) \neq -x(-t)$

not odd

neither even nor odd

find the even and odd Component of the discret signal.

1) $x(n) = \{-3, 1, \underset{\uparrow}{2}, -4, 2\}$ → It indicates Origin point

$x(-n) = \{+2, -4, 2, +1, -3\}$

$x_e(n) = \left\{\frac{-1}{2}, \frac{-3}{2}, \underset{\uparrow}{2}, \frac{-3}{2}, \frac{-1}{2}\right\} = \{-0.5, -1.5, \underset{\uparrow}{2}, -1.5, -0.5\}$

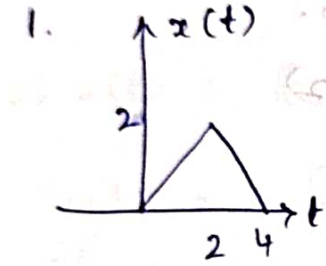
$x_o(n) = \left\{-\frac{5}{2}, \frac{5}{2}, \underset{\uparrow}{0}, -\frac{5}{2}, \frac{+5}{2}\right\} = \{-2.5, 2.5, 0, -2.5, 2.5\}$

2) $x(n) = \{5, 4, 3, 2, \underset{\uparrow}{1}\}$ $x(-n) = \{1, 2, 3, 4, 5\}$

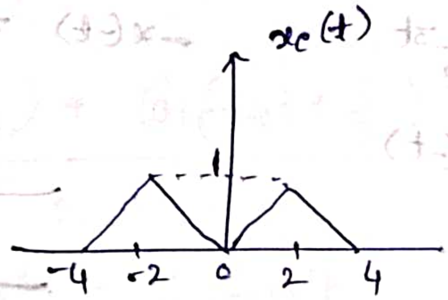
$x_e(n) = \{2.5, 2, 1.5, 1, \underset{\uparrow}{1}, 1, 1.5, 2, 2.5\}$

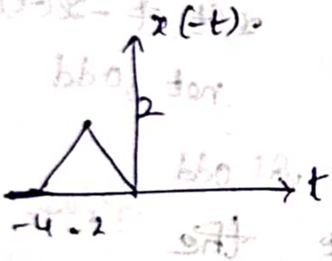
Sketch the Even and odd Components of a following Signal.

1.

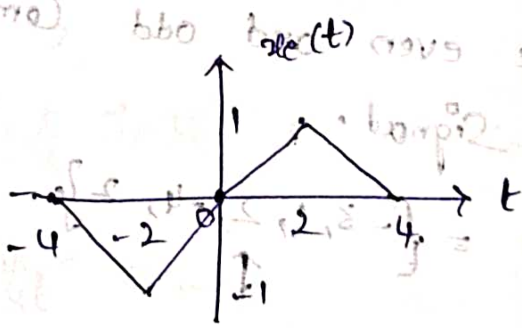


$$x_e(t) = \frac{x(t) + x(-t)}{2}$$ Even





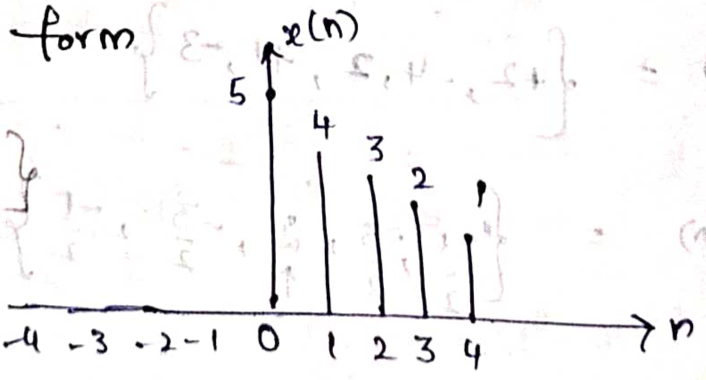
$$x_e(t) = \frac{x(t) - x(-t)}{2}$$

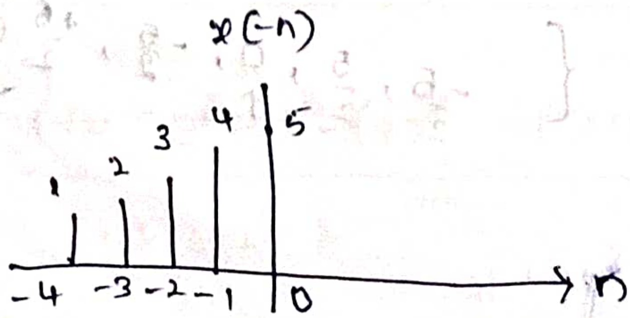


Draw the Wave form

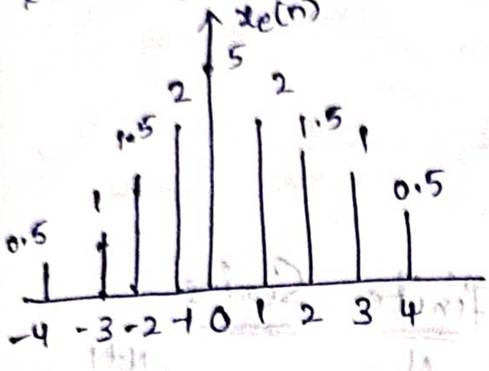
$x(n) = \{\underset{\uparrow}{5}, 4, 3, 2, 1\}$

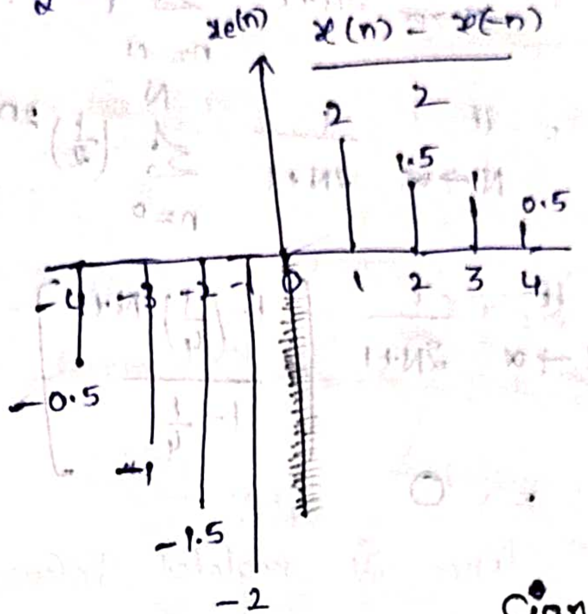
mathematical form





Even Component $x_e(n) = \frac{x(n) + x(-n)}{2}$, odd Component $= \frac{x(n) - x(-n)}{2}$





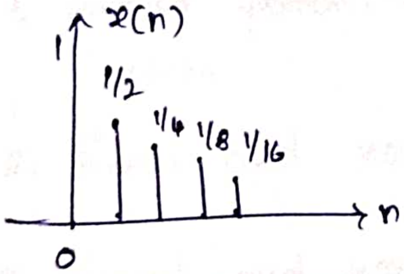
find the following signals are Energy signals or power signals.

Continuous time signal $e^{-at}u(t)$ → Exponentially decrease

1) $\left(\frac{1}{2}\right)^n u(n) = x(n)$ Decret signal

$$E = \sum_{n=-\infty}^{\infty} |x(n)|^2 \longrightarrow$$

Decret signal $\left(\frac{1}{a}\right)^n u(n)$ → Exponentially decreases



$$\boxed{E = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{2n}}$$

$= 1 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \cdots$

$\boxed{\text{Infinity Series } \left(\frac{a}{1-r}\right)}$

$\Rightarrow a = 1, \; r = \frac{1}{4}$

$\Rightarrow \frac{1}{1 - \frac{1}{4}} = \frac{4}{3}$ Joules

$$P = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x(n)|^2$$

$$= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=0}^{N} \left(\frac{1}{2}\right)^{2n}$$

$$= \lim_{N\to\infty} \frac{1}{2N+1} \left[\frac{1-\left(\frac{1}{4}\right)^{N+1}}{1-\frac{1}{4}}\right]$$

$$= 0$$

finite Series

$$\sum_{n=0}^{N} a^n = \frac{1-a^{N+1}}{1-a}$$

This term is neglected because when N is high then the value is very small (∴ so it is negligible)

2) $x(n) = \sin \frac{\pi}{3} n$ find Energy and power.

$$E = \sum_{n=-\infty}^{\infty} |x(n)|^2$$

$$= \sum_{n=-\infty}^{\infty} \left|\sin\left(\frac{\pi}{3}n\right)\right|^2$$

$$= \sum_{n=-\infty}^{\infty} \left|\frac{1-\cos 2\left(\frac{\pi}{3}n\right)}{2}\right|$$

$$= \frac{1}{2}\left[\sum_{n=-\infty}^{\infty} 1-\cos 2\left(\frac{\pi}{3}\right)n\right]$$

when n is increases then cos value is ∞

$1-\infty = \infty$

$$= \frac{1}{2}\cdot\infty = \infty$$

$$P = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x(n)|^2$$

$$= \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} \left| \sin\left(\frac{\pi}{3}\right) n \right|^2$$

finite

$$= \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} \left(\frac{1 - \cos \frac{2\pi}{3} n}{2} \right)$$

neg = pos

$\int_0^{2\pi} \cos\theta = 0$

$$= \frac{1}{2} \left[\lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} 1 - \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} \cos\left(\frac{2\pi}{3}\right) n \right]$$

$$= \frac{1}{2} \lim_{N \to \infty} \frac{1}{2N+1} (2N+1)$$

In finite full cycled $\cos\theta$ and $\sin\theta$ is zero

$$= \frac{1}{2}$$

Classification of System :-

I/P, O/P, impulse

$x(t)$, $y(t)$, $h(t)$

Response

Both Continuous time Systems and Discret time signals are classified as

1. lumped parameter System and Distributed System.
2. linear and non linear systems.
3. Causal and non Causal System.
4. Static and Dynamic System.
   (Static → memory less, Dynamic → memory)
5. Time invarient and Time Varient Systems.
6. Stable and unstable Systems.
7. Invertable and non invertable Systems.
8. FIR and IIR Systems.

lumped and distributed system :-

→ lumped systems are defined as the normal differential equations. It is known as lumped systems.

Ex:- Resistors, Capacitors, etc. (RLC)

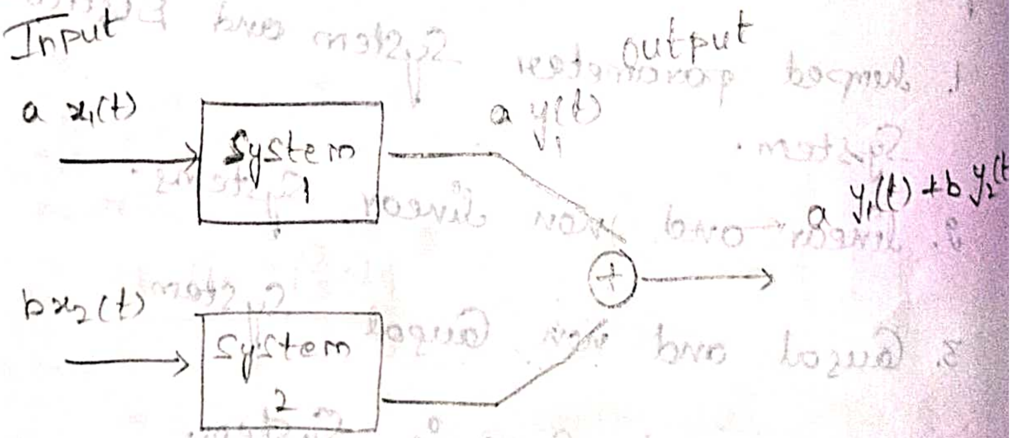
→ Distributive systems are defined by partial differentiation of Distributive parameters.

Ex:- Transmission lines.

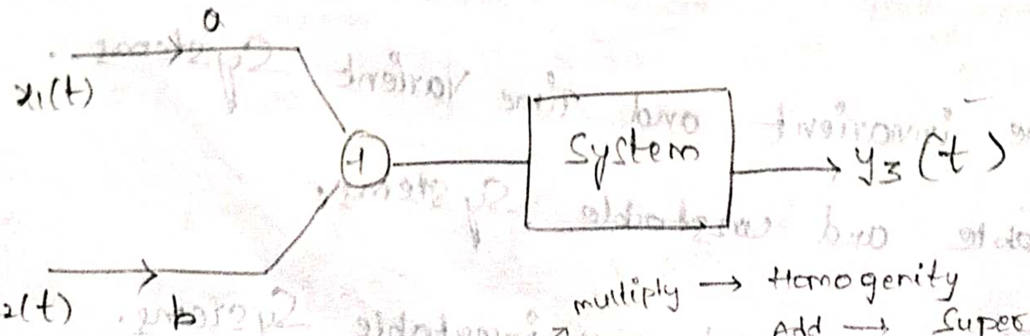
linear & non-linear System :-

— Electronic circuits

A system which satisfies the principle of superposition and the principle of Homogenity then the system is said to be linear system.



same



multiply → Homogenity

Add → Superposition

$\therefore\ y_3(t) = a\,y_1(t) + b\,y_2(t)$

Then it is said to be linear system.

check whether the following systems are linear or not :-

1. $y(t) = x(t^2)$

$y_1(t) = x_1(t^2) \rightarrow ①$

$y_2(t) = x_2(t^2) \rightarrow ②$

$y_3(t) = x_1(t^2) + x_2(t^2) \rightarrow ③$

check $y_3(t) = y_1(t) + y_2(t)$

$x_1(t^2) + x_2(t^2) = x_1(t^2) + x_2(t^2)$

It is equal.

linear.

2) $y(t) = e^{x(t)}$

$y_1(t) = e^{x_1(t)}$

$y_2(t) = e^{x_2(t)}$ — inplace of x we write x_1 and x_2

$y_3(t) = e^{[x_1 t + x_2 t]}$

check :- $y_3(t) = y_1(t) + y_2(t)$

$e^{x_1(t)} + e^{x_2(t)}$

It is not equal.

Non-linear

3. $y(t) = 2x^2(t)$

$y_1(t) = 2x_1^2(t)$

$y_2(t) = 2x_2^2(t)$

$y_3(t) = 2[x_1(t) + x_2(t)]^2$

check :- $y_3(t) = 2x_1^2(t) + 2x_2^2(t)$

$= 2[x_1^2(t) + x_2^2(t)]$

It is not equal.

non-linear

4. $y(n) = 2x(n) + 4$

$y_1(n) = 2x_1(n) + 4$

$y_2(n) = 2x_2(n) + 4$

$y_3(n) = 2[x_1(n) + x_2(n)] + 4$

check :- $y_3(n) = 2x_1(n) + 4 + 2x_2(n) + 4$

$= 2[x_1(n) + x_2 n] + 8$

It is not equal.

non-linear.

5. $y(n) = n^2 x(n)$

$y_1(n) = n^2 x_1(n)$

$y_2(n) = n^2 x_2(n)$

$y_3(n) = n^2[x_1(n) + x_2(n)]$

check :-

$y_3(n) = n^2 x_1(n) + n^2 x_2(n)$

$= n^2[x_1(n) + x_2(n)]$

It is equal.

linear

3. Causal and non Causal System :- Any System for which the response occurs only during or after the time in which excitation is applied Causal System.

(or)

ii) The System is Causal if the output at any time depends only on the values of input at the present and in the past.

→ The System is non Causal if the output at any time depends on the values of input at the future.

$y(t) = A x(t)$ → Causal System
A → not depend

Ex:- $y(t) = x(t+1)$

$t=0 \quad y(0) = x(1)$ → Non Causal System

$t=1 \quad y(1) = x(2)$

$t=-1 \quad y(-1) = x(0)$

Ex:- $y(n) = x\left(\frac{n}{2}\right)$

$t=0 \quad y(0) = x(0)$

$t=-1 \quad y(-1) = x\left(-\frac{1}{2}\right)$ - future

$t=1 \quad y(1) = x\left(\frac{1}{2}\right)$ - past

non Causal System

Ex:- $y(n) = x(2n)$ Non Causal System

$t=0 \quad y(0) = x(0)$

$t=1 \quad y(1) = x(2)$ - future

$t=-1 \quad y(-1) = x(-2)$ - future

Ex:- $y(t) = x(-t)$

$t=0 \quad y(0) = x(0)$ - past

$t=1 \quad y(1) = x(-1)$ - past

$t=-1 \quad y(-1) = x(1)$ - future

Non- Causal System

Ex:- $y(n) = x(n)+x(n-2)$

$t=0 \quad y(0) = x(0)+x(-2)$

$t=1 \quad y(1) = x(1)+x(-1)$

$t=-1 \quad y(-1) = x(-1)+x(-3)$

Causal System

Ex:- $y(t) = x(2-t)+x(t-4)$

future

$t=0 \quad y(0) = x(2)+x(-4)$

$t=1 \quad y(1) = x(1)+x(-3)$

$t=-1 \quad y(-1) = x(3)+x(-5)$

Non Causal System

Time invarient and Time Varient System :-

→ If An orbitary Excitation $x(n)$ & $x(t)$ of a System Causes a response $y(n)$, An orbitary excitation of the same System. $x(n-n_0)$ Causes a response $y(n-n_0)$ for any orbitary n_0 then the System to be time invarient.

$$x(t) \longrightarrow \boxed{\text{system}} \longrightarrow y(t)$$

$y(t) = \sin[x(t)]$ Internal quantity

Substitute $t \to t-t_0$

$y(t-t_0) = \sin[x(t-t_0)]$

subtract internal term to with ⓧ

$= y(t,t_0) = \sin[x(t-t_0)]$

Time Invarient

Eg:2 $y(t) = x\left(\frac{t}{2}\right)$

Replace $t \rightarrow t-t_0$

$y(t-t_0) = x\left(\frac{t-t_0}{2}\right)$

Now Subtract t_0 ~~with~~ from internal quantity

$y(t,t_0) = x\left(\frac{t}{2}-t_0\right)$

$y(t-t_0) \neq y(t,t_0)$

Time Varient.

Eg:3 $y(t) = x(-2t)$

$t \rightarrow t-t_0$

$y(t-t_0) = x(-2(t-t_0))$

$= x(-2t+2t_0)$

$y(t,t_0) = x(-2t-t_0)$

$y(t-t_0) \neq y(t,t_0)$

Time Varient.

Eg:4 $y(n) = x(n) + n\,x(n-2)$

Replace $n \rightarrow n-n_0$

$y(n-n_0) = x(n-n_0) + (n-n_0)\,x(n-n_0-2) \rightarrow ①$

$y(n,n_0) = x(n-n_0) + n\,x(n-n_0-2) \rightarrow ②$

Eq ① and Eq ② are not Same

$y(n-n_0) \neq y(n,n_0)$

Time Varient.

→ Resistors

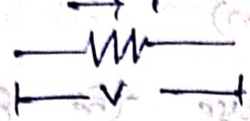
→ Capacitors Inductors (It stores the Energy)

Static and Dynamic System :- If any System Response at an orbitary time $t=t_0$, $y(t_0)$ depends only on at time Excitation $t=t_0$ i.e., $x(t_0)$.

$y(t_0)$ does not depends on any other Excitation [present & past] then the system is said to be memory less (and) or static system.

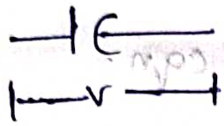
→ A System is said to be memory if the present i/p is depends on the past i/p and future i/p.

for static →
(memory less)



$V = iR$

for Dynamic →
(memory)



$V = \frac{1}{C} \int_{-\infty}^{t} i\,dt$

Eg: $y(t) = 5\,x(t)$ Here present o/p is depends on present i/p

so it is memory less i.e., static.

→ $y(t-5) = 5\,x(t-5)$

Same as above Reason.

Static.

→ $y(n) = 10\,x(n+1)$ Here present o/p is depends on future i/p

so it is Dynamic. Time advance

→ $y(n) = 10\,(n-10)$ Here present o/p is depends on past i/p

so it is Dynamic. Time Delay

→ $y(t) = \frac{d^2 x(t)}{dt^2} + 4x(t)$ → ① Continuous System

2 order derivative → 2 units delay

Differential eqns – Continuous System

Difference eqns – Discret System

Eq ① Reason is Dynamic System because it is in the form of differential Eqn.

→ $y(n) = x(n-2) + 4(x(n))$ → Discret System

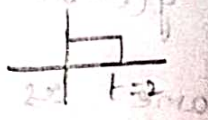
it is Dynamic because the Eqn is in the form of Difference eqn.

Stable and unstable Systems :-

Stability :- Any System for which the Response is bounded when the Excitation is bounded, then the System is Called Stable System.

i) BIBO (Bounded i/p Bounded o/p)

Calculate Stability in Impulse Response



$$S = \int_{-\infty}^{\infty} |h(\tau)|\, d\tau < \infty$$

In Both Cases at $t = 2$ A is 0 so the system is stable.

for Discret $$\sum_{k=-\infty}^{\infty} |h(k)| < \infty$$

Eq:- $y(t) = e^{x(t)}$ $\quad 0 \le |x(t)| \le 8$

$$x(t) \longrightarrow \boxed{\text{System}} \longrightarrow y(t)$$

$y(t)$ is bounded for all the values of $x(t)$.

→ If the Eqn Contains $h(t)$ → Impulse Response System.

If the Eqn Contains i/p and o/p → General Eqn.

→ $h(t) = (2 + e^{-3t})\, u(t)$

$$\int_{-\infty}^{\infty} |h(t)|\, dt$$

$$= \int_{-\infty}^{\infty} (2 + e^{-3t})\, u(t)\, dt = \int_{0}^{\infty} (2 + e^{-3t})\, dt$$

$$= 2(t)_0^{\infty} + \frac{e^{-3t}}{-3}\Big|_0^{\infty} = -\frac{1}{3}(0-1)$$

$$= \frac{1}{3} + \infty$$

$$= \infty$$

It is unstable.

Invertable and non-invertable Systems :- Any System is said to be Invertable if unique Excitation produce unique Response.

(or)

A System is said to be Invertable if distinct inputs leads to distinct Outputs.

Eg:- Half wave Rectifier

→ $y(t) = 4x(t)$

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$ → Invertable

→ $y(t) = (x(t))^2$

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$ → Non Invertable

→ Any System is said to be Non-Invertable if different inputs produce Same Output

Eg: full wave Rectifier

→ **FIR and IR Systems** :-

If the impulse Response Sequence is finite duration then the System is Called finite Impulse Response System.

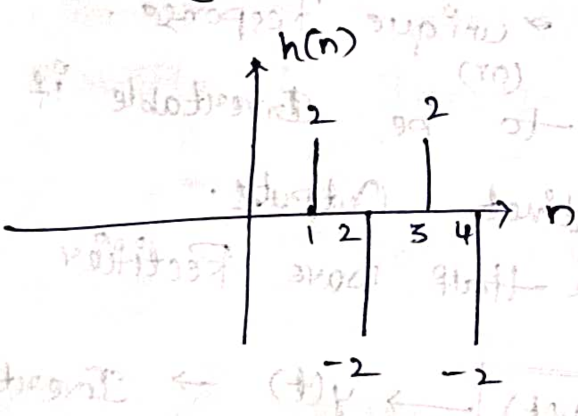
→ If the Impulse Response Sequence is infinite duration then the System is called Infinite Response System.

$$x(n) \longrightarrow \boxed{h(n)} \longrightarrow y(n)$$

FIR

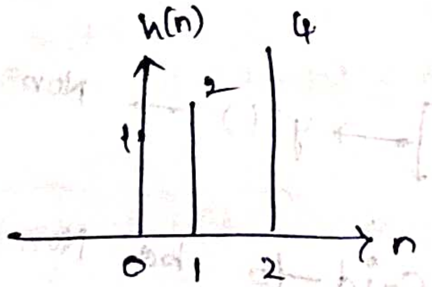
Ex:

$$h(n) = \begin{cases} +2 & \text{for } n = 1, 3 \\ -2 & \text{for } n = 2, 4 \\ 0 & \text{otherwise} \end{cases}$$



IR Eg:

$$h(n) = 2^n u(n)$$



FIR and IR filters does not Exist 't'.

It Exist only in discret Signals.

check whether the following systems are i) static or dynamic
ii) linear or non linear
iii) Causal or non Causal
iv) Time Invarient or Time Varient

→ $y(t) = at^2 x(t) + bt\, x(t-4)$

i) $t-4$ is past i/p
present o/p is depends on past i/p
dynamic.

ii) $y_1(t) = at^2 x_1(t) + bt\, x_1(t-4)$

$y_2(t) = at^2 x_2(t) + bt\, x_2(t-4)$

$y_3(t) = at^2 [x_1(t) + x_2(t)] + bt[x_1(t-4) + x_2(t-4)]$

check $y_3(t) = y_1(t) + y_2(t)$

$= at^2 [x_1(t) + x_2(t)] + bt[x_1(t-4) + x_2(t-4)]$

Verified.
This is linear system.

iii) present o/p is depends on present i/p $x(t)$ and past i/p $(x(t-4))$
so this is Causal System.

iv) $y(t) = at^2 x(t) + bt\, x(t-4)$

$y(t-t_0) = a(t-t_0)^2 x(t-t_0) + b(t-t_0)\, x(t-t_0-4)$

$y(t,t_0) = at^2 x(t-t_0) + bt\, x(t-4-t_0)$

$y(t,t_0) \neq y(t-t_0)$

Time Varient.

→ $y(n) = a^n u(n)$

i) $y(n) = a^n u(n)$ ⇒ we taken as $y(n) = a^n x(n)$

present i/p is depends on i/p present.

this is static.

ii) $y_1(n) = a^n x_1(n)$

$y_2(n) = a^n x_2(n)$

$y_3(n) = a^n [x_1(n) + x_2(n)]$

check $y_3(n) = y_1(n) + y_2(n)$

$= a^n [x_1(n) + x_2(n)]$

Verified.

It is linear.

iii) present o/p is depends on present i/p

this is Causal System.

iv) $y(n) = a^n x(n)$

$y(n-n_0) = a^{n-n_0} x(n-n_0)$

$y(n, n_0) = a^n x(n-n_0)$

this is time Varient because $y(n-n_0) \neq y(n, n_0)$

→ $y(n) = x^2(n) + \frac{1}{x^2(n-1)} \; x^2(n-1)$

i) $y(n) = x^2(n) + \frac{1}{x^2(n-1)}$

present o/p is depends on past i/p

this is dynamic.

ii) $y_1(n) = x_1^2(n) + \frac{1}{x_1^2(n-1)}$

$y_2(n) = x_2^2(n) + \frac{1}{x_2^2(n-1)}$

check $y_3(n) = x_1^2(n) + x_2^2(n) + \frac{1}{[x_1^2(n-1) + x_2^2(n-1)]}$

check $y_3(n) = x_1^2(n) + x_2^2(n) + \frac{1}{x_1^2(n-1)} + \frac{1}{x_2^2(n-1)}$

This is non linear.

iii) present o/p is depends on present i/p and past i/p.

This is Causal.

iv) $y(n) = x^2(n) + \frac{1}{x^2(n-1)}$

$y(n-n_0) = x^2(n-n_0) + \frac{1}{x^2(n-n_0-1)}$

$y(n,n_0) = x^2(n-n_0) + \frac{1}{x^2(n-1-n_0)}$

$y(n-n_0) = y(n,n_0)$

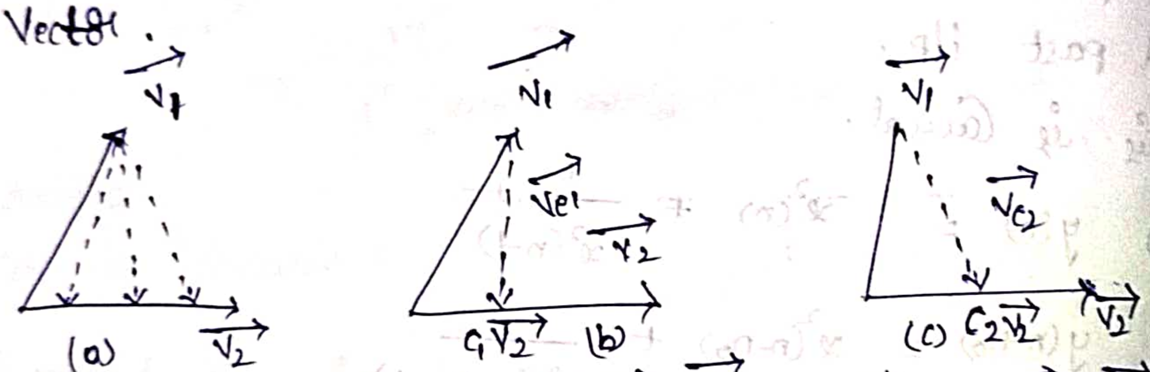
This is Time Invarient.

Signal Analysing :-

Analysing between Vectors And Signals :-
Analogy

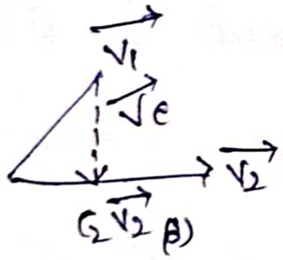
Vector is specified by magnitude and direction

→ the Vector $\vec{V_1}$ can be Expressed interms of Vector $\vec{V_2}$ in different ways by drawing a line from the End of $\vec{V_1}$ on to $\vec{V_2}$. In Each Representation $\vec{V_1}$ is Expressed in terms of $\vec{V_2}$ + An Error Vector.



$\vec{V_1} = C_1\vec{V_2} + \vec{V_{e_1}}$ $\qquad$ $\vec{V_1} = C_2\vec{V_2} + \vec{V_{e_2}}$

Some Component $\qquad$ Error Vector



$\vec{V_1} = C_2\vec{V_2} + \vec{V_e}$ → Minimum Error Vector & orthogonal

→ when the two Vectors are ⊥er to Each other then C will be zero.

→ If $C_{12} = 0$ then the vector $\vec{V_1}$ has no Component along the other Component $\vec{V_2}$.

→ Then the two Vectors are mutually ⊥er.
(90° phase shift)

→ The vectors which are Mutually ⊥er to Each other are called Orthogonal Vectors.

→ The Component of $\vec{V_1}$ along $\vec{V_2}$ is $\frac{\vec{V_1} \cdot \vec{V_2}}{V_2}$

$$\therefore C_{12} = \frac{\vec{V_1} \cdot \vec{V_2}}{V_2^2} \qquad = C_{12} V_2$$

→ If the two Vectors are ⊥er to Each other

Then $C_{12} = 0 \Rightarrow \boxed{\vec{V_1} \cdot \vec{V_2} = 0}$

Signals :- The Concept of Vector Comparision and Orthoganality can be Extended to signals.

Consider two Signals $x_1(t)$ and $x_2(t)$ where t is lies between t_1 to t_2 i.e., $t_1 < t < t_2$

signals are only in finite.

$$x_1(t) = C_{12} \, x_2(t) + x_e(t)$$

$$\boxed{x_e(t) = x_1(t) - C_{12} \, x_2(t)}$$

Common factor

→ To minimise the error $x_e(t)$ over the interval t_1 to t_2 to minimise the average value $x_e(t)$. Over this interval.

* Avg value of $x_e(t)$ over t_1 to t_2 is

$$\frac{1}{t_2 - t_1} \int_{t_1}^{t_2} x_e(t) \, dt$$

$$= \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} \left[x_1(t) - C_{12} \, x_2(t) \right] dt$$

This gives some false Results. so we take Mean Square. [∴ First square the signal (both +ve and -ve)]

→ Because of +ve and −ve errors present in the interval, that may cancel one another in the process of & Average and it gives false indication. To overcome this problem we choose to minimise the mean square of Error instead of mean error.

→ $\overline{(x_e(t))^2} \longrightarrow \epsilon$

$$\boxed{\epsilon = \frac{1}{t_2-t_1}\int_{t_1}^{t_2}\left[x_1(t) - C_{12}\,x_2(t)\right]^2 dt} \rightarrow \text{Mean Square error}$$

To find min value of mean square error.

$$\frac{d\epsilon}{dC_{12}} = 0 \Rightarrow \frac{d}{dC_{12}}\left(\frac{1}{t_2-t_1}\int_{t_1}^{t_2}(x_1(t) - C_{12}\,x_2(t))^2 dt\right)$$

$$= \frac{\cancel{1}}{t_2-t_1}\int_{t_1}^{t_2}$$

$$= \frac{1}{t_2-t_1}\frac{d}{dC_{12}}\left(\int_{t_1}^{t_2} x_1^2(t) - 2x_1(t)x_2(t)C_{12} + (C_{12}\,x_2(t))^2\right) =$$

change the order of Integration and differentiation

$$= \frac{1}{t_2-t_1}\left\{\int_{t_1}^{t_2}\frac{d}{dC_{12}}x_1^2(t)\,dt - \int_{t_1}^{t_2}\frac{d}{dC_{12}}(2x_1(t)\,C_{12})\,x_2(t)\,dt + \int_{t_1}^{t_2}\frac{d}{dC_{12}}(x_2(t)\ldots\,dt\right.$$

$$= \frac{1}{t_2-t_1}\left\{0 - 2\int_{t_1}^{t_2} x_1(t)\,x_2(t)\,dt + \int_{t_1}^{t_2} 2C_{12}\,x_2^2(t)\,dt = 0\right.$$

$$2\int_{t_1}^{t_2} x_2^2(t)\, C_{12}\, dt = 2\int_{t_1}^{t_2} x_1(t)\, x_2(t)\, dt$$

$$\boxed{C_{12} = \frac{\int_{t_1}^{t_2} x_1(t)\, x_2(t)\, dt}{\int_{t_1}^{t_2} x_2^2(t)\, dt}}$$

In this value Error signal will be minimum.

→ The Orthogonality Concept of two vectors $\vec{V_1}$ and $\vec{V_2}$ is applicable to two signals $x_1(t)$ and $x_2(t)$ over an interval t_1 to t_2.

→ The two signals are said to be orthogonal if $C_{12} = 0$

$$\therefore \frac{\int_{t_1}^{t_2} x_1(t)\, x_2(t)\, dt}{\int_{t_1}^{t_2} x_2^2(t)\, dt} = 0$$

Real valued signals

$$\therefore \boxed{\int_{t_1}^{t_2} x_1(t)\, x_2(t)\, dt = 0}$$

→ S.T the functions $x_1(t) = 2$ and $x_2(t) = \sqrt{3}(1-2t)$ are said to be Orthogonal over a interval of $(0,1)$

$$\int_{t_1}^{t_2} x_1(t)\, x_2(t)\, dt = 0 \leftarrow \text{formula}$$

$$\int_0^1 2 \cdot \sqrt{3}(1-2t)\, dt = \text{L.H.S}$$

$$2\sqrt{3}\int_0^1 (1-2t)\, dt$$

$$2\sqrt{3}\left[(t)_0^1 - \cancel{2}\left(\frac{t^2}{\cancel{2}}\right)_0^1 \right]$$

$$2\sqrt{3}\left[1-0-1+0\right]$$

$$= 0$$

$$= \text{R.H.S}$$

**

→ $x_1(t) = \sin n\omega_0 t$ and $x_2(t) = \cos m\omega_0 t$ are Orthogonal over any interval $\left(t_0, t_0+\frac{2\pi}{\omega_0}\right)$ where m and n are integers.

$$\text{L.H.S} = \int_{t_0}^{t_0+2\pi/\omega_0} \sin n\omega_0 t \cos m\omega_0 t\, dt$$

$$= \int_{t_0}^{t_0+2\pi/\omega_0} \frac{1}{2}\left[2 \sin n\omega_0 t \cos m\omega_0 t\right] dt$$

$$= \frac{1}{2}\int_{t_0}^{t_0+2\pi/\omega_0} \sin(n+m)\omega_0 t + \sin(n-m)\omega_0 t \, dt$$

$$= \frac{1}{2}\left[-\frac{\cos(n+m)\omega_0 t}{n+m} - \frac{\cos(n-m)\omega_0 t}{n-m}\right]_{t_0}^{t_0+\frac{2\pi}{\omega_0}}$$

$$= -\frac{1}{2}\left[\frac{\cos(n+m)\omega_0 t}{n+m} + \frac{\cos(n-m)\omega_0 t}{n-m}\right]_{t_0}^{t_0+\frac{2\pi}{\omega_0}}$$

$$= \frac{-1}{2}\left[\frac{\cos(n+m)\omega_0\left(t_0+\frac{2\pi}{\omega_0}\right) - \cos(n+m)\omega_0 t_0}{n+m} + \frac{\cos(n-m)\omega_0\left(t_0+\frac{2\pi}{\omega_0}\right) - \cos(n-m)\omega_0 t_0}{n-m}\right]$$

$= 0$ $\quad \because \cos[(n+m)\omega_0 t_0 + \overbrace{(n+m)}^{\text{integers}}2\pi]$

$= \cos(n+m)\omega_0 t_0$ $\quad \cos(2\pi+\theta)$ $\quad \cos\theta$ $\quad$ (IV qua)

*

→ prove that Complex Exponential signals are Orthogonal functions.

$x_1(t) = e^{jn\omega_0 t}$ $\qquad x_2(t) = e^{-jm\omega_0 t}$.

Complex Exponential functions are Orthogonal

if $\boxed{\displaystyle\int_{t_1}^{t_2} x_1(t)\, x_2^*(t)\, dt = 0}$

Complex Conjugate of $x_2(t)$

$$\therefore \int_{t_0}^{t_0+2\pi/w_0} e^{jnw_0t} \left(e^{jmw_0t}\right)^* dt$$

$$= \int_{t_0}^{t_0+2\pi/w_0} e^{jnw_0t} \cdot e^{-jmw_0t} \, dt$$

$$= \int_{t_0}^{t_0+2\pi/w_0} e^{j(n-m)w_0t} \, dt$$

$$= \left. \frac{e^{j(n-m)w_0t}}{j(n-m)w_0} \right|_{t_0}^{t_0+\frac{2\pi}{w_0}}$$

$$= \frac{e^{j(n-m)w_0\left(t_0+\frac{2\pi}{w_0}\right)} - e^{j(n-m)w_0t_0}}{j(n-m)w_0}$$

$$= \frac{e^{j(n-m)w_0t_0} \cdot e^{j(n-m)2\pi} - e^{j(n-m)w_0t_0}}{j(n-m)w_0}$$

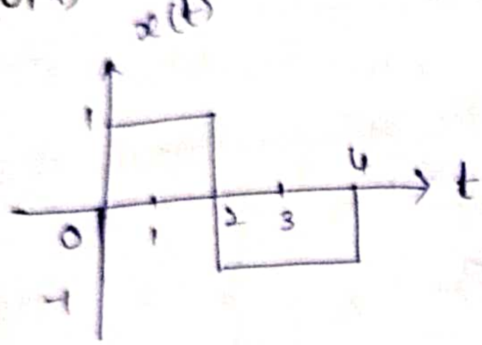
$$= 0$$

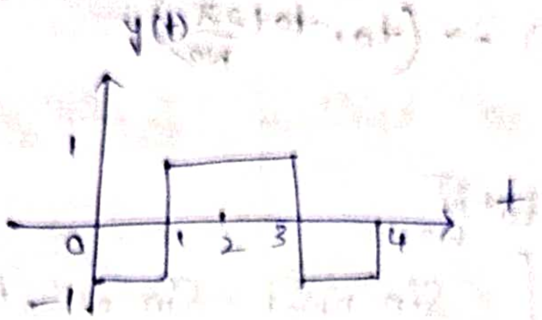
$$\boxed{\begin{aligned} e^{j\theta} &= \cos\theta + j\sin\theta \\ e^{j(n-m)2\pi} &= \cos 2\pi \overset{\text{Integers}}{(n-m)} + j \sin 2\pi (n-m) \\ &= 1 + 0 \\ &= 1 \end{aligned}}$$

→ Consider the following signals:

ST Both are Orthogonal over an interval of $(0,4)$

i)





$$x(t) = \begin{cases} 1 & 0 \le t < 2 \\ -1 & 2 \le t \le 4 \end{cases}$$

$$y(t) = \begin{cases} -1 & 0 \le t < 1 \\ 1 & 1 \le t < 3 \\ -1 & 3 \le t \le 4 \end{cases}$$

$$\int_{t_1}^{t_2} x(t)\,y(t)\,dt = \int_0^4 x(t)\,y(t)\,dt$$

$$= \int_0^1 1.(-1)dt + \int_1^2 1.1\,dt + \int_2^3 -1.+1\,dt + \int_3^4 (-1)(-1)dt$$

$$= -t\Big|_0^1 + t\Big|_1^2 + -t\Big|_2^3 + t\Big|_3^4$$

$$= (-1+0) + (2-1) + (-3+2) + (4-3)$$

$$= -1+1-1+1$$

$$= 0$$

→ S.T $\sin n\omega_0 t$ and $\sin m\omega_0 t$ are Orthogonal to each other for all integral values of $(m,n) \rightarrow \left(t_0, t_0+\frac{2\pi}{\omega_0}\right)$ [this statement means full cycles i.e., $(0,2\pi)$, $(0,4\pi)$]

$$= \int_{t_0}^{t_0+\frac{2\pi}{\omega_0}} \sin n\omega_0 t \, \sin m\omega_0 t \, dt$$

$$= \frac{1}{2}\int_{t_0}^{t_0+2\pi/\omega_0} \cos(n\omega_0 t - m\omega_0 t) - \cos(n\omega_0 t + m\omega_0 t)\, dt$$

$$= \frac{1}{2}\int_{t_0}^{t_0+2\pi/\omega_0} (\cos(n-m)\omega_0 t - \cos(n+m)\omega_0 t)\, dt$$

$$= \frac{1}{2}\left[\left(\frac{\sin(n-m)\omega_0 t}{(n-m)\omega_0} - \frac{\sin(n+m)\omega_0 t}{(n+m)\omega_0}\right]_{t_0}^{t_0+\frac{2\pi}{\omega_0}}\right]$$

$$= \frac{1}{2}\left[\left(\frac{\sin(n-m)\omega_0\left(t_0+\frac{2\pi}{\omega_0}\right) - \sin(n-m)\omega_0 t_0}{(n-m)\omega_0}\right) - \left(\frac{\sin(n+m)\omega_0\left(t_0+\frac{2\pi}{\omega_0}\right) - \sin(n+m)\omega_0 t_0}{(n+m)\omega_0}\right)\right]$$

$$\Rightarrow \frac{1}{2}\left\{ \cancel{\sin m\omega_0 t_0} \quad \sin(n-m)\omega_0 t_0 + \right.$$

$\sin[(n-m)\omega_0 t_0 + (n-m)2\pi]$

$= \sin(2\pi + \theta)$

$= \sin\theta$

$$= \frac{1}{2}\left[\frac{\sin(n-m)w_0t_0 - \sin(n-m)w_0t_0}{(n-m)w_0} - \frac{\sin(n+m)w_0t_0 - \sin(n+m)w_0t_0}{(n+m)w_0}\right]$$

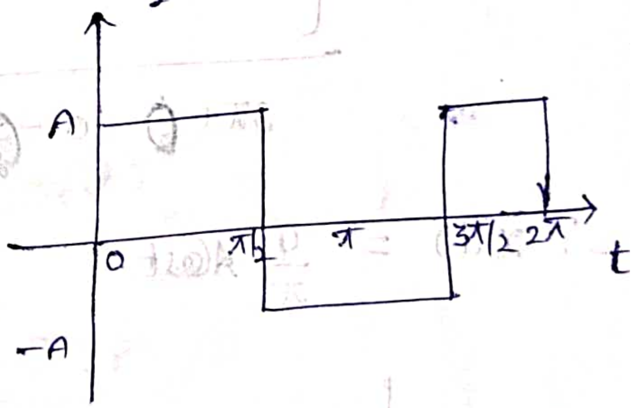
$$= 0$$

→ A Rectangular function is defined as

$$x(t) = \begin{cases} A & \text{for } 0 < t < \pi/2 \\ -A & \text{for } \pi/2 < t < 3\pi/2 \\ A & \text{for } 3\pi/2 < t < 2\pi \end{cases}$$

approximate the above function by $A\cos t$ b/w the interval $(0, 2\pi)$ such that the mean square error is minimum.

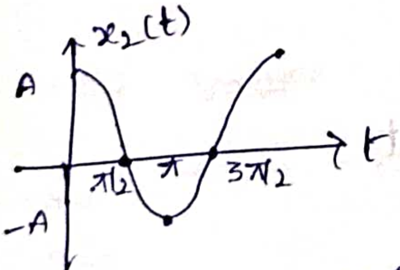
Rectangular function :-



Original Approximate Error

$$x_1(t) = C_{12}\,x_2(t) + x_e(t) \rightarrow ①$$

$$x(t) = A\cos t$$



Here C_{12} is calculate because to minimise the error

We have to calculate C_{12} by using

$$C_{12} = \frac{\int_{t_1}^{t_2} x_1(t)\cdot x_2(t)\,dt}{\int_{t_1}^{t_2} x_2^2 t\,dt}$$

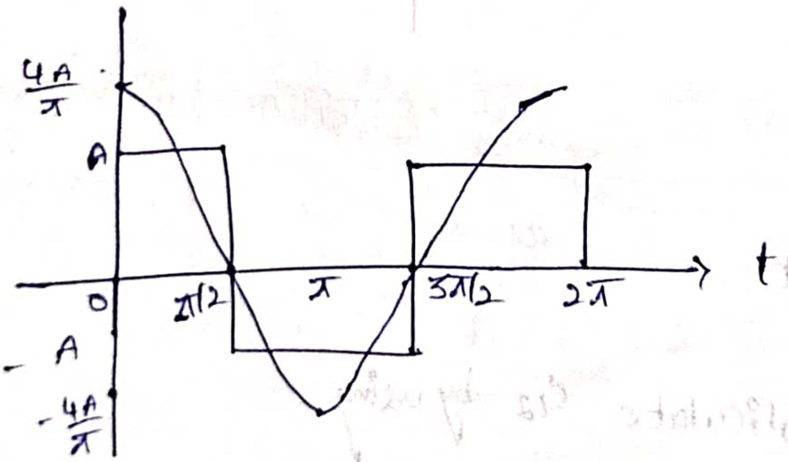
$$= \int_0^{\pi/2} A \cdot A\cos t\, dt + \int_{\pi/2}^{3\pi/2} -A \cdot A\cos t\, dt + \int_{3\pi/2}^{2\pi} A \cdot A\cos t\, dt$$

$$\overline{\int_0^{2\pi} (A\cos t)^2\, dt}$$

$$= \frac{A^2 \left[\sin t \Big|_0^{\pi/2} - \sin t \Big|_{\pi/2}^{3\pi/2} + \sin t \Big|_{3\pi/2}^{2\pi} \right]}{\frac{A^2}{2}\left(t + \frac{\sin 2t}{2}\right)\Big|_0^{2\pi}}$$

$$C_{12} = \frac{2\left[1 - (-1-1) + (0+1)\right]}{2\pi + 0 - 0 - 0} = \frac{2(4)}{2\pi} = \frac{4}{\pi}$$

$$\therefore x(t) = \frac{4}{\pi} A\cos t \qquad (\because Eq\ 1)$$



Orthogonal Signal Space :-

Consider a set of n functions $g_1(t)$, $g_2(t) \ldots g_n(t)$ which are orthogonal to one another over an interval of (t_1, t_2)

→ $\int_{t_1}^{t_2} g_j(t) \cdot g_k(t)\, dt = 0$ where $j \neq k$

→ let an arbitary signal $x(t)$ be approximated over an interval (t_1, t_2) is a linear combination of n mutuall orthogonal signals.

Here $g_1, g_2 + g_3 \ldots g_n$

(t_1, t_2)

$$x(t) = C_1 g_1(t) + C_2 g_2(t) + \ldots C_k g_k(t) \ldots C_n g_n(t)$$

$$= \sum_{r=1}^{n} C_r g_r$$

$$x(t) = \sum_{r=1}^{n} C_r g_r + x_e(t)$$

$$x_e(t) = x(t) - \sum_{r=1}^{n} C_r g_r$$

The values of $C_1, C_2 \ldots C_n$ are selected such that to minimise mean square error.

$$\epsilon = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} x_e^2(t)\, dt$$

Average square mean

$$\epsilon = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} \left(x(t) - \sum_{r=1}^{n} C_r g_r(t) \right)^2 dt$$

$C_1 C_2 \ldots C_n$ such that to minimise the mean square error.

$$\frac{\partial \epsilon}{\partial C_1} = \frac{\partial \epsilon}{\partial C_2} \cdots \frac{\partial \epsilon}{\partial C_j} \cdots \frac{\partial \epsilon}{\partial C_n} = 0$$

Consider the Eqn $\frac{\partial \epsilon}{\partial C_j} = 0$

$$= \frac{\partial}{\partial C_j}\left(\frac{1}{t_2 - t_1}\int_{t_1}^{t_2}\left[x(t) - \sum_{r=1}^{n} C_r g_r(t)\right]^2\right.$$

$$= \frac{\partial}{\partial C_j}\left\{\int_{t_1}^{t_2}\left(x(t)^2 + \sum_{r=1}^{n} C_r^2 g_r^2(t) - 2\, x(t)\sum_{r=1}^{n} C_r g_r\right) dt\right\}$$

$$= \int_{t_1}^{t_2} 2 C_j\, g_j^2(t)\, dt = \int_{t_1}^{t_2} 2 x(t)\, g_j(t)\, dt$$

$$\boxed{C_j = \frac{\int_{t_1}^{t_2} x(t)\, g_j(t)\, dt}{\int_{t_1}^{t_2} g_j^2(t)\, dt}}$$

$\therefore$ let $\int_{t_1}^{t_2} g_j^2(t)\, dt = k_j$

$$C_j = \frac{1}{k_j}\int_{t_1}^{t_2} x(t)\, g_j(t)\, dt$$

When we take multiple errors. Then the value of error should be small.

