Thought by Sonendra Gupta

Do not worry about your difficulties in mathematics, I assure you that they are very small.

FREE STUDY PACKAGE

ENGINEERING MATHEMATICS - 2

UNIT – 5

Contents

ENJOY MATHE-MAGIC

WITH SONENDRA GUPTA 9893455006

STUDENT'S NAME : ______________________

CLASS : ____________ BRANCH: __________

ROLL NO. : ____________ TIME : __________

EKLAVYA ENGINEERING HUB

B-191, infornt of Axis bank & SBI ATM, NEW ASHOKA GARDEN, BHOPAL

MOBILE:, 9329443789, 9893455006

Chapter – 1
VECTOR CALCULUS (VELOCITY AND ACCELERATION)

1. If be the unit vector in the direction of **r**, Show that $\hat{\mathbf{r}} \times d\,\hat{\mathbf{r}} = \dfrac{\mathbf{r}\times d\,\mathbf{r}}{r^2}$
2. A particle moves along the curve $x = t^3 + 1$, $y = t^2$, $z = 2t + 5$, where t is the time. Find the component of its velocity and acceleration at t =1 in the direction $\mathbf{i} + \mathbf{j} + 3\mathbf{k}$.
3. A particle moves along the curve $x = 3t^2$, $y = t^2 - 2t$, $z = t^3$, where t is the time. Find the velocity and acceleration at t =1
4. If $\mathbf{f} = (2x^2y - x^4)\,\mathbf{i} + (e^{xy} - y\sin x)\,\mathbf{j} + x^2\cos y\,\mathbf{k}$, verify $\dfrac{\partial^2 \mathbf{f}}{\partial x\,\partial y} = \dfrac{\partial^2 \mathbf{f}}{\partial y\,\partial x}$
5. If $\phi(x, y, z) = x y^2 z$ and $\mathbf{f} = x z\,\boldsymbol{i} - x y\,\mathbf{j} + y z^2\,\mathbf{k}$, show that $\dfrac{\partial^3}{\partial x^2\,\partial z}(\phi\,\mathbf{f})$ at (2, -1, 1) is $4\,\mathbf{i} + 2\,\mathbf{j}$.
6. If **a**, **b** are constant vectors, ω is a constant and **r** is a vector function of the scalar variable t is given by $\mathbf{r} = \mathbf{a}\cos\omega t + \mathbf{b}\sin\omega t$, Show that :

   (i) $\mathbf{r} \times \dfrac{d\mathbf{r}}{dt} = \omega\,\mathbf{a}\times\mathbf{b}$ (ii) $\dfrac{d^2\mathbf{r}}{dt^2} = -\omega^2\mathbf{r}$ **[RGPV Dec. 2007]**

9. If $\mathbf{r} = \cos nt\,\mathbf{i} + \sin nt\,\mathbf{j}$ where n is constant, show that :

   (i) $\mathbf{r} \times \dfrac{d\mathbf{r}}{dt} = n\,\mathbf{k}$ (ii) $\dfrac{d^2\mathbf{r}}{dt^2} = -n^2\mathbf{r}$

Chapter-2
Gradient, Divergence, Curl and Directional Derivative

1. Show that the following.

   (i) div $\mathbf{r} = 3$, (ii) Curl $\mathbf{r} = \mathbf{0}$ (iii) Curl $(\mathbf{a}\times\mathbf{r}) = 2\mathbf{a}$, (iv) div $(\mathbf{a}\times\mathbf{r}) = 0$

2. Show that

   (i) $r^n\,\mathbf{r}$ is a Solenoidal. If n = -3 **[RGPV June 2010]**

   (ii) $\nabla^2 f(r) = f''(r) + \dfrac{2}{r} f'(r)$ **[RGPV Dec. 2001, 02, 2010 June 05. 08]**

