UNIT - I

1) Determine whether the function $2xy + i(x^3 - y^2)$ is analytic.

2) Prove that an analytic function with constant imaginary part is constant.

3) show that Real parts of an analytic function satisfies Laplace equations

4) show that the function $f(z) = z$ is continuous

5) show that $f(z) = |z|^2$ is not analytic at any point.

6) Is the function $f(z) = xy + iy$ analytic?

7) Prove that $f(z) = \sin z$ is analytic.

8) Find $\lim_{z \to 0} \frac{z^2}{|z|}$

9) check analytic by using C-R equations for $f(z) = e^x(\cos y + i \sin y)$

1) Find the Analytic Function whose real part is

$$u(x,y) = \frac{\sin 2x}{\cosh 2y + \cos 2x}$$

2) show that the function $f(z) = z\bar{z}$ is differentiable but not analytic at origin.

3) Define analytic function and verify the whether $f(z) = \frac{x^3(1+i) - y^3(1-i)}{x^2+y^2}, (z \neq 0)$ and $f(0) = 0$ is analytic.

4) Define Harmonic function and verify whether $u(x,y) = e^{2x}(x\cos 2y - y\sin 2y)$ is harmonic and find its harmonic conjugate

5) show that $\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right] \log|f'(z)| = 0$, where $f(z)$ is analytic function

6) If $f(z) = u + iv$ is analytic and $v = \frac{2\sin x \sin y}{\cos 2x + \cosh 2y}$ Find u

7) show that for the function $f(z) = \begin{cases} \frac{z^5}{|z|^4}, & z \neq 0 \\ 0, & z = 0 \end{cases}$

C-R equation are satisfied at $z=0$, but $f(z)$ is not differentiable at 0.

8) show that the function $f(x,y) = x^3y - xy^3 + xy + x + y$ can be the imaginary part of an analytic function of $f(z)$ also find the real part of the complex function.

9) Find analytic $f(z)$ whose real part
$$u(x,y) = e^x[(x^2-y^2)(\cos y - 2xy\sin y)]$$

10) show that
$$f(z) = \begin{cases} \dfrac{(x^3-y^3)+i(x^3+y^3)}{x^2+y^2} & \text{if } z \neq 0 \\ 0 & \text{if } z = 0 \end{cases}$$
is not analytic at origin although C-R eq^n are satisfied at origin.

11) Find the conjugate harmonic of $u = e^{x^2-y^2}\cos 2xy$. Hence find $f(z)$ in terms of z.

12) Prove that $f'(z)$ does not exist at $z=0$ if
$$f(z) = \begin{cases} \dfrac{x^3y(y-ix)}{x^6+y^2} & \text{if } z \neq 0 \\ 0 & \text{if } z = 0 \end{cases}$$

13) Determine analytic function whose real part is
$$u = \frac{\sin 2x}{\cosh 2y - \cos 2x}$$

14) Find analytic function whose Real part
$$u(x,y) = x^3 - 3xy^2 + 3x^2 - 3y^2 + 2x + 1$$

15) If $f(z)$ is analytic function show that

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|Re f(z)|^2 = 2|f'(z)|^2$$

16) Determine analytic function whose real part $u=e^{x^2-y^2}\cos 2xy$

17) Prove that $f(z)=|z|^2$ is differentiable only at origin.

18) check $u(x,y)=e^{-x}(x\sin y - y\cos y)$ is harmonic or not If harmonic find its conjugate.

19) Find the Harmonic conjugate of $\log\sqrt{x^2+y^2}$

20) show that a analytic function $f(z)=u+iv$ form an orthogonal system.

21) Find the analytic function $f(z)=u+iv$ where $v(x,y)=e^{-x}(x\cos y+y\sin y)$

22) show that $f(z)=\begin{cases}\dfrac{xy^2(x+iy)}{x^2+y^4}, & \text{if } z\neq 0\\ 0, & \text{if } z=0\end{cases}$

satisfies C-R equations at the origin $f'(z)$ doesn't exist

23). Find analytic function $f(z)$ given that

$$u+v = \frac{2\sin 2x}{e^{2y}+e^{-2y}-2\cos 2x}$$

24) If $f(z)$ is analytic function show that

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4|f'(z)|^2$$

25) If $f(z)$ is analytic function with constant modulus then $f(z)$ is constant.

26) If $f(z) = u(r,\theta)+i\,v(r,\theta)$, is differentiable at $z = re^{i\theta} \neq 0$ then prove that

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \quad \frac{\partial v}{\partial r} = \frac{-1}{r}\frac{\partial u}{\partial \theta}$$

27) show that $u(x,y) = e^{-2xy}\sin(x^2-y^2)$ is harmonic. find its conjugate.

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UNIT – II

1. State and prove Cauchy's integral theorem?

2. Evaluate $\int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} dz$, where C is the circle $|z| = 3$ using cauchy's integral formula.

3. Evaluate $\int_C \frac{z}{z^2+1} dz$ where C is $\left|z + \frac{1}{z}\right| = 2$ using Cauchy's integral formula.

4. Evaluate $\int_C \frac{z-3}{z^2+2z+5}$ where C is the circle (i) $|z| = 1$ (ii) $|z+1-i| = 2$ (iii) $|z+1+i| = 2$

5. Evaluate $\int_C \frac{dz}{z^3(z+4)}$ where C is $|z| = 2$ using cauchy's integral formula.

6. Using cauchy's integral formula, evaluate $\int_C \frac{z^4}{(z+1)(z-i)^2} dz$ where C is the ellipse $9x^2+4y^2=36$

7. Evaluate $\int_C \frac{z^2-1}{(z^2+1)} dz$ along $C: |z-i| = 1$ using Cauchy's integral formula.

8. Evaluate, using Cauchy's integral formula $\int_C \frac{\log z}{(z-1)^3} dz$, where C is $|z-1| = 1/2$

9. Evaluate $\int_C \frac{z+1}{z^2+2z+4} dz$, where $C: |z+1+i| = 2$, using cauchy's integral formula.

10. Determine $\int_C \frac{z+1}{z^2+2z+4}\, dz$, where C is (i) $|z+1+i|=2$ (ii) $|z|=1$

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11. Evaluate $\int_C \frac{z^2-z+1}{z-1}\, dz$ where C is the circle (i) $|z|=2$

(ii) $|z|=1/2$

12. Verify Cauchy's theorem for the function $f(z)=3z^2+iz-4$ if C is the square with vertices at $1\pm i$, $-1\pm i$

13. Using Cauchy's integral formula, evaluate $\int_C \frac{z}{(z-1)(z-2)^2}\, dz$

where $C: |z-2|=1/2$

14. Evaluate $\int_C \frac{dz}{(z-2i)^2(z+2i)^2}$ C: being circumference of ellipse

$x^2+4(y-2)^2=4$ by Cauchy's integral formula.

15. Evaluate using Cauchy's integral formula

$\int_C \frac{z^3 e^{-z}}{(z-1)^3}\, dz$ where C is $|z-1|=1/2$

16. Evaluate $\int_C \frac{z+4}{z^2+2z+5}\, dz$ where C is the circle (i) $|z+1-i|=2$

(ii) $|z+1+i|=2$

17. Evaluate $\int_C \frac{e^z}{(z^2+\pi^2)^2}\, dz$ where C is $|z|=4$. Using Cauchy's integral formula.

18. Use Cauchy's integral formula to evaluate $\oint_C \frac{(z^4-3z^2+6)}{(z+i)^3}\, dz$

where C is the circle $|z|=2$

9. Evaluate $\int_c \frac{\cos z\, dz}{(z-\pi)}$, c: being circle $|z-1|=3$ by cauchy's integral formula.

20. Evaluate $\int_c \frac{e^{2z} dz}{(z-1)(z-2)}$, c: being circle $x^2+y^2=9$ by cauchy's integral formula.

21. $\int_c \frac{dz}{(z-a)^n}$, $(n=2,3,4\ldots)$ c: $|z-a|=R>0$ by cauchy's integral formula.

22. Using cauchy's integral formula, evaluate $\int_c \frac{\cosh \pi z}{z(z^2+1)} dz$, where c is $|z|=2$

23. Evaluate $\int_c \frac{1}{z^3(z+4)} dz$ where c is $|z|=2$ using cauchy's integral formula.

24. Use cauchy's integral formula to evaluate

$\int_c \frac{\cos \pi z^2}{(z-1)(z-2)} dz$ where c is the circle $|z|=3$

25. Evaluate $\int_c \frac{e^{2z}}{(z-1)(z-2)} dz$ where c: $|z|=3$ using cauchy's integral formula.

26. Evaluate $\int_c \frac{ze^z}{(z-a)^3} dz$ where 'a' lies within a closed curve by cauchy integral formula.

27. Evaluate $\int_c \left[\frac{e^z}{z^3} + \frac{z^4}{(z+i)^2}\right] dz$ where c: $|z|=2$ using cauchy's integral formula.

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UNIT – III

(1) Find the Laurent Series Expansion of the function $\frac{z^2-1}{z^2+5z+6}$ about $z=0$ in region $2<|z|<3$

(2) Expand $f(z) = \frac{(z-1)(z+2)}{(z+1)(z+4)}$ in the region

(i) $1<|z|<4$ (ii) $|z|<1$

(3) Explain different types of Singularities with examples

(4) Find Taylor's Expansion of $f(z) = \log(1+z)$ about the point $z=0$

(5) Find the Laurent Series of $f(z) = \frac{1}{z^2-4z+3}$, for $1<|z|<3$

(6) Discuss the type of Singularity of the function $f(z) = \frac{z-\sin z}{z^2}$

(7) Expand $f(z) = \log(1-z)$ by Taylor's Series $(z<1)$

(8) Expand $f(z) = \frac{1}{z^2 - z - 6}$ about

(i) $z = -1$ (ii) $z = 1$

(9) Find Laurent Series for $f(z) = \frac{1}{z^2(1-z)}$ and the region of Convergence.

(10) Expand $f(z) = \frac{z}{z^2+1}$ about $|z - 3i| > 2$ by Laurent's series

(11) Expand the Laurent Series of $\frac{z^2-1}{(z+2)(z+3)}$ for $|z|>3$

(12) Find Taylor's expansion of $f(z) = \cos z$ about the point $z = \pi/2$

(13) State and prove Laurent's theorem

(14) Obtain all the Laurent expansions of the function $\frac{7z-2}{(z+1)z(z-2)}$ about $z = -2$

(15) Expand $f(z) = \frac{1}{z(z^2-3z+2)}$ in $0<|z|<1$

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(16) Write Laurent's expansion for $f(z) = \frac{1}{(z+2)(1+z)^2}$ in

(i) $|z| < 2$ (ii) $|1+z| > 1$

(17) Write Laurent's expansion for $f(z) = \frac{1}{(z+2)(1+z^2)}$

(i) $|z| < 1$ (ii) $1 < |z| < 2$ (iii) $|z| > 2$

(18) Obtain Laurent's expansion for $f(z) = \frac{1}{(z+2)(1+z)^2}$

in $|1+z| > 1$

(19) Expand $f(z) = \cos z$ about $z = \pi/4$ as a Taylor's expansion.

(20) Find the Laurent Series of $f(z) = \frac{1}{z^2 - 4z + 3}$,

for $1 < |z| < 3$

(21) Find Taylor's expansion of $f(z) = \frac{2z^3+1}{z^2+z}$ about the point $z = i$

(22) Obtain Laurent's expansion for $f(z) = \frac{1}{(z+2)^2(z+1)}$

in $|z| > 2$

Zero of an analytic function :- (A function) A zero of an analytic function f(z) is a value of z, Such that $f(z) = 0$. (is Called).

Simple Zero :- A zero of Order 1 is Called Simple Zero.

Zero of Order m :- If an analytic function f(z) Can be Expressed as $f(z) = (z-a)^m \phi(z)$ where $\phi(z)$ is analytic and $\phi(a) \neq 0$ then $z=a$ is a Zero of Order m.

Singular point :- A point at which the given function fails to be analytic.

Types of Singularity :- (isola)

isolated Singularity :- A point (at) $z=a$ is called an isolated Singular point of an analytic function f(z). if f(z) is not analytic at $z=a$ if f(z) is analytic in the neighbourhood of $z=a$ which Contains no other Singularity.

pole of an analytic function :-

→ If $z=a$ is an isolated Singular point of an analytic function f(z) which is Expressed in laurent Series as

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-a)^n}$$

→ The negative (part) powers of $(z-a)$ in Laurent Series Expansion is Known as principal part.

