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DEPARTMENT OF MATHEMATICS

Transforms and Partial Differential Equations(TPDE)

MA8353

by

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MA8353 TRANSFORMS AND PARTIAL DIFFERENTIAL EQUATIONS L T P C

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UNIT I PARTIAL DIFFERENTIAL EQUATIONS(P.D.E.) 12

Formation of partial differential equations – Singular integrals - Solutions of standard types of first order partial differential equations - Lagrange's linear equation - Linear partial differential equations of second and higher order with constant coefficients of both homogeneous and non-homogeneous types.

UNIT II FOURIER SERIES(F.S.) 12

Dirichlet's conditions – General Fourier series – Odd and even functions – Half range sine series – Half range cosine series – Complex form of Fourier series – Parseval's identity – Harmonic analysis.

UNIT III APPLICATIONS OF P.D.E.(A.P.D.E.) 12

Classification of PDE – Method of separation of variables - Fourier Series Solutions of one dimensional wave equation – One dimensional equation of heat conduction – Steady state solution of two dimensional equation of heat conduction.

UNIT IV FOURIER TRANSFORMS(F.T.) 12

Statement of Fourier integral theorem - Fourier transform pair - Fourier sine and cosine transforms - Properties - Transforms of simple functions - Convolution theorem - Parseval's identity.

UNIT V Z-TRANSFORMS AND DIFFERENCE EQUATIONS(Z.T.) 12

$Z-$transforms - Elementary properties – Inverse $Z-$transform (using partial fraction and residues) – Initial and final value theorems - Convolution theorem - Formation of difference equations – Solution of difference equations using $Z-$ transform.

TOTAL : 60 PERIODS

OUTCOMES: Upon successful completion of the course, students should be able to:

- Understand how to solve the given standard partial differential equations.
- Solve differential equations using Fourier series analysis which plays a vital role in engineering applications.
- Appreciate the physical significance of Fourier series techniques in solving one and two dimensional heat flow problems and one dimensional wave equations.
- Understand the mathematical principles on transforms and partial differential equations would provide them the ability to formulate and solve some of the physical problems of engineering.
- Use the effective mathematical tools for the solutions of partial differential equations by using Z transform techniques for discrete time systems.

Contents

1 Partial Differential Equations(P.D.E.)

1.1 Part-A

1. Form the partial differential equation by eliminating the arbitrary constants a and b from $z = (x+a)^2 + (y+b)^2$.

Soln : Given $z = (x+a)^2 + (y+b)^2 \quad (1)$

$$p = \frac{\partial z}{\partial x} = 2(x+a) \Rightarrow \frac{p}{2} = (x+a) \quad (2)$$
$$q = \frac{\partial z}{\partial y} = 2(y+b) \Rightarrow \frac{q}{2} = (y+b) \quad (3)$$

Squaring (2) & (3) and substituting in equation (1), we get

$$\left(\frac{p}{2}\right)^2 = (x+a)^2$$
$$\left(\frac{q}{2}\right)^2 = (y+b)^2$$
$$\therefore (1) \Rightarrow z = \left(\frac{p}{2}\right)^2 + \left(\frac{q}{2}\right)^2$$
$$= \frac{p^2}{4} + \frac{q^2}{4}$$
$$= \frac{p^2+q^2}{4}$$
$$4z = p^2 + q^2$$

2. Form the partial differential equation by eliminating the arbitrary constants a and b from $(x-a)^2 + (y-b)^2 + z^2 = 1$.

Soln :

Given $(x-a)^2 + (y-b)^2 + z^2 = 1 \quad (1)$

Differentiate (1) partially w.r.t. x and y respectively,

$$2(x-a) + 2z\frac{\partial z}{\partial x} = 0$$
$$\Rightarrow (x-a) + z\frac{\partial z}{\partial x} = 0$$
$$\Rightarrow (x-a) = -zp$$
$$\Rightarrow (x-a)^2 = (zp)^2 \quad (2)$$
$$2(y-b) + 2z\frac{\partial z}{\partial y} = 0$$
$$\Rightarrow (y-b) + z\frac{\partial z}{\partial y} = 0$$
$$\Rightarrow (y-b) = -zq$$
$$\Rightarrow (y-b)^2 = (zq)^2 \quad (3)$$

substituting (2) & (3) in equation (1), we get

$$(zp)^2 + (zq)^2 + z^2 = 1$$
$$z^2(p^2+q^2+1) = 1$$

3. Form the partial differential equation of all the spheres whose centres are on the line $x = y = z$

Soln : The equation of all the spheres having their center on the line $x = y = z(=$‘a’$)$ and radius ‘b’ is given by

$$(x-a)^2 + (y-a)^2 + (z-a)^2 = b^2 \quad (1)$$

Differentiate (1) partially w.r.t. x and y

respectively,

$$2(x-a)+2(z-a)\frac{\partial z}{\partial x}=0$$
$$\Rightarrow (x-a)+(z-a)\frac{\partial z}{\partial x}=0$$
$$\Rightarrow (x-a)+(z-a)p=0$$
$$\Rightarrow a(1+p)=(x+zp)$$
$$\therefore a=\frac{(x+zp)}{(1+p)} \qquad (2)$$
$$2(y-a)+2(z-a)\frac{\partial z}{\partial y}=0$$
$$\Rightarrow (y-a)+(z-a)\frac{\partial z}{\partial y}=0$$
$$\Rightarrow (y-a)+(z-a)q=0$$
$$\Rightarrow a(1+q)=(y+zq)$$
$$\therefore a=\frac{(y+zq)}{(1+q)} \qquad (3)$$

Equating (2) & (3), we get

$$a=\frac{(x+zp)}{(1+p)}=\frac{(y+zq)}{(1+q)}$$
$$\Rightarrow x(1+q)-y(1+p)+z(p-q)=0$$

4. Form the partial differential equation by eliminating the arbitrary constants a and b from $(x-a)^2+(y-b)^2=4-z^2$.

Soln : Given $(x-a)^2+(y-b)^2+z^2=4$ (1)

Differentiate (1) partially w.r.t. x and y respectively,

$$2(x-a)+2z\frac{\partial z}{\partial x}=0$$
$$\Rightarrow (x-a)+z\frac{\partial z}{\partial x}=0$$
$$\Rightarrow (x-a)=-zp$$
$$\Rightarrow (x-a)^2=(zp)^2 \qquad (2)$$
$$2(x-a)+2z\frac{\partial z}{\partial y}=0$$
$$\Rightarrow (y-b)+z\frac{\partial z}{\partial y}=0$$
$$\Rightarrow (y-b)=-zq$$
$$\Rightarrow (y-b)^2=(zq)^2 \qquad (3)$$

Substituting (2) & (3) in equation (1), we get,

$$(zp)^2+(zq)^2+z^2=4$$
$$z^2(p^2+q^2+1)=4$$

5. Form the partial differential equation by eliminating the arbitrary constants 'a' and 'b' from $2z=(ax+y)^2+b$.

Soln : Given $2z=(ax+y)^2+b$ (1)

Differentiate (1) partially w.r.t. x and y respectively,

$$2p=2a(ax+y)+0 \qquad (2)$$
$$2q=2(ax+y) \qquad (3)$$
$$\frac{(2)}{(3)}\Rightarrow \frac{p}{q}=a \qquad (4)$$

From (3), $q=ax+y$

$$q=\frac{p}{q}x+y \qquad [\because \text{from } (4)]$$
$$\Rightarrow px+qy=q^2$$

6. Find the partial differential equations of all planes cutting equal intercepts with x and y-axes. (N/D '07)

Soln : The equation of the plane making equal intercepts $'a'$ with x and y-axes is given by

$$\frac{x}{a} + \frac{y}{a} + \frac{z}{b} = 1 \qquad (1)$$

Differentiate (1) partially w.r.t. x and y respectively,

$$\frac{1}{a} + \frac{\frac{\partial z}{\partial x}}{b} = 0 \Rightarrow \frac{1}{a} = -\frac{p}{b} \qquad (2)$$

$$\frac{1}{a} + \frac{\frac{\partial z}{\partial y}}{b} = 0 \Rightarrow \frac{1}{a} = -\frac{q}{b} \qquad (3)$$

Equating (2) and (3) we get, $p = q$.

7. Form the partial differential equation by eliminating the arbitrary constants a and b from $(x-a)^2 + (y-b)^2 = z^2 \cot^2 \alpha$.

Soln : Given $(x-a)^2 + (y-b)^2 = z^2 \cot^2 \alpha$ (1)

Differentiate (1) partially w.r.t. x and y respectively,

$$2(x-a) = 2z\frac{\partial z}{\partial x}\cot^2 \alpha$$

$$\Rightarrow (x-a) = z\frac{\partial z}{\partial x}\cot^2 \alpha$$

$$\Rightarrow (x-a) = -zp\cot^2 \alpha$$

$$\Rightarrow (x-a)^2 = \left(zp\cot^2 \alpha\right)^2 \qquad (2)$$

$$2(y-b) = 2z\frac{\partial z}{\partial y}\cot^2 \alpha$$

$$\Rightarrow (y-b) = z\frac{\partial z}{\partial y}\cot^2 \alpha$$

$$\Rightarrow (y-b) = -zq\cot^2 \alpha$$

$$\Rightarrow (y-b)^2 = \left(zq\cot^2 \alpha\right)^2 \qquad (3)$$

substituting (2) & (3) in equation (1), we get,

$$\left(zp\cot^2 \alpha\right)^2 + \left(zq\cot^2 \alpha\right)^2 = z^2 \cot^2 \alpha$$

Dividing by $\cot^2 \alpha$ on both sides,

$$z^2\left(p^2 \cot^2 \alpha + q^2 \cot^2 \alpha\right) = z^2$$

$$p^2 + q^2 = \tan^2 \alpha$$

8. Form the partial differential equation by eliminating the arbitrary function from $z = f\left(x^2+y^2\right)$.

Soln : Given $z = f\left(x^2+y^2\right)$ (1)

Differentiate (1) partially w.r.t. x and y respectively,

$$p = \frac{\partial z}{\partial x} = 2xf'\left(x^2+y^2\right)$$

$$\Rightarrow \frac{p}{2x} = f' \qquad (2)$$

$$q = \frac{\partial z}{\partial y} = 2yf'\left(x^2+y^2\right)$$

$$\Rightarrow \frac{q}{2y} = f' \qquad (3)$$

Equating (2) and (3) ,we get,

$$\frac{p}{2x} = \frac{q}{2y} \Rightarrow py - qx = 0$$

9. Form the partial differential equation by eliminating the arbitrary function from $z = f\left(\frac{xy}{z}\right)$.

Soln : Given $z = f\left(\frac{xy}{z}\right)$ (1)

Differentiate (1) partially w.r.t. x and y respectively,

$$p = \frac{\partial z}{\partial x} = f'\left(\frac{xy}{z}\right) \times \left(\frac{y}{z} - \frac{xy}{z^2}\frac{\partial z}{\partial x}\right)$$

$$\Rightarrow p = f' \times \left(\frac{y}{z} - \frac{xy}{z^2}p\right)$$

$$\Rightarrow f' = \frac{p}{\left(\frac{y}{z} - \frac{xy}{z^2}p\right)} \qquad (2)$$

Similarly $f' = \dfrac{q}{\left(\frac{x}{z} - \frac{xy}{z^2}q\right)}$ (3)

Equating (2) and (3) ,we get,

$$\frac{p}{\left(\frac{y}{z} - \frac{xy}{z^2}p\right)} = \frac{q}{\left(\frac{x}{z} - \frac{xy}{z^2}q\right)}$$

$$z(px - qy) = 0$$

10. Form the partial differential equation by eliminating the arbitrary function from $\phi\left(z^2 - xy, \frac{x}{z}\right) = 0$.

Soln : Given $\phi\left(z^2 - xy, \frac{x}{z}\right) = 0$ (1)

Let $u = z^2 - xy,\ v = \dfrac{x}{z}$

$$\begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial v}{\partial x} \\ \dfrac{\partial u}{\partial y} & \dfrac{\partial v}{\partial y} \end{vmatrix} = 0$$

$$\begin{vmatrix} 2zp - y & \dfrac{z - px}{z^2} \\ 2zq - x & \dfrac{-xq}{z^2} \end{vmatrix} = 0$$

$$px^2 - q\left(xy - 2z^2\right) = xz$$

11. Form the partial differential equation by eliminating the arbitrary function from $z = f\left(x^2 + y^2\right) + x + y$.

Soln : Given $z = f\left(x^2 + y^2\right) + x + y$ (1)

Differentiate (1) partially w.r.t. x and y respectively,

$$p = \frac{\partial z}{\partial x} = 2xf'\left(x^2 + y^2\right) + 1$$

$$\Rightarrow p = 2xf' + 1 \quad (2)$$

$$q = \frac{\partial z}{\partial y} = 2yf'\left(x^2 + y^2\right) + 1$$

$$\Rightarrow q = 2yf' + 1 \quad (3)$$

(2) $\times\ y$ − (3) $\times\ x$ gives,

$$py - qx = \left[2xyf' + y\right] - \left[2xyf' + x\right]$$

$$= y - x$$

$$py - qx = y - x$$

12. Find the complete integral of $p + q = pq$.

Soln : Given $p + q = pq$ (1)

Let $z = ax + by + c$ (2)

be a solution of the above equation.

Differentiate (2) partially w.r.t. x and y respectively,

$$\frac{\partial z}{\partial x} = p = a \quad ; \quad \frac{\partial z}{\partial y} = q = b$$

Equation (1) reduces to $a + b = ab$

$$ab - b = a$$

$$\Rightarrow b(a-1) = a$$

$$\Rightarrow b = \frac{a}{a-1}$$

The complete solution is given by $z = ax + \dfrac{a}{a-1}y + c$.

13. Find the complete integral of $\sqrt{p} + \sqrt{q} = 1$.

Soln : Given $\sqrt{p} + \sqrt{q} = 1$ (1)

Let $z = ax + by + c$ (2)

be a solution of the above equation.

Differentiate (2) partially w.r.t. x and y respectively,

$$\frac{\partial z}{\partial x} = p = a \quad ; \quad \frac{\partial z}{\partial y} = q = b$$

Equation (1) reduces to

$$\sqrt{a} + \sqrt{b} = 1 \Rightarrow b = \left(1 - \sqrt{a}\right)^2$$

The complete solution is given by

$$z = ax + \left(1 - \sqrt{a}\right)^2 y + c$$

14. Find the complete solution of the partial differential equation $p^2 + q^2 - 4pq = 0$.

Soln : Given $p^2 + q^2 - 4pq = 0$ (1)

Let $z = ax + by + c$ (2)

be a solution of the above equation.

Differentiate (2) partially w.r.t. x and y respectively,

$$\frac{\partial z}{\partial x} = p = a \quad ; \quad \frac{\partial z}{\partial y} = q = b$$

Equation (1) reduces to

$$a^2 + b^2 - 4ab = 0 \qquad [\text{Quadratic in } a]$$

$$\text{i.e., } b^2 - 4ab + a^2 = 0 \qquad [\text{Quadratic in } b]$$

$$\Rightarrow b = \frac{4a \pm \sqrt{16a - 4a^2}}{2}$$

$$= a \pm \sqrt{4a - a^2}$$

The complete solution is given by $z = ax + \left(a \pm \sqrt{4a - a^2}\right)y + c$

15. Write down the complete solution of $px+qy=\sqrt{1+p^2+q^2}+z$.

Soln: Given $px+qy=\sqrt{1+p^2+q^2}+z$ (1)

Rearranging the equation

$z=px+qy-\sqrt{1+p^2+q^2}$

The above equation is a Clairaut's equation.

Put $p=a$; $q=b$

The complete solution is given

$$z=ax+by-\sqrt{1+a^2+b^2}$$

16. Write down the complete solution of $z=px+qy+p^2-q^2$. (May/June 2006)

Soln : Given $z=px+qy+p^2-q^2$

The above equation is a Clairaut's equation.

Put $p=a$; $q=b$

The complete solution is given $z=ax+by+a^2-b^2$.

17. Solve $\left(D^3-3DD'^2+2D'^3\right)z=0$.

Soln : The auxiliary equation is given by

$$m^3-3m+2=0 \qquad (1)$$

Put $m=1$

$$\therefore (1) \Rightarrow 1^3-3.1+2=0$$

$m=1$ is a root of the auxiliary equation.

Solve by synthetic division method

1	1	0	-3	2
	0	1	1	-2
	1	1	-2	0

$\therefore m^2+m-2=0 \Rightarrow m=1,-2$ roots.

The roots of the auxiliary equation are $1,1,-2$.

The solution is given by

$z=f(y+x)+xg(y+x)+h(y-2x)$

18. Solve $\left(D^3-2D^2D'-4DD'^2+8D'^3\right)z=0$.

Soln : The auxiliary equation is given by

$$m^3-2m^2-4m+8=0 \qquad (1)$$

Put $m=2$

$$\therefore (1) \Rightarrow 2^3-2.2^2-4.2+8=0$$

$m=2$ is a root of the auxiliary equation.

Solve by synthetic division method

2	1	-2	-4	8
	0	2	0	-8
	1	0	-4	0

$\therefore m^2-4=0 \Rightarrow m=2,-2$ roots.

The roots of the auxiliary equation are $2,2,-2$.

The solution is given by

$z=f(y+2x)+xg(y+2x)+x^2h(y-2x)$.

19. Solve $\left(D^2-2DD'+D'^2\right)z=0$. (AU Apr/May 08)

Soln : The auxiliary equation is given by

$m^2-2m+1=0, \quad \therefore m=1,1$ roots.

The solution is given by

$z=f(y+x)+xg(y+x)$.

20. Find the particular Integral of $\left(D^3+2D^2D'\right)z=\sin(x+2y)$.

Soln : Particular Integral is given by

$$\begin{aligned}
\text{P.I.} &= \frac{1}{D^3+2D^2D'}\sin(x+2y) \\
&\qquad \{\text{Here } \sin(ax+by) \Rightarrow a=1; b=2\} \\
&= \frac{1}{-1^2D+2\left(-1^2D'\right)}\sin(x+2y) \\
&= -\frac{1}{D+2D'}\sin(x+2y) \\
&\qquad \left\{\text{Multiply Nr. \& Dr.by } \left(D-2D'\right)\right\} \\
&= -\frac{D-2D'}{D^2-4D'^2}\sin(x+2y) \\
&= -\frac{\cos(x+2y)-2\times 2\cos(x+2y)}{-1^2-4\left(-2^2\right)} \\
&= \frac{3\cos(x+2y)}{15} = \frac{\cos(x+2y)}{5}
\end{aligned}$$

1.2 Part-B

1.2.1 Examples of (Number of arbitrary constants > Number of independent variables)

Example 1.1. Form the partial differential equation of all spheres whose radii are the same.

Solution: The equation of all sphere with equal radius can be taken as

$$(x-a)^2+(y-b)^2+(z-c)^2=r^2 \tag{1}$$

where a, b, c are arbitrary constants and r is a given constant. Since the number of arbitrary constants is more than the number of independent variables, we will get the p.d.e. of order greater than 1.

Differentiating (1) partially w.r.t 'x'

$$2(x-a)+2(z-c)\frac{\partial z}{\partial x}=0 \qquad \text{i.e., } (x-a)+(z-c)p=0 \tag{2}$$

Differentiating (1) partially w.r.t 'y'

$$2(y-b)+2(z-c)\frac{\partial z}{\partial y}=0 \qquad \text{i.e., } (y-b)+(z-c)q=0 \tag{3}$$

Differentiating (2) partially w.r.t 'x'

$$1+(z-c)\frac{\partial^2 z}{\partial x^2}+\frac{\partial z}{\partial x}.\frac{\partial z}{\partial x}=0 \qquad \text{i.e., } 1+(z-c)r+p^2=0 \tag{4}$$

Differentiating (2) partially w.r.t 'y'

$$1+(z-c)\frac{\partial^2 z}{\partial y^2}+\frac{\partial z}{\partial y}.\frac{\partial z}{\partial y}=0 \qquad \text{i.e., } 1+(z-c)t+q2=0 \tag{5}$$

From (4) and (5), we get $z-c=\dfrac{-p^2-1}{r}$ and $z-c=\dfrac{-q^2-1}{t}$

$$\text{i.e., } z-c=\frac{-p^2-1}{r}=\frac{-q^2-1}{t}$$

$$t(p^2+1)=r(q^2+1)$$

Example 1.2. Obtain the partial differential equation by eliminating $\boldsymbol{a, b, c}$ from $\boldsymbol{\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1}$

Solution: Since the number of arbitrary constants is more than the number of independent variables, we will get the p.d.e of order greater than '1'.

Given $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1$ (1)

Differentiating (1) partially w.r.t 'x'

$$\frac{2x}{a^2}+\frac{2z}{c^2}\frac{\partial z}{\partial x}=0 \qquad \text{i.e., } \frac{x}{a^2}+\frac{z}{c^2}p=0 \tag{2}$$

Differentiating (1) partially w.r.t 'y'

$$\frac{2y}{b^2}+\frac{2z}{c^2}\frac{\partial z}{\partial y}=0 \qquad \text{i.e., } \frac{y}{b^2}+\frac{z}{c^2}q=0 \tag{3}$$

Differentiating (2) partially w.r.t. 'x'

$$\frac{1}{a^2}+\frac{1}{c^2}\left(z\frac{\partial^2 z}{\partial x^2}+\frac{\partial z}{\partial x}.\frac{\partial z}{\partial x}\right)=0 \qquad \text{i.e., } \frac{1}{a^2}+\frac{1}{c^2}\left(zr^2+p^2\right)=0 \tag{4}$$

Differentiating (3) partially w.r.t 'y'

$$\frac{1}{b^2}+\frac{1}{c^2}\left(z\frac{\partial^2 z}{\partial y^2}+\frac{\partial z}{\partial y}.\frac{\partial z}{\partial y}\right)=0 \qquad \text{i.e., } \frac{1}{b^2}+\frac{1}{c^2}\left(zt^2+q^2\right)=0 \tag{5}$$

Differentiating (2) partially w.r.t 'y'

$$0+\frac{1}{c^2}\left(z\frac{\partial^2 z}{\partial y\partial x}+\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}\right)=0 \qquad \text{i.e., } \frac{1}{c^2}(zs+pq)=0 \tag{5}$$

$$\text{i.e., } zs+pq=0$$

Note : We may also get different partial differential equations. The answer is not unique.

1.2.2 Examples of (Number of arbitrary functions > 1)

Example 1.3. Form PDE by eliminating arbitrary function f and g from $z = f(x+ay)+g(x-ay)$.

Solution: Given $z = f(x+ay)+g(x-ay)$ (1)

Differentiating (1) partially w.r.t 'x'

$$\frac{\partial z}{\partial x}=f'(x+ay)+g'(x-ay) \qquad \text{i.e., } p=f'+g' \tag{2}$$

Differentiating (1) partially w.r.t 'y'

$$\frac{\partial z}{\partial y}=af'(x+ay)-ag'(x-ay) \qquad \text{i.e., } q=af'-ag' \tag{3}$$

Differentiating (2) partially w.r.t 'x'

$$\frac{\partial^2 z}{\partial x^2}=f''+g'' \qquad \text{i.e., } r=f''+g'' \tag{4}$$

Differentiating (3) partially w.r.t 'y'

$$\frac{\partial^2 z}{\partial y^2}=a^2f''+a^2g'' \qquad \text{i.e., } t=a^2\left(f''+g''\right) \tag{5}$$

Using (4), we get $t = a^2r$.

Example 1.4. Eliminate the arbitrary function ϕ and ψ from $z = x\phi(y)+y\psi(x)$ and form the partial differential equation.

Solution: Given $z = x\phi(y)+y\psi(x)$ (1)

Differentiating (1) partially w.r.t 'x'

$$p=\phi(y)+y\psi'(x) \tag{2}$$

Differentiating (1) partially w.r.t 'y'

$$q=x\phi'(y)+\psi(x) \tag{3}$$

Differentiating (2) partially w.r.t 'y'

$$s = \phi'(y) + \psi'(x) \tag{4}$$

Now, $px + qy = x\phi(y) + xy\psi'(x) + xy\phi'(y) + y\psi(x)$

$$= x\phi(y) + y\psi(x) + xy(\phi'(y) + \psi'(x))$$

$px + qy = z + xys$ [using (1) and (4)]

Example 1.5. Form partial differential equation by eliminating arbitrary function f and g from $z = xf\left(\frac{y}{x}\right) + y\phi(x)$. **[UQ]**

Solution: Given $z = xf\left(\frac{y}{x}\right) + y\phi(x)$ (1)

Differentiating (1) partially w.r.t 'x'

$$\frac{\partial z}{\partial x} = xf'\left(\frac{y}{x}\right)\left(\frac{-y}{x^2}\right) + f\left(\frac{y}{x}\right) + y\phi'(x)$$

$$\frac{\partial z}{\partial x} = \frac{-y}{x}f'\left(\frac{y}{x}\right) + f\left(\frac{y}{x}\right) + y\phi'(x)$$

$$\text{i.e., } p = \frac{-y}{x}f' + f + y\phi' \tag{2}$$

Differentiating (1) partially w.r.t 'y'

$$\frac{\partial z}{\partial y} = xf'\left(\frac{y}{x}\right).\frac{1}{x} + \phi(x) \Rightarrow \frac{\partial z}{\partial y} = f'\left(\frac{y}{x}\right) + \phi(x)$$

$$\text{i.e., } q = f' + \phi \tag{3}$$

Now, $\frac{\partial^2 z}{\partial x \partial y} = f''\left(\frac{y}{x}\right).\left(\frac{-y}{x^2}\right) + \phi'(x)$ i.e., $s = \frac{-y}{x^2}f'' + \phi'$ (4)

$\frac{\partial^2 z}{\partial y^2} = f''\left(\frac{y}{x}\right).\frac{1}{x}$ i.e., $t = \frac{1}{x}f''$ (5)

$$\therefore px + qy = -yf' + xf + xy\phi' + yf' + y\phi \quad \text{[using (2) and (3)]}$$

$$px + qy = xf + y\phi + xy\phi'$$

From (1), $px + qy = z + xy\phi'$ (6)

Using (5) in (4), $s = \frac{-y}{x^2}(xt) + \phi' \Rightarrow \phi' = s + \frac{yt}{x}$

Sub. the value of ϕ' in (6), we get

$$px + qy = z + xy\left(s + \frac{yt}{x}\right)$$

$$px + qy = z + xys + y^2t$$

Example 1.6. Form a p.d.e by eliminating arbitrary function f and g from $z = f(x+y)g(x-y)$.

Solution: Given $z = f(x+y)g(x-y)$ i.e., $z = fg$ (1)

Differentiating (1) partially w.r.t 'x'

$$\frac{\partial z}{\partial x} = fg' + gf' \qquad \text{i.e., } p = fg' + gf' \tag{2}$$

Differentiating (1) partially w.r.t 'y'

$$\frac{\partial z}{\partial y} = f\left(-g'\right) + gf' \qquad \text{i.e., } q = gf' - fg' \tag{3}$$

$$\frac{\partial^2 z}{\partial x^2} = fg'' + g'f' + gf'' + f'g' \qquad \text{i.e., } r = fg'' + gf'' + 2f'g' \tag{4}$$

$$\begin{aligned}\frac{\partial^2 z}{\partial y^2} &= -[f(-g'') + g'f'] + gf'' + f'(-g')\\ t &= fg'' + gf'' - 2f'g'\end{aligned} \tag{5}$$

From (2) and (3), we get

$$\begin{aligned}p^2 - q^2 &= f^2g'^2 + g^2f'^2 + 2fgf'g' - (g^2f'^2 + f^2g'^2 - 2fgf'g')\\ &= 4fgf'g'\\ p^2 - q^2 &= 4zf'g' \qquad [using(1)]\end{aligned} \tag{6}$$

From (4) and (5), we get

$$r - t = 4f'g' \tag{7}$$

From (6) and (7), we get

$$p^2 - q^2 = z(r - t)$$

1.2.3 Solution of standard types of first order partial differential equations

1.2.3.1 Type II: $z = px + qy + f(p, q)$ (Clairaut's Form)

Example 1.7. Find the complete and singular integrals of $z = px + qy + pq$.

Solution: Given $z = px + qy + pq$ (1)

The given partial differential equation is in Clairaut's form.

$\therefore$ The complete integral is $z = ax + by + ab$ (2)

where a and b are arbitrary constants.

To find singular integral : Partially differentiating (2) w.r.t 'a' and 'b' then equating to zero, we get

$$x + b = 0$$

$$y + a = 0$$

$$\therefore a = -y \text{ and } b = -x$$

Substituting the values of 'a' and 'b' in (2), we get

$$z = -yx - xy + (-y)(-x) = -yx - xy + yx$$
$$\therefore z = -xy$$

which is the singular integral.

To find General Integral : Put $b = \beta(a)$ in complete integral equation (2), we get

$$z = ax + \beta(a)y + a\beta(a) \tag{3}$$

Differentiate (3) w.r.t. 'a', we get

$$z = x + \beta^{'}(a)y + a\beta^{'}(a) + \beta(a) \tag{4}$$

Eliminating 'a' from (3) and (4), we get general integral of the given p.d.e.

Example 1.8. Solve $z = px + qy + p^2q^2$.

Solution: The given equation is in Clairaut's form.

$\therefore$ The complete integral is $z = ax + by + a^2b^2$ (1)

To find singular integral: Partially differentiating (1) w.r.t a and b then equating to zero, we get

$$x = -2ab^2 \tag{2}$$
$$y = -2a^2b \tag{3}$$
$$xy = 4a^3b^3 \tag{4}$$
$$\text{From (2), } \frac{x}{b} = -2ab$$
$$\text{From (3), } \frac{y}{a} = -2ab$$

$$\text{From (1), } z = ab\left[\frac{x}{b} + \frac{y}{a} + ab\right] = ab(-2ab - 2ab + ab)$$
$$z = -3a^2b^2 \tag{5}$$
$$\text{Now, } z^3 = -27a^6b^6 \qquad \text{i.e., } z^3 = -27(a^3b^3)^2$$
$$\text{Using (4), } z^3 = -27\left(\frac{xy}{4}\right)^2 \qquad \Rightarrow z^3 = \frac{-27}{16}x^2y^2$$

which is the required singular integral.

The general integral is obtained as first example of this type.

Example 1.9. Solve $z = px + qy + p^2 + pq + q^2$.

Solution: The given equation is in Clairaut's form.

$\therefore$ The complete integral is $z = ax + by + a^2 + ab + b^2$ (1)

To find singular integral: Partially differentiating (1) w.r.t a and b then equating to zero, we get

$$x + 2a + b = 0 \tag{2}$$

$$y + a + 2b = 0 \tag{3}$$

Solving (2) and (3), we get

$$a = \frac{1}{3}(y - 2x), b = \frac{1}{3}(x - 2y)$$

Substituting the values of a and b in (1), we get

$$3z = xy - x^2 - y^2$$

which is the singular integral.

The general integral is obtained as first example of this type.

Example 1.10. Solve $z = px + qy + \sqrt{1 + p^2 + q^2}$.

Solution: The given equation is in Clairaut's form.

$\therefore$ The complete integral is $z = ax + by + \sqrt{1 + a^2 + b^2}$ (1)

To find singular integral: Partially differentiating (1) w.r.t a and b then equating to zero, we get

$$x + \frac{a}{\sqrt{1 + a^2 + b^2}} = 0 \Rightarrow x = \frac{-a}{\sqrt{1 + a^2 + b^2}} \tag{2}$$

$$y + \frac{b}{\sqrt{1 + a^2 + b^2}} = 0 \Rightarrow y = \frac{-b}{\sqrt{1 + a^2 + b^2}} \tag{3}$$

From (2) and (3), we get

$$x^2 + y^2 = \frac{a^2 + b^2}{1 + a^2 + b^2}$$

$$1 - (x^2 + y^2) = 1 - \frac{a^2 + b^2}{1 + a^2 + b^2}$$

$$1 - x^2 - y^2 = \frac{1}{1 + a^2 + b^2}$$

$$\sqrt{1 + a^2 + b^2} = \frac{1}{\sqrt{1 - x^2 - y^2}} \tag{4}$$

From (2) and (3), we get

$$a = \frac{-x}{\sqrt{1 - x^2 - y^2}} \tag{5}$$

and

$$b = \frac{-y}{\sqrt{1 - x^2 - y^2}} \tag{6}$$

Substituting (4), (5) and (6) in (1), we get

$$z = \frac{-x^2}{\sqrt{1-x^2-y^2}} - \frac{y^2}{\sqrt{1-x^2-y^2}} + \frac{1}{\sqrt{1-x^2-y^2}}$$
$$z = \sqrt{1-x^2-y^2}$$
$$z^2 = 1-x^2-y^2$$
$$\therefore x^2+y^2+z^2 = 1$$

which is the singular integral.

The general integral is obtained as first example of this type.

1.2.4 Examples of Lagrange's Linear Equations by Method of multipliers

Example 1.11. Solve $(y-z)p+(z-x)q = x-y$.

Solution: Given $(y-z)p+(z-x)q = x-y$

The subsidiary equations are $\dfrac{dx}{y-z} = \dfrac{dy}{z-x} = \dfrac{dz}{x-y}$ (1)

Each fraction of (1) $= \dfrac{dx+dy+dz}{y-z+z-x+x-y} = \dfrac{dx+dy+dz}{0}$

$$\Rightarrow dx+dy+dz = 0$$

Integrating, $x+y+z = a$

Using x, y, z as multipliers,

each fraction of (1) $= \dfrac{xdx+ydy+zdz}{x(y-z)+y(z-x)+z(x-y)} = \dfrac{xdx+ydy+zdz}{0}$

$$\Rightarrow xdx+ydy+zdz = 0$$

Integrating, $\dfrac{x^2}{2}+\dfrac{y^2}{2}+\dfrac{z^2}{2} = c$

i.e., $x^2+y^2+z^2 = b$

$\therefore$ The general solution is $f\left(x+y+z, x^2+y^2+z^2\right) = 0$.

Example 1.12. Solve $(mz-ny)\dfrac{\partial z}{\partial x}+(nx-lz)\dfrac{\partial z}{\partial y} = ly-mx$. **[UQ]**

Solution: Given $(mz-ny)p+(nx-lz)q = ly-mx$

The subsidiary equations are $\dfrac{dx}{mz-ny} = \dfrac{dy}{nx-lz} = \dfrac{dz}{ly-mx}$ (1)

Using x, y, z as multipliers, each fraction of (1)=

$$\frac{xdx+ydy+zdz}{x(mz-ny)+y(nx-lz)+z(ly-mx)} = \frac{xdx+ydy+zdz}{0}$$
$$\Rightarrow xdx+ydy+zdz = 0$$

Integrating, we get $\dfrac{x^2}{2}+\dfrac{y^2}{2}+\dfrac{z^2}{2} = c_1$

i.e., $x^2+y^2+z^2 = a$

Using l, m, n as multipliers, each fraction of (1) =

$$\frac{ldx + mdy + ndz}{l\,(mz - ny) + m\,(nx - lz) + n\,(ly - mx)} = \frac{ldx + mdy + ndz}{0}$$
$$ldx + mdy + ndz = 0$$

Integrating, $lx + my + nz = b$

$\therefore$ The general solution is $\phi(x^2 + y^2 + z^2, lx + my + nz) = 0$.

Example 1.13. Solve $x\left(y^2 - z^2\right)p + y\left(z^2 - x^2\right)q = z\left(x^2 - y^2\right)$. **[UQ]**

Solution: Given $x\left(y^2 - z^2\right)p + y\left(z^2 - x^2\right)q = z\left(x^2 - y^2\right)$

The subsidiary equations are

$$\frac{dx}{x\,(y^2 - z^2)} = \frac{dy}{y\,(z^2 - x^2)} = \frac{dz}{z\,(x^2 - y^2)}$$

Using x, y, z as multipliers, each fraction of (1)=

$$\frac{xdx + ydy + zdz}{x^2\,(y^2 - z^2) + y^2\,(z^2 - x^2) + z^2\,(x^2 - y^2)} = \frac{xdx + ydy + zdz}{0}$$
$$\Rightarrow xdx + ydy + zdz = 0$$

Integrating, we get $\dfrac{x^2}{2} + \dfrac{y^2}{2} + \dfrac{z^2}{2} = c_1$

i.e., $x^2 + y^2 + z^2 = a$

Using $\dfrac{1}{x}, \dfrac{1}{y}, \dfrac{1}{z}$ as multipliers, each fraction of (1) =

$$\frac{\frac{l}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{(y^2 - z^2) + (z^2 - x^2) + (x^2 - y^2)} = \frac{\frac{l}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{0}$$
$$\frac{l}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz = 0$$

Integrating, $\log x + \log y + \log z = \log b$

i.e., $xyz = b$

$\therefore$ The general solution is $\phi(x^2 + y^2 + z^2, xyz) = 0$.