Evaluation of mean Square Error :-

A ccording to the above Definition

$$\epsilon = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} \left(x(t) - \sum^{n} C_r g_r(t) \right)^2 dt$$

$$= \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} \left(x^2(t) + C_r^2 \sum_{r=1}^{n} g_r^2(t) - \sum_{r=1}^{n} 2 C_r g_r(t) \, x(t) \right) dt$$

Consider the term C_r we have

$$= \frac{\partial}{\partial C_r} \left[\int_{t_1}^{t_2} \left(x^2(t) + C_r^2 \sum_{r=1}^{n} g_r^2(t) - \sum_{r=1}^{n} 2 C_r g_r(t) \cdot x(t) \right] dt \right. \rightarrow ①$$

$$= \int_{t_1}^{t_2} 2 C_r g_r^2(t) \, dt = \int_{t_1}^{t_2} 2 x(t) g_r(t) \, dt$$

$$C_r K_r = \int_{t_1}^{t_2} x(t) \, g_r(t) \, dt \rightarrow ②$$

Sub Eq ② in Eq ①

$$\epsilon = \frac{\partial}{\partial C_r} \left(\frac{1}{t_2 - t_1} \int_{t_1}^{t_2} \left[x^2(t) + \sum_{r=1}^{n} C_r^2 K_r - \sum_{r=1}^{n} 2 C_r^2 K_r \right] dt \right.$$

$$= \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} \left[x^2(t) dt - \sum_{r=1}^{n} C_r^2 K_r \right]$$

$$= \frac{1}{t_2 - t_1} \left\{ \int_{t_1}^{t_2} x^2(t) dt - \left[C_1^2 K_1 + C_2^2 K_2 + \cdots C_n^2 K_n \right] \right\}$$

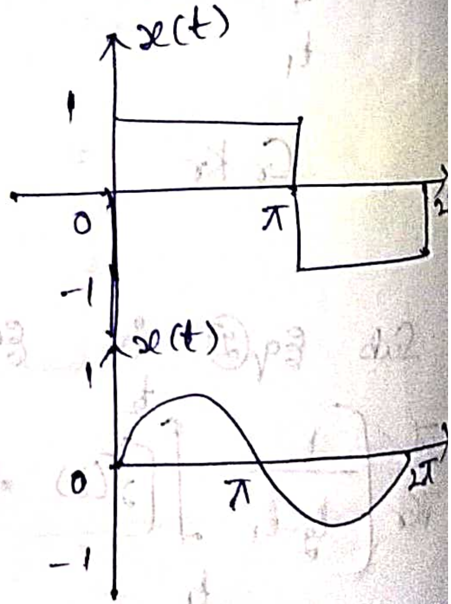
$$k_1 = \int_{t_1}^{t_2} g_1^2(t)\, dt \qquad k_2 = \int_{t_1}^{t_2} g_2^2(t)\, dt$$

**

→ A Rectangular function $x(t)$ is defined as

$$x(t) = \begin{cases} 1 & \text{for } 0<t<\pi \\ -1 & \text{for } \pi<t<2\pi \end{cases}$$

. Approximate the above Rectangular function by a single Sinusoidal Signal $\sin(t)$ over an interval $(0, 2\pi)$ such that Mean Square Error is minimum. Evaluate the mean Square Error in this approximation. also show that what happens when more number of Sinusoides used for this approximation.

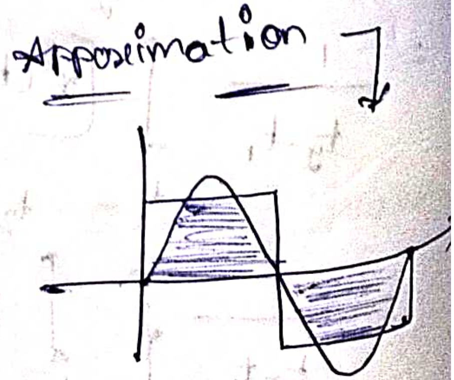


$x_2(t) = \sin t$

$\therefore\ x_1(t) = C_{12}\, x_2(t)$

$$C_{12} = \frac{\int_{t_1}^{t_2} x_1(t)\, x_2(t)\, dt}{\int_{t_1}^{t_2} x_2^2(t)\, dt}$$

$$= \frac{\int_{t_1}^{t_2} x(t)\, \sin t\, dt}{\int_{t_1}^{t_2} \sin^2 t\, dt}$$

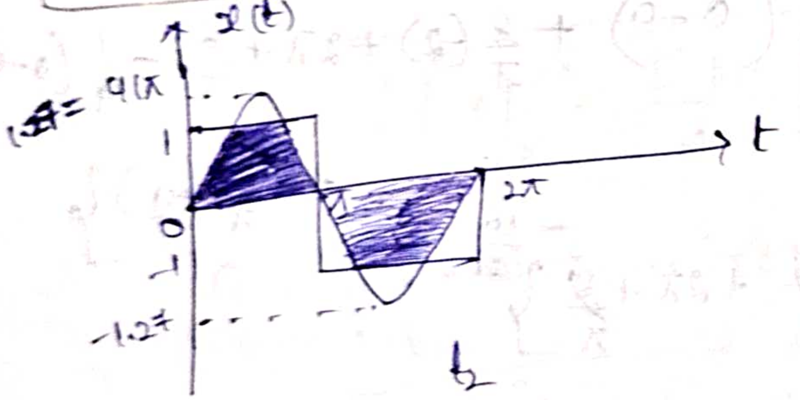


$$= \frac{\int_0^{\pi} 1 \cdot \sin t\,dt + \int_\pi^{2\pi} (-1)\sin t\,dt}{\int_0^{2\pi}\left(\frac{1-\cos 2t}{2}\right)dt}$$

$$= \frac{-\cos t\Big|_0^{\pi} + \cos t\Big|_\pi^{2\pi}}{\frac{1}{2}\left[t\Big|_0^{2\pi} - \frac{\sin 2t}{2}\Big|_0^{2\pi}\right]} = \frac{-(-1-1)+(1+1)}{\frac{1}{2}\left[2\pi - 0 - 0 + 0\right]} = \frac{4}{\pi} = 1.27$$

$$\boxed{x_1(t) = \frac{4}{\pi}\sin t}$$

$$\frac{1}{2\pi}\left(\int_0^{\pi} x_1^2(t)\,dt + \int_\pi^{2\pi} x_1^2(t)\,dt\right] - C_{12}^2$$



a^2, b^2 calculate then we get 0.189

$$\epsilon = \frac{1}{t_2 - t_1}\int_{t_1}^{t_2}\left[x_1(t) - C_{12}\,x_2(t)\right]^2 dt$$

Mean square error

$$= \frac{1}{2\pi}\int_0^{\pi}(1 - C_{12}\sin t)^2\,dt + \int_\pi^{2\pi}(-1 - C_{12}\sin t)^2\,dt$$

$$= \frac{1}{2\pi}\left\{\int_0^{\pi}\left(1 - \frac{4}{\pi}\sin t\right)^2 dt + \int_\pi^{2\pi}\left(-1 - \frac{4}{\pi}\sin t\right)^2 dt\right\}$$

$$= \frac{1}{2\pi}\left\{\int_0^{\pi}\left(1 + \frac{16}{\pi^2}\sin^2 t - \frac{8}{\pi}\sin t\right)dt + \int_\pi^{2\pi} 1 + \frac{16}{\pi^2}\sin^2 t + \frac{8}{\pi}\sin t\,dt\right.$$

$$= \frac{1}{2\pi}\left\{ (\pi - 0) + \frac{16}{\pi^2}\int_0^{\pi}\left(\frac{1-\cos 2t}{2}\right)dt + \frac{8}{\pi}\cos t\Big|_0^{\pi} + (2\pi - 0) + \right.$$

$$\frac{16}{\pi^2}\int_{\pi}^{2\pi} \frac{1-\cos 2t}{2}\,dt + \frac{8}{\pi} - \cos t\Big|_{\pi}^{2\pi}$$

$$= \frac{1}{2\pi}\left\{ \pi + \frac{16}{\pi^2}\left\{\left(\frac{1}{2}(\pi - 0)\right) - \frac{1}{2}\left(\frac{\sin 2t}{2}\right)\right|_0^{\pi} + \frac{8}{\pi}\left[-1-1\right] + 2\pi\right.$$

$$+ \frac{16}{\pi^2}\left\{\left(\frac{1}{2}(2\pi - \pi)\right) - \frac{1}{2}\left(\frac{\sin 2t}{2}\right)\right|_{\pi}^{2\pi} +$$

$$\frac{8}{\pi}\left[-(1+1)\right]$$

$$= \frac{1}{2\pi}\left\{ \pi + \frac{8}{\pi} - \frac{1}{4}(0-0) + \frac{8}{\pi}(-2) + 2\pi + \frac{8}{\pi} - \frac{1}{4}(0-0) + \frac{8}{\pi}(-2)\right\}$$

$$= \frac{1}{2\pi}\left\{ \pi + \cancel{\frac{8}{\pi}} - \cancel{\frac{16}{\pi}} + 2\pi + \cancel{\frac{8}{\pi}} - \frac{16}{\pi}\right\}$$

$$= \cancel{\frac{1}{2\pi}\left\{ 3\pi + \frac{16}{\pi}\right\} = \frac{3}{2} = 1.5}$$

$$= \frac{1}{2\pi}\left\{ 3\pi + \frac{16}{\pi}\right\} = \frac{1}{2\pi}\left\{\frac{3\pi^2+16}{\pi}\right\}$$

$$= \frac{3\pi^2+16}{2\pi^2} = 0.189$$

3.12

Approximation using finite Series of Sinusoidal Signal

$x(t) =$

$$G_r = \frac{\int_{t_1}^{t_2} x(t) \cdot g_r(t)\, dt}{\int_{t_1}^{t_2} g_r^2(t)\, dt}$$

$$= \frac{\int_0^{\pi} (1) \sin rt\, dt + \int_{\pi}^{2\pi} (-1) \sin rt\, dt}{\int_0^{2\pi} (\sin rt)^2\, dt}$$

$$= \frac{-\cos rt \Big|_0^{\pi} + \cos rt \Big|_{\pi}^{2\pi}}{\frac{1}{2}\left[t \Big|_0^{2\pi} - \frac{\sin 2rt}{2r} \Big|_0^{2\pi} \right]}$$

$$G_r = \frac{\frac{1}{r}\left[-\cos \pi r + 1 + \cos 2\pi r - \cos \pi r \right]}{\frac{1}{2}\left[2\pi - 0 - 0 \right]}$$

$$= \frac{\frac{1}{r}\left[-(-1)^r + 1 + 1 - (-1)^r \right]}{\pi}$$

$$= \frac{2}{r\pi}\left[1 - (-1)^r \right] \longrightarrow$$

If $r =$ Even Zero

If $r =$ odd Exist

$$G = \begin{cases} \frac{4}{\pi r} & \text{if } r \text{ is odd} \\ 0 & \text{if } r = \text{Even} \end{cases}$$

$$x(t) = \sum_{r=1}^{n} C_r \sin t$$

$$= \frac{4}{\pi}\sin t + \frac{4}{3\pi}\sin 3t + \frac{4}{5\pi}\sin 5t + \ldots$$

Calculate Mean Square Error

We have $$\epsilon = \frac{1}{t_2-t_1}\int_{t_1}^{t_2}\left[x^2(t) - \left\{C_1^2 K_1 + C_2^2 K_2 + \ldots\right\}\right]$$

$$= \frac{1}{t_2-t_1}\left[\int_{t_1}^{t_2} x^2(t) - \sum_{r=1}^{n} C_r^2 K_r\right]$$

$$= \frac{1}{2\pi}\left[\int_0^{2\pi}(1)^2 dt - \sum_{r=1}^{n} C_r^2 K_r\right]$$

$$= \frac{1}{2\pi}\left[\frac{1}{2\pi} - \left[\left(\frac{4}{\pi}\right)^2\pi - \left(\frac{4}{3\pi}\right)^2\cdot\pi \ldots\right]\right]$$

Consider two signals (8) one Sinusoidal Signal:-

$$\epsilon_1 = \frac{1}{2\pi}\left[2\pi - \left(\frac{4}{\pi}\right)^2\pi\right] = 0.189 = 18.9$$

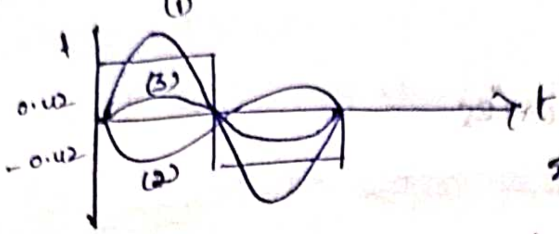
Consider two Sinusoidal Signals :-

$$\epsilon_2 = \frac{1}{2\pi}\left[2\pi - \left(\frac{4}{\pi}\right)^2\pi - \left(\frac{4}{3\pi}\right)^2\cdot\pi\right] = 0.099 = 9.9\%$$

Consider three Sinusoidal Signals :-

$$\epsilon_3 = \frac{1}{2\pi}\left[2\pi - \left(\frac{4}{\pi}\right)^2\pi - \left(\frac{4}{3\pi}\right)^2\pi - \left(\frac{4}{5\pi}\right)^2\pi\right]$$

$$= 0.066 = 6.6\%$$



Orthogonality in Complex function :-

Let us consider two complex functions, the function $x_1(t)$ is approximated using $x_2(t)$ over an interval (t_1, t_2) is

$$x_1(t) = C_{12}\, x_2(t)$$

$$C_{12} = \frac{\int_{t_1}^{t_2} x_1(t)\, x_2(t)\, dt}{\int_{t_1}^{t_2} x_2^2(t)\, dt}$$

$$= \frac{\int_{t_1}^{t_2} x_1(t)\, x_2^*(t)\, dt}{\int_{t_1}^{t_2} x_2(t) \cdot x_2^*(t)\, dt}$$

$x_2^*(t)$ is the Complex Conjugate of $x_2(t)$

Harmonic Analysis **Unit - II**

→ Fourier Series → It should be periodic.

To convert the wave form to Mathematical form.

Representation of fourier series :-

The Representation of signals over certain interval of time in terms of linear combination of orthogonal functions is called fourier series.

→ fourier series is applicable for periodic signals.

Classifications :-

Three types.

1. Trigonometric form. (sin & cos) 2. Cosine form 3. Exponential form

these are in the form of Trigonometric form / Cosine form / Exponential form

Existance of fourier series :-

→ A signal should be periodic.

→ function $x(t)$ must be single valued function (Every instant of time amplitude must be one time exist)

→ function $x(t)$ must be finite number of maxima and minima.

→ function $x(t)$ must be finite number of discontinuity. → Sudden raises / Decreases

→ function $x(t)$ must be (must be) absolutely integral over one period. $\int_0^T |x(t)|\,dt < \infty$

→ these Conditions are called as dirichlets Conditions

Trigonometric fourier Series :-

* A signal $x(t)$ is Represented by $x(t) = a_0 + a_1 \cos w_0 t + a_2 \cos 2w_0 t + a_3 \cos 3w_0 t + \dots b_1 \sin w_0 t + b_2 \sin 2w_0 t + b_3 \sin 3w_0 t + \dots$

$$= a_0 + \sum_{n=1}^{K} a_n \cos n w_0 t + b_n \sin n w_0 t \rightarrow \text{finite Series}$$

Here a_0, a_n, b_n → Fourier Series Coefficients

Evaluation of fourier Series (quo) Coefficients :-

a_0 Evaluation :-

Consider One period of time (t_0, t_0+T) or $(0, T)$ and integrate $(x(t))$ the Eqn of $x(t)$ both Sides.

$$\int_{t_0}^{t_0+T} x(t)\,dt = \int_{t_0}^{t_0+T} \left[\sum_{n=1}^{K} a_n \cos n w_0 t + b_n \sin n w_0 t\right] dt + \int_{t_0}^{t_0+T} a_0\,dt$$

$$= a_0 \cdot t \Big|_{t_0}^{t_0+T} + \sum_{n=1}^{K} \left[\int_{t_0}^{t_0+T} a_n \cos n w_0 t\,dt + \int_{t_0}^{t_0+T} b_n \sin n w_0 t\,dt\right]$$

Integration Over full cycle in Cos & sin $= 0$

$$= a_0 \cdot T +$$

$$a_0 = \frac{1}{T}\int_{t_0}^{t_0+T} x(t)\,dt \quad \text{or} \quad a_0 = \frac{1}{T}\int_{0}^{T} x(t)\,dt$$

a_n Evaluation :-

To Evaluation a_n and b_n we use the following results

$$\int_{t_0}^{t_0+T} \cos n\omega_0 t \cos m\omega_0 t \, dt = \begin{cases} 0 & \text{if } m \neq n \\ T/2 & \text{if } m = n \end{cases}$$ (0 not accepted)

$$\int_{t_0}^{t_0+T} \sin n\omega_0 t \sin m\omega_0 t \, dT = \begin{cases} 0 & \text{if } m \neq n \\ T/2 & \text{if } m = n \end{cases}$$

$$\int_{t_0}^{t_0+T} \sin n\omega_0 t \cdot \cos m\omega_0 t \, dt = 0 \quad \text{forall } m \& n.$$

To find the fourier Series Coefficients a_n, by multiplying the eqn of $x(t)$ by $\cos m\omega_0 t$ and integrate over one period.

$$\int_{t_0}^{t_0+T} x(t) \cos m\omega_0 t \, dt = \int_{t_0}^{t_0+T} a_0 \cos m\omega_0 t\,dt + \sum_{n=1}^{k} \left[\int_{t_0}^{t_0+T} a_n \cos n\omega_0 t \cos \omega_0 t \, dt + \int_{t_0}^{t_0+k} b_n \sin n\omega_0 t \cos m\omega_0 t \, dt \right]$$

Take $m = n$ then

$$\int_{t_0}^{t_0+T} x(t) \cos n\omega_0 t \, dt = a_n \frac{T}{2}$$

$$a_n = \frac{2}{T} \int_0^T x(t) \cos n\omega_0 t \, dt$$

To find the fourier Series Coefficients b_n, by multiplying the Eqn of $x(t)$ by $\sin m\omega_0 t$ and integrate Over one period.

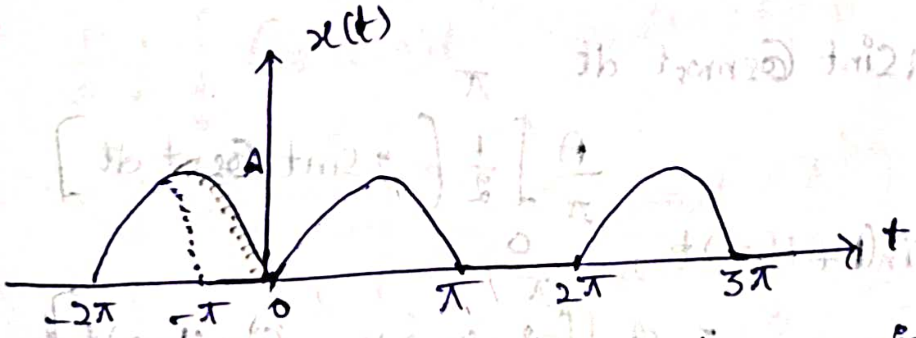
$$\int_{t_0}^{t_0+T} x(t) \sin m\omega_0 t \, dt = \int_{t_0}^{t_0+T} a_0 \sin m\omega_0 t\,dt +$$

$$= \sum_{n=1}^{K} \left[\int_{t_0}^{t_0+T} a_n \cos n\omega_0 t \cdot \sin m\omega_0 t \, dt + \int_{t_0}^{t_0+T} b_n \sin n\omega_0 t \, \sin m\omega_0 t\, d\right]$$

$$= b_n \cdot \frac{T}{2}$$

$$b_n = \frac{2}{T} \int_0^T x(t) \sin m\omega_0 t \, dt$$

→ find the fourier Series Expression of Half wave Rectified sine wave signal.



from the above wave form the period is 2π.

$$x(t) = \begin{cases} A \sin \omega t & 0 \le t \le \pi \\ 0 & \pi < t \le 2\pi \end{cases}$$

fundamental period $T = \frac{2\pi}{\omega}$ $\left(\because \omega_0 = \frac{2\pi}{T}\right)$

$\omega = 2\pi f$

$f = \frac{\omega}{2\pi}$

$T = \frac{2\pi}{\omega}$

$$w_0 = \frac{2\pi}{2\pi} = 1$$

$$\int_0^{\pi} A\sin t$$

The trigonometric F.S is $x(t) = a_0 + \sum_{n=1}^{k} a_n \cos n w_0 t + b_n \sin n w_0 t$

$$a_0 = \frac{1}{T}\int_0^T x(t)\, dt$$

$$= \frac{1}{2\pi}\int_0^{2\pi} A\sin t\, dt = \frac{A}{2\pi}(-\cos t)_0^{\pi}$$

$$= \frac{A}{2\pi}\left[-(-1-1)\right] = \frac{A}{\pi}$$

$$a_n = \frac{2}{T}\int_0^{2\pi} x(t) \cos n w_0 t\, dt$$

$$= \frac{2}{2\pi}\int_0^{2\pi} x(t) \cos n w_0 t\, dt$$

$$= \frac{2}{2\pi}\int_0^{\pi} A\sin t \cos n w_0 t\, dt$$

$$= \frac{A}{\pi}\left[\frac{1}{2}\left[\sin(1+w_0 n)t\right.\right.$$

$$\frac{A}{\pi}\int_0^{\pi}\frac{1}{2}\left[2\sin t \cos nt\, dt\right]$$

$$= \frac{A}{2\pi}\left[\int_0^{\pi}\sin(1+n)t + \sin(1-n)t\, dt\right]$$

$$= \frac{A}{2\pi}\left[\left[\frac{-\cos(1+n)t}{1+n}\right]_0^{\pi} + \left[-\frac{\cos(1-n)t}{1-n}\right]_0^{\pi}\right]$$

$$= -\frac{A}{2\pi}\left[\frac{1}{1+n}\left(\cos(1+n)\pi - 1\right) + \frac{1}{1-n}\left(\cos(1-n)\pi - 1\right.\right.$$

$$= \frac{-A}{2\pi}\left[\frac{(-1)^{1+n} - 1}{1+n} + \frac{(-1)^{1-n} - 1}{1-n}\right]$$

i) if $n =$ Even $\quad a_n = \frac{-A}{2\pi}\left[\frac{-1-1}{n+1} + \frac{-1-1}{1-n}\right]$

$$= \frac{-A}{2\pi}\left[\frac{-2+2n-2n-2}{1-n^2}\right]$$

$$= \frac{4A}{2\pi(1-n^2)} = \frac{2A}{\pi(1-n^2)}$$ Except $n = 1$

ii) if $n =$ odd

$$a_n = 0$$

$$b_n = \frac{2}{T}\int_0^{2\pi} x(t)\sin n\omega_0 t \, dt \qquad (\omega_0 = 1)$$

$$= \frac{2}{2\pi}\int_0^{2\pi} A \sin t \sin nt \, dt$$

$$= \frac{1}{\pi}\int_0^{\pi} A \sin t \sin nt \, dt$$

$$= \frac{A}{\pi}\frac{1}{2}\int_0^{\pi}\left(\cos(1-n)t - \cos(1+n)t\right)dt$$

$(-1)^{1+n}$

$$= \frac{A}{2\pi}\left[\frac{\sin(1-n)t}{1-n}\Big|_0^{\pi} - \frac{\sin(1+n)t}{1+n}\Big|_0^{\pi}\right]$$

$$= \frac{A}{2\pi}\left[\frac{1}{1-n}\left(\sin(1-n)\pi - 0\right) - \frac{\sin(1+n)\pi - 0}{1+n}\right]$$

$$= \frac{A}{2\pi}\left[\frac{\sin(1-n)\pi}{1-n} - \frac{\sin(1+n)\pi}{1+n}\right]$$ → It Exist $b = 1$. only one Condition

$=$ 0 for all values of 'n' Expect $n = 1$

$\therefore \frac{A}{2\pi}$ for $n = 1$

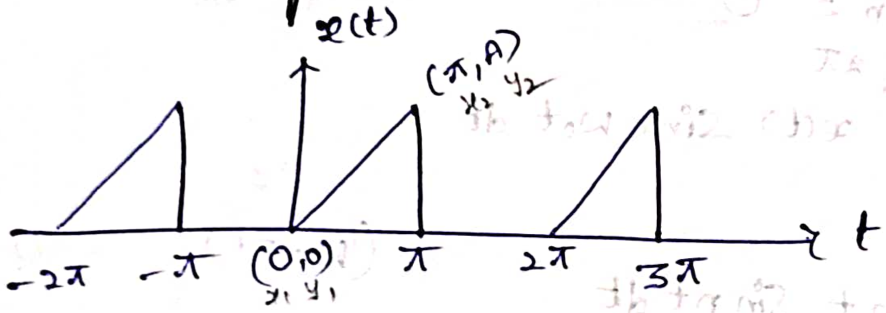
$\therefore$ The fourier Series is

$$x(t) = a_0 + \sum_{n=1}^{k} a_n \cos n w_0 t + b_n \sin n w_0 t$$

$$= a_0 + b_1 \sin w_0 t + \sum_{n=1}^{k} a_n \cos n w_0 t$$

$$= \frac{A}{\pi} + \frac{A}{2} \sin t + \sum_{n=\text{even}} \frac{2A}{\pi(1-n^2)} \cos nt$$

→ obtain the trigonometric fourier Series for the following wave forms.



period (T) = 2π

fundamental period $w_0 = \frac{2\pi}{T} = 1$

$$x(t) = \begin{cases} \left(\frac{A}{\pi}\right) t & \text{for } 0 \le t \le \pi \\ 0 & \text{for } \pi < t \le 2\pi \end{cases}$$

$y - y_1 = m(x - x_1)$

$x(t) - 0 = m(t - 0)$

$x(t) = \frac{A-0}{\pi - 0}(t)$

$x(t) = \frac{At}{\pi}$

$\therefore$ The trigonometric f.s is $x(t) = a_0 + \sum_{n=1}^{k} a_n \cos n w_0 t + b_n \sin n w_0 t$

$$a_0 = \frac{1}{T}\int_0^{T} x(t)\,dt$$

$$a_0 = \frac{1}{2\pi}\int_0^{\pi}\left(\frac{A}{\pi}\right)t\,dt = \frac{A}{2\pi^2}\left[\frac{t^2}{2}\Big|_0^{\pi}\right.$$

$$= \frac{A}{4\pi^2}\left[\pi^2 - 0\right]$$

$$= \frac{A}{4}$$

$$a_n = \frac{2}{T}\int_0^{2\pi} x(t)\cos n\omega_0 t\,dt$$

$$= \frac{2}{2\pi}\int_0^{\pi}\left(\frac{A}{\pi}\right)t\cos nt\,dt \qquad (\omega_0 = 1)$$

$$= \frac{A}{\pi^2}\int_0^{\pi} t\cos nt\,dt$$

$$= \frac{A}{\pi^2}\left[t\left(\frac{\sin nt}{n}\right)\Big|_0^{\pi} + \frac{\cos nt}{n^2}\Big|_0^{\pi}\right]$$

$$= \frac{A}{\pi^2}\left[\frac{t}{n}\left[0-0\right] + \frac{1}{n^2}\left[(-1)^n - 1\right]\right]$$

$$= \frac{A}{n^2\pi^2}\left[(-1)^n - 1\right]$$

If $n =$ odd $\qquad a_n = \dfrac{-2A}{\pi^2 n^2}$

$n =$ Even $\qquad a_n = 0$

$$b_n = \frac{2}{T}\int_0^{2\pi} x(t)\sin n\omega_0 t\,dt$$

$$= \frac{2}{2\pi}\int_0^{\pi}\left(\frac{A}{\pi}\right)t\sin nt\,dt \qquad (\omega_0 = 1)$$

$$\frac{A}{\pi^2}\int_0^\pi t \sin nt \, dt$$

$$= \frac{A}{\pi^2}\left[t\left(\frac{-\cos nt}{n}\right)\Bigg|_0^\pi + \frac{\sin nt}{n^2}\Bigg|_0^\pi \right]$$

$$= \frac{A}{\pi^2}\left[\left[-\frac{t}{n}\left[(-1)^n - 1\right]\right] + 0\right]$$

$$= \frac{A}{\pi^2}\left[\frac{-\left[(-1)^n - 1\right]}{n}\right]$$

$$= \frac{-A\cancel{\pi}(-1)^n}{n\pi^{\cancel{2}}} = -\frac{A(-1)^n}{n\pi^2}$$

OK

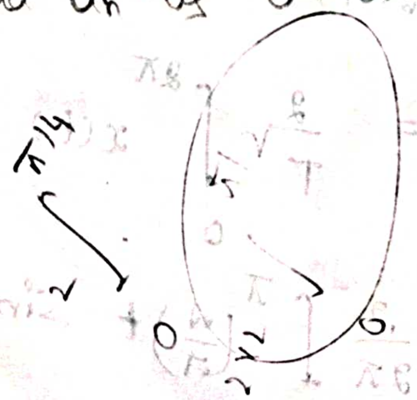
$$\therefore x(t) = \frac{A}{4} + \sum_{n=\text{odd}} \frac{-2A}{(\pi n)^2}\cos nt + \sum_{n=1} \frac{-A(-1)^n}{n\pi} \sin n w_0 t$$

Even → $b_n = 0$, a_0, a_n exist

odd → $a_0, a_n = 0$, b_n exist

* If the given signal is even signal [→ Cos signal] [Even Symmetry] then only a_0 and a_n is existing and $b_n = 0$

* If the given signal is odd signal [→ Sine signal] [odd Symmetry] then a_0 and a_n is 0 and b_n Exist.



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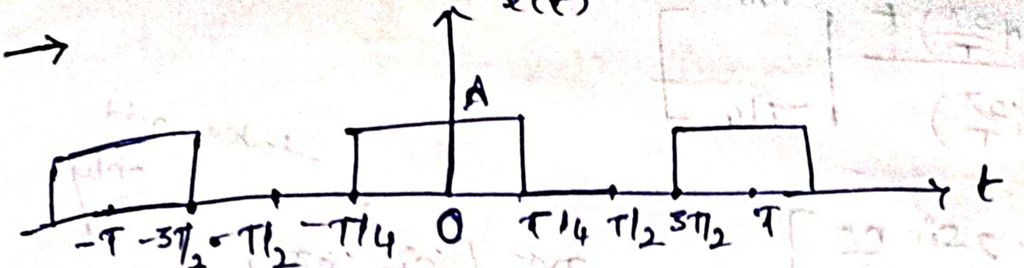


$x(t) = x(-t)$

The given Signal is Even Signal.

only a_0, a_n ⟶ Exist $\quad \omega = \frac{2\pi}{T}$

$b_0 = 0$

$$x(t) = \begin{cases} 0 & \text{for } -T/2 < t \le -T/4 \\ A & \text{for } -T/4 < t \le T/4 \\ 0 & \text{for } T/4 < t \le T/2 \end{cases}$$

$$a_0 = \frac{1}{T}\int_{t_0}^{t_0+T} x(t)\,dt = \frac{1}{T}\int_0^T x(t)\,dt$$

$$= \frac{1}{T}\int_{-T/4}^{T/4} A.dt = \frac{A}{T}\left[T/4 + T/4\right]$$

$$= \frac{A}{\cancel{T}}\cdot\frac{\cancel{2T}}{4_2} = \frac{A}{2}$$

$$a_n = \frac{2}{T}\int_0^T x(t)\,\cos n\omega_0 t\,dt$$

$$= \frac{2}{T}\int_{-T/4}^{T/4} A\cos n\omega_0 t\,dt \qquad \frac{2\pi}{T}$$

$$= \frac{2A}{T}\int_{-T/2}^{-T/2} \cos^n\left(\frac{2\pi}{T}\right) t\,dt$$

$$= \frac{2A}{T}\left[\frac{\sin n\left(\frac{2\pi}{T}\right)t}{n\left(\frac{2\pi}{T}\right)}\Bigg|_{-T/4}^{T/4}\right]$$

$$= \frac{\frac{2A}{T}}{\frac{2\pi n}{T}}\left[\cancel{1}+\cancel{1}\; 2\sin\frac{n\pi}{2}\right] = \frac{2A}{\pi n}\left[\sin\frac{n\pi}{2}\right]$$

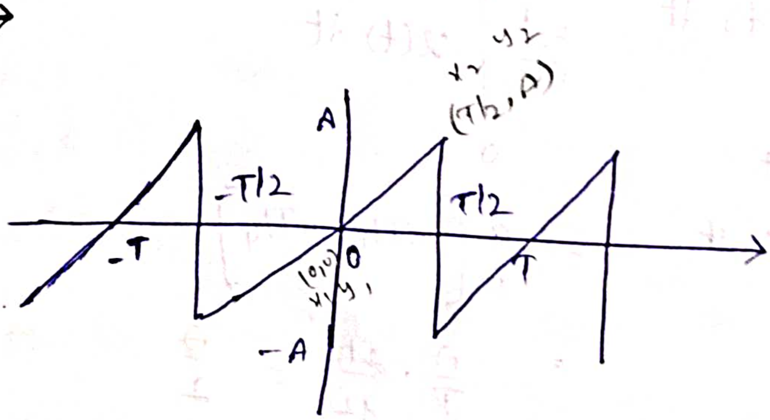
take odd only

$$\therefore x(t) = a_0 + \sum_{n=-\infty}^{\infty} a_n \cos n\omega_0 t + b_n \sin n\omega_0 t$$

$$= \frac{A}{2} + \sum_{n=-\infty}^{\infty} \frac{2A}{n\pi} \sin\left(\frac{n\pi}{2}\right) \cos n\omega_0 t$$

$$= \frac{A}{2} + \frac{2A}{\pi} \sum_{n=-\infty}^{\infty} \frac{1}{n} \sin\left(\frac{n\pi}{2}\right) \cos n\omega_0 t$$

→



Right side is invert image of left side

The above signal $x(t) = x(-t)$

$\therefore a_0 = 0$, $a_n = 0$ b_n Exist

$$b_n = \frac{2}{T}\int_0^T x(t) \sin n\omega_0 t \, dt$$

$$x(t) = \left(\frac{2A}{T}\right) t$$

$y - y_1 = m(x - x_1)$

$x(t) - 0 = m(t - 0)$

$x(t) = mt$

↓

$m = \frac{A - 0}{\frac{T}{2} - 0}$

$= \frac{2A}{T}$

$x(t) = (2A/T)t$

$$b_n = \frac{2}{T}\int_{-T/2}^{T/2} x(t)\ \sin n\omega_0 t\ dt$$

$$\Rightarrow \frac{2}{T}\cdot 2\int_{0}^{T/2}$$

$$= \frac{2}{T}\int_{-T/2}^{T/2}\left(\frac{2A}{T}\right)t\cdot \sin n\left(\frac{2\pi}{T}\right)t\ dt$$

$$= \frac{8A}{T^2}\left[\int_{0}^{T/2} t\cdot \sin n\left(\frac{2\pi}{T}\right)t\,dt\right]$$

$$= \frac{8A}{T^2}\left[-t\cdot\frac{\cos n\left(\frac{2\pi}{T}\right)t}{n\left(\frac{2\pi}{T}\right)}\Bigg|_{0}^{T/2} + \int 1\cdot\frac{\cos n\left(\frac{2\pi}{T}\right)t}{n\left(\frac{2\pi}{T}\right)}\,dt\right]$$

$$= \frac{8A}{T^2}\left[\frac{-\frac{T}{2}\cos n\pi}{n\left(\frac{2\pi}{T}\right)} + \frac{\sin n\left(\frac{2\pi}{T}\right)t}{\left[n\left(\frac{2\pi}{T}\right)\right]^2}\Bigg|_{0}^{T/2}\right]$$

$$= \frac{8A}{T^2}\left[\frac{-\frac{T}{2}(-1)^n}{n\frac{2\pi}{T}}\right] \qquad \frac{8A}{T^2}\times -\frac{T}{2}(-1)^n \times \frac{T}{n\times 2\pi} \qquad \frac{2A}{n\pi}(-1)^n$$

$$= \frac{2A}{n\pi}(-1)^n \rightarrow$$ Both even and odd exist

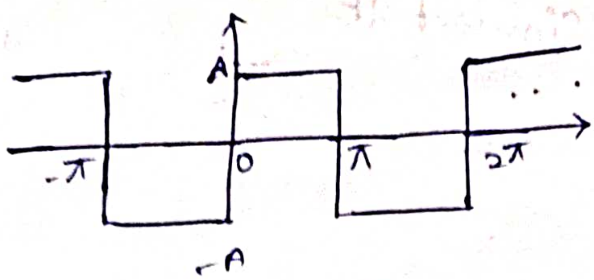
$$\therefore x(t) = a_0 + \sum_{n=-\infty}^{\infty} a_n \cos n\omega_0 t + b_n \sin n\omega_0 t$$

n = 0 means f = 0 i.e., DC Component

$$= \sum_{n=1}^{\infty} \frac{2A}{n\pi}(-1)^n.\ \sin n\left(\frac{2A}{T}\right)$$

< 0 not taken because those values are mirror images of other side.

→



$a_n = 0$, $a_0 = 0$, b_n only exist

$T = 2\pi$; $\omega_0 = \frac{2\pi}{T} = 1$

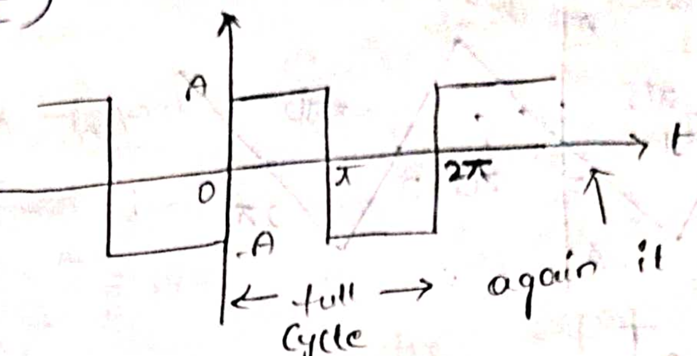
$$x(t) = \begin{cases} A & 0 \le x(t) < \pi \\ -A & \pi < x(t) \le 2\pi \end{cases}$$

$x(t) = -x(t \pm T/2)$ — HWS

$x(t) = -x(t \pm T/4)$ — QWS

Half wave Symmetry :- A periodic signal $x(t)$ with satisfies the following condition $x(t) = -x(t \pm \frac{T}{2})$

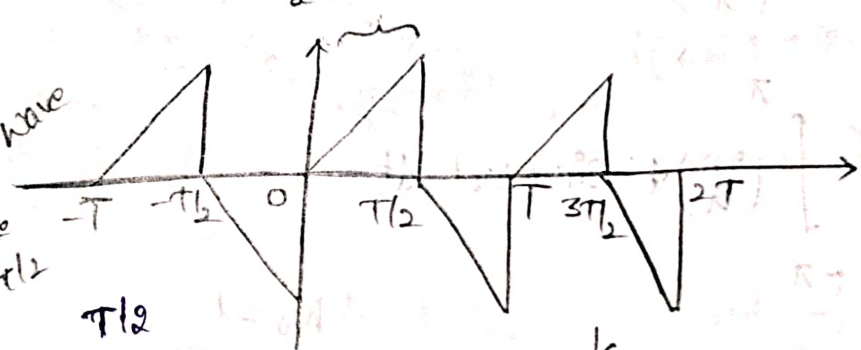
for Ex :-



again it Repeats Here

∴ full wave Symmetry

→ Example :-



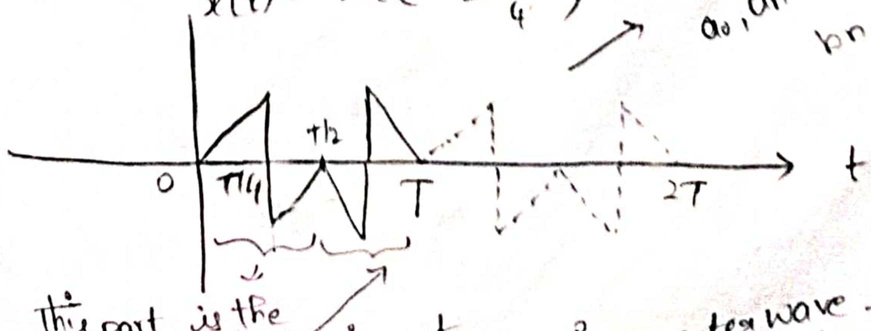
Because of half wave limits only take Half 0 to T/2

$$b_n = \frac{4}{T}\int_0^{T/2} x(t)\,\sin n\omega_0 t\,dt$$

this is odd so a_0, a_n both are '0', b_n Exist

quarter wave Symmetric Signal :-

$x(t) = -x(t \pm \frac{T}{4})$ → this is odd signal a_0, a_n both are '0' b_n only Exist

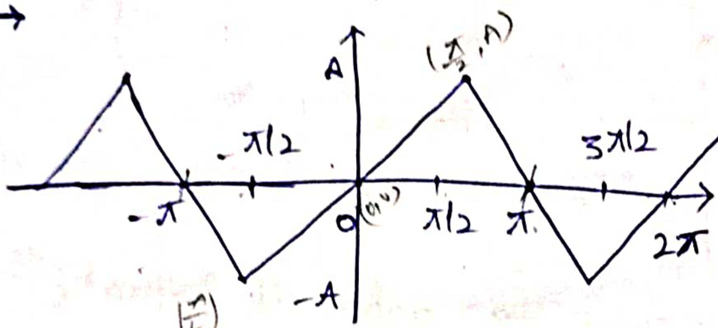


this part is the invert of this part

so it is quarter wave Symmetric Signal

$$b_n = \frac{8}{T}\int_0^{T/4} x(t)\,\sin n\omega_0 t\,dt$$

→



Odd signal

$a_0, a_n = 0$

b_n Exist

$A/t \quad -\pi < t \le -\pi/2$

$-\pi/2 < t \le 0$

$$b_n = \frac{2}{T}\int_0^T x(t)\,\sin n\omega_0 t\,dt$$

$T = \underline{2\pi}$

$\omega = \frac{2\pi}{2\pi} = 1$

$x(t) = \left(\frac{2A}{\pi}\right)t$

$(0,0)$

$(-\pi/2, -A)$

$(-\pi, 0)$

$(-\frac{\pi}{2}, -A)$

$y - y_1 = m(x - x_1)$

$x(t) - 0 = m(t - 0)$

$x(t) = mt$

$m = \frac{A-0}{\frac{\pi}{2}-0} = \frac{2A}{\pi}$

$$= \frac{\cancel{2}}{\cancel{2}\pi}\int_{-\pi}^{\pi}\left(\frac{2A}{\pi}\right)t\,\sin n\omega_0 t\,dt$$

$W_0 = 1$

$$\frac{2A}{\pi^2}\int$$

wave form Converted in to Exponential wave form then it is known as Exponential fourier Series.

Exponential fourier Series :- → It is obtained from Cosine fourier Series

It is most widely used Representation. the function $x(t)$ is Expressed as weighted sum of Complex Exponential functions.

$$x(t) = A_0 + \underbrace{\sum_{n=1}^{\infty} A_n \cos(n\omega_0 t + \theta_n)}_{\text{Cosine fourier Series}}$$

$$= A_0 + \sum_{n=1}^{\infty} A_n \left[\frac{e^{j(n\omega_0 t + \theta_n)} + e^{-j(n\omega_0 t + \theta_n)}}{2} \right]$$

$$= A_0 + \sum_{n=1}^{\infty} \frac{A_n}{2} \left[e^{jn\omega_0 t} \cdot e^{j\theta_n} + e^{-jn\omega_0 t} \cdot e^{-j\theta_n} \right]$$

$$= A_0 + \left[\sum_{n=1}^{\infty} \left(\frac{A_n}{2} e^{j\theta_n} \right) e^{jn\omega_0 t} + \sum_{n=1}^{\infty} \frac{A_n}{2} \left(e^{-j\theta_n} \right) \cdot e^{-jn\omega_0 t} \right]$$

$$= A_0 + \sum_{n=1}^{\infty} \frac{A_n}{2} e^{j\theta_n} \cdot e^{jn\omega_0 t} + \sum_{k=-1}^{\infty} \frac{A_k}{2} e^{j\theta_k} \cdot e^{jk\omega_0 t}]$$

let $A_0 = C_0$ Amplitude phase

$$\therefore \boxed{\frac{A_n}{2} e^{j\theta_n} = C_n} \rightarrow \text{formula to find } C_n$$

$$x(t) = C_0 + \sum_{n=1}^{\infty} C_n e^{jn\omega_0 t} + \sum_{k=-1}^{\infty} C_K e^{jK\omega_0 t}$$

$$\boxed{x(t) = \sum_{n=-\infty}^{\infty} C_n \, e^{jn\omega_0 t}}$$ → Exponential fourier Series.

Determination of fourier Series Coefficients :-

Consider the above Eqn and multiply the both Sides $e^{-jK\omega_0 t}$ integrate over one full Cycle.

$$\int_0^T x(t) \cdot e^{-jK\omega_0 t} dt = \int_0^T \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t} \cdot e^{-jK\omega_0 t} dt$$

$$= \sum_{n=-\infty}^{\infty} C_n \int_0^T e^{jn\omega_0 t} \cdot e^{-jK\omega_0 t} dt$$

If $n \neq K$ then $\int_0^T \underbrace{e^{jn\omega_0 t} \cdot e^{-jK\omega_0 t}}_{\text{orthogonal}} = 0$

$n = K$ then $\int_0^T e^{jn\omega_0 t} \cdot e^{-jK\omega_0 t} = T$

then we get Constant

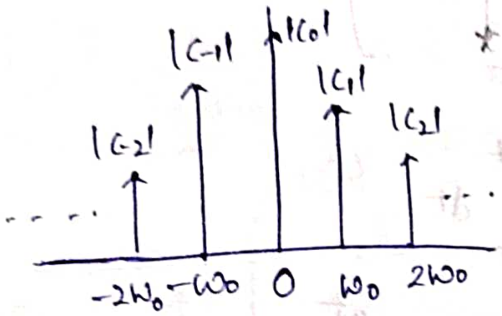
$$\int_0^T x(t) \cdot e^{-jn\omega_0 t} dt = \sum_{n=\infty}^{\infty} C_n \cdot T$$

$$\boxed{C_n = \frac{1}{T} \int_0^T x(t) \, e^{-jn\omega_0 t} dt}$$

Signal is expressed interms of time :- wave form.

Representation of given Signal in the frequency Domine → Spectrum

Spectrum :-



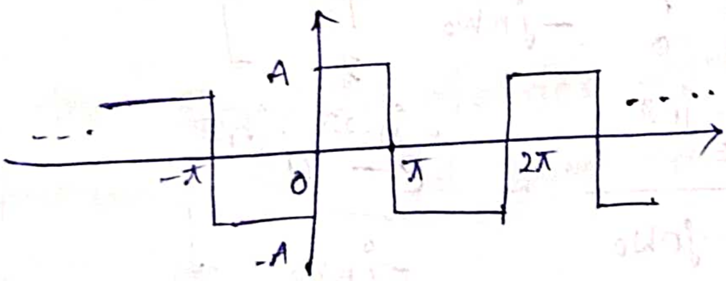
$$x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jnW_0 t}$$ neglect

put n=1

$C_1 \rightarrow W_0$

$C_2 \rightarrow 2W_0$

Problem :- obtain the Exponential fourier Series of following wave form and draw the frequency Spectrum



fundamental frequency $W_0 = \frac{2\pi}{T} = \frac{2\pi}{2\pi} = 1 \quad (T = 2\pi)$

$$\therefore \boxed{x(t) = C_0 + \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} C_n e^{jnW_0 t}}$$

$$C_0 = \frac{1}{T}\int_0^T x(t)\,dt$$

$$x(t) = \begin{cases} A & 0 < t \le \pi \\ -A & \pi < t \le 2\pi \end{cases}$$

$$C_0 = \frac{1}{2\pi}\int_0^{2\pi} A\,dt$$

$$= \frac{1}{2\pi}\left[\int_0^{\pi} A\,dt + \int_{\pi}^{2\pi} -A\,dt\right] = 0$$

$$C_n = \frac{1}{T}\int_0^{T} x(t)\,e^{-jn\omega_0 t}\,dt$$

$$= \frac{1}{2\pi}\left(\int_0^{\pi} A\,e^{-jn\omega_0 t}\,dt + \int_{\pi}^{2\pi} -A\,e^{-jn\omega_0 t}\,dt\right)$$

$$= \frac{A}{2\pi}\left[\frac{e^{-jn\omega_0 t}}{-jn\omega_0}\Bigg|_0^{\pi} - \frac{e^{-jn\omega_0 t}}{-jn\omega_0}\Bigg|_{\pi}^{2\pi}\right]$$

$$= \frac{A}{2\pi}\left[\frac{e^{-jn\pi} - 1}{-jn\omega_0} - \frac{e^{-jn2\pi} - (e)^{-jn\pi}}{-jn\omega_0}\right]$$

$(-1)^n$

$$= \frac{A}{2\pi}\left[\frac{(-1)^n - 1}{-jn\omega_0} - \frac{1-(-1)^n}{-jn\omega_0}\right]$$

$$= \frac{A}{2\pi}\left[\frac{2(-1)^n - 2}{-jn\omega_0}\right] = \frac{A}{\pi}\left[\frac{1-(-1)^n}{jn\omega_0}\right]$$

If n is Even $C_n = 0$

$\therefore$ odd $C_n = -j\dfrac{2A}{\pi n\omega_0}$

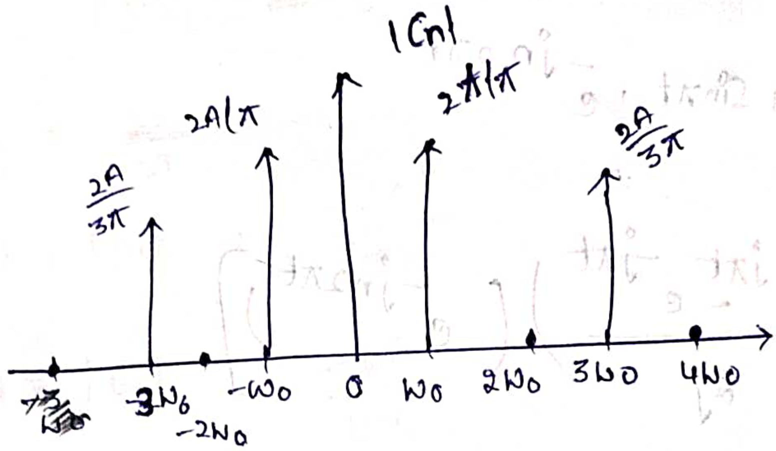
$$x(t) = \sum_{n=\text{odd}} \left(-j \frac{2A}{n\pi w_0} \right) \cdot e^{jnw_0t}$$

Spectrum :-

$C_0 = 0$

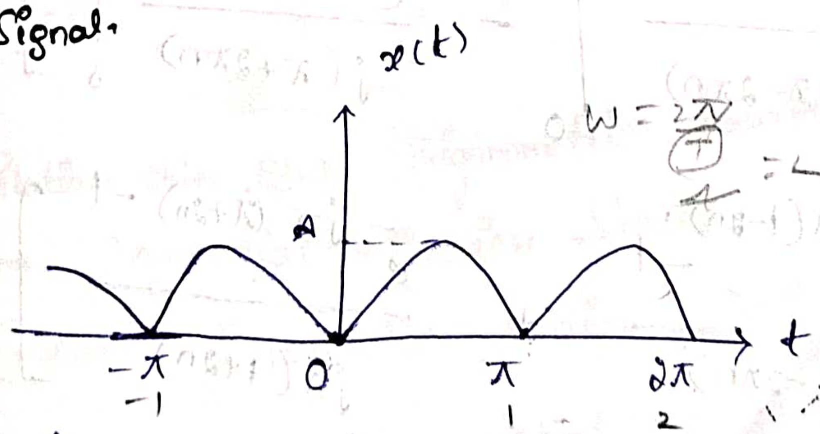
$|C_1| = |C_{-1}| = \frac{2A}{\pi w_0}$; $|C_2| = |C_{-2}| = 0$

$|C_3| = |C_{-3}| = \frac{2A}{3\pi w_0}$; $|C_4| = |C_{-4}| = 0$



frequency Spectrum

→ find the Exponential fourier Series and draw the Spectrum of full wave rectifier output Signal.