→ If the principal part Contains a finite number of terms then the (Sign.) Singular point $z=a$ is Called a pole of order m. (In denominated equal to 0)

Essential Singularity :-

→ If the principal part of $f(z)$ Contains an infinite number of terms then the point $z=a$ is Called Essential Singular point.

Removable Singularity :-

→ If the principle part of $f(z)$ Contains no terms then $z=a$ is Called Removable Singularity

Residue :-

→ The Coefficient of $\frac{1}{z-a}$ in the Laurent Series Expansion of $f(z)$ about the isolated Singular point $z=a$ is Called Residue of $f(z)$ at that point.

→ From Laurent Series Expansion

$$b_1 = \frac{1}{2\pi i} \oint_C f(z)\,dz$$

$$\Rightarrow \oint_C f(z)\,dz = 2\pi i \times b_1$$

$$= 2\pi i \times \left(\text{Residue of } f(z)\right)\Big|_{z=a}$$

→ The Residue of $f(z)$ at pole of order '1'.

→ let $z = a$ be a simple pole ~~of~~ (or pole of) order 'n' then its Residue can be defined as $\lim_{z\to a} f(z)(z-a)$.

→ If a pole of order 'm' then its Residue can be defined as $\frac{1}{(m-1)!} \lim_{z\to a} \frac{d^{m-1}}{dz^{m-1}}\left[f(z)(z-a)^m\right]$

→ find the poles and Residues of $f(z) = \frac{z}{z^2+1}$

$f(z) = \frac{z}{z^2+1} = \frac{z}{(z+i)(z-i)}$ pole find → Denominator equal to zero.

∴ poles of $f(z)$ are obtained by equating denominator to zero.

$(z-i)(z+i) = 0$

$\Rightarrow z = +i, -i$ are simple poles.

Residue of $f(z) = \frac{1}{(m-1)!} \lim_{z\to a} \frac{d^{m-1}}{dz^{m-1}}\left[f(z)(z-a)^m\right]$

here $m = 1$ → (∴ simple pole)

$R_1 \left[\operatorname{Res} f(z)\right]_{z=i}$

$R_1 = \lim_{z\to i} \frac{z}{\cancel{(z-i)}(z+i)} \cancel{(z-i)}$

$= \lim_{z\to i} \frac{z}{z+i}$

$= \frac{i}{2i} = \frac{1}{2}$.

$R_2 \left[\operatorname{Res} f(z)\right]_{z=-i}$ here also $m=1$

$R_2 = \lim_{z\to -i} \frac{z}{(z-i)\cancel{(z+i)}} \cancel{(z+i)} = \lim_{z\to -i} \frac{z}{z-i}$

$= \frac{-i}{-2i} = \frac{1}{2}$

$R_1 = \frac{1}{2} = R_2$

2. find the poles and Residues of an analytic function $f(z) = \frac{z \cdot e^z}{(z+1)(z-2)^3}$

∴ poles of $f(z)$ are obtained by equate denominator

$(z+1)(z-2)^3 = 0$

$z = -1$ is a simple pole

$z = 2$ is a pole of order 3.

$$\text{Residue} = \frac{1}{(m-1)!} \lim_{z \to a} \frac{d^{m-1}}{dz^{m-1}} f(z)(z-a)^m$$

$$R_1 = [\text{Res } f(z)]_{z=-1} = \lim_{z \to -1} f(z)(z+1)$$

$m = 1$

$$= \lim_{z \to -1} \frac{z \cdot e^z}{(z+1)(z-2)^3} \cdot (z+1)$$

$$= \lim_{z \to -1} \frac{ze^z}{(z-2)^3}$$

$$= \frac{-1e^{-1}}{(-1-2)^3} = \frac{1}{27e}$$

$R_2 = [\text{Res } f(z)]_{z=2}$ here $m = 3$

$$= \frac{1}{(3-1)!} \lim_{z \to 2} f(z)(z-2)^3$$

$$= \frac{1}{2!} \lim_{z \to 2} \frac{d^2}{dz^2} \left[\frac{ze^z}{(z+1)} \right]$$

$$\frac{1}{2} \lim_{z\to 2} \frac{d}{dz}\left[\frac{d}{dz}\left(\frac{ze^z}{z+1}\right)\right]$$

$$= \frac{1}{2} \lim_{z\to 2} \frac{d}{dz}\left[\frac{(z+1)\, e^z(z+1) - ze^z \cdot 1}{(z+1)^2}\right]$$

$$= \frac{1}{2} \lim_{z\to 2} \frac{d}{dz}\left[\frac{e^z(z^2+z+1)}{(z+1)^2}\right]$$

$$= \frac{1}{2} \lim_{z\to 2}\left[\frac{(z+1)^2\left[e^z(2z+1) + (z^2+z+1)e^z\right] - e^z(z^2+z+1)\,2(z+1)}{(z+1)^4}\right]$$

$$= \frac{1}{2} \lim_{z\to 2}\left[\frac{(z+1)\left(e^z(2z+1) + z^2+z+1\right) - 2e^z(z^2+z+1)}{(z+1)^3}\right]$$

$$= \frac{1}{2}\left[\frac{3\left(e^2(5) + 4+2+1\right) - 2e^2(4+2+1)}{(2+1)^3}\right]$$

$$= \frac{1}{2}\left[e^2\left(\frac{21+21-14}{27}\right)\right] = \frac{e^2}{2} \times \frac{28}{27} = \frac{19e^2}{2754}$$

$$= \frac{11e^2}{27}$$

3. find the pole and Residue $f(z)$

$$= \frac{z^2}{(z-1)(z-2)^2}$$

poles of $f(z)$ are obtained by Equate denominator

$(z-1)(z-2)^2 = 0$

$z = 1$ is a simple pole

$z = 2$ is a pole of order 2

$$\text{Residue} = \frac{1}{(m-1)!} \lim_{z\to a} \frac{d^{m-1}}{dz^{m-1}} f(z)(z-a)^m$$

$$R_1 = \left[\text{Res } f(z)\right]_{z=1} = \lim_{z \to 1} f(z)\, f(z-1)$$

$$= \lim_{z \to 1} \frac{z^2}{\cancel{(z-1)}(z-2)^2} \cdot \cancel{(z-1)}$$

$$= \lim_{z \to 1} \frac{z^2}{(z-2)^2}$$

$$= \frac{1}{(1-2)^2} = 1$$

$$R_2 = \left[\text{Res } f(z)\right]_{z=2} = \frac{1}{1!} \lim_{z \to 2} f(z)\, \cancel{f}(z-2)$$

$$= \lim_{z \to 2} \frac{z^2}{(z-1)\cancel{(z-2)^2}} \cancel{(z-2)^2}$$

$$= \lim_{z \to 2} \frac{d}{dz}\left[\frac{z^2}{(z-1)\cancel{(z-2)}}\right]$$

$$= \frac{\cancel{4}}{1}$$

$$= \lim_{z \to 2} \frac{(z-1)\,2z - z^2(1)}{(z-1)^2}$$

$$= \lim_{z \to 2} \frac{2z^2 - 2z - z^2}{(z-1)^2}$$

$$= \lim_{z \to 2} \frac{z^2 - 2z}{(z-1)^2}$$

$$= \frac{4-4}{1} = 0$$

4. Find the poles and Residues of $\dfrac{e^{2z}}{(z-1)(z+2)^2}$

$(z-1)(z+2)^2 = 0$

$z = 1$ is a simple pole

$z = -2$ is a pole of order 2

$$R_1 = \left[\text{Res } f(z)\right]_{z=1} = \frac{1}{(m-1)!} \lim_{z \to a} \frac{d^{m-1}}{dz^{m-1}} f(z)(z-a)^m$$

$$R_1 = \lim_{z \to 1} f(z)(z-1)$$

$$= \lim_{z \to 1} \frac{e^{2z}}{(z-1)(z+2)^2} \cdot (z-1)$$

$$= \lim_{z \to 1} \frac{e^{2z}}{(z+2)^2} = \frac{e^2}{9}$$

$$R_2 = \left[\text{Res } f(z)\right]_{z=-2} = \frac{1}{1!} \lim_{z \to -2} \frac{d}{dz} f(z)(z+2)^2$$

$$= \lim_{z \to -2} \frac{d}{dz} \frac{e^{2z}}{(z-1)\cancel{(z+2)^2}} \cancel{(z+2)^2}$$

$$= \lim_{z \to -2} \frac{d}{dz} \frac{e^{2z}}{z-1}$$

$$= \lim_{z \to -2} \frac{(z-1)2e^{2z} - e^{2z}}{(z-1)^2}$$

$$= \frac{(-2-1)2 \cdot e^{2(-2)} - e^{2(-2)}}{(-2-1)^2}$$

$$= \frac{-3 \cdot 2e^{-4} - e^{-4}}{9}$$

$$= \frac{-6e^{-4} - e^{-4}}{9} = \frac{-7e^{-4}}{9}$$

5. find the Residue & poles of $f(z) = \frac{z}{(z^2+4)^2}$

$z^2+4 = 0$ (or) $(z+2i)(z-2i)$

$z^2 = -4 \Rightarrow z = \pm 2i$ are the poles of order 2

$$R_1 = \left[\operatorname{Res} f(z)\right]_{z=2i} = \frac{1}{1!} \lim_{z\to 2i} \frac{d}{dz} f(z)(z-2i)$$

$$= \lim_{z\to 2i} \frac{d}{dz} \frac{z}{(z+2i)^2\cancel{(z-2i)^2}} \cancel{(z-2i)^2}$$

$$= \lim_{z\to 2i} \frac{d}{dz} \frac{z}{(z+2i)^2}$$

$$= \lim_{z\to 2i} \frac{(z+2i)^2 1 - z(1)}{(z+2i)^4} = \frac{(2i+2i)^2 - 2i}{(4i)^4}$$

$$= \frac{(4i)^2 - 2i}{(4i)^4}$$

$$\frac{1}{(z+2i)^2} - \frac{z}{(z+2i)^4} = \lim_{z\to 2i} \frac{\cancel{z^2} + (2i)^2 - \cancel{z} + 4zi}{(z+2i)^2}$$

$$= \frac{-16-2i}{-256}$$

$$\frac{1}{(4i)^2} - \frac{2i}{(4i)^4} = \frac{2i}{(4i)^2} = \frac{\cancel{2}i}{-\cancel{16}\,8} = \frac{-i}{8}$$

$$\frac{1}{-16} - \frac{2i}{}$$

$$= -\left(\frac{16+2i}{256}\right)$$

$$R_2 = \left[\operatorname{Res} f(z)\right]_{z=-2i} = \frac{1}{1!} \lim_{z\to -2i} \frac{d}{dz} f(z)(z+2i)$$

$$= \lim_{z\to 2i} \frac{d}{dz} \frac{z}{z-2i}$$

$$= \lim_{z\to 2i} \frac{(z-2i)^2 - z}{(z-2i)^4} = \frac{(-4i)^2 + 2i}{(-4i)^4}$$

$$= \frac{16+2i}{256}$$

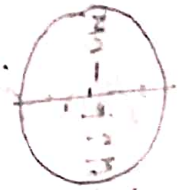
$$= \lim_{z\to 2i} \frac{-2i}{(-4i)^2}$$

$$= \frac{-2i}{-16} = \frac{i}{8}$$

$$= 0$$

6. If $f(z)$ is an analytic function with poles $z_1, z_2 \dots z_n$ and their corresponding residues $R_1, R_2 \dots R_n$ then $\oint f(z)\,dz = 2\pi i\,(R_1 + R_2 \dots R_n)$ If all the poles are lies inside a circle