Example 1.14. Solve $(y + z)p + (x + z)q = x + y$. **[UQ]**

Solution: Given $(y + z)p + (x + z)q = x + y$

The subsidiary equations are $\dfrac{dx}{y + z} = \dfrac{dy}{x + z} = \dfrac{dz}{x + y}$ (1)

Each fraction of (1) $= \dfrac{dx + dy + dz}{2\,(x + y + z)} = \dfrac{dx - dy}{-\,(x - y)} = \dfrac{dy - dz}{-\,(y - z)}$

$$\frac{d(x - y)}{(x - y)} = \frac{d(y - z)}{(y - z)}$$

Integrating, we get $\log(x - y) = \log(y - z) + \log a$

i.e., $\dfrac{x - y}{y - z} = a$

Taking $\dfrac{dx+dy+dz}{2\,(x+y+z)} = \dfrac{dx-dy}{-\,(x-y)}$, we have

$$\frac{1}{2}\frac{d(x+y+z)}{(x+y+z)} = -\frac{d(x-y)}{(x-y)}$$

Integrating, we get $\dfrac{1}{2}\log(x+y+z) = -\log(x-y) + \log b$

$$\sqrt{(x+y+z)}(x-y) = b$$

The general solution is $\phi\left(\dfrac{x-y}{y-z}, \sqrt{(x+y+z)}(x-y)\right) = 0.$

Example 1.15. Solve $(x^2-yz)p + (y^2-zx)q = z^2-xy$. **[UQ]**

Solution: The subsidiary equations are

$$\frac{dx}{(x^2-yz)} = \frac{dy}{(y^2-zx)} = \frac{dz}{z^2-xy} \tag{1}$$

Each fraction of (1) =

$$\frac{dx+dy+dz}{(x^2-yz)+(y^2-zx)+(z^2-xy)} = \frac{dx+dy+dz}{x^2+y^2+z^2-xy-yz-zx}$$

Using x, y, z as multipliers, each fraction of (1) =

$$\frac{xdx+ydy+zdz}{x^3+y^3+z^3-3xyz} = \frac{dx+dy+dz}{x^2+y^2+z^2-xy-yz-zx}$$

$$\frac{xdx+ydy+zdz}{(x+y+z)\,(x^2+y^2+z^2-xy-yz-zx)} = \frac{dx+dy+dz}{x^2+y^2+z^2-xy-yz-zx}$$

$$\frac{xdx+ydy+zdz}{x+y+z} = \frac{dx+dy+dz}{1}$$

$$xdx+ydy+zdz = (x+y+z)d(x+y+z)$$

Integrating, $\dfrac{x^2}{2}+\dfrac{y^2}{2}+\dfrac{z^2}{2} = \dfrac{(x+y+z)^2}{2}+c$

$$xy+yz+zx = a$$

Each fraction of (1) $= \dfrac{dx-dy}{(x^2-yz)-(y^2-zx)} = \dfrac{dy-dz}{(y^2-zx)-(z^2-xy)}$

i.e., $\dfrac{dx-dy}{(x^2-y^2)+z\,(x-y)} = \dfrac{dy-dz}{(y^2-z^2)+x\,(y-z)}$

$$\frac{d\,(x-y)}{(x-y)\,(x+y+z)} = \frac{d\,(y-z)}{(y-z)\,(x+y+z)}$$

$$\frac{d\,(x-y)}{(x-y)} = \frac{d\,(y-z)}{(y-z)}$$

Integrating, $\log(x-y) = \log(y-z) + \log b$

$$\frac{x-y}{y-z} = b$$

$\therefore$ The general solution is $\phi\left(xy+yz+zx, \dfrac{x-y}{y-z}\right) = 0.$

Example 1.16. Solve $y^2p - xyq = x(z-2y)$. **[UQ]**

Solution: The subsidiary equations are

$$p - xyq = x(z - 2y) \tag{1}$$

Taking $\dfrac{dx}{y^2} = \dfrac{dy}{-xy}$, we have

$$\frac{dx}{y} = \frac{dy}{-x}$$
$$xdx = -ydy$$

Integrating, $\dfrac{x^2}{2} = -\dfrac{y^2}{2} + c$

i.e., $x^2 + y^2 = a$

Taking $\dfrac{dy}{-xy} = \dfrac{dz}{x(z-2y)}$ we have

$$\frac{dy}{-y} = \frac{dz}{(z-2y)} \quad \Rightarrow (z-2y)dy = -ydz$$
$$\text{i.e., } ydz + zdy - 2ydy = 0 \quad \Rightarrow d(yz) - 2ydy = 0$$

Integrating, $yz - y^2 = b$

$\therefore$ The general solution is $\phi\left(x^2 + y^2, yz - y^2\right) = 0$.

Example 1.17. Solve $(x^2 - y^2 - z^2)p + 2xyq - 2xz = 0$. **[UQ]**

Solution: The subsidiary equations are $\dfrac{dx}{x^2 - y^2 - z^2} = \dfrac{dy}{2xy} = \dfrac{dz}{2xz}$ (1)

Taking $\dfrac{dy}{2xy} = \dfrac{dz}{2xz}$, we have

$$\frac{dy}{y} = \frac{dz}{z}$$
$$\log y = \log z + \log a$$
$$\text{i.e., } \frac{y}{z} = a$$

Using x, y, z as multipliers

Each fraction of (1) =

$$\frac{xdx + ydy + zdz}{x(x^2 - y^2 - z^2) + 2xy^2 + 2xz^2} = \frac{xdx + ydy + zdz}{x^3 + xy^2 + xz^2}$$
$$= \frac{xdx + ydy + zdz}{x(x^2 + y^2 + z^2)}$$

Taking $\dfrac{dy}{2xy} = \dfrac{xdx + ydy + zdz}{x(x^2 + y^2 + z^2)} \Rightarrow \quad \dfrac{dy}{y} = \dfrac{2(xdx + ydy + zdz)}{(x^2 + y^2 + z^2)}$

$$\frac{dy}{y} = \frac{d\left(x^2 + y^2 + z^2\right)}{(x^2 + y^2 + z^2)} \Rightarrow \quad \log y = \log(x^2 + y^2 + z^2) + \log b$$

i.e., $\dfrac{y}{x^2 + y^2 + z^2} = b$

$\therefore$ The general solution is $\phi\left(\dfrac{y}{z}, \dfrac{y}{x^2 + y^2 + z^2}\right) = 0$.

Example 1.18. Solve $(x-2z)p+(2z-y)q=y-x$. **[UQ]**

Solution: The subsidiary equations are $\dfrac{dx}{x-2z}=\dfrac{dy}{2z-y}=\dfrac{dz}{y-x}$ (1)

Each fraction of (1) $=\dfrac{dx+dy+dz}{x-2z+2z-y+y-x}=\dfrac{dx+dy+dz}{0}$

$$\text{i.e., } dx+dy+dz=0$$

$$\text{i.e., } x+y+z=a$$

Each fraction of (1) $=\dfrac{ydx+xdy}{xy-2yz+2xz-xy}=\dfrac{ydx+xdy}{2z(x-y)}$

Taking $\dfrac{ydx+xdy}{2z\,(x-y)}=\dfrac{dz}{y-x}$, we have

$$ydx+xdy=-2zdz$$

$$\text{i.e., } d(xy)=-2zdz$$

Integrating, $xy=-z^2+b$

$xy+z^2=b$

The general solution is $\phi\left(x+y+z,xy+z^2\right)=0$.

Example 1.19. Solve $(z^2-2yz-y^2)p+(xy+zx)q=xy-zx$. **[UQ]**

Solution: The subsidiary equations are

$$\frac{dx}{z^2-2yz-y^2}=\frac{dy}{xy+zx}=\frac{dz}{xy-zx} \tag{1}$$

Using x,y,z as multipliers each fraction of (1) $=\dfrac{xdx+ydy+zdz}{x\left(z^2-2yz-y^2\right)+y\left(xy+zx\right)+z\left(xy-zx\right)}=\dfrac{xdx+ydy+zdz}{0}$
we have

$$xdx+ydy+zdz=0$$

Integrating, we get $\dfrac{x^2}{2}+\dfrac{y^2}{2}+\dfrac{z^2}{2}=c_1$

$$x^2+y^2+z^2=a$$

Taking $\dfrac{dy}{xy+zx}=\dfrac{dz}{xy-zx}$, we have

$$\frac{dy}{y+z}=\frac{dz}{y-z}$$

$$\text{i.e., } (y-z)dy=(y+z)dz$$

$$ydy-zdy-ydz-zdz=0$$

$$ydy-(ydz+zdy)-zdz=0$$

$$ydy-d(yz)-zdz=0$$

Integrating, $\dfrac{y^2}{2} - yz - \dfrac{z^2}{2} = c_2$

i.e., $y^2 - z^2 - 2yz = b$

The general solution is $\phi\left(x^2 + y^2 + z^2, y^2 - z^2 - 2yz\right) = 0$.

Example 1.20. Solve $pz - qz = z^2 + (x+y)^2$.

Solution: The subsidiary equations are $\dfrac{dx}{z} = \dfrac{dy}{-z} = \dfrac{dz}{z^2 + (x+y)^2}$ (1)

Taking $\dfrac{dx}{z} = \dfrac{dy}{-z}$, we have $dx = -dy$

Integrating, $x + y = a$

Taking $\dfrac{dx}{z} = \dfrac{dz}{z^2 + (x+y)^2}$, we have

$$\frac{dx}{z} = \frac{dz}{z^2 + a^2} \qquad (\because x + y = a)$$

$$\text{i.e., } dx = \frac{zdz}{z^2 + a^2}$$

$$\text{Integrating, } x = \frac{1}{2}\log(z^2 + a^2) + c$$

$$2x = \log(z^2 + a^2) + 2c$$

i.e., $2x - \log\left[z^2 + (x+y)^2\right] = b$

The general solution is $\phi\left\{x + y, 2x - \log\left[z^2 + (x+y)^2\right]\right\} = 0$

1.2.5 Examples of Homogeneous Linear P.D.E.

Example 1.21. Solve the equation $\left(D^2 + 3DD' + 2D'^2\right) z = e^x \cosh y$

Solution: The auxillary equation is

$$m^2 + 3m + 2 = 0 \Rightarrow (m^2 - 1)(m^2 + 1) = 0$$

$$\text{i.e., } m = -1, -2$$

$\therefore$ C.F. is $f_1(y - x) + f_2(y - 2x)$

$$\begin{aligned}
\text{P.I. } &= \frac{1}{D^2 + 3DD' + 2D'^2} e^x \cosh y = \frac{1}{D^2 + 3DD' + 2D'^2} e^x \left(\frac{e^y + e^{-y}}{2}\right) \\
&= \frac{1}{2}\left[\frac{1}{D^2 + 3DD' + 2D'^2} e^{x+y} + \frac{1}{D^2 + 3DD' + 2D'^2} e^{x-y}\right] \\
&= \frac{1}{2}\left[\frac{1}{6} e^{x+y} + \frac{1}{0} e^{x-y}\right] \\
\text{P.I. } &= \frac{1}{2}\left[\frac{e^{x+y}}{6} + x\frac{1}{2D + 3D'} e^{x-y}\right] = \frac{1}{2}\left[\frac{e^{x+y}}{6} + x\frac{1}{-1} e^{x-y}\right] \\
\therefore \text{P.I. } &= \frac{1}{2}\left[\frac{e^{x+y}}{6} - xe^{x-y}\right]
\end{aligned}$$

$\therefore$ The complete solution $z = $ C.F. + P.I.

Example 1.22. **Solve** $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial y^2} = \cos 2x \cos 3y$

Solution: Given $\left(D^2 + D'^2\right) z = \cos 2x \cos 3y$

The auxillary equation is

$$m^2 + 1 = 0 \Rightarrow m^2 = -1$$

$$m = i, -i$$

$\therefore$ C.F. is $f_1(y + ix) + f_2(y - ix)$

$$\begin{aligned}
\text{P.I. } &= \frac{1}{D^2 + D'^2} \cos 2x \cos 3y = \frac{1}{D^2 + D'^2} \frac{1}{2} \left(\cos(2x + 3y) + \cos(2x - 3y)\right) \\
&= \frac{1}{2} \left(\frac{1}{D^2 + D'^2} \cos(2x + 3y) + \frac{1}{D^2 + D'^2} \cos(2x - 3y)\right) \\
&= \frac{1}{2} \left(\frac{1}{-4 - 9} \cos(2x + 3y) + \frac{1}{-4 - 9} \cos(2x - 3y)\right) \\
&= -\frac{1}{26} \left(\cos(2x + 3y) + \cos(2x - 3y)\right) = -\frac{1}{26} \left(2 \cos\left(\frac{4x}{2}\right) + \cos\left(\frac{6y}{2}\right)\right) \\
&= -\frac{1}{13} \left(\cos 2x cos 3y\right)
\end{aligned}$$

$\therefore$ The complete solution $z =$ C.F. + P.I.

Example 1.23. **Solve** $\left(D^2 + 3DD' + 2D'^2\right) z = x^2 y^2$

Solution: The auxillary equation is

$$m^2 + 3m + 2 = 0 \Rightarrow (m + 1)(m + 2) = 0$$

$$\text{i.e., } m = -1, -2$$

$\therefore$ C.F. $= f_1(y - x) + f_2(y - 2x)$

$$\begin{aligned}
\text{P.I. } &= \frac{1}{D^2 + 3DD' + 2D'^2} x^2 y^2 = \frac{1}{D^2 \left[1 + \frac{3D'}{D} + \frac{2D'^2}{D^2}\right]} x^2 y^2 \\
&= \frac{1}{D^2} \left[1 + \left(\frac{3D'}{D} + \frac{2D'^2}{D^2}\right)\right]^{-1} x^2 y^2 \\
&= \frac{1}{D^2} \left[1 - \left(\frac{3D'}{D} + \frac{2D'^2}{D^2}\right) + \left(\frac{3D'}{D} + \frac{2D'^2}{D^2}\right)^2 - \dots\right] x^2 y^2 \\
&= \frac{1}{D^2} \left[1 - \frac{3D'}{D} - \frac{2D'^2}{D^2} + \frac{9D'^2}{D^2}\right] x^2 y^2 = \frac{1}{D^2} \left[1 - \frac{3D'}{D} + \frac{7D'^2}{D^2}\right] x^2 y^2 \\
&= \frac{1}{D^2} \left[x^2 y^2 - \frac{3}{D}(2x^2 y) + \frac{7}{D^2}(2x^2)\right] = \frac{1}{D^2}(x^2 y^2) - \frac{1}{D^3}(6x^2 y) + \frac{1}{D^4}(14x^2) \\
&= \frac{1}{12} x^4 y^2 - \frac{1}{10} x^5 y + \frac{7}{180} x^6 = \frac{1}{2} \left[\frac{x}{5} \left[2 \cos(x + y) + 3 \cos(x + y)\right] - \frac{1}{6} \sin(x - y)\right]
\end{aligned}$$

$$\text{P.I. } = \frac{1}{2} \left[x \cos(x + y) - \frac{1}{6} \sin(x - y)\right] \qquad \left(\because \frac{1}{D^n} x^m = \frac{x^{m+n}}{(m + 1)(m + 1) \dots (m + n)}\right)$$

$\therefore$ The complete solution $z =$ C.F. + P.I.

Example 1.24. Solve $\left(D^2 + 6DD' + 5D'^2\right) z = xy^4$

Solution: The auxillary equation is

$$m^2 + 6m + 5 = 0 \Rightarrow (m+1)(m+5) = 0$$
$$\text{i.e., } m = -1, -5$$

$$\therefore \text{ C.F. } = f_1(y - x) + f_2(y - 5x)$$

$$\begin{aligned}
\text{P.I. } &= \frac{1}{D^2 + 6DD' + 5D'^2} xy^4 = \frac{1}{5D'^2\left[\frac{D^2}{5D'^2} + \frac{6}{5}\frac{D}{D'} + 1\right]} xy^4 \\
&= \frac{1}{5D'^2}\left[1 + \left(\frac{6D}{5D'} + \frac{D^2}{5D'^2}\right)\right]^{-1} xy^4 \\
&= \frac{1}{5D'^2}\left[1 - \left(\frac{6D}{5D'} + \frac{D^2}{5D'^2}\right) + \left(\frac{6D}{5D'} + \frac{D^2}{5D'^2}\right)^2 - \ldots\right] xy^4 \\
&= \frac{1}{5D'^2}\left[1 - \frac{6D}{5D'}\right] xy^4 = \frac{1}{5D'^2}\left[xy^4 - \frac{6y^4}{5D'}\right] \\
&= \frac{1}{5D'^2}(xy^4) - \frac{6}{25D'^3}(y^4) = \frac{xy^6}{150} - \frac{y^7}{875}
\end{aligned}$$

$\therefore$ The complete solution $z =$ C.F. + P.I.

Example 1.25. Solve $\left(D^2 - 4DD' + 4D'^2\right) z = e^{x-2y} \cos(2x - y)$

Solution: The auxillary equation is

$$m^2 - 4m + 4 = 0$$
$$\text{i.e., } m = 2, 2$$

$$\therefore \text{ C.F. } = f_1(y+2x) + xf_2(y+2x)$$

$$\begin{aligned}
\text{P.I. } &= \frac{1}{D^2-4DD'+4D'^2} e^{x-2y}\cos(2x-y) \\
&= e^{x-2y}\frac{1}{(D+1)^2-4(D+1)(D'-2)+4(D'-2)^2}\cos(2x-y) \\
&= e^{x-2y}\frac{1}{D^2+10D-4DD'-20D'+4D'^2+25}\cos(2x-y) \\
&= e^{x-2y}\frac{1}{-4+10D-8-20D'-4+25}\cos(2x-y) \\
&= e^{x-2y}\frac{1}{10D-20D'+9}\cos(2x-y) = e^{x-2y}\frac{((10D-20D')-9)\cos(2x-y)}{(10D-20D')^2-81} \\
&= e^{x-2y}\frac{(10D-20D'-9)\cos(2x-y)}{100D^2-400DD'+400D'^2-81} \\
&= e^{x-2y}\frac{(10D-20D'-9)}{-400-800-400-81}\cos(2x-y) \\
&= -\frac{e^{x-2y}}{1681}\left(-20\sin(2x-y)-20\sin(2x-y)-9\cos(2x-y)\right) \\
\text{P.I. } &= \frac{1}{1681}e^{x-2y}\left[40\sin(2x-y)+9\cos(2x-y)\right]
\end{aligned}$$

$\therefore$ The complete solution $z =$ C.F. + P.I.

Example 1.26. **Solve** $\boldsymbol{\left(D^2 - DD' - 2D'^2\right) z = (y-1)e^x}$

Solution: The auxillary equation is

$$m^2 - m - 2 = 0 \Rightarrow (m-2)(m+1) = 0$$
$$\text{i.e., } m = 2, -1$$

$$\therefore \text{ C.F. } = f_1(y+2x) + f_2(y-x)$$

$$\begin{aligned}
\text{P.I. } &= \frac{1}{D^2-DD'-2D'^2}(y-1)e^x = e^x\frac{1}{(D+1)^2-(D+1)D'-2D'^2}(y-1) \\
&= e^x\frac{1}{D^2+2D+1-DD'-D'-2D'^2}(y-1) \\
&= e^x\left[1+\left(D^2+2D-DD'-D'-2D'^2\right)\right]^{-1}(y-1) \\
&= e^x\left[1-\left(D^2+2D-DD'-D'-2D'^2\right)+\ldots\right](y-1) \\
&= e^x\left[1+DD'+D'\right](y-1) = e^x\left[y-1+D(1)+1\right] = e^xy
\end{aligned}$$

$\therefore$ The complete solution $z =$ C.F. + P.I.
i.e., $z = f_1(y+2x) + f_2(y-x) + ye^x$.

Example 1.27. **Solve** $\boldsymbol{\left(D^2 + DD' - 6D'^2\right) z = y\cos x}$

Solution: The auxillary equation is

$$m^2 + m - 6 = 0 \Rightarrow (m+3)(m-2) = 0$$
$$\text{i.e., } m = 2, -3$$

$\therefore$ C.F. $= f_1(y+2x) + f_2(y-3x)$

$$\begin{aligned}
\text{P.I.} &= \frac{1}{D^2+DD'-6D'^2} y\cos x = \frac{1}{(D-2D')(D+3D')} y\cos x \\
&= \frac{1}{D-2D'} \int (c+3x)\cos x dx \qquad \text{where } y = c+3x \\
&= \frac{1}{D-2D'} [(c+3x)\sin x + 3\cos x] \\
&= \frac{1}{D-2D'} [(y-3x+3x)\sin x + 3\cos x] \qquad \because c = y-3x \\
&= \frac{1}{D-2D'} (y\sin x + 3\cos x) \\
&= \int [(c_1-2x)\sin x + 3\cos x] dx \qquad \text{where } y = c_1 - 2x \\
&= [(c_1-2x)(-\cos x) - (-2)(-\sin x)] + 3\sin x \\
&= -(c_1-2x)\cos x - 2\sin x + 3\sin x = -y\cos x + \sin x \\
\text{P.I.} &= \sin x - y\cos x
\end{aligned}$$

$\therefore$ The complete solution $z =$ C.F. + P.I.

1.3 Assignment I[Partial Differential Equations]

1. (i) Form a partial differential equation by eliminating a and b from the expression $(x-a)^2 + (y-b)^2 + z^2 = c^2$.

   (ii) Find the partial differential equation of all planes which are at a constant distance a_1 from the origin.

2. (i) Form the partial differential equation by eliminating arbitrary function from $z = xf(2x+y) + g(2x+y)$.

   (ii) Form the PDE by eliminating the arbitrary function ϕ from $\phi\left(x^2+y^2+z^2, ax+by+cz\right) = 0$.

3. Solve the following:

   (i) $p(1+q)z = qz$

   (ii) $x^2p + y^2q = 0$ (iii) $xp + yq = 0$

4. Solve the following:

   (i) $z = px + qy + \log pq$ (ii) $z = px + qy + pq$

   (iii) $z = px + qy + \sqrt{1+p^2+q^2}$ (iv) $z = px + qy + p^2q^2$

5. Solve the following:

   (i) $p(1+q) = qz$ (ii) $x^4p^2 - yzq = z^2$

(iii) $x^4p^2 + y^2qz = 2z^2$

(iv) $z^2\left(p^2x^2 + q^2\right) = 1$

6. Solve the following:

(i) $p^2 + q^2 = x^2 + y^2$

(ii) $z^2\left(p^2 + q^2\right) = x^2 + y^2$

(iii) $4z^2q^2 = y + 2zp - x$

(iv) $p^2 + q^2 = z^2\left(x^2 + y^2\right)$

7. Solve the following:

(i) $x^2(y-z)p = y^2(z-x)q = z^2(x-y)$

(ii) $\left(x^2 - yz\right)p + \left(y^2 - zx\right)q = z^2 - xy$

(iii) $(y - xz)p + (yz - x)q = (x+y)(x-y)$

(iv) $\left(x^2 - y^2 - z^2\right)p + 2xyq = 2zx$

(v) $x\left(z^2 - y^2\right)p + y\left(x^2 - z^2\right)q = z\left(y^2 - x^2\right)$

(vi) $(3z - 4y)p + (4x - 2z)q = (2y - 3x)$

(vii) $x\left(y^2 + z\right)p + y\left(x^2 + z\right)q = z\left(x^2 - y^2\right)$

8. Solve the following:

(i) $\left(D^3 + D^2D' - DD'^2 - D'^3\right)z = e^{2x+y} + \cos(x+y)$

(ii) $\dfrac{\partial^3 z}{\partial x^3} - 2\dfrac{\partial^3 z}{\partial x^2 \partial y} = e^{x+2y} + 4\sin(x+y)$

(iii) $\left(D^3 - 7DD'^2 - 6D'^3\right)z = \cos(x+2y) + 4$

(iv) $\left(D^2 + 2DD' + D'^2\right)z = \sinh(x+y) + e^{x+2y}$

(v) $\left(D^2 + 2DD' + D'^2\right)z = x^2y + e^{x-y}$

(vi) $\left(D^2 - DD' - 2D'^2\right)z = 2x + 3y + e^{3x+4y}$

(vii) $\left(D^2 - D'^2\right)z = e^{x-y}\sin(x+2y)$

(viii) $\left(D^2 - 5DD' + 6D'^2\right)z = y\sin x$

(ix) $r + s - 6t = y\cos x$

9. Solve the following:

(i) $\left(D^2 - D'^2 - 3D + 3D'\right)z = xy + 7$

(ii) $\left(D^2 + 2DD' - 2D - 2D'^2\right)z = \sin(x+2y)$

(iii) $\left(2D^2 - DD' - D'^2 + 6D + 3D'\right)z = xe^y$

(iv) $\left(D^2 + 2DD' + D'^2 - 2D - 2D'\right)z = e^{3x+y} + 4$

(v) $\left(D^2 + D'^2 + 2DD' + 2D + 2D' + 1\right)z = e^{2x+y}$

2 Fourier Series(F.S.)

2.1 Part-A

1. Find the constant term in the Fourier series corresponding to $f(x) = \cos^2 x$ expressed in the interval $(-\pi, \pi)$.

   Soln :

$$\text{Given} f(x) = \cos^2 x = \frac{1 + \cos 2x}{2}$$
$$f(-x) = \cos^2(-x) = \cos^2 x = f(x)$$

Therefore the function is even.

Fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

Here the constant term is a_0

$$\begin{aligned}
\text{i.e.,}\quad a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} \cos^2 x \quad dx \\
&= \frac{2}{\pi}\int_0^{\pi} \cos^2 x dx \\
&= \frac{2}{\pi}\int_0^{\pi} \frac{1 + \cos 2x}{2} dx \\
&= \frac{1}{\pi}\left[x + \frac{\sin 2x}{2}\right]_{x=0}^{x=\pi} \\
&= \frac{1}{\pi}\left[\left(\pi + \frac{\sin 2\pi}{2}\right) - (0 + 0)\right] \\
&= 1.
\end{aligned}$$

2. If $f(x) = x^2 + x$ is expanded as a Fourier series in the interval $(-2, 2)$ to which value this series converges at $x = 2$?

   Soln : Here given $x = 2$ is a point of discontinuity in the extremum.

$$\begin{aligned}
f(x = 2) &= \frac{f(-2) + f(2)}{2} \\
&= \frac{\left(x^2 + x\right)_{x=-2} + \left(x^2 + x\right)_{x=2}}{2} \\
&= \frac{(4 - 2) + (4 + 2)}{2} = 4
\end{aligned}$$

3. If the Fourier series of $f(x) = x^2$ in $-\pi < x < \pi$ is equal to $f(x) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n \cos nx}{n^2}$, prove that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

   Soln : Given $f(x) = x^2$, $f(x) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n \cos nx}{n^2}$, (1)

   Put $x = \pi$ (a point of discontinuity in the extremum)

$$\begin{aligned}
f(x = \pi) &= \frac{f(x = -\pi) + f(x = \pi)}{2} = \pi^2 \\
&= \frac{\pi^2 + \pi^2}{2} \\
&= \pi^2
\end{aligned}$$

$$\therefore (1) \Rightarrow \pi^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n \cos n\pi}{n^2}$$
$$\Rightarrow \pi^2 - \frac{\pi^2}{3} = 4\sum_{n=1}^{\infty} \frac{(-1)^n (-1)^n}{n^2}$$
$$\Rightarrow \frac{2\pi^2}{3} = 4\sum_{n=1}^{\infty} \frac{(-1)^{2n}}{n^2}$$
$$\Rightarrow \frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

Hence Proved.

4. If $f(x) = |\cos x|$ is expanded in a Fourier series in the interval of $-\pi < x < \pi$, find a_0.

Soln : Given $f(x) = |\cos x|$

$$f(-x) = |-\cos x| = |\cos x| = f(x)$$

Therefore the function s even.

$\Rightarrow b_n = 0$.

$\therefore$ Fourier series in $-\pi < x < \pi$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

To find a_0,

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$
$$= \frac{2}{\pi}\int_0^{\pi} |\cos x|\,dx$$
$$= \frac{2}{\pi}\left\{\int_0^{\pi/2} \cos x dx + \int_{\pi/2}^{\pi} (-\cos x)\,dx\right\}$$
$$= \frac{2}{\pi}\left\{[\sin x]_0^{\pi/2} + [-\sin x]_{\pi/2}^{\pi}\right\}$$
$$= \frac{2}{\pi}[(1-0) + (-0+1)]$$
$$= \frac{4}{\pi}$$

5. State Dirichlet's condition in Fourier series and examine whether $\frac{1}{1-x}$ can be expanded in a Fourier series in any interval including the point $x = 1$.

Soln : The expansion of a function $f(x)$ in a trigonometry is possible if it satisfies the following condition in any interval.

a) $f(x)$ is well defined and single valued function.

b) $f(x)$ has finite number of points of continuity.

c) $f(x)$ has only finite number of maxima and minima.

$f(x)$ is not well defined at $x = 1$, since $f(x = 1) = \lim_{x\to 1}\left(\frac{1}{1-x}\right) = \infty$.
So $f(x)$ can not be expanded in a Fourier series in any interval including the point $x = 1$.

6. Find the constant term in the Fourier series corresponding to $f(x) = 2x - x^2$ expressed in the interval $(-2, 2)$.

Soln : Given $f(x) = 2x - x^2$

$$f(-x) = 2(-x) - (-x)^2 = -2x - x^2 \neq \begin{cases} f(x) \\ -f(x) \end{cases}$$

$\therefore$ The function is neither even nor odd.

To find constant a_0,

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$
$$= \frac{1}{\pi}\int_{-\pi}^{\pi} \left(2x - x^2\right)dx$$
$$= \frac{1}{\pi}\left[\frac{2x^2}{2} - \frac{x^3}{3}\right]_{-\pi}^{\pi}$$
$$= \frac{1}{\pi}\left[\left(\pi^2 - \pi^2\right) - \frac{1}{3}\left(\pi^3 + \pi^3\right)\right]$$
$$= -\frac{2\pi^3}{3}$$

7. The function $f(x) = \{-\pi, -\pi < x < 0 \& x, 0 < x < \pi$ is expanded as a Fourier series of period 2π. What is the sum of the series at $x = 0$ and $x = \frac{\pi}{2}$.

Soln : Given

$$f(x) = \begin{cases} -\pi, & -\pi < x < 0 \\ x, & 0 < x < \pi \end{cases}$$

Here $x = 0$ is a point of discontinuity in middle.

Sum of the series at $(x = 0)$ is

$$f(x=0) = \frac{f(0-) + f(0+)}{2}$$
$$= \frac{(-\pi) + 0}{2} = \frac{-\pi}{2}.$$

Here $x = \pi/2$ is a point of continuity in second interval.

Sum of the series at $(x = \pi/2)$ is

$$f(x = \pi/2) = \Big[x\Big]_{\frac{\pi}{2}} = \frac{\pi}{2}$$

8. Obtain the constant term in the Fourier series corresponding to $f(x) = \sqrt{(1-\cos x)}$ in the integral of $(-\pi, \pi)$.

Soln : Gn. $f(x) = \sqrt{(1-\cos x)}$

$$f(-x) = \sqrt{[1 - \cos(-x)]}$$
$$= \sqrt{1-\cos x} = f(x)$$

Therefore the function is even.

To find the constant term a_0,

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$
$$= \frac{2}{\pi}\int_{0}^{\pi} f(x)\,dx$$
$$= \frac{2}{\pi}\int_{0}^{\pi} \sqrt{1-\cos x}\,dx$$
$$= \frac{2}{\pi}\int_{0}^{\pi} \sqrt{2\sin^2\left(\frac{x}{2}\right)}\,dx$$
$$= \frac{2}{\pi}\int_{0}^{\pi} \sqrt{2}\sin\left(\frac{x}{2}\right)dx$$
$$= \left(\frac{2\sqrt{2}}{\pi}\right)\left[\frac{-\cos\left(\frac{x}{2}\right)}{\frac{1}{2}}\right]_0^{\pi}$$
$$= \frac{4\sqrt{2}}{\pi}\left[-\cos\frac{\pi}{2} + \cos 0\right]$$
$$= \frac{4\sqrt{2}}{\pi}[0+1]$$
$$= \frac{4\sqrt{2}}{\pi}$$

9. If the Fourier series for the function $f(x) = \begin{cases} 0, & 0 < x < \pi \\ \sin x, & \pi < x < 2\pi \end{cases}$ is $f(x) = \frac{-1}{\pi} + \frac{2}{\pi}\left[\frac{\cos 2x}{1.3} + \frac{\cos 4x}{3.5} + ...\right] + \frac{\sin x}{2}$. Deduce that $\frac{1}{1.3} - \frac{1}{3.5} + ... = \frac{\pi - 2}{4}$

Soln : Given $f(x) = \begin{cases} 0, & 0 < x < \pi \\ \sin x, & \pi < x < 2\pi \end{cases}$

Let $\frac{1}{1.3} - \frac{1}{3.5} + ... = \frac{\pi-2}{4}$ (1)

Put $x = \frac{\pi}{2}$ is a point of continuity,

$\therefore f\left(x = \frac{\pi}{2}\right) = 0.$

$$\therefore (1) \Rightarrow 0 = \frac{-1}{\pi} + \frac{2}{\pi}\left[-\frac{1}{1.3} + \frac{1}{3.5} - ...\right] + \frac{1}{2}$$
$$-\frac{1}{2} + \frac{1}{\pi} = \frac{2}{\pi}\left[-\frac{1}{1.3} + \frac{1}{3.5} - ...\right]$$
$$\frac{-\pi + 2}{2\pi} = -\frac{2}{\pi}\left[\frac{1}{1.3} - \frac{1}{3.5} + ...\right]$$
$$\left[\frac{\pi-2}{2\pi}\right]\frac{\pi}{2} = \left[\frac{1}{1.3} - \frac{1}{3.5} + ...\right]$$
$$\left[\frac{\pi-2}{4}\right] = \left[\frac{1}{1.3} - \frac{1}{3.5} + ...\right]$$

10. To what value , the half range sine series corresponding to $f(x) = x^2$ expressed in the interval (0, 2) converges at $x = 2$.

Soln : Given $f(x) = x^2$.

Here $x = 2$ is a point of discontinuity at extreme.

$$f(x=2) = \frac{f(x=-2)+f(x=2)}{2} = \frac{\left(x^2\right)_{x=-2}+\left(x^2\right)_{x=2}}{2} = \frac{(-2)^2+(2)^2}{2} = \frac{4+4}{2} = \frac{8}{2} = 4$$

11. If the Fourier series corresponding to $f(x) = x$ in the interval $(0, 2\pi)$ is $\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$ without finding the values of a_0, a_n, b_n find the value of $\frac{a_0^2}{2} + \sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right)$.

Soln : By Parseval's theorem

$$\frac{1}{\pi}\int_0^{2\pi}[f(x)]^2\,dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}\left(a_n^2+b_n^2\right)$$

$$\frac{a_0^2}{2} + \sum_{n=1}^{\infty}\left(a_n^2+b_n^2\right) = \frac{1}{\pi}\int_0^{2\pi}[f(x)]^2\,dx = \frac{1}{\pi}\int_0^{2\pi}[x]^2\,dx = \frac{1}{\pi}\left[\frac{x^3}{3}\right]_0^{2\pi} = \frac{1}{3\pi}8\pi^3 = \frac{8\pi^2}{3}$$

12. Find the half range sine series for $f(x) = e^x$ in $0 < x < 1$.

Soln : Given $f(x) = e^x$.