$w = \frac{2\pi}{T}$ ← negatives are converted into positives

so, Here $T = \pi$ (1)

$w_0 = \frac{2\pi}{T} = 2\pi$

In $\sin w_0 t \Rightarrow w_0 = \pi$ because of $\sin \pi t$ 2π half cycle

fundamental frequency $w_0 = \frac{2\pi}{T} = \frac{2\pi}{\pi_1} =$

$$C_0 = \frac{1}{\pi} \int_0^{\pi} x(t)\, dt$$

$x(t) = \begin{cases} A \sin w_0 t \\ A \sin \pi t & 0 < t < \pi \end{cases}$

$$= \frac{1}{\pi}\int_0^1 A\sin\pi t \, dt$$

$$= \frac{A}{\pi}\left[-\frac{\cos\pi t}{2}\Big|_0^1\right] = \frac{A}{2\pi}\{1+1\}$$

$$= \frac{2A}{\pi}$$

$$C_n = \frac{1}{T}\int_0^T x(t)\, e^{-jn\omega_0 t} dt$$

$$= \frac{1}{T}\int_0^1 A\sin\pi t \cdot e^{-jn2\pi t}$$

$$= A\left[\int_0^1 \left(\frac{e^{j\pi t}-e^{-j\pi t}}{2j}\right)\left(e^{-jn2\pi t}\right)\right]$$

$$= \frac{A}{2j}\left[\int_0^1 e^{j(\pi-2\pi n)t} - e^{-j(\pi+2\pi n)} dt\right]$$

$$= \frac{A}{2j}\left[\frac{e^{j(\pi-2\pi n)t}}{j(\pi-2\pi n)}\Big|_0^1 - \frac{e^{-j(\pi+2\pi n)}}{-j(\pi+2\pi n)}\Big|_0^1\right]$$

$$= \frac{A}{2j}\left[\frac{e^{j\pi(1-2n)}-1}{j(\pi-2\pi n)} + \frac{e^{-j\pi(\pi+2n)}-1}{j\pi(1+2n)}\right]$$

$$= -\frac{A}{2}\left[\frac{(-1)\cdot 1-1}{1-2n} + \frac{(-1)\cdot 1-1}{1+2n}\right]$$

$$= -\frac{A}{2\pi}\left[\frac{-2}{1-2n} - \frac{2}{1+2n}\right]$$

$$= \frac{2A}{2\pi}\left[\frac{\quad}{1-4n^2}\right]$$

$$= \frac{2A}{\pi(1-4n^2)}$$

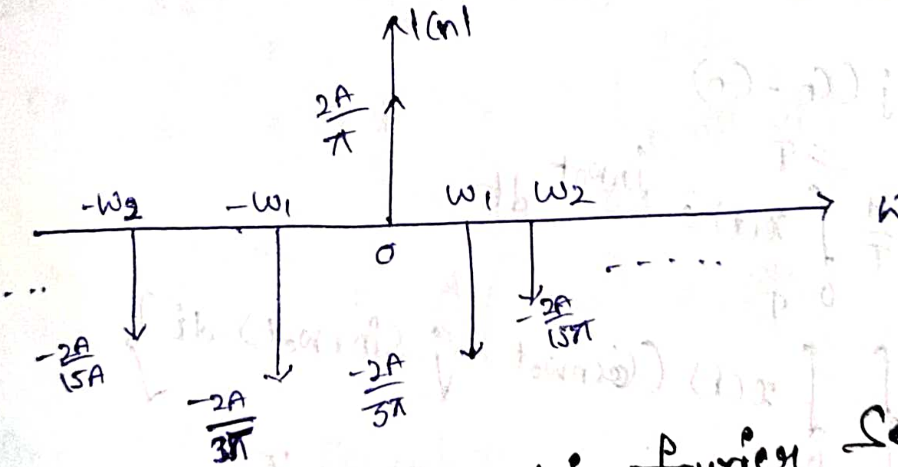
$$x(t) = C_0 + \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}$$

$$= \frac{2A}{\pi} + \sum_{n=-\infty}^{\infty} \frac{2A}{\pi(1-4n^2)} e^{jn\omega_0 t}$$

$$C_0 = \frac{2A}{\pi}$$

$$|C_1| = |C_{-1}| = \frac{-2A}{3\pi}$$

$$|C_2| = |C_{-2}| = \frac{-2A}{15\pi}$$



Relation ship B/W Trigonometric fourier Series and Exponential fourier Series

Consider Exponential fourier Series

$$x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}$$

$$= C_0 + \sum_{n=1}^{\infty} C_n e^{jn\omega_0 t} + \sum_{n=-\infty}^{-1} C_n e^{jn\omega_0 t}$$

$$= C_0 + \sum_{n=1}^{\infty} C_n e^{jn\omega_0 t} + \sum_{n=-\infty} C_n e^{-jn\omega_0 t}$$

$$= C_0 + \sum_{n=1}^{\infty} \left\{ C_n e^{jn\omega_0 t} + C_{-n} e^{-jn\omega_0 t} \right\}$$

$$= C_0 + \sum_{n=1}^{\infty} \Big[C_n (\cos n\omega_0 t + j \sin n\omega_0 t) + C_{-n}(\cos n\omega_0 t - j \sin n\omega_0 t) \Big]$$

$$= C_0 + \sum_{n=1}^{\infty} \Big[\cos n\omega_0 t \, (C_n + C_{-n}) + j \sin n\omega_0 t \, (C_n - C_{-n}) \Big]$$

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + b_n \sin n\omega_0 t$$

$$\therefore \ a_0 = C_0$$

$$a_n = C_n + C_{-n}$$

$$b_n = j(C_n - C_{-n})$$

Imp

$$\rightarrow \ C_n = \frac{1}{T} \int_0^T x(t) e^{-jn\omega_0 t} \, dt$$

$$= \frac{1}{T} \left\{ \int_0^T x(t) (\cos n\omega_0 t - j \sin n\omega_0 t) \, dt \right\}$$

$$= \frac{1}{2} \left[\frac{2}{T} \int_0^T x(t) \cos n\omega_0 t \, dt - j \frac{2}{T} \int_0^T x(t) \sin n\omega_0 t \, dt \right]$$

$$= \frac{1}{2} \left[a_n - j b_n \right]$$

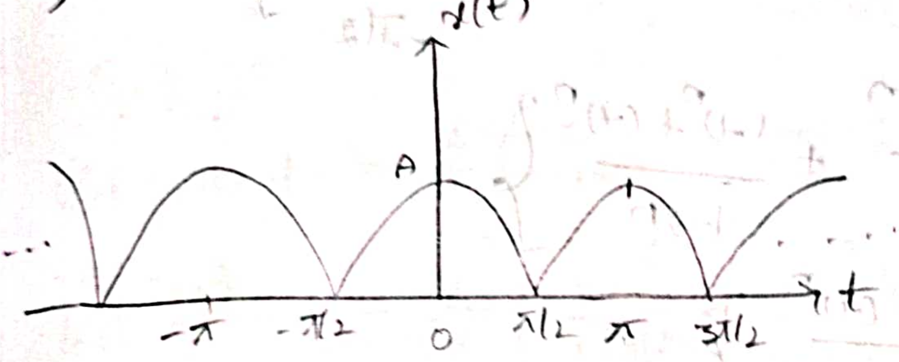
$$\rightarrow \ C_{-n} = \frac{1}{T} \int_0^T x(t) e^{jn\omega_0 t} \, dt$$

$$C_{-n} = \frac{1}{2} \left[a_n + j b_n \right]$$

→ Determine trigonometric fourier series of a full wave Rectifier Cosine function shown in figure.

b) Derive Corresponding Exponential fourier Series.

c) Draw the Complex fourier Spectrum.



$T = \pi$

$\omega_0 = \frac{2\pi}{T} = 2$

$x(t) = A\cos t \qquad -\frac{\pi}{2} < t < \pi/2$

$$a_0 = \frac{1}{T}\int_0^T x(t)\,dt = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} A\cos t\,dt$$

$$= \frac{A}{\pi}\sin t\Big|_{-\pi/2}^{\pi/2} = \frac{A}{\pi}\left[1+1\right] = \frac{2A}{\pi}$$

$$a_n = \frac{2}{T}\int_0^T x(t)\cos n\omega_0 t\,dt$$

$$= \frac{2}{\pi}\int_0^{\pi} A\cos t\cos n\omega_0 t\,dt$$

$$= \frac{2A}{\pi}\left(\cos t\,\frac{\sin n\omega_0 t}{n\omega_0}\Big|_0^{\pi} - \int -\sin t\,\frac{\sin n\omega_0 t}{n\omega_0}\right)$$

$$= \frac{2}{\pi}\cdot\frac{1}{2}\int_0^{\pi} 2A\cos t\cos n\omega_0 t\,dt$$

$$= \frac{2A}{\pi} \int_{-\pi/2}^{\pi/2} \cos(t + n w_0 n t) + \cos(t - n w_0 n t)\, dt$$

$$= \frac{A}{\pi} \int_{-\pi/2}^{\pi/2} \cos(1+2n)t + \cos(1-2n)t\, dt$$

$$= \frac{A}{\pi} \left\{ \left[\frac{\sin(1+2n)t}{1+2n} \right]_{-\pi/2}^{\pi/2} + \left. \frac{\sin(1-2n)t}{1-2n} \right|_{-\pi/2}^{\pi/2} \right\}$$

$$= \frac{A}{\pi} \left\{ \frac{(-1)^n + (-1)^n}{1+2n} + \frac{(-1)^n + (-1)^n}{1-2n} \right\}$$

$\sin(-\pi/2 - n\pi)$

$- \sin(\frac{\pi}{2} + n\pi)$

$- \cos n\pi$

$\downarrow$

$\cos n\pi = (-1)^n$

$$= \frac{2A}{\pi} \left\{ \frac{(-1)^n}{(1+2n)} + \frac{(-1)^n}{(1-2n)} \right\}$$

$\therefore \quad b_n = 0$

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n w_0 t + b_n \sin n w_0 t$$

$$= \frac{2A}{\pi} + \sum_{n=1}^{\infty} \left(\frac{(-1)^n}{1+2n} + \frac{(-1)^n}{1-2n} \right) \frac{2A}{\pi} \cos n w_0 t$$

Exponential series :-

$$a_n = C_0 = \frac{2A}{\pi}$$

$$C_n = \frac{1}{2} \left(a_n - j b_n \right]$$

$$= \frac{1}{2} \left(a_n \right]$$

$$= \frac{1}{2} \left[\frac{2A}{\pi} \left(\frac{(-1)^n}{1+2n} + \frac{(-1)^n}{1-2n} \right) \right]$$

$$= \frac{A}{\pi} \left(\frac{(-1)^n}{1+2n} + \frac{(-1)^n}{1-2n} \right)$$

$$\therefore x(t) = C_0 + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} C_n e^{jn\omega_0 t}$$

$$= \frac{2A}{\pi} + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{A}{\pi}\left(\frac{(-1)^n}{1+2n} + \frac{(-1)^n}{1-2n}\right) \cdot e^{jn\omega_0 t}$$

ii) frequency Domine

$$-\frac{1}{3}+1 \quad \frac{-1+3}{3}$$

$$C_0 = \frac{2A}{\pi}$$

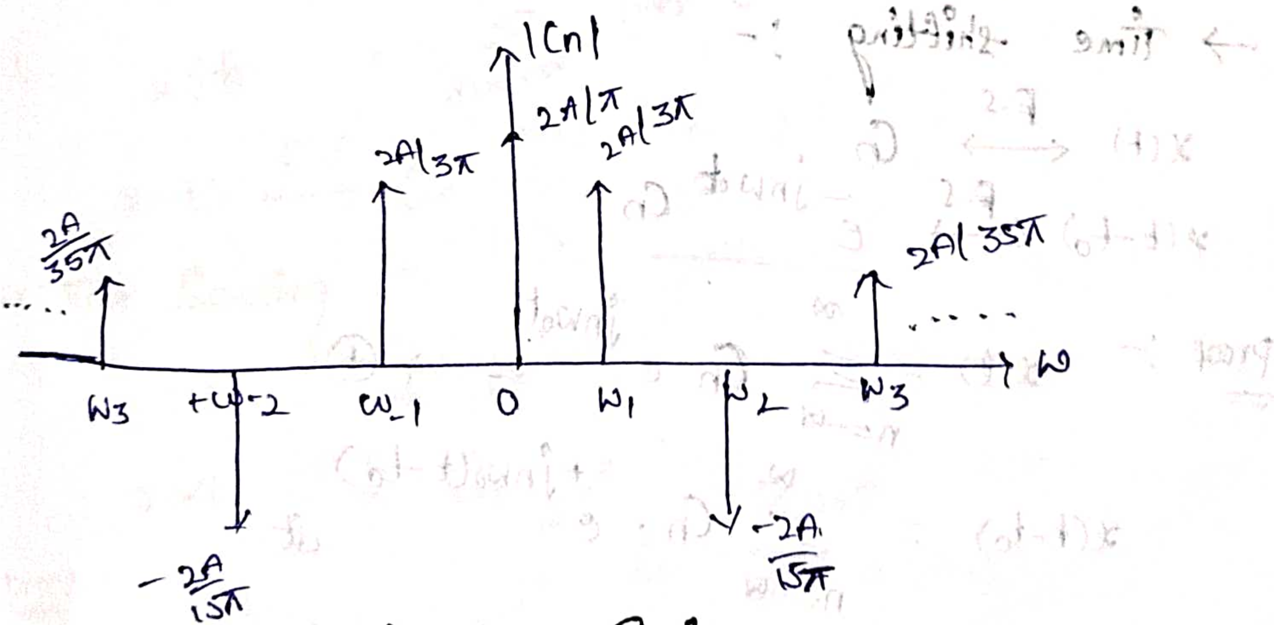
$$|C_1| = |C_{-1}| = \frac{A}{\pi}\left(\frac{-1}{1+2} + \frac{-1}{1-2}\right) = \frac{+2A}{3\pi}$$

$$|C_2| = |C_{-2}| = \frac{A}{\pi}\left(\frac{(-1)^2}{1+2(2)} + \frac{(-1)^2}{1-4}\right) = \frac{1}{5} + \frac{1}{-3}$$

$$= -\frac{2A}{15\pi}$$

4

$$|C_3| = |C_{-3}| = \frac{A}{\pi}\left(\frac{(-1)^3}{1+6} + \frac{(-1)^3}{1-6}\right) = \frac{2A}{35\pi}$$



properties of fourier Series :-

let us Consider two periodic Signals $x_1(t)$ and $x_2(t)$ with period T and the Corresponding fourier Series Coefficients C_n and D_n.

(i) Linearity :-

$\rightarrow x_1(t) \xleftrightarrow{F.S} C_n$

$x_2(t) \xleftrightarrow{F.S} D_n$

$A(x_1(t)) + B x_2(t) \xleftrightarrow{F.S} A C_n + B D_n$

proof :- $C_n = \frac{1}{T}\int_0^T x(t)\, e^{-jn\omega_0 t}\, dt$

$$F.S[A x_1(t) + B x_2(t)] = \frac{1}{T}\int_0^T (A x_1(t) + B x_2(t))\, e^{-jn\omega_0 t}\, dt$$

$$= \frac{1}{T}\int_0^T A x_1(t)\, e^{-jn\omega_0 t}\, dt + \frac{1}{T}\int_0^T B x_2(t)\, e^{-jn\omega_0 t}\, dt$$

$$= A C_n + B D_n.$$

$\rightarrow$ Time shifting :-

$x(t) \xleftrightarrow{F.S} C_n$

$x(t-t_0) \xleftrightarrow{F.S} e^{-jn\omega_0 t_0} C_n$

proof :- $x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t} \rightarrow ①$

$$x(t-t_0) = \sum_{n=-\infty}^{\infty} C_n \cdot e^{+jn\omega_0 (t-t_0)}\, dt$$

$$= \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t} \cdot e^{-jn\omega_0 t_0}$$

$$= \sum_{n=-\infty}^{\infty} (C_n e^{-jn\omega_0 t_0})\, e^{jn\omega_0 t} \rightarrow ②$$

→ Time Reversal :-

$$x(t) \xleftrightarrow{F.S} C_n$$

$$x(-t) \xleftrightarrow{F.S} C_{-n}$$

proof :- $$x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t} \rightarrow ①$$

$$x(-t) = \sum_{n=-\infty}^{\infty} C_n \cdot e^{-jn\omega_0 t}$$

Consider $n = -p$ then the above Eqn is

$$x(-t) = \sum_{p=\infty}^{-\infty} C_{-p} \cdot e^{jp\omega_0 t}$$

$$= \sum_{p=-\infty}^{\infty} C_{-p}\, e^{jp\omega_0 t}$$

$p = n$

$$x(-t) = \sum_{n=-\infty}^{\infty} C_{-n}\, e^{jn\omega_0 t} \rightarrow ②$$

$$x(-t) \xleftrightarrow{F.S} C_{-n}$$

→ Time Scaling :-

$$x(t) \xleftrightarrow{F.S} C_n$$

$$x(\alpha t) \xleftrightarrow{F.S} C_n$$

proof : $$x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t} \rightarrow ①$$

$$x(\alpha t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 \alpha t}$$

$$= \sum_{-\infty}^{\infty} C_n e^{jn(\alpha\omega_0)t} \rightarrow ②$$

frequency Coefficient is change, fourier Series Coefficient does not change (C_n)

Time Differentiation :-

$$x(t) \xleftrightarrow{F.S} C_n$$

$$\frac{d}{dt} x(t) \xleftrightarrow{F.S} jn\omega_0 \cdot C_n$$

proof :- $x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t} dt \rightarrow ①$

$$\frac{d}{dt} x(t) = \sum_{n=-\infty}^{\infty} C_n \frac{d}{dt} e^{jn\omega_0 t} dt$$

$$= \sum_{n=-\infty}^{\infty} C_n \cdot \frac{e^{jn\omega_0 t}}{\cancel{jn\omega_0}} (jn\omega_0)$$

$$= \sum_{n=-\infty}^{\infty} (C_n jn\omega_0) e^{jn\omega_0 t} \longrightarrow ②$$

$$\therefore \frac{d}{dt} x(t) \xleftrightarrow{F.S} C_n \cdot jn\omega_0$$

:- Fourier transforms :-

It is a transformation technique. ~~It~~ which transform Signal from time domine to Corresponding frequency domine and vice versa.

It is applicable for periodic and non-periodic signals.

→ fourier transform Can be developed by finding the fourier Series of a periodic function and Substitute ~~T~~ tend to ∞ $(T \rightarrow \infty)$

Continuous Time Fourier transform (CTFT) :-

$$F[x(t)] = X(j\omega) = \int_{-\infty}^{\infty} x(t) \cdot e^{-j\omega t}\, dt$$

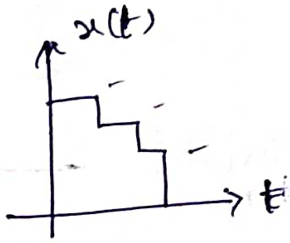
$$F^{-1}[X(j\omega)] = x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega t}\, d\omega$$

Existance of F.T (Dirichlet's Conditions) :-

i) Signal $x(t)$ must be absolutely integrable.

$$\int_{-\infty}^{\infty} x(t)\, dt < \infty$$

ii) $x(t)$ has finite number of Discontinuities in every finite interval.



3 times → finite

iii) $x(t)$ has finite number of maxima and minima in every finite interval of time.

max − 3, min − 2 } finite

→ The above three Conditions are Sufficient Conditions but not necessary Conditions.

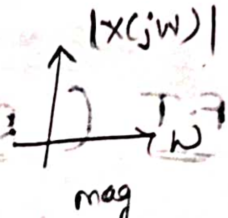
necessary Conditions are Compulsory Conditions.

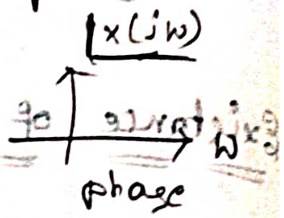
Fourier transform of Standard Signal :-

$$|X(j\omega)| = \sqrt{(X_1(j\omega))^2 + X_2(j\omega)^2}$$

$$\angle X(j\omega) = \tan^{-1}\left(\frac{b}{a}\right)$$

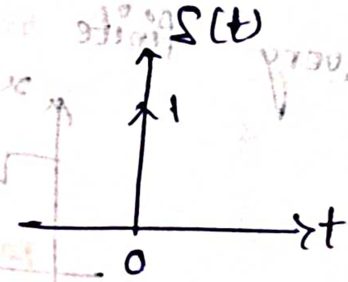
Combination of phase and magnitude
↓
Together known as frequency domine.





Impulse (~~Signa~~) function $\delta(t)$:-

$$\delta(t) = \begin{cases} 1 & \text{for } t = 0 \\ 0 & \text{for } t \neq 0 \end{cases}$$



$$X(j\omega) = \int_{-\infty}^{\infty} \delta(t) \cdot e^{-j\omega t}\, dt$$

$$= e^{-j\omega t}\Big|_{t=0} = 1$$

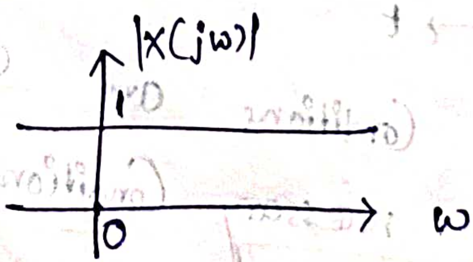
The fourier transform of Impulse function is 1

i.e., $1 + j(0)$

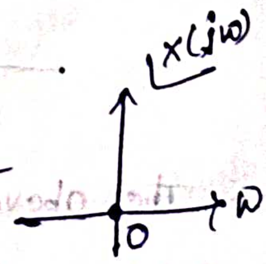
magnitude $|X(j\omega)| = 1$

$$\angle X(j\omega) = \tan^{-1}(0) = 0$$

magnitude plot →

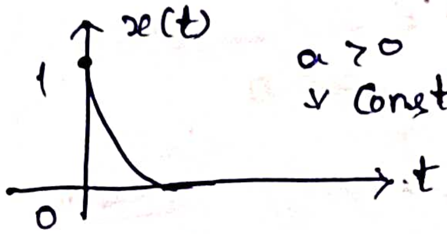


phase plot :-



Single Sided Exponential function :-

$$x(t) = e^{-at} u(t)$$ It Exist only in positive sides.



$a > 0$ Constant

fourier transform $X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t} dt$

$$= \int_{-\infty}^{\infty} e^{-at} u(t)\, e^{-j\omega t} dt$$

$$= \int_{0}^{\infty} e^{-at} \cdot e^{-j\omega t} dt = \int_{0}^{\infty} e^{-(a+j\omega)t} dt$$

$$= \left. \frac{e^{-(a+j\omega)}}{-(a+j\omega)} \right|_0^{\infty} = \frac{1}{a+j\omega}$$

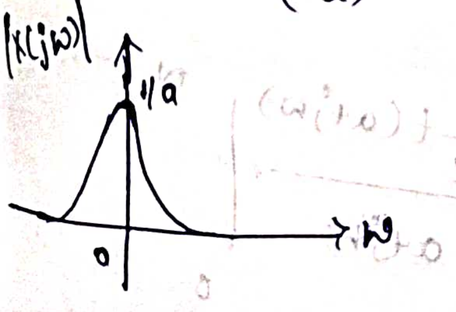
$$X(j\omega) - : \frac{a-j\omega}{a^2+\omega^2}$$

magnitude plot :- $|X(j\omega)| = \sqrt{\left(\frac{a}{a^2+\omega^2}\right)^2 + \left(\frac{\omega}{a^2+\omega^2}\right)^2}$

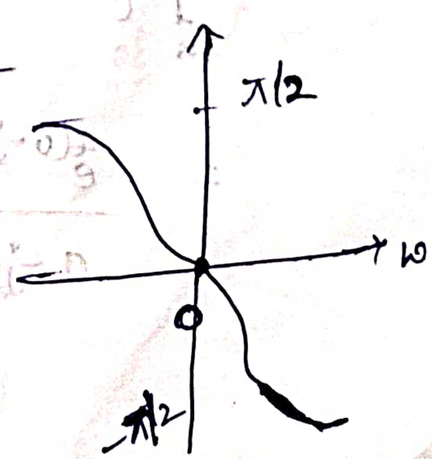
$$= \sqrt{\frac{a^2+\omega^2}{(a^2+\omega^2)^2}} = \frac{1}{\sqrt{a^2+\omega^2}}$$

phase plot :- $\angle X(j\omega) = \tan^{-1}\left(\frac{-\frac{\omega}{\cancel{a^2+\omega^2}}}{\frac{a}{\cancel{a^2+\omega^2}}}\right)$

$$= \tan^{-1}\left(-\frac{\omega}{a}\right) \Rightarrow -\tan^{-1}\left(\frac{\omega}{a}\right)$$

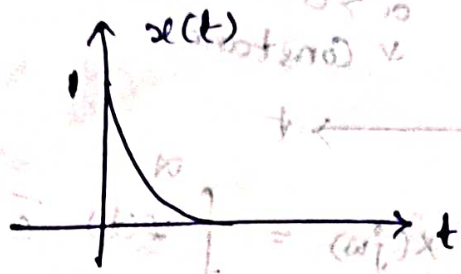


phase plot :-



Double Sided Exponential function :-

The Double sided Exponential function exists both sides of the axis.



$$x(t) = e^{-a|t|}$$

$|t| = +t$ for $t \geq 0$

$-t$ for $t < 0$

$$x(t) = \begin{cases} e^{-at} & t \geq 0 \\ e^{at} & t < 0 \end{cases}$$

$$x(t) = e^{-at} u(t) + e^{at} u(-t)$$

$$F[x(t)] = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

$$= \int_{-\infty}^{\infty} \left[e^{-at} u(t) + e^{at} u(-t) \right] e^{-j\omega t} dt$$

$$= \int_{-\infty}^{0} e^{+at} u(-t) e^{-j\omega t} dt + \int_{0}^{\infty} e^{-at} u(t) e^{-j\omega t} dt$$

$$= \int_{-\infty}^{0} e^{t(a-j\omega)} dt + \int_{0}^{\infty} e^{-t(a+j\omega)} dt$$

$$= \left. \frac{e^{t(a-j\omega)}}{a-j\omega} \right|_{-\infty}^{0} + \left. \frac{e^{-t(a+j\omega)}}{a+j\omega} \right|_{0}^{\infty}$$

$$F[x(t)] = \frac{1-0}{(a-j\omega)} + \frac{0-1}{-(a+j\omega)}$$

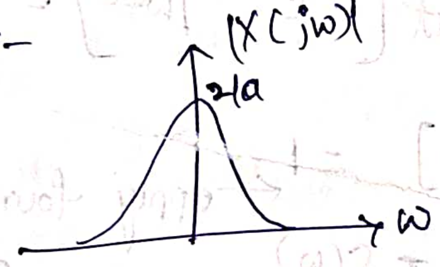
$$= \frac{1}{a-j\omega} + \frac{1}{a+j\omega} = \frac{2a}{a^2+\omega^2}$$

$$X(j\omega) = \frac{2a}{a^2+\omega^2}$$

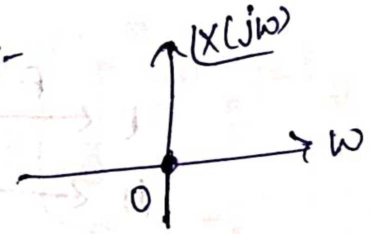
$$|X(j\omega)| = \frac{2a}{a^2+\omega^2}$$

$$\angle X(j\omega) = \tan^{-1}\left(\frac{0}{2a/a^2+\omega^2}\right) = 0$$

magnitude plot :-



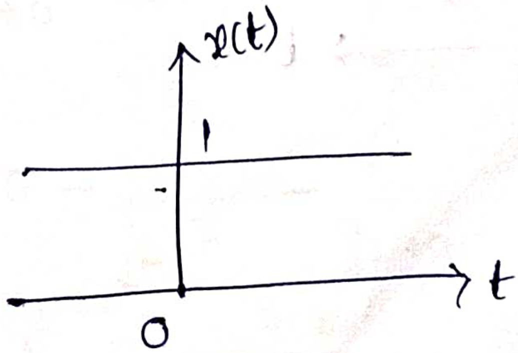
phase plot :-



Constant amplitude (DC term) :-

→ Consider $x(t) = 1$ It is not absolutely integrable. So, we cannot find fourier transform Directly.

→ The fourier transform of $x(t) = 1$ is calculated from inverse fourier transform of $\delta(\omega)$.



$x(t) = 1 \quad (k)$

$$\delta(\omega) = \begin{cases} 1 & \omega = 0 \\ 0 & \omega \neq 0 \end{cases}$$

$$\boxed{x(t) = F^{-1}(X(j\omega)) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega t}\, d\omega}$$

$$F^{-1}[X(j\omega)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(\omega)\, e^{j\omega t}\, d\omega$$

sub $\omega = 0$

$$= \frac{1}{2\pi}\left(e^{j\omega t} \Big|_{\omega=0} \right) = \frac{1}{2\pi}$$

$F^{-1}[2\pi(\delta(\omega))] = 1$ ← apply fourier transform on both sides

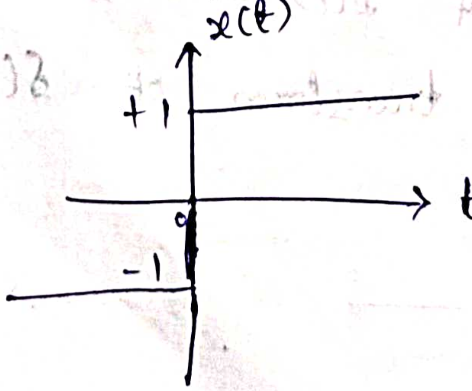
$F[1] = 2\pi\,\delta(\omega)$

$$1 \xleftrightarrow{F.T} 2\pi\,\delta(\omega)$$

$$A \longleftrightarrow 2\pi A\,\delta(\omega)$$

Signum function :-

$$\operatorname{sgn}(t) = \begin{cases} 1 & \text{for } t \geq 0 \\ -1 & \text{for } t < 0 \end{cases}$$



→ The above signal is not absolutely integrable. So, the fourier transform cannot form Directly.

→ let us consider the function

$$e^{-a|t|}\,\mathrm{sgn}(t) \quad \text{and} \quad \lim_{a\to 0}$$

$$\boxed{x(t) = \mathrm{Sgn}(t) = \lim_{a\to 0} e^{-a|t|}\,\mathrm{Sgn}(t)}$$

$$\mathrm{Sgn}(t) = \lim_{a\to 0}\left[e^{-at}u(t) + e^{at}u(-t)\right]$$

multiply +1 −1

Signum Both +ve and −ve

$$= \lim_{a\to 0} e^{-at}u(t) - e^{+at}u(-t)$$

$$F[\mathrm{sign}(t)] = \lim_{a\to 0}\int_{-\infty}^{\infty}\left[e^{-at}u(t) - e^{at}u(-t)\right]e^{-j\omega t}\,dt$$

$$= \lim_{a\to 0}\left[\int_{0}^{\infty} e^{-at}e^{-j\omega t}\,dt - \int_{-\infty}^{0} e^{at}e^{-j\omega t}\,dt\right]$$

$$= \lim_{a\to 0}\left[\int_{0}^{\infty} e^{-t(a+j\omega)}\,dt - \int_{-\infty}^{0} e^{t(a+j\omega)}\,dt\right]$$

$$= \lim_{a\to 0}\left[\frac{e^{-t(a-j\omega)}}{-(a-j\omega)}\Bigg|_{0}^{\infty} - \frac{e^{t(a+j\omega)}}{a+j\omega}\Bigg|_{-\infty}^{0}\right]$$

$$= \lim_{a\to 0}\frac{e^{-t(a-j\omega)}}{j\omega - a}\Bigg|_{0}^{\infty} - \frac{e^{t(a+j\omega)}}{j\omega+a}\Bigg|_{-\infty}^{0}$$

$$= 0 - \frac{1}{j\omega - a} - \frac{1}{j\omega + a} \neq \frac{j\omega+a - j\omega + a}{(j\omega)^2 - (a)^2}$$

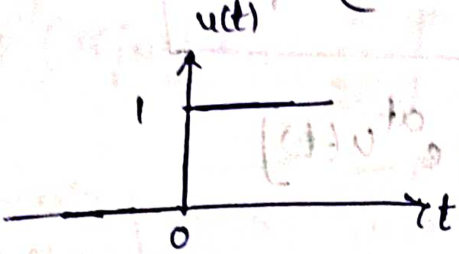
$$= \frac{-1}{j\omega} - \frac{1}{j\omega} = \frac{-2}{j\omega} \quad \text{but ans} \rightarrow \frac{2}{j\omega} \neq \frac{2a}{(j\omega)^2 - a^2}$$

magnitude $|X(j\omega)| = \frac{2}{\omega}$

phase angle $\angle X(j\omega) = -\tan^{-1}\left(\frac{2/\omega}{0}\right) = \pi/2$

→ **unit step function** :-

$$u(t) = \begin{cases} 1 & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases}$$

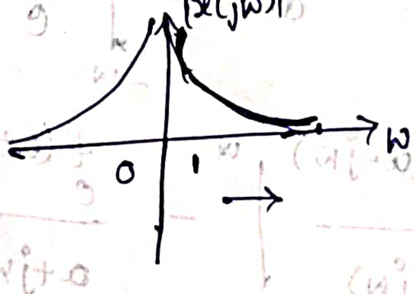


$u(t) = \frac{1}{2} + \frac{1}{2}\, sgn(t)$

$$F[u(t)] = F[1/2] + F[1/2\, Sgn(t)]$$
$$= \frac{1}{2}[2\pi\delta(\omega)] + \frac{1}{2}\left[\frac{2}{j\omega}\right]$$
$$= \pi\delta(\omega) + \frac{1}{j\omega}$$
$$= \pi\delta(\omega) - j\frac{1}{\omega}$$

$\delta(0) = 1$
$\delta(\text{else}) = 0$

$|X(j\omega)| = \sqrt{(\pi\delta(\omega))^2 + \left(\frac{1}{\omega}\right)^2}$



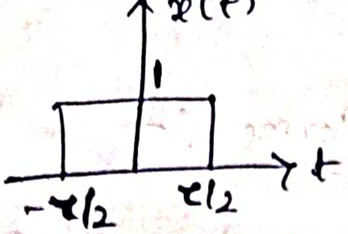
If ω is increased
decreases
$\omega = \alpha$
0

imp *

Rectangular pulse :- rect(t) & $\pi(t/\tau)$

$$rect(T) = \begin{cases} 1 & |t| \leq \tau/2 \\ 0 & \text{otherwise} \end{cases} \qquad -\tau/2 \leq t \leq \tau/2$$

If we apply fourier transform of Rectangular Signal then we will get sinc function.



$$x(j\omega) = \int_{-\infty}^{\infty} x(t)\cdot e^{-j\omega t}\, dt = \int_{-\tau/2}^{\tau/2} 1\cdot e^{j\omega t}\, dt$$

$$= \frac{e^{-j\omega t}}{-j\omega}\Bigg|_{-\tau/2}^{\tau/2} = -\frac{1}{j\omega}\left[e^{-j\omega\tau/2} - e^{j\omega\tau/2}\right]$$

N/D multiply by $\tau/2$

$$= +\frac{\tau/2}{j\omega\frac{\tau}{2}}\left[e^{j\omega\tau/2} - e^{-j\omega\tau/2}\right]$$

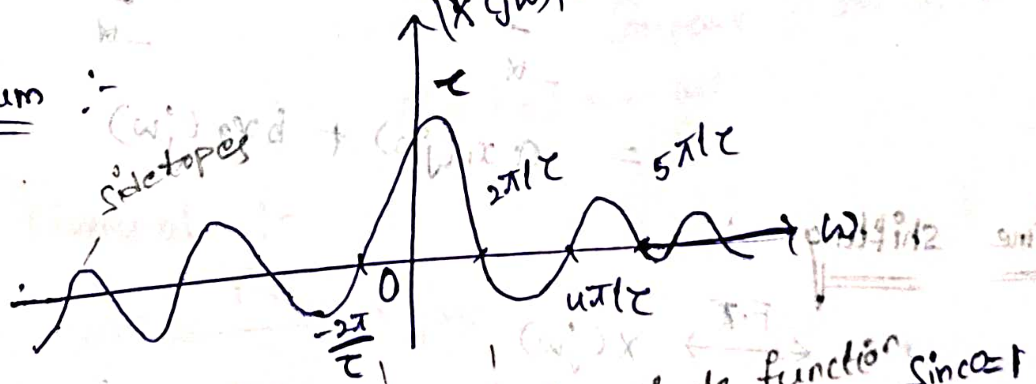
$$= \frac{\tau}{\omega\frac{\tau}{2}}\left[\frac{e^{j\omega\tau/2} - e^{-j\omega\tau/2}}{2j}\right]$$

$$= \tau\frac{(\sin\omega\tau/2)}{\omega\tau/2} = \tau\,\mathrm{Sinc}(\omega\tau/2)$$

$\frac{\sin\theta}{\theta} \rightarrow$ Sincfunc.

Here τ is Constant.

Spectrum :-



Max voltage is available in single function $\mathrm{Sinc}\,0 = 1$

(τ)

→ The Range of frequencies that is allowed by the System is known as Band width

Cosine wave :-

properties of fourier transforms :-

→ linearity :- It states that the fourier transform of weighted sum of two signals is equal to weighted sum of individual fourier transforms.

$$a\,x_1(t) \xleftrightarrow{F.T} a\,x_1(j\omega)$$

$$b\,x_2(t) \xleftrightarrow{F.T} b\,x_2(j\omega)$$

$$a\,x_1(t) + b\,x_2(t) \xleftrightarrow{F.T} a\,x_1(j\omega) + b\,x_2(j\omega)$$

∴ Apply fourier transform

$$F\left[a x_1(t) + b x_2(t)\right] = \int_{-\infty}^{\infty} x(t) \cdot e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} \left[a x_1(t) + b x_2(t)\right] e^{-j\omega t}\, dt$$

$$= a \int_{-\infty}^{\infty} x_1(t)\, e^{-j\omega t}\, dt + b \int_{-\infty}^{\infty} x_2(t)\, e^{-j\omega t}\, dt$$

$$= a\,x_1(j\omega) + b\,x_2(j\omega)$$

Time shifting :-

$$x(t) \xleftrightarrow{F.T} X(j\omega)$$

$$x(t-t_0) \longleftrightarrow e^{-j\omega t_0} X(j\omega)$$

$$X(j\omega) = F[x(t)] = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$F[x(t-t_0)] = \int_{-\infty}^{\infty} x(t-t_0)\, e^{-j\omega t}\, dt$$

let $t - t_0 = P$ $x(t-2) = e^{-j\omega 2} X(j\omega)$

$dt = dP$ (t_0 is Constant) $x(t+5) = e^{j\omega 5} X(j\omega)$

like this

$$F[x(t-t_0)] = \int_{-\infty}^{\infty} x(P)\, e^{-j\omega(P+t_0)}\, dP$$

$$= \underbrace{\int_{-\infty}^{\infty} x(P)\, e^{-j\omega P}\, dP}_{x(P)} \cdot e^{-j\omega t_0} \longrightarrow x(t) = X(j\omega)$$

Time $\rightarrow$

$$= e^{-j\omega t_0} X(j\omega)$$

Frequency Shifting :-

ignore (no problem)

frequency

$$x(t) \xleftrightarrow{F.T} X(j\omega)$$

$$e^{j\omega_0 t}\, x(t) \xleftrightarrow{F.T} X(\omega - \omega_0)$$

$e^{j5t} \longleftrightarrow X(\omega - 5)$

$e^{-j5t} \longleftrightarrow X(\omega + 5)$

$$X(j\omega) = F[x(t)] = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt \rightarrow (1)$$

$e^{-j\frac{1}{2}t} \longleftrightarrow X(\omega + \frac{1}{2})$

like this

$$F[e^{j\omega_0 t} x(t)] = \int_{-\infty}^{\infty} x(t)\, e^{j\omega_0 t} \cdot e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} x(t)\, e^{-j(\omega - \omega_0)t}\, dt \rightarrow (2)$$

Compare (1) and (2) we get

$$= X(\omega - \omega_0)$$

Time Reversal :-

$$x(t) \xleftrightarrow{F.T} X(j\omega)$$

$$x(-t) \xleftrightarrow{F.T} X(-j\omega)$$

$$X(j\omega) = F[x(t)] = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$F[x(-t)] = \int_{-\infty}^{\infty} x(-t)\, e^{-j\omega t}\, dt = \int_{-\infty}^{\infty} x(t)\, e^{-j(-\omega)t} = X(-j\omega)$$

Time Scaling :-

$$x(t) \xleftrightarrow{F.T} X(j\omega)$$

$$x(at) \xleftrightarrow{F.T} \frac{1}{|a|} X\left(\frac{j\omega}{a}\right)$$

$$X(j\omega) = F[x(t)] = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \rightarrow (1)$$

$$F[x(at)] = \int_{-\infty}^{\infty} x(at) e^{-j\omega t} dt$$

$$at = p$$

$$t = p/a \quad dt = \frac{1}{a} dp$$

$$F[x(at)] = \int_{-\infty}^{\infty} x(p) e^{-j\omega p/a} \cdot dp/a$$

$$= \frac{1}{a} \int_{-\infty}^{\infty} x(p) \cdot e^{-\left(\frac{j\omega}{a}\right) p} dp$$

Case : 1 $a > 0$

$$F[x(at)] = \frac{1}{a} \int_{-\infty}^{\infty} x(p) e^{-j(\omega/a)p} dp \rightarrow (2)$$

$$= \frac{1}{a} X\left(\frac{j\omega}{a}\right)$$

Case : 2 $a < 0$

$$F[x(at)] = -\frac{1}{a} \int_{-\infty}^{\infty} x(p) e^{-j(\omega/-a)p} \cdot (-dp)$$

$$F[x(a)] = \frac{1}{a} \int_{-\infty}^{\infty} x(p) e^{-j\left(\frac{\omega}{a}\right)p} \mp dp \rightarrow (3)$$

$$= \frac{1}{a} X\left(\frac{j\omega}{-a}\right)$$

//

$\therefore F[x(at)] = \frac{1}{|a|} X\left(\frac{j\omega}{a}\right)$

$x(2t) \longleftrightarrow \frac{1}{|2|} X\left(\frac{j\omega}{2}\right) = \frac{1}{2} X\left(\frac{j\omega}{2}\right)$

$x(-2t) \longleftrightarrow \frac{1}{|-2|} X\left(\frac{j\omega}{-2}\right) = \frac{1}{2} X\left(\frac{j\omega}{-2}\right)$

$x(-t/2) \longleftrightarrow \frac{1}{|-\frac{1}{2}|} X\left(\frac{j\omega}{-1/2}\right) = \frac{1}{1/2} X(-2j\omega)$

Differentiation in time domine :-

$$x(t) \xleftrightarrow{F.T} X(j\omega)$$

$$\frac{d}{dt} x(t) \xleftrightarrow{F.T} j\omega X(j\omega)$$

$$F^{-1}[X(j\omega)] = x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega \rightarrow (1)$$

$$\frac{d}{dt} x(t) = \frac{d}{dt}\left[\frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega\right]$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) \frac{d}{dt}\left(e^{j\omega t}\right) d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) \cdot e^{j\omega t} \cdot j\omega \, d\omega$$

$$= \frac{\cancel{j\omega}}{\cancel{2\pi}} \cancel{2\pi} F^{-1}\left[j\omega X(j\omega)\right] \quad (\because Eq (1))$$

$$F\left[\frac{d}{dt} x(t)\right] = j\omega X(j\omega)$$

Differentiation in frequency domine :-

$$x(t) \xleftrightarrow{F.T} X(j\omega)$$

$$t \cdot x(t) \xleftrightarrow{F.T} j\frac{d}{dt} X(j\omega)$$

$$F[x(t)] = X(j\omega) = \int_{-\infty}^{\infty} e^{-j\omega t} x(t)\, dt \rightarrow (1)$$

$$\frac{d}{d\omega} X(j\omega) = \int_{-\infty}^{\infty} x(t) \left(\frac{d}{d\omega} e^{-j\omega t}\right) dt$$

$$= \int_{-\infty}^{\infty} x(t) \cdot \underline{e^{-j\omega t}} \cdot -jt \; dt$$

$$j \frac{d}{d\omega} X(j\omega) = \int_{-\infty}^{\infty} t\, x(t)\, e^{-j\omega t} dt \rightarrow (2)$$

Compare (1) and (2)

$$= F[t\, x(t)]$$

<u>Time integration</u> :-

$$x(t) \xleftrightarrow{F.T} X(j\omega)$$

$$\int_{-\infty}^{\tau} x(\tau) d\tau \xleftrightarrow{F.T} \frac{1}{j\omega} X(j\omega)$$

This property is valid when initial conditions are zero.

Consider Inverse fourier Eqn

$$F^{-1}[X(j\omega)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega \rightarrow (1)$$

Replace $t \rightarrow \tau$

$$x(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega\tau} d\omega$$

integrate both sides $-\infty$ to τ.

$$\int_{-\infty}^{\tau} x(\tau)\, d\tau = \int_{-\infty}^{\tau} \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega\tau} d\omega \right] d\tau$$

change the Order of integration.

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\left[\int_{-\infty}^{\tau} e^{(j\omega)\tau} d\tau\right] d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega) \cdot \left.\frac{e^{j\omega\tau}}{j\omega}\right|_{-\infty}^{\tau} d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega) \cdot \frac{1}{j\omega} e^{j\omega\tau} d\omega \rightarrow (2)$$

$$= F^{-1}\left[\frac{1}{j\omega} X(j\omega)\right]$$

$$F\left[\int_{-\infty}^{\tau} x(\tau)\, d\tau\right] = \frac{1}{j\omega} X(j\omega)$$

→ The above Eqn is valid when the initial
nditions are zero.

i.e., $X(0) = 0$

If $X(0) \neq 0$ the time integrated signal includes
n impulse function.

$$\int_{-\infty}^{\infty} x(\tau)\, d\tau \xleftrightarrow{F.T} \frac{1}{j\omega} X(j\omega) + X(0) \cdot \pi\delta(\omega)$$

mp

onvolution property :- Convolution of

→ This theorem states that two signals in time
mine is equivalent to multiplication (of) in
requency domine.