3. Prove that div (grad r^n) $= n(n+1)\,r^{n-2}$ **[RGPV June. 2003, 08, Feb. 2008]**
4. Find the value of A when div (grad r^m) = A r^{m-2}, where $r^2 = x^2 + y^2 + z^2$ **[RGPV Dec. 2004]**
5. Show that the following vector is Solenoidal : $(x + 3y)\,\mathbf{i} + (y - 3z)\,\mathbf{j} + (x - 2z)\,\mathbf{k}$ **[RGPV Dec. 2003,10 Feb. 2010]**
6. Find div **F** and Curl **F** , when $F = \nabla(x^3 + y^3 + z^3 - 3xyz)$ is irrotational **[RGPV Dec. 2003, 07, June 07, 08, 2011, Feb. 10]**
7. If $\mathbf{F} = (x + y + 1)\,\mathbf{i} + \mathbf{j} - (x + y)\,\mathbf{k}$, show that **F** Curl **F** = 0 **[RGPV June. 2004, Feb 06]**
8. Determine the constant a, b, c so that **[RGPV Feb. 2005]**
   $\mathbf{F} = (x + 2y + az)\,\mathbf{i} + (bx - 3y - z)\,\mathbf{j} + (4x + cy + 2z)\,\mathbf{k}$, is irrotational. Find the scalar potential φ such that **F** = grad φ.
9. Prove that Curl (**A**) = **0**, where $\mathbf{A} = 2xyz^3\,\mathbf{i} + x^2z^3\,\mathbf{j} + 3x^2yz^2\,\mathbf{k}$
10. Prove that div ($\mathbf{R}/r^3$) = 0 , where $\mathbf{R} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$ and $r = |\boldsymbol{R}|$
11. Show that the vector field :
    $\mathbf{V} = (\sin y + z)\,\mathbf{i} + (x\cos y - z)\,\mathbf{j} + (x - y)\,\mathbf{k}$ is irrotational. **[RGPV Dec. 2005, June 2009]**
12. If $\mathbf{V} = e^{xyz}(\mathbf{i} + \mathbf{j} + \mathbf{k})$, find Curl **V**.
13. Prove that (i) Curl grad $r^m = \mathbf{0}$ (ii) $\text{Curl}\left(\dfrac{\mathbf{a}\times\mathbf{r}}{r^3}\right) = -\dfrac{\mathbf{a}}{r^3} + \dfrac{3(\mathbf{a}\cdot\mathbf{r})\,\mathbf{r}}{r^5}$
14. A vector field is given by $\mathbf{A} = (x^2 + xy^2)\,\mathbf{i} + (y^2 + x^2y)\,\mathbf{j}$. Show that the field is irrotational and find the scalar potential. **[RGPV July 2006]**
15. Prove that : $\nabla r^n = n\,r^{n-2}\boldsymbol{R}$, where $\boldsymbol{R} = x\,\boldsymbol{i} + y\,\boldsymbol{j} + z\,\boldsymbol{k}$; $|\boldsymbol{R}| = r$ **[RGPV Feb. 2007]**
16. A fluid motion is given by $\mathbf{V} = (y + z)\mathbf{i} + (z + x)\,\mathbf{j} + (x + y)\,\mathbf{k}$. Show that the motion is irrotational and hence find the velocity potential function.
    (**Hint** : $\boldsymbol{V} = \nabla\phi$, then find the value of φ)
17. If $\phi = x^3yz^2$, find grad φ at the point (1, 1, 1). **[RGPV Dec. 2007]**
18. Prove that the vector : $\mathbf{V} = 3y^4z^2\,\mathbf{i} + 4x^3z^2\,\mathbf{j} - 3x^2y^2\,\mathbf{k}$ is Solenoidal.
19. Find the directional derivative of $\phi = x^2y\,z + 4xz^2$ in the direction the vector $2\,\mathbf{i} - \mathbf{j} - 2\,\mathbf{k}$ at the point (1, -2, -1).
20. Evaluate the directional derivative of the function $\phi = x^2 - y^2 + 2z^2$ at the point P (1, 2, 3) in the direction of the line PQ where Q has coordinates (5, 0, 4).
21. Find the magnitude of D.D for the function $\phi = \dfrac{y}{x^2 + y^2}$ which makes an angle of 30^o with the positive direction of x – axis at the point (0, 1).
22. What is the greatest rate of increases of $u = xyz^2$ at the point (1, 0, 3) ? **[RGPV June. 2003, Feb. 2008]**
23. Find the values of constants a, b, c so that the D.D of $\phi = axy^2 + byz + cz^2x^3$ at (1, 2, -1) has a maximum magnitude 64 in the direction parallel to z-axis.
24. Find the D.D of $f(x, y, z) = x^2y^2z^2$ at the point (1, 1, -1) in the direction of the tangent to the curve $x = e^t$, $y = 2\sin t + 1$, $z = t - \cos t$, at t = 0.