The half range Fourier sine series in $(0, \ell = 1)$ is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{\ell}\right)$$

where

$$b_n = \frac{2}{\ell}\int_0^{\ell} f(x)\sin\left(\frac{n\pi x}{\ell}\right)dx$$

here $\ell = 1$

$$\therefore b_n = 2\int_0^1 e^x \sin(n\pi x)\,dx$$

$$= 2\left\{\frac{e^x}{1^2+(n\pi)^2}\left[\sin(n\pi x) - n\pi\cos(n\pi x)\right]\right\}_{x=0}^{x=1}$$

$$= 2\left\{\frac{e^1}{1^2+(n\pi)^2}(\sin n\pi - n\pi\cos n\pi) - \frac{1}{1^2+(n\pi)^2}(-n\pi)\right\}$$

$$= 2\left\{\frac{e^1}{1^2+(n\pi)^2}\left[-n\pi(-1)^n\right] - \frac{1}{1^2+(n\pi)^2}\left[-n\pi\right]\right\}$$

$$= 2\left\{\frac{n\pi}{1^2+(n\pi)^2}\left[-e(-1)^n+1\right]\right\}$$

Fourier series is

$$f(x) = \sum_{n=1}^{\infty}\left\{\frac{2n\pi}{1^2+(n\pi)^2}\left[-e(-1)^n+1\right]\right\}\sin(n\pi x)$$

13. The cosine series for $f(x) = x\sin x$ for $0 < x < \pi$ is given as the

$$x\sin x = 1 - \frac{1}{2}\cos x - 2\sum_{n=2}^{\infty}\frac{(-1)^n}{n^2-1}\cos nx$$

and deduce that $1 + 2\left[\frac{1}{1.3} - \frac{1}{3.5} + ...\right] = \frac{\pi}{2}$.

Soln : Given

$$x\sin x = 1 - \frac{1}{2}\cos x - 2\sum_{n=2}^{\infty}\frac{(-1)^n}{n^2-1}\cos nx$$

Put $x = \frac{\pi}{2}$(point of continuity)

$$\frac{\pi}{2}\sin\frac{\pi}{2} = 1 - \frac{1}{2}\cos\frac{\pi}{2} - 2\sum_{n=2}^{\infty}\frac{(-1)^n}{n^2-1}\cos n\frac{\pi}{2}$$

$$\frac{\pi}{2} = 1 - 2\sum_{n=2}^{\infty}\frac{(-1)^n}{n^2-1}\cos n\frac{\pi}{2}$$

$$\frac{\pi}{2} = 1 - 2\sum_{n=2}^{\infty}\frac{(-1)^n}{(n+1)(n-1)}\cos n\frac{\pi}{2}$$

$$\frac{\pi}{2} = 1 - 2\left[\frac{1}{(3)(1)}(-1) + \frac{-1}{(4)(2)}(0) + \frac{1}{(5)(3)}(1) + \ldots\right]$$

$$\therefore \frac{\pi}{2} = 1 + 2\left[\frac{1}{1.3} - \frac{1}{3.5} + \ldots\right]$$

Hence deduced.

14. If $f(x) = \sin^2 x, -\pi < x < \pi$, then find $b_1^2 + b_2^2 + b_3^2 + \cdots$.

Soln : Given $f(x) = \sin^2 x, -\pi < x < \pi$

$$f(-x) = \sin^2(-x) = (-\sin x)^2 = \sin^2 x = f(x)$$

$\therefore$ The function is even.

This implies $b_n = 0$.

Therefore, $\sum_{n=1}^{\infty} b_n^2 = \sum_{n=1}^{\infty}(0)^2 = 0$

This implies $b_1^2 + b_2^2 + b_3^2 + \cdots = 0$

15. If $f(x) = x - x^2$ in $(-1, 1)$, find the RMS value of $f(x)$.

Soln : Given $f(x) = x - x^2$.

The RMS value in an integral $(-1, 1)$.

$$y = \sqrt{\frac{1}{2}\int_{-1}^{1}[f(x)]^2dx} = \sqrt{\frac{1}{2}\int_{-1}^{1}[x-x^2]^2dx}$$

$$= \sqrt{\frac{1}{2}\int_{-1}^{1}[x^2+x^4-2x^3]dx} = \sqrt{\frac{1}{2}\left[\frac{x^3}{3}+\frac{x^5}{5}-2\frac{x^4}{4}\right]_{-1}^{1}}$$

$$= \sqrt{\frac{1}{2}\left[\left(\frac{1}{3}+\frac{1}{5}-\frac{1}{2}\right)-\left(\frac{-1}{3}+\frac{-1}{5}-\frac{1}{2}\right)\right]}$$

$$= \sqrt{\frac{1}{2}\left(\frac{2}{3}+\frac{2}{5}\right)} = \sqrt{\left(\frac{1}{3}+\frac{1}{5}\right)} = \sqrt{\frac{8}{15}}$$

16. State whether true or false: Fourier series of period 2 for $x\sin x$ in $(-1, 1)$ contains only sine terms. Justify your answer.

Soln : False, the Fourier series doesn't contain sine terms. Since,

$$f(-x) = (-x)\sin(-x) = (-x)(-\sin x) = x\sin x = f(x)$$

Therefore the function is even.

$\therefore$ Fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=0}^{\infty} a_n \cos\left(\frac{n\pi x}{\ell}\right).$$

So, it contains only cosine terms in the series.

17. The Fourier series for $f(x) = x^2$ in $-\pi < x < \pi$ will contain only cosine terms. State whether true or false. Justify your answer.

Soln : True , the Fourier series contain only cosine terms.

Since, $f(-x) = (-x)^2 = x^2 = f(x)$

Therefore the function is even.

$\therefore$ Fourier series is $f(x) = \frac{a_0}{2} + \sum_{n=0}^{\infty} a_n \cos nx$

So, it contains only cosine terms in the series.

18. Find the Half range Fourier sine series of

(i) $f(x) = x$ in $(0, 2)$

(ii) $f(x) = 1, 0 < x < \pi$. (Apr '04)

Soln :

(i) Given $f(x) = x$ in $(0, 2)$.

Half range Fourier sine series in $(0, \ell = 2)$ is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{\ell} = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{2}$$

where

$$b_n = \frac{2}{\ell}\int_0^{\ell} f(x)\,\sin\frac{n\pi x}{\ell}dx$$
$$= \frac{2}{2}\int_0^{2} f(x)\sin\frac{n\pi x}{2}dx = \int_0^{2} x\,\sin\frac{n\pi x}{2}dx$$
$$= \left\{(x)\left(\frac{-\cos\frac{n\pi x}{2}}{\frac{n\pi}{2}}\right) - (1)\left(\frac{-\sin\frac{n\pi x}{2}}{\left(\frac{n\pi}{2}\right)^2}\right)\right\}_0^2$$
$$= \left\{\left[-2\frac{(-1)^n}{\frac{n\pi}{2}} + 0\right] - [0+0]\right\}$$
$$= -4\frac{(-1)^n}{n\pi}$$

$\therefore$ Half range Fourier sine series is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{\ell}$$
$$= -\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin\frac{n\pi x}{2}$$

(ii) Given $f(x) = 1$ in $(0, \pi)$.

Half range Fourier sine series in $(0, \ell = \pi)$ is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{\ell}$$
$$= \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{\pi}$$
$$= \sum_{n=1}^{\infty} b_n \sin nx$$

where

$$b_n = \frac{2}{\ell}\int_0^{\ell} f(x)\,\sin\frac{n\pi x}{\ell}dx$$
$$= \frac{2}{\pi}\int_0^{\pi} 1\sin nx\,dx$$
$$= \frac{2}{\pi}\int_0^{\pi} \sin nx\,dx$$
$$= \frac{2}{\pi}\left[\frac{-\cos nx}{n}\right]_0^{\pi}$$
$$= \frac{-2}{n\pi}\left\{\left[(-1)^n\right] - [1]\right\}$$
$$= \begin{cases} 0, & n \text{ is even} \\ \dfrac{4}{n\pi}, & n \text{ is odd}\end{cases}$$

Half range Fourier sine series is,

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{\ell}$$
$$= \frac{4}{\pi}\sum_{n=\text{odd}}^{\infty}\frac{1}{n}\sin nx$$
$$= \frac{4}{\pi}\sum_{n=1,3,5,\cdots}^{\infty}\frac{1}{n}\sin nx$$

19. Find the constant term in the Fourier cosine series corresponding to $f(x) = \dfrac{\pi - x^2}{2}$ expressed in the interval $(0, 3)$.

Soln : Given $f(x) = \dfrac{\pi - x^2}{2}$.

Fourier cosine series in $(0, \ell = 3)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=0}^{\infty} a_n \cos\frac{n\pi x}{\ell}$$

$$= \frac{a_0}{2} + \sum_{n=0}^{\infty} a_n \cos\frac{n\pi x}{3}$$

$$\text{where, } a_0 = \frac{2}{\ell}\int_0^{\ell} f(x)dx$$
$$= \frac{2}{3}\int_0^{3} \frac{(\pi - x^2)}{2}dx$$
$$= \frac{1}{3}\left[\pi x - \frac{x^3}{3}\right]_0^3$$
$$= \frac{1}{3}\left[(3\pi - 9) - (0 - 0)\right]$$
$$= \pi - 3$$

20. State Parseval's theorem on Fourier coefficient.

Soln : If the Fourier series corresponding to $f(x)$ converges uniformly to $f(x)$ in an integral $(-\pi, \pi)$ then,

$$\frac{1}{\pi}\int_{-\pi}^{\pi} [f(x)]^2\, dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right)$$

21. State Parseval's identity for Half range Fourier cosine series (HRFCS) and Half range Fourier sine series (HRFSS)

Soln : Parseval's identity for Half range Fourier cosine series, since $b_n = 0$,

$$\frac{2}{\pi}\int_0^{\pi} [f(x)]^2\, dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}\left(a_n^2\right)$$

For Half range Fourier sine series, $a_0 = 0, a_n = 0$

$$\frac{2}{\pi}\int_0^{\pi} [f(x)]^2\, dx = \sum_{n=1}^{\infty}\left(b_n^2\right)$$

22. Define root mean square value of $f(x)$.

Soln : The root mean value of the function $y = f(x)$ over the interval (a, b) and it is defined as,

$$\overline{y} = \sqrt{\frac{1}{(b-a)}\int_a^b [f(x)]^2 dx}$$

23. Define complex form of Fourier series.

Soln : Let $f(x)$ be a function defined in interval $(0, 2\pi)$ and satisfies the Dirichlet's conditions, then we have a Fourier series,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

If $\cos nx = \dfrac{e^{ix} + e^{-ix}}{2}$ & $\sin nx = \dfrac{e^{ix} - e^{-ix}}{2i}$, then

The complex form of Fourier series is,

$$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{inx}$$

$$\text{where } C_n = \frac{1}{2\pi}\int_0^{2\pi} f(x)\, e^{-inx}dx$$

24. What is Harmonic analysis?

Soln : The process of finding Fourier series for a function $f(x)$ given by numerical values is known as "Harmonic Analysis" Consider the Fourier series in an interval of $f(x)$ as,

$$f(x) = \frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + \cdots + b_1 \sin x + b_2 \sin 2x + \cdots$$

where,

$$a_0 = 2\,[\text{Mean value of } f(x)] = \frac{2}{n}\left[\sum f(x)\right]$$
$$a_n = 2\,[\text{Mean value of } f(x)\cos nx] = \frac{2}{n}\left[\sum f(x)\cos nx\right]$$
$$b_n = 2\,[\text{Mean value of } f(x)\sin nx] = \frac{2}{n}\left[\sum f(x)\sin nx\right]$$

25. If $x^2 = \dfrac{\pi^2}{3} + 4\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx$, show that $\dfrac{\pi^4}{90} = \dfrac{1}{1^4} + \dfrac{1}{2^4} + \dfrac{1}{3^4} + \cdots$ in $-\pi < x < \pi$.

Soln : Given

$$f(x)=x^2=\frac{\pi^2}{3}+4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx$$

Here $\frac{a_0}{2}=\frac{\pi^2}{3}\Rightarrow a_0=\frac{2\pi^2}{3},a_n=\frac{4\,(-1)^n}{n^2}$

By Parseval's identity,

$$\frac{1}{\pi}\int_{-\pi}^{\pi}[f(x)]^2\,dx=\frac{a_0^2}{2}+\sum_{n=1}^{\infty}\left(a_n^2\right)$$

$$\frac{1}{\pi}\int_{-\pi}^{\pi}\left[x^2\right]^2\,dx=\frac{\left(\frac{2\pi^2}{3}\right)^2}{2}+\sum_{n=1}^{\infty}\left(\frac{4\,(-1)^n}{n^2}\right)^2$$

$$\frac{2}{\pi}\int_{0}^{\pi}x^4\,dx=\frac{\left(\frac{4\pi^4}{9}\right)}{2}+\sum_{n=1}^{\infty}\left(\frac{16\,(-1)^{2n}}{n^4}\right)$$

$$\frac{2}{\pi}\left(\frac{x^5}{5}\right)_0^{\pi}=\frac{4\pi^4}{18}+16\sum_{n=1}^{\infty}\left(\frac{1}{n^4}\right)$$

$$\frac{2}{5\pi}\left(\pi^5\right)=\frac{2\pi^4}{9}+16\sum_{n=1}^{\infty}\left(\frac{1}{n^4}\right)$$

$$\frac{2}{5}\left(\pi^4\right)=\frac{2\pi^4}{9}+16\sum_{n=1}^{\infty}\left(\frac{1}{n^4}\right)$$

$$\frac{2}{5}\left(\pi^4\right)-\frac{2\pi^4}{9}=16\sum_{n=1}^{\infty}\left(\frac{1}{n^4}\right)$$

$$\frac{1}{5}\left(\pi^4\right)-\frac{\pi^4}{9}=8\sum_{n=1}^{\infty}\left(\frac{1}{n^4}\right)$$

$$\pi^4\left(\frac{1}{5}-\frac{1}{9}\right)=8\sum_{n=1}^{\infty}\left(\frac{1}{n^4}\right)$$

$$\pi^4\left(\frac{9-5}{45}\right)=8\sum_{n=1}^{\infty}\left(\frac{1}{n^4}\right)$$

$$\pi^4\left(\frac{4}{45}\right)=8\sum_{n=1}^{\infty}\left(\frac{1}{n^4}\right)$$

$$\frac{\pi^4}{90}=\sum_{n=1}^{\infty}\left(\frac{1}{n^4}\right)$$

$$\frac{\pi^4}{90}=\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\cdots$$

26. Find the R.M.S. value of $y=x^2$ in $(-\pi,\pi)$. And $x+x^2$ in $(-\pi,\pi)$.

Soln : (i) Given $y=x^2(-\pi,\pi)$

The RMS value in an interval $(-\pi,\pi)$ is

$$y=\sqrt{\frac{1}{(b-a)}\int_a^b[f\,(x)]^2dx}$$

$$=\sqrt{\frac{1}{2\pi}\int_{-\pi}^{\pi}[x^2]^2dx}=\sqrt{\frac{1}{2\pi}\int_{-\pi}^{\pi}x^4dx}$$

$$=\sqrt{\frac{2}{2\pi}\int_{0}^{\pi}x^4dx}=\sqrt{\frac{1}{\pi}\left[\frac{x^5}{5}\right]_0^{\pi}}$$

$$=\sqrt{\frac{1}{5\pi}\left[(\pi^5)-(0^5)\right]}=\sqrt{\frac{\pi^5}{5\pi}}$$

$$=\sqrt{\frac{\pi^4}{5}}=\frac{\pi^2}{\sqrt{5}}$$

(ii) Given $y=x+x^2(-\pi,\pi)$ The RMS value in an interval $(-\pi,\pi)$ is

$$y=\sqrt{\frac{1}{(b-a)}\int_a^b[f\,(x)]^2dx}$$

$$=\sqrt{\frac{1}{2\pi}\int_{-\pi}^{\pi}[x+x^2]^2dx}$$

$$=\sqrt{\frac{1}{2\pi}\int_{-\pi}^{\pi}(x^2+x^4+2x^3)dx}$$

$$=\sqrt{\frac{2}{2\pi}\int_{0}^{\pi}(x^2+x^4)dx+\frac{1}{2\pi}\int_{-\pi}^{\pi}(2x^3)dx}$$

$$=\sqrt{\frac{1}{\pi}\left[\frac{x^3}{3}+\frac{x^5}{5}\right]_0^{\pi}+0}=\sqrt{\frac{1}{\pi}\left(\frac{\pi^3}{3}+\frac{\pi^5}{5}\right)}$$

$$=\sqrt{\frac{5\pi^3+3\pi^5}{15\pi}}=\sqrt{\frac{5\pi^2+3\pi^4}{15}}$$

27. What is the equivalent Fourier constant to the expression 2π mean value of $f(x)=\cos nx$ in $(c,c+2\pi)$.

Soln : If $f(x)$ is a periodic function with a period 2π and if $f(x)$ can be represented in a trigonometry series,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

where,

$$a_0 = \frac{1}{\pi} \int_{c}^{c+2\pi} f(x)dx$$

$$a_n = \frac{1}{\pi} \int_{c}^{c+2\pi} f(x) \cos nx dx$$

$$b_n = \frac{1}{\pi} \int_{c}^{c+2\pi} f(x) \sin nx dx$$

To find Fourier constant a_0:

$$\begin{aligned} a_0 &= \frac{1}{\pi} \int_{c}^{c+2\pi} f(x)dx = \frac{1}{\pi} \int_{c}^{c+2\pi} \cos nx dx \\ &= \frac{1}{\pi} \left(\frac{\sin nx}{n}\right)_{c}^{c+2\pi} \\ &= \frac{1}{n\pi} (\sin n(c+2\pi) - \sin nc) \\ &= \frac{1}{n\pi} [\sin nc \ \cos 2n\pi + \cos nc \sin 2n\pi - \sin nc] \\ &= \frac{1}{n\pi} [\sin nc(1) - \sin nc] = 0 \end{aligned}$$

2.2 Part-B

2.2.1 Examples under $(0, 2\pi)$

Example 2.1. Find the Fourier series of $f(x) = x^2$ in $(0, 2\pi)$ and with period 2π. $\left\{a_0 = \frac{8}{3}\pi^2, a_n = \frac{4}{n^2}, b_n = \frac{-4\pi}{n}\right\}$

Solution : Given $f(x) = x^2$ defined in the interval $(0, 2\pi)$.

$\therefore$ The Fourier series of $f(x)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \tag{1}$$

$$\text{where } a_0 = \frac{1}{\pi} \int_{0}^{2\pi} f(x)\, dx$$

$$a_n = \frac{1}{\pi} \int_{0}^{2\pi} f(x) \cos nx dx$$

$$b_n = \frac{1}{\pi} \int_{0}^{2\pi} f(x) \sin nx dx$$

$$\text{Now, } a_0 = \frac{1}{\pi}\int_0^{2\pi} x^2 dx = \frac{1}{\pi}\left[\frac{x^3}{3}\right] = \frac{1}{\pi}\left[\frac{8\pi^3}{3} - 0\right] = \frac{8\pi^2}{3}$$

$$\begin{aligned}
a_n =& \frac{1}{\pi}\int_0^{2\pi} x^2 \cos nx dx \\
=& \frac{1}{\pi}\left[x^2\left(\frac{\sin nx}{n}\right) - 2x\left(\frac{-\cos nx}{n^2}\right) + 2\left(\frac{-\sin nx}{n^3}\right)\right]_0^{2\pi} \\
=& \frac{1}{\pi}\left[x^2\frac{\sin nx}{n} + 2x\frac{\cos nx}{n^2} - 2\frac{\sin nx}{n^3}\right]_0^{2\pi} \\
=& \frac{1}{\pi}\left[\left(0 + \frac{4\pi}{n^2} - 0\right) - (0 + 0 - 0)\right] \qquad [\because \cos 2n\pi = 1] \\
\therefore a_n =& \frac{4}{n^2}
\end{aligned}$$

$$\begin{aligned}
b_n =& \frac{1}{\pi}\int_0^{2\pi} x^2 \sin nx dx \\
=& \frac{1}{\pi}\left[x^2\left(\frac{-\cos nx}{n}\right) - 2x\left(\frac{-\sin nx}{n^2}\right) + 2\left(\frac{\cos nx}{n^3}\right)\right]_0^{2\pi} \\
=& \frac{1}{\pi}\left[-x^2\frac{\cos nx}{n} + 2x\frac{\sin nx}{n^2} + 2\frac{\cos nx}{n^3}\right]_0^{2\pi} \\
=& \frac{1}{\pi}\left[\left(\frac{-4\pi^2}{n} + 0 + \frac{2}{n^3}\right) - \left(0 + 0 + \frac{2}{n^3}\right)\right] \\
=& \frac{1}{\pi}\left(\frac{-4\pi^2}{n}\right) \\
\therefore b_n =& \frac{-4\pi}{n}
\end{aligned}$$

Sub. the value of a_0, a_n, b_n in (1)

$$\begin{aligned}
f(x) =& \frac{1}{2}\left(\frac{8\pi^2}{3}\right) + \sum_{n=1}^{\infty}\left(\frac{4}{n^2}\cos nx - \frac{4\pi}{n}\sin nx\right) \\
=& \frac{4\pi^2}{3} + 4\sum_{n=1}^{\infty}\left(\frac{\cos nx}{n^2} - \frac{\pi \sin nx}{n}\right)
\end{aligned}$$

Example 2.2. Find the Fourier series of periodicity 2π for $f(x) = \begin{cases} x, & (0,\pi) \\ 2\pi - x, & (\pi, 2\pi) \end{cases}$

Solution: Given $f(x) = \begin{cases} x, & (0,\pi) \\ 2\pi - x, & (\pi, 2\pi) \end{cases}$

Since the function $f(x)$ is defined in the interval $(0, 2\pi)$.

$\therefore$ The Fourier series of $f(x)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \tag{1}$$

$$\text{where } a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx, a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx dx, b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx dx$$

$$\begin{aligned}
\text{Now, } a_0 &= \frac{1}{\pi}\left[\int_0^{\pi} x dx + \int_{\pi}^{2\pi} (2\pi - x)\,dx\right] \\
&= \frac{1}{\pi}\left[\left(\frac{x^2}{2}\right)_0^{\pi} + \left(2\pi x - \frac{x^2}{2}\right)_{\pi}^{2\pi}\right] \\
&= \frac{1}{\pi}\left[\left(\frac{\pi^2}{2} - 0\right) + \left(\left(4\pi^2 - 2\pi^2\right) - \left(2\pi^2 - \frac{\pi^2}{2}\right)\right)\right] \\
&= \frac{1}{\pi}\left[\frac{\pi^2}{2} + 2\pi^2 - \frac{3\pi^2}{2}\right] = \frac{1}{\pi}\left(\frac{\pi^2 + 4\pi^2 - 3\pi^2}{2}\right) \\
&= \frac{1}{\pi}\left(\frac{2\pi^2}{2}\right) \\
a_0 &= \pi
\end{aligned}$$

$$\begin{aligned}
a_n &= \frac{1}{\pi}\left[\int_0^{\pi} x\cos nx dx + \int_{\pi}^{2\pi} (2\pi - x)\cos nx dx\right] \\
&= \frac{1}{\pi}\left[\left[x\left(\frac{\sin nx}{n}\right) - 1\cdot\left(\frac{-\cos nx}{n^2}\right)\right]_0^{\pi} + \left[(2\pi - x)\left(\frac{\sin nx}{n}\right) - (-1)\left(\frac{-\cos nx}{n^2}\right)\right]_{\pi}^{2\pi}\right] \\
&= \frac{1}{\pi}\left[\left[\frac{x\sin nx}{n} + \frac{\cos nx}{n^2}\right]_0^{\pi} + \left[(2\pi - x)\frac{\sin nx}{n} - \frac{\cos nx}{n^2}\right]_{\pi}^{2\pi}\right] \\
&= \frac{1}{\pi}\left[\left[\left(0 + \frac{(-1)^n}{n^2}\right) - \left(0 + \frac{1}{n^2}\right)\right] + \left(0 - \frac{1}{n^2}\right) - \left(0 - \frac{(-1)^n}{n}\right)\right] \\
&= \frac{1}{\pi}\left[\frac{(-1)^n}{n^2} - \frac{1}{n^2} - \frac{1}{n^2} + \frac{(-1)^n}{n^2}\right] \\
&= \frac{1}{\pi}\left[\frac{2(-1)^n}{n^2} - \frac{2}{n^2}\right] \\
a_n &= \frac{2}{n^2\pi}\left[(-1)^n - 1\right]
\end{aligned}$$

$$b_n = \frac{1}{\pi}\left[\int_0^{\pi} x\sin nx dx + \int_{\pi}^{2\pi} (2\pi - x)\sin nx dx\right]$$
$$= \frac{1}{\pi}\left[\left[x\left(\frac{-\cos nx}{n}\right) - 1\left(\frac{-\sin nx}{n^2}\right)\right]_0^{\pi} + \left[(2\pi - x)\left(\frac{-\cos nx}{n}\right) - (-1)\left(\frac{-\sin nx}{n^2}\right)\right]_{\pi}^{2\pi}\right]$$
$$= \frac{1}{\pi}\left[\left[\frac{-x\cos nx}{n} + \frac{\sin nx}{n^2}\right]_0^{\pi} + \left[-(2\pi - x)\frac{\cos nx}{n} - \frac{\sin nx}{n^2}\right]_{\pi}^{2\pi}\right]$$
$$= \frac{1}{\pi}\left[\left[\left(\frac{-\pi(-1)^n}{n} + 0\right) - (0+0)\right] + \left[(0-0) - \left(\frac{-\pi(-1)^n}{n} - 0\right)\right]\right]$$
$$= \frac{1}{\pi}\left[\frac{-\pi(-1)^n}{n} + \frac{\pi(-1)^n}{n}\right]$$
$$b_n = 0$$

Sub. the value of a_0, a_n and b_n in (1)

$$f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty}\left(\frac{2}{n^2\pi}\left((-1)^n - 1\right)\cos nx + 0\right)$$
$$f(x) = \frac{\pi}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\left(\frac{(-1)^n - 1}{n^2}\right)\cos nx$$

Example 2.3. Find the Fourier series for $f(x) = x\sin x$ in $(0, 2\pi)$ and deduce $\frac{1}{(1)(3)} - \frac{1}{(3)(4)} + \frac{1}{(5)(7)} - \cdots = \frac{\pi - 2}{4}$.

$$\left\{a_0 = -2, a_n = \frac{-2}{1-n^2}(n \neq 1), a_1 = \frac{-1}{2}, b_n = 0\,(n \neq 1), b_1 = \pi \,\&\, x = \frac{\pi}{2}\right\}$$

Solution : Given $f(x) = x\sin x$ in $(0, 2\pi)$.

$\therefore$ The Fourier series of $f(x)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx) \tag{1}$$

where $a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx,\ a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx dx,\ b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx dx$

Now, $a_0 = \frac{1}{\pi}\int_0^{2\pi} x\sin x dx$

$$= \frac{1}{\pi}\left[x(-\cos x) - 1(-\sin x)\right]_0^{2\pi}$$
$$= \frac{1}{\pi}\left[-x\cos x + \sin x\right]_0^{2\pi}$$
$$= \frac{1}{\pi}\left[(-2\pi + 0) - (0 + 0)\right]$$
$$a_0 = -2$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} x\sin x\cos nx dx$$
$$= \frac{1}{\pi}\int_0^{2\pi} x\cos nx\sin x dx$$
$$= \frac{1}{\pi}\int_0^{2\pi} x\frac{1}{2}\left(\sin(n+1)x - \sin(n-1)x\right)dx$$
$$= \frac{1}{2\pi}\int_0^{2\pi} x\left(\sin(n+1)x - \sin(n-1)x\right)dx$$
$$= \frac{1}{2\pi}\left[x\left(\frac{-\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\right) - 1\left(\frac{-\sin(n+1)x}{(n+1)^2} + \frac{\sin(n-1)x}{(n-1)^2}\right)\right]_0^{2\pi}$$
$$= \frac{1}{2\pi}\left[x\left(\frac{\cos(n-1)x}{n-1} - \frac{\cos(n+1)x}{n+1}\right) + \frac{\sin(n+1)x}{(n+1)^2} - \frac{\sin(n-1)x}{(n-1)^2}\right]_0^{2\pi}$$
$$= \frac{1}{2\pi}\left[\left[2\pi\left(\frac{1}{n-1} - \frac{1}{n+1}\right) + 0 - 0\right] - [0+0-0]\right]$$
$$= \frac{1}{n-1} - \frac{1}{n+1} = \frac{n+1-(n-1)}{n^2-1}$$
$$a_n = \frac{2}{n^2-1}, n \neq 1$$

$$a_1 = \frac{1}{\pi}\int_0^{2\pi} x\sin x\cos x dx \qquad \left(\because a_n = \frac{1}{\pi}\int_0^{2\pi} x\sin x\cos nx dx\right)$$
$$= \frac{1}{2\pi}\int_0^{2\pi} x2\sin x\cos x dx$$
$$= \frac{1}{2\pi}\int_0^{2\pi} x\sin 2x dx$$
$$= \frac{1}{2\pi}\left[x\left(\frac{-\cos 2x}{2}\right) - 1\cdot\left(\frac{-\sin 2x}{4}\right)\right]_0^{2\pi}$$
$$= \frac{1}{2\pi}\left[\frac{-x\cos 2x}{2} + \frac{\sin 2x}{4}\right]_0^{2\pi} = \frac{1}{2\pi}\left[\left(\frac{-2\pi}{2} + 0\right) - (0+0)\right]$$
$$a_1 = \frac{-1}{2}$$

$$b_n = \frac{1}{\pi}\int\limits_0^{2\pi} x\sin x\sin nx dx$$

$$= \frac{1}{\pi}\int\limits_0^{2\pi} x\sin nx\sin x dx$$

$$= \frac{1}{\pi}\int\limits_0^{2\pi} x\frac{1}{2}\left(\cos\left(n-1\right)x - \cos\left(n+1\right)x\right) dx$$

$$= \frac{1}{\pi}\int\limits_0^{2\pi} x\left(\cos\left(n-1\right)x - \cos\left(n+1\right)x\right) dx$$

$$= \frac{1}{2\pi}\left[x\left(\frac{\sin\left(n-1\right)x}{n-1} - \frac{\sin\left(n+1\right)x}{n+1}\right) - 1\left(\frac{-\cos\left(n-1\right)x}{\left(n-1\right)^2} + \frac{\cos\left(n+1\right)x}{\left(n+1\right)^2}\right)\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[x\left(\frac{\sin\left(n+1\right)x}{n+1} - \frac{\sin\left(n-1\right)x}{n-1}\right) + \left(\frac{\cos\left(n-1\right)x}{\left(n-1\right)^2} - \frac{\cos\left(n+1\right)x}{\left(n+1\right)^2}\right)\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[\left(0 + \frac{1}{\left(n-1\right)^2} - \frac{1}{\left(n+1\right)^2}\right) - \left(0 + \frac{1}{\left(n-1\right)^2} - \frac{1}{\left(n+1\right)^2}\right)\right]$$

$$b_n = 0, n \neq 1$$

$$b_1 = \frac{1}{\pi}\int\limits_0^{2\pi} x\sin x\sin x dx = \frac{1}{\pi}\int\limits_0^{2\pi} x\sin^2 x dx$$

$$= \frac{1}{\pi}\int\limits_0^{2\pi} x\frac{1}{2}\left(1 - \cos 2x\right) dx$$

$$= \frac{1}{2\pi}\int\limits_0^{2\pi} \left(x - x\cos 2x\right) dx$$

$$= \frac{1}{2\pi}\left[\frac{x^2}{2} - \left[x\left(\frac{\sin 2x}{2}\right) - 1\left(\frac{-\cos 2x}{4}\right)\right]\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[\frac{x^2}{2} - \frac{x\sin 2x}{2} - \frac{\cos 2x}{4}\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[\left(\frac{4\pi^2}{2} - 0 - \frac{1}{4}\right) - \left(0 - 0 - \frac{1}{4}\right)\right]$$

$$= \frac{1}{2\pi}\left(2\pi^2\right)$$

$$b_1 = \pi$$

From (1),

$$\begin{aligned}
f(x) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \\
&= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \\
&= \frac{a_0}{2} + a_1 \cos x + \sum_{n=2}^{\infty} a_n \cos nx + b_1 \sin x + \sum_{n=2}^{\infty} b_n \sin nx \\
&= \frac{1}{2}(-2) - \frac{1}{2}\cos x + \sum_{n=2}^{\infty} \left(\frac{2}{n^2-1}\right) \cos nx + \pi \sin x + 0
\end{aligned}$$

i.e., $f(x) = -1 - \frac{1}{2}\cos x + 2\sum\limits_{n=2}^{\infty} \left(\frac{\cos nx}{n^2-1}\right) + \pi \sin x$

Example 2.4. Obtain Fourier series for $f(x) = e^{ax}$ in $(0, 2\pi)$.

Solution : Given $f(x) = e^{ax}$ in $(0, 2\pi)$.

The Fourier series of $f(x)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \tag{1}$$

where

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx,\ a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx,\ b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

Now, $a_0 = \frac{1}{\pi}\int\limits_0^{2\pi} e^{ax}dx = \frac{1}{\pi}\left[\frac{e^{ax}}{a}\right]_0^{2\pi} = \frac{1}{a\pi}\left[e^{ax}\right]_0^{2\pi} = \frac{1}{a\pi}\left[e^{a2\pi} - e^0\right]$

$$\therefore a_0 = \frac{1}{a\pi}\left(e^{2a\pi} - 1\right)$$

$$\begin{aligned}
a_n &= \frac{1}{\pi}\int_0^{2\pi} e^{ax}\cos nx\,dx = \frac{1}{\pi}\left[\frac{e^{ax}}{a^2+n^2}(a\cos nx + n\sin nx)\right]_0^{2\pi} \\
&= \frac{1}{\pi}\left[\frac{e^{2a\pi}}{a^2+n^2}(a+0) - \frac{1}{a^2+n^2}(a+0)\right] \\
&= \frac{1}{\pi}\left[\frac{ae^{2a\pi}}{a^2+n^2} - \frac{a}{a^2+n^2}\right] = \frac{a}{\pi(a^2+n^2)}\left[e^{2a\pi} - 1\right]
\end{aligned}$$

$$\therefore a_n = \frac{a\left(e^{2a\pi} - 1\right)}{\pi(a^2+n^2)}$$

$$\begin{aligned}
b_n =& \frac{1}{\pi}\int_0^{2\pi} e^{ax}\sin nx dx = \frac{1}{\pi}\left[\frac{e^{ax}}{a^2+n^2}\left(a\sin nx - \cos nx\right)\right]_0^{2\pi}\\
=& \frac{1}{\pi}\left[\frac{e^{2a\pi}}{a^2+n^2}\left(0-n\right) - \frac{1}{a^2+n^2}\left(0-n\right)\right]\\
=& \frac{1}{\pi}\left[\frac{-ne^{2a\pi}}{a^2+n^2} + \frac{n}{a^2+n^2}\right] = \frac{-n}{\pi\left(a^2+n^2\right)}\left[e^{2a\pi}-1\right]
\end{aligned}$$

$$\therefore b_n = \frac{-n\left(e^{2a\pi}-1\right)}{\pi\left(a^2+n^2\right)}$$

Sub. the value of a_0, a_n, b_n in (1)

$$\begin{aligned}
f(x) =& \frac{1}{2}\left(\frac{e^{2a\pi}-1}{a\pi}\right) + \sum_{n=1}^{\infty}\left[\frac{a\left(e^{2a\pi}-1\right)}{\pi\left(a^2+n^2\right)}\cos nx - \frac{n\left(e^{2a\pi}-1\right)}{\pi\left(a^2+n^2\right)}\sin nx\right]\\
=& \frac{e^{2a\pi}-1}{2a\pi} + \sum_{n=1}^{\infty}\left(\frac{e^{2a\pi}-1}{\pi\left(a^2+n^2\right)}\right)\left[a\cos nx - n\sin nx\right]\\
=& \frac{e^{2a\pi}-1}{2a\pi} + \frac{e^{2a\pi}-1}{\pi}\left[\sum_{n=1}^{\infty}\left(\frac{1}{a^2+n^2}\right)\left(a\cos nx - n\sin nx\right)\right]\\
=& \frac{e^{2a\pi}-1}{\pi}\left[\frac{1}{2a} + \sum_{n=1}^{\infty}\frac{1}{a^2+n^2}\left(a\cos nx - n\sin nx\right)\right]
\end{aligned}$$

2.2.2 Examples under $(-\pi, \pi)$

Example 2.5. Find the Fourier series for the function $f(x) = x^2$ in $[-\pi, \pi]$with period 2π and deduce

(i) $\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\ldots=\frac{\pi^2}{6}$ (ii) $\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\ldots=\frac{\pi^2}{12}$

(iii) $\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\ldots=\frac{\pi^2}{8}$ (iv) $\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\ldots=\frac{\pi^4}{90}$

$$\left\{a_0 = \frac{2}{3}\pi^2, a_n = \frac{4\left(-1\right)^n}{n^2}, b_n = 0(\because \text{ even})\right\}$$

$$\left\{\begin{array}{lll} (i)x=\pi(or-\pi) & (ii)x=0 & (iii)\ add\ i\ and\ ii\\ (iv)\ Use\ Parseval's\ identity & & \end{array}\right\}$$

Solution: Given $f(x) = x^2$ in $[-\pi, \pi]$.