Consider two signals :-

$$x_1(t) \xleftrightarrow{F.T} X_1(j\omega)$$

$$x_2(t) \xleftrightarrow{F.T} X_2(j\omega)$$

$$x_1(t) * x_2(t) \xleftrightarrow{F.T} X_1(j\omega) \cdot X_2(j\omega)$$

Convolution :-

$$x_1(t) * x_2(t) = \int_{-\infty}^{\infty} x_1(\tau) \cdot x_2(t-\tau)\, d\tau$$

time shifting

multiplication

$$= \int_{-\infty}^{\infty} x_1(t-\tau) \cdot x_2(\tau)\, d\tau$$ Integration

Proof:- $F[x_1(t)] = X_1(j\omega) = \int_{-\infty}^{\infty} x_1(t)\, e^{-j\omega t}\, dt$

$F[x_2(t)] = X_2(j\omega) = \int_{-\infty}^{\infty} x_2(t)\, e^{-j\omega t}\, dt$

$$F[x_1(t) * x_2(t)] = \int_{-\infty}^{\infty} [x_1(t) * x_2(t)]\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} x_1(\tau) \cdot x_2(t-\tau) \right] e^{-j\omega t}\, dt$$

Change the Order of Integration

$$= \int_{-\infty}^{\infty} x_1(\tau) \left[\int_{-\infty}^{\infty} x_2(t-\tau)\, e^{-j\omega t} \right] d\tau$$

Apply time shifting property:

$$= \int_{-\infty}^{\infty} \underbrace{x_1(\tau)\, e^{-j\omega\tau}}_{X_1(j\omega)} \times X_2(j\omega)\, d\tau$$

* $x(t) \longleftrightarrow X(j\omega)$

$x(t-t_0) \longleftrightarrow e^{-j\omega t_0} X(j\omega)$

$$= X_2(j\omega) \cdot X_1(j\omega)$$

Multiplication property :- Called as **modulation property**

* Mixing one signal with another signal is known as Modulation.

$$x_1(t) \xleftrightarrow{F.T} X_1(j\omega)$$

$$x_2(t) \xleftrightarrow{F.T} X_2(j\omega)$$

Telephone signal frequency = 300Hz - 3.4kHz

* Carrier signal ; Speech (20Hz - 20kHz) signal = Modulation

→ High frequency large signal is only used in communication signals

→ Carrier should be high frequency

$$x_1(t) \cdot x_2(t) \xleftrightarrow{F.T} \frac{1}{2\pi}\left[X_1(j\omega) * X_2(j\omega)\right]$$

$$F[x(t)] = X(j\omega) = \int_{-\infty}^{\infty} x(t) \cdot e^{-j\omega t} dt$$

$$F[x_1(t) \cdot x_2(t)] = \int_{-\infty}^{\infty} x_1(t)\, x_2(t)\, e^{-j\omega t} dt$$

$$= \int_{-\infty}^{\infty}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} X_1(\lambda)\, e^{j\lambda t} d\lambda\right] x_2(t) e^{-j\omega t} dt$$

change the Order of Integration.

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X_1(\lambda)\left(\left[\int_{-\infty}^{\infty} x_2(t)\, e^{j\lambda t} e^{-j\omega t} dt\right]\right) d\lambda$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} x_1(\lambda) \cdot \left[\int_{-\infty}^{\infty} x_1(t) \cdot e^{-j(\omega-\lambda)t} dt\right] d\lambda$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} x_1(\lambda)\, x_2(\omega-\lambda) d\lambda$$

$$= \frac{1}{2\pi}\left[x_1(\omega) * x_2(\omega)\right]$$

Duality property :-

$x(t) \xleftrightarrow{F.T} X(\omega)$

$X(t) \longleftrightarrow 2\pi\, x(-\omega)$

Time Domine Signals are denoted in Small letters. frequency Domine is in Capital letters.

$x(t) = F^{-1}[X(\omega)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega$

$$2\pi\, x(t) = \int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega$$

$t \to -t$

$$2\pi x(-t) = \int_{-\infty}^{\infty} X(\omega)\, e^{-j\omega t}\, d\omega$$

Interchange t and ω

$$2\pi x(-\omega) = \int_{-\infty}^{\infty} X(t)\, e^{-jt\omega}\, dt$$

$$2\pi x(-\omega) = \int_{-\infty}^{\infty} X(t)\, e^{-j\omega t}\, dt$$

$$= F[x(t)]$$

Conjugation property :-

$x(t) \xleftrightarrow{F.T} X(\omega)$

$x^*(t) \xleftrightarrow{F.T} X^*(-\omega)$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

Conjugate

$$X^*(\omega) = \int_{-\infty}^{\infty} x^*(t) \cdot e^{j\omega t}\, dt$$

$\omega \to -\omega$

$$X^*(-\omega) = \int_{-\infty}^{\infty} x^*(t) \cdot e^{-j\omega t}\, dt$$

$$= F[x^*(t)]$$

Parsevals Relation :-

$$x_1(t) \xleftrightarrow{F.T} X_1(j\omega)$$

$$x_2(t) \longleftrightarrow X_2(j\omega)$$

$$\boxed{\int_{-\infty}^{\infty} |x(t)|^2\, dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(j\omega)|^2\, d\omega}$$

$$\int_{-\infty}^{\infty} x_1(t) \cdot x_2^*(t)\, dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} X_1(j\omega) \cdot X_2^*(j\omega)\, d\omega$$

from the definition

$$\int_{-\infty}^{\infty} x_1(t)\, x_2^*(t)\, dt = \int\left(\frac{1}{2\pi}\left[\int_{-\infty}^{\infty} X_1(j\omega) e^{j\omega t}\, d\omega\right] x_2^* t\right) dt$$

change the order of Integration

$$\int_{-\infty}^{\infty} x_1(t) \cdot x_2^*(t)\, dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} X_1(j\omega)\left[\int_{-\infty}^{\infty} x_2^*(t)\, e^{j\omega t}\, dt\right] d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X_1(j\omega)\left[\int_{-\infty}^{\infty} x_2(t) \cdot e^{-j\omega t}\, dt\right]^* d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X_1(j\omega) \cdot X_2^*(j\omega)\, d\omega$$

1. find the fourier transform of following signal

i) $te^{-2t}u(t)$ ii) $e^{j2t}u(t)$ iii) $\cos w_0t\, u(t)$

iv) $\sin w_0t\, u(t)$ v) $e^{-3t}u(t-2)$ vi)

$\rightarrow x(t) = t \cdot e^{-2t}u(t)$

$$F[x(t)] = X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$x(t) \longleftrightarrow X(j\omega)$ $\rightarrow$ using Differentiation property

$t\, x(t) \longleftrightarrow j\frac{d}{d\omega}X(j\omega)$

$$F[e^{-at}u(t)] = \frac{1}{a+j\omega}$$

$$F[e^{-2t}u(t)] = \frac{1}{2+j\omega}$$

$$F[t \cdot e^{-2t}u(t)] = j\frac{d}{d\omega}\left(\frac{1}{2+j\omega}\right)$$

$$= j\left[\frac{-1}{(2+j\omega)^2}\, j\right]$$

$$= \frac{-(-1)}{(2+j\omega)^2}$$

$\rightarrow x(t) = e^{j2t}u(t)$

$x(t) \longleftrightarrow X(\omega)$ $\rightarrow$ using frequency shifting property

$e^{j\omega_0 t}x(t) \longleftrightarrow X(\omega-\omega_0)$

$F[u(t)] = \pi\delta(\omega) + \frac{1}{j\omega}$ — unit step function

frequency shifting property

$$F\left[e^{j2t}u(t)\right] = \pi\delta(\omega-2) + \frac{1}{j(\omega-2)}$$

$\rightarrow \cos\omega_0 t\, u(t) = x(t)$

$$= \frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2}\, u(t)$$

According to linearity property

$$a\,x(t) \xleftrightarrow{F.T} a\,X(j\omega)$$

$$F[x(t)] = \frac{1}{2}F\left[e^{j\omega_0 t}u(t)\right] + \frac{1}{2}F\left[e^{-j\omega_0 t}u(t)\right]$$

$$F[u(t)] = \pi\delta(\omega) + \frac{1}{j\omega} \longrightarrow \text{unit step function}$$

—using frequency shifting property

$$F\left[e^{j\omega_0 t}u(t)\right] = \pi\delta(\omega-\omega_0) + \frac{1}{j(\omega-\omega_0)}$$

$$F\left[e^{-j\omega_0 t}u(t)\right] = \pi\delta(\omega+\omega_0) + \frac{1}{j(\omega+\omega_0)}$$

$$\therefore F\left[\cos\omega_0 t\, u(t)\right] = \frac{1}{2}\left[\pi\delta(\omega-\omega_0) + \frac{1}{j(\omega-\omega_0)}\right] + \frac{1}{2}\left[\pi\delta(\omega+\omega_0) + \frac{1}{j(\omega+\omega_0)}\right]$$

$$= \frac{1}{2}\left[\pi\delta(\omega+\omega_0) + \pi\delta(\omega-\omega_0)\right] + \frac{1}{2}\left[\frac{2j\omega}{\omega_0^2-\omega^2}\right]$$

$$= \frac{\pi}{2}\left[\pi\delta(\omega+\omega_0) + \pi\delta(\omega-\omega_0)\right] + \frac{j\omega}{\omega_0^2-\omega^2}$$

$\rightarrow \sin w_0 t \, u(t) = x(t)$

$$P \quad \frac{e^{jw_0 t} - e^{-jw_0 t}}{2j} \, u(t) = x(t)$$

According to linearity property

$$a x(t) \xleftrightarrow{F.T} a X(jw)$$

$$F[x(t)] = \frac{1}{2} F\left[e^{jw_0 t} u(t)\right] - \frac{1}{2} F\left[e^{-jw_0 t} u(t)\right]$$

$$F[u(t)] = \pi \delta(w) + \frac{1}{jw} \rightarrow \text{unit Step function}$$

$$F\left[e^{jw_0 t} u(t)\right] = \pi \delta(w - w_0) + \frac{1}{j(w - w_0)}$$

$$F\left[e^{-jw_0 t} u(t)\right] = \pi \delta(w + w_0) + \frac{1}{j(w + w_0)}$$

$$\therefore F[\sin w_0 t \, u(t)] = \frac{1}{2j}\left[\pi \delta(w - w_0) + \frac{1}{j(w - w_0)}\right] - \frac{1}{2j}\left[\pi \delta(w + w_0) + \frac{1}{j(w + w_0)}\right]$$

$$= \frac{1}{2j}\left[\pi \delta(w + w_0) + \pi \delta(w - w_0)\right] + \frac{1}{2j}\left[\frac{1}{j(w + w_0)} - \frac{1}{j(w + w}\right.$$

$$= \frac{1}{2j}\left[\pi \delta(w + w_0) - \pi \delta(w - w_0)\right] + \frac{1}{2}\left[\frac{jw + jw_0 - jw + jw_0}{j(w_0^2 - w^2)}\right]$$

$$= \frac{1}{2j}\left[\pi \delta(w + w_0) - \pi \delta(w - w_0)\right] + \frac{1}{\cancel{2}j}\left(\frac{\cancel{2} j w_0}{\cancel{j}(w_0^2 - w^2)}\right)$$

$$= \frac{1}{2j}\left(\pi \delta(w + w_0) - \pi \delta(w - w_0)\right] + \frac{w_0}{w_0^2 - w^2}$$

$\rightarrow e^{-3t} u(t-2) = x(t)$

$x(t) \longleftrightarrow X(j\omega)$ — using time shifting property

$x(t-t_0) \longleftrightarrow e^{-j\omega t_0} X(\omega)$

$x(t) = e^{-3(t-2)} u(t-2)\, e^{-6}$

$F[x(t)] = e^{-6} F[e^{-3(t-2)} u(t-2)]$

$= e^{-6}\left[e^{-2j\omega} \dfrac{1}{(3+j\omega)}\right]$

$= \dfrac{e^{-6-j2\omega}}{3+j\omega} = \dfrac{e^{-2(3+j\omega)}}{(3+j\omega)}$

$e^{-at} u(t) \longleftrightarrow \dfrac{1}{a+j\omega}$

$e^{-a(t-t_0)} u(t-t_0) \longleftrightarrow e^{-j\omega t_0} \cdot \dfrac{1}{a+j\omega}$

$\rightarrow x(t) = u(-t)$

$\rightarrow$ Time Reversal property

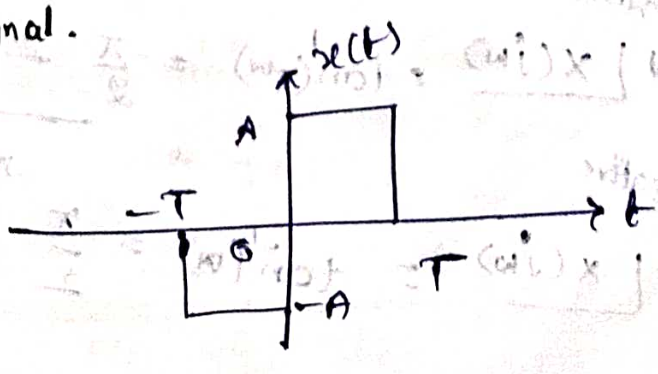
$x(t) \longleftrightarrow X(\omega)$

$x(-t) \longleftrightarrow X(-\omega)$

$F[u(t)] = \pi\delta(\omega) + \dfrac{1}{j\omega}$

$F[u(-t)] = \pi\delta(-\omega) + \dfrac{1}{-j\omega} = \pi\delta(-\omega) + \dfrac{j}{\omega}$

Find the Magnitude and phase spectrum of the following Signal.



$$x(t) = \begin{cases} A & 0 < t < T \\ -A & -T < t < 0 \\ 0 & \text{otherwise} \end{cases}$$

$$X(j\omega) = \int_{-T}^{0} -A\, e^{-j\omega t}\, dt + \int_{0}^{T} A \cdot e^{-j\omega t}\, dt$$

$$= -A \left. \frac{e^{-j\omega t}}{-j\omega} \right|_{-T}^{0} + A \cdot \left. \frac{e^{-j\omega t}}{-j\omega} \right|_{0}^{T}$$

$$= \frac{A}{j\omega}\left(1 - e^{j\omega T}\right) - \frac{A}{j\omega}\left(e^{-j\omega T} - 1\right)$$

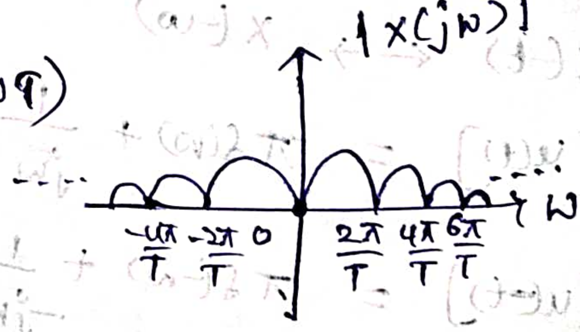
$$= \frac{A}{j\omega}\left[2 - 2\left(\frac{e^{j\omega T} + e^{-j\omega T}}{2}\right)\right]$$

$$= \frac{2A}{j\omega}\left[1 - \cos\omega T\right]$$

$$= 0 - j\frac{2A}{\omega}(1 - \cos\omega T)$$

__Magnitude__

$$|X(j\omega)| = \frac{2A}{\omega}(1 - \cos\omega T)$$



$1 - \cos\omega T = 0$

$\cos\omega T = 1$

$\omega T = \cos^{-1} 1$

$\omega T = 2n\pi$

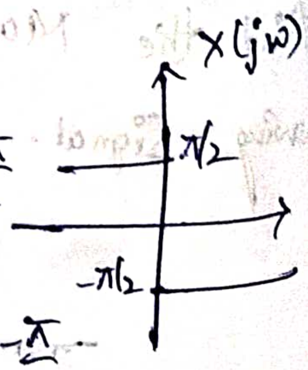
$\omega = \frac{2n\pi}{T}$

(i) for positive

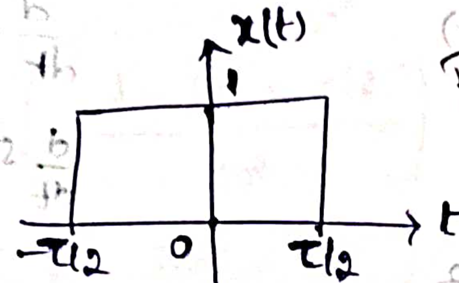
__Phase angle__ :- $\angle X(j\omega) = \tan^{-1}(-\infty) = -\frac{\pi}{2}$

(ii) for negative

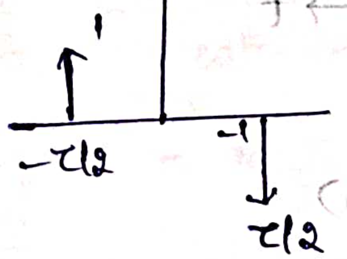
$\angle X(j\omega) = \tan^{-1}(\infty) = \frac{\pi}{2}$



→ find the fourier transform of given $x(t)$: by using differential property.



Rectangular function
fourier transform
Sinc function



$$\frac{d}{dt}x(t) = \delta(t+\tau/2) - \delta(t-\tau/2)$$

$$x(t) \longleftrightarrow X(j\omega)$$

$$\frac{d}{dt}x(t) \longleftrightarrow j\omega X(j\omega)$$

$$x(t-t_0) = e^{-j\omega t_0} X(\omega)$$

$$j\omega X(j\omega) = 1 \cdot e^{j\omega\tau/2} - e^{-j\omega\tau/2}$$

$$= 2\tau \frac{\left(e^{j\omega\tau/2} - e^{-j\omega\tau/2}\right)}{2\tau j\omega}$$

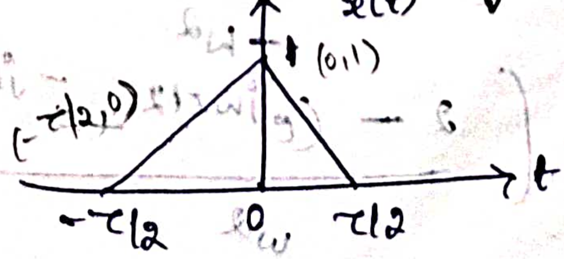
$$= \tau \left[\frac{e^{j\omega\tau/2} - e^{-j\omega\tau/2}}{2j\,(\omega\tau/2)}\right]$$

$$= \tau \frac{\sin \omega\tau/2}{\omega\tau/2}$$

$$= \tau \operatorname{sinc}(\omega\tau/2)$$

** IMP

→ find the fourier transform of given $x(t)$ by using differential property.

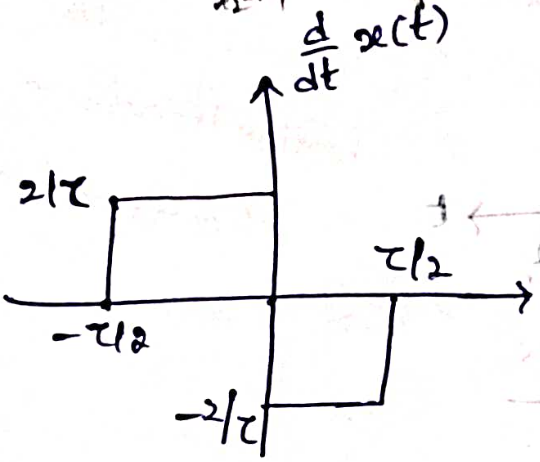


$$x(t) = \frac{1}{\tau/2}(t+\tau/2) = \frac{2}{\tau}(t+\tau/2)$$

slope $\frac{y_2-y_1}{x_2-x_1}$

$\frac{d}{dt}$ ramp = step

$\frac{d}{dt}$ step = impulse

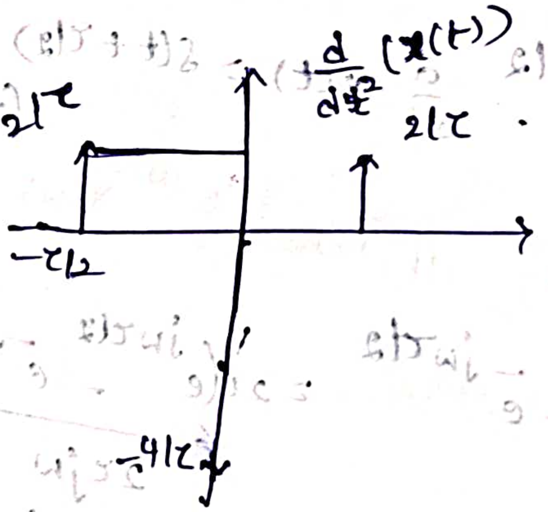


$y = mx + c$

$\frac{dy}{dx} = m \rightarrow \frac{2}{\tau}$

$y = -mx + c$

$\frac{dy}{dx} = -m \rightarrow -\frac{2}{\tau}$



$$\frac{d^2x(t)}{dt^2} = \frac{2}{\tau}\delta(t+\tau/2) - \frac{4}{\tau}\delta(t) + \frac{2}{\tau}\delta(t-\tau/2)$$

$$(j\omega)^2 X(j\omega) = \frac{2}{\tau}e^{j\omega\tau/2} - \frac{4}{\tau} + \frac{2}{\tau}e^{-j\omega\tau/2}$$

$$X(j\omega) = \frac{2}{\tau}\left[\frac{e^{j\omega\tau/2} - 2 + e^{-j\omega\tau/2}}{-\omega^2}\right]$$

$$= \frac{2}{\tau}\left[\frac{-2 + (e^{j\omega\tau/2} + e^{-j\omega\tau/2})}{-\omega^2}\right]$$

$$= \frac{2}{\tau}\left[\frac{2 - (e^{j\omega\tau/2} + e^{-j\omega\tau/2})}{\omega^2}\right]$$

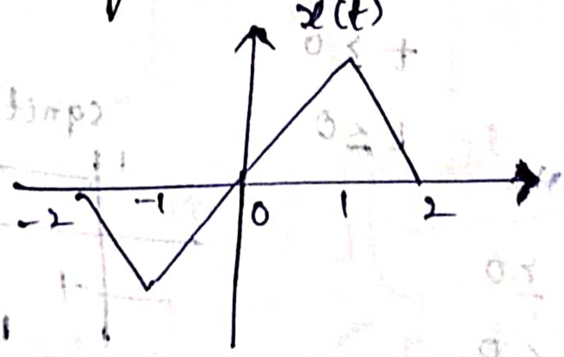
$$= \frac{4}{\tau}\left[\frac{1-\left(\frac{e^{j\omega\tau/2}+e^{-j\omega\tau/2}}{2}\right)}{\omega^2}\right]$$

$$= \frac{4}{\tau}\left(\frac{1-\cos\omega\tau/2}{\omega^2}\right)$$

$$= \frac{8}{\tau}\left(\frac{\sin^2\omega\tau/4}{\omega^2}\right) = \frac{8}{\tau}\left[\frac{\sin\omega\tau/4}{\omega\tau/4}\right]^2 \cdot \left(\frac{\tau}{4}\right)^2$$

$$= \frac{\tau}{2}\left[\operatorname{sinc}^2(\omega\tau/4)\right]$$

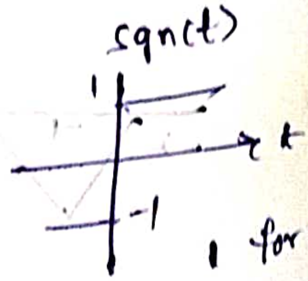
→ find the fourier transform of given signal x(t) by using differential property.



→ find the fourier transform of following

Signal $x(t) = e^{-a|t|} \, \mathrm{Sgn}(t)$

$$x(t) = e^{-at}(+1) \qquad t \ge 0$$
$$= e^{-a(-t)}(-1) \qquad t \le 0$$



$$x(t) = e^{-at} \qquad t \ge 0$$
$$= -e^{at} \qquad t < 0$$

1 for $t > 0$

-1 for $t < 0$

$$F[x(t)] = X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$X(j\omega) = \int_{-\infty}^{0} -e^{at}\, e^{-j\omega t}\, dt + \int_{0}^{\infty} e^{-at}\, e^{-j\omega t}\, dt$$

$$= -\int_{-\infty}^{0} e^{t(a-j\omega)}\, dt + \int_{0}^{\infty} e^{-t(a+j\omega)}\, dt$$

$$= -\left[\frac{e^{t(a-j\omega)}}{a-j\omega} \Bigg|_{-\infty}^{0} + \frac{e^{-t(a+j\omega)}}{-(a+j\omega)} \Bigg|_{0}^{\infty} \right]$$

$$= -\left(\frac{1-0}{a-j\omega}\right) + \frac{0-1}{-(a+j\omega)}$$

$$= \frac{1}{a+j\omega} - \frac{1}{a-j\omega} = \frac{a-j\omega-a-j\omega}{a^2+\omega^2} = \frac{-2j\omega}{a^2+\omega^2}$$

Compute the fourier transform of following Signal

$$x(t) = \begin{cases} t+\cos\pi t & |t|<1 \\ 0 & |t|>1 \end{cases}$$

$|t|<1$

$t<1$

$-t<1$

$t>-1$

$$F[x(t)] = X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-1}^{1} (1+\cos\pi t)\, e^{-j\omega t}\, dt$$

$$= \int_{-1}^{1} e^{-j\omega t}\, dt + \int_{-1}^{1} \left(\frac{e^{j\pi t}+e^{-j\pi t}}{2}\right) e^{-j\omega t}\, dt$$

$$= \left.\frac{e^{-j\omega t}}{-j\omega}\right|_{-1}^{1} + \frac{1}{2}\left(\int_{-1}^{1} e^{+t(j\pi-j\omega)} + \int_{-1}^{1} e^{-t(\pi j+j\omega)}\, dt\right)$$

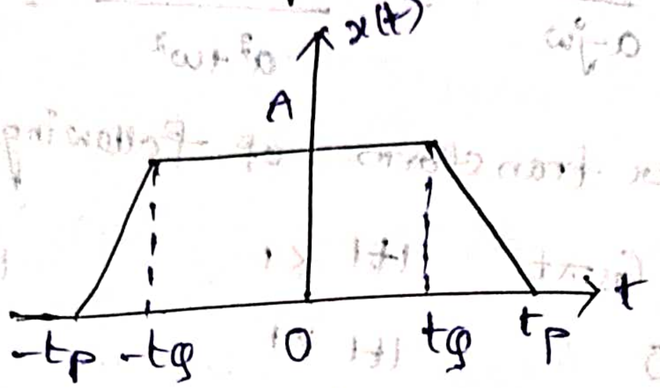
$$= \frac{1}{-j\omega} + \frac{1}{j\omega}$$

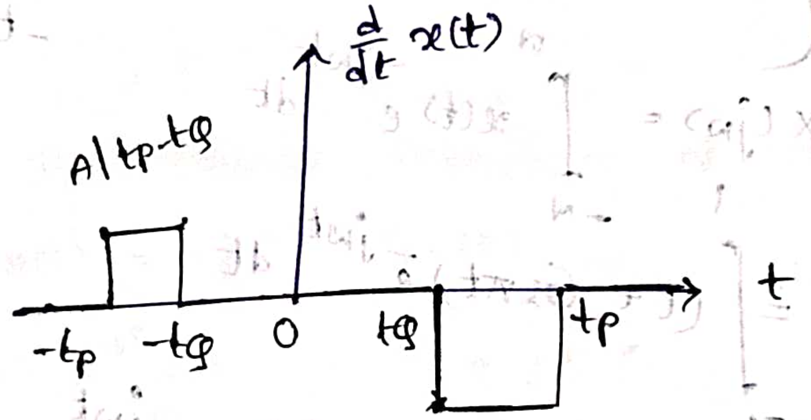
$$= \frac{e^{-j\omega}}{-j\omega} - \frac{e^{+j\omega}}{-j\omega} + \frac{1}{2}\left[\left.\frac{e^{jt(\pi-\omega)}}{j(\pi-\omega)}\right|_{-1}^{1} + \left.\frac{e^{-tj(\pi+\omega)}}{-j(\pi+\omega)}\right|_{-1}^{1}\right]$$

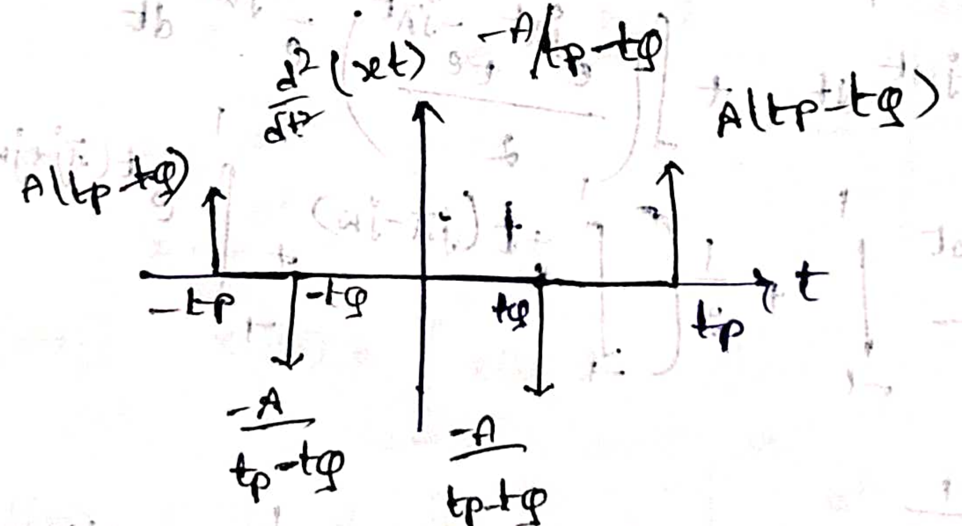
$$= \frac{e^{-j\omega}-e^{j\omega}}{(-j\omega)} + \frac{1}{2}\left[\frac{e^{j(\pi-\omega)}-e^{-j(\pi-\omega)}}{j(\pi-\omega)} + \frac{e^{-j(\pi+\omega)}-e^{j(\pi+\omega)}}{-j(\pi+\omega)}\right]$$

$$= \frac{2\sin\omega}{\omega} + \frac{\sin(\pi-\omega)}{\pi-\omega} + \frac{\sin(\pi+\omega)}{\pi+\omega}$$

→ find the fourier transform of tripizoidal pulse shown in figure.







$$\frac{d^2 x(t)}{dt^2} = \frac{A}{t_p - t_q}\,\delta(t+t_p) - \frac{A}{t_p-t_q}\,\delta(t+t_q) - \frac{A}{t_p-t_q}\,\delta(t-t_q) + \frac{A}{t_p-t_q}\,\delta(t-t_p)$$

Apply F.T

$$(j\omega)^2 \times (j(\omega)) = \frac{A}{t_p-t_q}\left[e^{+j\omega t_p} - e^{+j\omega t_q} - e^{-j\omega t_q} + e^{-j\omega t_p}\right]$$

$$= \frac{2A}{t_p - t_q}\left[\cos jw t_p - \cos jw t_q\right]$$

$$x(jw) = \frac{2A}{w^2(t_q - t_p)}\left[\cos wt_p - \cos wt_q\right]$$

Inverse fourier transforms :-

Definition :- Conversion of frequency Domine signal in to time domine signal is known as Inverse fourier transform.

$$F^{-1}[X(w)] = x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(w)\, e^{jwt}\, dw$$

$$\rightarrow X(w) = \frac{4(jw) + 6}{(jw)^2 + 6jw + 8}$$

$$= \frac{4(jw) + 6}{(jw+4)(jw+2)}$$

$$= \frac{A}{jw+4} + \frac{B}{jw+2}$$

$$4(jw) + 6 = A[(jw)+2] + B[(jw)+4]$$

put $jw = -2$

$$-8 + 6 = 2B \Rightarrow B = -1$$

put $jw = -4$

$$-16 + 6 = -2A \Rightarrow A = 5$$

$$x(\omega) = \frac{5}{(j\omega+4)} - \frac{1}{(j\omega+2)} \rightarrow ①$$

$5F^{-1}\left[\frac{1}{a+j\omega}\right]$

$5e^{-at}u(t)$

$5e^{-4t}u(t)$

$$\boxed{F[e^{-at}u(t)] = \frac{1}{a+j\omega}}$$

Apply I.F.T on both sides in Eq ①

$$x(t) = 5e^{-4t}u(t) - e^{-2t}u(t)$$

→ fourier transform is applicable only on Right side.

→ when fourier transform does not exist then we use laplace transform.

$$\rightarrow x(\omega) = \frac{1+3(j\omega)}{((j\omega)+3)^2}$$

$$1+3(j\omega) = \frac{A}{(j\omega+3)} + \frac{B}{(j\omega+3)^2}$$

$$1+3(j\omega) = A(j\omega+3) + B(1)$$

put $j\omega = -3$

$1-9 = B \Rightarrow B = -8$

$3 = A$

$$\therefore x(\omega) = \frac{3}{j\omega+3} - \frac{8}{(j\omega+3)^2}$$

$$= 3e^{-3t}u(t) - 8e^{-3t}u(t)$$

$x_1(t) = e^{-2t}u(t)$ by using ~~these~~ F.T find

$x_2(t) = e^{-3t}u(t)$

Convolution of signal.

$$x_1(t) * x_2(t) = \int_{-\infty}^{\infty} x_1(\tau) \cdot x_2(t-\tau)\, d\tau$$

$$= \int_{-\infty}^{\infty} e^{-2\tau}u(\tau) \cdot e^{-3(t-\tau)}\, d\tau$$

$$F[x_1(t) * x_2(t)] = X_1(j\omega) \cdot X_2(j\omega)$$

$$X_1(j\omega) = F[x_1(t)] = \frac{1}{2+j\omega}$$

$$X_2(j\omega) = F[x_2(t)] = \frac{1}{3+j\omega}$$

$$X_1(j\omega) \cdot X_2(j\omega) = \frac{1}{2+j\omega} \cdot \frac{1}{3+j\omega}$$

$$= \frac{A}{2+j\omega} + \frac{B}{3+j\omega}$$

$$1 = A(3+j\omega) + B(2+j\omega)$$

put $j\omega = -3$ put $j\omega = -2$

$1 = -B$ $1 = A$

$$\therefore X_1(j\omega) \cdot X_2(j\omega) = \frac{1}{2+j\omega} - \frac{1}{3+j\omega}$$

$$= e^{-2t}u(t) - e^{-3t}u(t)$$

$$= (e^{-2t} - e^{-3t})\,u(t)$$

→ using F.T find the convolution of following signals

$x_1(t) = t \cdot e^{-t} u(t)$

$x_2(t) = t \cdot e^{-2t} u(t)$

$$x_1(t) * x_2(t) = \int_{-\infty}^{\infty} x_1(\tau) \cdot x_2(t-\tau)\, d\tau$$

$$= \int_{-\infty}^{\infty} \tau e^{-\tau} u(\tau) \cdot \tau . e^{-2\tau} \cdot dt$$

$$F[x_1(t) * x_2(t)] = X_1(j\omega) \cdot X_2(j\omega)$$

$$X_1(j\omega) = F[x_1(t)] = \frac{1}{(1+j\omega)^2}$$

$$X_2(j\omega) = F[x_2(t)] = \frac{1}{(2+j\omega)^2}$$

$$X_1(j\omega) \cdot X_2(j\omega) = \frac{1}{(1+j\omega)^2} \cdot \frac{1}{(2+j\omega)^2} = \frac{1}{(1+j\omega)^2 (2+j\omega)^2}$$

$$\frac{1}{(1+j\omega)^2 (2+j\omega)^2} = \frac{A}{1+j\omega} + \frac{B}{(1+j\omega)^2} + \frac{C}{2+j\omega} + \frac{D}{(2+j\omega)^2}$$

$$= \frac{A(1+j\omega)(2+j\omega)^2 + B(2+j\omega)^2 + C(2+j\omega)(1+j\omega)^2 + D(1+j\omega)^2}{(1+j\omega)^2 (2+j\omega)^2}$$

$$1 = A(1+j\omega)(2+j\omega)^2 + B(2+j\omega)^2 + C(2+j\omega)(1+j\omega)^2 + D(1+j\omega)^2$$

put $j\omega = -1 \Rightarrow 1 = B \quad \therefore B = 1$

$j\omega = -2 \Rightarrow 1 = D \quad \therefore D = 1$

put $j\omega = 0$

$1 = 4A + 4B + 2C + D$

$\cancel{1} = 4A + 4 + 2C + \cancel{1}$

$2A + C = -2 \longrightarrow (1)$

put $j\omega = 1$

$1 = 18A + 9B + 12C + 4D$

$1 = 18A + 9 + 12C + 4$

$18A + 12C = -12$

$3A + 2C = -2 \longrightarrow (2)$

$-4 + 2C = -2$

$C = 2$

$$\begin{aligned} 4A + 2C &= -4 \\ 3A + 2C &= -2 \\ \hline A &= -2 \end{aligned}$$

$\therefore X_1(j\omega) \cdot X_2(j\omega) = \dfrac{-2}{1+j\omega} + \dfrac{1}{(1+j\omega)^2} + \dfrac{2}{2+j\omega} + \dfrac{1}{(2+j\omega)^2}$,

$= -2e^{-t}u(t) + te^{-t}u(t) + 2e^{-2t}u(t) + te^{-2t}u(t)$

$= (-2e^{-t} + te^{-t} + 2e^{-2t} + te^{-2t})u(t)$

$= [e^{-t}(t-2) + e^{-2t}(t+2)]\,u(t)$

Fourier transform of periodic signals :-

Conversion of time domine signal into frequency domine signal is known as fourier transform.

$$F[1] = 2\pi\delta(\omega)$$

$$F[1 \cdot e^{jn\omega_0 t}] = 2\pi\delta(\omega - n\omega_0)$$

$$X(\omega) = \sum_{n=-\infty}^{\infty} C_n 2\pi\delta(\omega - n\omega_0)$$

$\therefore F[x(t)] = X(\omega)$

$F[x(t) \cdot e^{j\omega_0 t}] = X(\omega - \omega_0)$

using frequency shifting

Express the given signal in Exponential F.S Coefficient.

$$x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}$$

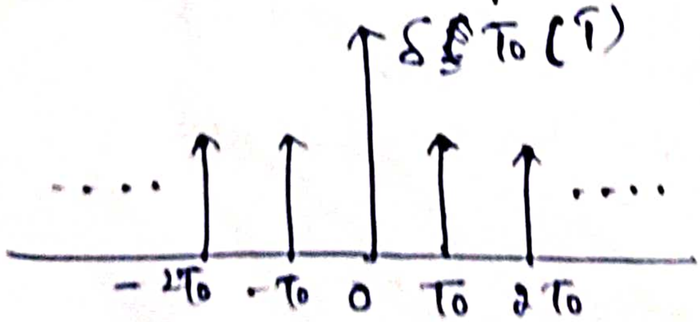
$$X(\omega) = \sum_{n=-\infty}^{\infty} C_n F[e^{jn\omega_0 t}]$$

Continuation

$$C_n = \frac{1}{T}\int_0^T x(t) e^{-jn\omega_0 t} dt$$

** IMP

→ find the F.T of given periodic impulse



period $T = T_0$

fundamental period $\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{T_0}$

$$x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}$$

$$C_n = \frac{1}{T}\int_0^T x(t)\, e^{-jn\omega_0 t}\, dt$$

$t = 0$, $x(t) = 1$

remaining values of t, $x(t) = 0$

$$= \frac{1}{T_0} e^{-jn\omega_0 t} \Big|_{t=0}$$

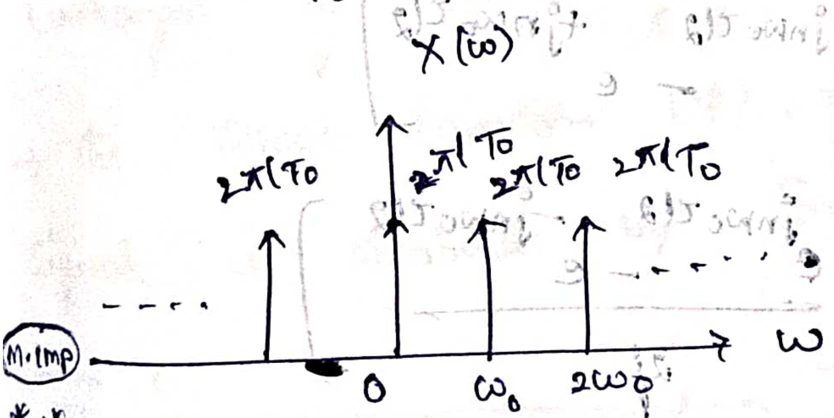
$$= \frac{1}{T_0}$$

$$x(t) = \sum_{n=-\infty}^{\infty} \frac{1}{T_0} e^{jn\omega_0 t}$$

Apply F.T

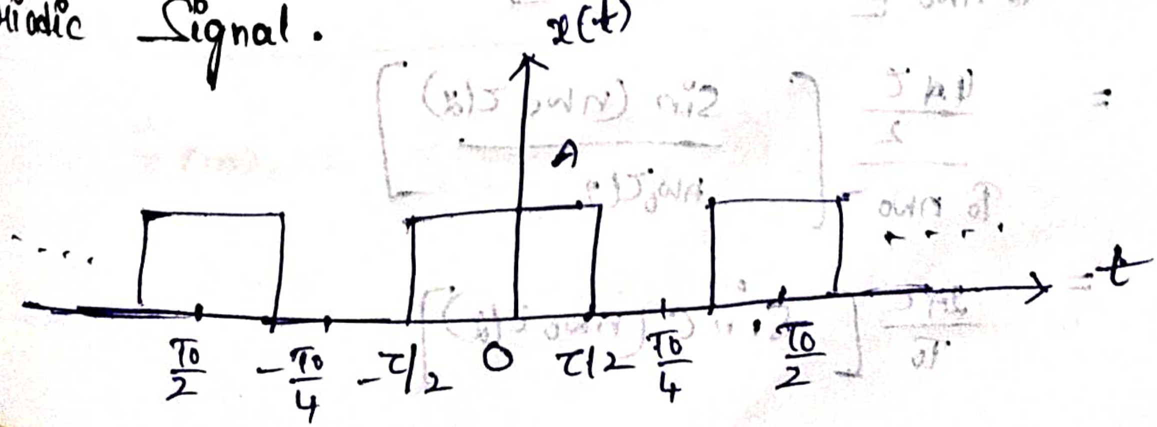
$$X(\omega) = \frac{1}{T_0} \sum_{n=-\infty}^{\infty} 2\pi\, \delta(\omega - n\omega_0)$$

$$= \frac{2\pi}{T_0}\Big(\delta(\omega) + \delta(\omega-\omega_0) + \delta(\omega-2\omega_0) \ldots + \delta(\omega+\omega_0) + \delta(\omega+2\omega_0) \ldots$$



M.Imp **

→ find the fourier transform of following periodic signal.



period $T = \left(\frac{T_0}{4} - \left(-\frac{T_0}{4}\right)\right) = T_0/2$

fundamental frequency $\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{\frac{T_0}{2}} = \frac{4\pi}{T_0}$

$$C_n = \frac{1}{T}\int_0^T x(t)\, e^{-jn\omega_0 t}\, dt$$

$$= \frac{1}{\frac{T_0}{2}} \int_{-\tau/2}^{\tau/2} A \cdot e^{-jn\frac{4\pi}{T_0} t}\, dt$$

$$= \frac{2A}{T_0}\left[\frac{e^{-jn\frac{4\pi}{T_0} t}}{-jn\frac{4\pi}{T_0}}\right]_{-\tau/2}^{\tau/2}$$

$$= \frac{2A}{T_0}\left[\frac{e^{-jn\omega_0 t}}{-jn\omega_0}\right]_{-\tau/2}^{\tau/2}$$

$$= \frac{-2A}{T_0\, jn\omega_0}\left[e^{-jn\omega_0 \tau/2} - e^{+jn\omega_0 \tau/2}\right]$$

$$= \frac{4A}{T_0\, n\omega_0}\left[\frac{e^{jn\omega_0\tau/2} - e^{-jn\omega_0\tau/2}}{2j}\right]$$

$$= \frac{4A}{T_0\, n\omega_0}\left[\sin(n\omega_0\tau/2)\right]$$

$$= \frac{\frac{4A\tau}{2}}{T_0\, n\omega_0}\left[\frac{\sin(n\omega_0\tau/2)}{n\omega_0\tau/2}\right]$$

$$= \frac{2A\tau}{T_0}\left[\operatorname{sinc}(n\omega_0\tau/2)\right]$$

$$X(\omega) = \sum_{n=-\infty}^{\infty} \frac{2A\tau}{T_0} \operatorname{Sinc}(n\omega_0\tau/2) \cdot 2\pi\delta(\omega - n\omega_0)$$

$$= \frac{4\pi A\tau}{T_0} \sum_{n=-\infty}^{\infty} \operatorname{Sinc}(n\omega_0\tau/2)\, \delta(\omega - n\omega_0)$$

$$x(t) \longrightarrow \boxed{1/\pi t} \longrightarrow \hat{x}(t)$$

$$X(\omega) \qquad\qquad \hat{X}(\omega)$$

HILBERT TRANSFORM :-

It Converts time domine to time domine only.

→ When the phase angles of all positive frequency signals are shifted by $-90°$ and negative signals are shifted by $+90°$, the Resulting function of time is called Hilbert transform of time.

→ The amplitude Spectrum of Signal is unchanged by Hilbert transform operation.

→ The Hilbert transform (or) $\hat{x}(t)$ of the signal is obtained by Convolving $x(t)$ with $\frac{1}{\pi t}$.

$$\boxed{\hat{x}(t) = x(t) * \frac{1}{\pi t}}$$

→ Convolution in time domine

$$\hat{X}(\omega) = X(\omega) \cdot F\left[\frac{1}{\pi t}\right]$$

↓ multiplication in frequency domine

$$\hat{x}(t) = \int_{-\infty}^{\infty} x(\tau) \cdot \frac{1}{\pi(t-\tau)} \, d\tau$$

$$= \boxed{\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{x(\tau)}{(t-\tau)} \, d\tau}$$

(8)

$$x_1(t) * x_2(t) = \int_{-\infty}^{\infty} x_1(\tau) x_2(t-\tau) \, d\tau$$

$$\therefore \hat{x}(t) = \int_{-\infty}^{\infty} x(t-\tau) \frac{1}{\pi\tau} \, d\tau$$

$$= \int_{-\infty}^{\infty} x_1(t-\tau) x_2(\tau) \, d\tau$$

$$= \boxed{\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{x(t-\tau)}{\tau} \, d\tau}$$

→ using Duality property

$$x(t) \longleftrightarrow X(\omega)$$

$$X(t) \longleftrightarrow 2\pi x(-\omega)$$

$$\mathrm{Sgn}(t) \overset{F.T}{\longleftrightarrow} \frac{2}{j\omega}$$

$\omega = t$

$$\frac{1}{2\pi}\left[\frac{2}{jt}\right] \longleftrightarrow \mathrm{Sgn}(-\omega)$$

$$\boxed{\frac{1}{\pi t} \longleftrightarrow +j\,\mathrm{Sgn}(-\omega) = -j\,\mathrm{Sgn}(\omega)}$$

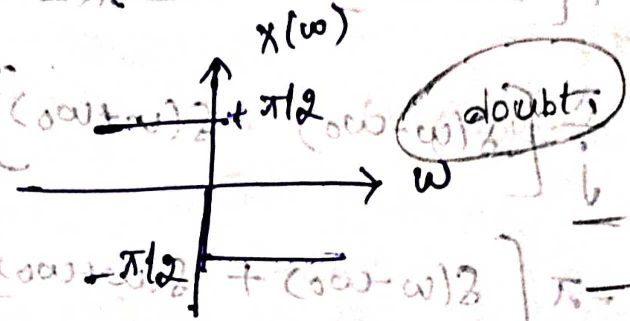
$$\hat{X}(\omega) = -j\,X(\omega) \qquad \omega > 0$$

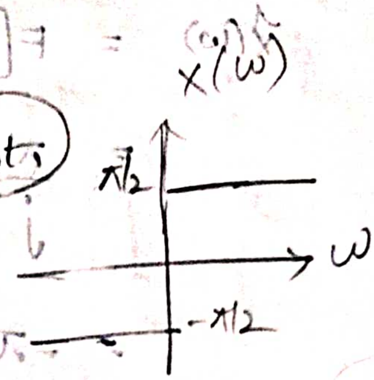
$$= j\,X(\omega) \qquad \omega < 0$$

Magnitude $|\hat{X}(\omega)| = X(\omega)$

→ If we apply Hilbert transform we observe that magnitude does not change.

$\underline{\hat{X}(\omega)} = -\pi/2$





Spectrum changes.

properties :-

→ Time to time domain only. but introduce shift

→ Two signals are orthogonal signals.

→ magnitude is same but spectrum changes.

→ $H[x(t)] = \hat{x}(t)$

$H[\hat{x}(t)] = -x(t)$

* Inverse transform of $\hat{x}(t)$ gives $H[x(t)]$

Applications :-

→ It is used in SSB.