→ Evaluate $\oint_C \frac{2e^z}{(z-1)^3}\,dz$ where C is a circle



$|z-1| = 3$

$$f(z) = \frac{2e^z}{(z-1)^3}$$

Equating denominator $(z-1)^3 = 0$

$z = 1$ is pole of order 3

Residue at $z = 1$

$$[\text{Res } f(z)]_{z=1} = \frac{1}{2!} \lim_{z \to 1} \frac{d^2}{dz^2} f(z)(z-1)^3$$

$$= \frac{1}{\cancel{2}} \lim_{z \to 1} \frac{d^2}{dz^2} \frac{\cancel{2}e^z}{\cancel{(z-1)^3}} \cancel{(z-1)^3}$$

$$= \lim_{z \to 1} \frac{d^2}{dz^2} e^z$$

$$= \lim_{z \to 1} e^z$$

$$= e$$

$$\therefore \oint_C \frac{2e^z}{(z-1)^3}\,dz = 2\pi i e$$

→ Evaluate $\oint_C \frac{dz}{(z^2+1)(z^2-4)}$ where C is a circle $|z| = 1.5$

$$\oint_C \frac{dz}{(z^2+1)(z^2-4)}$$

$$f(z) = \frac{1}{(z^2+1)(z^2-4)}$$

$$(z^2+1)(z^2-4) = 0$$

$z^2 = \pm i$, $z = \pm 2$ all are simple poles

$z = \pm 2$ lies outside the circle

$$R_1 = \left[\operatorname{Res} f(z)\right]_{z=i} = \lim_{z \to i} f(z)(z-a)$$

$$= \lim_{z \to i} \frac{1}{(z+i)\cancel{(z-i)}(z^2-4)} \cancel{(z-i)}$$

$$= \lim_{z \to i} \frac{1}{(z+i)(z^2-4)}$$

$$= \frac{-1}{10i}$$

$$R_2 = \left[\operatorname{Res} f(z)\right]_{z=-i} = \lim_{z \to -i} \frac{1}{(z-i)(z^2-4)}$$

$$= \frac{1}{10i}$$

$$\therefore \oint_C \frac{dz}{(z^2+1)(z^2-4)} = 2\pi i \left[-\frac{1}{10i} + \frac{1}{10i}\right]$$

$$= 0$$

→ Evaluate $\oint_C \frac{4-3z}{z(z-1)(z-2)} dz$ where c is a circle $|z| = \frac{3}{2}$ using Residue theorem.

$$\oint_C \frac{4-3z}{z(z-1)(z-2)} dz$$

$$f(z) = \frac{4-3z}{z(z-1)(z-2)}$$

$z(z-1)(z-2) = 0$

$z = 0, 1$ lies inside a circle

$z = 2$ lies outside a circles

$m = 1$

$$R_1 = [\text{Res } f(z)]_{z=0} = \lim_{z \to 0} \frac{4-3z}{\cancel{z}(z-1)(z-2)} \cdot \cancel{(z)}$$

$$= \frac{4}{(-1)(-2)} = 2$$

$$R_2 = [\text{Res } f(z)]_{z=1} = \lim_{z \to 1} \frac{4-3z}{z(\cancel{z-1})(z-2)} \cdot \cancel{(z-1)}$$

$$= \lim_{z \to 1} \frac{4-3z}{z(z-2)}$$

$$= \frac{4-3}{1(1-2)} = -1$$

$$\oint_C \frac{4-3z}{z(z-1)(z-2)} = 2\pi i [2-1]$$

$$= 2\pi i$$

→ Evaluate $\oint \frac{dz}{\sinh z}$ where C is a circle with $|z| = 4$

$\sinh z$ Can be written as $i \sin z$

$z = 0, \pm n\pi i$

$0, \pm \pi i, \pm 2\pi i \ldots$

$2\pi > 4$

$z = 0, \pm \pi i$ lies inside a circle.

$R_1 = [\text{Res } f(z)]_{z=0} =$

If $f(z)$ are in the form $\frac{\phi_1(z)}{\phi_2(z)}$ where $\phi_2(z) = 0$ but $\phi_1(z) \neq 0$ then its Residue Can be ~~residue~~ defined as $\frac{\phi_1(z_0)}{\phi_2'(z_0)}$

$f(z) = \frac{1}{i \sin z} \; \frac{\phi_1}{\phi_2}$

$R_1 = [\text{Res } f(z)]_{z=0}$ $= \left[\frac{1}{i \cos z}\right]_{z=0}$ $= \frac{1}{i}$

$\frac{1}{\cosh z}\Big|_{z=0} = \frac{1}{1} = 1$

$R_2 = [\text{Res } f(z)]_{z=\pi i} = \frac{1}{i \cos \pi i}\Big|_{z = \pi i}$

$=$

Evaluation of Real Definite integral by ~~Cauchy~~ Contour integration :-

The process of Evaluating Definite integral by making the part of integration about a suitable Curve in the Complex plane. is Called Contour integration.

Case i) Integration around unit radius.

Here, the integrals are of the form $\int_0^{2\pi} f(\sin\theta, \cos\theta)\, d\theta$ where f is Real Rational function of $\sin\theta$ and $\cos\theta$.

$$\int_0^{2\pi} f(\cos\theta, \sin\theta)\, d\theta$$

Consider $z = e^{i\theta}$

$$dz = ie^{i\theta} d\theta \Rightarrow dz = iz\, d\theta \Rightarrow d\theta = \frac{dz}{iz}.$$

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} = \frac{1}{2i}\left(z - \frac{1}{z}\right)$$

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{1}{2}\left[z + \frac{1}{z}\right]$$

$$\therefore \int_0^{2\pi} F(\cos\theta, \sin\theta)\, d\theta = \oint_{C\,|z|=1} F\left(\frac{1}{2}\left(z+\frac{1}{z}\right), \frac{1}{2i}\left(z-\frac{1}{z}\right)\right) \frac{dz}{iz}$$

→ ST By method of Residues $\displaystyle\int_0^{2\pi} \frac{d\theta}{a+b\cos\theta} = \frac{\pi}{\sqrt{a^2-b^2}}$

$a > b > 0$

$z = e^{i\theta} \quad dz = ie^{i\theta} d\theta \Rightarrow d\theta = \frac{dz}{iz}$

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{1}{2}\left[z + \frac{1}{z}\right]$$

$$= \frac{1}{2}\oint_0^{2\pi} \frac{d\theta}{a+b\cos\theta} = \frac{1}{2}\oint \frac{1}{a+b\left(\frac{1}{2}\left(z+\frac{1}{z}\right)\right)} \cdot \frac{dz}{iz}$$

$$= \frac{1}{2}\oint_C \frac{2}{2a+b\left(\frac{z^2+1}{z}\right)} \cdot \frac{dz}{iz}$$

$$= \oint_C \frac{\cancel{z}}{bz^2+2z+b} \cdot \frac{dz}{i\cancel{z}}$$

$$= \frac{1}{i}\oint_C \frac{1}{bz^2+2z+b}\, dz \qquad C ; |z| = 1$$

Let $f(z) = \dfrac{1}{i(bz^2+2z+b)}$

$bz^2 + 2z + b = 0$

$$z = \frac{-2a \pm \sqrt{4a^2 - 4b^2}}{2b}$$

$$z = \frac{-a \pm \sqrt{a^2-b^2}}{b}$$

$\alpha = \dfrac{-a - \sqrt{a^2-b^2}}{b} \qquad \beta = \dfrac{-a + \sqrt{a^2-b^2}}{b}$ are Simple poles

and $z = \beta$ lies inside a circle (C)

$$f(z) = \frac{1}{b(z-\alpha)(z-\beta)\, i}$$

$$\left[\text{Res } f(z)\right]_{z=\beta} = \lim_{z\to\beta} f(z)(z-\beta)$$

$= \frac{1}{b} \lim_{z \to \beta} \frac{1}{\beta (z-\alpha) i}$

$$R = \frac{1}{bi}\left[\frac{1}{\beta - \alpha}\right] = \frac{1}{bi}\left(\frac{1}{\frac{-a+\sqrt{a^2-b^2}}{b} - \left(\frac{-a-\sqrt{a^2-b^2}}{b}\right)}\right)$$

$$= \frac{1}{i\left(-a+\sqrt{a^2-b^2}+a+\sqrt{a^2-b^2}\right)}$$

$$= \frac{1}{2i\sqrt{a^2-b^2}}$$

$$\oint_C f(z)\, dz = 2\pi i \times R_1$$

$$= 2\pi i \times \frac{1}{2i\sqrt{a^2-b^2}} = \frac{\pi}{\sqrt{a^2-b^2}}$$

→ Evaluate $\int_0^{\pi} \frac{d\theta}{3+2\cos\theta} = \frac{\pi}{\sqrt{5}}$

W.K.T $\quad d\theta = \frac{dz}{iz} \quad$ & $\quad \cos\theta = \frac{z^2+1}{2z}$

$$\frac{1}{2}\int_0^{2\pi} \frac{d\theta}{3+2\cos\theta} = \frac{1}{2}\oint_C \frac{1}{3+2\left(\frac{z^2+1}{2z}\right)} \cdot \frac{dz}{iz} \qquad C;\ |z|=1$$

$$= \frac{1}{2i}\oint_C \frac{z}{3z+z^2+1} \cdot \frac{dz}{z}$$

$$= \frac{1}{2i}\oint_C \frac{1}{z^2+3z+1}\, dz$$

$$f(z) = \frac{1}{2i}\left(\frac{1}{z^2+3z+1}\right)$$

$z^2+3z+1 = 0$

$\alpha\beta = 1 \qquad \alpha+\beta = 3$

$$z = \frac{-3 \pm \sqrt{9-4}}{2} = \frac{-3 \pm \sqrt{5}}{2}$$

$$\alpha = \frac{-3+\sqrt{5}}{2} \qquad \beta = \frac{-3-\sqrt{5}}{2}$$

Assume $\sqrt{5} = 2.3$

$$= -3 - 2.3$$

$$\alpha = -3 + 2.3 = -0. \qquad -5.$$

$z = \alpha$ lies inside c and Simple pole.

$$f(z) = \frac{1}{2i \times \cancel{2} (z-\alpha)(z-\beta)}$$

$$= \frac{1}{\cancel{9}2i (z-\alpha)(z-\beta)}$$

$$[\text{Res } f(z)]_{z=\alpha} = \lim_{z \to \alpha} \frac{1}{\cancel{9}2i \cancel{(z-\alpha)}(z-\beta)} \cdot \cancel{(z-\alpha)}$$

$$= \frac{1}{\cancel{9}2i(\alpha-\beta)}$$

$$= \frac{1}{\cancel{9}2i\left(\frac{-3+\sqrt{5}}{2} - \frac{\cancel{+}3-\sqrt{5}}{2}\right)}$$

$$= \frac{1}{\cancel{2}i} \left[\frac{\cancel{2}}{-3+\sqrt{5}-3+\sqrt{5}} \right]$$

$$= \frac{1}{\cancel{2}i} \frac{1}{2\sqrt{5}}$$

$$\int f(z)\, dz = \cancel{2}\pi \cancel{i} \times \frac{1}{\cancel{2i}\sqrt{5}}$$

$$= \frac{\pi}{\cancel{2}\sqrt{5}}$$

Unit - 4

Partial Differentiation Eqns

partial differentiation Equation having partial Derivatives is called partial differential Eqn.

→ let z be a function of x and y the first order partial derivatives are denoted by p and q and same as the second order partial derivates are denoted by r, s, t.

formation of a partial Differential Eqn :-
A partial Differential Eqn is obtained by Eliminating arbitary Constants or by Eliminating two or more Arbitary functions having variables.

