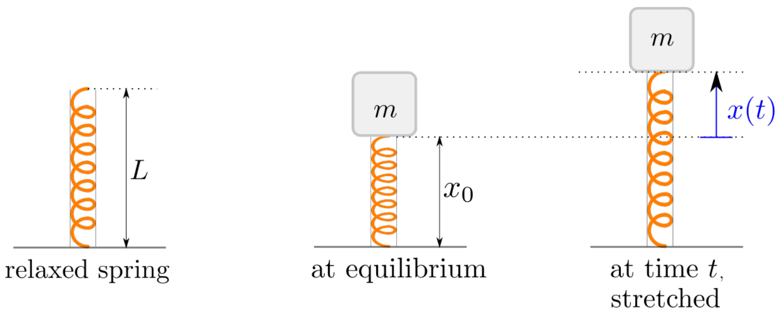
18.03 Recitation 11

Frequency response, Resonance

1. [Preparation: vertical spring-mass system]
A mass m (kg) is supported by a spring attached to the floor which is in a tube so that its motion is vertical only.



Suppose the relaxed length of a spring with spring constant k is L (meters), and the mass is in equilibrium position when the spring is at x_0 (meters) from the floor. Let $x(t)$ (meters) be the displacement of the mass from the equilibrium position at time t (seconds). We will choose $x > 0$ if the mass is above the x_0.

a. Use Newton's second law to write the weight of the mass mg in terms of k, x_0, L.

b. Now use Newton's second law to write a DE for the displacement $x(t)$ of the mass from the equilibrium position x_0 in terms of only k, x, x_0, L.

Real life application: Car suspension system

2. [The complex gain]

Recall the DE derived in the video describing the vertical position $x(t)$ (meters) of the car's body:

$$m\ddot{x} + b\dot{x} + kx = b\dot{y} + ky \qquad y = \cos(\omega t) \tag{1}$$

where the input $y(t)$ (meters) is the height of the road directly beneath the wheel at time t (seconds) and depends the bumps on the road and how fast the car is driving over them.

Find the complex gain $G(\omega)$ for the car suspension system, in terms of the system parameters m (kg), k (N/m), b (Ns/m), and ω (rad/s).

Mathlet activity

We will finish our analysis of a car's suspension system by investigating how the gain depends on the system parameters. Remember that the gain is THE property that determines how well the suspension absorbs the bumps in the road. We will investigate this using the Mathlet Amplitude and Phase: Second Order III.

On the left is the **schematic drawing** of a car's suspension. The **body of the car** is represented by the top orange block, which is coupled, by a spring and a damper, to the **surface of the road**, the blue bottom block here. As the car moves, it experiences the sinusoidal pattern of bumps of the road. Click on the play button >> below the graph to see the animation.

The **equation of motion** is equivalent to the equation of motion we derived in the previous video. Recall y is the sinusoidal driving function resulting from the car moving on the bumps,the input of the system, and x is the vertical position of the car, the response. Note that here in the mathet, the mass m of the car's body is normalized to be 1 and the amplitude of y is also normalized to be 1.

The **graphs** of both the input and the response plotted as functions of time. The blue curve is input graph $y(t)$, the height of the road at time t, and the orange curve is the graph of response $x(t)$, the vertical position of the car at time t. You can click on the play button to restart the animation.

We are interested in the gain $g(\omega)$ of the system, whose behavior over different frequencies of the input is shown by the **Bode plots**. Remember the gain is the ratio of the amplitudes of the response to the amplitude of the input. The graph of the phase lag $\phi(\omega)$ is also shown, but we are not interested in it in for the car's suspension system.

(Also take a look at the **Nyquist plot**, a plot of the trajectory of the complex gain G as ω varies.)

We can now determine how changing system parameters like b, k and ω will affect the gain.

3. [Driving very slow]

Let us study the effect of changing the driving frequency ω. This will tell us how a car would respond if it drove very slowly over a series of bumps, compared to if it drove very fast over the bumps. You can adjust the value of ω using the slider on the right under the Bode plots and see how the curves of the input and the response vary with ω.

a. From the Bode plot in the mathlet, what is $\lim_{\omega \to 0} g(\omega)$ for different values of the damping constant b and spring constant k? Is this consistent with what you expect from the complex gain formula?

b. What does this limit imply about the behaviour of the suspension system if you drive very slowly over a series of bumps?

4. [Driving very fast]

a. Find $\lim_{\omega \to \infty} g(\omega)$ using the complex gain (which you found in a previous problem). Verify the result by looking at the Bode plot on the mathlet at different values of b and k.

b. What does this imply about the behavior of the suspension system if you drove very fast over a series of bumps?

5. [Driving in resonance with bumps]

Fix the damping and spring constants $b = 1$ and $k = 2$.

a. From the Bode plot, what is the frequency ω_r at which the gain is maximum and what is the value of the maximum gain $g_{\max}$?

b. What does that mean for the car?

6. [Effect of damping on resonance]

Fix k, slowly vary b, and look at how the resonance peak on the Bode plot changes.

a. How does increasing the damping constant b affect the resonance peak?

b. If the suspension system does not have a dashpot, what would the resonant angular frequency? In other words, find the angular frequency ω_r at which $g(\omega)$ is maximum when $b = 0$. Use the complex gain formula that you have previously found and answer in terms of k and m.

c. Now what is the value of ω_r when $k = 2$, $m = 1$ and $b = 0$?

d. Compare this resonant frequency with no damping with the value of ω_r you read from mathlet for $k = 2$, $b = 1$ (and $m = 1$) in the previous problem. Is the resonant frequency with no damping a good approximation to the resonant frequency with damping?

7. [Effect of the spring on resonance]

Now fix b, slowly vary k, and again look at how the resonance peak on the Bode plot changes.

a. How does increasing the spring constant k affect the resonance peak?

8. [Minimum comfortable speed]

a. Set $k = 2$ and $b = 1$. Use the mathlet to figure out the minimum value of $\omega_{\min}$ so that $g < 1$. This would allow you to determine the minimum speed at which you should drive over a series of bumps to ensure a comfortable ride.

b. Now, slowly vary b while still fixing $k = 2$. How does increasing the damping constant b change $\omega_{\min}$? Is this what you expect from the complex gain formula?

Tuning the spring and damping constants

When designing a suspension system, engineers typically fix the value of ω at a value that reflects the typical road conditions that the car will encounter. The engineers then choose the parameters k and b to ensure that the vehicle will not be in resonance with bumps in the road under normal conditions. For example, if $omega$ is fixed at 2, it is obvious that choosing $k = 4$ and $b = 1$ would be a bad choice as the car would be in resonance under these conditions.

We have only touched upon a few of the interesting properties of the response of this driven oscillator. Continue playing with the mathlet so that you fully understand the behavior of the system.

For example, can you find the values of b, k, and ω that produce the largest possible gain in this mathlet? Have fun!