We know that, the Fourier series of $f(x)$ is given by (Refer above example),

$$f\left(x\right) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{\left(-1\right)^n}{n^2}\cos nx \qquad (1)$$

Deduction:

(i) Put $x = \pi$. [Here $x = \pi$ is a point of discontinuity which is one end of the given interval $(-\pi, \pi)$]

$\therefore$ Sum of the Fourier series of $f(x)$ is $\frac{f\left(-\pi\right)+f\left(\pi\right)}{2}$

i.e., $$\frac{f(-\pi)+f(\pi)}{2} = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}(-1)^n$$
$$\frac{\pi^2+\pi^2}{2} = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^{2n}}{n^2}$$
$$\frac{2\pi^2}{2} = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{1}{n^2}$$
$$\pi^2 - \frac{\pi^2}{3} = 4\sum_{n=1}^{\infty}\frac{1}{n^2}$$
$$\frac{2\pi^2}{3} = 4\sum_{n=1}^{\infty}\frac{1}{n^2}$$
$$\frac{\pi^2}{6} = \sum_{n=1}^{\infty}\frac{1}{n^2}$$
$$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots = \frac{\pi^2}{6} \tag{2}$$

(ii) Put $x = 0$

$$0 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}$$
$$\frac{-\pi^2}{3} = 4\left[\frac{-1}{1^2}+\frac{1}{2^2}-\frac{1}{3^2}+\cdots\right]$$
$$\frac{-\pi^2}{3} = -4\left[\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\cdots\right]$$
$$\frac{\pi^2}{12} = \frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\cdots$$
$$\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\cdots = \frac{\pi^2}{12} \tag{3}$$

(iii) $(1)+(2) \Rightarrow$

$$\frac{2}{1^2}+0+\frac{2}{3^2}+0+\cdots = \frac{\pi^2}{6}+\frac{\pi^2}{12}$$
$$2\left(\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots\right) = \frac{3\pi^2}{12}$$
$$\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots = \frac{\pi^2}{8}.$$

Example 2.6. Expand $f(x) = |\cos x|$ in a Fourier series for in the interval $(-\pi, \pi)$.

Solution : Given $f(x) = |\cos x|$

$$f(-x) = |\cos(-x)| = |\cos x|$$
$$f(-x) = f(x)$$

$\therefore f(x)$ is an even function in $-\pi < x < \pi$.

$\therefore$ The Fourier series for the even function $f(x)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \tag{1}$$

$$\text{where } a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)dx, a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx dx$$

Now,

$$\begin{aligned}
a_0 =& \frac{2}{\pi}\int_0^{\pi} |\cos x| dx \\
=& \frac{2}{\pi}\left[\int_0^{\frac{\pi}{2}} |\cos x|\, dx + \int_{\frac{\pi}{2}}^{\pi} |\cos x|\, dx\right] \\
=& \frac{2}{\pi}\left[\int_0^{\frac{\pi}{2}} \cos x dx + \int_{\frac{\pi}{2}}^{\pi} (-\cos x) dx\right] \\
=& \frac{2}{\pi}\left[\int_0^{\frac{\pi}{2}} \cos x dx - \int_{\frac{\pi}{2}}^{\pi} \cos x dx\right] \\
=& \frac{2}{\pi}\left[(\sin x)_0^{\frac{\pi}{2}} - (\sin x)_{\frac{\pi}{2}}^{\pi}\right] \\
=& \frac{2}{\pi}[(1-0)-(0-1)] = \frac{2}{\pi}[1+1] \\
\therefore a_0 =& \frac{4}{\pi}
\end{aligned}$$

$$
\begin{aligned}
a_n =& \frac{2}{\pi}\int_0^{\pi} |\cos x| \cos nx dx \\
=& \frac{2}{\pi}\left[\int_0^{\frac{\pi}{2}} |\cos x| \cos nx dx + \int_{\frac{\pi}{2}}^{\pi} |\cos x| \cos nx dx\right] \\
=& \frac{2}{\pi}\left[\int_0^{\frac{\pi}{2}} \cos x \cos nx dx + \int_{\frac{\pi}{2}}^{\pi} (-\cos x) \cos nx dx\right] \\
=& \frac{2}{\pi}\left[\int_0^{\frac{\pi}{2}} \cos nx \cos x dx - \int_{\frac{\pi}{2}}^{\pi} \cos nx \cos x dx\right] \\
=& \frac{2}{\pi}\left\{\int_0^{\frac{\pi}{2}} \frac{1}{2}\left[\cos (n+1) x + \cos (n-1) x\right] dx - \int_{\frac{\pi}{2}}^{\pi} \frac{1}{2}[\cos (n+1) x + \cos (n-1) x] dx\right\} \\
& \left[\because \cos A \cos B = \frac{1}{2} \cos (A+B) + \cos (A-B)\right] \\
=& \frac{1}{\pi}\left\{\int_0^{\frac{\pi}{2}} \left[\cos (n+1) x + \cos (n-1) x\right] dx - \int_{\frac{\pi}{2}}^{\pi} \left[\cos (n+1) x + \cos (n-1) x\right] dx\right\} \\
=& \frac{1}{\pi}\left\{\left(\frac{\sin (n+1) x}{n+1} + \frac{\sin (n-1) x}{n-1}\right)_0^{\frac{\pi}{2}} - \left(\frac{\sin (n+1) x}{n+1} + \frac{\sin (n-1) x}{n-1}\right)_{\frac{\pi}{2}}^{\pi}\right\} \\
=& \frac{1}{\pi}\left\{\left[\left(\frac{\sin (n+1) \frac{\pi}{2}}{n+1} + \frac{\sin (n-1) \frac{\pi}{2}}{n-1}\right) - 0\right] - \left[0 - \left(\frac{\sin (n+1) \frac{\pi}{2}}{n+1} + \frac{\sin (n-1) \frac{\pi}{2}}{n-1}\right)\right]\right\} \\
=& \frac{1}{\pi}\left[\frac{2 \sin (n+1) \frac{\pi}{2}}{n+1} + \frac{2 \sin (n-1) \frac{\pi}{2}}{n-1}\right] \\
=& \frac{2}{\pi}\left[\frac{\sin\left(\frac{n\pi}{2} + \frac{\pi}{2}\right)}{n+1} + \frac{\sin\left(\frac{n\pi}{2} - \frac{\pi}{2}\right)}{n-1}\right] \\
=& \frac{2}{\pi}\left[\frac{\sin \frac{n\pi}{2} \cos \frac{\pi}{2} + \cos n\frac{\pi}{2} \sin \frac{\pi}{2}}{n+1} + \frac{\sin \frac{n\pi}{2} \cos \frac{\pi}{2} - \cos n\frac{\pi}{2} \sin \frac{\pi}{2}}{n-1}\right] \\
=& \frac{2}{\pi}\left[\frac{\cos \frac{n\pi}{2}}{n+1} - \frac{\cos \frac{n\pi}{2}}{n-1}\right] \\
=& \frac{2 \cos \frac{n\pi}{2}}{\pi}\left[\frac{1}{n+1} - \frac{1}{n-1}\right] = \frac{2 \cos \frac{n\pi}{2}}{\pi}\left[\frac{-1-(+1)}{n^2-1}\right] \\
\therefore a_n =& \frac{-4 \cos \frac{n\pi}{2}}{\pi\left(n^2-1\right)}, n \neq 1
\end{aligned}
$$

$$\text{Now, } a_1 = \frac{2}{\pi}\int_0^{\pi} |\cos x| \cos x dx \quad \left[\begin{array}{ll} \because a_n & = \dfrac{2}{\pi}\displaystyle\int_0^{\pi} |\cos x| \cos nx dx \\ \therefore a_1 & = \dfrac{2}{\pi}\displaystyle\int_0^{\pi} |\cos x| \cos x dx \end{array}\right]$$

$$\begin{aligned}
&= \frac{2}{\pi}\left[\int_0^{\frac{\pi}{2}} \cos x \cos x dx + \int_{\frac{\pi}{2}}^{\pi} -\cos x \cos x dx\right] \\
&= \frac{2}{\pi}\left[\int_0^{\frac{\pi}{2}} \cos^2 x dx - \int_{\frac{\pi}{2}}^{\pi} \cos^2 x dx\right] \\
&= \frac{2}{\pi}\left[\int_0^{\frac{\pi}{2}} \frac{1}{2}(1+\cos 2x)\, dx - \int_{\frac{\pi}{2}}^{\pi} \frac{1}{2}(1+\cos 2x)\, dx\right] \\
&= \frac{1}{\pi}\left[\int_0^{\frac{\pi}{2}} (1+\cos 2x)\, dx - \int_{\frac{\pi}{2}}^{\pi} \frac{1}{2}(1+\cos 2x)\, dx\right] \\
&= \frac{1}{\pi}\left[\left(x + \frac{\sin 2x}{2}\right)_0^{\frac{\pi}{2}} - \left(x + \frac{\sin 2x}{2}\right)_{\frac{\pi}{2}}^{\pi}\right] \\
&= \frac{1}{\pi}\left[\left[\left(\frac{\pi}{2} - 0\right) - 0\right] - \left[(\pi + 0) - \left(\frac{\pi}{2} + 0\right)\right]\right] \\
&= \frac{1}{\pi}\left[\frac{\pi}{2} - \frac{\pi}{2}\right] \\
\therefore a_1 &= 0
\end{aligned}$$

From (1), the Fourier series of $f(x)$ is

$$\begin{aligned}
f(x) &= \frac{a_0}{2} + a_1 \cos x + \sum_{n=2}^{\infty} a_n \cos nx \\
&= \frac{1}{2}\left(\frac{4}{\pi}\right) + 0 + \sum_{n=2}^{\infty} \frac{-4\cos\frac{n\pi}{2}}{\pi\left(n^2-1\right)} \cos nx \\
&= \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{\cos\frac{n\pi}{2}}{n^2-1} \cos nx.
\end{aligned}$$

2.2.3 Half range Fourier series in $(0, \pi)$

Example 2.7. Expand $x(\pi - x)$ in half range sine series in the interval $(0, \pi)$.

Solution: Given $f(x) = x(\pi - x)$ in $(0, \pi)$

$$= \pi x - x^2$$

The sine series of $f(x)$ is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx \qquad (1)$$

$$\begin{aligned}
\text{where } b_n &= \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx dx = \frac{2}{\pi}\int_0^{\pi}\left(\pi x - x^2\right)\sin nx dx \\
&= \frac{2}{\pi}\left[\left(\pi x - x^2\right)\left(\frac{-\cos nx}{n}\right) - (\pi - 2x)\left(\frac{-\sin nx}{n^2}\right) + (-2)\left(\frac{\cos nx}{n^3}\right)\right]_0^{\pi} \\
&= \frac{2}{\pi}\left[-\left(\pi x - x^2\right)\frac{\cos nx}{n} + (\pi - 2x)\frac{\sin nx}{n^2} - 2\frac{\cos nx}{n^3}\right]_0^{\pi} \\
&= \frac{2}{\pi}\left[\left(0 + 0 - \frac{2(-1)^n}{n^3}\right) - \left(0 + 0 - \frac{2}{n^3}\right)\right] \\
&= \frac{2}{\pi}\left[\frac{2}{n^3} - \frac{2(-1)^n}{n^3}\right] \\
b_n &= \frac{4}{\pi n^3}\left[1 - (-1)^n\right]
\end{aligned}$$

Sub. the value of b_n in (1)

$$\begin{aligned}
f(x) &= \sum_{n=1}^{\infty} \frac{4}{\pi n^3}\left(1 - (-1)^n\right)\sin nx \\
&= \frac{4}{\pi}\sum_{n=1}^{\infty}\left(\frac{1 - (-1)^n}{n^3}\right)\sin nx \\
&= \frac{4}{\pi}\left[\frac{2}{1^3}\sin x + 0 + \frac{2}{3^3}\sin 3x + 0 + \cdots\right] \\
&= \frac{8}{\pi}\left[\frac{\sin x}{1^3} + \frac{\sin 3x}{3^3} + \frac{\sin 5x}{5^3} + \cdots\right]
\end{aligned}$$

2.2.4 Examples under $(0, 2\ell)$

2.2.5 Examples under $(-\ell, \ell)$

2.2.6 Examples under $(0, \ell)$

2.2.7 Complex form of Fourier Series

Example 2.8. Derive complex form for $f(x) = e^{ax}$ in $(0, 2\pi)$.

$$\left\{c_n = \frac{(a + in)\left(e^{2a\pi} - 1\right)}{2\pi\left(a^2 + n^2\right)}\right\}$$

Example 2.9. Find the complex form of the series for the function $f(x) = x$ in $(-\ell, \ell)$. $\left\{c_n = \frac{-\ell(-1)^n}{in\pi}\right\}$

2.2.8 Harmonic Analysis

2.2.8.1 Examples under π form(Radian form)

Example 2.10. The table of values of the function $y = f(x)$ is given below:

x	0	$\pi/3$	$2\pi/3$	π	$4\pi/3$	$5\pi/3$	2π
$f(x)$	1.0	1.4	1.9	1.7	1.5	1.2	1.0

. Find the Fourier series upto 2^{nd} harmonic to represent $y = f(x)$ in terms of x in $(0, 2\pi)$.

Solution : Let

$$f(x) = \frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + b_1 \sin x + b_2 \sin 2x \qquad (1)$$

be the F.S. upto second harmonic.

Since the first and last values of y are same in the given table, leave the first column (or) last column of the table. Hence only the first six column values will be used.

x	y	$\cos x$	$\cos 2x$	$\sin x$	$\sin 2x$	$y\cos x$	$y\cos 2x$	$y\sin x$	$y\sin 2x$
0	1.0	1	1	0	0	1	1	0	0
$\frac{\pi}{3}$	1.4	0.5	-0.5	0.866	0.866	0.7	-0.7	1.2124	1.2124
$\frac{2\pi}{3}$	1.9	-0.5	-0.5	0.866	-0.866	-0.95	-0.95	1.6454	-1.6454
π	1.7	-1	1	0	0	-1.7	1.7	0	0
$\frac{4\pi}{3}$	1.5	-0.5	-0.5	-0.866	0.866	-0.75	-0.75	-1.299	1.299
$\frac{5\pi}{3}$	1.2	0.5	-0.5	-0.866	-0.866	0.6	-0.6	-1.0392	-1.0392
Sum	8.7	-	-	-	-	-1.1	-0.3	0.5196	-0.1732

$$\begin{aligned}
a_0 &= 2 \times \frac{1}{6}\left[\sum y\right] = 2 \times \frac{1}{6}[8.7] = 2.9 \Rightarrow \frac{a_0}{2} = \frac{2.9}{2} = 1.45 \\
a_1 &= 2 \times \frac{1}{6}\left[\sum y\cos x\right] = 2 \times \frac{1}{6}[-1.1] = -0.37 \\
a_2 &= 2 \times \frac{1}{6}\left[\sum y\cos 2x\right] = 2 \times \frac{1}{6}[-0.3] = -0.1 \\
b_1 &= 2 \times \frac{1}{6}\left[\sum y\sin x\right] = 2 \times \frac{1}{6}[0.5196] = 0.17 \\
b_2 &= 2 \times \frac{1}{6}\left[\sum y\sin 2x\right] = 2 \times \frac{1}{6}[-0.1732] = -0.06
\end{aligned}$$

Hence the required Fourier Series upto second harmonic for the data is

$$(1) \Rightarrow y = f(x) = 1.45 - 0.37\cos x - 0.1\cos 2x + 0.17\sin x - 0.06\sin 2x$$

2.2.8.2 Examples under θ° form(Degree form)

Example 2.11. Find an emprical form of the function

$f(x) = a_0 + a_1 \cos x + b_1 \sin x$ with period 2π.

x^o:	0	60	120	180	240	300	360
$y = f(x)$	40	31	–13	20	3.7	–21	40

Solution: Since the last value of y is a repetition of the first, only the first six values will be used.

x^o:	y	$\cos x$	$\sin x$	$y\cos x$	$y\sin x$
0	40	1	0	40	0
60	31	0.5	0.866	15.5	26.846
120	-13.7	-0.5	0.866	6.85	-11864
180	20	-1	0	-20	0
240	3.7	-0.5	-0.866	-1.85	-3.204
300	-21	0.5	0.5	-10.5	18.186
	$\sum y = 60$			$\sum y\cos x = 30$	$\sum y\sin x = 329.964$

$$
\begin{aligned}
a_0 =& 2\times\frac{1}{6}\left[\sum y\right] = 2\times\frac{1}{6}60 = 20\\
a_1 =& 2\times\frac{1}{6}\left[\sum y\cos x\right] = 2\times\frac{1}{6}[30] = 10\\
b_1 =& 2\times\frac{1}{6}\left[\sum y\sin x\right] = 2\times\frac{1}{6}[29.964] = 9.988\\
&\therefore f(x) = 20 + 10\cos x + 9.988\sin x
\end{aligned}
$$

2.2.8.3 Examples under T form: ($\theta = 2\pi x/T$)

Example 2.12. The following table gives the vibration of periodic current over a period. Find Fourier series upto 1^{st} harmonic.

T(sec)	0	$T/6$	$T/3$	$T/2$	$2T/3$	$5T/6$	T
I(Amp)	1.98	1.3	1.05	1.3	–0.88	–0.25	1.98

Solution : Here first and last values are same. Hence omit the last value.

When x varies from 0 to T

θ varies from 0 to 2π

Let $f(x) = F(\theta) = \dfrac{a_0}{2} + a_1\cos\theta + b_1\sin\theta.$

X:	$\theta = \frac{2\pi X}{T}$	y	$\cos\theta$	$\sin\theta$	$y\cos\theta$	$y\sin\theta$
0	0	1.98	1.0	0	1.98	0
$\frac{T}{6}$	$\frac{\pi}{3}$	1.3	0.5	0.866	0.65	1.1258
$\frac{T}{3}$	$\frac{2\pi}{3}$	1.05	-0.5	0.866	-0.525	0.9093
$\frac{T}{2}$	π	1.3	-1	0	-1.3	0
$\frac{2T}{3}$	$\frac{4\pi}{3}$	-0.88	-0.5	-0.866	0.44	0.762
$\frac{5T}{6}$	$\frac{5\pi}{3}$	-0.25	0.5	-0.866	-0.125	0.2165
	$\sum y = 4.5$				$\sum y\cos\theta = 1.12$	$\sum y\sin\theta = 3.013$

$$a_0 = 2\times\frac{1}{6}\left[\sum y\right] = 2\times\frac{1}{6}4.5 = 1.5 \Rightarrow \frac{a_0}{2} = 0.75$$
$$a_1 = 2\times\frac{1}{6}\left[\sum y\cos x\right] = 2\times\frac{1}{6}[1.12] = 0.373$$
$$b_1 = 2\times\frac{1}{6}\left[\sum y\sin x\right] = 2\times\frac{1}{6}[3.013] = 1.004$$
$$\therefore f(x) = 0.75 + 0.373\cos\frac{2\pi x}{T} + 1.004\sin\frac{2\pi x}{T}$$

2.2.8.4 Problems under ℓ form: (2ℓ = Number of data)

Example 2.13. Find the first harmonic of the Fourier series for $f(x)$ for the data.

x	0	*1*	*2*	*3*	*4*	*5*
f(x)	9	18	24	28	26	20

Solution : Here the length of the interval is $2\ell = 6$, i.e., $\ell = 3$.

$\therefore$ THe Fourier series upto second harmonic can be represented by

$$y = \frac{a_0}{2} + \left(a_1\cos\frac{\pi x}{\ell} + b_1\sin\frac{\pi x}{\ell}\right) + \left(a_2\cos\frac{2\pi x}{\ell} + b_2\sin\frac{2\pi x}{\ell}\right)$$
$$y = \frac{a_0}{2} + \left(a_1\cos\frac{\pi x}{3} + b_1\sin\frac{\pi x}{3}\right) + \left(a_2\cos\frac{2\pi x}{3} + b_2\sin\frac{2\pi x}{3}\right) \qquad (1)$$

X:	y	$\cos\frac{\pi x}{3}$	$\sin\frac{\pi x}{3}$	$y\cos\frac{\pi x}{3}$	$y\sin\frac{\pi x}{3}$	$y\cos\frac{2\pi x}{3}$
0	9	1	0	9	0	9
1	18	0.5	0.866	9	15.588	-9
2	24	0.5	0.866	-12	20785	-12
3	28	-1	0	-28	0	28
4	26	-0.5	-0.866	-13	-22517	-13
5	20	0.5	-.866	10	-17.321	-10
	$\sum y = 125$		$\sum y\cos\frac{\pi x}{3} = -25$	$\sum y\sin\frac{\pi x}{3} = -3.465$	$\sum y\cos\frac{2\pi x}{3} = -7$	$\sum y\sin\frac{2\pi x}{3} = 0.00$

$$\begin{aligned}
a_0 &= 2 \times \frac{1}{6}\left[\sum y\right] = 2 \times \frac{1}{6}125 = 41.67 \Rightarrow \frac{a_0}{2} = 20.84 \\
a_1 &= 2 \times \frac{1}{6}\left[\sum y \cos\frac{\pi x}{3}\right] = 2 \times \frac{1}{6}\left[-25\right] = -8.33 \\
a_2 &= 2 \times \frac{1}{6}\left[\sum y \cos\frac{2\pi x}{3}\right] = 2 \times \frac{1}{6}\left[-7\right] = -2.333 \\
b_1 &= 2 \times \frac{1}{6}\left[\sum y \sin\frac{\pi x}{3}\right] = 2 \times \frac{1}{6}\left[-3.465\right] = -1.16 \\
b_2 &= 2 \times \frac{1}{6}\left[\sum y \sin\frac{2\pi x}{3}\right] = 2 \times \frac{1}{6}\left[0.0004\right] = 0.00013
\end{aligned}$$

$$\therefore y = f(x) = 20.84 - 8.33\cos\frac{\pi x}{3} - 1.161.004\sin\frac{\pi x}{3} - 2.333\cos\frac{2\pi x}{3} + 0.00013\sin\frac{2\pi x}{3}$$

2.3 Assignment II[Fourier series]

1. Find the Fourier series of $f(x) = \frac{1}{2}(\pi - x)$ in the interval $(0, 2\pi)$. Hence deduce that $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots \infty = \frac{\pi}{4}$.

2. Find the Fourier series expansion of $f(x) = e^x$ in the interval $(0, 2\ell)$.

3. Obtain the Fourier series for the function $f(x) = \begin{cases} 1 - x, & -\pi < x < 0 \\ 1 + x, & 0 < x < \pi \end{cases}$. Hence deduce that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\infty = \frac{\pi^2}{8}$.

4. Find the Fourier series for the function given by $f(x) = \begin{cases} 1 + \frac{2x}{\ell}, & -\ell \le x \le 0 \\ 1 - \frac{2x}{\ell}, & 0 \le x \le \ell \end{cases}$. Hence deduce that $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$.

5. Express $f(x)$ as a Fourier sine series where $f(x) = \begin{cases} \frac{1}{4} - x, 0 < x < \frac{1}{2} \\ x - \frac{3}{4}, \frac{1}{2} < x < 1 \end{cases}$.

6. Find the half range Fourier cosine series for the function $f(x) = x(\pi - x)$ in $0 < x < \pi$. Deduce that $\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \cdots \infty = \frac{\pi^4}{90}$.

7. Find the complex form of the Fourier series of $f(x) = e^{ax}, -\pi < x < \pi$.

8. Obtain a Fourier series upto the second harmonics from the data

x:	0	$\pi/3$	$2\pi/3$	π	$4\pi/3$	$5\pi/3$	2π
f(x):	0.8	0.6	0.4	0.7	0.9	1.1	0.8

9. The following table gives the vibration of periodic current over a period. Find the Fourier upto 2^{nd} harmonic.

T(sec):	0	$T/6$	$T/6$	$T/2$	$2T/3$	$5T/6$	T
I(Amp):	1.98	1.3	1.05	1.3	-0.88	-0.25	1.98

10. Find the Fourier series as for as the second harmonic to represent the function given in the following table

x:	0	1	2	3	4	5
f(x):	9	18	24	28	26	20

3 Applications of P.D.E.(A.P.D.E.)

3.1 Part-A

1. Classify the following partial differential equations.

   (i) $y^2U_{xx}-2xyU_{xy}+x^2U_{yy}+2U_x-3U=0$

   (ii) $y^2U_{xx}+U_{yy}+U_x^2+U_y^2+7=0$

   Soln :

   (i)Comparing the given equation with the general second order linear PDE,

   $AU_{xx}+BU_{xy}+CU_{yy}+DU_x+EU_y+Fu=0.$

   Here $A=y^2, B=-2xy, C=x^2$

   $$\begin{aligned}\therefore B^2-4AC &= (-2xy)^2-4\left(x^2\right)\left(y^2\right)\\ &= 4x^2y^2-4y^2x^2\\ &= 0\end{aligned}$$

   The given partial differential equation is Parabolic.

   (ii) Here $A=y^2, B=0, C=1$

   $$\therefore B^2-4AC=-4y^2<0$$

   The given partial differential equation is Elliptic.

2. Classify the following partial differential equations.

   (i) $\dfrac{\partial^2u}{\partial x^2}=\dfrac{\partial^2u}{\partial y^2}$

   (ii) $\dfrac{\partial^2u}{\partial x\partial y}=\left(\dfrac{\partial u}{\partial x}\right)\left(\dfrac{\partial u}{\partial y}\right)+xy$

   Soln :

   (i) Here $A=1, B=0, C=-1.$

   $$\therefore B^2-4AC=-4(-1)=4>0$$

   The given partial differential equation is Hyperbolic.

   (ii) Here $A=0, B=1, C=0.$

   $$\therefore B^2-4AC=1>0$$

   The given partial differential equation is Hyperbolic.

3. Classify the partial differential equation $\dfrac{\partial^2u}{\partial x^2}+x\dfrac{\partial^2u}{\partial y^2}=0$

   Soln : Here $A=1, B=0, C=x.$

   $$\therefore B^2-4AC=-4x$$

   The equation is Elliptic if $x>0$,
   Hyperbolic if $x<0$
   and Parabolic if $x=0$.

4. What is the constant a^2 in the wave equation $\dfrac{\partial^2y}{\partial t^2}=a^2\dfrac{\partial^2y}{\partial x^2}$?

   Soln : Here

   $$\begin{aligned}a^2 &= \frac{T}{M}\\ &= \frac{\text{The tension in the string}}{\text{Mass per unit length of the string}}\end{aligned}$$

5. Write the possible solutions of the one dimensional heat equation $\dfrac{\partial u}{\partial t}=a^2\dfrac{\partial^2u}{\partial x^2}$.

   Soln :

$$u(x,t) = (A\cos px + B\sin px)\, e^{-\alpha^2 p^2 t}$$
$$u(x,t) = \left(Ae^{px} + Be^{-px}\right) e^{\alpha^2 p^2 t}$$
$$u(x,t) = (Ax + B)$$

6. Write the possible solutions of the one dimensional wave $\dfrac{\partial^2 y}{\partial t^2} = a^2 \dfrac{\partial^2 y}{\partial x^2}$.

Soln :

$$y(x,t) = (A\cos px + B\sin px)(C\cos pat + D\sin pat)$$
$$y(x,t) = \left(Ae^{px} + Be^{-px}\right)\left(Ce^{pat} + De^{-pat}\right)$$
$$y(x,t) = (A_3 x + B_3)(C_3 t + D_3)$$

7. Write any two solutions of the Laplace equation $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$

Soln :

$$u(x,y) = (A\cos px + B\sin px)\left(Ce^{py} + De^{-py}\right)$$
$$u(x,y) = \left(Ae^{px} + Be^{-px}\right)(C\cos py + D\sin py)$$
$$u(x,y) = (Ax + B)(Cy + D)$$

8. A tightly stretched string of length 2ℓ is fastened at both ends. The mid point of the string is displaced to a distance b and released from rest in this position write the initial conditions.

Soln : The initial conditions are

(i) $\left(\dfrac{\partial y}{\partial t}\right)_{t=0} = 0, 0 \le x \le \ell$

(ii) $y(x,0) = \begin{cases} \dfrac{bx}{\ell}, & 0 \le x \le \ell \\ \dfrac{b(2\ell - x)}{\ell}, & \ell \le x \le 2\ell \end{cases}$

$\left[\text{Since equation of } OA \text{ is } \dfrac{y}{b} = \dfrac{x}{\ell} \Rightarrow y = \dfrac{bx}{\ell}\right]$

$\left[\text{Since equation of } AB \text{ is } \dfrac{y-b}{b} = \dfrac{\ell - x}{\ell} \Rightarrow y = \dfrac{b}{\ell}(2\ell - x)\right]$

9. Write the boundary conditions and initial conditions for solving the vibration of string equation, if the string is subjected to initial displacement $f(x)$ and initial velocity $g(x)$.

Soln : The boundary and initial conditions are

(i) $y(x=0,t) = 0$

(ii) $y(x=\ell,t) = 0$

(iii) $\dfrac{\partial y}{\partial t}(x, t=0) = g(x)$

(iv) $y(x, t=0) = f(x)$

10. In one dimensional heat equation $u_t = \alpha^2 u_{xx}$, what does α^2 stands for?

Soln :

$$\alpha^2 = \frac{k}{\rho c} = \frac{\text{Thermal Conductivity}}{\text{(Density Specific Heat)(Specific Heat)}} = \text{diffusivity of the material}$$

11. State any two laws which are assumed to derive one dimensional heat equation.

Soln :

(1) Heat flows from higher temperature to lower temperature.

(2) The amount of heat required to produce a given temperature change in a body is proportional to the mass of the body and to the temperature change.

(3) The rate at which heat flows through an area is proportional to the area and to the temperature gradient normal to the area.

12. In steady state conditions derive the solution of one dimensional heat flow equation.

Soln :

The temperature function u will be a function of x alone in the steady state condition.

i.e., $\frac{\partial u}{\partial t} = 0$, under steady state condition.

$\therefore$ The heat flow equation $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$ reduces to $\frac{\partial^2 u}{\partial x^2} = 0$

Integrating we get, $u(x) = ax + b$.

13. State one dimensional heat equation with the initial and boundary conditions.

Soln : The one dimensional heat flow equation is $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$

The boundary conditions are

$$\text{(i) } u(x=0,t) = k_1^\circ C \text{ for all } t > 0$$

$$\text{(ii) } u(x=\ell,t) = k_2^\circ C \text{ for all } t > 0$$

$$\text{(iii) } u(x,t=0) = f(x) \text{ in } (0,\ell)$$

$$\longrightarrow \text{ initial condition}$$

14. An insulated rod of length 60cm has its ends at A and B maintained at 20°C and 80°C respectively. Find the steady state solution of the heat equation in one dimension along the rod.

Soln :

Method : 1

$$u = \frac{u_{(x=60\text{cm})} - u_{(x=0\text{cm})}}{\text{length of the rod}}\, x + u_{(x=0\text{cm})}$$

$$= \frac{80-20}{60}\, x + 20$$

$\therefore u = x + 20$

(OR)

Method : 2

The equation of heat flow in one dimension is given by

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$$

In the steady state equation reduces to $\frac{\partial^2 u}{\partial x^2} = 0$.

Integrating we get, $u = ax + b$ (1)

When $x = 0, u = 20, \therefore b = 20$

When $x = 60, u = 80 \Rightarrow 80 = 60a + b$

$80 = 60a + 20 \Rightarrow 60a = 60 \Rightarrow a = 1$

$$\therefore (1) \Rightarrow u = x + 20$$

15. A rod 30cm long has its ends a and b kept at 200c and 800c respectively until steady state conditions prevail. Find the steady state temperature in the rod.

Soln :

Method : 1

$$u = \frac{u_{(x=30\text{cm})} - u_{(x=0\text{cm})}}{\text{length of the rod}}\, x + u_{(x=0\text{cm})}$$

$$= \frac{800-200}{30}\, x + 20$$

$\therefore u = 20x + 200$

(OR)

Method : 2

The equation of heat flow in one dimension is given by $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$

In the steady state equation reduces to $\frac{\partial^2 u}{\partial x^2} = 0$.

Integrating we get, $u = ax + b$

When $x = 0, u = 200, \therefore b = 200$

When $x = 30, u = 800 \Rightarrow 800 = 30a + b$

$$800 = 30a + 200$$

$$\Rightarrow 30a = 600$$

$$\Rightarrow a = 20$$

$$\therefore u = 20x + 200$$

16. What is the basic difference between the solutions of one dimensional wave equation and one dimensional heat equation.

Soln :

#	**one dimensional wave equation** $\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^y}{\partial x^2}$	**one dimensional heat equation** $\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}$
1.	It is a hyperbolic p.d.e.	It is a parabolic p.d.e.
2.	The suitable solution of one dim. wave eqn. is **periodic** w.r.t. time 't'.	The suitable solution of one dim. heat eqn. is **non-periodic** w.r.t. time 't'.

17. State any two assumptions made in the derivation of one-dimensional wave equation.

Soln :

(i) The motion takes entirely in one plane and in this plane each particle moves in a direction perpendicular to the equilibrium position of the string.

(ii) The tension in the string is constant.

(iii) The gravitational force may be neglected in comparison with the tension t.

(iv) The effect of friction is negligible.

18. Define temperature gradient.

Soln :

The rate of change of temperature with respect to distance is called temperature gradient and is denoted as $\frac{\partial u}{\partial x}$.

19. Distinguish between steady and unsteady states in heat conduction problems.

Soln :

In unsteady state the temperature at any point of the body depends on the position of the point and also the time t.

In steady state, the temperature at any point depends only on the position of the point and is independent of time $'t'$.

20. Write down the two dimensional heat equation both in transient and steady states.

Soln :

Transient state : $\frac{\partial u}{\partial t} = \alpha^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$

Steady state : $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

$\left[\because \frac{\partial u}{\partial t} = 0 \text{ in Transient state} \right]$

3.2 Part-B

3.2.1 One dimensional wave equation

3.2.1.1 Zero Velocity Problems

Example 3.1. A tightly stretched string of length ℓ is fastened at both ends. Motion is started by displacing the string into the form $y = k\left(\ell x - x^2\right)$, from which it is released at time $t = 0$. Find the displacement at any point on the string at a distance x from one end at time t. [UQ]

Solution : The displacement function $y(x,t)$ of the string is the solution of wave equation

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}. \quad \text{(I)}$$

where $a^2 = \dfrac{T}{M} = \dfrac{\text{Tension}}{\text{Mass}}$

Solving the equation (I) by method of separation of variable, we get the following three possible solutions.

$$(1) y(x,t) = \left(A_1 e^{px} + A_2 e^{-px}\right)\left(A_3 e^{apt} + A_4 e^{-apt}\right)$$
$$(2) y(x,t) = \left(A_5 \cos px + A_6 \sin px\right)\left(A_7 \cos apt + A_8 \sin apt\right)$$
$$(3) y(x,t) = \left(A_9 x + A_{10}\right)\left(A_{11} t + A_{12}\right)$$

out of these solutions, we have to select that solution which suits the physical nature of the problem and the boundary conditions.