→ It is used, in designation of band pass filter.

find Hilbert T.F of $\sin\omega_0 t$

$$\hat{x}(t) = \sin n\omega_0 t + \frac{1}{\pi t}$$

after F-T

$$\hat{x}(\omega) = F[\sin \omega_0 t] \cdot F\left[\frac{1}{\pi t}\right]$$

$$= \frac{\pi}{j}\left[\delta(\omega-\omega_0) - \delta(\omega+\omega_0)\right] -j\, \mathrm{Sgn}(\omega)$$

$$= -\pi\left[\delta(\omega-\omega_0) + \delta(\omega+\omega_0)\right]$$

Apply IFT

$$\hat{x}(t) = -\cos\omega_0 t.$$

$$F[\sin\omega_0 t] = F\left[\frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j}\right]$$

$$= \frac{1}{2j}\left[F[e^{j\omega_0 t}] - F[e^{-j\omega_0 t}]\right]$$

$$= \frac{1}{2j}\left[2\pi\delta(\omega-\omega_0) - 2\pi\,\delta(\omega+\omega_0)\right]$$

$$= \frac{\pi}{j}\left[\delta(\omega-\omega_0) - \delta(\omega+\omega_0)\right]$$

Find Hilbert T.F of $\cos\omega_0 t$

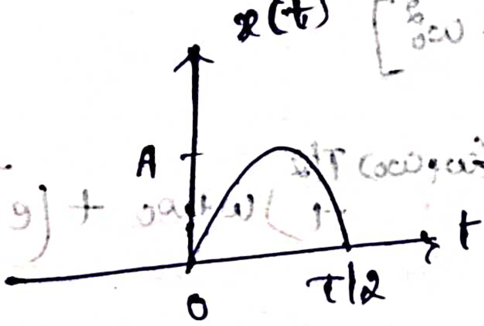
$$\hat{x}(t) = \cos n\omega_0 t + \frac{1}{\pi t}$$

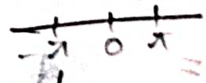
$$\hat{x}(\omega) = F[\cos n\omega_0 t] \cdot F\left(\frac{1}{\pi t}\right]$$

$$=$$

→ find the fourier transform of given Sinusoidal pulse (2018)

It Contains 5 then Cost or sint based on signal





$$x(t) = \begin{cases} A \sin \omega_0 t & 0 \le t \le T/2 \\ 0 & \text{otherwise} \end{cases}$$

like this Cos wt or Sinwt based on Signal

$$F[x(t)] = X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$= \int_{0}^{T/2} A \sin \omega_0 t \cdot e^{-j\omega t}\, dt$$

$$\frac{A}{2j}\int_0^{T/2} \left(e^{j\omega_0 t} - e^{-j\omega_0 t}\right) e^{-j\omega t}\, dt$$

$$\frac{A}{2j}\int_0^{T/2} e^{j\omega_0 t} \cdot e^{-j\omega t} - e^{-j\omega_0 t}\, e^{-j\omega t}\, dt$$

$$\frac{A}{2j}\int_0^{T/2} e^{jt(\omega+\omega_0)} - e^{-jt(\omega+\omega_0)}\, dt$$

$$\frac{A}{2j}\left[\frac{e^{j(\omega+\omega_0)t}}{j(\omega+\omega_0)}\Bigg|_0^{T/2} - \frac{e^{-j(\omega+\omega_0)t}}{-j(\omega+\omega_0)}\Bigg|_0^{T/2}\right]$$

$$= \frac{A}{2j}\left[\frac{e^{j(\omega+\omega_0)T/2} - 1}{j(\omega+\omega_0)} + \frac{e^{-j(\omega+\omega_0)T/2} - 1}{j(\omega+\omega_0)}\right]$$

$$= \frac{A}{2j}\left[\frac{\left(e^{j(\omega+\omega_0)T/2} - 1\right)(\omega+\omega_0) + \left(e^{-j(\omega+\omega_0)T/2} - 1\right)-\omega+\omega_0}{j\left[-\omega^2+\omega_0^2\right]}\right]$$

$$= \frac{A}{2(\omega^2+\omega_0^2)}\left[\left(e^{j(\omega+\omega_0)T/2} - 1\right)\omega+\omega_0 + \left(e^{-j(\omega+\omega_0)T/2} - 1\right)\omega+\omega_0\right]$$

$$= \frac{A}{2(\omega_0^2-\omega^2)}\left[\left(e^{j(\omega_0-\omega)T/2} - 1\right)\omega+\omega_0 + \left(e^{-j(\omega+\omega_0)T/2} - 1\right)(\omega_0-\omega)\right]$$

d

$$\int e^{cx} \sin bx = \frac{e^{cx}}{c^2+b^2}\left[c \sin bx - b \cos bx\right]$$

$$\int e^{cx} \cos bx = \frac{e^{cx}}{c^2+b^2}\left[c \cos bx + b \sin bx\right]$$

$$\frac{-2A}{\omega_0^2-\omega^2}\, e^{-j\omega T/4}\left(\sin \omega T/4 + \omega_0/j\right)$$

Unit-3

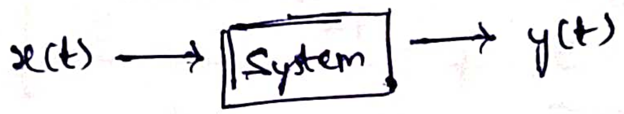
Analysis of LTI Systems :-

System :- It is defined as an entity that acts as on an input signal and transform it into output signal called as a System.

→ A System which is both linear and time invariant called linear Time invariant System.

Response of linear Systems :-

impulse response



$$y(t) = x(t) * h(t)$$

$$Y(\omega) = X(\omega) \cdot H(\omega)$$

$$Y(\omega) = \delta(\omega) \cdot H(\omega)$$

$$\boxed{Y(\omega) = H(\omega)}$$

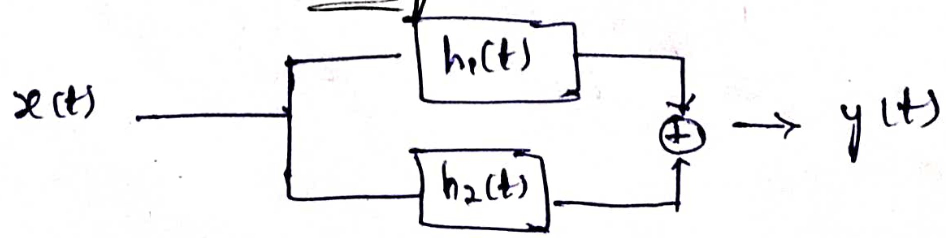
properties of LTI Systems :-

1. **Commutative property** :-

$$y(t) = x(t) * h(t) = h(t) * x(t)$$

$$= \int_{-\infty}^{\infty} x(\tau) h(t-\tau) \cdot d\tau = \int_{-\infty}^{\infty} h(\tau) x(t-\tau) \cdot d\tau$$

2. **Distributive property** :-



$x(t) * h_1(t) + x(t) * h_2(t) = y(t)$

$$x(t) \longrightarrow \boxed{h_1(t) + h_2(t)} \longrightarrow y(t)$$

$$x(t) * (h_1(t) + h_2(t)) = y(t)$$

3. **Associative property** :-

$$x(t) \longrightarrow \boxed{h_1(t)} \longrightarrow \boxed{h_2(t)} \longrightarrow y(t)$$

$$y(t) = (x(t) * h_1(t)) * h_2(t)$$

$$x(t) \longrightarrow \boxed{h_1(t) * h_2(t)} \longrightarrow y(t)$$

$$y(t) = x(t) * (h_1(t) * h_2(t))$$

$$\therefore\ y(t) = (x(t) * h_1(t)) * h_2(t) = x(t) * (h_1(t) * h_2(t))$$

$\therefore$ LTI Systems obeys 3 properties.

Transfer function of LTI System :- $\dfrac{\text{output Signal}}{\text{input Signal}}$

$$\underset{\substack{\text{Input} \\ X(j\omega)}}{\longrightarrow} \overset{h(t)}{\underset{\substack{H(j\omega)\ \text{(or)} \\ H(s)}}{\boxed{\text{System}}}} \longrightarrow \underset{\substack{y(t)}}{\text{output}}$$

o/p spectrum $Y(j\omega)$ (or) $Y(s)$

It is defined as (output) the Ratio of Output (Voltage) Spectrum by input (voltage) Spectrum

(or)

o/p signal in frequency (or) i/p signal in frequency domine

$$\text{T.F}\ H(j\omega)\ \text{(or)}\ H(s) = \frac{Y(j\omega)}{X(j\omega)} = \frac{Y(s)}{X(s)}$$

$Y(j\omega) = X(j\omega) \cdot H(j\omega)$ Response

$F^{-1}[Y(j\omega)] = \overline{y(t)}^{\,o/p} = F^{-1}[X(j\omega) \cdot H(j\omega)]$

Magnitude Spectrum is Even function

phase Spectrum is odd function $\angle H(\omega) = -\angle H(-\omega)$

↑

properties of Transfer function

(2017) Calculate 5) Consider System function of LTI system is $\frac{1}{j\omega+2}$. what is the o/p of the System when the i/p Signal is $(0.8)^t u(t)$.

$H(j\omega) = \frac{1}{j\omega+2}$

$x(t) = (0.8)^t u(t)$

$y(t) = ?$

$h(t) = F^{-1}[H(j\omega)] = F^{-1}\left(\frac{1}{j\omega+2}\right) = e^{-2t} u(t)$

$y(t) = x(t) * h(t)$

$= \int_{-\infty}^{\infty} x(t-\tau) \cdot h(\tau)\, d\tau$

for simplify

$= \int_{0}^{\infty} e^{-2\tau} (0.8)^t$

$= (0.8)^t \int_{0}^{\infty} ((0.8)e^2)^{-\tau} d\tau$

Consider $[(0.8)e^{-2}]^{-1}$ is taken as 'a'.

$$y(t) = (0.8)^t \cdot \int_0^\infty a^\tau \, d\tau$$

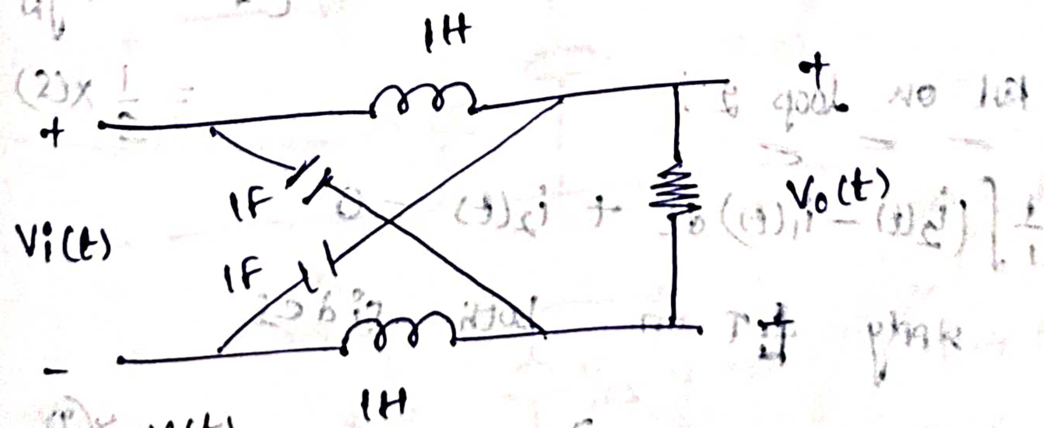
$$= (0.8)^t \cdot \frac{a^\tau}{\log a}\Bigg|_0^\infty$$

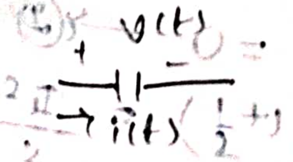
$$= (0.8)^t \left[\frac{((0.8)\cdot e^{-2})^{-\tau}}{\log(0.8e^{-2})^{-1}}\right]_0^\infty$$

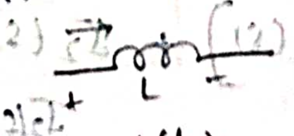
$$= \frac{(0.8)^t}{\log(0.8e^{-2})^{-1}}\left[0-1\right] = \frac{(0.8)^t}{\log(0.8e^{2})}$$

$$= \frac{(0.8)^t}{\log 0.8 + \log e^2} = \frac{(0.8)^t}{\log 0.8 + 2}$$

→ find the transfer function of the following lattice network.

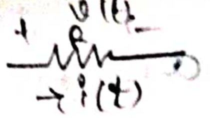




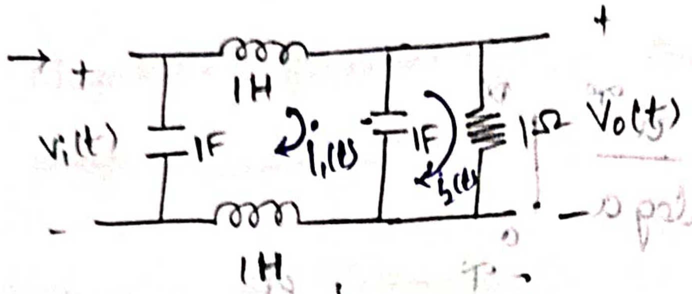


$v(t) = \frac{1}{C}\int i(t)\,dt$ | $v(t) = L\frac{di(t)}{dt}$

$i(t) = C\frac{dv(t)}{dt}$ | $i(t) = \frac{1}{L}\int v(t)\,dt$



$v(t) = R\,i(t)$



Apply KVL :- loop 1

$$-V_i(t) - 1.\frac{d\,i_1(t)}{dt} + \frac{1}{1}\int (i_1(t) - i_2(t))dt + 1.\frac{d\,i_1(t)}{dt} = 0$$

$$V_i(t) = \frac{d i_1(t)}{dt} + \int (i_1(t) - i_2(t))\,dt + \frac{d i_1(t)}{dt}$$

Take laplace transform on both sides

$$Vi(s) = s.I_1(s) + \frac{1}{s}[I_1(s) - I_2(s)] + s\,I_1(s) \rightarrow 1$$

$F\left[\frac{d}{dt}x(t)\right] = j\Omega X(j\Omega)$

$L\left(\frac{d}{dt}x(t)\right) = s\,X(s)$

$L\left[\int x(t)\,dt\right] = \frac{1}{j\Omega}X(j\Omega)$

$= \frac{1}{s}X(s)$

Apply KVL on loop 2 :-

$$\frac{1}{1}\int (i_2(t) - i_1(t))\,dt + i_2(t) = 0$$

Apply L.T. on both sides

$$\frac{1}{s}\left[I_2(s) - I_1(s)\right] + I_2(s) = 0 \rightarrow 2$$

$$I_2(s)\left(1+\frac{1}{s}\right) = \frac{I_1(s)}{s} \rightarrow 2$$

Take ohms law at o/p loop :-

$$V_0(t) = i_2(t).1$$

$$V_0(t) = I_2(s) \rightarrow 3$$

Sub Eq ③ in Eq ②

$$V_{os}\left(1+\frac{1}{s}\right) = \frac{1}{s} I_1(s)$$

$$I_1(s) = V_o(s)\cdot(1+s) \rightarrow ④$$

Sub Eq ④ in Eq ①

$$V_i(s) = I_1(s)\left[s+\frac{1}{s}+s\right] - \frac{1}{s} I_2(s)$$

Sub Eq ④ in above eqn

$$V_i(s) = V_{os}(1+s)\left(\frac{2s^2+1}{s}\right) - \frac{1}{s}V_o(s)$$

$$V_i(s) = V_{os}\left[\frac{(1+s)(1+2s^2)}{s} - \frac{1}{s}\right]$$

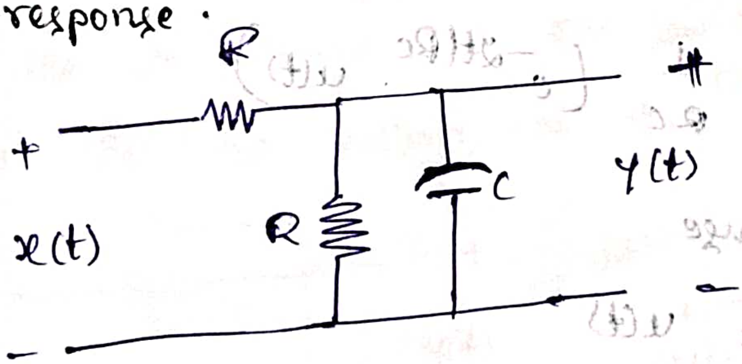
$$\frac{1+2s^2+s+2s^3-1}{s}$$

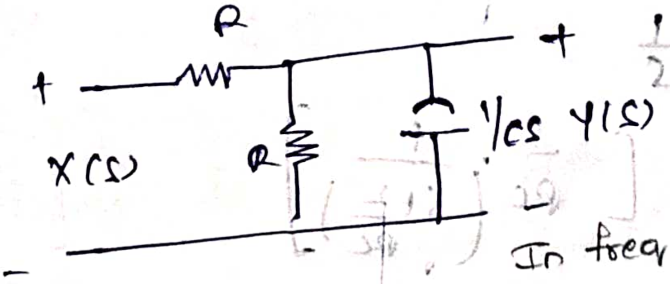
$$= \frac{2s^3+2s^2+s}{s}$$

$$= 2s^2+2s+1$$

$$\frac{V_{os}}{V_i(s)} = \frac{1}{2s^2+2s+1}$$

→ Consider the following circuit and find the impulse response and step response.





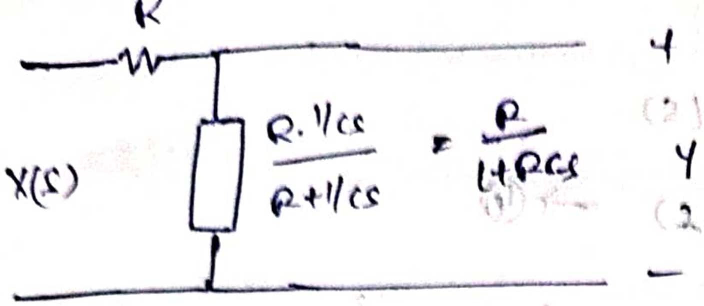
$R \rightarrow R$

$C \rightarrow 1/cs$

$L \rightarrow Ls$

In frequency domine All are consider as impedences

∴ Resistor



$$Y(s) = \left[\frac{X(s)\cdot \frac{R}{1+Rcs}}{R+\frac{R}{1+Rcs}}\right] = \frac{X(s)\cdot R}{R(2+Rcs)}$$

$$= X(s)\cdot\frac{1}{2+Rcs}$$

$$= X(s)\cdot\frac{1}{RC}\left[\frac{1}{s+2/RC}\right]$$

(i) find impulse Response

$x(t) = \delta(t)$

$X(s) = 1 \rightarrow \frac{1}{a+j\omega} = \frac{1}{a+s}$

$X(s) = \frac{1}{RC}\left[\frac{1}{s+\frac{2}{RC}}\right]$

$y(t) = \frac{1}{RC}\left(e^{-2t/RC}\,u(t)\right)$

(ii) Step Response

$x(t) = u(t)$

$X(s) = \frac{1}{s}$

$$Y(s) = \frac{1}{s}\left[\frac{1}{RC}\left(\frac{1}{s+\frac{2}{RC}}\right)\right] = \frac{1}{RC}\left[\frac{1}{s\left(s+\frac{2}{RC}\right)}\right]$$

$$= \frac{1}{RC}\left[\frac{A}{s}+\frac{B}{s+\frac{2}{RC}}\right]$$

$$\left(\frac{1}{RC}\cdot\frac{1}{s+\frac{2}{RC}}\right) = \frac{A}{s} + \frac{B}{s+\frac{2}{RC}}$$

$$1 = A\left(s+\frac{2}{RC}\right) + B(s)$$

$s=0$

$$1 = A\left(\frac{2}{RC}\right) \Rightarrow A = +\frac{RC}{2}$$

RC = 0 Compare

$$0 = A(s) + B(s)$$

$$A+B = 0$$

$$B = -\frac{RC}{2}$$

$$Y(s) = \frac{1}{RC}\left[\frac{RC/2}{s} + \frac{-RC/2}{s+2/RC}\right]$$

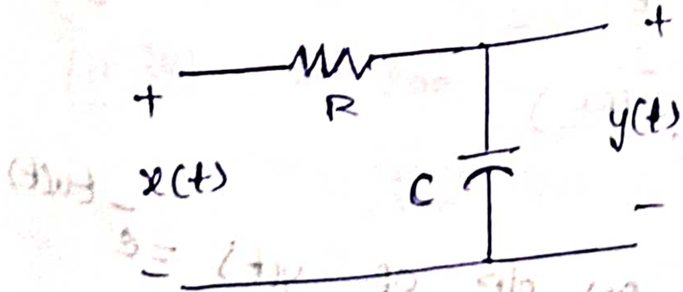
$$= \frac{1}{2}\left[\frac{1}{s} - \frac{1}{s+2/RC}\right]$$

ILT

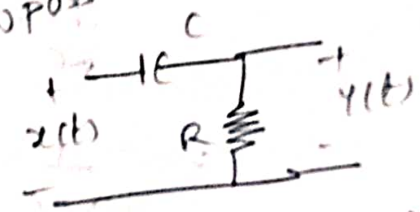
$$y(t) = \frac{1}{2}\left[u(t) - e^{\frac{-2t}{RC}}u(t)\right]$$

$$= \frac{1}{2}\left(1-e^{\frac{-2t}{RC}}\right]u(t)$$

→ find the o/p voltage of RC low pass circuit (low frequency will be passing) (and) for the i/p voltage of $t.e^{-t/RC} = x(t)$



→ low pass circuit



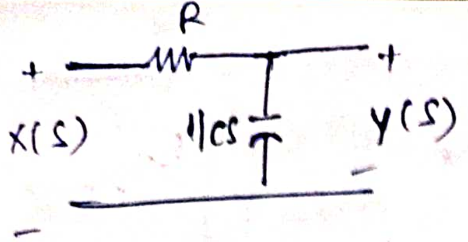
↓

High pass circuit

(→ Capacitance is high at high frequencies.

→ at low frequency capacitance is low)

$C = \frac{1}{j\omega C} = \frac{1}{j2\pi fC}$ f is ↑ C is ↓ (S.C) $C = \frac{1}{j\omega C} = \frac{1}{j2\pi fC}$ if $f=0$ $C=\infty$ (O.C)



$x(t) = t \cdot e^{-t/RC}$

$$Y(S) = \frac{X(S) \cdot \frac{1}{CS}}{R + 1/CS} \rightarrow \text{V.D.L}$$

$$H(S) = \frac{Y(S)}{X(S)} = \frac{1}{1+RCS}$$

$$X(S) = \frac{1}{(S+1/RC)^2} \quad \left(\rightarrow \because e^{-at} = \frac{1}{a+j\omega} = \frac{1}{a+s} \right)$$

$$Y(S) = \frac{1}{(S+1/RC)^2} \cdot \frac{1}{1+RCS}$$

$$Y(S) = \frac{1}{RC(S+1/RC)^3}$$

$$y(t) = L^{-1}[Y(S)]$$

$$= L^{-1}\left[\frac{1}{RC}\left(\frac{1}{(S+1/RC)^3}\right)\right]$$

$$= \frac{1}{RC} L^{-1}\left(\frac{1}{(S+1/RC)^3}\right)$$

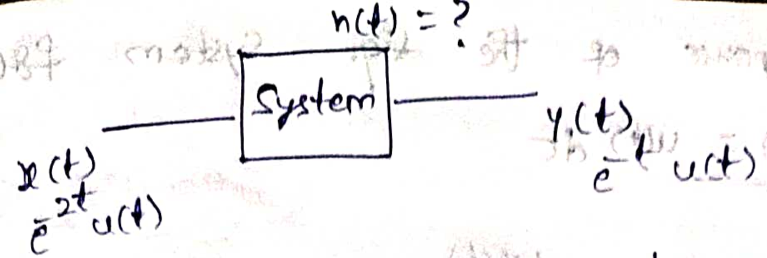
If contains 2 then $\downarrow$ $L^{-1}\left(\frac{1}{S+a}\right) = e^{-at} \cdot t$

it contains 3 then $e^{-at} \cdot \frac{t^2}{2}$

$$= \frac{1}{RC} \cdot e^{-t/RC} \frac{t^2}{2}$$

$$= \frac{1}{2RC} t^2 (e^{-t/RC})$$

→ A system produce an o/p of $y(t) = e^{-t}u(t)$ and i/p of $e^{-2t}u(t)$. Determine impulse & frequency Response of the function.



$$X(S) = \frac{1}{S+2}, \quad Y(S) = \frac{1}{S+1}$$

$$T.F = H(S) = \frac{Y(S)}{X(S)} = \frac{\frac{1}{S+1}}{\frac{1}{S+2}} = \frac{S+2}{S+1} = 1 + \frac{1}{S+1}$$

$$h(t) = L^{-1}[H(S)]$$

$$= L^{-1}\left[\frac{S+2}{S+1}\right]$$

$L[\delta(t)] = 1$

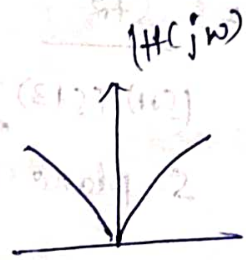
$L^{-1}(1) = \delta(t)$

$$= L^{-1}\left(1 + \frac{1}{S+1}\right) = L^{-1}(1) + L^{-1}\left(\frac{1}{S+1}\right)$$

$$= \delta(t) + e^{-t}u(t)$$

$$H(S) = \frac{S+2}{S+1}$$

$$H(j\omega) = \frac{2+j\omega}{1+j\omega}$$



$$|H(j\omega)| = \frac{\sqrt{4+\omega^2}}{\sqrt{1+\omega^2}}$$

$$\underline{/H(j\omega)} = \frac{\tan^{-1}\left(\frac{\omega}{2}\right)}{\tan^{-1}\left(\frac{\omega}{1}\right)}$$

$$= \tan^{-1}\left(\frac{\omega}{2}\right) - \tan^{-1}\left(\frac{\omega}{1}\right)$$

physical Exist (stable System)

→ Consider a stable LTI System is characterized by differential Eqn

$$\frac{d^2y(t)}{dt^2} + 4\frac{dy(t)}{dt} + 3y(t) = \frac{dx(t)}{dt} + 2x(t)$$

Find the Response of the ~~figr~~ System for the i/p Signal $x(t) = u(t)\ dt$

Differential eqn → Continuous System (t)

$$\frac{d^2 y(t)}{dt} + 4\frac{d\,y(t)}{dt} + 3y(t) = \frac{d\,x(t)}{dt} + 2x(t)$$

Difference eqn → Discret System (n)

$3y(n-2) + 5y(n-1)$

$$\left[s^2 y(s) - y(0) - \dot{y}(0)\right] + \left[s\,Y(s) - Y(0)\right] + 3Y(s)$$

$$= \left[s\,X(s) + 2X(s)\right]$$

neglet all initial Conditions

Second / 1st order differentiation is present that means initial Conditions are present

$$s^2 y(s) + s\,Y(s) + 3X(s) = sX(s) + 2X(s)$$

$y(t) \leftrightarrow Y(j\omega)\ ;\ Y(s)$

$\frac{d}{dt} y(t) \leftrightarrow j\omega Y(t)$

$sY(s)$

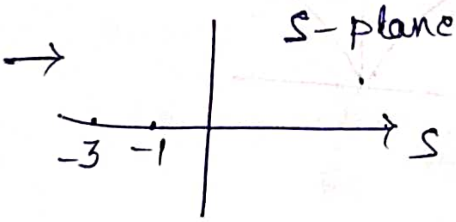
$$\frac{Y(s)}{X(s)} = \frac{s+2}{s^2+4s+3}$$

$$= \frac{s+2}{(s+1)(s+3)} \quad \begin{matrix} -N \text{ (No of zeroes)} \\ -D \text{ (No of poles)} \end{matrix}$$

$s = -1, -3$

No of poles > No of Zero's

→ S-plane



→ If the poles are present in left half of S-plane then it is stable.

→ If the poles are present in right half of S-plane then it is unstable.

$$Y(s) = X(s)\cdot\frac{s+2}{(s+1)(s+3)}$$

But $X(s) = \frac{1}{s+1}$ $\left(\because e^{-t}u(t) \to \frac{1}{s+1}\right)$ (Generally not Exist)

$$Y(s) = \frac{1}{s+1}\left(\frac{s+2}{(s+1)(s+3)}\right)$$

$$Y(s) = \frac{s+2}{(s+1)^2(s+3)}$$

$$= \frac{A}{s+1} + \frac{B}{(s+1)^2} + \frac{C}{s+3}$$

$$\frac{A(s+1)(s+3) + B(s+3) + C(s+1)^2}{(s+1)^2(s+3)}$$

$$= \frac{(s+2)}{(s+1)^2(s+3)}$$

$$s+2 = A(s+1)(s+3) + B(s+3) + C(s+1)^2$$

put $s = -1$

$1 = 2B \Rightarrow B = \frac{1}{2}$

sub $s = -3$

$-1 = 4C \Rightarrow C = -\frac{1}{4}$

put $s = 0$

$2 = 3A + 3B + C$

$2 = 3A + 3\cdot\frac{1}{2} + -\frac{1}{4}$

$3A = 2 + \frac{1}{4} - \frac{3}{2}$

$3A = \frac{8+1-6}{4} \Rightarrow \frac{1}{4}$

$$Y(s) = \frac{\frac{1}{4}}{s+1} + \frac{1/2}{s+1} - \frac{1/4}{s+3}$$

$$y(t) = \frac{1}{4} e^{-t} u(t) + \frac{1}{2} t e^{-t} u(t) - \frac{1}{4} e^{-3t} u(t).$$

→ Consider a stable LTI System characterised by following differential eqn $\frac{dy(t)}{dt} + 2y(t) = x(t)$ find the impulse Response.

$$\frac{dy(t)}{dt} + 2y(t) = x(t)$$

$$sY(s) + 2Y(s) = X(s)$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{1}{s+2}$$

$(X(s) = 1 \therefore$ because input is impulse $)$

$$Y(s) = \frac{1}{s+2}$$

$$L^{-1}\left[\frac{1}{s+2}\right]$$

$$= e^{-2t}u(t)$$

Signal Band width :- The band of frequencies that contains most of the signal Energy is known as Band width of the signal.

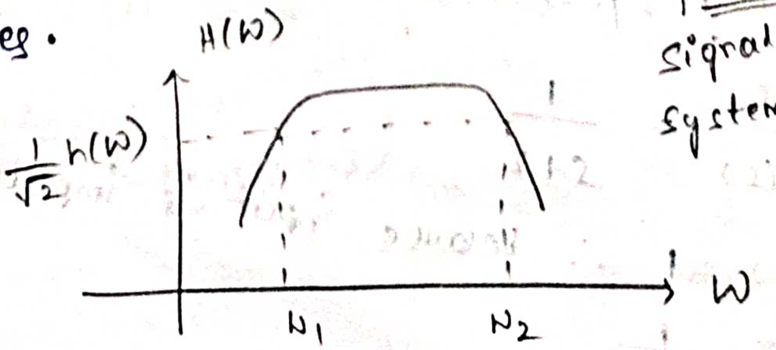
→ The B.W is selected such that it contains around 95% of total Energy depending on the precition.

The Range of frequency that is allowed by the system is known as B.W.

→ Spectral Energy is lies b/w $-\infty$ to ∞.

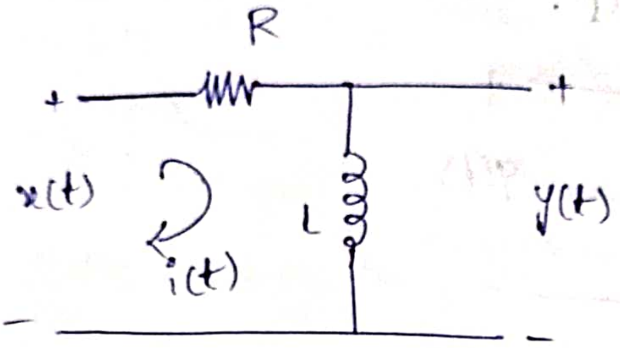
System Band width :- for distortionless transmission we need system has infinite B.W. due to physical limitations it is impossible to construct infinite B.W.

→ So the Distortionless transmission is achieved by the system with finite B.W is the magnitude is constant over the Range of frequencies.

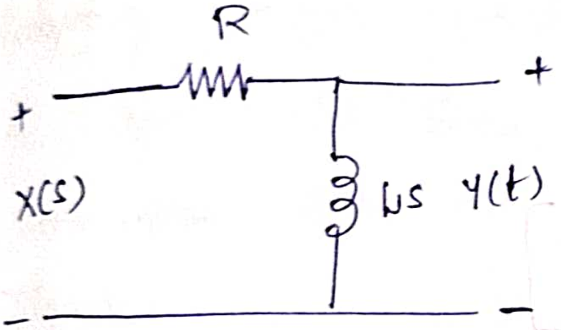


Perfect
signal B.W - low
system B.W - High

→ find the impulse Response and transfer function of the following RL filter shown in figure and what is its frequency Response



At low frequency
o/p is 0
At high frequency
✓ o/p is infinity
j = 2πfL



$$Y(s) = X(s)\left[\frac{Ls}{R+Ls}\right]$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{Y(s)}{L(s+R/L)} = \frac{s}{s+RL} = \frac{j\omega}{j\omega+R/L}$$

$$= F^{-1}\left[1-\frac{R}{L}\left(\frac{1}{j\omega R/L}\right)\right]$$

$j\omega R/L)\ j\omega\ (1$
$\frac{j\omega R/L}{-R/L}$

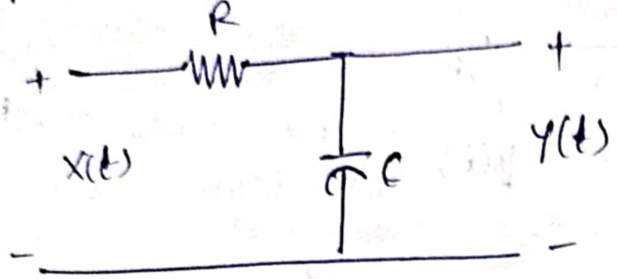
$$= \delta(t) - \frac{R}{L}e^{-\frac{R}{L}t}u(t)$$

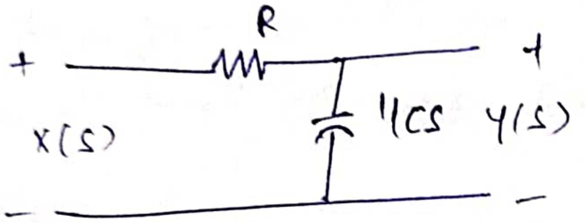
$$H(j\omega) = \frac{j\omega}{j\omega + R/L}$$

$$|H(j\omega)| = \frac{\sqrt{0^2+\omega^2}}{\sqrt{\omega^2+(R/L)^2}} = \frac{\omega}{\sqrt{\omega^2+(R/L)^2}}$$

$$\angle H(j\omega) = \frac{\pi}{2} + \left(-\tan^{-1}\left(\frac{\omega}{R/L}\right)\right) = \frac{\pi}{2} - \tan^{-1}\left(\frac{\omega}{R/L}\right)$$

→ find the impulse Response and transfer function of the following filter and Sketch the frequency Response characteristics.





$$Y(s) = X(s)\left[\frac{1/Cs}{R + 1/Cs}\right]$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{1}{1+RCs} = \frac{1}{RC\left(s+\frac{1}{RC}\right)}$$

Sub $s = j\omega$

$$H(j\omega) = \frac{1}{RC\left(j\omega + \frac{1}{RC}\right)}$$

$$h(t) = F^{-1}\left[H(j\omega)\right] = \frac{1}{RC} e^{-\frac{1}{RC}t} u(t)$$

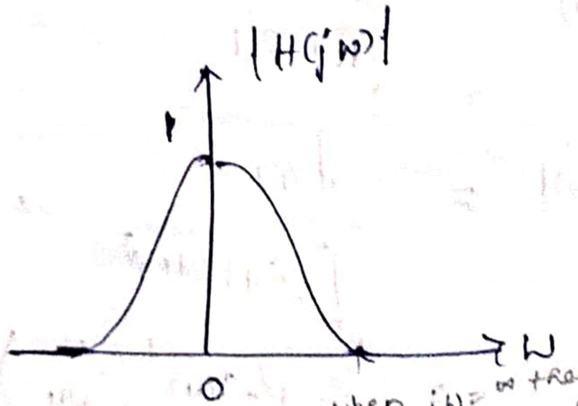
$$|H(j\omega)| = \frac{1}{RC}\frac{1}{\sqrt{\omega^2 + (1/RC)^2}}$$

mega micro

10^{+6} 10^{-6}

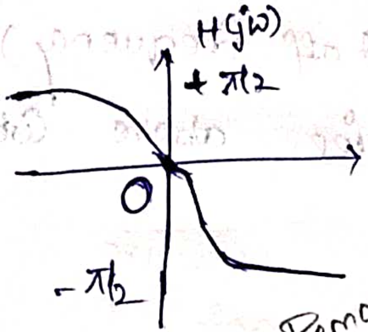
let $RC = 1$

$$|H(j\omega)| = \frac{1}{\sqrt{1+\omega^2}}$$



when $j\omega = \infty$ then we get 0

$\angle H(j\omega) = -\tan^{-1}(\omega)$ $\frac{1}{1+j\omega}$ $-\tan^{-1}\left(\frac{\omega}{1}\right)$



→ to Remove unwanted frequencies.

Ideal filter
↓
The filter cut off frequency is sharp and phase is linear

Ideal filter characteristics :- A filter is a on frequency selective network. It allows the transmissi of signals of certain frequencies with less attenuation and it rejects all other frequency signals.

→ at Zeroth order it Removes unwanted frequency to get 95% pure filter. As we increases the order (i.e., first order, second order, ...) we get strong frequency

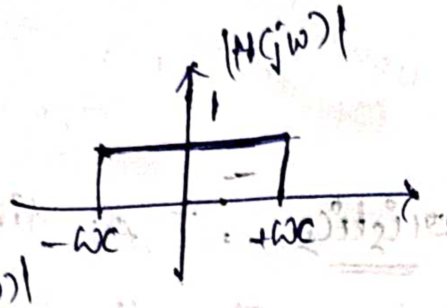
→ In ideal filter has very sharp cut off characteristics.

→ Filters are classified according to their frequency response characteristics. these are

1. low pass filter (LPF)
2. High pass filter (HPF)
3. Band pass filter (BPF)
4. Band stop filter (BSF) (or) (BEF) or (BRF)
   Elimination Reject
5. APF (All pass filter) → phase

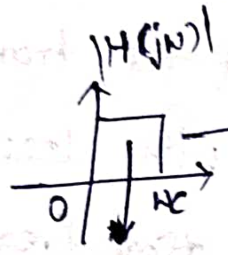
Ideal

low pass filter :- It tronsmit all the signals below Certain frequency W_c (Cut off frequency) and reject all other frequencies above Cut off frequencies



below WC gain = 1
i.e., i/p = o/p

After WC, gain = 0
i.e., o/p = 0



→ Stop band

pass Band

$$\therefore \ |H(j\omega)| = 1 \qquad -W_c \le |W| \le W_c$$
$$= 0 \qquad \text{otherwise}$$

$$|H(j\omega)| = 1 \qquad |W| \le W_c$$
$$= 0 \qquad \text{otherwise}$$

$|W| \le W_c$

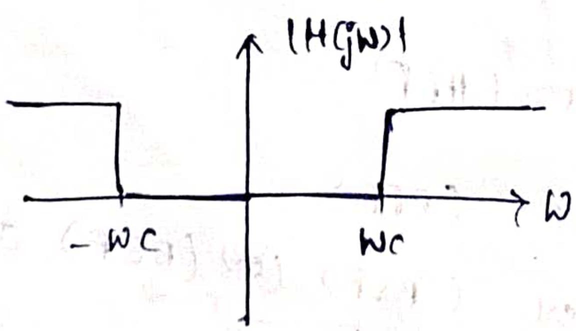
Noise
Distortion -

phase Response is linear

Ideal High pass filter :-

whatever signal is applied i/p that should be pass as o/p

→ It tronsmit all the frequency signal above Cut off frequency with no distortion and attenuate ~~with~~ all the frequencies below Cut off frequency



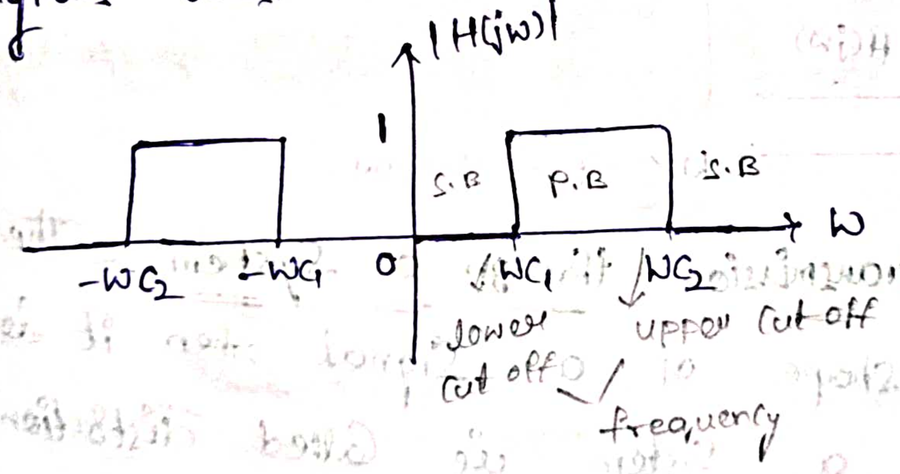
$$|H(j\omega)| = \begin{cases} 0 & |\omega| \le \omega_c \\ 1 & |\omega| \ge \omega_c \end{cases} \quad \text{or} \quad \begin{cases} 1 & |\omega| \ge \omega_c \\ 0 & \text{otherwise} \end{cases}$$

| frequency $> \omega_c$
gain $= 1$, $i/p = o/p$

frequency $< \omega_c$
gain $= 0$, $o/p = 0$

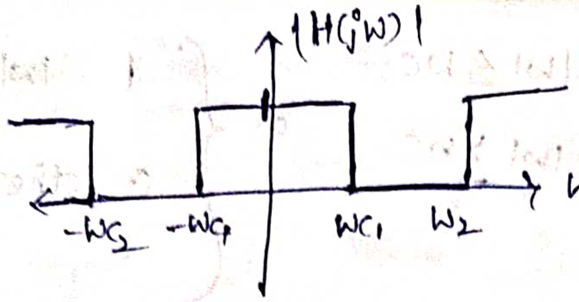
Ideal → Digital modulation techniques

→ **Band pass filter** :- In this filter pass all the frequency signals with in certain frequency band $(\omega_2 - \omega_1)$ and attenuate all the frequency signals outside this band.



$$|H(j\omega)| = \begin{cases} 1 & \omega_{c_1} \le |\omega| \le \omega_{c_2} \\ 0 & \text{otherwise} \end{cases} \quad \text{or} \quad \begin{cases} 0 & |\omega| < \omega_{c_1},\ |\omega| > \omega_{c_2} \\ 1 & \omega_{c_1} \le \omega \le \omega_{c_2} \end{cases}$$

→ **Ideal Band stop filters** :- In this filter stop all the frequency signals with in certain frequency band $(\omega_2 - \omega_1)$ and pass all the frequency signals outside this band.

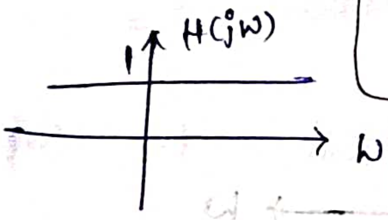


$$|H(jw)| = \begin{cases} 1 & |w| \le wc_1 ,\ |w| \ge wc_2 \\ 0 & \text{otherwise} \end{cases} \quad (81)$$

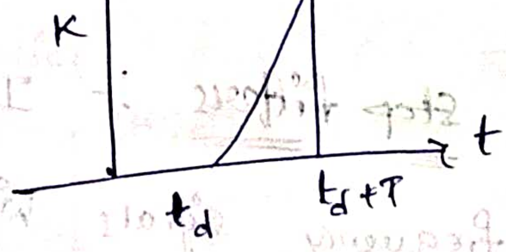
$$= \begin{cases} 1 & |w| \le wc_1 ,\ |w| \ge wc_2 \\ 0 & wc_1 \le |w| \le wc_2 \end{cases}$$

Ideal

All pass filter :- It pass all the filters with gain is equal to 1. (i.e., where ever i/p signal is passed in to o/p signal)



Distortion less transmission through a system :- The change of shape of a signal when it is passed through a system is called distortion. (when ever magnitude is constant and phase is linear)



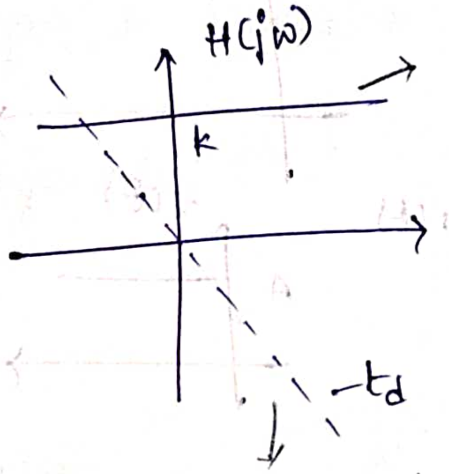
$$y(t) = k\, x(t-t_d)$$

$$Y(jw) = K e^{-jwt_d} X(jw)$$

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = k \cdot e^{-j\omega t_d} \Rightarrow |H(j\omega)| \cdot e^{j\phi(\omega)}$$

$$|H(j\omega)| = k$$

$$\angle H(j\omega) = -\omega t_d \rightarrow \text{like } y = -mx \text{ (no Distortion)}$$

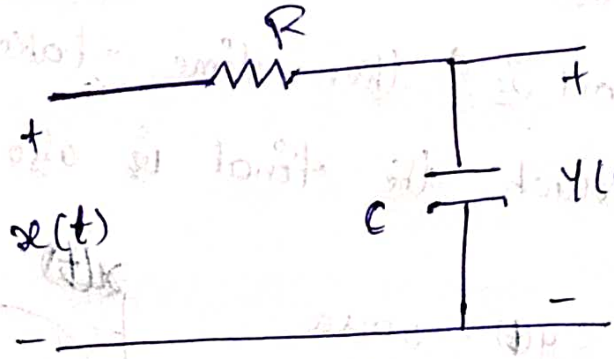


→ Magnitude plot (Constant)

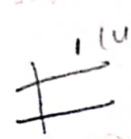
↓ phase plot ($\because y = -mx$)

Relationship between Band width and Rise time :-

Rise time → the time taken by the Capacitor to Rise from 10% to 90% of the final Voltage



$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{\frac{1}{j\omega C}}{R + \frac{1}{j\omega C}} = \frac{1}{1 + sRC} = \frac{1}{RC\left(s + \frac{1}{RC}\right)}$$

$x(t) =$ 

$$\frac{Y(s)}{X(s)} = \frac{1}{RC\left(s + \frac{1}{RC}\right)}$$

$L\{1\} = \frac{1}{s}$

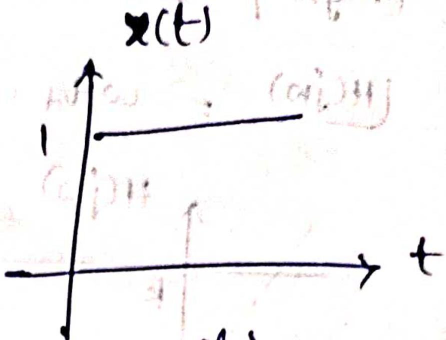
$X(s) = \frac{1}{s}$

$$Y(s) = X(s) \frac{1}{RC\left(s + \frac{1}{RC}\right)} = \frac{1}{s} \cdot \frac{1}{RC\left(s + \frac{1}{RC}\right)}$$

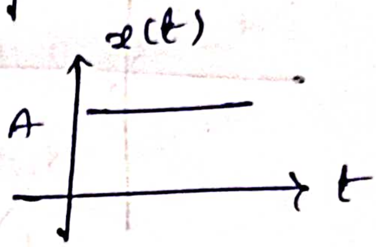
$$y(s) = \left(\frac{A}{s} + \frac{B}{\left(s+\frac{1}{RC}\right)}\right) \cdot \frac{1}{RC}$$

$$y(t) = \left(1 - e^{-t/RC}\right) u(t)$$

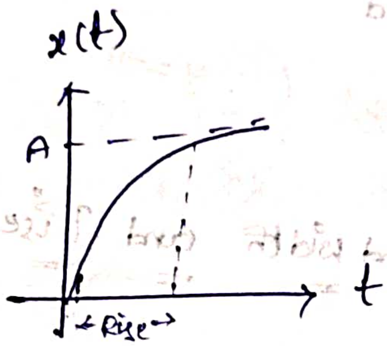
$$\Rightarrow y(t) = \left(1 - e^{-t/RC}\right) u(t)$$



$$\Rightarrow y(t) = A\left(1 - e^{-t/RC}\right) u(t)$$



Waveform :-

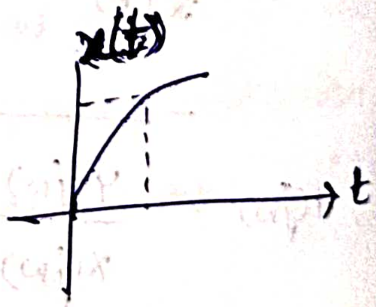


If the time Constant is ↑ then time taken by the Capacitor to reach the final is also increases.

→ RC = 1 m sec y(t) = 5 msec

RC = 10 m sec y(t) = 50 msec

RC = 0.1 msec y(t) = 0.5 msec



Rise :- It is defined as the time taken by the Capacitor to rise from 10% to 90% of the final Voltage.