Elimination of arbitary Constants :- let $f(x,y,z,a,b) = 0$ be an Eqn having arbitary Constants a & b. and z is a function of x and y. Differentiate the given eqn w.r.t x and y. we get

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial x} = 0$$

$$\frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial y} = 0$$

from the above three Eqns we get a partial differential Eqn of the form $f(x, y, z, p, q) = 0$

1. $z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ a, b Constants $\rightarrow$ (1)

diff (1) w.r.t x & y partially

$$\frac{\partial z}{\partial x} = \frac{2x}{a^2} \Rightarrow \frac{p}{2} = \frac{x}{a^2} \rightarrow (2)$$

$$\frac{\partial z}{\partial y} = \frac{2y}{b^2} \Rightarrow \frac{q}{2} = \frac{y}{b^2} \rightarrow (3)$$

Sub Eq (2) and Eq (3) in Eq (1)

$$z = x\left(\frac{p}{2}\right) + y\left(\frac{q}{2}\right) \Rightarrow px + qy = 2z$$

2. $(x-a)^2 + (y-b)^2 = z^2 \cot^2\alpha$ a, b are Constants $\rightarrow$ (1)

diff (1) w.r.t x and y partially

$$2(x-a)(1) = 2z \frac{\partial z}{\partial x} \cot^2\alpha \rightarrow (2)$$

$$(x-a)(1) = z \frac{\partial z}{\partial x} \cot^2\alpha \rightarrow (2) = x - a = zp\cot^2 z$$

$$2(y-b)(1) = 2z \frac{\partial z}{\partial y} \cot^2\alpha \rightarrow (3)$$

$$y - b = z\frac{\partial z}{\partial y}\cot^2\alpha \rightarrow (3) \qquad y-b = zq\cot^2 z$$

Sub Eq (2) and Eq (3) in Eq (1)

$$(zp\cot^2\alpha)^2 + (zq\cot^2\alpha)^2 = z^2\cot^2\alpha$$

$$p^2\cot^2\alpha + q^2\cot^2\alpha = 1$$

$$p^2 + q^2 = \tan^2\alpha$$

3. $\log(az-1) = x+ay+b \rightarrow (1)$

$$\frac{1}{(az-1)} a\frac{\partial z}{\partial x} = 1+0+0$$

$$\Rightarrow \frac{ap}{az-1} = 1 \Rightarrow ap = az-1 \rightarrow (2)$$

$$\frac{1}{az-1}\left(a\frac{\partial z}{\partial y}\right) = 0+a\cdot 1+0$$

$$\frac{\cancel{a}q}{\cancel{a}z-1} = \cancel{a} \Rightarrow q = az-1 \rightarrow (3)$$

$$\Rightarrow a = \frac{q+1}{z}$$

Sub 'a' in eq (2)

$$p\left(\frac{q+1}{z}\right) = z\left(\frac{q+1}{z}\right)-1$$

$$\frac{pq+p}{z} = q+1-1$$

$$pq+p = qz$$

$$pq-qz+p = 0$$

→ form the partial Differential eqn of all planes passing through the Origin.

eqn of plane passing through the origin

$$ax+by+cz = 0 \rightarrow (1)$$

diff (1) w.r.t x and y partially

$$a+0+c\frac{\partial z}{\partial x} = 0$$

$$a = -cp \rightarrow (2)$$

$$0 + b \cdot 1 + c\frac{\partial z}{\partial y} = 0$$

$$b = -cq \rightarrow (3)$$

Sub Eq (2) and (3) in Eq (1)

$$-cpx - cqy + cz = 0$$

$$px + qy - z = 0$$

$$px + qy = z$$

→ form the partial differential Eqn of all planes having Equal intercepts on x, y axis.

$$\frac{x}{a} + \frac{y}{a} + \frac{z}{c} = 1 \rightarrow (1)$$

Eqn having Equal intercepts on x, y axis.

diff (1) w.r.t x and y partially

$$\frac{1}{a} + 0 + \frac{1}{c}\frac{\partial z}{\partial x} = 0$$

$$\frac{1}{a} = -\frac{1}{c}p \rightarrow (2)$$

$$0 + \frac{1}{a} + \frac{1}{c}\frac{\partial z}{\partial y} = 0$$

$$\frac{1}{a} = -\frac{1}{c}q \rightarrow (3)$$

Equating Eq (2) and Eq (3)

$$-\frac{p}{c} = -\frac{q}{c} \Rightarrow p - q = 0$$

→ form the partial differential Eqn

Elimination of arbitrary function :-

Let $u = u(x,y,z)$ and $v = v(x,y,z)$ be independent functions of the variables x, y, z and let $\phi(u,v) = 0$ be an arbitary relation among them. By eliminating the arbitary function ϕ from $\phi(u,v) = 0$. we get an partial differential equation which is of the form $Pp + Qq = R$. for the consider the determinant as

$$\begin{vmatrix} \frac{\partial u}{\partial x} + p\frac{\partial u}{\partial z} & \frac{\partial v}{\partial x} + p\frac{\partial v}{\partial z} \\ \frac{\partial u}{\partial y} + q\frac{\partial u}{\partial z} & \frac{\partial v}{\partial y} + q\frac{\partial v}{\partial z} \end{vmatrix} = 0$$

1. form the partial differentiation equation by eliminating the arbitory function ϕ from

$\phi(x+y+z,\ x^2+y^2+z^2) = 0$

$\phi(x+y+z,\ x^2+y^2+z^2) = 0$

Given $\phi(x+y+z,\ x^2+y^2+z^2)$

let $u = x+y+z$ $\quad$ $v = x^2+y^2+z^2$

$u_x = 1$ $\quad$ $v_x = 2x$

$u_y = 1$ $\quad$ $v_y = 2y$

$u_z = 1$ $\quad$ $v_z = 2z$

$$\begin{vmatrix} u_x + pu_z & v_x + pv_z \\ u_y + qu_z & v_y + qv_z \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 1+p & 2x+p(2z) \\ 1+q & 2y+q(2z) \end{vmatrix} = 0$$

$$2(1+p)(y+qz) - 2(1+q)(x+pz) = 0$$

$$y + \cancel{pqz} + qz + py - x - pz - qx - \cancel{pqz} = 0$$

$$py - qx + qz - pz = x - y$$

$$p(y-z) + q(z-x) = x - y$$

2) $\phi(z^2 - xy,\ x^2+y^2+z^2) = 0$

$$\phi(z^2 - xy,\ x^2+y^2+z^2) = 0$$

$u = z^2 - xy$ $\quad\quad$ $V = x^2+y^2+z^2$

$u_x = -y$ $\quad\quad$ $V_x = 2x$

$u_y = -x$ $\quad\quad$ $V_y = 2y$

$u_z = 2z$ $\quad\quad$ $V_z = 2z$

$$\begin{vmatrix} u_x + pu_z & V_x + pV_z \\ u_y + qu_z & V_y + qV_z \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} -y + p(2z) & 2x + p(2z) \\ -x + q(2z) & 2y + q(2z) \end{vmatrix} = 0$$

$$= (-y + 2pz)(2y + 2qz) - (-x + 2qz)(2x + 2pz) = 0$$

$$= -2y^2 - 2ypz + 4p\cancel{y}z + 4pqz^2 + 2x^2 + 2pxz - 4qxz - 4\cancel{pq}z^2 = 0$$

$$= 4pyz + 2pxz - 4qxz - 2yqz = 2y^2 - 2x^2$$

$$= 2pyz + pxz - 2qxz - yqz = y^2 - x^2$$

$$= pz(2y + x) - qz(2x + y) = y^2 - x^2$$

3) form the partial differential Equation by Eliminating the orbitary function from $z = f(x^2+y^2)$

Given $z = f(x^2+y^2) \longrightarrow ①$

diff ① wrt x and y on b.s partially

$\frac{\partial z}{\partial x} = f'(x^2+y^2)\,2x$ $\qquad$ $\frac{\partial z}{\partial y} = f'(x^2+y^2)(2y)$

$\frac{p}{2x} = f'(x^2+y^2) \longrightarrow ②$ $\qquad$ $\frac{q}{2y} = f'(x^2+y^2) \longrightarrow ③$

Eq ② = Eq ③

$\frac{p}{2x} = \frac{q}{2y} \Rightarrow py - qx = 0$

4) find the partial differential Eqn by Eliminating arbitary function from $lx+my+nz = \phi(x^2+y^2+z^2)$

Given $lx+my+nz = \phi(x^2+y^2+z^2) \longrightarrow ①$

diff ① w.r.t x and y partially on b.s

$l + n\frac{\partial z}{\partial x} = \phi'(x^2+y^2+z^2)\left(2x + 2z\cdot\frac{\partial z}{\partial x}\right)$

$l+np = \phi'(x^2+y^2+z^2)(2x+2zp) \longrightarrow ②$

$m+n\frac{\partial z}{\partial y} = \phi'(x^2+y^2+z^2)\left(2y+2z\frac{\partial z}{\partial y}\right)$

$m+nq = \phi'(x^2+y^2+z^2)(2y+2zq) \longrightarrow ③$

$\frac{②}{③} \Rightarrow \frac{l+np}{m+nq} = \frac{2x+2zp}{2y+2zq}$

$(l+np)(2y+2zq) = (m+nq)(2x+2zp)$

5) form the P.D.E Eliminating arbitary function of ϕ

i) $\phi(xy + 2z ; x+y+z)$

ii) $z = xy + f(x^2+y^2)$

i) $\phi(xy+2z^2 ; x+y+z)$

$u = xy + 2z^2$ $\qquad$ $V = x+y+z$

$u_x = y$ $\qquad$ $V_x = 1$

$u_y = x$ $\qquad$ $V_y = 1$

$u_z = 4z$ $\qquad$ $V_z = 1$

$$\begin{vmatrix} u_x + pu_z & V_x + pV_z \\ u_y + qu_z & V_y + qV_z \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} y + p(4z) & 1+p \\ x + q(4z) & 1+q \end{vmatrix} = 0$$

$$\Rightarrow (1+q)(y+4pz) - (1+p)(x+4qz) = 0$$

$$\Rightarrow y + 4pz + qy + \cancel{4pqz} - x - 4qz - px - \cancel{4pqz} = 0$$

$$\Rightarrow 4pz - 4qz - px + qy = x-y$$

$$p(4z-x) + q(y-4z) = x-y$$

ii) $z = xy + f'(x^2+y^2) \longrightarrow ①$

diff ① w.r.t x and y partially on b.s

$$\frac{\partial z}{\partial x} = y + f'(x^2+y^2)(2x)$$

$$\frac{p-y}{2x} = f'(x^2+y^2) \longrightarrow ②$$

$$\frac{\partial z}{\partial y} = x + f'(x^2+y^2)(2y)$$

$$\frac{q-x}{2y} = f'(x^2+y^2) \longrightarrow (3)$$

$$(2) = (3)$$

$$\frac{p-y}{2x} = \frac{q-x}{2y}$$

$$(p-y)2y = (q-x)2x$$

Solution of Partial differential Equation :-

Solution of Non-linear partial Differential Equation :-

A partial differential Equation which involve first order derivatives with degree higher order 1 and product of p and q is called Non-linear partial differential Eqn.

→ Non-linear partial differential Eqns are of four types.

1. Eqn of the form $f(p,q) = 0$ and its solution is defined as

let $z = ax+by+c$

put $p = a$ and $q = b$ in above Eqn then

It becomes $f(a,b) = 0$.

write b in terms of a then $\therefore b = f(a)$

$$z = ax + f(a)y + c$$

Ex:- $p+q = 1 \longrightarrow (1)$

let $z = ax+by+c \xrightarrow{\text{soln of}} (1)$

put $p = a \quad q = b$ in Eq (1)

$a+b = 1$

$b = 1-a$

$\therefore$ Soln is $z = ax + (1-a)y + c$

$\rightarrow \quad pq = p+q \longrightarrow (1)$

let $z = ax+by+c$ be soln of (1)

put $p = a, \quad q = b$ in (1)

$ab = a+b$

$b(a-1) = a$

$b = \dfrac{a}{a-1}$

$\therefore \quad z = ax + \left(\dfrac{a}{a-1}\right)y + c$

$\rightarrow \quad q = 3p^2 \longrightarrow (1)$

let $z = ax+by+c$ be soln of Eq (1)

put $p = a$ and $q = b$ in Eq (1)

$b = 3a^2$

$\therefore \quad z = ax + 3ya^2 + c$

$\rightarrow \quad p^2 - q^2 = 1 \longrightarrow (1)$

let $z = ax+by+c$ be soln of Eq (1)

put $p = a$ and $q = b$ in Eq (1)

$a^2 - b^2 = 1 \Rightarrow b^2 = 1-a^2 \Rightarrow b = \sqrt{1-a^2}$

$\therefore \quad z = ax \pm \sqrt{1-a^2}\,y + c$

→ $p^2+q^2 = npq \longrightarrow (1)$

let $z = ax+by+c$ be a soln of Eq (1)

put $p=a$ and $q=b$ in (1)