Boundary conditions

$$(i) y(x=0,t) = 0, \forall t > 0$$
$$(ii) y(x=\ell,t) = 0, \forall t > 0$$

Initial conditions

$$(iii) \frac{\partial y}{\partial t}(x,t=0) = 0, \forall x \in (0,\ell)$$
$$(iv) y(x,t=0) = f(x) = k\left(\ell x - x^2\right), \forall x \in (0,\ell)$$

The suitable solution which satisfies above boundary conditions is

$$y(x,t) = \left(c_1 \cos px + c_2 \sin px\right)\left(c_3 \cos apt + c_4 \sin apt\right) \tag{1}$$

Applying condition (i) in equation (1), we get

We have $(i) \Rightarrow \quad y(x=0,t) = 0, \forall t > 0$

$$(1) \Rightarrow \quad \left(c_1 + 0\right)\left(c_3 \cos pat + c_4 \sin pat\right) = 0$$

Here $\left(c_3 \cos pat + c_4 \sin pat\right) \neq 0 \quad (\because$ it is defined $\forall t > 0)$

$$\boxed{\therefore c_1 = 0}$$

$$\therefore \text{ Now } (1) \Rightarrow y(x,t) = c_2 \sin px \left(c_3 \cos pat + c_4 \sin pat\right) \tag{2}$$

Applying condition (ii) in equation (2), we get

We have $(ii) \Rightarrow \quad y(x=\ell,t) = 0, \forall t > 0$

$$(2) \Rightarrow \quad \left(c_2 \sin p\ell\right)\left(c_3 \cos pat + c_4 \sin pat\right) = 0$$

Here $\left(c_3 \cos pat + c_4 \sin pat\right) \neq 0 \quad (\because$ it is defined $\forall t > 0)$

$$\therefore c_2 \neq 0 \quad (\because \text{ it gives trivial solution})$$

$$\therefore \sin p\ell = 0$$
$$= \sin n\pi$$

$$\boxed{\therefore p = \frac{n\pi}{\ell}}$$

$$\therefore \text{ Now } (2) \Rightarrow y(x,t) = c_2 \sin\left(\frac{n\pi x}{\ell}\right)\left(c_3 \cos \frac{n\pi}{\ell}at + c_4 \sin \frac{n\pi}{\ell}at\right) \tag{3}$$

Before applying condition (iii) in (3), differentiate (3) w.r.t. 't'

$$\frac{\partial y}{\partial t}(x,t) = c_2 \sin \frac{n\pi x}{\ell}\left[-c_3\left(\frac{n\pi}{\ell}a\right)\sin \frac{n\pi}{\ell}at + c_4\left(\frac{n\pi}{\ell}a\right)\cos \frac{n\pi}{\ell}at\right] \tag{4}$$

Applying condition (iii) in equation (4), we get

$$\text{We have } (iii) \Rightarrow \quad \frac{\partial y}{\partial t}(x, t=0) = 0, \forall x \in (0,\ell)$$

$$(4) \Rightarrow \quad c_2 \sin\frac{n\pi x}{\ell}\left[c_4\left(\frac{n\pi}{\ell}a\right)\right] = 0$$

$$\text{Here,} \quad c_2 \neq 0 \quad (\because \text{ it gives trivial solution})$$

$$\sin\frac{n\pi x}{\ell} \neq 0 \quad (\because \text{ it is defined } \forall x \in (0,\ell))$$

$$\frac{n\pi}{\ell}a \neq 0 \quad (\because \text{ all are constants})$$

$$\boxed{\therefore c_4 = 0}$$

$$\begin{aligned}\therefore \text{ Now } (3) \Rightarrow y(x,t) &= c_2 \sin\left(\frac{n\pi}{\ell}x\right) c_3 \cos\left(\frac{n\pi}{\ell}at\right)\\ &= c_2c_3 \sin\left(\frac{n\pi}{\ell}x\right)\cos\left(\frac{n\pi}{\ell}at\right)\\ &= c_n \sin\left(\frac{n\pi}{\ell}x\right)\cos\left(\frac{n\pi}{\ell}at\right)\end{aligned}$$

where $c_n = c_2c_3$

$\therefore$ By superposition principle(i.e., adding all such solutions), the most general solution is

$$y(x,t) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{\ell}\right)\cos\left(\frac{n\pi a}{\ell}t\right) \tag{5}$$

Applying condition (iv) in equation (5), we get

$$\text{We have } (iv) \Rightarrow \quad y(x, t=0) = f(x), \forall x \in (0,\ell)$$

$$(5) \Rightarrow \quad \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{\ell}\right) = f(x) = k\left(\ell x - x^2\right) \tag{6}$$

To find c_n,

Expand $f(x)$ in a half range Fourier sine series in the interval $(0,\ell)$.

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{\ell} \tag{7}$$

where $b_n = \dfrac{2}{\ell}\displaystyle\int_0^\ell f(x)\sin\frac{n\pi x}{\ell}dx$

From (6) and (7), we have $c_n = b_n$

$$\begin{aligned}
\therefore c_n =& \frac{2}{\ell}\int_0^{\ell} f(x)\sin\frac{n\pi x}{\ell}dx \\
=& \frac{2}{\ell}\int_0^{\ell} k(\ell x - x^2)\sin\frac{n\pi x}{\ell}dx \\
=& \frac{2k}{\ell}\left[\left(0+0-\frac{2\ell^3}{n^3\pi^3}(-1)^n\right)-\left(0+0-\frac{2\ell^3}{n^3\pi^3}\right)\right] \\
& (\sin n\pi = 0\,, \cos n\pi = (-1)^n) \\
=& \frac{4k\ell^2}{n^3\pi^3}[1-(-1)^n] \\
\therefore c_n =& \begin{cases} 0, & \text{if } n \text{ is even} \\ \dfrac{8k\ell^2}{n^3\pi^3}, & \text{if } n \text{ is odd} \end{cases}
\end{aligned}$$

$\therefore$ The most general solution (5) reduces to

$$y(x,t) = \sum_{n=\text{odd}}^{\infty} \frac{8k\ell^2}{n^3\pi^3}\sin\left(\frac{n\pi}{\ell}x\right)\cos\left(\frac{n\pi a}{\ell}t\right)$$

This is the required displacement function $y(x,t)$.

Example 3.2. A string is stretched and fastened to two points at a distance 'ℓ' apart. Motion is started by displacing the string in the form $y = k\sin\dfrac{\pi x}{\ell}$ from which it is released at time $t = 0$. Show that the displacement of any point on the string at a distance x from one end at time t is given by $k\sin\dfrac{\pi x}{\ell}\cos\dfrac{\pi a t}{\ell}$.

Solution : The displacement function $y(x,t)$ of the string is the solution of wave equation

$$\frac{\partial^2 y}{\partial t^2} = a^2\frac{\partial^2 y}{\partial x^2}. \tag{I}$$

where $a^2 = \dfrac{T}{M} = \dfrac{\text{Tension}}{\text{Mass}}$

Solving the equation (I) by method of separation of variable, we get the following three possible solutions.

$$\begin{aligned}
&(1) y(x,t) = \left(A_1e^{px} + A_2e^{-px}\right)\left(A_3e^{apt} + A_4e^{-apt}\right) \\
&(2) y(x,t) = \left(A_5\cos px + A_6\sin px\right)\left(A_7\cos apt + A_8\sin apt\right) \\
&(3) y(x,t) = \left(A_9x + A_{10}\right)\left(A_{11}t + A_{12}\right)
\end{aligned}$$

out of these solutions, we have to select that solution which suits the physical nature of the problem and the boundary conditions.

Boundary conditions

$$\begin{aligned}
(i) y(x=0,t) &= 0, \forall t > 0 \\
(ii) y(x=\ell,t) &= 0, \forall t > 0
\end{aligned}$$

Initial conditions

$$(iii)\frac{\partial y}{\partial t}(x,t=0)=0, \forall x \in (0,\ell)$$

$$(iv) y(x,t=0)=f(x)=k\sin\frac{\pi x}{\ell}, \forall x \in (0,\ell)$$

The suitable solution which satisfies above boundary conditions is

$$y(x,t)=(c_1\cos px+c_2\sin px)(c_3\cos apt+c_4\sin apt) \quad (1)$$

Applying condition (i) in equation (1), we get

We have $(i) \Rightarrow \quad y(x=0,t)=0, \forall t>0$

$$(1) \Rightarrow \quad (c_1+0)(c_3\cos pat+c_4\sin pat)=0$$

Here $(c_3\cos pat+c_4\sin pat)\neq 0 \quad (\because$ it is defined $\forall t>0)$

$$\therefore c_1=0$$

$$\therefore \text{ Now } (1) \Rightarrow y(x,t)=c_2\sin px\,(c_3\cos pat+c_4\sin pat) \quad (2)$$

Applying condition (ii) in equation (2), we get

We have $(ii) \Rightarrow \quad y(x=\ell,t)=0, \forall t>0$

$$(2) \Rightarrow \quad (c_2\sin p\ell)(c_3\cos pat+c_4\sin pat)=0$$

Here $(c_3\cos pat+c_4\sin pat)\neq 0 \quad (\because$ it is defined $\forall t>0)$

$\therefore c_2\neq 0 \quad (\because$ it gives trivial solution)

$$\therefore \sin p\ell=0$$
$$=\sin n\pi$$
$$\therefore p=\frac{n\pi}{\ell}$$

$$\therefore \text{ Now } (2) \Rightarrow y(x,t)=c_2\sin\left(\frac{n\pi x}{\ell}\right)\left(c_3\cos\frac{n\pi}{\ell}at+c_4\sin\frac{n\pi}{\ell}at\right) \quad (3)$$

Before applying condition (iii) in (3), differentiate (3) w.r.t. 't'

$$\frac{\partial y}{\partial t}(x,t)=c_2\sin\frac{n\pi x}{\ell}\left[-c_3\left(\frac{n\pi}{\ell}a\right)\sin\frac{n\pi}{\ell}at+c_4\left(\frac{n\pi}{\ell}a\right)\cos\frac{n\pi}{\ell}at\right] \quad (4)$$

Applying condition (iii) in equation (4), we get

We have $(iii) \Rightarrow \quad \frac{\partial y}{\partial t}(x,t=0)=0, \forall x \in (0,\ell)$

$$(4) \Rightarrow \quad c_2\sin\frac{n\pi x}{\ell}\left[c_4\left(\frac{n\pi}{\ell}a\right)\right]=0$$

Here, $c_2\neq 0 \quad (\because$ it gives trivial solution)

$\sin\frac{n\pi x}{\ell}\neq 0 \quad (\because$ it is defined $\forall x\in(0,\ell))$

$\frac{n\pi}{\ell}a\neq 0 \quad (\because$ all are constants)

$$\therefore c_4=0$$

$$\therefore \text{ Now } (3) \Rightarrow y(x,t)=c_2\sin\left(\frac{n\pi}{\ell}x\right)c_3\cos\left(\frac{n\pi}{\ell}at\right)$$
$$=c_2c_3\sin\left(\frac{n\pi}{\ell}x\right)\cos\left(\frac{n\pi}{\ell}at\right)$$
$$=c_n\sin\left(\frac{n\pi}{\ell}x\right)\cos\left(\frac{n\pi}{\ell}at\right)$$

where $c_n=c_2c_3$

$\therefore$ By superposition principle(i.e., adding all such solutions), the most general solution is

$$y(x,t) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{\ell}\right) \cos\left(\frac{n\pi a}{\ell} t\right) \tag{5}$$

Applying condition (iv) in equation (5), we get

$$\text{We have } (iv) \Rightarrow \qquad y(x, t=0) = f(x), \forall x \in (0,\ell)$$

$$(5) \Rightarrow \qquad \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{\ell}\right) = f(x) = k \sin\frac{\pi x}{\ell} \tag{6}$$

To find c_n,

Expand $f(x)$ in a half range Fourier sine series in the interval $(0,\ell)$.

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{\ell} \tag{7}$$

where $b_n = \dfrac{2}{\ell}\displaystyle\int_0^{\ell} f(x) \sin\frac{n\pi x}{\ell} dx$

From (6) and (7), we have $c_n = b_n$

$$k \sin\frac{\pi x}{\ell} = \sum_{n=1}^{\infty} c_n \sin\left(\frac{\pi x}{\ell}\right)$$

$$k \sin\frac{n\pi x}{\ell} = c_1 \sin\left(\frac{\pi x}{\ell}\right) + c_2 \sin\left(\frac{2\pi x}{\ell}\right) + c_3 \sin\left(\frac{3\pi x}{\ell}\right) + \dots$$

Comparing the like coefficients, we get

$$\therefore c_1 = k, c_2 = c_3 = c_4 = \cdots = 0$$

$\therefore$ The most general solution (5) reduces to

$$\therefore y(x,t) = k \sin\frac{\pi x}{\ell} \cos\frac{\pi a t}{\ell}$$

This is the required displacement function $y(x,t)$.

Example 3.3. [Section problem]

A uniform string is fixed at the ends $x = 0$ and $x = \ell$ are fixed. One end is taken at the origin and at a distance 'b' from this end the string is displaced a distance 'h' transversely and is released from rest in that position. Find the displacement of any of the string at any subsequent time.

Solution : This is of Zero Velocity example format as previous.

The displacement function $y(x,t)$ of the string is the solution of wave equation $\dfrac{\partial^2 y}{\partial t^2} = a^2 \dfrac{\partial^2 y}{\partial x^2}$. (I)

where $a^2 = \dfrac{T}{M} = \dfrac{\text{Tension}}{\text{Mass}}$

Solving the equation (I) by method of separation of variable, we get the following three possible solutions.

$$(1) y(x,t) = \left(A_1 e^{px} + A_2 e^{-px}\right)\left(A_3 e^{apt} + A_4 e^{-apt}\right)$$
$$(2) y(x,t) = (A_5 \cos px + A_6 \sin px)(A_7 \cos apt + A_8 \sin apt)$$
$$(3) y(x,t) = (A_9 x + A_{10})(A_{11} t + A_{12})$$

out of these solutions, we have to select that solution which suits the physical nature of the problem and the boundary conditions.

Boundary conditions

$$(i) y(x=0,t) = 0, \forall t > 0$$
$$(ii) y(x=\ell,t) = 0, \forall t > 0$$

Initial conditions

$$(iii) \frac{\partial y}{\partial t}(x, t=0) = 0, \forall x \in (0,\ell)$$

$(iv) y(x,t=0) = f(x), \forall x \in (0,\ell),$ where $f(x)$ is to be find later.

The suitable solution which satisfies above boundary conditions is

$$y(x,t) = (c_1 \cos px + c_2 \sin px)(c_3 \cos apt + c_4 \sin apt) \tag{1}$$

As in previous example, the solution of (I) satisfying the boundary conditions (i),(ii) and (iii) is

$$y(x,t) = c_n \sin\frac{n\pi x}{\ell}\cos\frac{n\pi at}{\ell} \qquad \text{(where } c_n = c_2c_3)$$

$\therefore$ By superposition principle(i.e., adding all such solutions), the most general solution is

$$y(x,t) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{\ell}\right)\cos\left(\frac{n\pi a}{\ell}t\right) \tag{5}$$

Applying condition (iv) in equation (5), we get

$$\text{We have } (iv) \Rightarrow \qquad y(x,t=0) = f(x), \forall x \in (0,\ell)$$
$$(5) \Rightarrow \qquad \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{\ell}\right) = f(x) \tag{6}$$

To find c_n,

Expand $f(x)$ in a half range Fourier sine series in the interval $(0,\ell)$.

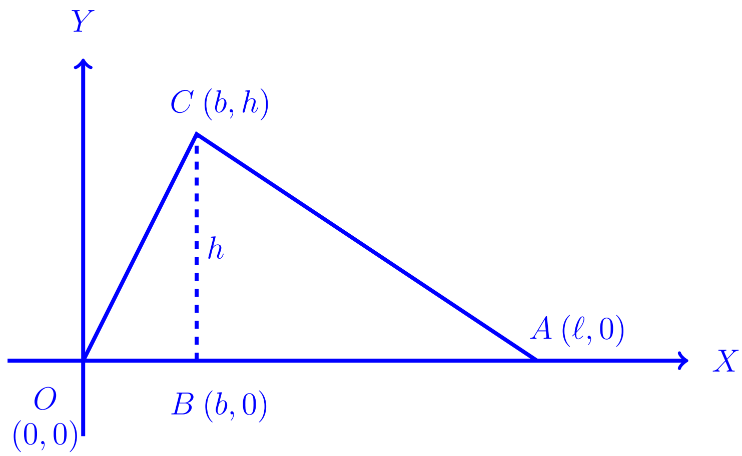
$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{\ell} \tag{7}$$

where $b_n = \dfrac{2}{\ell}\displaystyle\int_0^{\ell} f(x)\sin\frac{n\pi x}{\ell}dx$

From (6) and (7), we have $c_n = b_n$

$$c_n = \frac{2}{\ell}\int_0^{\ell} f(x)\sin\frac{n\pi x}{\ell}dx \tag{8}$$

where $f(x)$ is not yet find, we will find $f(x)$ as follows
Taking $OA = \ell$, length of the string with B as its points of bisection.



The initial position of the string is given by the equations to the lines OB and BA.

To find y = f (x)= $\begin{cases} \textbf{Equation of OC}, & 0 < x < b \\ \textbf{Equation of CA}, & b < x < \ell \end{cases}$

Equation of the line OC joining $(0, 0)$ & (b, h) is given by

$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$$
$$\frac{y - 0}{h - 0} = \frac{x - 0}{b - 0}$$
$$\therefore y = \frac{h}{b}x, 0 \le x \le b$$

$\therefore y = \frac{h}{b}x, 0 \le x \le b$

Equation of the line CA joining (b, h) & $(\ell, 0)$ is given by

$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$$
$$\frac{y - h}{0 - h} = \frac{x - b}{\ell - b} \Rightarrow \frac{y - h}{-h} = \frac{x - b}{\ell - b}$$
$$(y - h)(\ell - b) = h(x - b) \Rightarrow y\ell - yb - h\ell + hb = -hx + hb$$
$$y(\ell - b) = h(\ell - x) \Rightarrow y(b - \ell) = h(x - \ell)$$
$$\therefore y = \frac{h}{b - \ell}[x - \ell], b \le x \le \ell$$

$\therefore$ The required equation of the string is

$$y = f(x) = \begin{cases} \frac{h}{b}x & , 0 \le x \le b \\ \frac{h}{b - \ell}[x - \ell] & , b \le x \le \ell \end{cases}$$

$\therefore (8) \Rightarrow$

$$c_n = \frac{2}{\ell}\left[\int_0^b \frac{hx}{b}\sin\frac{n\pi x}{\ell}dx + \int_b^\ell \frac{h(x-\ell)}{b-\ell}\sin\frac{n\pi x}{\ell}dx\right]$$

$$= \frac{2h}{\ell}\left\{\frac{1}{b}\left[x\left(\frac{-\cos\frac{n\pi x}{\ell}}{\frac{n\pi}{\ell}}\right) - (1)\left(\frac{-\sin\frac{n\pi x}{\ell}}{\left(\frac{n\pi}{\ell}\right)^2}\right)\right]_0^b\right.$$

$$\left. + \frac{1}{b-\ell}\left[(x-\ell)\left(\frac{-\cos\frac{n\pi x}{\ell}}{\frac{n\pi}{\ell}}\right) - (1)\left(\frac{-\sin\frac{n\pi x}{\ell}}{\left(\frac{n\pi}{\ell}\right)^2}\right)\right]_b^\ell\right\}$$

$$= \frac{2h}{\ell}\left\{\frac{1}{b}\left[-x\frac{\ell}{n\pi}\left(\cos\frac{n\pi x}{\ell}\right) + \left(\frac{\ell}{n\pi}\right)^2\left(\sin\frac{n\pi x}{\ell}\right)\right]_0^b\right.$$

$$\left. + \frac{1}{b-\ell}\left[-(x-\ell)\frac{\ell}{n\pi}\left(\cos\frac{n\pi x}{\ell}\right) + \left(\frac{\ell}{n\pi}\right)^2\left(\sin\frac{n\pi x}{\ell}\right)\right]_b^\ell\right\}$$

$$= \frac{2h}{\ell}\left\{\frac{1}{b}\left[-\frac{b\ell}{n\pi}\left(\cos\frac{n\pi b}{\ell}\right) + \left(\frac{\ell}{n\pi}\right)^2\left(\sin\frac{n\pi b}{\ell}\right)\right] - [0+0]\right.$$

$$\left. + \frac{1}{b-\ell}\left[[0+0] - \left[-(b-\ell)\frac{\ell}{n\pi}\left[\cos\frac{n\pi b}{\ell}\right] + \left[\frac{\ell}{n\pi}\right]^2\left[\sin\frac{n\pi b}{\ell}\right]\right]\right]\right\}$$

$$= \frac{2h}{\ell}\left\{\frac{-\ell}{n\pi}\left[\cos\frac{n\pi b}{\ell}\right] + \left[\frac{\ell}{n\pi}\right]^2\frac{1}{b}\left[\sin\frac{n\pi b}{\ell}\right] + \frac{\ell}{n\pi}\left[\cos\frac{n\pi b}{\ell}\right]\right.$$

$$\left. - \frac{1}{b-\ell}\left[\frac{\ell}{n\pi}\right]^2\left[\sin\frac{n\pi b}{\ell}\right]\right\}$$

$$= \frac{2h}{\ell}\left[\left(\frac{\ell}{n\pi}\right)^2\frac{1}{b}\left(\sin\frac{n\pi b}{\ell}\right) - \frac{1}{b-\ell}\left(\frac{\ell}{n\pi}\right)^2\left(\sin\frac{n\pi b}{\ell}\right)\right]$$

$$= \frac{2h}{\ell}\left[\left(\frac{\ell}{n\pi}\right)^2\left(\sin\frac{n\pi b}{\ell}\right)\left(\frac{1}{b} - \frac{1}{b-\ell}\right)\right]$$

$$= \frac{-2h\ell^2}{b(b-\ell)n^2\pi^2}\left(\sin\frac{n\pi b}{\ell}\right)$$

$\therefore$ The most general solution (5) reduces to

i.e., (4) $\Rightarrow y(x,t) = \dfrac{-2h\ell^2}{b(b-\ell)\pi^2}\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}\left(\sin\frac{n\pi b}{\ell}\right)\sin\left(\frac{n\pi x}{\ell}\right)\cos\left(\frac{n\pi a}{\ell}t\right)$

This is the required displacement solution $y(x,t)$.

3.2.1.2 [Non - Zero Velocity Problem]

Example 3.4. A tightly stretched string with fixed end points $x = 0$ and $x = \ell$ is initially at rest in its equilibrium position. If it is set vibrating by giving to each of its points a velocity $\lambda x(\ell - x)$, find $y(x,t)$.

Solution : The displacement function $y(x,t)$ of the string is the solution of wave equation

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}. \tag{I}$$

where $a^2 = \frac{T}{M} = \frac{\text{Tension}}{\text{Mass}}$

Solving the equation (I) by method of separation of variable, we get the following three possible solutions.

$$(1) y(x,t) = \left(A_1 e^{px} + A_2 e^{-px}\right)\left(A_3 e^{apt} + A_4 e^{-apt}\right)$$
$$(2) y(x,t) = \left(A_5 \cos px + A_6 \sin px\right)\left(A_7 \cos apt + A_8 \sin apt\right)$$
$$(3) y(x,t) = \left(A_9 x + A_{10}\right)\left(A_{11} t + A_{12}\right)$$

out of these solutions, we have to select that solution which suits the physical nature of the problem and the boundary conditions.

Boundary conditions

$$(i) y(x=0,t) = 0, \forall t > 0$$
$$(ii) y(x=\ell,t) = 0, \forall t > 0$$

Initial conditions

$$(iii) y(x,t=0) = 0, \forall x \in (0,\ell)$$
$$(iv) \frac{\partial y}{\partial t}(x,t=0) = f(x) = \lambda\left(\ell x - x^2\right), \forall x \in (0,\ell)$$

The suitable solution which satisfies above boundary conditions is

$$y(x,t) = \left(c_1 \cos px + c_2 \sin px\right)\left(c_3 \cos apt + c_4 \sin apt\right) \tag{1}$$

Applying condition (i) in equation (1), we get

We have $(i) \Rightarrow$ $\quad y(x=0,t) = 0, \forall t > 0$

$$(1) \Rightarrow \quad (c_1 + 0)\left(c_3 \cos pat + c_4 \sin pat\right) = 0$$

Here $\left(c_3 \cos pat + c_4 \sin pat\right) \neq 0 \quad (\because$ it is defined $\forall t > 0)$

$$\therefore c_1 = 0$$

$$\therefore \text{ Now } (1) \Rightarrow y(x,t) = c_2 \sin px \left(c_3 \cos pat + c_4 \sin pat\right) \tag{2}$$

Applying condition (ii) in equation (2), we get

We have $(ii) \Rightarrow$ $\quad y(x=\ell,t) = 0, \forall t > 0$

$$(2) \Rightarrow \quad (c_2 \sin p\ell)\left(c_3 \cos pat + c_4 \sin pat\right) = 0$$

Here $\left(c_3 \cos pat + c_4 \sin pat\right) \neq 0 \quad (\because$ it is defined $\forall t > 0)$

$\therefore c_2 \neq 0 \quad (\because$ it gives trivial solution$)$

$$\therefore \sin p\ell = 0$$
$$= \sin n\pi$$
$$\therefore p = \frac{n\pi}{\ell}$$

$$\therefore \text{ Now } (2) \Rightarrow y(x,t) = c_2 \sin\left(\frac{n\pi x}{\ell}\right)\left(c_3 \cos \frac{n\pi}{\ell} at + c_4 \sin \frac{n\pi}{\ell} at\right) \tag{3}$$

Applying condition (iii) in (3), we get

We have $(iii) \Rightarrow y(x,t=0)=0$

$\therefore (3) \Rightarrow \left[c_2 \sin\frac{n\pi x}{\ell}(c_3)\right]=0$

$$\begin{aligned} c_2 &\neq 0 && (\because \text{ it gives trivial solution}) \\ \sin\frac{n\pi x}{\ell} &\neq 0 && (\because \text{ it is defined for all } x) \\ \therefore c_3 &= 0 \end{aligned}$$

$$\begin{aligned} \therefore \text{ Now } (3) \Rightarrow y(x,t) &= c_2 \sin\left(\frac{n\pi}{\ell}x\right) c_4 \sin\left(\frac{n\pi}{\ell}at\right) \\ &= c_2c_4 \sin\left(\frac{n\pi}{\ell}x\right) \sin\left(\frac{n\pi}{\ell}at\right) \\ &= c_n \sin\left(\frac{n\pi}{\ell}x\right) \sin\left(\frac{n\pi}{\ell}at\right) \end{aligned}$$

where $c_n = c_2c_4$

$\therefore$ By superposition principle(i.e., adding all such above solutions), the most general solution is

$$y(x,t)=\sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{\ell}\right)\sin\left(\frac{n\pi a}{\ell}t\right) \tag{4}$$

Before applying condition (iv), differentiate (4) w.r.t. 't'

$$\frac{\partial y}{\partial t}(x,t)=\sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{\ell}\right)\frac{n\pi a}{\ell}\cos\left(\frac{n\pi a}{\ell}t\right) \tag{5}$$

Applying condition (iv) in equation (5), we get

$$\begin{aligned} \text{We have } (iv) \Rightarrow \quad & \frac{\partial y}{\partial t}(x,t=0)=f(x), \forall x \in (0,\ell) \\ (5) \Rightarrow \quad & \sum_{n=1}^{\infty} c_n\frac{n\pi a}{\ell}\sin\left(\frac{n\pi x}{\ell}\right)=f(x)=\lambda\left(\ell x-x^2\right) \end{aligned} \tag{6}$$

To find c_n,

Expand $f(x)$ in a half range Fourier sine series in the interval $(0,\ell)$

$$f(x)=\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{\ell} \tag{7}$$

where $b_n=\frac{2}{\ell}\int_0^{\ell} f(x)\sin\frac{n\pi x}{\ell}dx$

From (6) and (7), we have $b_n = c_n\frac{n\pi a}{\ell}$

$$c_n\frac{n\pi a}{\ell} = \frac{2}{\ell}\int_0^{\ell} f(x)\sin\frac{n\pi x}{\ell}dx$$

$$=\frac{2}{\ell}\int_0^\ell \lambda(lx-x^2)\sin\frac{n\pi x}{\ell}dx$$

$$=\frac{2\lambda}{\ell}\left[\left(0+0-\frac{2\ell^3}{n^3\pi^3}(-1)^n\right)-\left(0+0-\frac{2\ell^3}{n^3\pi^3}\right)\right]$$

$$(\because \sin n\pi=0\,,\cos n\pi=(-1)^n)$$

$$=\frac{4\lambda\ell^2}{n^3\pi^3}[1-(-1)^n]$$

$$\therefore c_n=\begin{cases}0 & ,\text{ if } n \text{ is even}\\ \dfrac{8\lambda\ell^3}{n^4\pi^4} & ,\text{ if } n \text{ is odd}\end{cases}$$

$\therefore$ The most general solution is

i.e., (4) $\Rightarrow y(x,t)=\dfrac{8\lambda\ell^3}{\pi^4}\displaystyle\sum_{n=\text{odd}}^{\infty}\frac{1}{n^4}\sin\left(\frac{n\pi x}{\ell}\right)\sin\left(\frac{n\pi a}{\ell}t\right)$

This is the required displacement $y(x,t)$.

Example 3.5. A tightly stretched string with fixed end points $x=0$ and $x=\ell$ is initially in a position with velocity ν given by $\nu=y_0\sin^3\left(\frac{\pi x}{\ell}\right)$. Find the displacement $y(x,t)$ in terms of Fourier coefficients of $f(x)$.

Solution : The displacement function $y(x,t)$ of the string is the solution of wave equation

$$\frac{\partial^2 y}{\partial t^2}=a^2\frac{\partial^2 y}{\partial x^2}. \tag{I}$$

where $a^2=\dfrac{T}{M}=\dfrac{\text{Tension}}{\text{Mass}}$

Solving the equation (I) by method of separation of variable, we get the following three possible solutions.

$$(1)y(x,t)=\left(A_1e^{px}+A_2e^{-px}\right)\left(A_3e^{apt}+A_4e^{-apt}\right)$$

$$(2)y(x,t)=\left(A_5\cos px+A_6\sin px\right)\left(A_7\cos apt+A_8\sin apt\right)$$

$$(3)y(x,t)=\left(A_9x+A_{10}\right)\left(A_{11}t+A_{12}\right)$$

out of these solutions, we have to select that solution which suits the physical nature of the problem and the boundary conditions.

Boundary conditions

$$(i)y(x=0,t)\quad=0,\forall t>0$$

$$(ii)y(x=\ell,t)\quad=0,\forall t>0$$

Initial conditions

$$(iii)y(x,t=0)=0,\forall x\in(0,\ell)$$

$$(iv)\frac{\partial y}{\partial t}(x,t=0)=f(x)=\nu=y_0\sin^3\left(\frac{\pi x}{\ell}\right)$$

The suitable solution which satisfies above boundary conditions is

$$y(x,t)=(c_1\cos px+c_2\sin px)\left(c_3\cos apt+c_4\sin apt\right) \tag{1}$$

Applying condition (i),(ii) and (iii) as in previous problem, we get

$\therefore$ By superposition principle(i.e., adding all such above solutions), the most general solution is

$$y(x,t) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{\ell}\right) \sin\left(\frac{n\pi a}{\ell}t\right) \tag{4}$$

Before applying condition (iv), differentiate (4) w.r.t. 't'

$$\frac{\partial y}{\partial t}(x,t) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{\ell}\right) \frac{n\pi a}{\ell} \cos\left(\frac{n\pi a}{\ell}t\right) \tag{5}$$

Applying condition (iv) in equation (5), we get

$$\text{We have } (iv) \Rightarrow \quad \frac{\partial y}{\partial t}(x,t=0) = f(x)$$

$$(5) \Rightarrow \quad \sum_{n=1}^{\infty} c_n \frac{n\pi a}{\ell} \sin\left(\frac{n\pi x}{\ell}\right) = f(x) = \nu = y_0 \sin^3\left(\frac{\pi x}{\ell}\right) \tag{6}$$

To find c_n,

Expand $f(x)$ in a half range Fourier sine series in the interval $(0,\ell)$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{\ell} \tag{7}$$

where $b_n = \frac{2}{\ell}\int\limits_0^{\ell} f(x) \sin\frac{n\pi x}{\ell} dx$

From (6) and (7), we have $b_n = c_n \frac{n\pi a}{\ell}$

To find c_n,

$$y_0 \sin^3\left(\frac{\pi x}{\ell}\right) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{\ell}\right)$$

$$y_0\left[\frac{3\sin\left(\frac{\pi x}{\ell}\right) - \sin 3\left(\frac{\pi x}{\ell}\right)}{4}\right] = b_1 \sin\left[\frac{\pi}{\ell}x\right] + b_2 \sin\left[\frac{2\pi}{\ell}x\right] + b_3 \sin\left[\frac{3\pi}{\ell}x\right] + \dots$$

$$\left[\because \sin^3 A = \frac{3\sin A - \sin 3A}{4}\right]$$

$$\frac{3y_0}{4}\sin\left[\frac{\pi x}{\ell}\right] - \frac{y_0}{4}\sin\left[\frac{3\pi x}{\ell}\right] = b_1 \sin\left[\frac{\pi}{\ell}x\right] + b_2 \sin\left[\frac{2\pi}{\ell}x\right] + b_3 \sin\left[\frac{3\pi}{\ell}x\right] + \dots$$

Comparing the like coefficients, we get

$$\therefore b_1 = \frac{3y_0}{4} \qquad \text{and } b_3 = \frac{-y_0}{4}, \qquad b_n = 0 \text{ for } n \neq 2,4,5,6$$

$$\therefore c_1 = b_1\frac{\ell}{\pi a} = \frac{3y_0}{4}\frac{\ell}{\pi a} \qquad \text{and } c_3 = b_3\frac{\ell}{3\pi a} = \frac{-y_0}{4}\frac{\ell}{3\pi a}, \quad c_n = 0 \text{ for } n \neq 2,4,5,6$$

$$\left[\because b_n = c_n\frac{n\pi a}{\ell} \Rightarrow c_n = b_n\frac{\ell}{n\pi a}\right]$$

$\therefore$ The most general solution is

$$\text{i.e.,}(4) \Rightarrow y(x,t) = c_1 \sin\left[\frac{\pi}{\ell}x\right] \sin\left[\frac{\pi a}{\ell}t\right] + c_3 \sin\left[\frac{3\pi}{\ell}x\right] \sin\left[\frac{3\pi a}{\ell}t\right]$$
$$= \frac{3y_0}{4}\frac{\ell}{\pi a} \sin\left[\frac{\pi}{\ell}x\right] \sin\left[\frac{\pi a}{\ell}t\right] + \frac{-y_0}{4}\frac{\ell}{3\pi a} \sin\left[\frac{3\pi}{\ell}x\right] \sin\left[\frac{3\pi a}{\ell}t\right]$$
$$= \frac{3y_0\ell}{4\pi a} \sin\left[\frac{\pi}{\ell}x\right] \sin\left[\frac{\pi a}{\ell}t\right] - \frac{y_0\ell}{12\pi a} \sin\left[\frac{3\pi}{\ell}x\right] \sin\left[\frac{3\pi a}{\ell}t\right]$$

This is the required displacement $y(x,t)$.

3.2.2 One Dimensional heat flow equation

3.2.2.1 Steady state conditions and zero boundary conditions

Example 3.6. A rod of length ℓ has its ends A and B kept at $0°C$ and $120°C$ respectively until steady state conditions prevail. If the temperature at B is reduced to $0°C$ and kept so while that of A is maintained, find the resulting temperature distribution $u(x,t)$ taking origin at A.

Solution : The temperature function $u(x,t)$ satisfies the one dimensional heat equation is

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}. \qquad \text{(I)}$$

where $a^2 = \dfrac{k}{\rho c} = \dfrac{\text{Thermal conductivity}}{\text{(Density)(Specific heat)}}$

Solving the equation (I) by method of separation of variable, we get the following three possible solutions.

$$(1) u(x,t) = \left(A_1 e^{px} + A_2 e^{-px}\right) A_3 e^{a^2p^2t}$$
$$(2) u(x,t) = \left(A_4 \cos px + A_5 \sin px\right) A_6 e^{-a^2p^2t}$$
$$(3) u(x,t) = \left(A_7 + A_8 x\right) A_9$$

out of these solutions, we have to select that solution which suits the physical nature of the problem and the boundary conditions.

First let us find the temperature distribution at any distance x, before the end A and B are reduced to zero. Prior to the temperature change at the ends A and B, when $t = 0$, the heat flow was independent of time (steady state conditions). When the temperature u depends only on x and not on t,

When steady state conditions prevail

(I) reduces to

$$\frac{\partial u}{\partial t} = 0 \Rightarrow \frac{\partial^2 u}{\partial x^2} = 0 \qquad \text{(II)}$$

The general solution of (II) is $u = ax + b$ (III)

where a and b are arbitrary constants.

Given that $u = 0$ when $x = 0$

$\therefore$ From (III), we get $0 = a(0) + b \Rightarrow b = 0$

Also given that $u = 120$ when $x = \ell$

$$\text{From (III)}, a\ell + b = 120$$
$$a\ell + 0 = 120$$
$$a = \frac{120}{\ell}x$$

$\therefore$ (III) becomes $u = \dfrac{120}{\ell}x$, which is initial temprature distribution.