→

Let us Consider the transfer function of ideal low pass filter is :-

from the definition of low pass filter

$$H(j\omega) = |H(j\omega)| \cdot e^{-j\omega t_d}$$

Cut off frequency

$$H(j\omega) = \begin{cases} 1 & |\omega| \leq \omega_c \\ 0 & |\omega| > \omega_c \end{cases}$$

$$H(\omega) = \begin{cases} e^{-j\omega t_d} & -\omega_c \leq \omega \leq \omega_c \\ 0 & \text{other wise} \end{cases}$$

Apply Inverse transform of $H(\omega)$

$$h(t) = F^{-1}(H(\omega))$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} H(\omega)\, e^{j\omega t}\, dt$$

$$h(t) = \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} e^{-j\omega t_d} \cdot e^{j\omega t}\, d\omega$$

$$= \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} e^{j\omega(t-t_d)}\, d\omega$$

$$= \frac{1}{2\pi} \left[\frac{e^{j\omega(t-t_d)}}{j(t-t_d)} \Bigg|_{-\omega_c}^{\omega_c} \right]$$

$$= \frac{1}{2\pi j(t-t_d)}\left[e^{jW_c(t-t_d)} - e^{-jw(t-t_d)} \right]$$

$$= \frac{W_c}{W_c} \frac{\sin w_c(t-t_d)}{\pi (t-t_d)}$$

$$= \frac{w_c}{\pi}\left[\sin c^{W_c}(t-t_d) \right]$$ → This is called impulse Response
→ It is used in step function

Apply the step signal to i/p of low pass filter

$$y(t) = x(t) * h(t)$$

$$= u(t) * h(t)$$

$$= \int_{-\infty}^{\infty} u(t-\tau) h(\tau)\, d\tau$$

$$= \int_{-\infty}^{t} h(\tau)\, d\tau$$

$$= \int_{-\infty}^{t} \frac{W_c}{\pi} \operatorname{sinc} W_c(\tau - t_d)\, d\tau$$

let $W_c(\tau - t_d) = x \Rightarrow W_c\, d\tau = dx$

$$d\tau = \frac{dx}{W_c}$$

$$y(t) = \int_{-\infty}^{W_c(t-t_d)} \frac{\cancel{W_c}}{\pi} \operatorname{sinc} x \cdot \frac{dx}{\cancel{W_c}}$$

$$= \frac{1}{\pi}\left[\operatorname{sinc} x \Big|_{-\infty}^{W_c(t-t_d)} \right]$$

$$= \frac{1}{\pi}\left[\operatorname{sinc} W_c(t-t_d) - \operatorname{sinc}(-\infty) \right]$$

Continuation

telephone network → 300Hz - 3.4kHz

Actual audio signal → 0 to 5kHz

Voice Signal → 20 - 20kHz

Continuation ↓

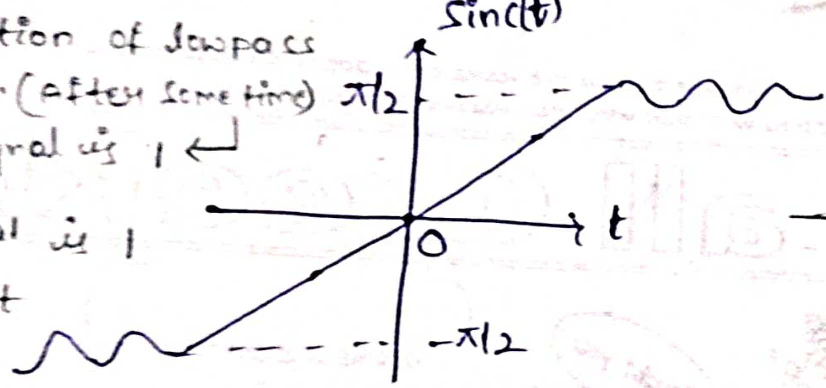
Sinc function is always odd function and

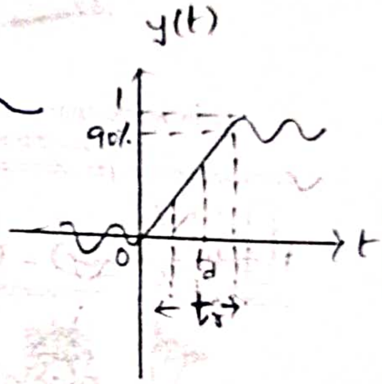
$\text{Sinc}(0) = 0$ and $\text{sinc}(\infty) = \frac{\pi}{2}$ and $\text{Sin}(-\infty) = -\frac{\pi}{2}$

According to Definition of lowpass filter. (after some time) i/p signal is 1

o/p signal is 1

i.e., unit step





At $W_c \to \infty$ the Response is

$$y(t) = \frac{1}{\pi}\left[\text{Sinc } W_c(t-t_d) + \frac{\pi}{2}\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi}{2} + \frac{\pi}{2}\right] = 1$$

At $W_c \to -\infty$ the Response is

$$y(t) = \frac{1}{\pi}\left[-\frac{\pi}{2} + \frac{\pi}{2}\right] = 0$$

→ The Rise time is defined as the time Required for response to Reach from 0% to 100% of final value.

→ The Rise time is Calculated for the above graph by Considering $t = t_d$ delay

$$\frac{d}{dt} y(t)\Big|_{t=t_d} = \frac{1}{t_r}$$

$$= \frac{1}{\pi}\left[\frac{\sin W_c(t-t_d)}{W_c(t-t_d)} \cdot W_c\right]$$

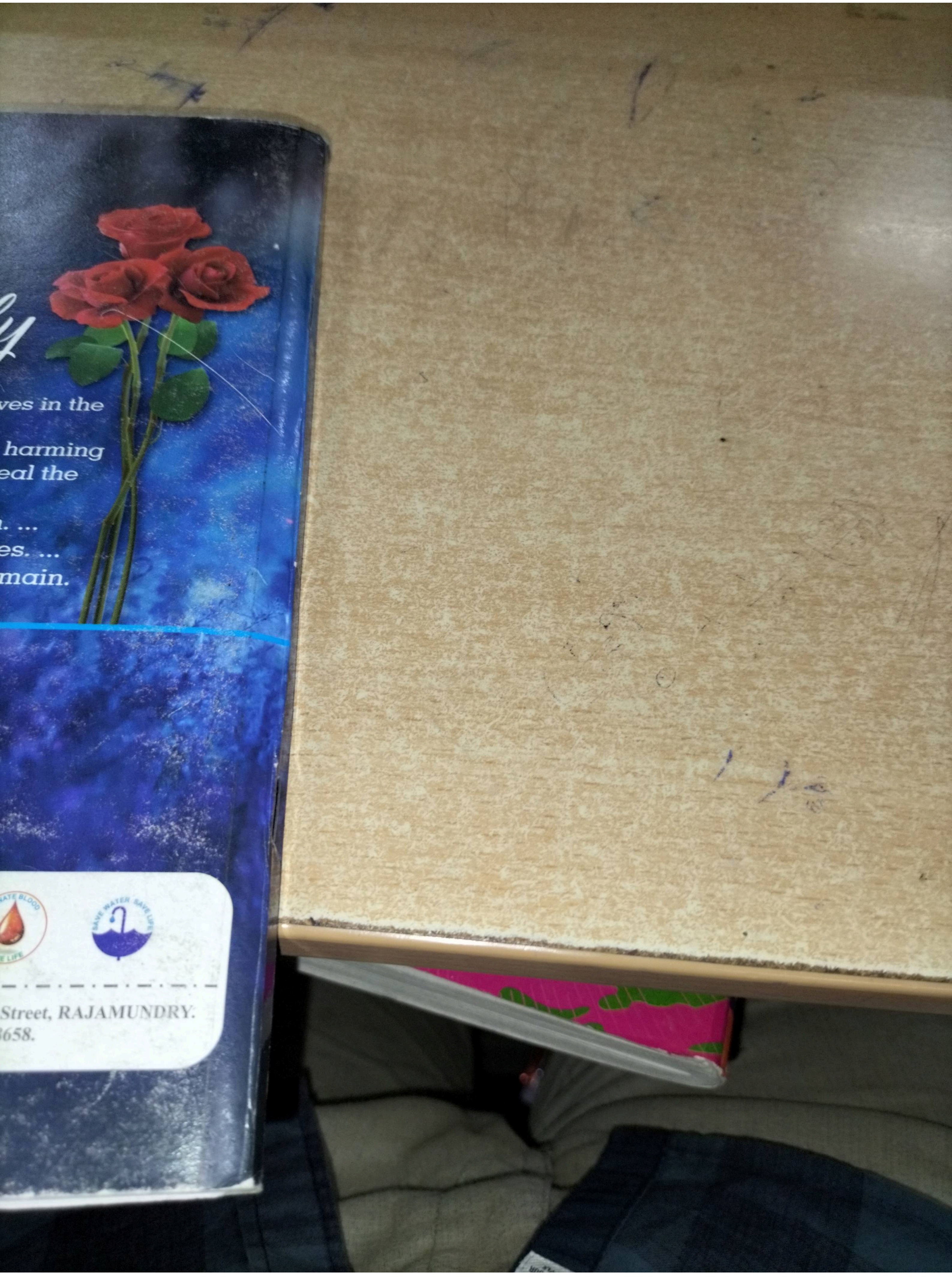
$$= \frac{W_c}{\pi}(1)$$

→ In low pass filter cut off frequency is called as Band width

$$t_r = \frac{\pi}{B.W}$$

$$\boxed{\text{Band width} \times \text{Rise time} = \text{Constant}}$$

= End =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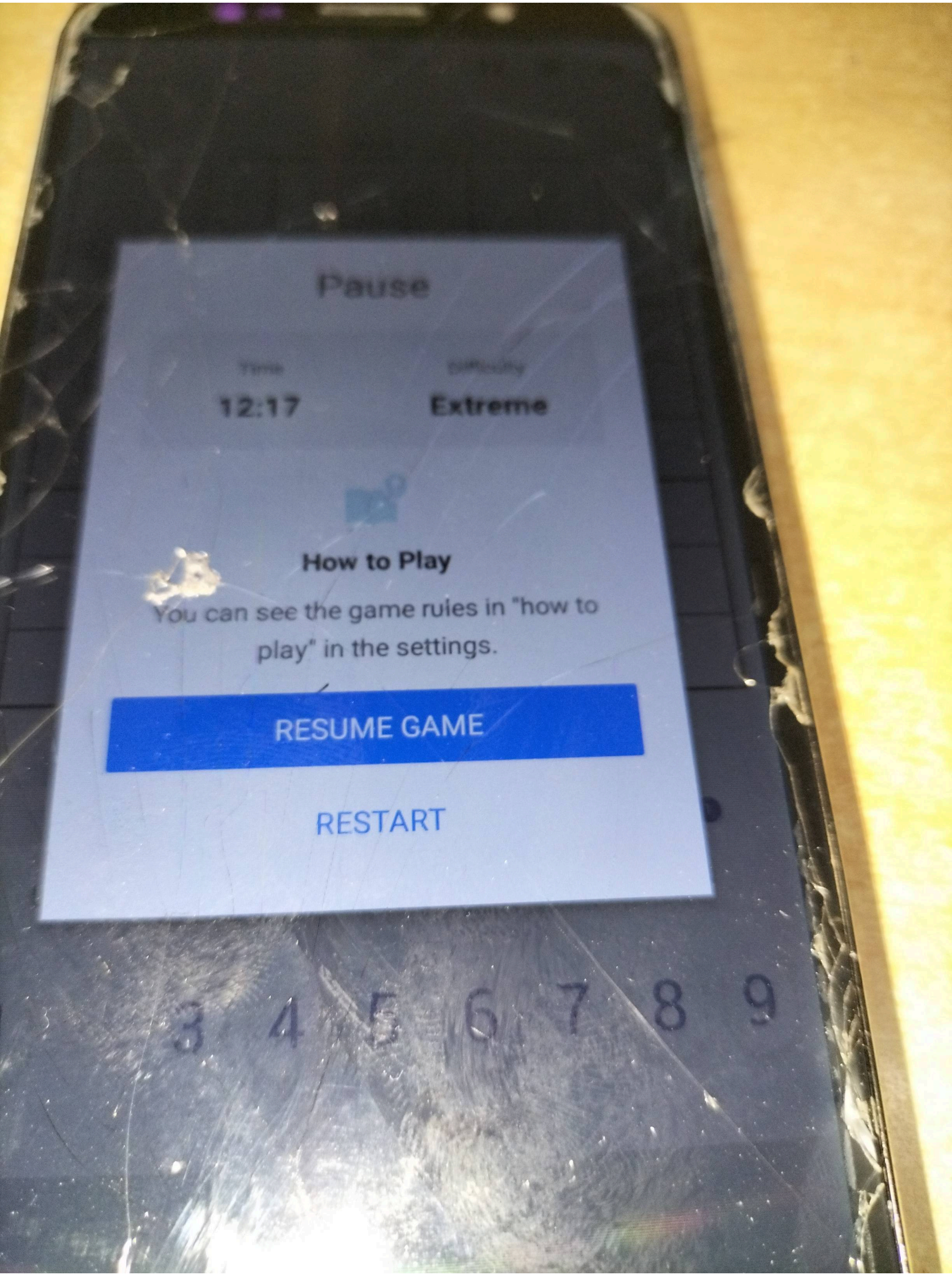
3658.


Srinivasa
GOLD
lovely
pleasing the senses or mind aesthetically.


lovely


Srinivasa
GOLD
lovely
pleasing the senses or mind aesthetically.


Pause
12:17
Extreme
How to Play
You can see the game rules in "how to play" in the settings.
RESUME GAME
RESTART
3 4 5 6 7 8 9

UNIT-5

LAPLACE TRANSFORMS AND Z-TRANSFORMS

LAPLACE TRANSFORMS:

28/10 1, 2, 6, 10, 11, 12, 14, 20, 21, 23, 25, 26, 28, 33, 35, 38, 41, 42, 48, 50, 55,

→ Laplace transform represents continuous time signals in terms of complex exponentials i.e e^{-st}. It is used to analyze the signals or functions which are not absolutely integrable.

→ More effectively continuous time signals can be analyzed using Laplace transform.

→ Laplace transform provides broader characterization compared to Fourier Transform

DEFINITION:

→ To transform a time domain signal x(t) to S-domain, multiply the signal by e^{-st} and then integrate from $-\infty$ to ∞.

→ The transformed signal is represented as X(S) and transformation is denoted by letter $\mathcal{L}$

Laplace transform is given as for continuous time signal x(t)

i.e

$$\boxed{X(S) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt} \rightarrow ①$$

where 's' is complex in nature and given as $S = \sigma + j\omega$.

realpart / attenuation constant

imaginary / complex frequency

→ If x(t) is defined for $t \geq 0$ (i.e x(t) is causal) then

$$\boxed{\mathcal{L}\{x(t)\} = X(S) = \int_{0}^{\infty} x(t)\, e^{-st}\, dt} \rightarrow ②$$

TYPES OF LAPLACE TRANSFORM:

(i) Bilateral or two sided Laplace transform: If the integration is taken from $-\infty$ to ∞ as shown in eq(1) then it is called Bilateral L.T

(ii) Unilateral or one sided Laplace transform: If the integration is taken from 0 to ∞ as shown in eq(2) then it is called Unilateral LT

→ Useful in analysis of networks and solving differential equations

INVERSE LAPLACE TRANSFORM:

→ The s- domain signal X(s) can be transformed to time domain signal x(t) by using inverse laplace transform and is defined as

$$\boxed{L^{-1}[X(s)] = x(t) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} X(s)e^{st}\,ds}$$

→ The signal x(t) and X(s) are called Laplace transform pair

$$x(t) \underset{L^{-1}}{\overset{L}{\rightleftarrows}} X(s)$$

RELATION BETWEEN FOURIER TRANSFORM AND LAPLACE TRANSFORM:

Fourier transform is given as

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)\,e^{-j\omega t}\,dt \rightarrow (1)$$

→ FT can be calculated only if x(t) is absolutely integrable i.e

$$\int_{-\infty}^{\infty} |x(t)|\,dt < \infty \longrightarrow (2)$$

→ Laplace transform is written as

$$X(s) = \int_{-\infty}^{\infty} x(t)\,e^{-(\sigma+j\omega)t}\,dt \quad \text{putting } s=\sigma+j\omega$$

$$= \int_{-\infty}^{\infty} x(t)\,e^{-\sigma t}\cdot e^{-j\omega t}\,dt$$

$$= \left\{\int_{-\infty}^{\infty} x(t)\,e^{-\sigma t}\right\} e^{-j\omega t}\,dt \rightarrow (3)$$

comparing eq(3) with eq(1), laplace transform of x(t) is basically fourier transform of $x(t)e^{-\sigma t}$.

→ If $\sigma=0$, then above equation i.e $s=j\omega$ the above eqn

$$X(s) = \int_{-\infty}^{\infty} x(t)\,e^{-j\omega t}\,dt = X(j\omega) \text{ when } s=j\omega$$

→ It is basically fourier transform on imaginary ($j\omega$) axis in s-plane.

CONVERGENCE / REGION OF CONVERGENCE (ROC):

→ From eqn $\{\int_{-\infty}^{\infty}(x(t)e^{-\sigma t})\, e^{-j\omega t}dt\}$ we know that Laplace transform is basically the fourier transform of $x(t)e^{-\sigma t}$. If fourier transform of $x(t)e^{-\sigma t}$ exists then laplace transform of $x(t)$ exists.

→ $\int_{-\infty}^{\infty}|x(t)e^{-\sigma t}|\,dt < \infty$ must be absolutely integrable for fourier transform to exists.

→ Laplace transform of $x(t)$ will exist, if above condition is satisfied.

→ The range of values of 'σ' for which laplace transform converges is called ROC or region of convergence.

(or)

→ The Laplace transform of a signal given by $\int_{-\infty}^{\infty}x(t)e^{-st}dt$. The values of 's' for which the integral $\int_{-\infty}^{\infty}x(t)e^{-st}dt$ converges is called ROC.

PROBLEMS:

(1) Calculate the Laplace transform and ROC for $x(t) = e^{-at}u(t)$ (Right sided causal sgl)

Sol

$$x(t) = e^{-at}u(t) \quad \text{where } a>0$$
$$= e^{-at} \quad \text{for } t\geq 0$$

$$L[x(t)] = X(s) = \int_{-\infty}^{\infty}x(t)e^{-st}dt = \int_{-\infty}^{\infty}e^{-at}\cdot u(t)\, e^{-st}dt$$

$$= \int_{0}^{\infty}e^{-at}e^{-st}\,dt = \int_{0}^{\infty}e^{-(s+a)t}dt$$

$$= \frac{1}{-(s+a)}\left[e^{-(s+a)t}\right]_0^{\infty} = \frac{-1}{s+a}\left[0-1\right]$$

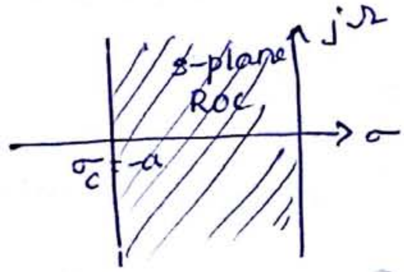
$$= \frac{1}{s+a}$$

ROC: $s > -a$

$$= -\left[\frac{e^{-(s+a)t}}{s+a}\right]_0^{\infty}$$

$$= -\left\{\lim_{t\to\infty}\left[\frac{e^{-(s+a)t}}{s+a}\right] - \lim_{t\to 0}\left[\frac{e^{-(s+a)t}}{s+a}\right]\right\}$$

$$X(s) = +\left[\frac{e^{-(s+a)\infty}}{-(s+a)} - \frac{e^{-(s+a)0}}{-(s+a)}\right].$$



$$s = j\omega + \sigma$$

$$L\{x(t)\} = \frac{e^{-K\times\infty}e^{-j\Omega\times\infty}}{s+a} + \frac{1}{s+a}$$

where $K = \sigma + a = \sigma - (-a)$.

when $\sigma > a$, $k = \sigma - (-a) =$ positive $\therefore e^{-K\infty} = e^{-\infty} = 0$

When $\sigma < -a$, $k = \sigma - (-a) =$ Negative $\therefore e^{-K\infty} = e^{+\infty} = \infty$

$\therefore$ $X(s)$ converges when $\sigma > -a$ and does not converge for $\sigma < -a$.

$\therefore$ When $\sigma > -a$, the $X(s)$ is given by

$$L\{x(t)\} = X(s) = \frac{-e^{-k\times\infty}e^{-j\Omega\times\infty}}{s+a} + \frac{1}{s+a} = \frac{-0\times e^{-j\Omega\times\infty}}{s+a} + \frac{1}{s+a} = \frac{1}{s+a}$$

2)

Properties of L.T :

(1) Amplitude Scaling

If $L[x(t)] = X(s)$ then $L[A\,x(t)] = A\,X(s)$

Proof:

$$X(s) = L\{x(t)\} = \int_{-\infty}^{\infty} x(t)\cdot e^{-st}\,dt$$

$$L\{A\,x(t)\} = \int_{-\infty}^{\infty} A\cdot x(t) e^{-st}\,dt = A\int_{-\infty}^{\infty} x(t) e^{-st}\,dt$$

$$= A\,X(s).$$

(2) Linearity:

If $L\{x_1(t)\} = X_1(s)$ and $L\{x_2(t)\} = X_2(s)$ then $L\{a_1x_1(t) + a_2x_2(t)\} = a_1X(s) + a_2X_2(s)$

proof

$$X_1(s) = L\{x_1(t)\} = \int_{-\infty}^{\infty} x_1(t) e^{-st}\,dt$$

$$X_2(s) = L\{x_2(t)\} = \int_{-\infty}^{\infty} x_2(t) e^{-st}\,dt$$

$$L\{a_1x_1(t) + a_2x_2(t)\} = \int_{-\infty}^{\infty} [a_1x_1(t) + a_2x_2(t)] e^{-st}\,dt$$

$$= a_1\int_{-\infty}^{\infty} x_1(t) e^{-st}\,dt + a_2\int_{-\infty}^{\infty} x_2(t) e^{-st}\,dt$$

$$= a_1X_1(s) + a_2X_2(s)$$

3) Time differentiation:

If $L\{x(t)\} = X(s)$ then $L\left\{\frac{d}{dt}x(t)\right\} = sX(s) - x(0)$; where $x(0)$ is value of $x(t)$ at $t=0$

proof

$$X(s) = L\{x(t)\} = \int_0^{\infty} x(t) e^{-st}\,dt$$

$$\therefore L\left\{\frac{d}{dt}x(t)\right\} = \int_0^{\infty} \frac{dx(t)}{dt}\, e^{-st} = \int_0^{\infty} e^{-st}\frac{dx(t)}{dt}\,dt$$

$\left\{ \because \int uv = u\int v - \int [du \int v] \right.$

$u = e^{-st} \mid v = \frac{dx(t)}{dt}$

$$= [e^{-st}\cdot x(t)]_0^{\infty} - \int_0^{\infty} -se^{-st} x(t)\,dt$$

$$= e^{-\infty}x(\infty) - e^{0}x(0) + s\int_0^{\infty} x(t) e^{-st}\,dt$$

$$= s\int_0^{\infty} x(t) e^{-st}\,dt - x(0) = sX(s) - x(0)$$

(4) Time Integration:

If $L[x(t)] = X(s)$ then $L\left\{\int x(t)\,dt\right\} = \frac{X(s)}{s} + \frac{\left[\int x(t)\,dt\right]_{t=0}}{s}$

proof

$$X(s) = L\{x(t)\} = \int_0^\infty x(t)e^{-st}dt$$

$$L\left\{\int x(t)\,dt\right\} = \int_0^\infty \left[\int x(t)\,dt\right] e^{-st}dt$$

$$= \left[\left[\int x(t)\,dt\right]\frac{e^{-st}}{-s}\right]_0^\infty - \int_0^\infty x(t)\cdot\frac{e^{-st}}{-s}\,dt$$

$$= \left[\int x(t)\,dt\right]\Big|_{t=\infty}\frac{e^{-\infty}}{-s} - \left[\int x(t)\,dt\right]\Big|_{t=0}\frac{e^{0}}{-s} + \frac{1}{s}\int_0^\infty x(t)e^{-st}dt$$

$$= \frac{1}{s}\left[\int x(t)\,dt\right]\Big|_{t=0} + \frac{1}{s}\int_0^\infty x(t)\,e^{-st}dt$$

$$= \frac{X(s)}{s} + \frac{\left[\int x(t)\,dt\right]\Big|_{t=0}}{s}$$

5) Frequency shifting:

If $L[x(t)] = X(s)$ then $L\left\{e^{\pm at}x(t)\right\} = X(s \mp a)$

proof

$$X(s) = L\{x(t)\} = \int_{-\infty}^\infty x(t)e^{-st}dt$$

$$\therefore L\{e^{\pm at}x(t)\} = \int_{-\infty}^\infty e^{\pm at}x(t)\cdot e^{-st}dt$$

$$= \int_{-\infty}^\infty x(t)\cdot e^{-(s\mp a)t}dt$$

$$= X(s\mp a).$$

6) Time shifting

If $L\{x(t)\} = X(s)$ then $L\{x(t\pm a)\} = e^{\pm as}X(s)$

Proof:
$$X(s) = L\{x(t)\} = \int_{-\infty}^{\infty} x(t) e^{-st} dt$$

$$L\{x(t \pm a)\} = \int_{-\infty}^{\infty} x(t \pm a) e^{-st} dt = \int_{-\infty}^{\infty} x(\tau) e^{-s(\tau \mp a)} d\tau$$

put $t \pm a = \tau$
$\therefore t = \tau \mp a$
$dt = d\tau$

$$= \int_{-\infty}^{\infty} x(\tau) e^{-s\tau} \times e^{\pm as} d\tau = e^{\pm as} \int_{-\infty}^{\infty} x(\tau) e^{-s\tau} d\tau$$

$$= e^{\pm as} \int_{-\infty}^{\infty} x(t) e^{-st} dt = e^{\pm as} (X(s))$$

(7) Frequency differentiation:

If $L\{x(t)\} = X(s)$ then $L\{t x(t)\} = -\frac{d}{ds} X(s)$

Proof
$$X(s) = L\{x(t)\} = \int_{-\infty}^{\infty} x(t) e^{-st} dt$$

$$\frac{d}{ds} X(s) = \frac{d}{ds} \int_{-\infty}^{\infty} x(t) e^{-st} dt$$

$$= \int_{-\infty}^{\infty} x(t) \left(\frac{d}{ds} e^{-st}\right) dt = \int_{-\infty}^{\infty} x(t) (-t e^{-st}) dt$$

$$= \int_{-\infty}^{\infty} (-t x(t)) e^{-st} dt = L\{-t x(t)\} = -L\{t x(t)\}$$

$$\therefore L\{t \times x(t)\} = -\frac{d}{ds} X(s).$$

(8) Frequency Integration:

If $L\{x(t)\} = X(s)$ then $L\left\{\frac{1}{t} x(t)\right\} = \int_{s}^{\infty} X(s) ds$.

Proof
$$X(s) = L\{x(t)\} = \int_{-\infty}^{\infty} x(t) e^{-st} dt$$

On integrating above eqn w.r.to s b/w limits s to ∞

$$\int_{s}^{\infty} X(s) ds = \int_{s}^{\infty} \left[\int_{-\infty}^{\infty} x(t) e^{-st} dt\right] ds = \int_{-\infty}^{\infty} x(t) \left[\int_{s}^{\infty} e^{-st} ds\right] dt$$

$$= \int_{-\infty}^{\infty} x(t) \left[\frac{e^{-st}}{-t}\right]_{s}^{\infty} dt = \int_{-\infty}^{\infty} x(t) \left[\frac{e^{-\infty}}{-t} - \frac{e^{-st}}{-t}\right] dt$$

$$= \int_{-\infty}^{\infty} x(t) \left[0 + \frac{e^{-st}}{t}\right] dt = \int_{-\infty}^{\infty} \left[\frac{1}{t} x(t)\right] e^{-st} dt = L\left\{\frac{1}{t} x(t)\right\}.$$

9) Time Scaling:

If $L[x(t)] = X(s)$ then $L\{x(at)\} = \frac{1}{|a|} X\left(\frac{s}{a}\right)$

Proof $X(s) = L\{x(t)\} = \int_{-\infty}^{\infty} x(t)\, e^{-st} dt$

$$\therefore L\{x(at)\} = \int_{-\infty}^{\infty} x(at)\, e^{-st} dt = \int_{-\infty}^{\infty} x(\tau)\, e^{-s\left(\frac{\tau}{a}\right)} \frac{d\tau}{a} \quad \left\{ \begin{array}{l} \text{put } at = \tau \\ \therefore dt = \frac{d\tau}{a} \end{array} \right.$$

$$= \frac{1}{a}\int_{-\infty}^{\infty} x(\tau)\, e^{-\left(\frac{s}{a}\right)\tau} d\tau$$

$$= \frac{1}{a} X\left(\frac{s}{a}\right)$$

The above transform is applicable for positive values of 'a'.

If 'a' happens to be negative

$$L\{x(at)\} = \frac{-1}{a} X\left(\frac{s}{a}\right)$$

In general, $L\{x(at)\} = \frac{1}{|a|} X\left(\frac{s}{a}\right)$.

(10) Periodicity:

If $x(t) = x(t+nT)$ and $x_1(t)$ be one period of $x(t)$ and $L\{x_1(t)\} = \int_0^T x_1(t) e^{-st} dt$

then $L\{x(t+nT)\} = \frac{1}{1-e^{-sT}} \int_0^T x_1(t) e^{-st} dt$

Proof

$$L\{x(t+nT)\} = \int_0^{\infty} x(t+nT) e^{-st} dt$$

$$= \int_0^T x_1(t) e^{-st} dt + \int_T^{2T} x_1(t-T)\, e^{-s(t+T)} dt + \int_{2T}^{3T} x_1(t-2T) e^{-s(t+2T)} dt + \dots$$

$$+ \dots + \int_{PT}^{(P+1)T} x_1(t-PT) e^{-s(t+PT)} dt + \dots$$

$$= \sum_{P=0}^{\infty} \int_{PT}^{(P+1)T} x_1(t-PT)\, e^{-s(t+PT)} dt$$

$$= \sum_{P=0}^{\infty} \int_0^T x_1(t) e^{-st} \cdot e^{-PsT} dt = \int_0^T x_1(t)\, e^{-st} \left(\sum_{P=0}^{\infty} e^{-PsT}\right) dt$$

$$= \int_0^T x_1(t)\, e^{-st} \left(\sum_{P=0}^{\infty} e^{-sT}\right)^P dt$$

$$= \int_0^T x_1(t) e^{-st} \left(\frac{1}{1-e^{-sT}}\right) dt = \frac{1}{1-e^{-sT}} \int_0^T x_1(t) e^{-st} dt.$$

INVERSE LAPLACE TRANSFORM: (PARTIAL FRACTION EXPANSION)

The inverse laplace transform by partial fraction method of all three cases.

Case i) when s-domain signal $X(s)$ has distinct poles

Let $X(s) = \dfrac{K}{s(s+P_1)(s+P_2)}$

By partial fraction

$$X(s) = \frac{K_1}{s} + \frac{K_2}{s+P_1} + \frac{K_3}{s+P_2}$$

The residues K_1, K_2, K_3 are given by

$$K_1 = X(s) \times s \Big|_{s=0}, \quad K_2 = X(s) \times (s+P_1)\Big|_{s=-P_1}$$

$$K_3 = X(s) \times (s+P_2)\Big|_{s=-P_2}.$$

$$L^{-1}\{X(s)\} = L^{-1}\left\{\frac{K_1}{s} + \frac{K_2}{s+P_1} + \frac{K_3}{s+P_2}\right\}$$

$$x(t) = K_1 L^{-1}\left\{\frac{1}{s}\right\} + K_2 L^{-1}\left\{\frac{1}{s+P_1}\right\} + K_3 L^{-1}\left\{\frac{1}{s+P_2}\right\}$$

$$= K_1 u(t) + K_2 e^{-P_1(t)} u(t) + K_3 e^{-P_2 t} u(t).$$

Case ii) when s-domain signal $X(s)$ has multiple poles

$$X(s) = \frac{K}{s(s+P_1)(s+P_2)^2}$$

$$X(s) = \frac{K_1}{s} + \frac{K_2}{s+P_1} + \frac{K_3}{(s+P_2)^2} + \frac{K_4}{s+P_2}$$

The residues K_1, K_2, K_3, K_4 are given by

$$K_1 = X(s) \times s\Big|_{s=0} \qquad K_2 = X(s) \times (s+P_1)\Big|_{s=-P_1}$$

$$K_3 = X(s) \times (s+P_2)^2\Big|_{s=-P^2} \qquad K_4 = \frac{d}{ds}\left[X(s) \times (s+P_2)^2\right]_{s=-P_2}$$

$$L^{-1}\{X(s)\} = L^{-1}\left\{\frac{K_1}{s} + \frac{K_2}{s+P_1} + \frac{K_3}{(s+P_2)^2} + \frac{K_4}{s+P_2}\right\}$$

$$= k_1 u(t) + k_2 e^{-P_1 t} u(t) + k_3 t\, e^{-P_2 t} u(t) + k_4 e^{-P_2 t} u(t)$$

In general

$$X(s) = \frac{k}{s(s+P_1)(s+P_2)^q} \quad \text{then}$$

$$X(s) = \frac{k_1}{s} + \frac{k_2}{s+P_1} + \frac{k_3}{(s+P_2)^q} + \frac{k_{12}}{(s+P_2)^{q-1}} + \cdots + \frac{k_{(q-1)2}}{s+P_2}$$

$$k_{r2} = \frac{1}{r!}\frac{d^r}{dq^r}\left[X(s) \times (s+2)^q\right] \; ; r = 1, 2 \ldots q-1.$$

Case iii) When s-domain signal $X(s)$ has complex conjugate poles

Let $X(s) = \dfrac{k}{(s+P_1)(s^2+bs+c)}$

$$X(s) = \frac{k_1}{s+P_1} + \frac{k_2 s + k_3}{s^2+bs+c}$$

$$k_1 = X(s) \times (s+P_1)_{s=-P_1}$$

$$X(s) = \frac{k_1}{s+P_1} + \frac{k_2 s + k_3}{s^2 + 2\times\frac{b}{2}\cdot s + \left(\frac{b}{2}\right)^2 + c - \left(\frac{b}{2}\right)^2}$$

Arranging s^2+bs in $(x+y)^2$.

$$= \frac{k_1}{s+P_1} + \frac{k_2 s + k_3}{\left(s+\frac{b}{2}\right)^2 + \left(c - \frac{b^2}{4}\right)^2}$$

put $\frac{b}{2} = a$, $c - \frac{b^2}{4} = \Omega_0^2$

$$X(s) = \frac{k_1}{s+P_1} + k_2 \cdot \frac{s + a + \frac{k_3}{k_2} - a}{(s+a)^2 + \Omega_0^2}$$

$$= \frac{k_1}{s+P_1} + k_2 \cdot \frac{s + a + k_4}{(s+a)^2 + \Omega_0}$$

$\left(\because \text{put } \frac{k_3}{k_2} - a = k_4\right)$

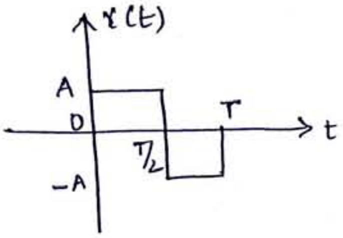
$$\therefore X(s) = \frac{k_1}{s+P_1} + k_2 \cdot \frac{s+a}{(s+a)^2+\Omega_0^2} + k_5 \frac{\Omega_0}{(s+a)^2+\Omega_0^2}$$

$\left(\frac{k_2 k_4}{\Omega_0} = k_5\right).$

$$x(t) = k_1 e^{-P_1 t} u(t) + k_2 e^{-at} \cos \Omega_0 t\, u(t) + k_5 e^{-at} \sin \Omega_0 t\, u(t).$$

LAPLACE TRANSFORM OF CERTAIN SIGNALS USING WAVEFORM SYNTHESIS :

(a)



proof $x(t) = A$ for $0<t<T/2$

$= -A$ for $T/2<t<T$

$$L\{x(t)\} = X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt = \int_0^T x(t)\, e^{-st}\, dt$$

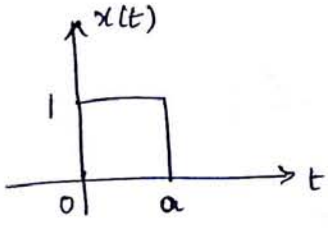
$$= \int_0^{T/2} A e^{-st}\, dt + \int_{T/2}^{T} (-A) e^{-st}\, dt = \left[\frac{Ae^{-st}}{-s} \right]_0^{T/2} + \left[\frac{-Ae^{-st}}{-s} \right]_{T/2}^{T}$$

$$= \left[\frac{Ae^{-sT/2}}{-s} - \frac{Ae^{0}}{-s} \right] + \left[\frac{Ae^{-sT}}{s} - \frac{Ae^{-sT/2}}{s} \right]$$

$$= -\frac{Ae^{-\frac{sT}{2}}}{s} + \frac{A}{s} + \frac{Ae^{-sT}}{s} - \frac{Ae^{-sT/2}}{s}$$

$$= \frac{A}{s}\left[1 - e^{-sT/2} \right]^2$$

(b)



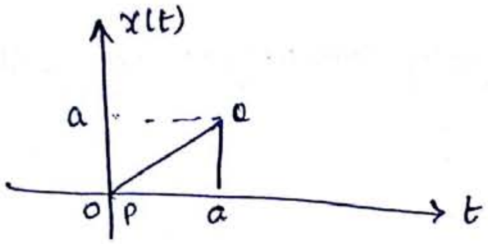
proof $x(t) = 1$ for $0<t\leq a$

$= 0$ for $t>a$

$$L\{x(t)\} = X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt = \int_0^a 1\times e^{-st}\, dt = \int_0^a e^{-st}\, dt$$

$$= \left[\frac{e^{-st}}{-s} \right]_0^a = \frac{e^{-as}}{-s} - \frac{e^{0}}{-s} = \frac{-e^{-as}}{s} + \frac{1}{s} = \frac{1}{s}(1 - e^{-as})$$

(c)



Consider the eqn of straight line $\frac{y-y_1}{y_1-y_2} = \frac{x-x_1}{x_1-x_2}$.

Here $y = x(t)$, $x = t$

Consider points P and Q as shown in figure

$P = [0, 0]$, $Q[a, a]$
$t_1, x_1(t)$ $t_2, x_2(t)$

$$\therefore \frac{x(t)-0}{0-a} = \frac{t-0}{0-a} \Rightarrow x(t) = t$$

$\therefore$ $x(t) = t$ for $t = 0$ to a

$= 0$ for $t > a$

$$L\{x(t)\} = X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st} dt = \int_0^a e^{-st} \cdot t\, dt$$

$$= \left[t \times \frac{e^{-st}}{-s} - \int 1 \cdot \frac{e^{-st}}{-s} dt \right]_0^a$$

$$= \left[-\frac{te^{-st}}{s} - \frac{e^{-st}}{s^2} \right]_0^a$$

$$= \left[-\frac{ae^{-sa}}{s} - \frac{e^{-sa}}{s^2} + 0 + \frac{e^0}{s^2} \right]$$

$$= \frac{1}{s^2} - \frac{e^{-as}}{s^2} - \frac{ae^{-as}}{s}$$

$$= \frac{1}{s^2}\left[1 - e^{-as}(1+as)\right]$$

Problems on Laplace transforms:

(i) Determine the Laplace transform of continuous time signals and their ROC

(a) $x(t) = A\,u(t)$

$x(t) = A$ for $t \geq 0$ bcoz $u(t) = 1$ for $t \geq 0$
$= 0$ for $t < 0$

Laplace transform

$$L\{x(t)\} = X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st} dt = \int_{0}^{\infty} A \cdot e^{-st} dt = A\left[\frac{e^{-st}}{-s}\right]_0^{\infty}$$

$$= A\left[\frac{e^{-(\sigma + j\omega)t}}{-s}\right]_0^{\infty}$$

$$= A\left[\frac{e^{-(\sigma + j\omega)\times\infty}}{-s} + \frac{e^0}{s}\right] = A\left[\frac{e^{-\sigma\times\infty} \cdot e^{-j\omega\times\infty}}{-s} + \frac{e^0}{s}\right]$$

when $\sigma > 0$; $e^{-\sigma\times\infty} = e^{-\sigma\infty} = e^{-\infty} = 0$

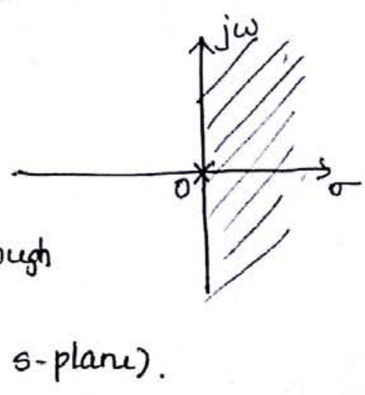
when $\sigma < 0$; $e^{-\sigma\times\infty} = e^{\infty} = \infty$

$\therefore$ we can say that $X(s)$ converges when $\sigma > 0$

when $\sigma > 0$, the $X(s)$ is given by

$$X(s) = A\left[\frac{0 \times e^{-j\omega\times\infty}}{-s} + \frac{1}{s}\right] = \frac{A}{s}$$

$\therefore L\{A\,u(t)\} = \frac{A}{s}$; with ROC as all point is s-plane to the right of line passing through $\sigma = 0$



(or ROC is right half of s-plane).

(2) $x(t) = t\,u(t)$

Sol

$$x(t) = t \begin{cases} u(t) = 1 & \text{for } t \geq 0 \\ = 0 & \text{otherwise} \end{cases}$$

$$\int uv = u\int v - \int \left[du \int v \right]$$

$$L\{x(t)\} = X(s) = \int_0^{\infty} x(t)\, e^{-st}\, dt$$

$$= \int_0^{\infty} t \cdot e^{-st}\, dt = \cancel{t}\left[\frac{e^{-st}}{-s}\right]_0^{\infty} - \int \left[\frac{e^{-st}}{-s}\right]_0^{\infty}$$

$$= \cancel{t}\cdot\left[\frac{e^{-st}}{-s}\right]_0^{\infty} - \left[\frac{e^{-st}}{s^2}\right]_0^{\infty}$$

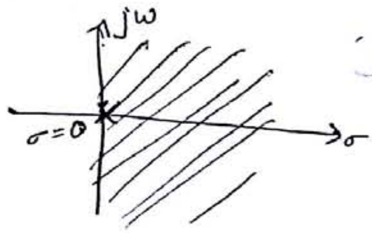
$$= t\left[\frac{e^{-(\sigma+j\omega)t}}{-s}\right]_0^{\infty} - \left[\frac{e^{-(\sigma+j\omega)t}}{s^2}\right]_0^{\infty}$$

$$= \left[\infty \times \frac{e^{-(\sigma+j\omega)\infty}}{-s} - 0 - \frac{e^{-(\sigma+j\omega)\infty}}{s^2} + \frac{e^0}{s^2}\right]$$

$$= \left[\infty \times \frac{e^{-\sigma\times\infty} \times e^{-j\omega\times\infty}}{-s} - \frac{e^{-\sigma\times\infty}\cdot e^{-j\omega\times\infty}}{s^2} + \frac{1}{s^2}\right]$$

when $\sigma > 0$, positive i.e $e^{-\sigma\times\infty} = e^{-\sigma\infty} = 0$

when $\sigma < 0$, negative i.e $e^{-\sigma\times\infty} = e^{\sigma\infty} = \infty$



It converges when $\sigma > 0$

$$= \left[\infty \times \frac{0 \times e^{-j\omega\times\infty}}{-s} - \frac{0 \times e^{-j\omega\times\infty}}{s^2} + \frac{1}{s^2}\right] = \frac{1}{s^2}$$

$\therefore$ ROC is right half of s-plane.

3) Given that $x(t) = e^{-3t}u(t)$:

Sol

$x(t) = e^{-3t}$ for $t \geq 0$

$$L\{x(t)\} = X(s) = \int_{-\infty}^{\infty} x(t)\cdot e^{-st}\, dt$$

$$= \int_0^{\infty} x(t) \cdot e^{-st} dt = \int_0^{\infty} e^{-3t} \cdot e^{-st} dt = \int_0^{\infty} e^{-(s+3)t} dt.$$

$$= \left[\frac{e^{-(s+3)t}}{-(s+3)} \right]_0^{\infty}$$

$$= \left[\frac{e^{-(s+3)\infty}}{-(s+3)} + \frac{e^{-(s+3)\cdot 0}}{s+3} \right] \Rightarrow \frac{e^{-(\sigma+j\omega+3)\times\infty}}{-(s+3)} + \frac{1}{s+3}$$

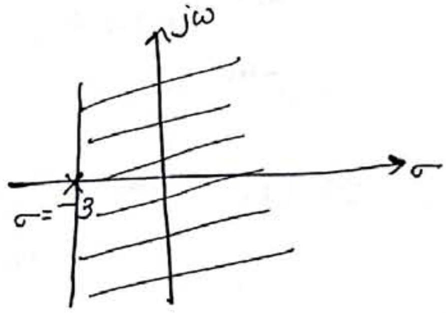
$$= \left[\frac{e^{-(\sigma+3)\infty} \cdot e^{-j\omega\times\infty}}{-(s+3)} + \frac{1}{s+3} \right]$$

$\sigma + 3 = \sigma - (-3)$

if $\sigma > -3$ = positive $\therefore e^{-k\times\infty} = e^{-k\infty} = 0$

if $\sigma < -3$ = negative $\therefore e^{-k\times\infty} = e^{k\infty} = \infty$

$\therefore$ It converges at $\sigma > -3$.



$$x(s) = \left[\frac{0 \times e^{-j\omega\times\infty}}{-(s+3)} + \frac{1}{s+3} \right] = \frac{1}{s+3}$$

$L\{e^{-3t}u(t)\} = \frac{1}{s+3}$; with Roc as all pts in s-plane to right of line passing through $\sigma = -3$

(4) $x(t) = e^{-3t} u(-t)$

Sol $x(t) = e^{-3t}$ for $t \leq 0$

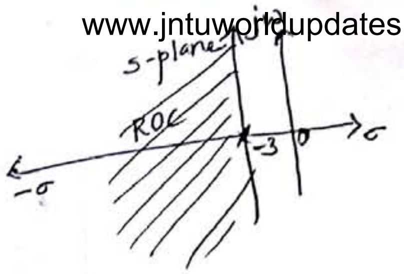
$$L\{x(t)\} = x(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt = \int_{-\infty}^{0} e^{-st} \cdot e^{-3t} dt$$

$$= \int_{-\infty}^{0} e^{-(s+3)t} dt$$

$$= \left. \frac{e^{-(s+3)t}}{-(s+3)} \right]_{-\infty}^{0} = \frac{e^0}{-(s+3)} + \frac{e^{+(s+3)\times\infty}}{+(s+3)} \Big]$$

$$= \frac{-1}{s+3} + \frac{e^{+(\sigma+j\omega+3)\times\infty}}{s+3}$$

$$= \frac{-1}{s+3} + \frac{e^{+(\sigma+3)\times\infty} \cdot e^{-j\omega\times\infty}}{s+3}$$



$k = \sigma + 3 = \sigma - (-3)$

when $\sigma > -3$; ~~.....~~ positive $+(\sigma+3)\times\infty = \infty$

when $\sigma < -3$; negative $e^{-k\infty} = e^{k\infty} = 0$

$\therefore$ ROC it converges when $\sigma < -3 \Rightarrow \frac{-1}{s+3} + \frac{0 \times e^{-j\omega\times\infty}}{s+3}$

$L\{x(t)\} = L\{e^{-3t}u(-t)\} = \frac{-1}{s+3}$ { ROC as all pts in s plane ~~to~~ the left passing through $\sigma = -3$

---

(5) $x(t) = e^{-4|t|}$

$= e^{-4t}$ for $t \geq 0$

$= e^{+4t}$ for $t \leq 0$

$$L\{x(t)\} = X(s) = \int_{-\infty}^{\infty} x(t) \cdot e^{-st} dt = \int_{-\infty}^{0} e^{4t} \cdot e^{-st} dt + \int_{0}^{\infty} e^{-4t} \cdot e^{-st} dt$$

$$= \int_{-\infty}^{0} e^{-(s-4)t} dt + \int_{0}^{\infty} e^{-(s+4)t} dt$$

$$= \left[\frac{e^{-(s-4)t}}{-(s-4)}\right]_{-\infty}^{0} + \left[\frac{e^{-(s+4)t}}{-(s+4)}\right]_{0}^{\infty}$$

$$= \left[\frac{1}{-(s-4)} + \frac{e^{+(s-4)\times\infty}}{s-4} - \frac{e^{-(s+4)\times\infty}}{s+4} + \frac{1}{s+4}\right]$$

$$= \left[\frac{-1}{s-4} + \frac{e^{(\sigma+j\omega-4)\times\infty}}{s-4} - \frac{e^{-(\sigma+j\omega+4)\times\infty}}{s+4} + \frac{1}{s+4}\right]$$

$$= \left[\frac{-1}{s-4} + \frac{e^{(\sigma-4)\times\infty} \cdot e^{+j\omega\times\infty}}{s-4} - \frac{e^{-(\sigma+4)\times\infty} \cdot e^{-j\omega\times\infty}}{s+4} + \frac{1}{s+4}\right]$$

$\sigma - 4 \Rightarrow \sigma - (+4)$

when $\sigma > 4$ positive $e^{k\times\infty} = \infty$

when $\sigma < 4$ negative $e^{-k\infty} = 0$

It converges when $\sigma < +4$

$\sigma + 4 \rightarrow \sigma - (-4)$

when $\sigma > -4$ positive $e^{-k\infty} = 0$

when $\sigma < -4$ negative $e^{+k\infty} = \infty$

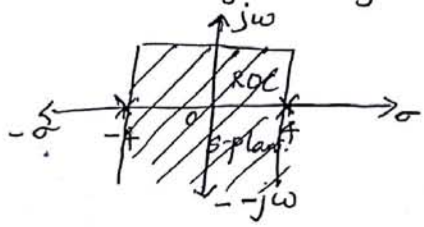
It converges when $\sigma > -4$

when σ lies between -4 and $+4$, the $X(s)$ is given by

$$X(s) = \left[\frac{-1}{s-4} + \frac{e^{(\sigma-4)\times\infty} e^{j\omega\times\infty}}{s-4} - \frac{e^{-(\sigma+4)\times\infty} \cdot e^{-j\omega\times\infty}}{s+4} + \frac{1}{s+4} \right]$$

$$= \frac{-1}{s-4} + \frac{1}{s+4} \Rightarrow \frac{-s-4+s-4}{s^2-16} = \frac{-8}{s^2-16}.$$

with ROC as all points in splane in between the lines passing through $\sigma = -4$ and $\sigma = 4$.