$a^2+b^2 = nab \Rightarrow a^2+b^2-nab = 0$

$\cancel{nab-b^2 = a^2}$

$$b = \frac{na \pm \sqrt{n^2a^2-4a^2}}{2}$$

$$= \frac{na \pm a\sqrt{n^2-4}}{2}$$

$$\therefore z = ax + \frac{na \pm a\sqrt{n^2-4}}{2} + c$$

→ Standard form of type 2 :- Eqn of the form of $f(z,p,q)=0$ and its solution can be defined as :-

let $z = \phi(u)$

let $u = x+ay$

Sub $p = \frac{dz}{du}$ & $q = a\frac{dz}{du}$

$q = \frac{\partial z}{\partial y}$ $\quad p = \frac{\partial z}{\partial x} = \frac{dz}{du}\cdot\frac{\partial u}{\partial x} = \frac{dz}{du}$

$$q = \frac{\partial z}{\partial y} = \frac{dz}{du}\cdot\frac{\partial u}{\partial y} = a\frac{dz}{du}$$

Ex:- $p^2+q^2 = 1 \longrightarrow (1)$

let $z = \phi(u)$ be soln. for (1)

let $u = x+ay$

Sub $p = \frac{dz}{du}$ and $q = a\frac{dz}{du}$ in (1)

$$z\left(\frac{dz}{du}\right)^2 + \left(a\frac{dz}{du}\right)^2$$

$$\left(\frac{dz}{du}\right)^2(1+a^2) = z$$

$$\left(\frac{dz}{du}\right)\sqrt{1+a^2} = \sqrt{z}$$

$$\frac{dz}{\sqrt{z}} = \frac{1}{\sqrt{1+a^2}}\,du$$

$$\int\frac{dz}{\sqrt{z}} = \frac{1}{\sqrt{1+a^2}}\int du + c$$

$$2\sqrt{z} = \frac{1}{\sqrt{1+a^2}}\,u + c$$

$$2\sqrt{z} = \frac{x+ay}{\sqrt{1+a^2}} + c$$

→ Solve the partial differential Eqn $p^2+pq = z^2 \rightarrow (1)$

let $z = \phi(u)$ be the soln of Eq (1)

Consider $u = x+ay$

put $p = \frac{dz}{du}$ and $q = a\frac{dz}{du}$ in Eq (1)

$$\left(\frac{dz}{du}\right)^2 + \left(\frac{dz}{du}\right)\left(a\frac{dz}{du}\right) = z^2$$

$$\left(\frac{dz}{du}\right)^2\left[1+a\right] = z^2$$

S.R on both sides

$$\frac{dz}{du}\sqrt{1+a} = z$$

$$\frac{dz}{z} = \frac{1}{\sqrt{1+a}}\,du$$

$$\int\frac{1}{z}\,dz = \int\frac{1}{\sqrt{1+a}}\,du + c$$

$$\log z = \frac{1}{\sqrt{1+a}}\,u + c$$

$\log z = \frac{x+ay}{\sqrt{1+a}} + c$

→ Solve $p + pq = qz \rightarrow ①$

let $z = \phi(u)$ be the soln for Eq ①

Consider $u = x + ay$

put $p = \frac{dz}{du}$ and $q = a\frac{dz}{du}$ in Eq ①

$$\frac{dz}{du} + \frac{dz}{du}\left(a\frac{dz}{du}\right) = \left(a\frac{dz}{du}\right)\frac{dz}{du} z$$

$$1 + a\frac{dz}{du} = a\frac{dz}{du} z$$

$$\frac{dz}{du} = \frac{az-1}{a} = z - \frac{1}{a}$$

$$\frac{1}{z-\frac{1}{a}} dz = du$$

$$\int \frac{dz}{z-\frac{1}{a}} = \int du + c$$

$$\log\left(z - \frac{1}{a}\right) = u + c$$

$$\log\left(z - \frac{1}{a}\right) = x + ay + c$$

→ Solve $z^2(p^2+q^2+1) = 1 \rightarrow ①$

let $z = \phi(u)$ be the soln for Eq ①

Consider $u = x + ay$

put $p = \frac{dz}{du}$ and $q = a\frac{dz}{du}$ in Eq ①

$$z^2\left(\left(\frac{dz}{du}\right)^2 + \left(a\frac{dz}{du}\right)^2 + 1\right) = 1$$

$$\left(\frac{dz}{du}\right)^2 + \left(a\frac{dz}{du}\right)^2 + 1 = \frac{1}{z^2}$$

$$\left(\frac{dz}{du}\right)^2 + \left(a\frac{dz}{du}\right)^2 = \frac{1}{z^2}-1$$

$$\left(\frac{dz}{du}\right)^2(1+a^2) = \frac{1}{z^2}-1$$

$$\frac{dz}{du}\sqrt{1+a^2} = \sqrt{\frac{1}{z^2}-1}$$

$$\frac{1}{\sqrt{\frac{1}{z^2}-1}}\,dz = \frac{1}{\sqrt{1+a^2}}\,du$$

$$\int\frac{1}{\sqrt{\frac{1}{z^2}-1}}\,dz = \int\frac{1}{\sqrt{1+a^2}}\,du$$

$$\int\frac{z\,dz}{\sqrt{1-z^2}} \cancel{\sqrt{\frac{1}{z^2}-1}} = \frac{x+ay}{\sqrt{1+a^2}}+c$$

$$-\frac{1}{2}\int\frac{-2z\,dz}{\sqrt{1-z^2}} = \frac{x+ay}{\sqrt{1+a^2}}+c \qquad z^2=$$

$$-\frac{1}{2}\log(\sqrt{1-z^2}) = \frac{x+ay}{\sqrt{1+a^2}}+c$$

→ Solve $1+p^2 = qz \rightarrow (1)$

let $z=\phi(u)$ be a soln for Eq (1)

Consider $u=x+ay$

put $p=\frac{dz}{du}$ and $q=a\frac{dz}{du}$ in Eq (1)

$$1+\left(\frac{dz}{du}\right)^2 = \left(a\frac{dz}{du}\right)z$$

$$\left(\frac{dz}{du}\right)^2 - az\left(\frac{dz}{du}\right)+1 = 0$$

$$\frac{dz}{du} = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$\frac{dz}{du} = \frac{az \pm \sqrt{a^2z^2-4}}{2}$$

$$\frac{dz}{az \pm \sqrt{a^2z^2-4}} = \frac{du}{2}$$

$$\int \frac{1\,dz}{az \pm \sqrt{a^2z^2-4}} = \int \frac{1}{2}\,du$$

$$\boxed{\int \sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\log\left(x+\sqrt{x^2-a^2}\right)}$$

$$\cancel{\int \frac{1}{az}\,dz} \int \frac{1}{az \pm \sqrt{a^2z^2-4}} \times \frac{az \mp \sqrt{a^2z^2-4}}{az \mp \sqrt{a^2z^2-4}} = \frac{u}{2} + c$$

$$= \int \frac{az \mp \sqrt{a^2z^2-4}}{\cancel{a^2z^2} - \cancel{a^2z^2} + 4}\,dz = \frac{u}{2} + c$$

$$\frac{1}{4}\int az \pm \sqrt{a^2z^2-4}\,dz = \frac{x+ay}{2} + c$$

$$\frac{a}{4}\cdot\frac{z^2}{2} \pm \frac{1}{4}\int \sqrt{a^2z^2-4}\,dz = \frac{x+ay}{2} + c$$

$$\frac{az^2}{8} \pm \frac{1}{4}$$

InComplete Here

Standard form of Type : 4

Eqns are of the form $z = px + qy + f(p,q)$.

$\downarrow$ clairauts form

The solution can be defined as :-

$$z = ax + by + c$$

put $p = a$ $q = b$ in G.E

Soln $z = ax + by + f(a,b)$

$$z = px + qy + pq$$

$$z = ax + by + ab$$

Ex: $z = px + qy + \cancel{p^2}q^2$

$$z = ax + by + a^2 b^2$$

1. solve $z = px + qy + \log pq$

$$z = ax + by + \log ab$$

2. solve $z = px + qy + 2\sqrt{pq}$

$$z = ax + by + 2\sqrt{ab}$$

3. solve $pqz = p^2(qx + p^2) + q^2(py + q^2)$

$$abz = a^2(bx + a^2) + b^2(ay + a^2)$$

$$z = \frac{a^2(bx + a^2) + b^2(ay + a^2)}{ab}$$

Type ÷ 3

Eqns are in the form of $f_1(x,p) = f_2(y,q) = a$

The solution can be defined as :-

p, q

$z = f(x,y)$

$dz = p\,dx + q\,dy$

Integrate we get the soln.

→ $p - q = x^2 + y^2$

$f_1(x,p) = f_2(y,q) = a$

$p - x^2 = q + y^2 = a$

$p - x^2 = a \quad , \quad q + y^2 = a$

$p = a + x^2 \quad , \quad q = a - y^2$

$z = f(x,y)$

$dz = p\,dx + q\,dy$

$a + x^2\,dx + a - y^2\,dy$

Integrate on B.S

$z = \int a + x^2\,dx + \int a - y^2\,dy$

$z = ax + \frac{x^3}{3} + ay - \frac{y^3}{3} + c$

→ $pq + qx = y$

Eqn is in standard form 3

$pq + qx = y$

$q(p + x) = y$

$p + x = y/q$

$p + q = a \qquad \cancel{\frac{y}{q}} = \cancel{a} \qquad y = aq$

$p = a - x \qquad q = \frac{y}{a}$

$z = f(x, y)$

$dz = p\,dx + q\,dy \qquad y/a$

$dz = (a - x)\,dx + \left(\frac{y}{a}\right) dy$

Integrate B.S

$$z = ax - \frac{x^2}{2} + \frac{y^2}{2a} + c$$

→ $py + qx + pq = 0$

The eqn is in standard form (3)

$qx + pq = -py$

$q(p + x) = -py$

$\frac{p + x}{p} = -\frac{y}{q}$

$\frac{p + x}{p} = a \qquad -\frac{y}{q} = a$

$p + x = ap \qquad y = -aq$

$p(a - 1) = x \qquad q = -\frac{y}{a}$

$p = \frac{x}{a - 1}$

$z = f(x, y)$

$dz = p\,dx + q\,dy$

Integrate on B.S

$\int dz = \int p\,dx + \int q\,dy$

$$z = \frac{1}{a-1} \cdot \frac{x^2}{2} - \frac{1}{a} \frac{y^2}{2} + c$$

$$z = \frac{x^2}{2(a-1)} - \frac{y^2}{2a} + c$$

linear P.D.E :-

~~$pq\neq q$~~

$$Pp + Qq = R$$

$$A.E = \frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

Sol is $\phi(u,v) = 0$

multipliers l, m, n

$l\,dx + m\,dy + n\,dz$

$lP + mQ + nR = 0$

→ $px + qy = z$

The eqn is in the form of $Pp + Qq = R$

$$P = x, \quad Q = y, \quad R = z$$

$$A.E = \frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$= \frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z}$$

let $\frac{dx}{x} = \frac{dy}{y}$

$$\int \frac{dx}{x} = \int \frac{dy}{y}$$

$\log x = \log y + \log u$ $\qquad$ $\log y = \log z + \log v$

$\qquad\qquad\qquad\qquad\qquad\qquad$ $v = y/z$

$\Rightarrow u = x/y$

$$\frac{dy}{y} = \frac{dz}{z}$$

$$\int \frac{dy}{y} = \int \frac{dz}{z}$$

Soln is $\phi(u,v) = 0$

$\Rightarrow \phi\left(\frac{x}{y}, \frac{y}{z}\right) = 0$

→ Solve $z = p^2 + q^2 \longrightarrow ①$

The above Eqn is in Type ②

let $z = \phi(u)$ be the soln of Eq ①

Consider $u = x + ay$

put $p = \dfrac{dz}{du}$ and $q = a\dfrac{dz}{du}$ in Eq ①

$$z = \left(\frac{dz}{du}\right)^2 + a^2\left(\frac{dz}{du}\right)^2$$

$$z = \left(\frac{dz}{du}\right)^2 (1 + a^2)$$

S.R on Both sides

$$\sqrt{z} = \frac{dz}{du}\sqrt{1+a^2}$$

$$\frac{1}{\sqrt{z}}\,dz = \frac{1}{\sqrt{1+a^2}}\,du$$

Integrate on B.S

$$\int \frac{1}{\sqrt{z}}\,dz = \int \frac{1}{\sqrt{1+a^2}}\,du$$

$$2\sqrt{z} = \frac{x+ay}{\sqrt{1+a^2}} + c$$

→ Solve $p^2 + p = q^2 \longrightarrow ①$

The above Eqn is in Type ①

let $z = ax + by + c$ be the soln of Eq ①

put $p = a$ and $q = b$ in Eq ①

$$a^2 + b = b^2$$

$$\cancel{b^2 b / z / a}$$

$$b^2 - b - a^2 = 0$$

$$b = \frac{1 \pm \sqrt{1+4a^2}}{2}$$

$$\therefore \quad z = ax + y\left[\frac{1 \pm \sqrt{1+4a^2}}{2}\right] + c$$

$$\therefore \quad z = ax + \frac{1 \pm \sqrt{1+4a^2}}{2}\, y + c$$

→ Solve $pq = p+q \rightarrow (1)$

The above Eqn is in Type (1)