25. Find a unit vector normal to the surface $xy^3z^2 = 4$ at the point (-1, -1, 2).
[RGPV June 2007, 09, Feb. 2010, Dec. 2010]

26. Find the directional derivative of $\phi = 5x^2y - 5y^2z + 2.5\, z^2x$ at the point P (1, 1, 1) in the direction of the line $\frac{x-1}{2} = \frac{y-3}{-2} = \frac{z}{1}$

27. Find the unit vector normal to the surface $x^4 - 3xyz + z^2 + 1 = 0$ at the point (1, 1, 1).
[RGPV June 2011]

28. Find the directional derivative of the function $\phi (x, y, z) = xy^2 + yz^3$ at the point (2, -1, 1) in the direction of the vector $\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$ **[RGPV June 2010]**

29. Show that the vector field given by: $\boldsymbol{F} = (x^2 - yz)\boldsymbol{i} + (y^2 - xz)\boldsymbol{j} + (z^2 - xy)\boldsymbol{k}$ is irrotational and find the scalar potential.

Chapter - 3
Vector Integration

1. Given that $\mathbf{r}(t) = \begin{cases} 2\mathbf{i} - \mathbf{j} + 2\mathbf{k}, & \text{when } t = 2 \\ 4\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}, & \text{when } t = 3 \end{cases}$, Show that $\int_2^3 \mathbf{r} \cdot \frac{d\mathbf{r}}{dt}\, dt = 10.$
[RGPV Feb. 2006]

2. If $\mathbf{r}(t) = 5t^2\,\mathbf{i} + t\,\mathbf{j} - t^3\,\mathbf{k}$, Show that $\int_1^2 \mathbf{r} \times \frac{d^2\mathbf{r}}{dt^2}\, dt = -14\,\mathbf{i} + 75\,\mathbf{j} - 15\,\mathbf{k}$ **[RGPV July. 2006]**

3. Prove that $\int_1^2 [\,\mathbf{A}.(\mathbf{B}\times\mathbf{C})]\, dt = 0$, Where $\mathbf{A} = t\,\mathbf{i} - 3\,\mathbf{j} + 2t\,\mathbf{k}$, $\mathbf{B} = \mathbf{i} - 2\,\mathbf{j} + 2\,\mathbf{k}$, $\mathbf{C} = 3\,\mathbf{i} + t\,\mathbf{j} - \mathbf{k}$

4. If $\mathbf{r} \times d\mathbf{r} = \mathbf{0}$, Show that $\hat{\mathbf{r}} =$ constant. **[RGPV Feb. 2007]**

Chapter - 4
Line Integral, Surface integral and Volume integral

LINE INTEGRAL

1. Evaluate $\int_c \boldsymbol{F}.d\boldsymbol{r}$ where $\mathbf{F} = (x^2 - y^2)\,\mathbf{i} + xy\,\mathbf{j}$ and C is the arc of the curve $y = x^3$ in the xy-plane from (0, 0) to (2, 8).

2. Evaluate $\int_c \boldsymbol{F}.d\boldsymbol{r}$, where $\mathbf{F} = 3xy\,\mathbf{i} - y^2\,\mathbf{j}$ and C is the arc of the parabola $y = 2x^2$ from (0, 0) to (1, 2).

3. Using the line integral, compute the work done by the force $\mathbf{F} = (2y + 3)\,\mathbf{i} + x z\,\mathbf{j} + (y z - x)\,\mathbf{k}$ when it moves a particle from the point (0, 0, 0) to the point (2, 1, 1) along the curve $x = 2t^2$, $y = t$, $z = t^3$

4. Find the total work done in moving a particle in a force field by $\mathbf{F} = 3xy\,\mathbf{i} - 5z\,\mathbf{j} + 10x\,\mathbf{k}$ along the curve $x = t^2 + 1$, $y = 2t^2$, $z = t^3$ from t = 1 to t = 2. **[RGPV Dec. 2002, Feb. 2008]**

5. If $F = 2y\,\mathbf{i} - z\,\mathbf{j} + x\,\mathbf{k}$, evaluate $\int_c \boldsymbol{F}.d\boldsymbol{r}$ along the curve $x = \cos t$, $y = \sin t$, $z = 2\cos t$ from t = 0, to $t = \pi / 2$.