After steady state conditions

When the temperature at A and B are reduced to $0°C$, the state is not more steady state. For this transient state, the boundary conditions are

$$(i) u(x = 0, t) = 0, \forall t > 0$$
$$(ii) u(x = \ell, t) = 0, \forall t > 0$$
$$(iii) u(x, t = 0) = f(x) = \frac{120}{\ell}x, 0 < x < \ell$$

The suitable solution which satisfies above boundary conditions is

$$u(x, t) = (A\cos px + B\sin px)\, e^{-a^2p^2t} \tag{1}$$

Applying condition (i) in equation (1) we get

We have

$$(i) \Rightarrow u(x = 0, t) = 0$$
$$(1) \Rightarrow Ae^{-a^2p^2t} = 0$$
$$\text{Here } e^{-a^2p^2t} \neq 0 \qquad (\because \text{ it is defined for all } t)$$
$$\therefore A = 0$$

$$\therefore \text{ Now } (1) \Rightarrow u(x, t) = B\sin px e^{-a^2p^2t} \tag{2}$$

Applying condition (ii) in equation (2) we get

We have

$$(ii) \Rightarrow u(x = \ell, t) = 0$$
$$(2) \Rightarrow B\sin px e^{-a^2p^2t} = 0$$

$$\text{Here } e^{-a^2p^2t} \neq 0 \qquad (\because \text{ it is defined for all } t)$$
$$B \neq 0 \qquad (\because \text{ it gives trivial solution })$$
$$\therefore \sin p\ell = 0$$
$$= \sin n\pi$$
$$\therefore p = \frac{n\pi}{\ell}$$

$\therefore$ Now (2) $\Rightarrow u(x,t) = B\sin\frac{n\pi x}{\ell}e^{-a^2\frac{n^2\pi^2}{\ell^2}t}$

$\therefore$ By superposition principle(i.e., adding all such above solutions), the most general solution is

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{\ell}e^{-a^2\frac{n^2\pi^2}{\ell^2}t} \tag{3}$$

Applying condition (iii) in equation (3), we get

We have $(iii) \Rightarrow u(x,t=0) = f(x) = \frac{120}{\ell}x, 0 < x < \ell$

$$\therefore (3) \Rightarrow \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{\ell} = f(x) = \frac{120}{\ell}x \tag{4}$$

To find B_n,

Expand $f(x)$ in a half range Fourier sine series in the interval $(0,\ell)$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{\ell} \tag{5}$$

where $b_n = \frac{2}{\ell}\int_0^{\ell} f(x)\sin\frac{n\pi x}{\ell}dx$

From (4) and (5), we have $B_n = b_n$

$$\therefore B_n = \frac{2}{\ell}\int_0^{\ell} f(x)\sin\frac{n\pi x}{\ell}dx$$
$$= \frac{2}{\ell}\int_0^{\ell} \frac{120}{\ell}x\sin\frac{n\pi x}{\ell}dx$$
$$= \frac{240}{\ell^2}\left[x\left(\frac{-\cos\frac{n\pi x}{\ell}}{\frac{n\pi}{\ell}}\right) - (1)\left(\frac{-\sin\frac{n\pi x}{\ell}}{\left(\frac{n\pi}{\ell}\right)^2}\right)\right]_0^{\ell}$$
$$= \frac{240}{\ell^2}\left[\left[-\ell\frac{\ell}{n\pi}(-1)^n + 0\right] - (0+0)\right]$$
$$\therefore B_n = \frac{240}{n\pi}(-1)^{n+1}$$

$\therefore$ The required most general solution (3) is

$$u(x,t) = \sum_{n=1}^{\infty} \frac{240}{n\pi}(-1)^{n+1}\sin\frac{n\pi x}{\ell}e^{\frac{-a^2n^2\pi^2t}{\ell^2}}$$

3.2.2.2 Steady state conditions and non-zero boundary conditions

Example 3.7. Two ends A and B of rod of length 20 cm have the temperatures at $30°C$ and $80°C$ respectively until steady state conditions prevail. Then the temperature at the ends A and B changed to $40°C$ and $60°C$ respectively. Find the resulting temperature distribution $u(x,t)$ taking origin at A.

Solution : The temperature function $u(x,t)$ satisfies the one dimensional heat equation is

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}. \qquad \text{(I)}$$

where $a^2 = \dfrac{k}{\rho c} = \dfrac{\text{Thermal conductivity}}{\text{(Density)(Specific heat)}}$

Solving the equation (I) by method of separation of variable, we get the following three possible solutions.

$$(1) u(x,t) = \left(A_1 e^{px} + A_2 e^{-px}\right) A_3 e^{a^2 p^2 t}$$

$$(2) u(x,t) = \left(A_4 \cos px + A_5 \sin px\right) A_6 e^{-a^2 p^2 t}$$

$$(3) u(x,t) = \left(A_7 + A_8 x\right) A_9$$

out of these solutions, we have to select that solution which suits the physical nature of the problem and the boundary conditions.

First let us find the temperature distribution at any distance x, before the end A and B are reduced to zero. Prior to the temperature change at the ends A and B, when $t = 0$, the heat flow was independent of time (steady state conditions). When the temperature u depends only on x and not on t,

When steady state conditions prevail

(I) reduces to

$$\frac{\partial u}{\partial t} = 0 \Rightarrow \frac{\partial^2 u}{\partial x^2} = 0 \qquad \text{(II)}$$

The general solution of (II) is $u = ax + b$ (III)

Where a and b are arbitrary constants. Given that $u = 30$ when $x = 0$

$\therefore$ From (III), we get $30 = a(0) + b \Rightarrow b = 30$

Also given that $u = 80$ when $x = 20$

From (III),

$$20a + b = 80$$

$$20a + 30 = 80$$

$$a = \frac{5}{2}$$

$\therefore$ (III) becomes $u = \frac{5}{2}x + 30$, which is initial temperature distribution. (IV)

After steady state conditions with non-zero boundary conditions

When the temperature at A and B are reduced to $40°C$ and $60°C$, the state is not more steady state. For this transient state, the boundary conditions are

$$(i) u(x = 0, t) = 40, \forall t > 0$$

$$(ii) u(x = \ell, t) = 60, \forall t > 0 \text{ where } \ell = 20$$

$$(iii) u(x, t = 0) = f(x) = \frac{5}{2}x + 30, 0 < x < \ell$$

Here we have non zero boundary values. In such case, the temperature function $u(x, t)$ is given by

$$u(x, t) = u_s(x) + u_t(x, t) \qquad \text{(V)}$$

Where $u_s(x)$ is a solution of (I) involving x only and satisfying the boundary conditions (i) and (ii). $u_t(x, t)$ is a function defined by (V) satisfying (I).

i.e., $u_s(x)$ is a steady state solution of (I) and $u_t(x, t)$ may therefore regarded as transient solution which decreases with increase of time t.

$u_s(x)$ satisfies (I).

To find $u_s(x)$

The general solution of (I) is $u_s(x) = a_1 x + b_1$ (VI)

Where a_1 and b_1 are arbitrary constants.

By the condition (i), we have $u = 40$ when $x = 0$

From (VI), we get $u_s(0) = b_1 = 40$

By the condition (ii), we have $u = 60$ when $x = \ell$

From (VI), $u_s(\ell) = a_1\ell + b_1 = 60$

$$a_1\ell + 40 = 60$$

$$a_1 = \frac{20}{\ell}$$

$\therefore$ (VI) becomes $u_s(x) = \frac{20}{\ell}x + 40$ (VII)

To find $u_t(x, t)$ [zero boundary conditions]

Hence the boundary conditions for the transient solution $u_t(x, t)$ by using (V) are

$$(iv) u_t(x=0,t) = u(0,t) - u_s(0) = 40 - 40 = 0, \forall t > 0$$
$$(v) u_t(x=\ell,t) = u(\ell,t) - u_s(\ell) = 60 - 60 = 0, \forall t > 0$$
$$\begin{aligned}(vi) u_t(x,t=0) &= f(x) = u(x,0) - u_s(x)\\ &= \left(\frac{5}{2}x + 30\right) - \left(\frac{20}{\ell}x + 40\right)\\ &= \left(\frac{5}{2}x + 30\right) - (x + 40) \qquad (\because \ell = 20)\\ &= \frac{3}{2}x - 10, 0 < x < \ell = 20\end{aligned}$$

The suitable solution which satisfies above boundary conditions is

$$u_t(x,t) = (A\cos px + B\sin px)\, e^{-a^2p^2t} \tag{1}$$

Applying condition (iv) in equation (1) we get

We have

$$\begin{aligned}(iv) &\Rightarrow u_t(x=0,t) = 0\\ (1) &\Rightarrow Ae^{-a^2p^2t} = 0\\ \text{Here } e^{-a^2p^2t} &\neq 0 \qquad (\because \text{ it is defined for all } t)\\ \therefore A &= 0\end{aligned}$$

$$\therefore \text{ Now } (1) \Rightarrow u_t(x,t) = B\sin px e^{-a^2p^2t} \tag{2}$$

Applying condition (v) in equation (2) we get

We have

$$(v) \Rightarrow u_t(x=\ell,t) = 0$$
$$(2) \Rightarrow B\sin p\ell e^{-a^2p^2t} = 0$$
$$\begin{aligned}\text{Here } e^{-a^2p^2t} &\neq 0 \qquad (\because \text{ it is defined for all } t)\\ B &\neq 0 \qquad (\because \text{ it gives trivial solution })\\ \therefore \sin p\ell &= 0\\ &= \sin n\pi\\ \therefore p &= \frac{n\pi}{\ell}\end{aligned}$$

$$\therefore \text{ Now } (2) \Rightarrow u_t(x,t) = B\sin\frac{n\pi x}{\ell}e^{-a^2\frac{n^2\pi^2}{\ell^2}t}$$

$\therefore$ By superposition principle(i.e., adding all such above solutions), the most general solution is

$$u_t(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{\ell}e^{-a^2\frac{n^2\pi^2}{\ell^2}t} \tag{3}$$

Applying condition (vi) in equation (3), we get

We have $(vi) \Rightarrow u_t(x,t=0) = f(x) = \frac{3}{2}x - 10, 0 < x < \ell = 20$

$$\sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{\ell} = f(x) = \frac{3}{2}x - 10 \tag{4}$$

To find B_n,

Expand $f(x)$ in a half range Fourier sine series in the interval $(0, \ell = 20)$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{\ell} \tag{5}$$

where $b_n = \frac{2}{\ell} \int_0^{\ell} f(x) \sin \frac{n\pi x}{\ell} dx$

From (4) and (5), we have $B_n = b_n$

$$\therefore B_n = \frac{2}{\ell} \int_0^{\ell} f(x) \sin \frac{n\pi x}{\ell} dx = \frac{2}{\ell} \int_0^{\ell} \left(\frac{3}{2}x - 10\right) \sin \frac{n\pi x}{\ell} dx$$

$$= \frac{2}{\ell} \left[\left(\frac{3}{2}x - 10\right) \left(\frac{-\cos \frac{n\pi x}{\ell}}{\frac{n\pi}{\ell}} \right) - \left(\frac{3}{2}\right) \left(\frac{-\sin \frac{n\pi x}{\ell}}{\left(\frac{n\pi}{\ell}\right)^2} \right) \right]_0^{\ell}$$

$$= \frac{2}{\ell} \left[\left(\frac{3\ell}{2} - 10 \right) \left[\left(\frac{\ell}{n\pi} \right) (-\cos n\pi) \right] + 0 - \left(\frac{10\ell}{n\pi} + 0 \right) \right]$$

$$= \frac{1}{10} \left[-\frac{400}{n\pi} (-1)^n - \frac{200}{n\pi} \right] \qquad (\because \ell = 20)$$

$$\therefore B_n = -\frac{20}{n\pi} \left[1 + 2(-1)^n \right]$$

$\therefore$ The required most general solution (3) is

$$u_t(x,t) = \sum_{n=1}^{\infty} \left\{ -\frac{20}{n\pi} \left[1 + 2(-1)^n \right] \right\} \sin \frac{n\pi x}{\ell} e^{\frac{-a^2 n^2 \pi^2 t}{\ell^2}}$$

Hence $u(x,t) = x + 40 - \frac{20}{\pi} \sum_{n=1}^{\infty} \left\{ \frac{\left[1 + 2(-1)^n \right]}{n} \right\} \sin \frac{n\pi x}{\ell} e^{\frac{-a^2 n^2 \pi^2 t}{\ell^2}}$

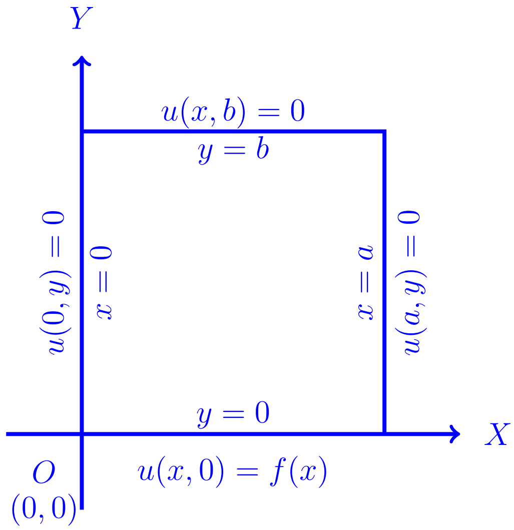
3.2.3 Two dimensional heat flow equations

3.2.3.1 Examples of two dimensional finite plates

Examples of Type I : $f(x) \neq 0$ parallel to X axis

Example 3.8. Solve for the steady state temperature at any point of a rectangular plate of sides a and b insulated on the lateral surface and satisfy $u(0, y) = 0, u(a, y) = 0, u(x, b) = 0$ and $u(x, 0) = x(a - x)$.

Solution :



Let $u(x, y)$ be the temperature distribution satisfying the two dimensional heat flow equation in steady state conditions is

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0 \tag{I}$$

Solving the equation (I) by method of separation of variable, we get the following three possible solutions.

$$(1) u(x,t)=\left(C_1 e^{px}+C_2 e^{-px}\right)\left(C_3\cos py+C_4\sin py\right)$$
$$(2) u(x,t)=\left(C_5\cos px+C_6\sin px\right)\left(C_7 e^{py}+C_8 e^{-py}\right)$$
$$(3) u(x,t)=\left(C_9+C_{10}x\right)\left(C_{11}+C_{12}y\right)$$

out of these solutions, we have to select that solution which suits the physical nature of the problem and the boundary conditions.

For this problem, the boundary and initial conditions are

$$(i) u(x=0,y)=0 \text{ for } 0<y<b$$
$$(ii) u(x=a,y)=0 \text{ for } 0<y<b$$
$$(iii) u(x,y=b)=0 \text{ for } 0<x<a$$
$$(iv) u(x,y=0)=f(x)=x(a-x)$$
$$=\left(ax-x^2\right) \text{ for } 0<x<a$$

The suitable solution which satisfies above boundary conditions is

$$u(x,y)=\left(A\cos px+B\sin px\right)\left(Ce^{py}+De^{-py}\right) \tag{1}$$

Applying condition (i) in equation (1) we get

$$\text{We have } (i) \Rightarrow u(x=0,y)=0$$
$$(1) \Rightarrow A\left(Ce^{py}+De^{-py}\right)=0$$
$$\text{Here } Ce^{py}+De^{-py}\neq 0 \qquad (\because \text{ it is defined } \forall y)$$
$$\therefore A=0$$

$$\therefore \text{ Now } (1) \Rightarrow u(x,y) = B\sin px\left(Ce^{py} + De^{-py}\right) \tag{2}$$

Applying condition (ii) in equation (2) we get

$$\begin{aligned}
\text{We have } (ii) \Rightarrow u(x=a,y) &= 0\\
(2) \Rightarrow B\sin pa\left(Ce^{py} + De^{-py}\right) &= 0\\
\text{Here } Ce^{py} + De^{-py} &\neq 0 && (\because \text{ it is defined } \forall y)\\
\therefore B &\neq 0 && (\because \text{ it gives trivial solution })\\
\sin pa &= 0\\
&= \sin n\pi\\
\therefore pa &= n\pi\\
\text{i.e., } p &= \frac{n\pi}{a}
\end{aligned}$$

$$\therefore \text{ Now } (2) \Rightarrow u(x,y) = B\sin\left(\frac{n\pi x}{a}\right)\left(Ce^{\frac{n\pi y}{a}} + De^{-\frac{n\pi}{a}y}\right) \tag{3}$$

Applying condition (iii) in equation (3), we get

$$\begin{aligned}
\text{We have } (iii) \Rightarrow u(x,y=b) &= 0\\
(3) \Rightarrow B\sin\left(\frac{n\pi x}{a}\right)\left(Ce^{\frac{n\pi}{a}b} + De^{-\frac{n\pi}{a}b}\right) &= 0\\
\text{Here } \sin\left(\frac{n\pi x}{a}\right) &\neq 0 && (\because \text{ it is defined } \forall x)\\
B &\neq 0 && (\because \text{ it gives trivial solution })\\
\therefore Ce^{\frac{n\pi}{a}b} + De^{-\frac{n\pi}{a}b} &= 0\\
De^{-\frac{n\pi}{a}b} &= -Ce^{\frac{n\pi}{a}b}\\
D &= -C\frac{e^{\frac{n\pi}{a}b}}{e^{-\frac{n\pi}{a}b}}
\end{aligned}$$

$$\begin{aligned}
\therefore (3) \Rightarrow u(x,y) &= B\sin\left(\frac{n\pi x}{a}\right)\left(Ce^{\frac{n\pi}{a}y} - C\frac{e^{\frac{n\pi}{a}b}}{e^{-\frac{n\pi}{a}b}}e^{-\frac{n\pi y}{a}}\right)\\
&= BC\sin\left(\frac{n\pi x}{a}\right)\left(\frac{e^{\frac{n\pi}{a}(y-b)} - e^{\frac{n\pi}{a}(b-y)}}{e^{-\frac{n\pi}{a}b}}\right)\\
&= BCe^{\frac{n\pi}{a}b}\sin\left(\frac{n\pi x}{a}\right)\left(e^{-\frac{n\pi}{a}(b-y)} - e^{\frac{n\pi}{a}(b-y)}\right)\\
&= -BCe^{\frac{n\pi}{a}b}\sin\left(\frac{n\pi x}{a}\right)\left(e^{\frac{n\pi}{a}(b-y)} - e^{-\frac{n\pi}{a}(b-y)}\right)\\
&= -2BCe^{\frac{n\pi}{a}b}\sin\left(\frac{n\pi x}{a}\right)\sinh\frac{n\pi}{a}(b-y)
\end{aligned}$$

$\therefore$ By superposition principle(i.e., adding all such above solutions), the most general solution is

$$u(x,y) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{a}\right)\sinh\frac{n\pi}{a}(b-y) \tag{4}$$

Applying condition (iv) in equation (4), we get

$$\text{We have } (iv) \Rightarrow u(x, y=0) = f(x) = \left(ax - x^2\right)$$

$$(4) \Rightarrow \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{a}\right) \sinh \frac{n\pi}{a} b = f(x)$$

$$\therefore \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{a}\right) = f(x) \tag{5}$$

$A_n = B_n \sinh \dfrac{n\pi}{a} b$

To find A_n,

Expand $f(x)$ in a half range Fourier sine series in the interval $(0, a)$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{a} \tag{6}$$

where $b_n = \dfrac{2}{a} \displaystyle\int_0^a f(x) \sin \frac{n\pi x}{a} dx$

From (5) and (6), we have $A_n = b_n$

$$A_n = \frac{2}{a} \int_0^a f(x) \sin \frac{n\pi x}{a} dx$$

$$= \frac{2}{a} \int_0^a \left(ax - x^2\right) \sin \frac{n\pi x}{a} dx$$

$$= \frac{2}{a} \left[(ax - x^2) \left(\frac{-\cos \frac{n\pi x}{a}}{\frac{n\pi}{a}} \right) - (a - 2x) \left(\frac{-\sin \frac{n\pi x}{a}}{\left(\frac{n\pi}{a}\right)^2} \right) + (-2) \left(\frac{\cos \frac{n\pi x}{a}}{\left(\frac{n\pi}{a}\right)^3} \right) \right]_0^a$$

$$= \frac{2}{a} \left[(ax - x^2) \frac{a}{n\pi} \sin \frac{n\pi x}{a} + (a - 2x) \left(\frac{a}{n\pi}\right)^2 \cos \frac{n\pi x}{\ell} + 2 \left(\frac{\ell}{n\pi}\right)^3 \sin \frac{n\pi x}{\ell} \right]_0^a$$

$$= \frac{2}{a} \left\{ \left[0 + 0 - 2 \left(\frac{a}{n\pi}\right)^3 (-1)^n \right] - \left[0 + 0 - 2 \left(\frac{a}{n\pi}\right)^3 \right] \right\}$$

$$= \frac{2}{a} \left[2 \left(\frac{a}{n\pi}\right)^3 \right] [1 - (-1)^n]$$

$$= \frac{4a^2}{n^3\pi^3} [1 - (-1)^n]$$

$$\therefore A_n = \begin{cases} \dfrac{8a^2}{n^3\pi^3} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$$

$$\therefore B_n = \frac{A_n}{\sinh \frac{n\pi b}{a}} = \begin{cases} \dfrac{8a^2}{n^3\pi^3} \sinh \dfrac{n\pi b}{a} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$$

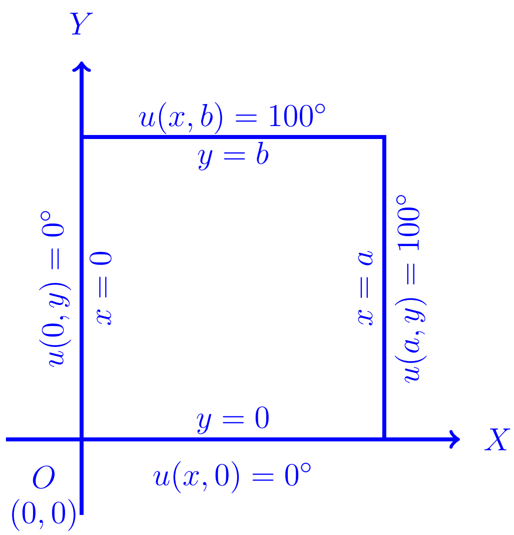
$\therefore$ The required most general solution (4) is

$$
\begin{aligned}
u(x,y) &= \sum_{n=\text{odd}}^{\infty} \frac{8a^2}{n^3\pi^3} \sinh\frac{n\pi b}{a} \sin\frac{n\pi x}{a} \sinh\frac{n\pi}{a}(b-y) \\
&= \frac{8a^2}{\pi^3} \sum_{n=\text{odd}}^{\infty} \frac{1}{n^3} \sinh\frac{n\pi b}{a} \sin\frac{n\pi x}{a} \sinh\frac{n\pi}{a}(b-y) \\
&= \frac{8a^2}{\pi^3} \sum_{n=1}^{\infty} \frac{\sin\frac{(2n-1)\pi x}{a}}{(2n-1)^3 \sinh\frac{n\pi b}{a}} \sinh\frac{(2n-1)\pi}{a}(b-y)
\end{aligned}
$$

Examples of Type II($f(y) \neq 0$ parallel to Y axis) and III($f(x) \neq 0$ & $f(y) \neq 0$)

Example 3.9. Solve Find the steady state temperature distribution at a point of a rectangular plate, if, two adjacent edges are kept at $0°C$ and other at $100°C$.

Solution : Let X & Y axes with $0°C$ and $X = a$ & $Y = b$ with $100°C$. Then



Let $u(x,y)$ be the temperature distribution satisfying the two dimensional heat flow equation in steady state conditions is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{I}$$

Solving the equation (I) by method of separation of variable, we get the following three possible solutions.

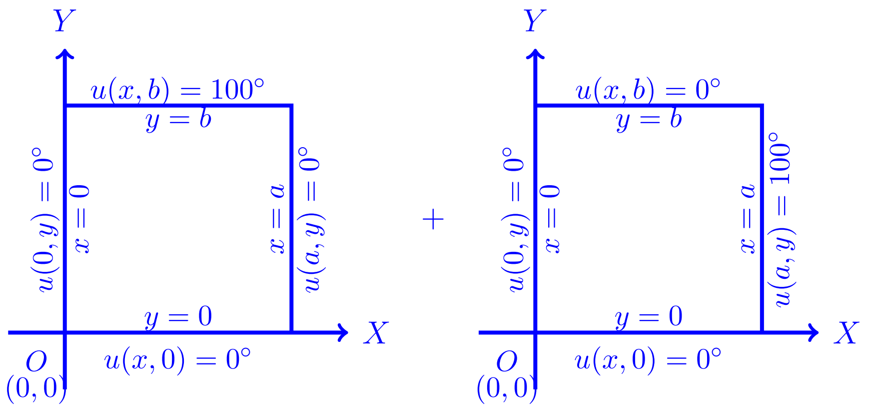
$$
\begin{aligned}
&(1) u(x,t) = \left(C_1 e^{px} + C_2 e^{-px}\right)\left(C_3 \cos py + C_4 \sin py\right) \\
&(2) u(x,t) = \left(C_5 \cos px + C_6 \sin px\right)\left(C_7 e^{py} + C_8 e^{-py}\right) \\
&(3) u(x,t) = \left(C_9 + C_{10} x\right)\left(C_{11} + C_{12} y\right)
\end{aligned}
$$

out of these solutions, we have to select that solution which suits the physical nature of the problem and the boundary conditions.

Let the temperature along the adjacent edges AB and BC be $100°C$ and the temperature along the other edges OC and OA be $0°C$.

The Boundary conditions are

$$(i)u(x=0,y)=0 \text{ for } 0<y<b$$
$$(ii)u(x,y=0)=0 \text{ for } 0<x<a$$
$$(iii)u(x=a,y)=100 \text{ for } 0<y<b$$
$$(iv)u(x,y=b)=100 \text{ for } 0<x<a$$



$\therefore u(x,y) = u_1(x,y) + u_2(x,y)$.

Where $u_1(x,y)$ and $u_2(x,y)$ are solutions of (I) and further $u_1(x,y)$ is the temperature at the edge BC kept at $100^\circ C$ and the other three sides at $0^\circ C$ while $u_2(x,y)$ is the temperature at the edge AB maintained at $100^\circ C$ and the other three edges at $0^\circ C$.

Boundary conditions for the function $u_1(x,y)$ and $u_2(x,y)$ as $\widehat{A}$ and $\widehat{B}$ are

	$\widehat{A}$		$\widehat{B}$
(i)	$u_1(x=0,y)=0$ in $0<y<b$	(v)	$u_2(x,y=0)=0$ in $0<x<a$
(ii)	$u_1(x=a,y)=0$ in $0<y<b$	(vi)	$u_2(x,y=b)=0$ in $0<x<a$
(iii)	$u_1(x,y=0)=0$ in $0<x<a$	(vii)	$u_2(x=0,y)=0$ in $0<y<b$
(iv)	$u_1(x,y=b)=100$ in $0<x<a$	(viii)	$u_2(x=a,y)=100$ in $0<y<b$

The suitable solution which satisfies above boundary conditions $\left(\widehat{A}\right)$ is

$$u_1(x,y) = (A\cos px + B\sin px)\left(Ce^{py} + De^{-py}\right) \tag{1}$$

Applying condition (i),(ii),(iii) as in previous problem, we get

$\therefore$ By superposition principle(i.e., adding all such above solutions), the most general solution is

$$u(x,y) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{a}\right)\sinh\frac{n\pi y}{a} \tag{4}$$

Applying condition (iv) in equation (4), we get

$$\text{We have } (iv) \Rightarrow u(x,y=b) = f(x) = 100$$
$$(4) \Rightarrow \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{a}\right)\sinh\frac{n\pi b}{a} = f(x) = 100$$
$$\therefore \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{a}\right) = f(x) = 100 \tag{5}$$

where $B_n = C_n \sinh \frac{n\pi b}{a}$

To find B_n,

Expand $f(x)$ in a half range Fourier sine series in $(0, a)$, we get

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{a} \tag{6}$$

where $b_n = \frac{2}{a} \int_0^a f(x) \sin \frac{n\pi x}{a} dx$

From (5) and (6), we have $B_n = b_n$

$$B_n = \frac{2}{a} \int_0^a 100 \sin \frac{n\pi x}{a} dx = \frac{200}{a} \int_0^a \sin \frac{n\pi x}{a} dx$$

$$= \frac{200}{a} \left[\frac{-\cos \frac{n\pi x}{a}}{\frac{n\pi}{a}} \right]_0^a$$

$$= -\left(\frac{200}{n\pi}\right) [\cos n\pi - 1] = -\frac{200}{n\pi} [(-1)^n - 1]$$

$$\therefore B_n = \begin{cases} \frac{400}{n\pi} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$$

$$\therefore C_n = \frac{B_n}{\sinh \frac{n\pi b}{a}} = \begin{cases} \frac{400}{n\pi} \frac{1}{\sinh \frac{n\pi b}{a}} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$$

$\therefore$ The required solution (4) is

$$u_1(x, y) = \sum_{n=\text{odd}}^{\infty} \frac{400}{n\pi} \frac{1}{\sinh \frac{n\pi b}{a}} \sin \frac{n\pi x}{a} \sinh \frac{n\pi y}{a}$$

$$= \frac{400}{\pi} \sum_{n=1}^{\infty} \frac{\sin \frac{(2n-1)\pi x}{a}}{(2n-1) \sinh \frac{(2n-1)\pi b}{a}} \sinh \frac{(2n-1)\pi}{a} y$$

This is the required solution of $u_1(x, y)$.

Similarly, the suitable solution which satisfies above boundary conditions $\left(\widehat{B}\right)$ is

$$u_2(x, y) = \left(Ae^{px} + Be^{-px}\right)(C \cos py + D \sin py) \tag{7}$$

Similarly by applying condition (v),(vi),(vii) and (viii), we get the required solution of $u_2(x, y)$ as

$$u_2(x, y) = \sum_{n=\text{odd}}^{\infty} \frac{400}{n\pi} \frac{1}{\sinh \frac{n\pi a}{b} \sinh \frac{n\pi a}{b}} \sin \frac{n\pi y}{b} \sinh \frac{n\pi x}{b}$$

$$= \frac{400}{\pi} \sum_{n=1}^{\infty} \frac{\sin \frac{(2n-1)\pi y}{b}}{(2n-1) \sinh \frac{(2n-1)\pi a}{b}} \sinh \frac{(2n-1)\pi}{b} x$$

This is the required solution of $u_2(x, y)$.

$\therefore$ Finally, the required most general solution $= u(x, y) = u_1(x, y) + u_2(x, y)$.

3.2.3.2 Examples of two dimensional infinite plates

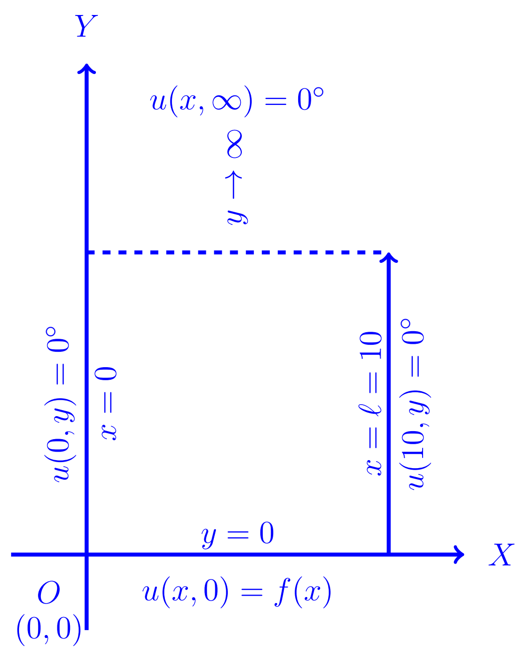
Examples of Type (IV) : $f(x) \neq 0$ parallel to X axis

Example 3.10. A rectangular plate with insulated surfaces is 10 cm wide and so long compared to its width that it may be considered infinite in length without introducing an appreciable error. If the temperature along the short edge $y = 0$ is given by

$$u(x,0) = \begin{cases} 20x, & 0 < x < 5 \\ 20(10-x), & 5 < x < 10 \end{cases}$$

while the two long edges $x = 0$ and $x = 10$ as well as the other short edge are kept at $0°C$, find the temperature function $u(x,y)$ in steady state.

Solution :



Let $u(x,y)$ be the temperature at any point $P(x,y)$ in the steady state. Then $u(x,y)$ satisfying the two dimensional heat flow equation in steady state conditions is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{I}$$

Solving the equation (I) by method of separation of variable, we get the following three possible solutions.

$$(1) u(x,t) = \left(C_1 e^{px} + C_2 e^{-px}\right)(C_3 \cos py + C_4 \sin py)$$
$$(2) u(x,t) = (C_5 \cos px + C_6 \sin px)\left(C_7 e^{py} + C_8 e^{-py}\right)$$
$$(3) u(x,t) = (C_9 + C_{10}x)(C_{11} + C_{12}y)$$

out of these solutions, we have to select that solution which suits the physical nature of the problem and the boundary conditions.

For this problem, the boundary and initial conditions are

$$(i) u(x=0,y)=0 \text{ for all } y$$
$$(ii) u(x=10,y)=0 \text{ for all } y$$
$$(iii) u(x,y\to\infty)=0 \text{ for } 0<x<10$$
$$(iv) u(x,y=0)=f(x)=\begin{cases} 20x, & 0<x<5 \\ 20(10-x), & 5<x<10 \end{cases}$$

The suitable solution which satisfies above boundary conditions is

$$u(x,y)=(A\cos px+B\sin px)\left(Ce^{py}+De^{-py}\right) \tag{1}$$

Applying condition (i),(ii),(iii) as in previous problem, we get

$\therefore$ By superposition principle(i.e., adding all such above solutions), the most general solution is

$$u(x,y)=\sum_{n=1}^{\infty} E_n \sin\left(\frac{n\pi}{10}x\right)e^{-\frac{n\pi y}{10}} \tag{4}$$

Applying condition (iv) in equation (4), we get

We have $(iv) \Rightarrow u(x,y=0)=f(x)$

$$(4) \Rightarrow \sum_{n=1}^{\infty} E_n \sin\left(\frac{n\pi x}{\ell}\right)=f(x)=\begin{cases} 20x, & 0<x<5 \\ 20(10-x), & 5<x<10 \end{cases} \tag{5}$$

To find E_n,

Expand $f(x)$ in a half range series in $(0,10)$, we have

$$f(x)=\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{10} \tag{6}$$

where $b_n=\dfrac{2}{10}\displaystyle\int_0^{10} f(x)\sin\frac{n\pi x}{10}dx$

From (5) and (6), we have $E_n=b_n$

$$\begin{aligned} E_n &= \frac{2}{10}\int_0^{10} f(x)\sin\frac{n\pi x}{10}dx \\ &= \frac{1}{5}\left[\int_0^5 20x\sin\frac{n\pi x}{10}dx+\int_5^{10} 20(10-x)\sin\frac{n\pi x}{10}dx\right] \\ &= 4\left[\int_0^5 x\sin\frac{n\pi x}{10}dx+\int_5^{10}(10-x)\sin\frac{n\pi x}{10}dx\right] \end{aligned}$$

$$= 4\left\{\left[x\frac{10}{n\pi}\left(-\cos\frac{n\pi x}{10}\right) - (1)\left(\frac{10}{n\pi}\right)^2\left(-\sin\frac{n\pi x}{10}\right)\right]_0^5\right.$$
$$\left.+\left[(10-x)\frac{10}{n\pi}\left(-\cos\frac{n\pi x}{10}\right) - (-1)\left(\frac{10}{n\pi}\right)^2\left(-\sin\frac{n\pi x}{10}\right)\right]_5^{10}\right\}$$
$$= 4\left\{\left[-\frac{50}{n\pi}\left(\cos\frac{n\pi}{2}\right) + \frac{100}{n^2\pi^2}\left(\sin\frac{n\pi}{2}\right)\right] - [0+0]\right.$$
$$\left.+[0+0] - \left[\frac{-50}{n\pi}\left(\cos\frac{n\pi}{2}\right) - \frac{100}{n^2\pi^2}\left(\sin\frac{n\pi}{2}\right)\right]\right\}$$
$$\therefore E_n = \frac{800}{n^2\pi^2}\sin\frac{n\pi}{2}$$

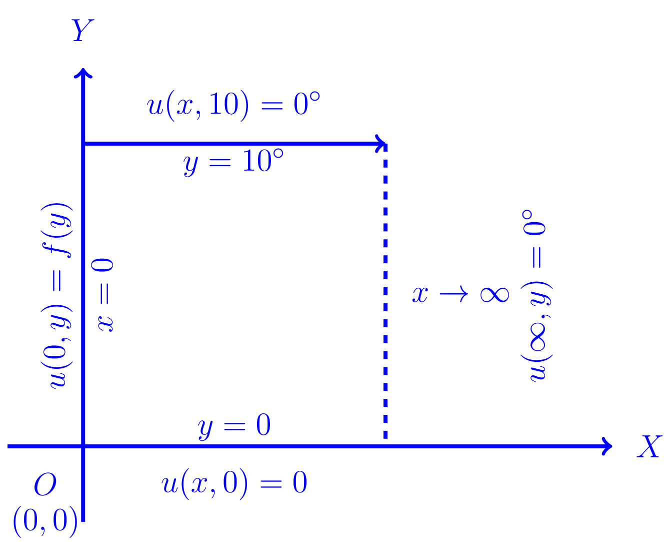
$\therefore$ The required most general solution (4) is

$$u(x,y) = \sum_{n=1}^{\infty}\frac{800}{n^2\pi^2}\sin\frac{n\pi}{2}\sin\frac{n\pi x}{10}e^{-\frac{n\pi y}{10}}$$
$$= \frac{800}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi}{2}\sin\frac{n\pi x}{10}e^{-\frac{n\pi y}{10}}$$

Examples of Type V : $f(y) \neq 0$ parallel to Y axis

Example 3.11. An infinitely long rectangular plate with insulated surface is 10 cm wide. The two long edges and one short edge are kept at zero temperature, while the other short edge $x = 0$ is kept at temperature $u = \begin{cases} 20y, & 0 < y < 5 \\ 20(10-y), & 5 < y < 10 \end{cases}$. Find the temperature function $u(x,y)$ in steady state.