Let $z = ax + by + c$

put $p = a,\ q = b$ in Eq (1)

$$ab = a + b$$

$$b(a-1) = a$$

$$b = \frac{a}{a-1}$$

$\therefore$ from (1) $z = ax + \left(\frac{a}{a-1}\right) y + c$

→ Solve $p(1+q) = qz \rightarrow (1)$

The above Eqn is in Type (2)

Let $z = \phi(u)$ be the Soln of Eq (1)

Consider $u = x + ay$

put $p = \frac{dz}{du}$ and $q = a\frac{dz}{du}$ in Eq (1)

$$\left(\frac{dz}{du}\right)\left(1 + a\frac{dz}{du}\right) = \left(a\frac{dz}{du}\right) z$$

$$1 + a\frac{dz}{du} = a\frac{dz}{du} z$$

$$1 + a\frac{dz}{du} = az$$

$$\frac{1}{a} + \frac{dz}{du} = z$$

$$\frac{1}{z}\,dz = a\,du$$

$$-\frac{1}{a} + z = \frac{dz}{du}$$

$$-\int \frac{1}{a}\,du = -\int z\,dz$$

$$-\frac{1}{a}u = -\frac{z^2}{2} + c$$

$$-\frac{x+ay}{a} = -\frac{z^2}{2} + c$$

$$\frac{z^2}{2} - \frac{x+ay}{a} - c = 0$$

→ Solve $p^3+q^3 = 8z \rightarrow (1)$

The above Eqn is in Type (2)

$z = \phi(u)$ be the Soln of Eq (1)

$$u = x+ay$$

$$p = \frac{dz}{du} \qquad q = a\frac{dz}{du}$$

$$\left(\frac{dz}{du}\right)^3 + \left(a\frac{dz}{du}\right)^3 = 8z$$

$$\left(\frac{dz}{du}\right)^3 + a^3\left(\frac{dz}{du}\right)^3 = 8z$$

$$\left(\frac{dz}{du}\right)^3(1+a^3) = 8z$$

$$\left(\frac{dz}{du}\right)\sqrt[3]{1+a^3} = 2\sqrt[3]{z}$$

$$\frac{dz}{2(z)^{1/3}} = \frac{du}{(1+a^3)^{1/3}}$$

Integrate on B.S

$$\int \frac{dz}{2(z)^{1/3}} = \int \frac{du}{(1+a^3)^{1/3}}$$

$\Rightarrow$

$$\int (z)^{-1/3} dz = 2\int \frac{du}{(1+a^3)^{1/3}}$$

$$\frac{z^{-\frac{1}{3}+1}}{-\frac{1}{3}+1} = 2 \cdot \frac{u}{(1+a^3)^{1/3}} + c$$

$$= \frac{2(x+ay)}{(1+a^3)^{1/3}} + c$$

$$\frac{3z^{2/3}}{2} = \frac{2(x+ay)}{(1+a^3)^{1/3}} + c$$

$$\frac{3z^{2/3}}{2} = \frac{2(x+ay)}{(1+a^3)^{1/3}} + c$$

$\rightarrow$ $p+q = \sin x + \sin y \rightarrow (1)$

The above Eqn is in Type (3)

$$f_1(p,x) = f_2(y,q) = a$$

$$p - \sin x = -q + \sin y = a$$

$$p = \sin x + a, \quad q = \sin y - a$$

$$z(x,y)$$

$$dz = p\,dx + q\,dy$$

$$\int dz = \int(\sin x + a)\,dx + \int(\sin y - a)\,dy$$

Integrate on both Sides,

$$z = -\cos x + ax - \cos y - ay + c$$

$$z = -\cos x + ax - \cos y - ay + c$$

$\rightarrow \quad p^2+q^2 = x^2+y^2 \rightarrow (1)$

The above Eqn is in Type (3)

$$f_1(x,p) = f_2(y,q) = a$$

Here $\quad p^2-x^2 = y^2-q^2 = a$

$$p^2 = a+x^2 \quad , \quad y^2-q^2 = a$$

$$p = \sqrt{a+x^2} \qquad q^2 = y^2-a$$

$$q = \sqrt{y^2-a}$$

$$z = f(x,y)$$

$$dz = p\,dx + q\,dy$$

$$dz = (\sqrt{a+x^2})\,dx + (\sqrt{y^2-a})\,dy$$

Integrate on b.s

$$\int dz = \int \sqrt{a+x^2}\,dx + \int \sqrt{y^2-a}\,dy$$

$$\therefore z = \frac{\frac{2}{3}(a+x^2)^{3/2}}{2x} + \frac{\frac{2}{3}(y^2-a)^{3/2}}{2y} + c$$

$$z = \frac{1}{3}\left[\frac{(a+x^2)^{3/2}}{x} + \frac{(y^2-a)^{3/2}}{y}\right] + c$$

$\rightarrow \quad q = px+p^2 \rightarrow (1)$

the above Eqn is in Type (3)

$$f_1(x,p) = f_2(y,q) = a$$

$$px+p^2 - q = 0 \Rightarrow px+p^2 = a$$

$$p^2 + px - a = 0$$

$$p = \frac{-x \pm \sqrt{x^2+4a}}{2}$$

$\therefore \quad q = a$

$z = f(x, y)$

$dz = p\,dx + q\,dy$

$$dz = \frac{-x \pm \sqrt{x^2+4a}}{2}\,dx + a\,dy$$

Integrate on both sides

$$z = -\int \frac{x}{2}\,dx \pm \int \frac{\sqrt{x^2+4a}}{2}\,dx + \int a\,dy$$

$$z = -\frac{x^2}{4} \pm \frac{1}{2}\left[\frac{x}{2}\sqrt{x^2+4a} + \frac{4a}{2}\log\left|x+\sqrt{x^2+4a}\right|\right] + ay + c$$

$$z = -\frac{x^2}{4} \pm \left[\frac{x}{4}\sqrt{x^2+4a} \pm a\log\left(x+\sqrt{x^2+4a}\right)\right] + ay + c$$

$\rightarrow \quad z = px + qy + (p^2 - q^2) \longrightarrow (1)$

The above Eqn is in Type (4)

put $p = a$ and $q = b$ in Eq (1)

$$z = ax + by + (a^2 - b^2)$$

$$pqz = p^2(qx + p^2) + q^2(py + q^2)$$

$$z = px + qy + \frac{p^3}{q} + \frac{q^3}{p} \rightarrow (1)$$

The above Eqn is in Type - (4)

put $p = a$ and $q = b$ in that Eq (1)

$$z = ax + by + \frac{a^3}{b} + \frac{b^3}{a}$$

Linear Differential Eqns :-

→ It is in the form of $Pp + Qq = R$.

→ It is defined as soln $\phi(u, v) = 0$.

A.E $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

l, m, n

$lP + mQ + nR = 0$

$\int l\,dx + m\,dy + n\,dz$

→ $px - qy = y^2 - x^2$

The Eqn is in the form of $Pp + Qq = R$

Here $P = x$, $Q = -y$ and $R = y^2 - x^2$

A.E $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

$\frac{dx}{x} = \frac{dy}{-y} = \frac{dz}{y^2 - x^2}$

$x^2 - y^2 + y^2 - x^2$

let $\frac{dx}{x} = \frac{dy}{-y}$

multipliers are $x, y, 1$

$\int \frac{dx}{x} = -\int \frac{dy}{y}$

$\Rightarrow \frac{dx}{x} = \frac{dy}{-y} = \frac{dz}{y^2 - x^2} =$

$\log x = -\log y + \log u$

$\frac{x\,dx + y\,dy + dz}{0}$

$u = xy$

$\int x\,dx + y\,dx + dz = v$

$\frac{x^2}{2} + \frac{y^2}{2} + z = v$

Soln $\phi\left(xy, \frac{x^2}{2}, \frac{y^2}{2}, z\right) = 0$

$\rightarrow x^2(y-z)p + y^2(z-x)q = z^2(x-y)$

The Eqn is in the form of $Pp + Qq = R$

Here $P = x^2(y-z)$, $Q = y^2(z-x)$, $R = z^2(x-y)$

A.E $\dfrac{dx}{P} = \dfrac{dy}{Q} = \dfrac{dz}{R}$

$$\Rightarrow \frac{dx}{x^2(y-z)} = \frac{dy}{y^2(z-x)} = \frac{dz}{z^2(x-y)}$$

$\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ are multipliers

$$\frac{\left(\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z}\right)}{x(y-z) + y(z-x) + z(x-y)} = \frac{\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z}}{xy - zx + yz - xy + zx - yz}$$

$$= \frac{\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{dz}}{0} = \int \frac{dx}{x} + \frac{dy}{y} + dz = 0$$

$\log x + \log y + \log z = c_1$

$xyz = c_1$

$\frac{1}{x^2}, \frac{1}{y^2}, \frac{1}{z^2}$ as multipliers

$$= \left(\frac{dx}{x^2} + \frac{dy}{y^2} + \frac{dz}{z^2}\right) \Big/ y - z + z - x + x - y$$

$$= \frac{dx}{x^2} + \frac{dy}{y^2} + \frac{dz}{z^2} \Big/ 0$$

$$\int \frac{dx}{x^2} + \int \frac{dy}{y^2} + \int \frac{dz}{z^2} = 0$$

$$-\frac{1}{x} - \frac{1}{y} - \frac{1}{z} = -c_2$$

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = c_2 \qquad \therefore \phi\left(xyz, \frac{1}{x}, \frac{1}{y}, \frac{1}{z}\right) = 0$$

$\rightarrow \quad y^2p - xyq = x(z-2y)$

It is in the form of $Pp + Qq = R$

Here $P = y^2$, $Q = -xy$ $R = x(z-2y)$

A.E $\quad \frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

$$\frac{dx}{y^2} = \frac{dy}{-xy} = \frac{dz}{x(z-2y)}$$

$$\frac{dx}{y^2} = \frac{dy}{-xy}$$

$$\int x\,dx = -\int y\,dy$$

$$\frac{x^2}{2} = -\frac{y^2}{2} + c$$

$$x^2+y^2 - c$$

$\rightarrow \quad \frac{dy}{-xy} = \frac{dz}{x(z-2y)}$

multipliers are 2, z and y

$$= \frac{2dx + z\,dy + y\,dz}{2y^2 - xyz + xyz - 2y^2}$$

$$= \frac{2dx + z\,dy + y\,dz}{0}$$

$$= \int 2\,dx + \int z\,dy + \int y\,dz = 0$$

$$2x + zy + yz = 0$$

$\rightarrow \quad y^2 p - xyq = x(z-2y)$

It is in the form of $Pp + Qq = R$

Here $P = y^2$, $Q = -xy$, $R = x(z-2y)$

$$A.E = \frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$A.E = \frac{dx}{y^2} = \frac{dy}{-xy} = \frac{dz}{x(z-2y)}$$

$$\rightarrow \quad \frac{dx}{y^2} = \frac{dy}{-xy}$$

$$\int x\,dx = -\int y\,dy$$

$$x^2 + y^2 = c_1\, u$$

$$\rightarrow \quad \frac{dy}{-xy} = \frac{dz}{x(z-2y)}$$

$$\frac{dy}{-y} = \frac{dz}{z-2y}$$

$$(z-2y)\,dy = -y\,dz$$

$$z\,dy - 2y\,dy = -y\,dz$$

$$2y\,dy = z\,dy + y\,dz$$

$$\int 2y\,dy = \int z\,dy + y\,dz$$

$$2\cdot\frac{y^2}{2} = zy + c_2\, v$$

$$y^2 - zy = c_2\, v$$

$$\phi(x^2+y^2,\ y^2 - zy)$$

→ Solve $x(y-z)p + y(z-x)q = z(x-y)$

It is in the form of $Pp + Qq = R$

Here $P = x(y-z)$; $Q = y(z-x)$, $R = z(x-y)$.