6. Find the value of $\int_c \boldsymbol{F}.d\boldsymbol{r}$, where $F = e^x \sin y\,\mathbf{i} + e^x \cos y\,\mathbf{j}$, the vertices of the rectangle C are (0, 0), (1, 0) (1,π /2) and (0, π/2).

SURFACE AND VOLUME INTEGRAL

7. Evaluate $\iint_s \boldsymbol{F}\,\hat{\boldsymbol{n}}\,.ds$, where $\mathbf{F} = (x + y^2)\,\mathbf{i} - 2x\,\mathbf{j} + 2yz\,\mathbf{k}$ and S is the surface of the plane $2x + y + 2z = 6$ in the first octant. **[RGPV June 2007, Dec. 2010]**

8. Evaluate $\iint_s \boldsymbol{F}\,\hat{\boldsymbol{n}}\,.ds$, where $\mathbf{F} = yz\,\mathbf{i} + zx\,\mathbf{j} + xy\,\mathbf{k}$ and S is the part of the surface of the sphere $x^2 + y^2 + z^2 = 1$which lies in the first octant.

9. Evaluate $\iint_s \mathbf{A}\,\hat{\mathbf{n}}\,.ds$, where $\mathbf{A} = 12x^2y\,\mathbf{i} - 3yz\,\mathbf{j} + 2y\,\mathbf{k}$ and S is the portion of the plane $x + y + z = 1$ included in the first octant.

10. Evaluate $\iint_s (\nabla \times \boldsymbol{F})\,\hat{\boldsymbol{n}}\,.ds$, where $\mathbf{F} = y^2\,\mathbf{i} + y\,\mathbf{j} - xz\,\mathbf{k}$ and S is the upper half of the sphere $x^2 + y^2 + z^2 = a^2$, and $z \geq 0$.

11. Evaluate $\iint_s \boldsymbol{F}\,\hat{\boldsymbol{n}}\,.ds$, where $\mathbf{F} = 18z\,\mathbf{i} - 12\,\mathbf{j} + 3y\,\mathbf{k}$ and S is the surface of the plane $2x + 3y + 6z = 12$ in the first octant. **[RGPV Feb. 2006]**

12. If $\mathbf{F} = 2xz\,\mathbf{i} - x\,\mathbf{j} + y^2\,\mathbf{k}$, then evaluate $\iiint_V \mathbf{F}\, dV$, where V is the region bounded by the surfaces x = 0, y = 0, x = 2, y = 6, $z = x^2$, z = 4 **[RGPV Feb. 2007]**

13. If $\mathbf{F} = (2x^2 - 3z)\,\mathbf{i} - 2xy\,\mathbf{j} - 4x\,\mathbf{k}$, then evaluate $\iiint_V \nabla\ \mathbf{F}\, dV$, where V is the closed region bounded by the planes x = 0, y = 0, = 0 and 2x + 2y + z = 4 **[RGPV July 2006, June 2008]**

Chapter - 5
GAUSS DIVERGENCE THEOREM
[relation between surface and volume integral]