Solution :



Let $u(x,y)$ be the temperature at any point $P(x,y)$ in the steady state. Then $u(x,y)$ satisfying the two dimensional heat flow equation in steady state conditions is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{I}$$

Solving the equation (I) by method of separation of variable, we get the following three possible solutions.

$$(1) u(x,t) = \left(C_1 e^{px} + C_2 e^{-px}\right)\left(C_3 \cos py + C_4 \sin py\right)$$
$$(2) u(x,t) = \left(C_5 \cos px + C_6 \sin px\right)\left(C_7 e^{py} + C_8 e^{-py}\right)$$
$$(3) u(x,t) = \left(C_9 + C_{10} x\right)\left(C_{11} + C_{12} y\right)$$

out of these solutions, we have to select that solution which suits the physical nature of the problem and the boundary conditions.

For this problem, the boundary and initial conditions are

$$(i) u(x, y = 0) = 0 \text{ for all } x$$
$$(ii) u(x, y = 10) = 0 \text{ for all } x$$
$$(iii) u(x \to \infty, y) = 0 \text{ for } 0 < y < 10$$
$$(iv) u(x = 0, y) = f(y) = \begin{cases} 20y, & 0 < y \leq 5 \\ 20(10 - y), & 5 < y \leq 10 \end{cases}$$

The suitable solution which satisfies above boundary conditions is

$$u(x,y) = \left(Ae^{px} + Be^{-px}\right)\left(C \cos py + D \sin py\right) \tag{1}$$

Applying condition (i) in equation (1) we get

$$\text{We have } (i) \Rightarrow u(x, y = 0) = 0$$
$$(1) \Rightarrow \left(Ae^{px} + Be^{-px}\right) C = 0$$
$$\text{Here } Ae^{px} + Be^{-px} \neq 0 \qquad (\because \text{ it is defined } \forall x)$$
$$\therefore C = 0$$

$$\therefore \text{ Now } (1) \Rightarrow u(x,y) = \left(Ae^{px} + Be^{-px}\right) D \sin py \tag{2}$$

Applying condition (ii) in equation (2) we get

$$\text{We have } (ii) \Rightarrow u(x, y = 10) = 0$$
$$(2) \Rightarrow \left(Ae^{px} + Be^{-px}\right) D \sin 10p = 0$$
$$\text{Here } Ae^{px} + Be^{-px} \neq 0 \qquad (\because \text{ it is defined } \forall x)$$
$$D \neq 0 \qquad (\because \text{ it gives trivial solution })$$
$$\therefore \sin 10p = 0$$
$$= \sin n\pi$$
$$\therefore p = \frac{n\pi}{10}$$

$$\therefore \text{ Now } (2) \Rightarrow u(x,y) = \left(Ae^{\frac{n\pi x}{10}} + Be^{-\frac{n\pi x}{10}}\right) D \sin \frac{n\pi y}{10} \tag{3}$$

Applying condition (iii) in equation (3), we get

$$\begin{aligned}
\text{We have } (iii) &\Rightarrow u(x \to \infty, y) = 0 \\
(3) \Rightarrow \left(Ae^{\infty} + Be^{-\infty}\right) D \sin \frac{n\pi y}{10} &= 0 \\
\left(Ae^{\infty}\right) D \sin \frac{n\pi y}{10} &= 0 \\
\text{Here } \sin \frac{n\pi y}{10} &\neq 0 \qquad (\because \text{ it is defined } \forall y) \\
B &\neq 0 \qquad (\because \text{ it gives trivial solution }) \\
\therefore A &= 0 \qquad (\because e^{\infty} = \infty)
\end{aligned}$$

$$\begin{aligned}
\therefore \text{ Now } (3) \Rightarrow u(x,y) &= \left(Be^{\frac{n\pi x}{10}}\right)\left(D\frac{\sin n\pi y}{10}\right) \\
&= BD \sin \frac{n\pi y}{10}\left(e^{-\frac{n\pi x}{10}}\right) \\
&= E \sin \frac{n\pi y}{10} e^{-\frac{n\pi x}{10}}
\end{aligned}$$

where $E = BD$

$\therefore$ By superposition principle(i.e., adding all such above solutions), the most general solution is

$$u(x,y) = \sum_{n=1}^{\infty} E_n \sin \frac{n\pi y}{10} e^{-\frac{n\pi x}{10}} \tag{4}$$

Applying condition (iv) in equation (4), we get

$$\begin{aligned}
\text{We have } (iv) &\Rightarrow u(x=0,y) = f(y) \\
(4) \Rightarrow \sum_{n=1}^{\infty} E_n \sin \frac{n\pi y}{10} &= f(y)
\end{aligned} \tag{5}$$

To find E_n,

Expand $f(y)$ in a half range Fourier sine series in the interval $(0, 10)$

$$f(y) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi y}{10} \tag{6}$$

where $b_n = \dfrac{2}{10}\displaystyle\int_0^{10} f(y) \sin \frac{n\pi y}{10} dy$

From (5) and (6), we have $E_n = b_n$

$$E_n = \frac{1}{5}\left[\int_0^5 20y \sin\frac{n\pi y}{10}dy + \int_5^{10} 20(10-y)\sin\frac{n\pi y}{10}dy\right]$$

$$= 4\left[\int_0^5 y \sin\frac{n\pi y}{10}dy + \int_5^{10} (10-y)\sin\frac{n\pi y}{10}dy\right]$$

$$= 4\left\{\left[y\frac{10}{n\pi}\left(-\cos\frac{n\pi y}{10}\right) - (1)\left(\frac{10}{n\pi}\right)^2\left(-\sin\frac{n\pi y}{10}\right)\right]_0^5\right.$$

$$\left.+\left[(10-y)\frac{10}{n\pi}\left(-\cos\frac{n\pi y}{10}\right) - (-1)\left(\frac{10}{n\pi}\right)^2\left(-\sin\frac{n\pi y}{10}\right)\right]_5^{10}\right\}$$

$$= 4\left\{\left[-\frac{50}{n\pi}\left(\cos\frac{n\pi}{2}\right) + \frac{100}{n^2\pi^2}\left(\sin\frac{n\pi}{2}\right)\right] - [0+0]\right.$$

$$\left.= +[0+0] - \left[\frac{-50}{n\pi}\left(\cos\frac{n\pi}{2}\right) - \frac{100}{n^2\pi^2}\left(\sin\frac{n\pi}{2}\right)\right]\right\}$$

$$\therefore E_n = \frac{800}{n^2\pi^2}\sin\frac{n\pi}{2}$$

$\therefore$ The required most general solution (4) is

$$u(x,y) = \sum_{n=1}^{\infty}\frac{800}{n^2\pi^2}\sin\frac{n\pi}{2}\sin\frac{n\pi y}{10}e^{-\frac{n\pi x}{10}}$$

$$= \frac{800}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi}{2}\sin\frac{n\pi y}{10}e^{-\frac{n\pi x}{10}}$$

3.3 Assignment III[Applications of Partial Differential Equations]

1. A string is stretched and fastened to two points ℓ apart. Motion is started by displacing the string into the form $y = k(\ell x - x^2)$ from which it is released at time $t = 0$. Find the displacement of any point of the string at a distance x from one end at any time t.

2. A tightly stretched string of length '2ℓ' has its ends fastened at $x = 0, x = 2\ell$. The mid point of the string is then taken to height 'b' and then released from rest in that position. Find the lateral displacement of a point of the string at time 't' from the instant of release.

3. A tightly stretched string of length 'ℓ' is initially at rest in its equilibrium position and each of its points is given the velocity $V_0 \sin^3\left(\frac{\pi x}{\ell}\right)$ Find the displacement of $y(x,t)$.

4. A string is stretched between two fixed points at a distance 2ℓ apart and the points of the string are given initial velocities ν where $\nu = \begin{cases} \frac{cx}{\ell} & \text{in } 0 \le x \le \ell \\ \frac{c}{\ell}(2\ell - x) & \text{in } \ell \le x \le 2\ell \end{cases}$ x being the distance from one end point. Find the displacement of the string at any subsequent time.

5. The ends of A and B of a rod ℓ c.m. long have their temperature kept at $30°C$ and $80°C$, until steady state conditions prevail. The temperature of the end B is suddenly reduced to $60°C$ and that of A is increased to $40°C$. Find the temperature distribution in the rod after time t.

6. Find the solution of the one dimensional diffusion equation satisfying the boundary conditions:
   (i) u is bounded as $t \to \infty$ (ii) $\left[\frac{\partial u}{\partial x}\right]_{x=0} = 0$ for all t
   (iii) $\left[\frac{\partial u}{\partial x}\right]_{x=a} = 0$ for all t (iv) $u(x, 0) = x(a - x), 0 < x < a$

7. Find the steady state temperature distribution in a rectangular plate of sides a and b insulated at the lateral surface and satisfying the boundary conditions $u(0, y) = u(a, y) = 0$ for $0 \leq y \leq b$; $u(x, b) = 0$ and $u(x, 0) = x(a - x)$ for $0 \leq x \leq a$.

8. A square plate is bounded by the lines $x = 0, y = 0, x = 20$ and $y = 20$. Its faces are insulated. The temperature along the upper horizontal edge is given by while the other two edges are kept at 0^circC. Find the steady state temperature distribution in the plate.

9. A rectangular plate with insulated surface is 10 cm wide and so long compared to its width that it may be considered infinite in length without introducing appreciable error. The temperature at short edge $y = 0$ is given by $\begin{cases} 20x, & 0 \leq x \leq 5 \\ 20(10 - x), & 5 \leq x \leq 10 \end{cases}$ and all the other three edges are kept at $0°C$. Find the steady-state temperature at any point of the plate.

10. An infinitely long rectangular plate with insulated surface is 10 cm wide. The two long edges and one short edge are kept at zero temperature, while the other short edge $x = 0$ is kept at temperature $u = \begin{cases} 20y, & 0 < y \leq 5 \\ 20(10 - y), & 5 < y \leq 10 \end{cases}$. Find the temperature function $u(x, y)$ in steady state.

11. A rectangular plate with insulated surface is 20 cm wide and so long compared to its width rectangular plate with insulated surface is 20 cm wide and so long compared to its width that it may be considered infinite in length without introducing an appreciable error. If the temperature of the short edge $x = 0$ is given by and the two long edges as well as the other short edge are kept at $0°C$. Find the steady state temperature distribution in the plate.

4 Fourier Transforms(F.T.)

4.1 Part-A

1. If $F(s)$ is the Fourier transform of $f(x)$, write the formula for the Fourier transform of $f(x)\cos ax$.

   Soln :

$$F\left[f\left(x\right)\cos ax\right]$$
$$= \frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty} e^{isx} f\left(x\right)\cos ax \ dx$$
$$= \frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty} e^{isx} f\left(x\right)\left(\frac{e^{iax}+e^{-iax}}{2}\right) \ dx$$
$$= \frac{1}{2\sqrt{2\pi}}\int\limits_{-\infty}^{\infty} \left[e^{i(s+a)x}+e^{i(s-a)x}\right] f\left(x\right) \ dx$$
$$= \frac{1}{2}\left[\frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty} e^{i(s+a)x} f\left(x\right) \ dx + \frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty} e^{i(s-a)x} f\left(x\right) \ dx\right]$$
$$= \frac{1}{2}\left[F\left(s+a\right)+F\left(s-a\right)\right]$$

2. If the Fourier transform of $f(x)$ is $F(s)$. What is the Fourier transform of $f(x-a)$.

   Soln :

$$F\left[f\left(x-a\right)\right] = \frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty} e^{isx} f\left(x-a\right)dx,$$

   Set $x-a=t$. Then $dx=dt$.

   When $x=-\infty, t=-\infty$ and when $x=\infty, t=\infty$.

$$F\left[f\left(x-a\right)\right] = \frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty} e^{is(a+t)} f\left(t\right)dt$$
$$= e^{ias}\frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty} e^{ist} f\left(t\right)dt$$
$$= e^{ias}F\left(s\right)$$

3. If the Fourier transform of $f\left(x\right) = \begin{cases} 1, |x|<a \\ 0, |x|>a \end{cases}$ is $\sqrt{\frac{2}{\pi}}\frac{\sin as}{s}$, evaluate $\int\limits_{0}^{\infty}\left(\frac{\sin t}{t}\right)^2 dt$.

   Soln : Given $F\left[f\left(x\right)\right] = F\left(s\right) = \sqrt{\frac{2}{\pi}}\frac{\sin as}{s}$.

   By Parseval's identity,

$$\int\limits_{-\infty}^{\infty}\left|f\left(x\right)\right|^2 dx = \int\limits_{-\infty}^{\infty}\left|F\left(s\right)\right|^2 ds$$
$$\int\limits_{-a}^{a}\left(1\right)^2 dx = \int\limits_{-\infty}^{\infty}\left(\sqrt{\frac{2}{\pi}}\frac{\sin as}{s}\right)^2 ds$$
$$2a = \frac{2}{\pi}\int\limits_{-\infty}^{\infty}\left(\frac{\sin^2 as}{s^2}\right) ds$$

   Set $as=t$. Then $ds=dt/a$.

   When $s=-\infty, t=-\infty$ and when $s=\infty, t=\infty$.

$$a = \frac{1}{\pi}\int_{-\infty}^{\infty}\left(\frac{\sin^2 t}{(t/a)^2}\right)dt/a$$

$$a = \frac{a}{\pi}\int_{-\infty}^{\infty}\left(\frac{\sin^2 t}{t^2}\right)dt$$

$$1 = \frac{1}{\pi}\int_{-\infty}^{\infty}\left(\frac{\sin^2 t}{t^2}\right)dt$$

$$\Rightarrow 1 = \frac{2}{\pi}\int_{0}^{\infty}\left(\frac{\sin^2 t}{t^2}\right)dt$$

$$\int_{0}^{\infty}\left(\frac{\sin^2 t}{t^2}\right)dt = \frac{\pi}{2}$$

4. Find the Fourier transform of $f(x) = \begin{cases} x, |x| < a \\ 0, |x| > a \end{cases}$.

Soln :

$$F[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{isx} f(x)\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{a} e^{isx}(0)\,dx + \frac{1}{\sqrt{2\pi}}\int_{-a}^{a} e^{isx}x dx$$

$$+ \frac{1}{\sqrt{2\pi}}\int_{a}^{\infty} e^{isx}(0)\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-a}^{a} e^{isx}x dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-a}^{a} x(\cos sx + i\sin sx)dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-a}^{a} x\cos sx dx + \frac{i}{\sqrt{2\pi}}\int_{-a}^{a} x\sin sx dx$$

$$= 0 + \frac{2i}{\sqrt{2\pi}}\int_{0}^{a} x\sin sx dx$$

$$= i\sqrt{\frac{2}{\pi}}\left[\frac{\sin as - as\cos as}{s^2}\right]$$

5. If $F(s)$ is the Fourier transform of $f(x)$, prove that the Fourier transform of $f(ax)$ is $\frac{1}{|a|}F(s/a), a \neq 0$.

WKT

Soln : $F[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{isx}dx$

$$F[f(ax)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(ax)e^{isx}dx$$

Set $t = ax$. Then $dt = adx$.

When $x = -\infty, t = -\infty$ and when $x = \infty, t = \infty$.

$$F[f(ax)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{ist/a}\frac{dt}{a}$$

$$= \frac{1}{a\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{ist/a}dt$$

$$= \frac{1}{a\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{i\left(\frac{s}{a}\right)t}dt$$

$$= \frac{1}{a\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{i\left(\frac{s}{a}\right)x}dx$$

[$\because$ x is a dummy variable]

$$= \frac{1}{a}F(s/a), a \neq 0$$

Similarly if $a < 0$,

$$F[f(ax)] = \frac{-1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(ax)e^{isx}dx$$

$$= \frac{-1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{i\left(\frac{s}{a}\right)t}\frac{dt}{a}$$

$$= \frac{-1}{a}F(s/a)$$

$\therefore$ In general, $F[f(ax)] = \frac{1}{|a|}F(s/a)$

6. If the Fourier transform of $e^{-\frac{x^2}{2}}$ is $e^{-\frac{s^2}{2}}$, what is the Fourier transform of $xe^{-\frac{x^2}{2}}$?

Soln : If $F(s) = F[f(x)]$, then

$$F[xf(x)] = -i\frac{d}{ds}F(s)$$
$$F\left[xe^{-\frac{x^2}{2}}\right] = -i\frac{d}{ds}F\left(e^{-\frac{x^2}{2}}\right)$$
$$= -i\frac{d}{ds}\left[e^{-\frac{s^2}{2}}\right]$$
$$= -ise^{-\frac{s^2}{2}}$$

7. Write the Fourier transform pair.

Soln : Fourier transform:

The Fourier transform of $f(x)$ is given by

$$F[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \mathrm{f}(x)\, e^{isx}\, dx \qquad (1)$$

Inverse Fourier transform:

The Inverse Fourier transform of $F(s)$ is given by

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F[f(x)]\, e^{-isx}\, ds \qquad (2)$$

The above equations (1) and (2) are jointly called as Fourier Transform pair.

8. Write the Fourier sine and cosine transform pair.

Soln :

Fourier Sine transform pair:

Fourier Sine transform of $f(x)$ is

$$F_S[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\sin sx\, dx = F_S(s).$$

Inverse Fourier Sine transform of $F_S[f(x)] = F_S[s]$ is

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} F_S[f(x)]\sin sx\, ds$$

Fourier Cosine transform pair:

Fourier Cosine transform of $f(x)$ is

$$F_C[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\cos sx\, dx = F_C(s)$$

Inverse Fourier Cosine transform of $F_C[f(x)] = F_C[s]$ is

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} F_C[f(x)]\cos sx\, ds$$

9. If $F_S(s)$ is the Fourier sine transform of $f(x)$, prove that the Fourier sine transform of $f(ax)$ is $\frac{1}{a}F_S(s/a)$.

Soln : WKT

$$F_S[f(ax)] = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(ax)\sin sxdx$$

Set $ax = t$. Then $dx = dt/a$.

When $x = 0, t = 0$ and when $x \to \infty, t \to \infty$.

$$F_S[f(ax)] = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(t)\sin\left(\frac{st}{a}\right)\frac{dt}{a}$$
$$= \frac{1}{a}\sqrt{\frac{2}{\pi}}\int_0^{\infty} f(t)\sin\left(\frac{s}{a}\right)t\, dt$$
$$= \frac{1}{a}F_S\left[\frac{s}{a}\right]$$

10. State Fourier integral theorem.

Soln : If $f(x)$ is piecewise continuously differentiable and absolutely integrable in $(-\infty, \infty)$, then

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)\, e^{is(x-t)}\, dt\, ds$$

(or)

$$f(x) = \frac{1}{\pi}\int_0^{\infty}\int_{-\infty}^{\infty} f(t)\cos\lambda(x-t)\, dt\, d\lambda$$

11. Obtain the Fourier cosine transform of $e^{-ax}, a > 0$..

Soln :
$F_C\left(e^{-ax}\right)$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}\cos sx dx$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{e^{-ax}}{a^2+s^2}\left(-a\cos sx + s\sin sx\right)\right]_0^\infty$$

$$= \sqrt{\frac{2}{\pi}}\left(\frac{a}{a^2+s^2}\right)$$

12. Obtain the Fourier sine transform of $e^{-ax}, a > 0$.

Soln :
$F_S\left(e^{-ax}\right)$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}\sin sx dx$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{e^{-ax}}{a^2+s^2}\left(-a\sin sx - s\cos sx\right)\right]_0^\infty$$

$$= \sqrt{\frac{2}{\pi}}\left(\frac{s}{a^2+s^2}\right)$$

13. Obtain the Fourier sine transform of $1/x$.

Soln : $F_S\left(1/x\right) = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{1}{x}\sin sx dx$

Set $sx = t$. Then $dx = dt/s$.

When $x = -\infty, t = -\infty$ and when $x = \infty, t = \infty$.

$$F_S\left(1/x\right) = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{1}{\frac{t}{s}}\sin t\frac{dt}{s}$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{1}{t}\sin t dt \qquad \left[\because \int_0^\infty \frac{\sin t}{t}dt = \frac{\pi}{2}\right]$$

$$= \sqrt{\frac{2}{\pi}\frac{\pi}{2}} = \sqrt{\frac{\pi}{2}}$$

14. Obtain the Fourier transform of the derivative of a function.

Soln : We have to prove
$F\left[f'\left(x\right)\right] = -isF\left(s\right)$, if $f\left(x\right) \to 0$ as $x \to \pm\infty$

W.K.T.

$$F\left[f\left(x\right)\right] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty \mathrm{f}\left(x\right)e^{isx}dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty \mathrm{f'}\left(x\right)e^{isx}\ dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty e^{isx}\ \mathrm{d}\left[f\left(x\right)\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left\{\left[e^{isx}f\left(x\right)\right]_{-\infty}^\infty - \int_{-\infty}^\infty f\left(x\right)e^{isx}\left(is\right)dx\right\}$$

$$= \frac{1}{\sqrt{2\pi}}\left\{\left[0-0\right]_{-\infty}^\infty - is\int_{-\infty}^\infty f\left(x\right)e^{isx}dx\right\}$$

$$= \frac{-is}{\sqrt{2\pi}}\left\{\int_{-\infty}^\infty f\left(x\right)e^{isx}dx\right\}$$

$$F\left[f'\left(x\right)\right] = -isF\left(s\right)$$

In general,

$$F\left[f^n\left(x\right)\right] = \left(-is\right)^nF\left(s\right)$$

if $f, f', f'', ...f^{n-1} \to 0$ as $x \to \pm\infty$

15. Prove that $F(e^{iax}f(x)) = F(s+a)$, where $F[f(x)] = F(s)$.

Soln : WKT $F\left[f\left(x\right)\right] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty \mathrm{f}\left(x\right)e^{isx}\ dx$

$$F\left[e^{iax}f(x)\right] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{iax}\mathrm{f}(x)\,e^{isx}\,dx$$
$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \mathrm{f}(x)\,e^{i(s+a)x}\,dx$$
$$= F(s+a)$$

16. Find the Fourier transform of $e^{-a|x|}, a > 0$.

Soln :

$F\left(e^{-a|x|}\right)$
$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-a|x|}e^{isx}dx$$
$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-a|x|}(\cos sx + i\sin sx)dx$$
$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-a|x|}\cos sxdx$$
$$+ \frac{i}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-a|x|}\sin sxdx$$
$$= \frac{2}{\sqrt{2\pi}}\int_{0}^{\infty} e^{-a|x|}\cos sxdx$$
$$= \sqrt{\frac{2}{\pi}}\left(\frac{a}{a^2+s^2}\right)$$

17. Obtain the Fourier cosine transform of $f(x) = \begin{cases} x, & 0 < x < 1 \\ 2-x, & 1 < x < 2 \\ 0, & x > 2 \end{cases}$.

Soln :

$$F[f(x)]$$
$$= \sqrt{\frac{2}{\pi}}\int_{0}^{\infty} f(x)\cos sxdx$$
$$= \sqrt{\frac{2}{\pi}}\left[\int_{0}^{1} x\cos sxdx + \int_{1}^{2}(2-x)\cos sxdx\right.$$
$$\left.+\int_{2}^{\infty}(0)\cos sxdx\right]$$
$$= \sqrt{\frac{2}{\pi}}\left\{(x)\left(\frac{\sin sx}{s}\right) - (1)\left(\frac{-\cos sx}{s^2}\right)\right\}_0^1$$
$$+\sqrt{\frac{2}{\pi}}\left\{(2-x)\left(\frac{\sin sx}{s}\right) - (-1)\left(\frac{-\cos sx}{s^2}\right)\right\}_1^2$$
$$= \sqrt{\frac{2}{\pi}}\left(\frac{2\cos s - 1 - \cos 2s}{s^2}\right)$$

18. If $F[f(x)] = F(s)$ then find the value of $F_S[f(x)\cos ax]$.

Soln :

$$F_s[f(x)\cos ax] = \sqrt{\frac{2}{\pi}}\int_{0}^{\infty} f(x)\cos ax\cos sxdx$$
$$= \sqrt{\frac{2}{\pi}}\int_{0}^{\infty} f(x)\cos sx\cos axdx$$
$$= \frac{1}{2}\left\{\sqrt{\frac{2}{\pi}}\int_{0}^{\infty} f(x)\left[\cos(s-a)x\right.\right.$$
$$\left.\left.+\cos(s+a)x\right]dx\right\}$$
$$= \frac{1}{2}\left\{\sqrt{\frac{2}{\pi}}\int_{0}^{\infty} f(x)\left[\cos(s-a)x\right]dx\right\}$$
$$+\frac{1}{2}\left\{\sqrt{\frac{2}{\pi}}\int_{0}^{\infty} f(x)\left[\cos(s+a)x\right]dx\right\}$$
$$= \frac{1}{2}\left[F_c(s+a) + F_c(s-a)\right]$$

19. Find the Fourier sine transform of e^{-x}.

Soln : Same as Problem 12^{th} with $a = 1$.

20. Find the Fourier cosine transform of $f(x)$, if $f(x) = \begin{cases} \cos x, & 0 < x < a \\ 0, & x > a \end{cases}$

Soln :

$$F_C[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sx dx = \sqrt{\frac{2}{\pi}}\int_0^a \cos x\cos sx dx = \sqrt{\frac{2}{\pi}}\int_0^a \left[\frac{\cos(s+1)x+\cos(s-1)x}{2}\right]dx$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{\sin(s+1)x}{(s+1)}+\frac{\sin(s-1)x}{(s-1)}\right]_0^a = \frac{1}{\sqrt{2\pi}}\left[\frac{\sin(s+1)a}{(s+1)}+\frac{\sin(s-1)a}{(s-1)}\right]$$

21. If the Fourier cosine transform of e^{-x} is $\sqrt{\frac{2}{\pi}}\left(\frac{1}{s^2+1}\right)$, prove that $\int_0^\infty \frac{dx}{(x^2+1)^2} = \frac{\pi}{4}$ using Parseval's identity.

Soln : Given

$$F_C[f(x)] = F_C\left(e^{-x}\right) = F_C(s) = \sqrt{\frac{2}{\pi}}\left(\frac{1}{s^2+1}\right)$$

By Parseval's identity,

$$\int_0^\infty |f(x)|^2 dx = \int_0^\infty |F_C(s)|^2 ds$$

$$\int_0^\infty \left|e^{-x}\right|^2 dx = \int_0^\infty \left|\sqrt{\frac{2}{\pi}}\left(\frac{1}{s^2+1}\right)\right|^2 ds$$

$$\int_0^\infty e^{-2x}dx = \frac{2}{\pi}\int_0^\infty \frac{1}{(s^2+1)^2}ds$$

$$\left(\frac{e^{-2x}}{-2}\right)_0^\infty = \frac{2}{\pi}\int_0^\infty \frac{1}{(x^2+1)^2}dx$$

(s is a dummy variable)

$$\int_0^\infty \frac{1}{(x^2+1)^2}dx = \frac{\pi}{4}$$

22. The area covered by the curve $\frac{\sin x}{x}$ between $x=0$ and $x=\infty$ with $x-$ axis is $\frac{\pi}{2}$. Find $F_S\left(\frac{1}{x}\right)$.

Soln : Given $\int_0^\infty \frac{\sin x}{x}dx = \frac{\pi}{2}$

$$\therefore F_S\left(\frac{1}{x}\right) = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{1}{x}\sin sx dx$$

Set $sx = t$. Then $dx = dt/s$.

When $x = -\infty, t = -\infty$ and when $x = \infty, t = \infty$.

$$F_S(1/x) = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{1}{t/s}\sin t\ dt/s = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{1}{t}\sin t\ dt$$

$$\left[\because \int_0^\infty \frac{\sin t}{t}dt = \frac{\pi}{2}\right]$$

$$= \sqrt{\frac{2}{\pi}\frac{\pi}{2}} = \sqrt{\frac{\pi}{2}}$$

23. Find the Fourier sine transform of $f(x) = \begin{cases} \sin x, & 0<x<a \\ 0, & a>0 \end{cases}$.

Soln :

$$F_S[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin sx dx = \sqrt{\frac{2}{\pi}}\int_0^a \sin x\sin sx dx = \sqrt{\frac{2}{\pi}}\int_0^a \left[\frac{\cos(s-1)x-\cos(s+1)x}{2}\right]dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{\sin(s-1)x}{(s-1)} - \frac{\sin(s+1)x}{(s+1)} \right]_0^a$$
$$= \sqrt{\frac{2}{\pi}} \left[\frac{\sin(s-1)a}{(s-1)} - \frac{\sin(s+1)a}{(s+1)} \right]$$

24. Find the Fourier sine transform of $f(x) = \begin{cases} x, & 0 \le x < \pi \\ 0, & x \ge \pi \end{cases}$.

Soln :

$F_S[f(x)]$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin sx dx$$
$$= \sqrt{\frac{2}{\pi}} \int_0^\pi x \sin sx dx$$
$$= \sqrt{\frac{2}{\pi}} \left[(x) \left(\frac{-\cos sx}{s} \right) - (1) \left(\frac{-\sin sx}{s^2} \right) \right]_0^\pi$$
$$= \sqrt{\frac{2}{\pi}} \left[(\pi) \left(\frac{-\cos s\pi}{s} \right) + \left(\frac{\sin s\pi}{s^2} \right) \right]$$

25. If the Fourier sine transform of e^{-x} is $\sqrt{\frac{2}{\pi}} \left(\frac{s}{s^2+1} \right)$, prove that $\int_0^\infty \frac{x^2 dx}{(x^2+1)^2} = \frac{\pi}{4}$ using Parseval's identity.

Soln : Given

$$F_S[f(x)] = F_S\left(e^{-x}\right)$$
$$= F_S(s)$$
$$= \sqrt{\frac{2}{\pi}} \left(\frac{s}{s^2+1} \right)$$

By Parseval's identity,

$$\int_0^\infty |f(x)|^2 dx = \int_0^\infty |F_S(s)|^2 ds$$
$$\int_0^\infty \left|e^{-x}\right|^2 dx = \int_0^\infty \left| \sqrt{\frac{2}{\pi}} \left(\frac{s}{s^2+1} \right) \right|^2 ds$$
$$\int_0^\infty e^{-2x} dx = \frac{2}{\pi} \int_0^\infty \frac{s^2}{(s^2+1)^2} ds$$
$$\left(\frac{e^{-2x}}{-2} \right)_0^\infty = \frac{2}{\pi} \int_0^\infty \frac{x^2}{(x^2+1)^2} dx$$

($\because s$ is a dummy variable)

$$\int_0^\infty \frac{x^2}{(x^2+1)^2} dx = \frac{\pi}{4}$$

26. State the convolution theorem and Parseval's Identity for Fourier transforms.

Soln : Convolution theorem for Fourier transform:

$$F[f(x) * g(x)] = F[f(x)].F[g(x)]$$
$$= F(s).G(s)$$

Parseval's Identity for Fourier transform:

Suppose $F[f(x)] = F(s)$ & $F[g(x)] = G(s)$, then

1)If $f(x) \ne g(x)$,

$$\int_{-\infty}^\infty F[f(x)].F[g(x)] \, ds = \int_{-\infty}^\infty f(x).g(x) \, dx$$

2) If $f(x) = g(x)$,

$$\int_{-\infty}^\infty |F[f(x)]|^2 \, ds = \int_{-\infty}^\infty |f(x)|^2 \, dx$$

4.2 Part-B

4.2.1 Examples under Fourier Transform Pair

Example 4.1. Find the Fourier transform of $f(x) = \begin{cases} 1-|x| & \text{for } |x|<1 \\ 0 & \text{otherwise} \end{cases}$. Hence show that $\int_0^\infty \frac{\sin^2 x}{x^2} dx = \frac{\pi}{2}$

Solution : The given function can be written as $f(x) = \begin{cases} 1-|x| & \text{for } -1<x<1 \\ 0 & \text{otherwise} \end{cases}$.

$$F(s) = F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} (1-|x|)(cos\, sx + isin\, sx) dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} (1-|x|) \cos sx dx + i \int_{-1}^{1} (1-|x|) \sin sx dx$$

$$= \frac{2}{\sqrt{2\pi}} \int_{0}^{1} (1-|x|) \cos sx dx$$

$$[\because (1-|x|) \sin sx \text{ is odd}]$$

$$= \sqrt{\frac{2}{\pi}} \int_0^1 (1-x) \cos sx dx$$

$$= \sqrt{\frac{2}{\pi}} \left[(1-x) \frac{\sin sx}{s} - (-1) \left(\frac{-cossx}{s^2} \right) \right]_0^1$$

$$= \sqrt{\frac{2}{\pi}} \left[(1-x) \frac{\sin sx}{s} - \frac{\cos sx}{s^2} \right]_0^1$$

$$= \sqrt{\frac{2}{\pi}} \left[\left(0 - \frac{\cos s}{s^2} \right) - \left(0 - \frac{1}{s^2} \right) \right]$$

$$F(s) = \sqrt{\frac{2}{\pi}} \left(\frac{1-\cos s}{s^2} \right)$$

By inversion formula for Fourier transform

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s) e^{-isx} ds$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sqrt{\frac{2}{\pi}} \left(\frac{1 - \cos s}{s^2} \right) (\cos sx - i \sin sx) ds$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \left[\left(\frac{1 - \cos s}{s^2} \right) \cos sx - i \left(\frac{1 - \cos s}{s^2} \right) \sin sx \right] ds$$

$$f(x) = \frac{2}{\pi} \int_{0}^{\infty} \left(\frac{1 - \cos s}{s^2} \right) \cos sx ds$$

$$\left[\because \left(\frac{1 - \cos s}{s^2} \right) \cos sx \text{ is even and } \left(\frac{1 - \cos s}{s^2} \right) \sin sx \text{ is odd} \right]$$

$$\int_{0}^{\infty} \left(\frac{1 - \cos s}{s^2} \right) \cos sx ds = \frac{\pi}{2} f(x)$$

Put $x = 0$,

$$\int_{0}^{\infty} \left(\frac{1 - \cos s}{s^2} \right) ds = \frac{\pi}{2}.1$$

$$\int_{0}^{\infty} \frac{2 \sin^2 \frac{s}{2}}{s^2} ds = \frac{\pi}{2}$$

$$\int_{0}^{\infty} \frac{\sin^2 \frac{s}{2}}{\left(\frac{s}{2}\right)^2} \frac{ds}{2} = \frac{\pi}{2}$$

Put $\frac{s}{2} = t \Rightarrow \frac{ds}{2} = dt$

$$\therefore \int_{0}^{\infty} \frac{\sin^2 t}{t^2} dt = \frac{\pi}{2}$$

Example 4.2. Find the Fourier transform of $f(x) = \begin{cases} 1 - x^2 & \text{if } |x| < 1 \\ 0 & \text{if } |x| > 1 \end{cases}$ and hence evaluate $\int_{0}^{\infty} \left(\frac{\sin x - x \cos x}{x^3} \right) \cos \frac{x}{2} dx$

Solution : The given function can be written as

$$f(x) = \begin{cases} 1 - x^2 & \text{if } -1 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$F(s)=F[f(x)]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{isx}dx$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-1}^{1}(1-x^2)(cossx+isinsx)dx$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-1}^{1}\left[(1-x^2)\cos sx+i(1-x^2)\sin sx\right]dx$$

$$=\frac{2}{\sqrt{2\pi}}\int_{0}^{1}(1-x^2)\cos sx dx$$

$$\left[\because \left(1-x^2\right)\cos sx \text{ is even and } \left(1-x^2\right)\sin sx \text{ is odd}\right]$$

$$=\sqrt{\frac{2}{\pi}}\left[(1-x^2)\frac{\sin sx}{s}-(-2x)\left(\frac{-\cos sx}{s^2}\right)+(-2)\left(\frac{-\sin sx}{s^3}\right)\right]_0^1$$

$$=\sqrt{\frac{2}{\pi}}\left[(1-x^2)\frac{\sin sx}{s}-2x\frac{\cos sx}{s^2}+(2)\left(\frac{\sin sx}{s^3}\right)\right]$$

$$=\sqrt{\frac{2}{\pi}}\left[\frac{-2\cos s}{s^2}+\frac{2\sin s}{s^3}\right]$$

$$F(s)=2\sqrt{\frac{2}{\pi}}\left(\frac{\sin s-s\cos s}{s^3}\right)$$

By inversion formula for Fourier transform

$$f(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}F(s)e^{-isx}ds$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}2.\sqrt{\frac{2}{\pi}}\left(\frac{\sin s-s\cos s}{s^3}\right)e^{-isx}ds$$

$$=\frac{2}{\pi}\int_{-\infty}^{\infty}\left[\left(\frac{\sin s-s\cos s}{s^3}\right)\cos sx-i\left(\frac{\sin s-s\cos s}{s^3}\right)\sin sx\right]ds$$

$$f(x)=\frac{4}{\pi}\int_{0}^{\infty}\left(\frac{\sin s-s\cos s}{s^3}\right)\cos sx ds$$

$$\left[\because \left(\frac{\sin s-s\cos s}{s^3}\right)\cos sx \text{ is even and } \left(\frac{\sin s-s\cos s}{s^3}\right)\sin sx \text{ is odd}\right]$$

$$\therefore \int_{0}^{\infty}\left(\frac{\sin s-s\cos s}{s^3}\right)\cos sx ds=\frac{\pi}{4}f(x)$$

Put $x=\frac{1}{2}$

$$\int_0^{\infty} \left(\frac{\sin s - s\cos s}{s^3}\right) \cos\frac{s}{2} ds = \frac{\pi}{4}\left(1 - \frac{1}{4}\right)$$
$$= \frac{3\pi}{16}$$

Example 4.3. Find the Fourier transform of $e^{-a^2x^2}$. Hence prove that $e^{-\frac{x^2}{2}}$ is self reciprocal with respect to the Fourier transform.