A.E $\Rightarrow \dfrac{dx}{P} = \dfrac{dy}{Q} = \dfrac{dz}{R}$

A.E $\Rightarrow \dfrac{dx}{x(y-z)} = \dfrac{dy}{y(z-x)} = \dfrac{dz}{z(x-y)}$

→ multipliers are 1, 1, 1

$$= \frac{dx + dy + dz}{xy - xz + yz - xy + zx - zy}$$

$$= \frac{dx + dy + dz}{0}$$

$$u = \int dx + \int dy + \int dz$$

$$u = x + y + z.$$

→ multipliers are $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$

$$\Rightarrow \frac{\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z}}{y - z + z - x + x - y}$$

$$= \frac{\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z}}{0}$$

$$\int \frac{dx}{x} + \int \frac{dy}{y} + \int \frac{dz}{z} = v$$

$$\log x + \log y + \log z = \log v$$

$$xyz = v$$

$$\phi(xyz, x + y + z) = 0$$

→ Solve $z(x-y) = px^2 - qy^2$

It is in the form of $Pp + Qq = R$

Here $P = x^2$, $Q = -y^2$, $R = z(x-y)$

A.E ⇒ $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

$$\frac{dx}{x^2} = \frac{dy}{-y^2} = \frac{dz}{z(x-y)}$$

$$\frac{dx}{x^2} = \frac{dy}{-y^2}$$

$$\int \frac{1}{x^2}\, dx = -\int \frac{1}{y^2}\, dy$$

$$-\frac{1}{x} = \frac{1}{y} + u$$

$$\frac{1}{x} + \frac{1}{y} = u$$

→ multipliers are $-\frac{1}{x}, -\frac{1}{y}, \frac{1}{z}$

$$\therefore \frac{-\frac{dx}{x} - \frac{dy}{y} + \frac{dz}{z}}{-x+y+x-y} = \frac{-\frac{dx}{x} - \frac{dy}{y} + \frac{dz}{z}}{0}$$

$$-\int \frac{1}{x}\, dx - \int \frac{1}{y}\, dy + \int \frac{1}{z}\, dz = v$$

$$\therefore -\log x - \log y + \log z = \log v$$

$$-(\log x + \log y) + \log z = \log v$$

$$-\log xy + \log z = \log v$$

$$\log\left(\frac{z}{xy}\right) = \log v$$

$\therefore \quad v = \frac{z}{xy}$

$$\phi\left(\frac{1}{x}+\frac{1}{y}, \frac{z}{xy}\right) = 0$$

→ Solve $(x^2 - yz)p + (y^2 - zx)q = z^2 - xy$

it is in the form of $Pp + Qq = R$

Here $P = x^2 - yz$, $Q = y^2 - zx$, $R = z^2 - xy$

A.E $\Rightarrow \frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

$$\frac{dx}{x^2-yz} = \frac{dy}{y^2-zx} = \frac{dz}{z^2-xy}$$

$$\frac{dx-dy}{(x-y)\cancel{(x+y+z)}} = \frac{dy-dz}{(y-z)\cancel{(x+y+z)}} = \frac{dz-dx}{(z-x)\cancel{(x+y+z)}}$$

$$\frac{dx-dy}{x-y} = \frac{dy-dz}{(y-z)} = \frac{dz-dx}{(z-x)}$$

$$\frac{dx}{x-y} - \frac{dy}{x-y} = \frac{dy}{y-z} - \frac{dz}{y-z} = \frac{dz}{z-x} - \frac{dx}{z-x}$$

→ $\frac{dx}{x-y} - \frac{dy}{x-y} = \frac{dy}{y-z} - \frac{dz}{y-z}$

$\int \frac{1}{x-y}\,dx - \int \frac{1}{x-y}\,dy = \int \frac{dz}{y-z} - \int \frac{dz}{y-z}$

$\log(x-y) + \log(x-y) = \log(y-z) + \log(y-z) + \log u$

$\log (x-y)^2 - \log (y-z)^2 = \log u$

$\left(\frac{x-y}{y-z}\right)^2 = u \Rightarrow \frac{x-y}{y-z} = u$

→ $\frac{dy}{y-z} - \frac{dz}{y-z} = \frac{dz}{z-x} - \frac{dx}{z-x}$

$\int \frac{dy}{y-z} - \int \frac{dz}{y-z} = \int \frac{dz}{z-x} - \int \frac{dx}{z-x}$

$\log(y-z) + \log(y-z) = \log(z-x) + \log(z-x) + \log v$

$\Rightarrow \log (y-z)^2 - \log (z-x)^2 = \log v$

$\left(\frac{y-z}{z-x}\right)^2 = v$

$\Rightarrow \frac{y-z}{z-x} = v$

$$\therefore \boxed{\phi(u,v) = \left(\frac{x-y}{y-z}, \frac{y-z}{z-x}\right) = 0}$$

→ Solve $px^2 + qy^2 = z(x+y)$

It is in the form of $Pp + Qq = R$

Here $P = x^2$, $Q = y^2$, $R = z(x+y)$

A.E ⇒ $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

$$\frac{dx}{x^2} = \frac{dy}{y^2} = \frac{dz}{z(x+y)}$$

$$\frac{dx}{x^2} = \frac{dy}{y^2}$$

$$\int \frac{1}{x^2}\,dx = \int \frac{1}{y^2}\,dy$$

$$-\frac{1}{x} = -\frac{1}{y} + c$$

$$\frac{1}{x} - \frac{1}{y} = u$$

multipliers are $-\frac{1}{x}, -\frac{1}{y}, \frac{1}{z}$

$$\therefore \frac{-\frac{dx}{x} - \frac{dy}{y} + \frac{dz}{z}}{-x-y+x+y}$$

$$\Rightarrow \frac{-\frac{dx}{x} - \frac{dy}{y} + \frac{dz}{z}}{0}$$

$$-\int \frac{1}{x}\,dx - \int \frac{1}{y}\,dy + \int \frac{1}{z}\,dz = V$$

$$-\log x - \log y + \log z = \log v$$

$$-(\log xy) + \log z = \log v$$

$$\log\left(\frac{z}{xy}\right) = \log v$$

$\therefore \quad v = \frac{z}{xy}$

$$\phi\left(\frac{1}{x} - \frac{1}{y}, \frac{z}{xy}\right) = 0$$

$\rightarrow \quad p^3 = qz \longrightarrow (1)$.

The above Eqn is in Type-2

let $\quad z = \phi(u)$

$\quad\quad u = x + ay$

Sub $\quad p = \frac{dz}{du}, \quad q = a\frac{dz}{du}$

$$\left(\frac{dz}{du}\right)^3 = \left(a\frac{dz}{du}\right)z$$

$$\left(\frac{dz}{du}\right)^2 = az$$

Square Root on B.S

$$\left(\frac{dz}{du}\right) = \sqrt{a}\sqrt{z}$$

$$\frac{1}{\sqrt{z}}\,dz = \sqrt{a}\,du$$

Integrate on B.S

$$\int \frac{1}{\sqrt{z}}\,dz = \int \sqrt{a}\,du$$

$$2\sqrt{z} = u\sqrt{a} + c$$

$$2\sqrt{z} = (x+ay)\sqrt{a} + c$$

$z = y^2 + 2f\left(\frac{1}{x} + \log y\right)$

$\Rightarrow$ Given that $z = y^2 + 2f\left(\frac{1}{x} + \log y\right)$

diff w.r.t x and y partially on B.S

$$\frac{\partial z}{\partial x} = 0 + 2f'\left(\frac{1}{x} + \log y\right)\left(-\frac{1}{x^2}\right)$$

$$-\frac{px^2}{2} = f'\left(\frac{1}{x} + \log y\right) \rightarrow (1)$$

$$\frac{\partial z}{\partial y} = 2y + 2f'\left(\frac{1}{x} + \log y\right)\left(\frac{1}{y}\right)$$

$$\frac{q}{2y} = q = 2\left[y + f'\left(\frac{1}{x} + \log y\right)\frac{1}{y}\right]$$

$$\left(\frac{q}{2} - y\right)y = f'\left(\frac{1}{x} + \log y\right) \rightarrow (2)$$

Eq (1) = Eq (2)

$$-\frac{px^2}{2} = \left(\frac{q}{2} - y\right)y$$

$$-\frac{px^2}{\cancel{2}} = \left(\frac{q - 2y}{\cancel{2}}\right)y$$

$$-px^2 = qy - 2y^2$$

$$px^2 + qy - 2y^2 = 0$$

Higher Order P.D.E

It is in the form of $f\left(x_1, x_2 \cdots x_n, \frac{\partial z}{\partial x_1}, \frac{\partial z}{\partial x_2} \cdots \frac{\partial z}{\partial x_n}, \frac{\partial^2 z}{\partial x_1}, \frac{\partial^2 z}{\partial x_1 \partial x_2} \cdots\right) = 0$

$$z = f(x, y)$$

$$D = \frac{\partial}{\partial x} \qquad D' = \frac{\partial}{\partial y}$$

$$f(D, D')z = \phi(x, y) \longrightarrow ①$$

$$\frac{\partial^2 z}{\partial x^2} + 2\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = e^{2x+3y}$$

$$(D^2 + 2DD' + (D')^2)z = e^{2x+3y}$$

$\therefore f(D, D')z = \phi(x, y)$

Sol^n $z = z_c + z_p$

for z_c if $f(D, D')z = 0$ is homogenous

$y_c \longrightarrow$ AE $f(m, 1) = 0$

$$D = m \quad D' = 1$$

Homogenous L.P.D.E

Non-Homogenous L.P.D.E

One dimensional wave eqn

Two dimensional laplace Eqn

Homogenous linear partial differential Equations

Case :- i

$f(D, D')z = \varphi(x, y)$

Soln $z = z_c + z_p$

for z_c if $f(D, D') = 0$ is homogenous

$z_c \longrightarrow$ AE $f(m, 1) = 0$

$D = m \; , \; D' = 1$

we get different values of m

Case (i) :- if m are equal

$$z_c = \phi_1(y + mx) + x\phi_2(y + mx) + x^2\phi_3(y + mx) + \ldots$$

Case (ii) :- If m are distinct

$$z_c = \phi_1(y + mx) + \phi_2(y + mx) + \phi_3(y + mx) + \ldots$$

Case (iii) :- If roots are Complex

$a_1 + ib_1 \, , \; a_2 + ib_2 + \ldots \; a_n + ib_n$

$$z_c = \phi_1(y + (a_1 + ib_1)x) + \phi_2(y + (a_2 + ib_2)x) + \ldots$$

Case :- II

→ for finding particular Integral :-

1. If $\varphi(x, y) = e^{ax+by}$

$$z_p = \frac{1}{f(D, D')} \cdot e^{ax+by}$$

$f(D, D')z = \varphi(x, y)$

$$z_p = \frac{1}{f(D, D')} \varphi(x, y)$$

put $D = a$ & $D' = b$ in $f(D, D')$

if $f(a,b) \neq 0$

$$Z_p = \frac{e^{ax+by}}{f(a,b)}$$

if $f(a,b) = 0$

$$Z_p = \frac{\text{multiply numerator with } x}{\text{diff the denominator with } D}$$

Consider $(D^2 + 7DD' + 12D'^2) z = e^{2x+y}$

The given Eqn is in H.L.P.D.E

($\because$ It is in Same Order)