1. Show that $\iint_S (ax\,\boldsymbol{i} + by\,\boldsymbol{j} + cz\,\boldsymbol{k})\,\hat{\boldsymbol{n}}\, dS = \frac{4}{3}\pi(a + b + c)$, where S is the surface of the sphere $x^2 + y^2 + z^2 = 1$
2. Verify Gauss's theorem and show that
$$\iint_S [(x^3 - yz)\,\boldsymbol{i} - 2x^2y\,\boldsymbol{j} + 2\,\boldsymbol{k}]\,\hat{\boldsymbol{n}}\, dS = \frac{a^5}{3}$$
, where S denotes the surface of the cube bounded by planes x = 0, x = a, y = 0, y = a, z = 0, z = a.
3. Verify Divergence theorem for $F = x^2\,\mathbf{i} + z\,\mathbf{j} + yz\,\mathbf{k}$, taken over the cube bounded by x = 0, x = 1, y = 0, y = 1, z = 0 and z = 1 **[RGPV Dec. 2003, June 2008, Feb. 2010]**
4. Verify Divergence theorem for $F = x^2\,\mathbf{i} + z\,\mathbf{j} + yz\,\mathbf{k}$, taken over the cube bounded by x = 0, x = a, y = 0, y = b, z = 0 and z = c **[RGPV Feb. 2007]**
5. Verify divergence theorem for $\mathbf{F} = 4x\,\mathbf{i} - 2y^2\,\mathbf{j} + z^2\,\mathbf{k}$ and S is the surface bounding the region $x^2 + y^2 = 4$, z = 0 and z = 3. **[RGPV Dec. 2004, 2010 June 2004]**
6. Find the value of $\iint_s \boldsymbol{F}\,\hat{\boldsymbol{n}}\,.ds$ when $\mathbf{F} = x\,\mathbf{i} - y\,\mathbf{j} + (z^2 - 1)\,\mathbf{k}$ and S is a closed surface bounded by plane z = 0, z = 1 and cylinder $x^2 + y^2 = 4$. Verify the Gauss theorem also
7. Evaluate $\iint_s \boldsymbol{F}\,\hat{\boldsymbol{n}}\,.ds$, over the surface of the region above xy-plane bounded by curve $z^2 = x^2 + y^2$ and the plane z = 4, if $\mathbf{F} = 4xz\,\mathbf{i} + xyz^2\,\mathbf{j} + 3z\,\mathbf{k}$ **[RGPV June 2011]**
8. Evaluate by Gauss divergence theorem $\iint_S (y^2z^2\,\boldsymbol{i} + z^2x^2\,\boldsymbol{j} + z^2y^2\,\boldsymbol{k})\,\hat{\boldsymbol{n}}\, dS$, where S is the part of the sphere $x^2 + y^2 + z^2 = 1$ above XOY plane bounded by xy-plane.

Chapter - 6
STOKE'S and GREEN THEOREM
[relation between surface and Line integral]

1. Verify Stoke's theorem for the vector field $\mathbf{F} = (x^2 - y^2)\,\mathbf{i} + 2xy\,\mathbf{j}$, integrated round the rectangle in the plane z = 0 and bounded by the lines x = 0, y = 0, x = a and y = b **[RGPV June 2002, July 06, Dec. 06]**
2. Verify Stoke's theorem for the vector field $\mathbf{F} = (x^2 + y^2)\,\mathbf{i} - 2xy\,\mathbf{j}$, integrated round the rectangle in the plane z = 0 and bounded by the line, x = ± a, y = 0 and y = b **[RGPV Feb. 2005, Feb. 2010, June 2011]**
3. Verify Stoke's theorem for the vector field $\mathbf{F} = x^2\,\mathbf{i} + xy\,\mathbf{j}$, integrated round the square whose sides are x = 0, y = 0, x = a and y = a in the plane z = 0.
4. By using Stoke's theorem, evaluate $\int_C (zy\,dx + zx\,dy + xy\,dz)$, where C is the curve $x^2 + y^2 = 1$, $z = y^2$ **[RGPV June 2005]**
5. Apply Stoke's theorem to valuate $\int_C [(x + y)\,dx + (2x - z)\,dy + (z + y)\,dz]$ where C is the boundary of the triangle with vertices (2, 0, 0), (0, 3, 0) and (0, 0, 6). **[RGPV Dec. 2003, Sept. 2009]**
6. Apply Stoke's theorem to evaluate $\int_C [(x + 2y)\,dx + (x - z)\,dy + (y - z)\,dz]$, where C is the boundary of the triangle with vertices (2, 0, 0), (0, 3, 0) and (0, 0, 6) oriented in the anticlockwise direction
7. Verify the Stoke's theorem for the vector $\mathbf{F} = (2x - y)\,\mathbf{i} - yz^2\,\mathbf{j} - y^2z\,\mathbf{k}$, over the upper half surface of $x^2 + y^2 + z^2 = 1$, bounded by its projection on xy-plane. **[RGPV Dec. 2005, 07, June 07, 08]**
8. Use Stoke's theorem to evaluate $\oint_C \boldsymbol{V}\, d\boldsymbol{r}$, where $\mathbf{V} = y^2\,\mathbf{i} + xy\,\mathbf{j} + xz\,\mathbf{k}$, and C is the bounding curve of the hemisphere $x^2 + y^2 + z^2 = 9$, z > 0, oriented in the positive direction.
9. Apply Stoke's theorem to evaluate $\int_C (y\,dx + z\,dy + x\,dz)$, where C is the curve of intersection of $x^2 + y^2 + z^2 = a^2$ and x + z = a **[RGPV June. 2004, Feb. 2008]**