⑥ Determine the laplace transform of following signals.

$x(t) = \sin\omega_0 t \, u(t)$

Sol

$x(t) = \sin\omega_0 t$ for $t \geq 0$

$$L\{x(t)\} = X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt \qquad \because \sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

$$= \int_{0}^{\infty} \sin\omega_0 t \cdot e^{-st} dt$$

$$= \int_0^{\infty} \frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j} \cdot e^{-st} dt$$

$$= \frac{1}{2j}\left[\int_0^{\infty} e^{-(s-j\omega_0)t} dt - \int_0^{\infty} e^{-(s+j\omega_0)t} dt \right]$$

$$= \frac{1}{2j}\left\{ \left[\frac{e^{-(s-j\omega_0)t}}{-(s-j\omega_0)} \right]_0^{\infty} - \left[\frac{e^{-(s+j\omega_0)t}}{-(s+j\omega_0)} \right]_0^{\infty} \right\}$$

$$= \frac{1}{2j}\left[\frac{e^{-\infty}}{-(s-j\omega_0)} + \frac{e^0}{s-j\omega_0} + \frac{e^{-\infty}}{s+j\omega_0} - \frac{e^0}{s+j\omega_0} \right]$$

$$= \frac{1}{2j}\left[\frac{1}{s-j\omega_0} - \frac{1}{s+j\omega_0} \right] = \frac{1}{2j}\left[\frac{\cancel{s}+j\omega_0-\cancel{s}+j\omega_0}{s^2-j^2\omega_0^2} \right] = \frac{1}{2j} \cdot \frac{2j\omega_0}{s^2+\omega_0^2}.$$

$$= \omega_0 / s^2+\omega_0^2.$$

(7) $x(t) = \cos\omega_0 t\, u(t)$

Sol $x(t) = \cos\omega_0 t \quad ; t \geq 0$

$$L\{x(t)\} = X(S) = \int_0^\infty x(t) \cdot e^{-st} dt = \int_0^\infty \frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2} \cdot e^{-st} dt$$

$$= \frac{1}{2} \int_0^\infty \left[e^{-(s-j\omega_0)t} + e^{-(s+j\omega_0)t} \right] dt$$

$$= \frac{1}{2} \left[\frac{e^{-(s-j\omega_0)t}}{-(s-j\omega_0)} + \frac{e^{-(s+j\omega_0)t}}{-(s+j\omega_0)} \right]_0^\infty$$

$$= \frac{1}{2} \left[\frac{e^{-\infty}}{-(s-j\omega_0)} + \frac{e^0}{s-j\omega_0} - \frac{e^{-\infty}}{s+j\omega_0} + \frac{e^0}{s+j\omega_0} \right]$$

$$= \frac{1}{2} \left[\frac{1}{s-j\omega_0} + \frac{1}{s+j\omega_0} \right] = \frac{1}{2} \left[\frac{s+j\cancel{\omega_0} + s - j\cancel{\omega_0}}{s^2 - j^2\omega_0^2} \right] = \frac{1}{\cancel{2}} \cdot \frac{\cancel{2}s}{s^2+\omega_0^2}$$

$$L\{\cos\omega_0 t\, u(t)\} = \frac{s}{s^2+\omega_0^2}$$

8) $x(t) = \cosh\omega_0 t \ \ u(t)$

Sol $x(t) = \cosh\omega_0 t$ for $t \geq 0$ $\qquad \because \cosh\theta = \dfrac{e^\theta + e^{-\theta}}{2}$

$$L\{x(t)\} = X(S) = \int_{-\infty}^\infty x(t) \cdot e^{-st} dt$$

$$= \int_0^\infty \left(\frac{e^{\omega_0 t} + e^{-\omega_0 t}}{2} \right) e^{-st} dt$$

$$= \frac{1}{2} \int_0^\infty e^{\omega_0 t} \cdot e^{-st} dt + \frac{1}{2} \int_0^\infty e^{-\omega_0 t} \cdot e^{-st} dt$$

$$= \frac{1}{2} \left[\int_0^\infty e^{-(s-\omega_0)t} dt + \int_0^\infty e^{-(s+\omega_0)t} dt \right]$$

$$= \frac{1}{2} \left\{ \left[\frac{e^{-(s-\omega_0)t}}{-(s-\omega_0)} \right]_0^\infty + \left[\frac{e^{-(s+\omega_0)t}}{-(s+\omega_0)} \right]_0^\infty \right.$$

$$= \frac{1}{2} \left[\frac{e^{-\infty}}{-(s-\omega_0)} + \frac{1}{s-\omega_0} + \frac{e^{-\infty}}{-(s+\omega_0)} + \frac{1}{s+\omega_0} \right]$$

$$= \frac{1}{2}\left[\frac{1}{s-\omega_0} + \frac{1}{s+\omega_0}\right]$$

$$= \frac{1}{2}\left[\frac{s+\cancel{\omega_0}+s-\cancel{\omega_0}}{s^2-\omega_0^2}\right] = \frac{s}{s^2-\omega_0^2}$$

9) $x(t) = e^{-at}\sin\omega_0 t\, u(t)$

Sol $x(t) = e^{-at}\sin\omega_0 t$ for $t \geq 0$

$$L\{x(t)\} = X(s) = \int_0^\infty x(t).e^{-st}dt = \int_0^\infty e^{-at}\sin\omega_0 t\, e^{-st}\,dt = \int_0^\infty e^{-at}\left(\frac{e^{j\omega_0 t}-e^{-j\omega_0 t}}{2j}\right)e^{-st}dt$$

$$= \frac{1}{2j}\left[\int_0^\infty e^{-(s+a-j\omega_0)t} - e^{-(s+a+j\omega_0)t}\right]dt$$

$$= \frac{1}{2j}\left[\frac{e^{-(s+a-j\omega_0)t}}{-(s+a-j\omega_0)}\right]_0^\infty - \left[\frac{e^{-(s+a+j\omega_0)t}}{-(s+a+j\omega_0)}\right]_0^\infty$$

$$= \frac{1}{2j}\left[\frac{e^{-\infty}}{-(s+a-j\omega)} + \frac{e^0}{s+a-j\omega_0} + \frac{e^{-\infty}}{s+a+j\omega} - \frac{e^0}{s+a+j\omega_0}\right]$$

$$= \frac{1}{2j}\left[\frac{1}{s+a-j\omega_0} - \frac{1}{s+a+j\omega_0}\right] = \frac{1}{2j}\left[\frac{\cancel{s}+\cancel{a}+j\omega_0-\cancel{s}-\cancel{a}+j\omega_0}{(s+a)^2+\omega_0^2}\right]$$

$$= \frac{\cancel{2}\cancel{j}\omega_0}{\cancel{2}\cancel{j}\cdot(s+a)^2+\omega_0^2}$$

$$= \frac{\omega_0}{(s+a)^2+\omega_0^2}$$

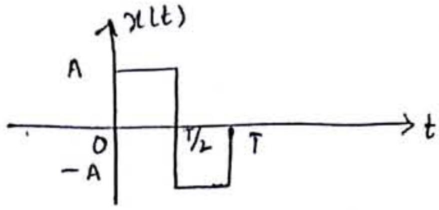
(10) $x(t) = e^{-at}\cos\omega_0 t\ u(t)$

Sol $x(t) = e^{-at}\cos\omega_0 t$ for $t \geq 0$

$$|||^{ly} = \frac{s+a}{(s+a)^2+\omega_0^2}$$

(11) Determine the Laplace transform of following signals.

Sol)



$x(t) = A$ for $0 < t \leq T/2$

$= -A$ for $T/2 < t < T$

$$L\{x(t)\} = X(s) = \int_0^{T/2} A \cdot e^{-st} dt + \int_{T/2}^{T} -A \cdot e^{-st} dt$$

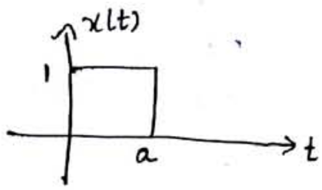
$$= A\left[\frac{e^{-st}}{-s}\right]_0^{T/2} - A\left[\frac{e^{-st}}{-s}\right]_{T/2}^{T}$$

$$= A\left[\frac{e^{-sT/2}}{-s} + \frac{1}{s}\right] - A\left[\frac{e^{-sT}}{-s} + \frac{e^{-sT/2}}{s}\right]$$

$$= -\frac{Ae^{-sT/2}}{s} + \frac{A}{s} + \frac{Ae^{-sT}}{s} - \frac{Ae^{-sT/2}}{s}$$

$$= \frac{A}{s}\left[1 + e^{-sT} - 2e^{-sT/2}\right] = \frac{A}{s}\left[1 - e^{-sT/2}\right]^2.$$

(12)



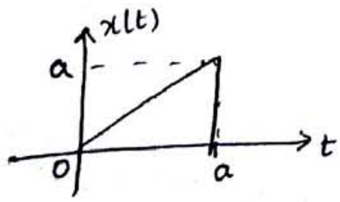
$x(t) = 1$ for $0 < t < a$

$= 0$ otherwise

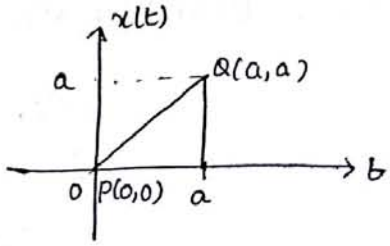
$$L\{x(t)\} = X(s) = \int_0^a e^{-st} dt = \left[\frac{e^{-st}}{-s}\right]_0^a$$

$$= \frac{e^{-as}}{-s} + \frac{1}{s} \Rightarrow \frac{1}{s}\left[1 - e^{-as}\right]$$

(13)



$$\frac{y-y_1}{y_1-y_2} = \frac{x-x_1}{x_1-x_2}.$$



here $(x_1,y_1) = (0,0)$, $(x_2,y_2) = (a,a)$

$x = t$, $y = x(t)$.

$$\Rightarrow \frac{x(t)-0}{0-a} = \frac{t-0}{0-a} \Rightarrow \frac{x(t)}{-a} = \frac{t}{-a} \Rightarrow x(t) = t \quad \text{for } 0<t<a$$
$$= 0 \quad \text{otherwise}$$

$$L\{x(t)\} = X(s) = \int_{-\infty}^{\infty} x(t)\cdot e^{-st}\,dt$$

$$= \int_0^a t e^{-st}\,dt = \left\{ t\left[\frac{e^{-st}}{-s}\right]_0^a - \int \left[\frac{e^{-st}}{-s}\right]_0^a \right\}$$

$$= t\cdot\frac{e^{-st}}{-s}\Big]_0^a - \frac{e^{-st}}{s^2}\Big]_0^a$$

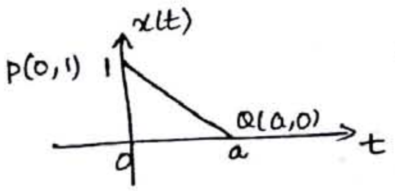
$$= a\cdot\frac{e^{-sa}}{-s} - \frac{e^{-as}}{s^2} + \frac{1}{s^2}$$

$$= \frac{1}{s^2}\left[1 - e^{-as} - s\cdot e^{-as}\cdot a\right]$$

$$= \frac{1}{s^2}\left[1 - e^{-as} - as\cdot e^{-as}\right]$$

$$= \frac{1}{s^2}\left[1 - e^{-as}(1+as)\right]$$

(14)



$(x_1,y_1) = (0,1)$, $(x_2,y_2) = (a,0)$, $x = t$, $y = x(t)$.

$$\frac{y-y_1}{y_1-y_2} = \frac{x-x_1}{x_1-x_2} \Rightarrow \frac{x(t)-1}{1-0} = \frac{t-0}{0-a} \Rightarrow x(t)-1 = \frac{-t}{a}$$

$x(t) = 1 - t/a$ for 0 to a

$= 0$ for $t>a$

$$L\{x(t)\} = X(s) = \int_{-\infty}^{\infty} x(t)\cdot e^{-st}dt$$

$$= \int_0^a \left(1-\frac{t}{a}\right) e^{-st}dt = \int_0^a e^{-st}dt - \int_0^a \frac{t}{a} e^{-st}dt$$

$$= \left.\frac{e^{-st}}{-s}\right]_0^a - \frac{1}{a}\left[\int_0^a t\cdot e^{-st}dt\right]$$

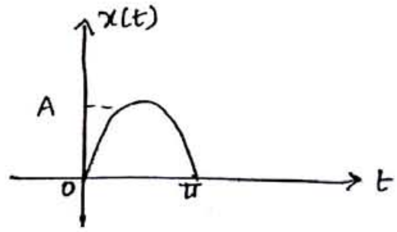
$$= \frac{e^{-as}}{-s} + \frac{1}{s} - \frac{1}{a}\left[t\cdot\frac{e^{-st}}{-s} - \frac{e^{-st}}{s^2}\right]_0^a$$

$$= \frac{e^{-as}}{-s} + \frac{1}{s} - \frac{1}{a}\cdot a\cdot\frac{e^{-as}}{-s} + \frac{1e^{-sa}}{as^2} - \frac{1}{as^2}\Big]$$

$$= \cancel{\frac{e^{-as}}{-s}} + \frac{1}{s} + \cancel{\frac{e^{-as}}{s}} + \frac{e^{-as}}{as^2} - \frac{1}{as^2}$$

$$= \frac{1}{s} + \frac{e^{-as}}{as^2} - \frac{1}{as^2} = \frac{1}{as^2}\left[e^{-as} + as - 1\right]$$

(15) Determine the laplace transform of sine pulse



Sol $x(t) = A\sin t$ for $0<t<\pi$

$= 0$ for $t>\pi$

$$L\{x(t)\} = X(s) = \int_0^{\pi} A\cdot\sin t\, e^{-st}dt$$

$$= A\left[\frac{e^{jt}-e^{-jt}}{2j} e^{-st}\right]dt = \frac{A}{2j}\left[\int_0^{\pi} e^{-(s-j)t}dt - \int_0^{\pi} e^{-(s+j)t}dt\right]_0$$

$$= \frac{A}{2j}\left[\frac{e^{-(s-j)t}}{-(s-j)}\right]_0^{\pi} - \frac{A}{2j}\left[\frac{e^{-(s+j)t}}{-(s+j)}\right]_0^{\pi}$$

$$= \frac{A}{2j}\left[\frac{e^{-(s+j)t}}{s+j} - \frac{e^{-(s-j)t}}{s-j}\right]_0^{\pi} = \frac{A}{2j}\left[\frac{(s-j)e^{-st}\cdot e^{-jt} - (s+j)e^{-st}e^{+jt}}{s^2-j^2}\right]_0^{\pi}$$

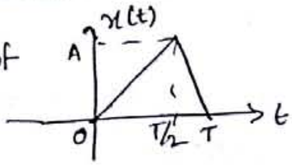
$$= \frac{A}{2j(s^2+1)}\left[(s-j)e^{-st}e^{-jt} - (s+j)e^{-st}e^{jt}\right]_0^{\pi}$$

$$= \frac{A}{2j(s^2+1)}\left[(s-j)e^{-s\pi}e^{-j\pi} - (s+j)e^{-s\pi}e^{j\pi} - (s-j)e^{-0} + (s+j)e^{0}\right]$$

$$= \frac{A}{2j(s^2+1)}\left[-(s-j)e^{-s\pi} + (s+j)e^{-s\pi} - (s-j) + (s+j)\right] = \frac{A}{2j(s^2+1)}(2je^{-\pi s} + 2j)$$

$$\Rightarrow A/(s^2+1)(e^{-\pi s}+1)$$

(16) Determine laplace transform of



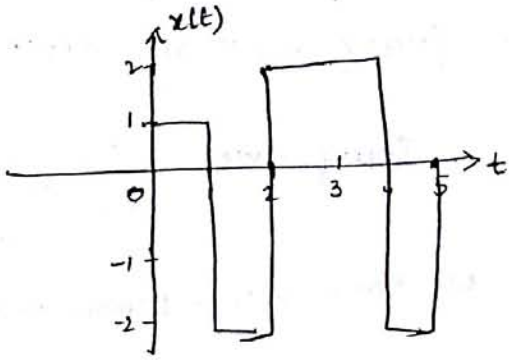
Sol

$$x(t) = \frac{2At}{T} \; ; \; 0<t<T/2$$

$$= 2A - \frac{2At}{T} \quad T/2<t<T.$$

$$L\{x(t)\} = X(s) = \frac{2A}{Ts^2}\left(1 - e^{-sT/2}\right)^2.$$

(17) Determine laplace transform of



Sol

$$x(t) = \begin{cases} 1 & \text{for } 0<t<1 \\ -2 & \text{for } 1<t<2 \\ 2 & \text{for } 2<t<4 \\ -2 & \text{for } 4<t<5 \\ 0 & \text{for } t>5 \end{cases}$$

$$L\{x(t)\} = X(s) = \int_0^1 e^{-st}\,dt + \int_1^2 -2e^{-st}\,dt + \int_2^4 2e^{-st}\,dt + \int_4^5 -2e^{-st}\,dt$$

$$= \left.\frac{e^{-st}}{-s}\right]_0^1 - 2\left[\frac{e^{-st}}{-s}\right]_1^2 + 2\left[\frac{e^{-st}}{-s}\right]_2^4 - 2\left[\frac{e^{-st}}{-s}\right]_4^5$$

$$= \frac{e^{-s}}{-s} + \frac{1}{s} + \frac{2e^{-2s}}{s} - \frac{e^{-s}}{s} - \frac{2e^{-4s}}{s} + \frac{2e^{-2s}}{s} + \frac{2e^{-5s}}{s} - \frac{2e^{-4s}}{s}$$

$$= \frac{1}{s}\left[1 - 2e^{-s} + 4e^{-2s} - 4e^{-4s} + 2e^{-5s}\right]$$

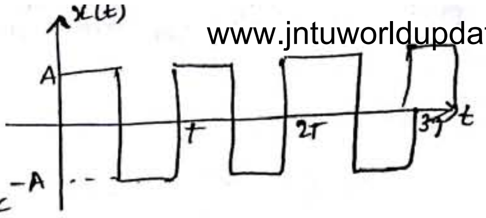
(18) Determine the Laplace transform of $\delta(t)$.

Sol

$$\delta(t) = 1 \text{ for } t=0$$
$$= 0 \text{ otherwise.}$$

$$L\{x(t)\} = X(s) = \left.\frac{e^{-st}}{-s}\right|_{t=0} =$$

Determine the Laplace transform of periodic square wave



Sol: The given waveform satisfy the condition

$x(t+nT) = x(t)$ and so it is periodic

$x_1(t) = A$ for $t = 0$ to $T/2$

$= -A$ for $t = T/2$ to T

From periodicity property of Laplace transform

If $X(S) = L\{x(t)\}$ and if $x(t) = x(t+nT)$ then $X(S) = \frac{1}{1-e^{-ST}} \int_0^T x_1(t) e^{-st} dt$

$$\therefore L\{x(t)\} = X(S) = \frac{1}{1-e^{-ST}} \int_0^T x_1(t) e^{-st} dt$$

we know $x_1(t)$ = laplace transform $= \frac{A}{S}\left[1 - e^{-ST/2}\right]^2$

$$X(S) = \frac{1}{1-e^{-ST}} \left[\frac{A}{S}\left(1 - e^{-ST/2}\right)\right]^2$$

$$= \frac{1}{\left(1-e^{-ST/2}\right)\left(1+e^{-ST/2}\right)} \left[\frac{A}{S}\left(1-e^{-ST/2}\right)^2\right]$$

$$= \frac{A}{S}\left[\frac{1-e^{-ST/2}}{1+e^{-ST/2}}\right]$$

Initial Value Theorem:

The initial value theorem states that, if x(t) and its derivative are Laplace transformable then $\underset{t\to 0}{Lt}\, x(t) = \underset{s\to\infty}{Lt}\, sX(s)$.

i.e $x(0) = \underset{t\to 0}{Lt}\, x(t) = \underset{s\to\infty}{Lt}\, sX(s)$.

Proof

We know that

$$L\left\{\frac{dx(t)}{dt}\right\} = sX(s) - x(0)$$

On taking limits $s\to\infty$ on both sides of equation.

$$\underset{s\to\infty}{Lt}\, L\left\{\frac{dx(t)}{dt}\right\} = \underset{s\to\infty}{Lt}\left[sX(s) - x(0)\right]$$

$$\Rightarrow \underset{s\to\infty}{Lt} \int_0^\infty \frac{dx(t)}{dt} e^{-st} dt = \underset{s\to\infty}{Lt}\left[sX(s) - x(0)\right]$$

$$\Rightarrow \int_0^\infty \frac{dx(t)}{dt}\left(\underset{s\to\infty}{Lt}\, e^{-st}\right) dt = \underset{s\to\infty}{Lt}\, sX(s) - x(0)$$

$$\Rightarrow 0 = \underset{s\to\infty}{Lt}\, sX(s) - x(0)$$

$$\therefore x(0) = \underset{s\to\infty}{Lt}\, sX(s).$$

$$\boxed{\therefore \underset{t\to 0}{Lt}\, x(t) = \underset{s\to\infty}{Lt}\, sX(s)}$$

Final value theorem:

The final value theorem states that if x(t) and its deriative are laplace transformable then $\underset{t\to\infty}{Lt}\, x(t) = \underset{s\to 0}{Lt}\, sX(s)$.

i.e Final value of signal $x(\infty) = \underset{t\to\infty}{Lt}\, x(t) = \underset{s\to 0}{Lt}\, sX(s)$.

proof

$$L\left\{\frac{dx(t)}{dt}\right\} = sX(s) - x(0)$$

By taking limits $s\to 0$ on both sides of above eqn we get

$$\underset{s\to 0}{Lt}\, L\left\{\frac{dx(t)}{dt}\right\} = \underset{s\to 0}{Lt}\left\{sX(s) - x(0)\right]$$

$$\lim_{s\to 0}\int_0^\infty \frac{dx(t)}{dt} e^{-st} dt = \lim_{s\to 0}\left[sX(s) - x(0)\right]$$

$$\int_0^\infty \frac{dx(t)}{dt}\left(\lim_{s\to 0} e^{-st}\right) dt = \lim_{s\to 0} sX(s) - x(0)$$

Not fn of 's'

Unit - V : Z- transforms :-

If the signal is Discret time signal then Z-transforms and D.T. fourier transforms are appliCable

→ If the signal is said to be Dicret time fourier transform then $x(n)$ should be in absolutely Integrable.

$$\text{DTFT} \rightarrow X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}, \quad \sum_{n=-\infty}^{\infty} |x(n) \cdot r^{-n}| < \infty$$

$$\rightarrow Z[x(n)] = X(Z) = \sum_{n=-\infty}^{\infty} x(n)\, Z^{-n} \quad (\because Z = r \cdot e^{j\omega})$$

$$= \sum_{n=-\infty}^{\infty} x(n)\, (r e^{j\omega})^{-n}$$

$$X(Z) = \sum_{n=-\infty}^{\infty} (x(n)\, r^{-n})\, e^{-j\omega n}$$

→ $x(n)$ is not absolutely integrable then Z-transforms are appliCable.

Z- transform :- $x(n) \xleftrightarrow{Z.T} X(Z)$

Time domine — Transform & frequency domine

$$Z[x(n)] = X(Z) = \sum_{n=-\infty}^{\infty} x(n)\, Z^{-n}$$

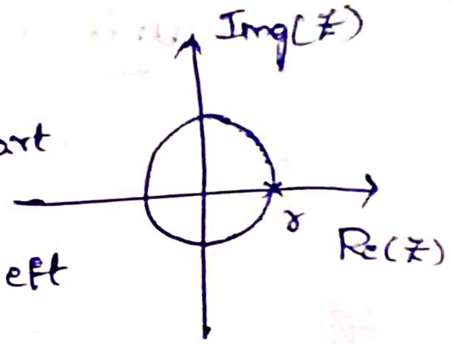
This is Two sided Z- transform.

$$= \sum_{n=0}^{\infty} x(n)\, Z^{-n} \rightarrow \text{Single sided } Z\text{- transform}$$

ROC of Z.T :-

$Z = r\, e^{jw}$

Real part (r), Imaginary part (e^{jw})



$|Z| < r$ (inside a circle) ROC is left side.

→ $|Z| > r$ (out side a circle) Right side ROC.

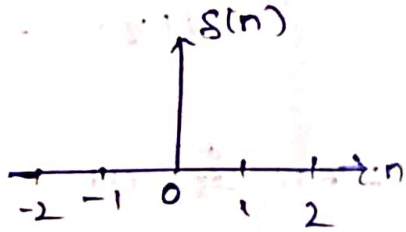
ROC :- The Range of Values of Z in which Z-transform Exist is Called Region of Convergence

→ find the Z-transform of following Signals :-

1) **Impulse Sequence** :-

W.K.T $Z[x(n)] = X(Z) = \sum_{n=-\infty}^{\infty} x(n) Z^{-n}$

$x(n) = \delta(n) = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$



$\therefore Z[x(n)] = X(Z) = Z[\delta(n)] = \sum_{n=-\infty}^{\infty} \delta(n) Z^{-n}$

$= 1 \cdot Z^{-n}\Big|_{n=0} = 1$

$Z^{-1}[1] = \delta(n)$, ROC = Entire Z-plane

2) **Step Sequence** :- $u(n) = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$

$Z[u(n)] = X(Z) = \sum_{n=-\infty}^{\infty} u(n) Z^{-n}$

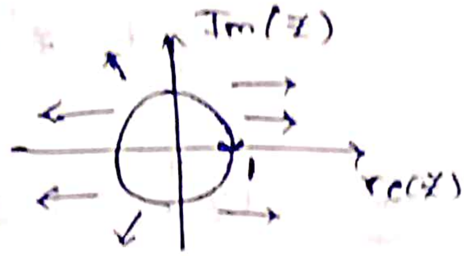
$= \sum_{n=0}^{\infty} Z^{-n} = 1 + Z^{-1} + Z^{-2} + \cdots$

$= \frac{1}{1 - Z^{-1}}$

ROC :- $|1-z^{-1}| > 0$ (Denominator must be in +ve)

$|z^{-1}| < 1$

$\frac{1}{z} < 1 \Rightarrow |z| > 1$

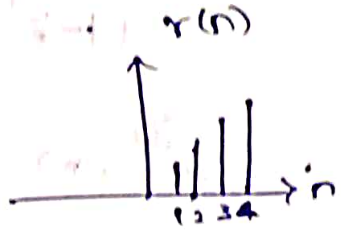


$\therefore z[u(n)] = \frac{1}{1-z^{-1}}$, ROC is $|z| > 1$

$= \frac{z}{z-1}$

Ramp Sequence :- r(n)

$$r(n) = \begin{cases} n & n \geq 0 \\ 0 & n < 0 \end{cases}$$



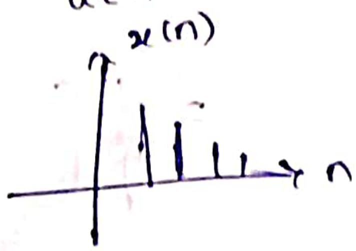
$$z[x(n)] = X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$= \sum_{n=0}^{\infty} n \cdot z^{-n} = 0 + z^{-1} + 2z^{-2} + 3z^{-3} + \cdots$$

$$= z^{-1}(1 + 2z^{-1} + 3z^{-2} + \cdots)$$

$=$

4) Exponential Sequence :- $x(n) = e^{-j\omega n} u(n)$



$$X(z) = \sum_{n=0}^{\infty} e^{-j\omega n} z^{-n}$$

$$= \sum_{n=0}^{\infty} (z^{-1} e^{-j\omega})^n$$

$$= 1 + (z^{-1} e^{-j\omega}) + (\quad)^2 + \cdots$$

$$= \frac{1}{1-(z^{-1}e^{-j\omega})}$$

ROC :- $|z^{-1}e^{-j\omega}| < 1$

$|z^{-1}| < 1$

$|z| > 1$ (Roc is outside circle).

$$X(z) = \frac{1}{(1-z^{-1}e^{-j\omega})} = \frac{z}{z-e^{-j\omega}}$$

$$\therefore \; e^{-j\omega}u(n) \longleftrightarrow \frac{z}{z-e^{-j\omega}}$$

5*

5) $x(n) = \sin \omega n \; u(n)$

$$= \frac{e^{j\omega n} - e^{-j\omega n}}{2j}$$

$$z[x(n)] = X(z) = \sum_{n=0}^{\infty} \frac{e^{j\omega n} - e^{-j\omega n}}{2j} \cdot z^{-n}$$

$$= \frac{1}{2j}\left[\sum_{n=0}^{\infty}(z^{-1}e^{j\omega})^n - \sum_{n=0}^{\infty}(z^{-1}e^{-j\omega})^n\right]$$

Roc $|z| > 1$

$$= \frac{1}{2j}\left[\frac{1}{1-z^{-1}e^{j\omega}} - \frac{1}{1-z^{-1}e^{-j\omega}}\right]$$

$$= \frac{1}{2j}\left[\frac{z}{z-e^{j\omega}} - \frac{z}{z-e^{-j\omega}}\right]$$

$$= \frac{1}{2j}\left[\frac{z(z-e^{-j\omega}) - z(z-e^{j\omega})}{(z-e^{j\omega})(z-e^{-j\omega})}\right]$$

$$= \frac{1}{2j}\left[\frac{z\,(e^{jw} - e^{-jw})}{z^2 - z(e^{jw} + e^{-jw}) + 1} \right]$$

$$= \frac{z \sin w}{z^2 - 2z \cos w + 1}$$

$$\therefore \sin wn\, u(n) \rightleftharpoons \frac{z \sin w}{z^2 - 2z\cos w + 1}$$

5*

6) $x(n) = \cos wn\, u(n)$

$$\cos wn\, u(n) \longrightarrow \frac{z\,(z - \cos w)}{z^2 - 2z\cos w + 1}$$

→ find the Z-transform of following Signals.

1. $x(n) = a^n u(n)$

$$x(n) = a^n \quad n \geq 0$$
$$= 0 \quad n < 0$$

2. $x(n) = +a^n u(n)$

$$X(z) = \sum_{n=-\infty}^{\infty} a^n u(n) z^{-n}$$

$$= \sum_{n=0}^{\infty} a^n z^{-n}$$

$$= \sum_{n=0}^{\infty} (a z^{-1})^n$$

$$= 1 + (a z^{-1}) + (a z^{-1})^2 + \dots$$

ROC :- $a z^{-1} < 1$

$\frac{a}{z} < 1$

$\frac{a}{z} < \frac{1}{1}$

$|z| > a$

3. $x(n) = -a^{n} u(-n-1)$

$= -a^{n} \qquad -n-1 \geq 0 \Rightarrow n \leq -1$

$0 \qquad n \geq 0$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} = \sum_{n=-\infty}^{\infty} -a^{n} u(-n-1) z^{-n}$$

$$= \sum_{n=-\infty}^{-1} -a^{n} z^{-n}$$

$$= -\sum_{n=1}^{\infty} \cancel{(a z)} \; -a^{n} z^{-n}$$

$$= -\sum_{n=1}^{\infty} (a^{-1} z)^{n} + 1 - 1$$

$$= -\left[\sum_{n=0}^{\infty} (a^{-1} z)^{n} - 1\right]$$

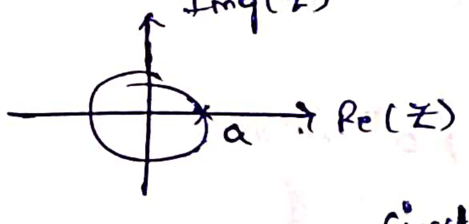
$$= -\left(\frac{1}{1-a^{-1} z} - 1\right) = -\left(\frac{a}{a-z} - 1\right)$$

$$= \frac{z}{z-a}$$

ROC :- $|a^{-1} z| < 1$

$\left|\frac{z}{a}\right| < 1$

$|z| < |a|$



Inside a circle

$\therefore x(n) = a^{n} u(n) \longleftrightarrow \frac{z}{z-a}$, ROC $\rightarrow |z| > |a|$

$= -a^{n} u(n) \longleftrightarrow \frac{z}{z-a}$, ROC $|z| < |a|$

→ Determine the Z ~~proble~~ transform of following Eqns.

$x(n) = a^n u(n) - b^n u(-n-1)$

$$x(z) = \sum_{n=-\infty}^{\infty} [a^n u(n) - b^n u(-n-1)]\, z^{-n}$$

$$= \sum_{n=0}^{\infty} a^n z^{-n} - \sum_{n=-\infty}^{-1} b^n z^{-n}$$

$$= \sum_{n=0}^{\infty} (a z^{-1})^{+n} - \sum_{n=-\infty}^{-1} b^n z^{-n}$$

$$= \sum_{n=0}^{\infty} (a z^{-1})^{+n} - \sum_{n=1}^{\infty} b^{-n} z^{n}$$

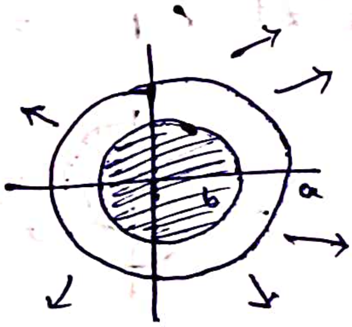
$$= \sum_{n=0}^{\infty} (a z^{-1})^{+n} - \sum_{n=1}^{\infty} (b^{-1} z)^{n}$$

$$= \frac{1}{(1-(a/z^{-1}))} -$$

**

Condition : 1 $|b| < |a|$

Z-transform not Exist.

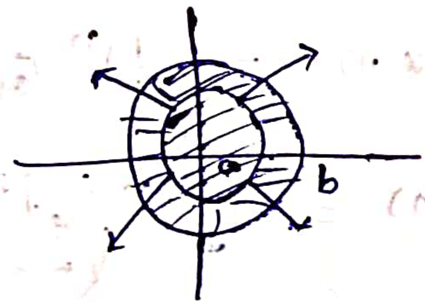


Condition : 2 $|b| > |a|$ - overlap area is not present.

$\therefore x(n) = a^n u(n) - b^n u(-n-1)$

$$= \frac{z}{z-a} + \frac{z}{z-b}$$

$\therefore |a| < z < |b|$



$|a| < z < |b|$

→ $x(n) = 7\left(\frac{1}{3}\right)^n$.

→ $x(n) = \left(\frac{1}{2}\right)^n u(-n) - 2^n u(-n-1)$

$$X(Z) = \sum_{n=-\infty}^{\infty}\left[\left(\frac{1}{2}\right)^n u(-n) - 2^n u(-n-1)\right] Z^{-n}$$

$$= \sum_{n=-\infty}^{0}\left(\frac{1}{2}\right)^n Z^{-n} - \sum_{n=-1}^{-\infty} 2^n Z^{-n}$$

$$= \sum_{n=-\infty}^{0}\left(\frac{1}{2}\right)^{-n} Z^{n} - \sum_{n=+1}^{\infty} 2^{-n} Z^{n} + 1 - 1$$

$$= \sum_{n=0}^{\infty}\left[\left(\frac{1}{2}\right)^{-1} Z\right]^n - \sum_{n=0}^{\infty}\left(2^{-1} Z\right)^n - 1$$

$$= \frac{1}{1-\left(\left(\frac{1}{2}\right)^{-1} Z\right)} - \left[\frac{1}{1-(2^{-1}Z)} + 1\right] = \frac{1}{1-\left(\frac{1}{2}^{-1}\right)Z} - \frac{2^{-1}Z}{1-2^{-1}Z}$$

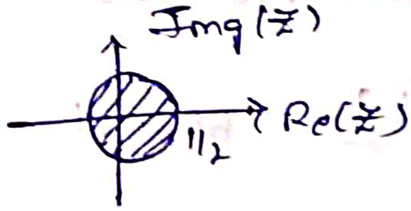
ROC :- $\left|\left(\frac{1}{2}\right)^{-1} Z\right| < 1 \Rightarrow |Z| < \frac{1}{2}$

$|2^{-1} Z| < 1 \Rightarrow |Z| < 2$

$$= \frac{1}{1-2Z} - \frac{0.5Z}{1-0.5Z}$$

$$= \frac{(1-0.5Z) - 0.5Z(1-2Z)}{(1-2Z)(1-0.5Z)}$$

Common ROC is $|Z| < 1/2$.



→ $x(n) = \left(\frac{1}{4}\right)^n \cos\left(\frac{\pi}{3} n\right) u(n)$

$$x(n) = \left(\frac{1}{4}\right)^n \left(\frac{e^{j\pi/3\, n} + e^{-j\pi/3\, n}}{2}\right) u(n)$$

$$= \frac{1}{2}\left[\left(\frac{1}{4} e^{j\pi/3}\right)^n + \left(\frac{1}{4} \cdot e^{-j\pi/3}\right)^n\right] u(n)$$

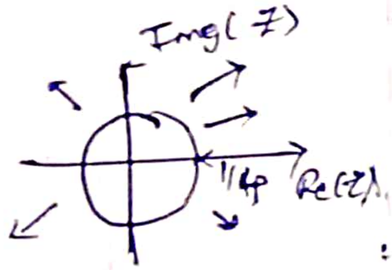
$$X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$$

$$= \frac{1}{2} \sum_{n=0}^{\infty} \left(\frac{1}{4} e^{j\pi/3}\right)^{n} z^{-n} + \sum_{n=0}^{\infty} \left(\frac{1}{4} e^{-j\pi/3}\right)^{n} z^{-n}$$

$$= \frac{1}{2}\left[\sum_{n=0}^{\infty} \left(\frac{1}{4} e^{j\frac{\pi}{3}} z^{-1}\right)^{n} + \sum_{n=0}^{\infty} \left(\frac{1}{4} e^{-j\pi/3} z^{-1}\right)^{n}\right]$$

$$= \frac{1}{2}\left[\frac{1}{1-\left(\frac{1}{4} e^{j\pi/3} z^{-1}\right)} + \frac{1}{1-\left(\frac{1}{4} e^{-j\pi/3} z^{-1}\right)}\right] \rightarrow (1)$$

<u>ROC</u> :- $\left|\frac{1}{4} e^{j\pi/3} z^{-1}\right| < 1$



$$\frac{1}{4} z^{-1} < 1 \Rightarrow |z| > \frac{1}{4}$$

from (1) $= \frac{1}{2}\left[\frac{4z}{4z - e^{j\pi/3}} + \frac{4z}{4z - e^{-j\pi/3}}\right]$

$$= \frac{1}{2}\left(\frac{4z\left[4z - e^{-j\pi/3}\right] + 4z\left[4z - e^{j\pi/3}\right]}{\left(4z - e^{j\pi/3}\right)\left(4z - e^{-j\pi/3}\right)}\right)$$

$$= \frac{1}{2}\left[\frac{16z^{2} - 4z e^{-j\pi/3} + 16z^{2} - 4z e^{j\pi/3}}{4z - \left(\cos\frac{\pi}{3} + j\sin\frac{\pi}{3}\right)\left(4z - \left(\cos\frac{\pi}{3} - j\sin\frac{\pi}{3}\right)\right)}\right]$$

$$= \frac{1}{2} \cdot \frac{24z^{2} - 4z e^{-j\pi/3} - 4z e^{j\pi/3}}{4z - \left(\cos\frac{\pi}{3} + j\sin\frac{\pi}{3}\right)\left(4z - \left(\cos\frac{\pi}{3} - j\sin\frac{\pi}{3}\right)\right)}$$

$$= \frac{z(z - 1/8)}{(\quad)}$$

→ find Z-transform and ROC of $x(n) = a^{|n|}$, $|a| < 1$.

Sol:- $Z[x(n)] = X(Z) = \sum_{n=-\infty}^{\infty} x(n) Z^{-n}$

$$= \sum_{n=-\infty}^{\infty} a^{|n|} Z^{-n}$$

$$= \sum_{n=-\infty}^{-1} a^{-n} Z^{-n} + \sum_{n=0}^{\infty} a^{n} Z^{-n}$$

$$\left(\because a^{|n|} = a^{n}, \; n > 0 \atop = a^{-n}, \; n < 0 \right)$$

$$= \sum_{n=1}^{\infty} (aZ)^{n} + \sum_{n=0}^{\infty} (aZ^{-1})^{n}$$

$$= \sum_{n=0}^{\infty} (aZ)^{n} - 1 + \sum_{n=0}^{\infty} (aZ^{-1})^{n}$$

$$= \frac{1}{1-(aZ)} - 1 + \frac{1}{1-aZ^{-1}}$$

$$= \frac{\cancel{1} - \cancel{1} + aZ}{1-aZ} + \frac{Z}{Z-a}$$

$$= -\frac{Z}{Z-\frac{1}{a}} + \frac{Z}{Z-a}$$

$$= \frac{-Z(Z-a) + Z\left(Z-\frac{1}{a}\right)}{(Z-a)\left(Z-\frac{1}{a}\right)}$$

$$= \frac{-\cancel{Z^2} + aZ + \cancel{Z^2} - \frac{Z}{a}}{(Z-a)\left(Z-\frac{1}{a}\right)}$$

$$= \frac{Z\left(a-\frac{1}{a}\right)}{(Z-a)\left(Z-\frac{1}{a}\right)}$$

Roc :- $|az| < 1$. $\quad |az^{-1}| < 1$

$|z| < \frac{1}{a}$ $\quad |z| > a$

$\therefore$ $a > |z| < \frac{1}{a}$

$\rightarrow$ $x(n) = \left(\frac{1}{2}\right)^n u(n-2)$ Roc and Z-transform

$$Z[x(n)] = X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$= \sum_{n=2}^{\infty} \left(\frac{1}{2}\right)^n z^{-n}$$

$$= \sum_{n=2}^{\infty} \left(\frac{1}{2} \cdot z^{-1}\right)^n - \sum_{n=0}^{1} \left(\frac{1}{2} z^{-1}\right)^n$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{2} z^{-1}\right)^n - 1 - \frac{1}{2} z^{-1}$$

$$= \frac{1}{1-\left(\frac{1}{2} z^{-1}\right)} - 1 - \frac{1}{2} z^{-1}$$

$$= \frac{z}{z - \frac{1}{2}} - 1 - \frac{1}{2z}$$

$$= \frac{2z}{2z-1} - 1 - \frac{1}{2z}$$

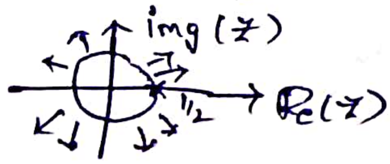
$$= \frac{(2z) - 2z + 1}{2z-1} - \frac{1}{2z}$$

$$= \frac{2z - 2z + 1}{2z(2z-1)}$$

$$= \frac{1}{4z\left(z - \frac{1}{2}\right)}$$

Roc : $\left|\frac{1}{2} z^{-1}\right| < 1$

$|z| > \frac{1}{2}$



Note:- $\sum_{n=0}^{N-1} \alpha^n = \frac{1-\alpha^N}{1-\alpha}$ when $\alpha \neq 1$

$= N$ when $\alpha = 1$

$\sum_{n=0}^{5} a^n = \frac{1-a^6}{1-a}$ It is for finite series.

properties of Z-transform :-

linearity :- $x_1(n) \longleftrightarrow X_1(z)$

$x_2(n) \longleftrightarrow X_2(z)$

$a x_1(n) + b x_2(n) \longleftrightarrow a X_1(z) + b X_2(z)$

Proof :-

$Z[x(n)] = X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$

$Z[a x_1(n) + b x_2(n)] = \sum_{n=-\infty}^{\infty} a x_1(n) z^{-n} + \sum_{n=-\infty}^{\infty} b x_2(n) z^{-n}$

$= a X_1(z) + b X_2(z)$

2. **Time shifting property :-**

$x(n) \longleftrightarrow X(z)$

$x(n-n_0) \longleftrightarrow z^{-n_0} X(z)$ (neglect initial conditions)

proof :-

$Z[x(n-n_0)] = \sum_{n=-\infty}^{\infty} x(n-n_0) z^{-n}$

$n-n_0 = p \Rightarrow n = p+n_0$

$= \sum_{p=-\infty}^{\infty} x(p) z^{-(p+n_0)}$

$= \sum_{p=-\infty}^{\infty} x(p) z^{-p} \cdot z^{-n_0}$

$$= z^{-n_0} \cdot X(z)$$

3. Time Reversal property :-

$$x(n) \longleftrightarrow X(z)$$

$$x(-n) \longleftrightarrow X(1/z)$$

proof :-

$$Z[x(-n)] = \sum_{n=-\infty}^{\infty} x(-n)\, z^{-n}$$

$$-n = p \Rightarrow n = -p$$

$$\sum_{n=-\infty}^{\infty} x(n)\, z^{-n} = X(z)$$

$$= \sum_{p=\infty}^{-\infty} x(p)\, z^{p}$$

$$= \sum_{p=\infty}^{-\infty} x(p) (z^{-1})^{-p}$$

$$= X(z^{-1}) = X\left(\frac{1}{z}\right)$$

~~4. Time Reversal property :-~~

~~$x(n) \longleftrightarrow X(z)$~~

4. Multiplication by Exponential Sequence property :-

$$x(n) \longleftrightarrow X(z)$$

$$a^n \cdot x(n) \longleftrightarrow X(z/a)$$

proof :- $X(z) = Z[x(n)] = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$

$$Z[a^n x(n)] = \sum_{n=-\infty}^{\infty} a^n x(n)\, z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} x(n) \left(\frac{z}{a}\right)^{-n}$$

$$= X\left(\frac{z}{a}\right)$$

Time Differentiation property :- (In frequency domine)

$$x(n) \longleftrightarrow X(z)$$

$$n \cdot x(n) \longleftrightarrow -z \frac{d}{dz} X(z)$$

Proof :-

$$X(z) = z[x(n)] = \sum_{n=-\infty}^{\infty} x(n) z^{-n} \rightarrow (1)$$

Apply diff w.r.t z on both sides

$$\frac{d}{dz} X(z) = \frac{d}{dz} \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} x(n) \cdot z^{-n-1} (-n)$$

$$= -\sum_{n=-\infty}^{\infty} n[x(n)] \cdot z^{-n} \cdot z^{-1}$$

$$-z \frac{d}{dz} X(z) = \sum_{n=-\infty}^{\infty} n\, x(n) z^{-n} \rightarrow (2)$$

Compare (1) and (2)

$$-z \frac{d}{dz} X(z) = z[n\, x(n)]$$

$$\therefore \quad n\, x(n) \overset{Z.T}{\longleftrightarrow} -z \frac{d}{dz} X(z)$$

Convolution property :-

$$x_1(n) \longleftrightarrow X_1(z)$$

$$x_2(n) \longleftrightarrow X_2(z)$$

$$x_1(n) * x_2(n) \longleftrightarrow X_1(z) \cdot X_2(z)$$

proof :- $z[x(n)] = X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$

$$z[x_1(n) * x_2(n)] = \sum_{n=-\infty}^{\infty} (x_1(n) * x_2(n)) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} \left[\sum_{k=-\infty}^{\infty} x_1(k)\, x_2(n-k) \right] Z^{-n}$$

change the order of summation.

$$= \sum_{k=-\infty}^{\infty} x_1(k) \left(\sum_{n=-\infty}^{\infty} x_2(n-k)\, Z^{-n} \right)$$

Apply time shifting property $Z[x(n-n_0)] = X(Z) \cdot Z^{-n_0}$

$$= \underbrace{\sum_{k=-\infty}^{\infty} x_1(k)\, Z^{-k}}\, X_2(Z)$$

$$= X_1(Z) \cdot X_2(Z)$$

→ **Initial value theorem** :- → The signal should be taken as causal signal.

$$x(n) \longleftrightarrow X(Z)$$

$$\lim_{n \to 0} x(n) \longleftrightarrow x(0) = \lim_{Z \to \infty} X(Z)$$

proof :- $Z[x(n)] = X(Z) = \sum_{n=+0}^{\infty} x(n) Z^{-n}$

$$= x(0)Z^{0} + x(1) Z^{-1} + x(2) Z^{-2} + \cdots\cdots$$

Take $Z \to \infty$ on Both sides

$$\lim_{Z \to \infty} X(Z) = \lim_{Z \to \infty} \left[x(0) + \frac{x(1)}{Z} + \frac{x(2)}{Z^2} + \cdots \right]$$

$$= x(0) + 0 + 0 + \cdots\cdots$$

$$\lim_{n \to 0} x(n) = x(0) = \lim_{Z \to \infty} X(Z)$$

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Final value theorem :-

$$x(n) \longleftrightarrow X(Z)$$

$$\lim_{n \to \infty} x(n) = x(\infty) = \lim_{Z \to 1} (Z-1) X(Z)$$

Proof :- The Signal Should be Causal Signal.

$$Z[x(n)] = X(Z) = \sum_{n=0}^{\infty} x(n) Z^{-n}$$

Apply time shifting property.

$$\therefore Z[x(n+1)] = Z^{1} X(Z) - Z x(0) = \sum_{n=0}^{\infty} x(n+1) Z^{-n}$$

Sub $X(Z)$ on both Sides.

$$Z^{1} X(Z) - Z x(0) - X(Z) = \sum_{n=0}^{\infty} x(n+1) Z^{-n} - \sum_{n=0}^{\infty} X(Z) Z^{-n}$$

$$X(Z)[Z-1] - Z x(0) = \sum_{n=0}^{\infty} x(n+1) - x(n) \cdot Z^{-n}$$

Apply $\lim_{Z \to 1}$ on both Sides. (2)

$$\lim_{Z \to 1} X(Z)(Z-1) - x(0) = \lim_{Z \to 1} \sum_{n=0}^{\infty} (x(n+1) - x(n)) Z^{-n}$$

$$= \cancel{x(1)} - x(0) + \cancel{x(2)} - \cancel{x(1)} + x(3) - \cancel{x(2)} + \dots + x(\infty) - x(\infty - 1)$$

$$\lim_{Z \to 1} X(Z)(Z-1) - \cancel{x(0)} = x(\infty) - \cancel{x(0)}$$

$$\lim_{Z \to 1} X(Z)(Z-1) = x(\infty)$$

→ find the initial and final values using Both initial and final Value theorem.

a) $X(Z) = \dfrac{Z+2}{4(Z-1)(Z+0.7)}$ find $x(\infty)$

$$x(\infty) = \lim_{z\to 1} (z-1)\, X(z) = \lim_{z\to 1} \cancel{(z-1)} \cdot \frac{z+2}{4\cancel{(z-1)}(z+0.7)}$$

$$= \frac{3}{4(1.7)} = \frac{3}{6.8}$$

$$= 0.44$$

b) $X(z) = \dfrac{z+1}{(z-0.6)^2}$ find $x(\infty)$

$$x(\infty) = \lim_{z\to 1} (z-1)\, X(z) = \lim_{z\to 1} (z-1) \cdot \frac{z+1}{(z-0.6)^2}$$

$$= \lim_{z\to 1} (z-1) \frac{(z+1)}{(z-0.6)(z-0.6)}$$

$$= 0$$

→ find initial value $x(0)$ if $X(z)$ is given

$$X(z) = \frac{z+3}{(z+1)(z+2)}$$

$$x(0) = \lim_{z\to\infty} X(z) = \lim_{z\to\infty} \frac{z+3}{(z+1)(z+2)}$$

$$= \lim_{z\to\infty} \frac{\cancel{z}(1+3/z)}{z\cancel{z}(1+1/z)(1+2/z)}$$

$$= \frac{1}{z}\left[\frac{1+0}{1}\right] = 0$$

→ $X(z) = \dfrac{z^2+2z+2}{(z+1)(z+0.5)}$ $\quad x(0) = ?$

$$x(0) = \lim_{z\to\infty} X(z) = \lim_{z\to\infty} \frac{z^2+2z+2}{(z+1)(z+0.5)}$$

$$= \lim_{z\to\infty} \frac{\cancel{z^2}\left(1+\frac{2}{z}+\frac{2}{z^2}\right)}{\cancel{z^2}\left(1+\frac{1}{z}\right)\left(1+\frac{0.5}{z}\right)}$$

$$= \frac{1+0}{1+0} = 1$$

$\rightarrow x(n) = n\, 2^n \sin\left(\frac{\pi}{2} n\right) u(n)$

$$Z\left[\sin \omega n \; u(n)\right] = \frac{z \sin\omega}{z^2 - 2z\cos\omega + 1}$$

$\therefore$ ~~x(n)~~

$$\boxed{Z\left[\sin\left(\frac{\pi}{2} n\right) u(n)\right] = \frac{z \sin \pi/2}{z^2 - 2z\cos \pi/2 + 1} = \frac{z}{z^2+1}}$$

$$\underline{a^n x(n) \longleftrightarrow X(z/a)}$$

$$Z\left[2^n \cdot \sin(\pi/2) n \; u(n)\right] = \frac{z/2}{(z/2)^2+1} = \frac{2z}{z^2+4}$$

$$\underline{n(x(n)) \longleftrightarrow -z \cdot \frac{d}{dz} X(z)}$$

$$Z\left[n \cdot 2^n \sin \pi/2\, n \cdot u(n)\right] = -z \frac{d}{dz}\left[\frac{2z}{z^2+4}\right]$$

$$= -z\left[\frac{(z^2+4)(2) - 2z\,(2z)}{(z^2+4)^2}\right]$$

$$= -z\left[\frac{2z^2+8-4z^2}{(z^2+4)^2}\right]$$

$$= -z\left[\frac{-2z^2+8}{(z^2+4)^2}\right]$$

$$= \frac{2z(z^2-4)}{(z^2+4)^2}$$

Inverse Z transform :-

$X(z) \longleftrightarrow$ given

$x(n) = ?$

$$z^{-1}[X(z)] = x(n) = \frac{1}{2\pi j}\oint_C X(z)\cdot z^{n-1}\,dz$$

In practical Cases :-

We have four methods in inverse Z- transform :-

1. power Series method [long division method].
2. partial fraction method.
3. Inversion Integral method [Residue method].
4. Convolution method.