Soln is $z = Z_c + Z_p$

for $Z_c \Rightarrow$ let $f'(D, D') = 0$

$$(D^2 + 7DD' + 12D'^2) z = 0$$

Auxilary Eqn $f(m, 1) = 0$

$$m^2 + 7m + 12 = 0$$

$$m^2 + 4m + 3m + 12 = 0$$

$$m = -3, -4$$

Values of m are distinct

Values of m are distinct

$$z_c = \phi_1(y+4x) + \phi_2(y-3x)$$

for z_p

$$z_p = \frac{1}{f(D,D')}\varphi(x,y)$$

$$= \frac{1}{D^2+7DD'+12D'^2} e^{2x+y}$$

put $D=2$ $D'=1$

$$z_p = \frac{1}{4+14+12} e^{2x+y}$$

$$= \frac{e^{2x+y}}{30}$$

Soln as $z = \phi_1(y-4x)+\phi_2(y-3x) + \frac{e^{2x+y}}{30}$

→ $(D-D')z = e^{x+y}$

The given Eqn is in H.L.P.D.E

i.e. It is in the form of $f(D,D')z = \varphi(x,y)$

Soln is $z = z_c + z_p$

for z_c let $f(D,D') = 0$

$$(D-D')z = 0$$

Auxilary Eqn $f(m,1) = 0$

$$m-1 = 0$$

$$m = 1$$

$$z_c = \phi_1(y+x)$$

for z_p $z_p = \frac{1}{f(D,D')}\varphi(x,y)$

$$= \frac{1}{D-D'} \cdot e^{x+y}$$

$\Rightarrow$ put $D=1$ $D'=1$

~~Sis~~ $f(1,1)=0$

$$Z_p = \frac{x}{\frac{\partial}{\partial D}(D-D')} \cdot e^{x+y}$$

$$= \frac{x}{1-0} \cdot e^{x+y} = xe^{x+y}$$

Solⁿ is $Z = \phi_1(y+x) + xe^{x+y}$

$\rightarrow (D^2 - 6DD' + 9D'^2)Z = e^{2x+3y}$

It is in the form of $f(D,D')Z = g(x,y)$

$\therefore$ The given Eqn is in H.L.P.D.E

Soln is $Z = Z_c + Z_p$

for Z_c let $f(D,D') = 0$

$$(D^2 - 6DD' + 9D'^2)Z = 0$$

Auxiliary Eqn $f(m,1) = 0$

$$m^2 - 6m + 9 = 0$$

$$m^2 - 3m - 3m + 9 = 0$$

$$m(m-3) - 3(m-3) = 0$$

$$(m-3)(m-3) = 0$$

$$m = 3, 3$$

$$Z_c = \phi_1(y+3x) + x\phi_2(y+3x)$$

for Z_p $\quad Z_p = \dfrac{1}{f(D,D')} \cdot g(x,y)$

$$= \frac{1}{D^2-6DD'+9D'^2} \cdot e^{2x+3y}$$

put $D=2$ and $D'=3$

$$= \frac{1}{4-36+81} e^{2x+3y}$$

$$= \frac{1}{85-36} e^{2x+3y}$$

$$= \frac{e^{2x+3y}}{49}$$

$\therefore$ Soln is $Z = \phi_1(y+3x) + x\,\phi_2(y+3x) + \dfrac{e^{2x+3y}}{49}$

Case ii)

$\rightarrow$ $f(D,D')\,Z = g(x,y)$

$\downarrow$ Homogenous

$\downarrow$ Sin $(ax+by)$ (or) Cos $(ax+by)$

for $Z_p = \dfrac{1}{f(D,D')}\, g(x,y)$

put $D^2 = -a^2$, $D'^2 = -b^2$ and $DD' = -ab$

$\rightarrow \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x \partial y} - \frac{6\partial^2 z}{\partial y^2} = \cos(x+y)$ find P.I, of given D.E ?

let $D = \frac{\partial}{\partial x}$; $D' = \frac{\partial}{\partial y}$

The Eqn Can be written as

$$(D^2 + DD' - 6D'^2) z = \cos(x+y)$$

particular Integral z_p Can be defined as :-

$$\frac{1}{f(D,D')} \varphi(x,y)$$

$$z_p = \frac{1}{D^2+DD'-6D'^2} \cos(x+y)$$

$a=1 \quad b=1$

put $D^2 = -a^2 = -1$, $D'^2 = -b^2 = -1$, $DD' = -ab = -1$

$$z_p = \frac{1}{-1-1+6} \cos(x+y)$$

$$= \frac{\cos(x+y)}{4}$$

$\rightarrow (D^3 - 4D^2D' + 4DD'^2) z = \sin(3x+2y)$

It is in the form of $f(D,D') z = \varphi(x,y)$.

and it is Homogenous.

soln is $z = z_c + z_p$

let $f(D,D') z = 0$

$$(D^3 - 4D^2D' + 4DD'^2) z = 0$$

$$f(m,1) = 0$$

$m^3 - 4m^2 + 4m = 0$

$m(m-2)^2 = 0$

$m = 0, 2, 2$

$z_c = \phi_1(y + 0(x)) + \phi_2(y + 2x) + x\,\phi_3(y + 2x)$

$\therefore z_c = \phi_1(y) + \phi_2(y + 2x) + x\,\phi_3(y + 2x)$

$$z_p = \frac{1}{D^3 - 4D^2D' + 4DD'^2} \sin(3x + 2y)$$

$a = 3, \quad b = 2$

put $D^2 = -a^2 = -9, \; D'^2 = -b^2 = -4, \; DD' = -ab = -6$

$$z_p = \frac{1}{-9D + 24D - 16D} \sin(3x + 2y)$$

$$z_p = \frac{1}{-D} \sin(3x + 2y)$$

$$= -\int (\sin 3x + 2y)\, dx$$

$$= -\left[-\frac{\cos(3x + 2y)}{3} + c \right]$$

$$= \frac{\cos(3x + 2y)}{3} + c$$

$\therefore z = \phi_1(y) + \phi_2(y + 2x) + x\,\phi_3(y + 2x) + \frac{\cos(3x + 2y)}{3} + c$

$\rightarrow (D^2 + 2D^2D' + DD'^2)\, z = e^{2x+y} + \sin(x + 2y)$

it is in the form of $f(D, D')\, z = \phi(x, y)$ and

it is in Homogenous.

Soln is $z = z_c + z_p$.

for $z_c \Rightarrow$ let $f(D,D')z = 0$

$$(D^3 + 2D^2D' + DD'^2)z = 0$$

$$f(m,1) = 0$$

$$m^3 + 2m^2 + m = 0$$

$$m(m^2+2m+1) = 0$$

$$m(m+1)^2 = 0$$

$$m = 0, -1, -1$$

$$z_c = \phi_1(y+0(x)) + \phi_2(y-x) + x\phi_3(y-x)$$

$$z_c = \phi_1(y) + \phi_2(y-x) + x\phi_3(y-x)$$

for z_p :- $z_p = \dfrac{1}{f(D,D')}\phi(x,y)$

$$= \frac{1}{D^3+2D^2D'+DD'^2} e^{2x+y} + \sin(x+2y)$$

$$= \frac{1}{D^3+2D^2D'+DD'^2} e^{2x+y} + \frac{1}{D^3+2D^2D'+DD'^2}\sin(x+2y)$$

$\Rightarrow D = a = 2$ $\qquad D^2 = -a^2 = -1$, $D'^2 = -b^2 = -2$

$D' = b = 1$ $\qquad DD' = -4$

$$z_p = \frac{1}{8+8+2} e^{2x+y} + \frac{1}{D(D^2+2DD'+D'^2)}\sin(x+2y)$$

$$= \frac{e^{2x+y}}{18} + \frac{1}{D(-1-4-4)}\sin(x+2y)$$

$$= \frac{e^{2x+y}}{18} - \frac{1}{9D}\sin(x+2y)$$

$$= \frac{e^{2x+y}}{18} - \frac{1}{9}\int \sin(x+2y)\,dx$$

$$= \frac{e^{2x+y}}{18} + \frac{1}{9}\cos(x+2y) + C$$

$\therefore z = \phi_1(y) + \phi_2(y-x) + x\phi_3(y-x) + \frac{e^{2x+y}}{18} + \frac{\cos(x+2y)}{9} + c$

Case :- 3

$\rightarrow z_p = \varphi(x,y) = \sin(ax+by)$ or $\cos(ax+by)$

$\varphi(x,y) = x^m y^n$

$z_p = \frac{1}{f(D,D')} \varphi(x,y)$

$= \frac{1}{f(D,D')} x^m y^n$

write $f(D,D') = (1 \pm \phi(D,D'))$

$\rightarrow (D^2 - 2DD')z = x^3 y$ Solve ?

The given Eqn is in the form of $f(D,D') = \varphi(x,y)$

and it is Homogeneous.

$\rightarrow$ The soln is $z = z_c + z_p$

for z_c => Consider $f(D,D') = 0$

$D^2 - 2DD' = 0$

$f(m,1) = 0$

$m^2 - 2m = 0$

$m(m-2) = 0$

$m = 0, 2$

$z_c = \phi_1(y + (0)x) + \phi_2(y + 2x)$

$\therefore z_c = \phi_1(y) + \phi_2(y + 2x)$

for z_p => $\frac{1}{f(D,D')} x^m y^n \varphi(x,y)$

$z_p = \frac{1}{D^2 - 2DD'} x^3 y$

$$z_p = \frac{1}{(2DD' - D^2)} x^3 y$$

$$= \frac{1}{D^2\left(1 - \frac{2D'}{D}\right)} x^3 y$$

$$= \frac{1}{D^2}\left[1 - \frac{2D'}{D}\right]^{-1} x^3 y$$

$$\therefore (1-x)^{-1} = 1 + x + x^2 + \dots$$

$$z_p = \frac{1}{D^2}\left[1 + \frac{2D'}{D} + \frac{4D'^2}{D^2} + \dots\right] x^3 y$$

$$\frac{2D'}{D} = \frac{2}{D}\left[\frac{\partial}{\partial y} x^3 y\right] = \frac{2x^3}{D} = 2\int x^3\, dx = \frac{x^4}{2}$$

$$z_p = \frac{1}{D^2}\left(x^3 y + \frac{x^4}{2}\right)$$

$$= \iint \left(x^3 y + \frac{x^4}{2}\right) dx\, dx$$

$$= \frac{x^5}{20} y + \frac{x^6}{120} + C$$

$\rightarrow (D^3 - 7DD'^2 - 6D'^3)z = e^{2x+y} + \cos(x+y) + x^2 y$

(Incomplete)

The given Eqn is in the form of $f(D, D') = g(x, y)$ and it is Homogeneous.

$\rightarrow$ The Soln is $z = z_c + z_p$

for $z_c \Rightarrow$ Consider $f(D, D') = 0$

$$D^3 - 7DD'^2 - 6D'^3 = 0$$

$$f(m, 1) = 0$$

$$m^3 - 7m - 6 = 0$$

Non Homogenous Differential Eqn :-

$f(D,D') = \varphi(x,y)$

Soln $z = z_c + z_p$

for $z_c \Rightarrow$ let $f(D,D')z = 0$

Resolve it linear factor as

$(D - m_1D' - a_1)(D - m_2D' - a_2) \cdots (D - m_nD' - a_n)z = 0$

Case : 1 If linear factors are Repeated

then $z_c = e^{ax}\left[\phi_1(y+mx) + x\,\phi_2(y+mx) + \cdots\right]$

Case : 2 If linear factors are not Repeated

then $z_c = e^{a_1x}\phi_1(y+m_1x) + e^{a_2x}\phi_2(y+m_2x) + \cdots$

1. solve $(D+D'-1)(D+2D'-2)z = 0 \rightarrow (1)$

The given Eqn is in the form of N.H.L.P.D.E of Higher order.

So, Compare $(D-m_1D'-a_1)(D-m_2D'-a_2)z = 0$ with Eq (1)

Here $m_1 = -1$ $m_2 = -2$

$a_1 = 1$ $a_2 = 2$

$z_c = e^{x}[\phi_1(y-x)] + e^{2x}\phi_2(y-2x)$

2. solve $(D^2+D-D'^2-D')z = 0$

The given Eqn is in the form of N.H.L.P.D.E of higher order.

$(D^2-D'^2+D-D')z = 0$

$$\left[(D+D')(D-D') + (D-D')\right] Z = 0$$

$$D-D'\left[D+D'+1\right]Z = 0$$

$$\left[(D-D'-0)\ (D+D'+1)\right]Z = 0$$

Compare with $\left[(D-m_1D'-a_1)\ (D-m_2D'-a_2)\right]Z = 0$

Here $m_1 = 1$ $a_1 = 0$

$m_2 = -1$ $a_2 = -1$

$$Z_C = e^{0(x)}\phi_1(y+x) + e^{-x}\phi_2(y-x)$$

$$Z_C = \phi_1(y+x) + e^{-x}\phi_2(y-x)$$

3. Solve $(D^2 + DD' - 2D'^2 + 2D + 2D')Z = 0$

The given Eq.n is in the form of N.H.L.P.D.E of Higher order.

Incomplete

$$(D^2 + DD' - 2D'^2 + 2D + 2D')Z = 0$$