1. power Series method [long division method] :-

→ Roc should be either inside circle & outside circle

$|z| < \alpha$ & $|z| > \alpha$

(i) Roc $|z| > \alpha$, the given Sequence $x(n)$ should be a Causal Sequence. (R I-R)

$$\therefore X(z) = \frac{N(z)}{D(z)}$$

$N(z)$ & $D(z)$ must be decending powers of z.

(or) Increasing (Acending) powers of z^{-1}.

(ii) $|z| < a$ - non Causal Sequence (R-L)

$$\therefore X(z) = \frac{N(z)}{D(z)}$$

N(z), D(z) should be in ~~inpower~~ increasing powers of z & Decreasing powers of z^{-1}

1. find the Inverse Z transform using long division method.

$$X(z) = \frac{z^2+2z}{z^3-3z^2+4z+1} \quad \text{Roc } |z|>1$$

The given sequence is Causal Sequence.

$$X(z) = \frac{N(z)}{D(z)} = \frac{z^2+2z}{z^3-3z^2+4z+1}$$

$$
\begin{array}{r|l}
z^3-3z^2+4z+1 & z^2+2z \;\left(z^{-1}+5z^{-2}+11z^{-3}+12z^{-4}\right. \\
 & \underline{z^2-3z+4+z^{-1}} \\
 & \quad 5z-4-z^{-1} \\
 & \quad \underline{5z-15+20z^{-1}+5z^{-2}} \\
 & \qquad 11-21z^{-1}-5z^{-2} \\
 & \qquad \underline{11-33z^{-1}+44z^{-2}+11z^{-3}} \\
 & \qquad\quad 12z^{-1}-49z^{-2}-11z^{-3} \\
 & \qquad\quad \underline{12z^{-1}-36z^{-2}+48z^{-3}+12z^{-4}} \\
 & \qquad\qquad -13z^{-2}-59z^{-3}-12z^{-4}
\end{array}
$$

$$X(z) = \frac{N(z)}{D(z)} = z^{-1} + 5z^{-2} + 11z^{-3} + 12z^{-4} - 13z^{-5} \cdots$$

$z^{-1}[X(z)]$

$$= x(n) = \{1, 5, 11, 12, -13, \ldots\}$$

(8)

$$\rightarrow X(z) = \frac{z^2+2z}{z^3-3z^2+4z+1} = \frac{\cancel{z^3}(z^{-1}+2z^{-2})}{\cancel{z^3}(1-3z^{-1}+4z^{-2}+z^{-3})}$$

2. Find the Inverse z-transform using long division method

$$X(z) = \frac{z}{2z^2-3z+1}. \quad |z| < \frac{1}{2}$$

$|z| < \frac{1}{2}$, The given Eqn is in non-Causal Sequence

$$X(z) = \frac{N(z)}{D(z)} = \text{Increasing powers of } z$$

$$X(z) = \frac{z}{1-3z+2z^2}$$

$$
\begin{array}{r|l}
1-3z+2z^2 & z \quad (z+3z^2+7z^3+15z^4 \\
 & \underline{z - 3z^2 + 2z^3} \\
 & \quad 3z^2 - 2z^3 \\
 & \quad \underline{3z^2 - 9z^3 + 6z^4} \\
 & \qquad 7z^3 - 6z^4 \\
 & \qquad \underline{7z^3 - 21z^4 + 14z^5} \\
 & \qquad\quad 15z^4 - 14z^5 \\
 & \qquad\quad \underline{15z^4 - 45z^5 + 30z^6} \\
 & \qquad\qquad 30z^5 - 30z^6
\end{array}
$$

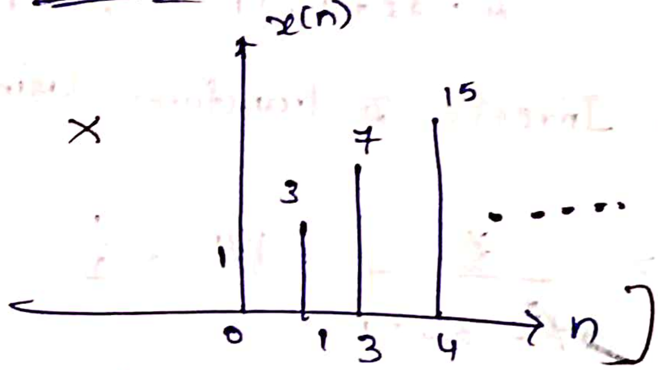
$$X(z) = z + 3z^2 + 7z^3 + 15z^4$$

15, 7, 3, 1, 0

$\therefore Z^{-1}[X(Z)] = x(n) = \{1, 3, 7, 15, \ldots\}$

$Z^{-1}[X(Z)] = x(n) = \delta(n+1) + 3\,\delta(n+2) + 7\,\delta(n+3) + 15\,\delta(n+4) + \ldots$

Waveform Representation :-



$\rightarrow X(Z) = \dfrac{1}{2 - 4Z^{-1} + 2Z^{-2}}$ (i) ROC $|Z| > 1$

(ii). ROC : $|Z| < \frac{1}{2}$

$X(Z) = \dfrac{Z^2}{2Z^2 - 4Z + 2}$

(i) ROC $\rightarrow |Z| > 1$ the given Sequence is Causal Sequence.

$X(Z) = \dfrac{N(Z)}{D(Z)}$

$= \dfrac{Z^2}{2Z^2 - 4Z + 2}$

$$2z^2-4z+2 \overline{\big) \, z^2} \; \left(\frac{1}{2}+z^{-1}+\frac{3}{2z^2}+\frac{2}{z^3}\right.$$

$$z^2 - 2z + 1$$
$$- \quad + \quad -$$
$$\overline{}$$
$$2z - 1$$
$$2z - 4 + 2z^{-1}$$
$$- \quad + \quad -$$
$$\overline{}$$
$$3 - 2z^{-1}$$
$$3 - 6z^{-1} + 3z^{-2}$$
$$- \quad + \quad -$$
$$\overline{}$$
$$4z^{-1} - 3z^{-2}$$
$$4z^{-1} - 8z^{-2} + 4z^{-3}$$
$$- \quad + \quad -$$
$$\overline{}$$
$$5z^{-2} - 4z^{-3}$$

$$X(z) = \frac{1}{2} + z^{-1} + 1.5z^{-2} + 2z^{-3} + \cdots$$

$$z^{-1}[X(z)] = x(n) = \left\{ \underset{\uparrow}{0.5}, 1, 1.5, 2, \cdots \right\}$$

(ii) ROC $\rightarrow$ $|z| < \frac{1}{2}$ The given Sequence is non-Causal Sequence.

$$X(z) = \frac{N(z)}{D(z)} = \frac{z^2}{2-4z+2z^2}$$

$$2-4z+2z^2 \overline{\big) \, z^2} \; \left(\frac{z^2}{2}+z^3+\frac{3}{2}z^4+2z^5\right.$$

$$z^2 - 2z^3 + z^4$$
$$- \quad + \quad -$$
$$\overline{}$$
$$2z^3 - z^4$$
$$2z^3 - 4z^4 + 2z^5$$
$$- \quad + \quad -$$
$$\overline{}$$
$$3z^4 - 2z^5$$
$$3z^4 - 6z^5 + 3z^6$$
$$- \quad + \quad -$$
$$\overline{}$$
$$4z^5 - 3z^6$$
$$-4z^5 \mp 6z^6 \pm 4z^7$$
$$\overline{}$$
$$3z^6 - 4z^7$$

$$X(z) = \frac{1}{2}z^2 + z^3 + \frac{3}{2}z^4 + 2z^5 + \cdots$$

$$z^{-1}[X(z)] = x(n) = \{\ldots 2, 1.5, 1, 0.5\}$$

→ partial fraction Expansion method :-

$$\frac{X(z)}{z} = \frac{Nr(z)}{Dr(z)}$$

$$\frac{X(z)}{z} = \frac{C_1}{(z-P_1)} + \frac{C_2}{(z-P_2)} + \cdots \frac{C_N}{(z-P_N)}$$

Here $P_1, P_2 \ldots P_N$ are the Coefficients.

$$C_K = (z-P_K) \cdot \frac{X(z)}{z}\Bigg|_{z=P_K} \quad \text{where } K = 1, 2 \ldots N$$

$$\therefore X(z) = \frac{z \cdot C_1}{z-P_1} + \frac{z \cdot C_2}{z-P_2} + \cdots + \frac{z}{z-P_N} C_N$$

$$\frac{z}{z-a} \longleftrightarrow a^n u(n) \quad |z| > a$$

$$-a^n u(-n-1) \quad |z| < a$$

→ find the I.Z.T using partial fraction Expansion method.

$$X(z) = \frac{z^{-1}}{3 - 4z^{-1} + z^{-2}} \qquad ROC\, |z| > 1$$

$$X(z) = \frac{\frac{1}{z}}{3 - \frac{4}{z} + \frac{1}{z^2}} = \frac{z}{3z^2 - 4z + 1}$$

$$\frac{X(z)}{z} = \frac{1}{3z^2-4z+1}$$

$$\frac{X(z)}{z} = \frac{1}{3z(z-1)-1(z-1)}$$

$$\frac{X(z)}{z} = \frac{1}{(z-1)(3z-1)}$$

$$\frac{1}{(z-1)(3z-1)} = \frac{1}{(z-1)\cdot 3\left(z-\frac{1}{3}\right)}$$

$$= \frac{A}{z-1} + \frac{B}{z-\frac{1}{3}}$$

$$1 = A\left(z-\tfrac{1}{3}\right) + B(z-1)$$

put $z = \frac{1}{3} \Rightarrow 1 = B\left(\frac{1}{3}-1\right) \Rightarrow -\frac{2}{3}B = 1 \Rightarrow B = -\frac{3}{2}$

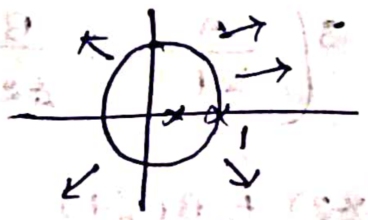
put $z = 1 \Rightarrow 1 = \left(1-\frac{1}{3}\right)A \Rightarrow 1 = \frac{2}{3}A \Rightarrow A = \frac{3}{2}$

$$\therefore \frac{X(z)}{z} = \frac{1}{3}\left(\frac{+\frac{3}{2}}{(z-1)} + \frac{-\frac{3}{2}}{\left(z-\frac{1}{3}\right)}\right)$$

$$\frac{X(z)}{z} = \frac{1}{2(z-1)} - \frac{1}{2\left(z-\frac{1}{3}\right)}$$

$$X(z) = \frac{1}{2}\cdot\frac{z}{z-1} - \frac{1}{2}\cdot\frac{z}{z-\frac{1}{3}}$$

ROC $|z| > 1$



poles are $1, \frac{1}{3} =< 1$

Here poles are present at $1, 1/3$

Here two poles are Causal poles

$$X(Z) = \frac{1}{2}(1)^n u(n) - \frac{1}{2}\left(\frac{1}{3}\right)^n u(n)$$

$$= \frac{1}{2}\left[1 - \left(\frac{1}{3}\right)^n\right] u(n)$$

→ $X(Z) = \dfrac{4 - 3Z^{-1} + 3Z^{-2}}{(Z+2)(Z-3)^2}$ a) $|Z| > 3$ b) $|Z| < 2$ (c) $2 < |Z| < 3$

→ $X(Z) = \dfrac{3 \cdot Z^{-1}}{(1-Z^{-1})(1-2Z^{-1})}$ find I.Z.T a) $|Z| > 2$ b) $|Z| < 1$ c) $1 < |Z| < 2$

$$X(Z) = \frac{3Z^{-1}}{(1-Z^{-1})(1-2Z^{-1})}$$

$$= \frac{\frac{3}{Z}}{\left(1-\frac{1}{Z}\right)\left(1-\frac{2}{Z}\right)}$$

$$= \frac{\frac{3}{Z}}{\frac{(Z-1)(Z-2)}{Z^2}}$$

$$= \frac{3Z}{(Z-1)(Z-2)}$$

$$\frac{X(Z)}{Z} = \frac{3}{(Z-1)(Z-2)}$$

$$\frac{X(Z)}{Z} = 3\left(\frac{A}{Z-1} + \frac{B}{Z-2}\right)$$

$$1 = A(Z-2) + B(Z-1)$$

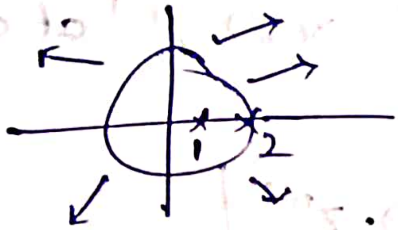
put $Z=2$ => $1 = B$ => $B = 1$

put $Z=1$ => $1 = -A$ => $A = -1$

$$\frac{X(z)}{z} = \frac{-3}{z-1} + \frac{3}{z-2}$$

$$X(z) = \frac{-3z}{z-1} + \frac{3z}{z-2}$$

a) $|z| > 2$



we get two poles they are, 1 and 2

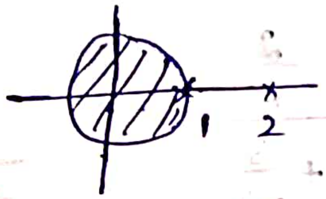
Causal

$$X(z) = \frac{-3z}{z-1} + \frac{3z}{z-2}$$

$$= -3(1)^n u(n) + 3(2)^n u(n)$$

$$= -3u(n)\left[1 - 2^n\right]$$

(b) $|z| < 1$



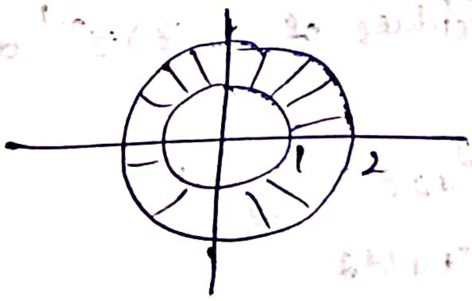
poles are 1 and 2

Non Causal and anti-Causal poles

$$\therefore X(z) = -3\left[-(1)^n u(-n-1)\right] + 3\left[-(2)^n u(-n-1)\right]$$

$$= -3u(-n-1)\left[-(1)^n - (-2)^n\right] = -3u(-n-1)\left[1-2^n\right]$$

(c) $1 < |z| < 2$



$$X(z) = \frac{-3z}{z-1} + \frac{3z}{z-2}$$

pole is is Causal pole and pole 2 is non-Causal pole.

$$X(z) = -3\left[(1)^n u(n)\right] + 3\left[-(2)^n u(-n-1)\right]$$

$$= -3\left[u(n)\right] + 3\left[-2^n u(-n-1)\right] \Rightarrow -3\left(u(n) + 2^n u(-n-1)\right)$$

Residue (Inversion integration) method :-

→ not necessary

$$Z^{-1}[X(z)] = x(n) = \frac{1}{2\pi j} \oint_c X(z)\, z^{n-1}\, dz$$

$x(n) = \Sigma$ Residue of $X(z) . z^{n-1}$ at all poles inside C.

$$= \Sigma (z-z_i)^{\wedge} X(z) \cdot z^{n-1} \Big|_{z=z_i}.$$

using Residue Method find the I. Z. T

$$X(z) = \frac{1 + 2z^{-1}}{1+4z^{-1}+3z^{-2}} \quad \text{Roc } |z| > 3$$

$$X(\tfrac{1}{z}) = \frac{1 + \frac{2}{z}}{1+\frac{4}{z} + \frac{3}{z^2}} = \frac{\frac{z+2}{z}}{\frac{z^2+4z+2}{z^2}}$$

$$X(z) = \frac{z(2+z)}{z^2+4z+2} = \frac{z^2+2z}{z^2+4z+2}.$$

$x(n) = \Sigma$ Residues of $X(z) z^{n-1}$ at all poles inside C

$$= \Sigma \frac{z^2+2z}{z^2+4z+2} z^{n-1}.$$

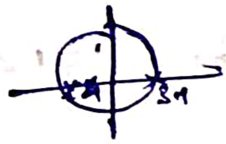
$$= \Sigma \frac{z(z+2)}{(z+1)(z+3)} . z^{n-1}$$

$$= \Sigma \frac{z^n(z+2)}{(z+1)(z+3)} \quad \text{at poles } z=-1, -3$$

$$x(n) = \frac{\cancel{(z+1)}\, z^n (z+2)}{\cancel{(z+1)}(z+3)}\Bigg|_{z=-1} + \frac{\cancel{(z+3)}\, z^n (z+2)}{(z+1)\cancel{(z+3)}}\Bigg|_{z=-3}$$

$$= (-1)^n \cdot \frac{1}{2}\, u(n) + (-3)^n \cdot \frac{-1}{2} \cdot u(n) \qquad \text{ROC}: |z| > 3$$

$$= \frac{1}{2} u(n) \left[(-1)^n + (-3)^n \right]$$



2. using inversion integration method & Contour Integration method or Residue method find I.Z.T

$$X(z) = \frac{z(z+1)}{z^2 - 3z + 2}$$

a) ROC $|z| > 2$

b) $|z| < 1$

c) $1 < |z| < 3$

$$X(z) = \frac{z(z+1)}{(z-1)(z-2)}$$ two poles, at $z = 1, 2$

find Residue of pole $z = 1$ at $X(z) \cdot z^{n-1}$

$$= \frac{\cancel{z}(z+1)}{(z-1)(z-2)} \cdot z^n \cdot \cancel{z}^{-1}$$

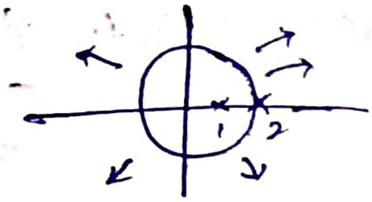
$$= \frac{(z+1) \cdot z^n}{(z-1)(z-2)}$$

$$x(n) = \cancel{(z-1)} \frac{(z+1) z^n}{\cancel{(z-1)}(z-2)}\Bigg|_{z=1} + \cancel{(z-2)} \frac{(z+1) z^n}{(z-1)\cancel{(z-2)}}\Bigg|_{z=2}$$

$$= \frac{2 \cdot (1)^n}{-1} + \frac{3 \cdot 2^n}{1}$$

$$= -2 + 3\, 2^n$$

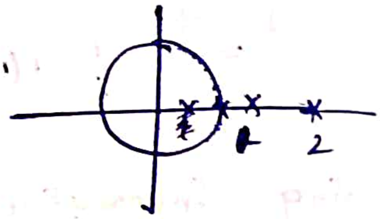
a) ROC : $|z| > 2$



$x(n) = \sum \text{Residue of } X(z)\, z^{n-1}$

$= -2\,u(n) + 3(2^n)\,u(n)$

b) ROC : $|z| < 1$

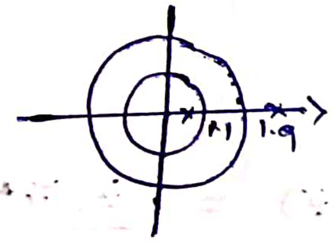


Both poles are outside C

(Exterior)

$\therefore\ x(n) = 2\,u(-n-1) - 3\,2^n\,u(-n-1)$

c) $1 < |z| < 2$



pole '1' is interior and pole '2' is Exterior pole.

$x(n) = \cancel{2\,u(-n-1) + 3 \cdot 2^n\,u(n)}$

$-2u(n) - 3\,2^n\,u(-n-1)$

Interior poles → +ve

Exterior poles → −ve

Convolution method :- (Convolution)

time domine — frequency domine

$$x_1(n) * x_2(n) \xleftrightarrow{Z \cdot T} X_1(z) \cdot X_2(z)$$

$x(n) =$

$$x_1(n) * x_2(n) = \sum_{k=-\infty}^{\infty} x_1(k)\, x_2(n-k)$$

find the I.Z.T using Convolution method

$$X(z) = \frac{z^2}{(z-2)(z-3)}$$

$$X(z) = \frac{z}{z-2} \cdot \frac{z}{z-3}$$

$$x_1(n) = Z^{-1}\left[\frac{z}{z-2}\right] = 2^n u(n)$$

$$x_2(n) = Z^{-1}\left[\frac{z}{z-3}\right] = 3^n u(n)$$

$$x(n) = x_1(n) * x_2(n)$$

$$= \sum_{n=-\infty}^{\infty} x_1(k)\, x_2(n-k)$$

$$= \sum_{k=-\infty}^{\infty} 2^k u(k) \cdot 3^{n-k} u(n-k)$$

$$= \sum_{k=0}^{n} 2^k \cdot 3^{n-k}$$

$$= 3^n \sum_{k=0}^{n} \left(\frac{2}{3}\right)^k$$

$$= 3^n \cdot \frac{1-\left(\frac{2}{3}\right)^{n+1}}{1-\frac{2}{3}} \qquad \left(\because \frac{1-\alpha^N}{1-\alpha}\right)$$

$$= \frac{3^{n+1}\left[1-\left(\frac{2}{3}\right)^{n+1}\right]}{3} \cdot 3$$

$$= 3^{n+1} u(n) - 2^{n+1} u(n)$$

Application for Z-transform :-

delay

→ A System discribed by Difference Equation

$$y(n) - \frac{3}{4} y(n-1) + \frac{1}{8} y(n-2) = x(n) + x(n-1)$$

with $y(-1) = 0$, $y(-2) = -1$, find 1) Natural Response

i/p is step signal 2) forced Response

3) frequency Response

a) Natural Response :- i/p signal = 0 then it is N.S

i/p applied then it is F.R

$$CS = CF + PI$$

$x(n) = 0$

$$y(n) - \frac{3}{4} y(n-1) + \frac{1}{8} y(n-2) = 0 \quad \text{Taking I.C}$$

$$Y(z) - \frac{3}{4}\left[z^{-1} Y(z) + y(-1)\right] + \frac{1}{8}\left[z^{-2} Y(z) + z^{-1} y(-1) + y(-2)\right]$$

$$Y(z) - \frac{3}{4} z^{-1} Y(z) + \frac{1}{8} z^{-2} Y(z) - \frac{1}{8} = 0$$

$$Y(z) = \frac{1/8}{1 - 3/4 z^{-1} + \frac{1}{8} z^{-2}} = \frac{\frac{1}{8} z^2}{z^2 - \frac{3}{4} z + \frac{1}{8}}$$

$$\therefore \frac{Y(z)}{z} = \frac{\frac{1}{8} z}{(z-\frac{1}{2})(z-1/4)}$$

$$\frac{Y(z)}{z} = \frac{A}{z-1/2} + \frac{B}{z-1/4}$$

$A = 1/4$, $B = -1/8$

$$Y(z) = \frac{1/4\, z}{z-1/2} - \frac{1/8\, z}{z-1/4}$$

$$y(n) = \frac{1}{4}\left(\frac{1}{2}\right)^n u(n) - \frac{1}{8}\left(\frac{1}{4}\right)^n u(n)$$

b) forced Response :- (neglect initial Conditions)

i/p Signal = Step Signal = u(t)

$$y(n) - \frac{3}{4} y(n-1) + \frac{1}{8} y(n-2) = u(n) + u(n-1)$$

$$Y(Z) - \frac{3}{4} Z^{-1} Y(Z) + \frac{1}{8} Z^{-2} Y(Z) = \frac{Z}{Z-1} + \frac{1}{Z-1}$$

$$\therefore Y(Z)\left[1 - \frac{3}{4} Z^{-1} + \frac{1}{8} Z^{-2}\right] = \frac{Z^2 - \cancel{2Z} - 1}{(Z-1)^2} = \frac{(Z+1)\cancel{(Z-1)}}{(Z-1)^{\cancel{2}}}$$

$$= \frac{Z+1}{Z-1}$$

$$Y(Z)\left[1 - \frac{3}{4} Z^{-1} + \frac{1}{8} Z^{-2}\right] = \frac{Z+1}{Z-1}$$

$$Y(Z) = \frac{Z+1}{(Z-1)\left[1 - \frac{3}{4} Z^{-1} + \frac{1}{8} Z^{-2}\right]}$$

$$= \frac{Z^2 (Z+1)}{(Z-1)(Z-1/2)(Z-1/4)}$$

$$\frac{Y(Z)}{Z} = \frac{Z(Z+1)}{(Z-1)(Z-1/2)(Z-1/4)}$$

$$y(n) = \frac{16}{3} u(n) - 6 \cdot \left(\frac{1}{2}\right)^n u(n) + \frac{5}{3}\left(\frac{1}{4}\right)^n u(n)$$

This is Called as forced Response or Step Response.

c) frequency Response :-

$$y(n) - \frac{3}{4}y(n-1) + \frac{1}{8}y(n-2) = x(n) + x(n-1)$$

~~$x(n) = \delta(n)$~~

$$Y(z) - \frac{3}{4}Y(z)\cdot z^{-1} + \frac{1}{8}z^{-2}Y(z) = X(z) + z^{-1}X(z)$$

$$Y(z)\cdot\left[1 - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}\right] = X(z)\cdot\left[1 + z^{-1}\right]$$

T.F $$H(z) = \frac{Y(z)}{X(z)} = \frac{1+z^{-1}}{\left(1 - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}\right)}$$

$$= \frac{z + z^2}{z^2 - \frac{3}{4}z + \frac{1}{8}}$$

$z = r\cdot e^{jw}$

$r = 1$

$$= \frac{z(z+1)}{z^2 - \frac{3}{4}z + \frac{1}{8}}$$

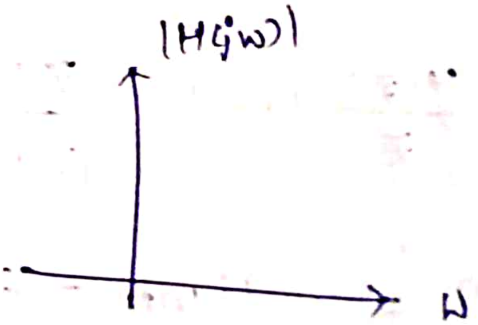
Sub $z = e^{jw}$

$$\therefore H(e^{jw}) = \frac{e^{jw}(e^{jw}+1)}{(e^{jw})^2 - \frac{3}{4}(e^{jw}) + \frac{1}{8}}$$

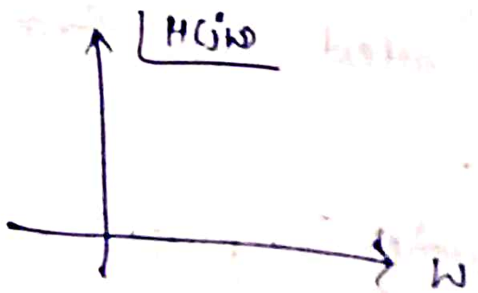
This is called as frequency Response.

It has two plots :-

Magnitude plot :



Phase plot :-

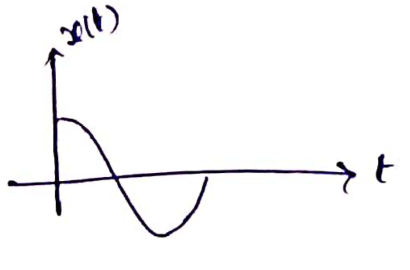


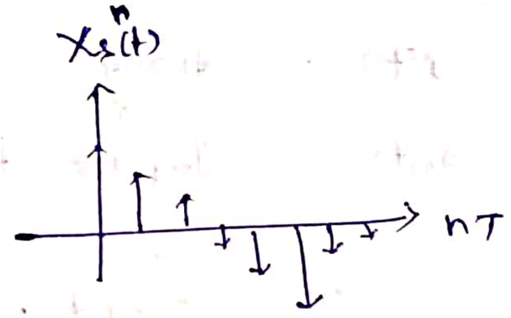
low pass signal
message signal or

Sampling theorem

Introduction :- It is a process of Converting analog signals into discrete signals. is Called Sampling.

discrete → digital (Encoding)





f_s → Sampling frequency = Samples / sec

$T_s = \frac{1}{f_s}$ = Sampling time

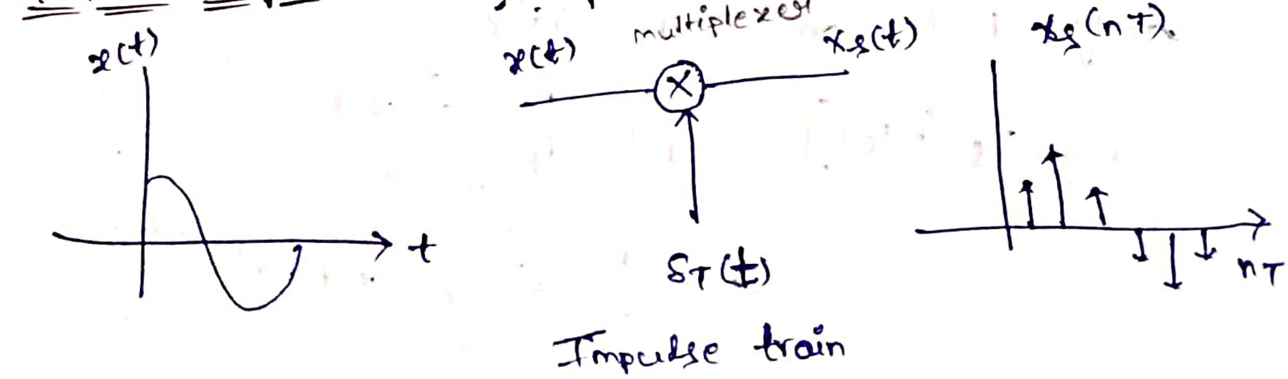
Sampling theorem for low pass signals (or) Band limited Signals :-

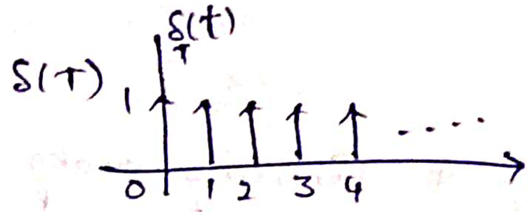
Definition :- It states that Consider a band limited Signal of highest frequency is "f_m", is Sampled $x(nT_s)$. if Sampling frequency $\boxed{f_s \geq 2f_m}$

$f_s = 2f_m$ (Critical Sampling) , $f_s < 2f_m$

(500) (200)

Graphical Signal :-





$T_s \rightarrow$ Sample period , $\frac{1}{T_s} = f_s \rightarrow$ Sampling frequency

Multiplexer :-

$x(t)$ = Analog

$\delta_T(t)$ = Impulse train

$$= \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$$

$$= \sum_{n=-\infty}^{\infty} C_n e^{jnw_s t}$$

$$w_s = 2\pi f_s = \frac{2\pi}{T_s}$$

$$\boxed{C_n = \frac{1}{T} \int_0^T \delta(t) e^{-jnw_0 t}}$$

$$\delta(t) = \frac{1}{T} \sum_{n=-\infty}^{\infty} e^{jnw_s t}$$

$$x_s(t) = x(t) \cdot \delta_{T_s}(t)$$

$$\boxed{x_s(nT_s) = \frac{1}{T_s} \sum_{n=-\infty}^{\infty} x(t) e^{jnw_s t}} \rightarrow \text{Sampled signal}$$

Apply F.T on both sides

$$F[x_s(nT_s)] = F\left[\frac{1}{T_s} \sum_{n=-\infty}^{\infty} x(t) e^{jnw_s t}\right]$$

$$= \frac{1}{T_s} \sum_{n=-\infty}^{\infty} F\left[x(t) e^{jnw_s t}\right]$$

$$X_s(\omega) = \frac{1}{T_s} \sum_{n=-\infty}^{\infty} X(\omega - n\omega_0)$$

$\because F[x(t)] \longleftrightarrow X(\omega)$

$F[x(t) \cdot e^{j\omega_0 t}] \longleftrightarrow X(\omega - \omega_0)$

$$X_s(f) = f_s \sum_{n=-\infty}^{\infty} X(f - nf_s) \quad f_s \geq 2f_m$$

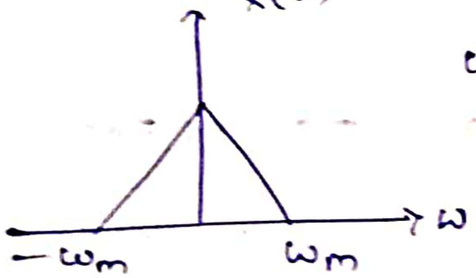
$f_m \rightarrow$ highest frequency in $x(t)$

$$X_s(\omega) = f_s \sum_{n=-\infty}^{\infty} X(\omega - \omega_s)$$

$$X_s(\omega) = f_s \left[\dots + X(\omega + \omega_s) + X(\omega) + X(\omega - \omega_s) + X(\omega - 2\omega_s) + \dots \right]$$

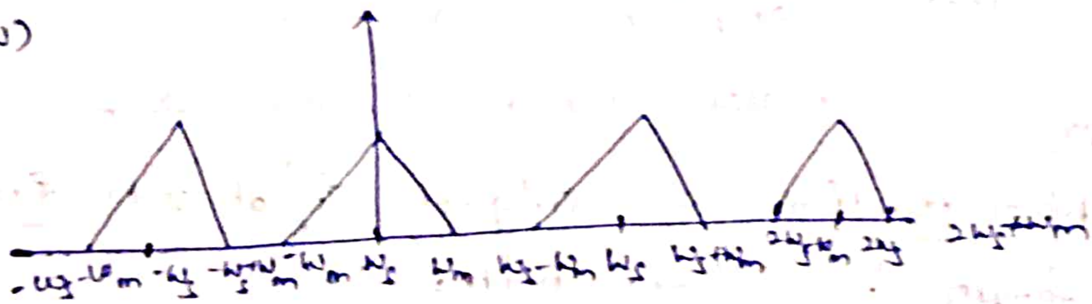
$$x(t) \xleftrightarrow{F.T} X(\omega)$$

Spectrum :-



$\omega_m \rightarrow$ highest frequency component.

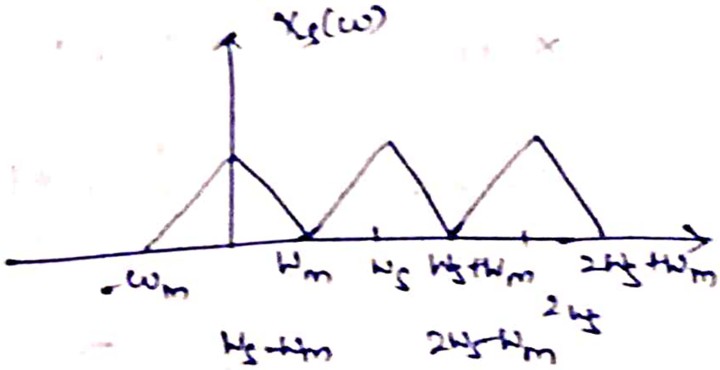
for $X_s(\omega)$



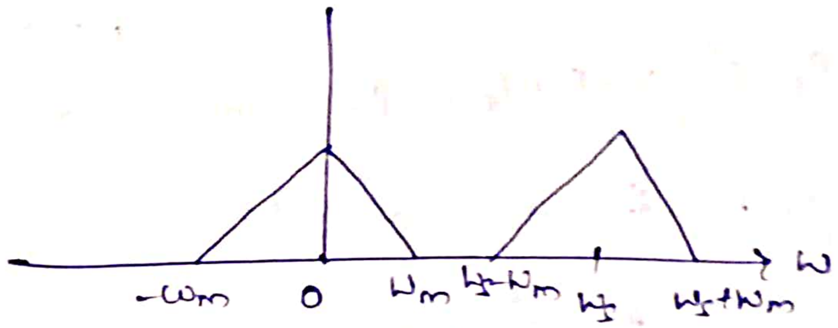
Condition :- 1

$\omega_s = 2\omega_m$

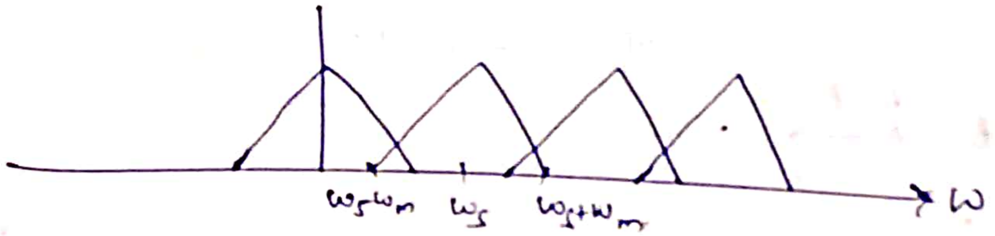
(Critical Sampling)



Condition : 2 $\omega_s > 2\omega_m$ (over Sampling)



Condition : 3 $\omega_s < 2\omega_m$ (under Sampling)



i) Critical Sampling → $\omega_s = 2\omega_m \Rightarrow f_s = 2f_m$

ii) over Sampling → $\omega_s > 2\omega_m \Rightarrow f_s > 2f_m$

iii) under Sampling → $\omega_s < 2\omega_m \Rightarrow f_s < 2f_m$

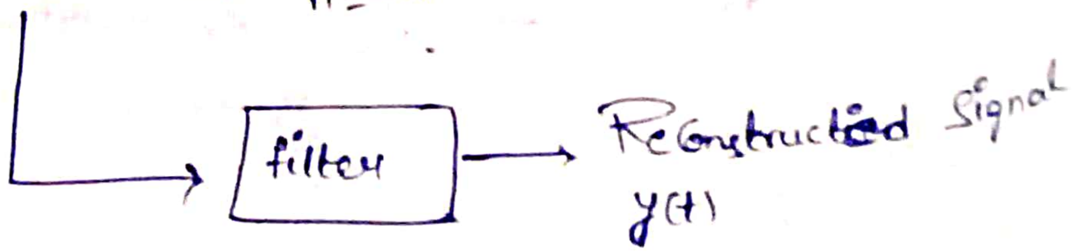
↳ also Called as aliasing effect

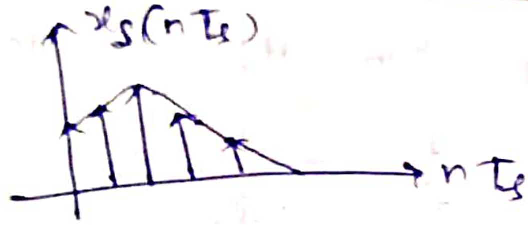
MP

Reconstruction of Original Signal $x(t)$ from Samples in "Band limited (LP) :-

$$X_s(t) = \sum_{n=-\infty}^{\infty} x(t)\,\delta(t - nT_s)$$

$$X_s(nT_s) = \sum_{n=-\infty}^{\infty} x(nT_s)\,\delta(t - nT_s)$$





$$y(t) = x_s(nT_s) * h(t)$$

$$= \left[\sum_{n=-\infty}^{\infty} x(nT_s)\,\delta(t-nT_s)\right] * h(t)$$

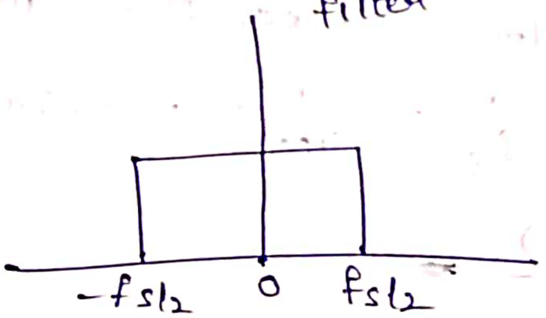
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} x(nT_s)\,\delta(t-nT_s)\Big\}\, h(t-\tau)\,d\tau$$

$$= \sum_{n=-\infty}^{\infty} x(nT_s) \int_{\tau=-\infty}^{\infty} \delta(\tau-nT_s)\, h(t-\tau)\,d\tau$$

$$y(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\, h(t-nT_s)$$

filter

$x(nT_s)$ $\rightarrow$



$\rightarrow$ $y(t)$

$$\boxed{y(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\,.\,\text{sinc}\, f_s(t-nT_s)}$$

Reconstructed Signal

Sample signal

→ **Nyquist rate & Nyquist Interval :-**

It is a minimum sampling rate required to perform sampling at trans & reconstruction is possible in the trans side without distortion is called Nyquist rate.

Nyquist rate $\Rightarrow$ $f_s = 2f_m$ HZ (Samples/sec)

Nyquist Interval $T_s = \frac{1}{f_s} = \frac{1}{2f_m}$ (sec) $(\because f_s = 2f_m)$

→ Find Nyquist Rate & interval. Corresponding to the following signal.

1. $x(t) = 1 + \cos 2000\pi t + \sin 4000\pi t$

$x(t) = 1 + \cos 2\pi(1000t) + \sin 2\pi(2000t)$

Compare with $x(t) = 1 + \cos 2\pi f_{m_1} t + \sin 2\pi f_{m_2} t$

$\therefore$ Here $f_{m_1} = 1000$ HZ, $f_{m_2} = 2000$ HZ

$f_m = 2000$ HZ (Highest frequency)

Nyquist Rate $f_s = 2f_m = 2 \times 2000$ HZ $= 4000$ HZ

Nyquist interval $T_s = \frac{1}{f_s} = \frac{1}{4000} = 0.25$ msec

2. $x(t) = \frac{\sin(4000\pi t)}{\pi t}$

$x(t) = \frac{\sin 2\pi(2000t)}{\pi t} = \frac{\sin 2\pi f_m t}{\pi t}$

$f_m = 2000$ HZ

$f_s = 2 \times 2000 = 4000$ HZ

$T_s = \frac{1}{f_s} = 0.25$ msec

3. $x(t) = \left(\frac{\sin 4000\pi t}{\pi t}\right)^2$ $1 - \cos$

$x(t) = \frac{\sin^2 2\pi(2000t)}{\pi t} = \frac{1 - \cos 8000\pi t}{(\pi t)^2}$

$= \frac{1 - \cos 2\pi(4000)t}{(\pi t)^2}$

(2)

$$= \frac{1 - \cos 2\pi f_m t}{(\pi t)^2}$$

$f_m = 4000\ Hz$

$f_s = 8000\ Hz, \quad T_s = 0.125\ mm\ sec$

* The maximum difference b/w two successful samples is known as Nyquist Interval.

4. $x(t) = 2\ \text{Sinc}(100\pi t)$

$= 2\ \text{Sinc}(2\pi(50t))$

$f_m = 50\ Hz$

$f_s = 100\ Hz, \quad T_s = \frac{1}{100} = 0.01\ \text{Sec} = 0.001\ mm\ sec$

5. $x(t) = \frac{1}{2}\ \text{Sinc}(100\pi t) + \frac{1}{3}\ \text{Sinc}(50\pi t)$

$x(t) = \frac{1}{2}\ \text{Sinc}\ 2\pi(50t) + \frac{1}{3}\ \text{Sinc}\ 2\pi(25t)$

$f_{m_1} = 50 \quad f_{m_2} = 25$

$f_m = 50\ Hz$

$f_s = 100\ Hz \quad T_s = \frac{1}{100} = 0.01\ \text{Sec} = 0.001\ mm\ sec$

6. $x(t) = \text{Sinc}(100\pi t) + 3\ \text{Sinc}^2(60\pi t)$

$x(t) = \text{Sinc}\ 2\pi(50t) + 3 \left[\frac{\sin 60\pi t}{60\pi t}\right]^2$

$= \text{Sinc}\ 2\pi(50t) + 3\left(\frac{1 - \cos 120\pi t}{(60\pi t)^2}\right)$

$= \text{Sinc}\ 2\pi(50t) + 3\left[\frac{1 - \cos 2\pi(60t)}{(60\pi t)^2}\right]$

$f_{m_1} = 50 \quad f_{m_2} = 60$

$f_m = 60$

$f_s = 120\ Hz, \quad T_s = \frac{1}{120} = 8.3\ mm\ sec$

7. $x(t) = \operatorname{sinc}(80\pi t) \cdot \operatorname{sinc}(120\pi t)$

$$= \frac{\sin 80\pi t}{80\pi t} \cdot \frac{\sin 120\pi t}{120\pi t}$$

$$= \frac{1}{2}\left[\frac{\cos(80-120)\pi t - \cos(80+120)\pi t}{(80\pi)(120\pi)}\right]$$

$$= \frac{1}{2}\left[\frac{\cos 40\pi t}{(80\pi)(120\pi)}\right] - \frac{1}{2}\left[\frac{\cos 200\pi t}{(80\pi)(120\pi)}\right]$$

$$= \frac{1}{2}\left[\frac{\cos 2\pi(20t)}{(80\pi)(120\pi)}\right] - \frac{1}{2}\left[\frac{\cos 2\pi(100t)}{(80\pi)(180\pi)}\right]$$

$f_{m_1} = 20 \qquad f_{m_2} = 100$

$f_m = 100 \text{ HZ}$

$f_s = 2 \times 100 = 200 \text{ HZ}$

$T_s = \frac{1}{200} = 5 \text{ mm Sec}$