Solution : Given $f(x) = e^{-a^2x^2}$.

The Fourier transform of $f(x)$ is given by

$$F(s) = F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx$$
$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a^2x^2} e^{isx} dx$$
$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(a^2x^2 - isx)} dx$$
$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left[(ax)^2 - 2(ax)\left(\frac{is}{2a}\right)\right]} dx$$
$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left[(ax)^2 - 2(ax)\left(\frac{is}{2a}\right) + \left(\frac{is}{2a}\right)^2 - \left(\frac{is}{2a}\right)^2\right]} dx$$
$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left[\left(ax - \frac{is}{2a}\right)^2 + \frac{s^2}{4a^2}\right]} dx$$
$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(ax - \frac{is}{2a}\right)^2} e^{-\frac{s^2}{4a^2}} dx$$
$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{s^2}{4a^2}} \int_{-\infty}^{\infty} e^{-\left(ax - \frac{is}{2a}\right)^2} dx$$

Put $ax - \dfrac{is}{2a} = t \Rightarrow dx = \dfrac{dt}{a}$

$$\therefore F(s) = \frac{1}{\sqrt{2\pi}} e^{-\frac{s^2}{4a^2}} \int_{-\infty}^{\infty} e^{-t^2} \frac{dt}{a}$$
$$= \frac{1}{a\sqrt{2\pi}} e^{-\frac{s^2}{4a^2}} \int_{-\infty}^{\infty} e^{-t^2} dt$$

$$= \frac{1}{a\sqrt{2\pi}} e^{-\frac{s^2}{4a^2}} 2 \int_0^\infty e^{-t^2} dt$$

$$= \frac{1}{a\sqrt{2\pi}} e^{-\frac{s^2}{4a^2}} 2 \left(\frac{\sqrt{\pi}}{2}\right)$$

$$\therefore F(s) = \frac{1}{a\sqrt{2}} e^{-\frac{s^2}{4a^2}}$$

$$F[f(x)] = \frac{1}{a\sqrt{2}} e^{-\frac{s^2}{4a^2}}$$

$$F\left(e^{-a^2x^2}\right) = \frac{1}{a\sqrt{2}} e^{-\frac{s^2}{4a^2}} \qquad \left[\because f(x) = e^{-a^2x^2}\right]$$

Setting $a = \dfrac{1}{\sqrt{2}}$

$$F\left(e^{-\frac{x^2}{2}}\right) = e^{-\frac{s^2}{2}}$$

$\therefore f(x) = e^{-\frac{x^2}{2}}$ is self reciprocal.

4.3 Parseval's Identity for Fourier transform:

Example 4.4. Show that the Fourier transform of $f(x) = \begin{cases} a^2 - x^2, \ |x| < a \\ 0, \ |x| > a \end{cases}$ as $2\sqrt{\frac{2}{\pi}}\left(\frac{\sin as - as\cos as}{s^3}\right)$. Hence deduce (i) $\int_0^\infty \left(\frac{\sin s - s\cos s}{s^3}\right) ds = \frac{\pi}{4}$ (ii) $\int_0^\infty \left(\frac{\sin s - s\cos s}{s^3}\right)^2 ds = \frac{\pi}{15}$.

$$\left\{\begin{array}{l} x = 0, a = 1 \text{ in } (i), \\ \text{Parseval's with } a = 1 \text{ in } (ii) \end{array}\right\}$$

Example 4.5. Find the Fourier transform of $f(x) = \begin{cases} a - |x|, \ |x| < a \\ 0, \ |x| > a \end{cases}$ and hence deduce that (i) $\int_0^\infty \left(\frac{\sin t}{t}\right)^2 dt = \frac{\pi}{2}$. (ii) $\int_0^\infty \left(\frac{\sin t}{t}\right)^4 dt = \frac{\pi}{3}$.

$$\left\{\begin{array}{l} F(s) = \sqrt{\frac{2}{\pi}} \frac{1}{s^2}\left(2\sin^2\frac{as}{2}\right), f(x) = \frac{2}{\pi}\int_{-\infty}^{\infty}\left(\frac{\sin\frac{as}{2}}{s}\right)^2 e^{-isx} ds, \\ x = 0, a = 2 \text{ in } (i), \text{Parseval's with } a = 2 \text{ in } (ii) \end{array}\right\}$$

4.3.1 Examples under Fourier Sine & Cosine Transform:

Example 4.6. Find the Fourier sine and Cosine Transform of $e^{-ax}, x \geq 0$.

Solution : The Fourier sine transform of $f(x)$ is given by

$$F_S(s) = F_S[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin sx\,dx$$
$$= \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}\sin sx\,dx$$
$$= \sqrt{\frac{2}{\pi}}\left[\frac{e^{-ax}}{a^2+s^2}(-a\sin sx - s\cos sx)\right]_0^\infty$$
$$= \sqrt{\frac{2}{\pi}}\frac{s}{a^2+s^2}$$

The Fourier Cosine transform of $f(x)$ is given by

$$F_c(s) = F_c[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sx\,dx$$
$$= \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-x}\cos sx\,dx$$
$$= \sqrt{\frac{2}{\pi}}\left[\frac{e^{-x}}{a^2+s^2}(-a\cos sx + s\sin sx)\right]_0^\infty$$
$$= \sqrt{\frac{2}{\pi}}\frac{a}{a^2+s^2}$$

4.3.2 Examples under Convolution & Parseval's identity:

Example 4.7. Evaluate $\displaystyle\int_0^\infty \frac{dx}{(x^2+a^2)(x^2+b^2)}$. $\left\{\dfrac{\pi}{2ab(a+b)}\right\}$

Example 4.8. Evaluate $\displaystyle\int_0^\infty \frac{x^2dx}{(x^2+a^2)(x^2+b^2)}$. $\left\{\dfrac{\pi}{2(a+b)}\right\}$

Example 4.9. Evaluate $\displaystyle\int_0^\infty \frac{dx}{(x^2+a^2)^2}$. $\left\{\dfrac{\pi}{4a^3}\right\}$

Example 4.10. Evaluate $\displaystyle\int_0^\infty \frac{x^2dx}{(x^2+a^2)^2}$. $\left\{\dfrac{\pi}{4a}\right\}$

4.4 Assignment IV[Fourier Transforms]

1. Find the Fourier integral representation of $f(x)$ defined as $f(x) = \begin{cases} 0, & x < 0 \\ \frac{1}{2}, & x = 0 \\ e^{-x}, & x > 0 \end{cases}$.

2. Find the Fourier transform of the function $f(x)$ defined by $f(x) = \begin{cases} 1 - x^2; if\ |x| < 1 \\ 0; if\ |x| \geq 1 \end{cases}$. Hence prove that

   (i) $\int\limits_0^\infty \left(\frac{\sin s - s\cos s}{s^3}\right) \cos\left(\frac{s}{2}\right) ds = \frac{3\pi}{16}$

   (ii) $\int\limits_0^\infty \left(\frac{\sin s - s\cos s}{s^3}\right)^2 ds = \frac{\pi}{15}$.

3. Find F.T. of $f(x) = \begin{cases} 1 - |x|\,; if\ |x| < 1, \\ 0; if\ |x| > 1, \end{cases}$ and hence find the value of $\int\limits_0^\infty \left(\frac{\sin t}{t}\right) dt$ and $\int\limits_0^\infty \left(\frac{\sin t}{t}\right)^4 dt$

4. Find the Fourier transform of $e^{-a^2x^2}$. Hence prove $e^{-x^2/2}$ is self reciprocal.

5. Find the Fourier sine and cosine transform of $f(x) = \begin{cases} x; 0 < x < 1 \\ 2 - x; 1 < x < 2 \\ 0; x > 2 \end{cases}$.

6. Prove that $e^{-x^2/2}$ is self reciprocal under Fourier cosine transform.

7. Find the Fourier sine and cosine transform of x^{n-1} and hence prove $\frac{1}{\sqrt{x}}$ is self reciprocal under Fourier sine and cosine transforms.

8. Find the Fourier sine transform of e^{-ax} and hence evaluate Fourier cosine transforms of xe^{-ax} and $e^{-ax}\sin ax$.

9. Find F.S.T. and F.C.T. of $e^{-ax}, a > 0$. Hence evaluate $\int\limits_0^\infty \frac{x^2}{(a^2+x^2)^2} dx$ and $\int\limits_0^\infty \frac{dx}{(x^2+a^2)(x^2+b^2)}$

10. State and prove convolution theorem and Parseval's identity for Fourier transforms.

5 Z-Transforms(Z.T.)

5.1 Part-A

1. Find $Z\left[a^n\right]$.

   Soln : We know that

$$Z\left[x\left(n\right)\right]=\sum_{n=0}^{\infty}x\left(n\right)z^{-n}$$

$$\therefore Z\left[a^n\right]=\sum_{n=0}^{\infty}a^nz^{-n}$$

$$=\sum_{n=0}^{\infty}\left(\frac{a}{z}\right)^n$$

$$=1+\frac{a}{z}+\left(\frac{a}{z}\right)^2+\cdots$$

$$=\left[1-\frac{a}{z}\right]^{-1}$$

$$=\left[\frac{z-a}{z}\right]^{-1}$$

$$\therefore Z\left[a^n\right]=\left[\frac{z}{z-a}\right]$$

2. Find $Z\left[u\left(n-1\right)\right]$.

   Soln :

$$Z\left[u\left(n-1\right)\right]=\sum_{n=0}^{\infty}1.z^{-n}$$

$$=\frac{1}{z}+\left(\frac{1}{z^2}\right)+\cdots$$

$$=\frac{1}{z}\left[1+\frac{1}{z}+\left(\frac{1}{z}\right)^2+\cdots\right]$$

$$=\frac{1}{z}\left[1-\frac{1}{z}\right]^{-1}$$

$$=\frac{1}{z}\left[\frac{z-1}{z}\right]^{-1}=\frac{1}{z}\left[\frac{z}{z-1}\right]$$

$$=\frac{1}{z-1}$$

3. Prove $Z\left[n\right]=\dfrac{z}{\left(z-1\right)^2}$.

   Soln : We know that

$$Z\left[x\left(n\right)\right]=\sum_{n=0}^{\infty}x\left(n\right)z^{-n}$$

$$Z\left[n\right]=\sum_{n=0}^{\infty}nz^{-n}$$

$$=\sum_{n=0}^{\infty}\left(\frac{n}{z^n}\right)=0+\frac{1}{z}+\frac{2}{z^2}+\frac{3}{z^3}+\ldots$$

$$=\frac{1}{z}\left[1+2\left(\frac{1}{z}\right)+3\left(\frac{1}{z}\right)^2+\ldots\right]$$

$$=\frac{1}{z}\left[\frac{z-1}{z}\right]^{-2}=\frac{1}{z}\left[\frac{z}{z-1}\right]^2=\frac{z}{\left(z-1\right)^2}$$

4. Find $Z\left[\dfrac{1}{n\left(n+1\right)}\right]$.

   Soln : Let $\dfrac{1}{n\left(n+1\right)}=\dfrac{A}{n}+\dfrac{B}{n+1}$ (1)

$$1=A\left(n+1\right)+B\left(n\right)$$

Put $n=0$, we get $A=1$

Put $n=-1$, we get $B=-1$

$$(1)\Rightarrow\frac{1}{n\left(n+1\right)}=\frac{1}{n}-\frac{1}{n+1}$$

$$\therefore Z\left[\frac{1}{n\left(n+1\right)}\right]=Z\left[\frac{1}{n}-\frac{1}{n+1}\right]$$

$$=Z\left[\frac{1}{n}\right]-Z\left[\frac{1}{n+1}\right]$$

$$=\log\frac{z}{z-1}-zlog\frac{z}{z-1}$$

$$=\left(1-z\right)log\frac{z}{z-1}$$

5. Prove that $Z[n\ \mathrm{f}(\mathrm{n})] = -z\dfrac{d}{dx}F(z)$..

Soln : We know that

$$Z[x(n)] = \sum_{n=0}^{\infty} x(n) z^{-n}$$

$$\frac{d}{dz}[F(z)] = \sum_{n=0}^{\infty} (-n) f(n) z^{-n-1}$$

$$= -\sum_{n=0}^{\infty} nf(n) \frac{z^{-n}}{z}$$

$$z\frac{d}{dz}[F(z)] = -\sum_{n=0}^{\infty} nf(n) z^{-n}$$

$$= -Z[nf(n)]$$

$$Z[nf(n)] = -z\frac{d}{dz}[F(z)]$$

6. Find $Z\left[e^{-t}t^2\right]$.

Soln : We know that

$$Z\left[e^{-at}f(t)\right] = Z[f(t)]_{z\to ze^{aT}}$$

$$Z\left[e^{-t}t^2\right] = \left[Z\left(t^2\right)\right]_{z\to ze^{T}}$$

$$= \left[\frac{T^2 z(z+1)}{(z-1)^3}\right]_{z\to ze^{T}}$$

$$= \frac{T^2 ze^T\left(ze^T+1\right)}{\left(ze^T-1\right)^3}$$

7. Find $Z[1]$.

Soln : We know that

$$Z[x(n)] = \sum_{n=0}^{\infty} x(n) z^{-n}$$

$$Z[1] = \sum_{n=0}^{\infty} 1.z^{-n} = \sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n$$

$$= 1 + \frac{1}{z} + \left(\frac{1}{z}\right)^2 + \cdots$$

$$= \left[1 - \frac{1}{z}\right]^{-1}$$

$$= \left[\frac{z-1}{z}\right]^{-1}$$

$$= \left[\frac{z}{z-1}\right]$$

8. Show that $Z\left[\dfrac{1}{n!}\right] = e^{1/z}$.

Soln : We know that

$$Z[x(n)] = \sum_{n=0}^{\infty} x(n) z^{-n}$$

$$Z\left[\frac{1}{n!}\right] = \sum_{n=0}^{\infty}\left(\frac{1}{n!}\right) z^{-n}$$

$$= \sum_{n=0}^{\infty}\left(\frac{1}{n!}\right)\left(\frac{1}{z}\right)^n$$

$$= 1 + \frac{1}{1!}\left(\frac{1}{z}\right) + \frac{1}{2!}\left(\frac{1}{z}\right)^2 + \cdots$$

$$= 1 + \frac{\left(\frac{1}{z}\right)}{1!} + \frac{\left(\frac{1}{z}\right)^2}{2!} + \cdots$$

$$= e^{1/z}$$

9. Find $Z\left[\dfrac{1}{3^n}\right]$.

Soln : We know that $Z[a^n] = \dfrac{z}{z-a}$

$$Z\left[\frac{1}{3^n}\right] = Z\left[\left(\frac{1}{3}\right)^n\right]$$

$$= \frac{z}{z-\frac{1}{3}}$$

$$= \frac{3z}{3z-1}$$

10. Find $Z\left[e^{3t-5}\right]$.

Soln :

$$Z\left[e^{3t-5}\right] = Z\left[e^{3t}e^{-5}\right]$$

$$= e^{-5}Z\left[e^{3t}\right]$$

$$= e^{-5}Z\left[e^{3nT}\right]$$

$$= e^{-5}Z\left[\left(e^{3T}\right)^n\right]$$

$$= \frac{e^{-5}z}{z-e^{3T}}$$

11. Find $Z\left[4(3)^n + 2(-1)^n\right]$.

Soln : We know that

$$Z[a^n] = \frac{z}{z-a}$$

$$Z\left[4.3^n + 2(-1)^n\right] = 4Z[3^n] + 2Z\left[(-1)^n\right]$$

$$= \frac{4z}{z-3} + \frac{2z}{z+1}$$

12. Find $Z\left[e^{at}\sin bt\right]$.

Soln : We know that

$$Z\left[e^{-at}f(t)\right] = Z\left[f(t)\right]_{z\to ze^{aT}}$$
$$Z\left[e^{-at}\sin bt\right] = \left[Z(\sin bt)\right]_{z\to ze^{aT}}$$
$$= \left[\frac{z\sin bT}{z^2 - 2z\cos bT + 1}\right]_{z\to ze^{aT}}$$
$$= \frac{ze^{aT}\sin bT}{z^2e^{2aT} - 2ze^{aT}\cos bT + 1}$$

13. Find $Z\left[a^n t\right]$.

Soln : We know that

$$Z\left[a^n f(t)\right] = F\left[\frac{z}{a}\right]$$
$$Z\left[a^n t\right] = \left[Z(t)\right]_{z\to z/a}$$
$$= \left[\frac{Tz}{(z-1)^2}\right]_{z\to z/a}$$
$$= \frac{T\frac{z}{a}}{\left(\frac{z}{a}-1\right)^2}$$
$$= \left[\frac{Taz}{(z-a)^2}\right]$$

14. Find State and prove initial value theorem in Z-transform.

Soln :

Statement : If $Z\left[f(n)\right] = F(z)$,, then $\lim_{n\to 0} f(0) = \lim_{z\to\infty} F(z)$

Proof :

$$F[z] = \sum_{n=0}^{\infty} f(n)z^{-n}$$
$$= f(0)z^{-0} + f(1)z^{-1} + f(2)z^{-2} + \cdots$$
$$= f(0) + \frac{f(1)}{z} + \frac{f(2)}{z^2} + \cdots$$

By applying limits,

$$\lim_{n\to 0} F(z) = \lim_{z\to\infty}\left[f(0) + \frac{f(1)}{z} + \frac{f(2)}{z^2} + \cdots\right]$$
$$= f(0)\quad\left[\because \lim_{z\to\infty}\frac{1}{z} = \lim_{z\to\infty}\frac{1}{z^2} = 0\right]$$

We can write $f(0)$ as $\lim_{n\to 0} f(0)$

$$\lim_{z\to\infty} F(z) = \lim_{n\to 0} f(0)$$

15. Define unit step sequence.

Soln : The unit step sequence $u(n)$ has values

$$u(n) = \begin{cases} 1 & \text{for } n > 0 \\ 0 & \text{for } n < 0 \end{cases}$$

16. State the Damping rule.

Soln : The geometric factor a^{-n} when $|a| < 1$, damps the function u_n. Hence we use the name damping rule.

If $Z(u_n) = U(z)$, then $Z\left(a^{-n}u_n\right) = U(az)$

17. If $f(z) = \left[\frac{2z}{z - e^{-T}}\right]$. Find $\lim_{t\to\infty} f(t)$ and $f(0)$.

Soln : By initial value theorem

$$f(0) = \lim_{z\to\infty} F(z)$$
$$= \lim_{z\to\infty}\frac{2z}{z - e^{-T}}$$
$$= \lim_{z\to\infty}\frac{2z}{z\left(1 - \frac{e^{-T}}{z}\right)}$$
$$= 2$$

By final value theorem

$$\lim_{t\to\infty} f(t) = \lim_{z\to 1}(z-1)F(z)$$
$$= \lim_{z\to 1}(z-1)\frac{2z}{z - e^{-T}}$$
$$= 0$$

18. If $f[z] = \frac{5z}{(z-2)(z-3)}$. Find find $f(0)$.

Soln : By initial value theorem :

$$f(0) = \lim_{z\to\infty} F(z)$$
$$= \lim_{z\to\infty}\frac{5z}{(z-2)(z-3)} = \frac{\infty}{\infty}$$
$$= \lim_{z\to\infty}\frac{5}{(z-2)+(z-3)}$$

[by L'Hospital's rule]

$$= \lim_{z\to\infty}\frac{5}{2z-5} = 0$$

19. Find $Z^{-1}\left[\log\left(\frac{z}{z+1}\right)\right]$.

Soln :

Let $F(z) = \log\left(\frac{z}{z+1}\right)$

$$= \log\left(\frac{1/y}{(1/y)+1}\right) \qquad \left[\text{by } z = \frac{1}{y}\right]$$
$$= \log\left(\frac{\frac{1}{y}}{\frac{1+y}{y}}\right)$$
$$= \log\left(\frac{1}{1+y}\right)$$
$$= \log\left[(1+y)^{-1}\right]$$
$$= -\log(1+y)$$
$$= -y + \frac{1}{2}y^2 - \frac{1}{3}y^3 + \ldots + \frac{(-1)^n}{n}z^{-n}f(n)$$
$$= Z^{-1}[f(n)]$$
$$= \begin{cases} 0, & \text{for } n = 0 \\ \frac{(-1)^n}{n}, & \text{otherwise} \end{cases}$$

20. Form $y_n = a2^n + b(-2)^n$, derive a difference equation by eliminating the constants.

Soln : Given $y_n = a2^n + b(-2)^n$

$$y_{n+1} = a2^{n+1} + b(-2)^{n+1}$$
$$= 2a2^n - 2b(-2)^n$$
$$y_{n+2} = a2^{n+2} + b(-2)^{n+2}$$
$$= 4a2^n + 4b(-2)^n$$

Eliminating a and b, we get

$$\begin{vmatrix} y_n & 1 & 1 \\ y_{n+1} & 2 & -2 \\ y_{n+2} & 4 & 4 \end{vmatrix} = 0$$

$$\Rightarrow y_{n+2} - 4y_n = 0$$

which is the desired difference equation.

21. Find $Z\left[\frac{a^n}{n!}\right]$ by Z-transform.

Soln : By of change of scale property

$$Z[a^n f_n] = F\left[\frac{z}{a}\right]$$

$$\text{But, } Z\left[\frac{1}{n!}\right] = e^{1/z}$$

$$\therefore Z\left[\frac{a^n}{n!}\right] = e^{a/z} \qquad \left[\text{Replacing } z \text{ by } \frac{z}{a}\right]$$

22. Find $Z\left[e^{-iat}\right]$ by z-transform.

Soln :

$$Z\left[e^{-iat}\right] = Z\left[e^{-iat}\cdot 1\right]$$
$$= \{Z(1)\}_{z\to ze^{iaT}} \qquad [\text{By shifting thm.}]$$
$$= \left[\frac{z}{z-1}\right]_{z\to ze^{iaT}} \qquad \left[\because Z(1) = \frac{z}{z-1}\right]$$
$$= \frac{ze^{iaT}}{ze^{iaT}-1}$$

where T is the sampling period

23. Find the Z-transform of $(n+1)(n+2)$.

Soln :

$$Z[(n+1)(n+2)]$$
$$= Z[n^2+3n+2]$$
$$= Z[n^2] + 3Z[n] + Z[2]$$
$$= \frac{z(z+1)}{(z-1)^3} + \frac{3z}{(z-1)^2} + \frac{z}{z-1}$$
$$= \frac{z^3+2z^2-z}{(z-1)^3}$$

24. Find the Z-transform of $(n+2)$.

Soln : $Z(n+2) = Z(n) + Z(2)$

$$= \frac{z}{(z-1)^2} + \frac{2z}{(z-1)}$$

25. State final value theorem in Z-transform.

Soln : If $Z[f(n)] = F(z)$, then $\lim\limits_{n\to\infty} f(0) = \lim\limits_{z\to 1}(z-1)F(z)$

26. Express $Zf(n+1)$ in terms of $\overline{f}(z)$.

Soln : Using the formula of Z-transform of f_{n+1}.

We get $Zf(n+1) = Z\left[\overline{f} - f_0\right]$.

27. Find the value of $Zf(n)$ when $f(n) = na^n$.

Soln : We know that

$$Z\left[nf_n\right] = -z\frac{d}{dz}\left[Z\left(f_n\right)\right]$$

$$= -z\frac{d}{dz}\left[Z\left[a^n\right]\right] \qquad \left[\text{Here } f_n = a^n\right]$$

$$= -z\frac{d}{dz}\left[\frac{z}{z-a}\right]$$

$$= -z\left[\frac{(z-a)-z}{(z-a)^2}\right]$$

$$= -z\left[\frac{-a}{(z-a)^2}\right]$$

$$\therefore Z\left[na^n\right] = \frac{az}{(z-a)^2}$$

5.2 Part-B

Table of $Z-$ transform of standard functions.

S.No.	$x(n)$	$Z[x(n)]$
1.	1	$\frac{z}{z-1}$
2.	k	$\frac{kz}{z-1}$
3.	$-k$	$\frac{kz}{z+1}$
4.	a^n	$\frac{z}{z-a}$
5.	n	$\frac{z}{(z-1)^2}$
6.	na^n	$\frac{az}{(z-a)^2}$
7.	n^2	$\frac{z^2+z}{(z-1)^3}$
8.	n^2a^n	$\frac{az(z+a)}{(z-a)^3}$
9.	n^p	$-z\frac{d}{dz}\left\{Z[n^{p-1}]\right\}$
10.	$\frac{1}{n}$	$\log\left(\frac{z}{z-1}\right)$
11.	$\frac{1}{n+1}$	$z\log\frac{z}{z-1}$
12.	$\frac{1}{n!}$	$e^{\frac{1}{z}}$
13.	$\frac{a^n}{n!}$	$e^{\frac{a}{z}}$

14.	$\delta(n)$	1
15.	$\delta(n-k)$	$\dfrac{1}{z^k}$
16.	$a^n\delta(n-k)$	$\left(\dfrac{a}{z}\right)^k$
17.	$u(n)$	$\dfrac{z}{z-1}$
18.	$u(n-1)$	$\dfrac{1}{z-1}$
19.	$\cos n\theta$	$\dfrac{z(z-\cos\theta)}{z^2-2z\cos\theta+1}$
20.	$a^n\cos n\theta$	$\dfrac{z(z-a\cos\theta)}{z^2-2az\cos\theta+a^2}$
21.	$\sin n\theta$	$\dfrac{z\sin\theta}{z^2-2z\cos\theta+1}$
22.	$a^n\sin n\theta$	$\dfrac{az\sin\theta}{z^2-2az\cos\theta+a^2}$
23.	$\cos hn\theta$	$\dfrac{z(z-\cos h\theta)}{z^2-2z\cos h\theta+1}$
24.	$a^n\cos hn\theta$	$\dfrac{z(z-a\cos h\theta)}{z^2-2az\cos h\theta+a^2}$
25.	$\sin hn\theta$	$\dfrac{z\sin h\theta}{z^2-2z\cos h\theta+1}$
26.	$a^n\sin hn\theta$	$\dfrac{az\sin h\theta}{z^2-2az\cos h\theta+a^2}$

5.3 Inverse Z-transform

Definition. If $Z[x(n)]=X(z)$, then $Z^{-1}[X(z)]=x(n)$.

Example. We know that $Z[1]=\dfrac{z}{z-1}$.

$\therefore Z^{-1}\left[\dfrac{z}{z-1}\right]=1.$

The following table gives the inverse $Z-$transform of standard functions.

S.No.	$X(z)$	$x(n)$
1.	$\dfrac{z}{z-1}$	1
2.	$\dfrac{z}{z+1}$	$(-1)^n$
3.	$\dfrac{z}{z-a}$	a^n

4.	$\dfrac{z}{(z-1)^2}$	n
5.	$\dfrac{z^2+z}{(z-1)^3}$	n^2
6.	$\dfrac{az}{(z-a)^2}$	na^n
7.	$\dfrac{z^2}{z^2+a^2}$	$a^n \cos \dfrac{n\pi}{2}$
8.	$\dfrac{az}{z^2+a^2}$	$a^n \sin \dfrac{n\pi}{2}$
9.	$\dfrac{z^2}{(z-a)^2}$	$(n+1)a^n$
10.	$\dfrac{1}{z-a}$	a^{n-1}
11.	$\dfrac{1}{z+a}$	$(-a)^{n-1}$
12.	$\dfrac{1}{(z-a)^2}$	$(n-1)a^{n-2}$
13.	$\dfrac{1}{(z-a)^2}$	$\dfrac{1}{2}(n-1)(n-2)a^{n-3}$
14.	$\dfrac{z}{z+a}$	$(-a)^n$
15.	$\dfrac{z^3}{(z-a)^3}$	$\dfrac{1}{2!}(n+1)(n+2)a^n$

5.3.1 Evaluation of inverse Z-transform using the method of Convolution

Statement. If $Z^{-1}[X(z)] = x(n)$ and $Z^{-1}[Y(z)] = y(n)$, then

$$Z^{-1}[X(z) \times Y(z)] = x(n) * y(n) = \sum_{k=0}^{n} x(k)y(n-k)$$

Example 5.1. Use convolution theorem to evaluate $Z^{-1}\left[\dfrac{z^2}{(z-a)(z-b)}\right]$ [Dec 2011]

Solution.

$$\begin{aligned}
Z^{-1}\left[\frac{z^2}{(z-a)(z-b)}\right] &= Z^{-1}\left[\frac{z}{z-a}\frac{z}{z-b}\right] \\
&= Z^{-1}\left[\frac{z}{z-a}\right] * Z^{-1}\left[\frac{z}{z-b}\right] \\
&= a^n * b^n \\
&= \sum_{k=0}^{n} a^k b^{n-k} \\
&= \sum_{k=0}^{n} a^k b^n b^{-k} \\
&= b^n \sum_{k=0}^{n} \left(\frac{a}{b}\right)^k \\
&= b^n \left[1 + \frac{a}{b} + \left(\frac{a}{b}\right)^2 + \cdots + \left(\frac{a}{b}\right)^n\right] \\
&= b^n \left[\frac{\left(\frac{a}{b}\right)^{n+1} - 1}{\frac{a}{b} - 1}\right] \qquad \left[\text{G.P. with C.R. } \frac{a}{b}\right] \\
&= b^n \frac{a^{n+1} - b^{n+1}}{b^{n+1}} \times \frac{b}{a-b} = \frac{a^{n+1} - b^{n+1}}{a-b}.
\end{aligned}$$

Example 5.2. Using convolution theorem, find $Z^{-1}\left[\dfrac{z^2}{(z-1)(z-3)}\right]$.

[Dec 2013, May 2011]

Solution.
$$\begin{aligned} Z^{-1}\left[\frac{z^2}{(z-1)(z-3)}\right] &= Z^{-1}\left[\frac{z}{z-1}\frac{z}{z-3}\right] \\ &= Z^{-1}\left[\frac{z}{z-1}\right] * Z^{-1}\left[\frac{z}{z-3}\right] \\ &= 1^n * 3^n \\ &= \sum_{k=0}^{n} 1^k 3^{n-k} \\ &= \sum_{k=0}^{n} \frac{3^n}{3^k} \\ &= 3^n \sum_{k=0}^{n} \left(\frac{1}{3}\right)^k \\ &= 3^n \left[1 + \frac{1}{3} + \left(\frac{1}{3}\right)^2 + \cdots + \left(\frac{1}{3}\right)^n\right] \\ &= 3^n \left[\frac{1-\left(\frac{1}{3}\right)^{n+1}}{1-\frac{1}{3}}\right] \\ &= 3^n \frac{3^{n+1}-1}{3^{n+1}} \times \frac{3}{2} = \frac{3^{n+1}-1}{2}. \end{aligned}$$

Example 5.3. Using convolution theorem, find the inverse $Z-$transform of $\dfrac{z^2}{(z+a)^2}$. [Dec 2012]

Solution.
$$\begin{aligned} Z^{-1}\left[\frac{z^2}{(z+a)^2}\right] &= Z^{-1}\left[\frac{z}{z+a}\frac{z}{z+a}\right] \\ &= Z^{-1}\left[\frac{z}{z+a}\right] * Z^{-1}\left[\frac{z}{z+a}\right] \\ &= (-a)^n * (-a)^n \\ &= \sum_{k=0}^{n} (-a)^k (-a)^{n-k} \\ &= \sum_{k=0}^{n} (-a)^k \frac{(-a)^n}{(-a)^k} \\ &= (-a)^n \sum_{k=0}^{n} 1 \\ &= (-a)^n \left[1 + 1 + \cdots + 1 \quad (n+1) \text{ terms}\right] \\ &= (n+1)(-a)^n. \end{aligned}$$

Example 5.4. Using convolution theorem, find the inverse $Z-$transform of $\dfrac{8z^2}{(2z-1)(4z-1)}$ [Jun

2012]

Solution.
$$
\begin{aligned}
Z^{-1}\left[\frac{8z^2}{(2z-1)(4z-1)}\right] &= Z^{-1}\left[\frac{8z^2}{8\left(z-\frac{1}{2}\right)\left(z-\frac{1}{4}\right)}\right]\\
&= Z^{-1}\left[\frac{z^2}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{4}\right)}\right]\\
&= Z^{-1}\left[\frac{z}{z-\frac{1}{2}}\times\frac{z}{z-\frac{1}{4}}\right]\\
&= Z^{-1}\left[\frac{z}{z-\frac{1}{2}}\right] * Z^{-1}\left[\frac{z}{z-\frac{1}{4}}\right]\\
&= \left(\frac{1}{2}\right)^n * \left(\frac{1}{4}\right)^n\\
&= \sum_{k=0}^{n}\left(\frac{1}{2}\right)^k\left(\frac{1}{4}\right)^{n-k}\\
&= \sum_{k=0}^{n}\left(\frac{1}{2}\right)^k\left(\frac{1}{4}\right)^{n}\left(\frac{1}{4}\right)^{-k}\\
&= \left(\frac{1}{4}\right)^n\sum_{k=0}^{n}\left(\frac{\frac{1}{2}}{\frac{1}{4}}\right)^k\\
&= \left(\frac{1}{4}\right)^n\sum_{k=0}^{n}2^k\\
&= \left(\frac{1}{4}\right)^n\frac{2^{n+1}-1}{2-1}\\
&= \frac{1}{4^n}\left(2^{n+1}-1\right).
\end{aligned}
$$

Example 5.5. Using convolution theorem, find the inverse $Z-$transform of $\left(\frac{z}{z-4}\right)^3$. [May 2010]

Solution.
$$\begin{aligned}
Z^{-1}\left[\left(\frac{z}{z-4}\right)^3\right] &= Z^{-1}\left[\frac{z}{z-4}\frac{z}{(z-4)^2}\right] \\
&= Z^{-1}\left[\frac{z}{z-4}\right] * Z^{-1}\left[\frac{z^2}{(z-4)^2}\right] \\
&= 4^n * (n+1)\,4^n \\
&= \sum_{k=0}^{n}(k+1)\,4^k \times 4^{n-k} \\
&= 4^n\sum_{k=0}^{n}(k+1) \\
&= 4^n\left[1+2+3+\cdots(n+1)\right] \\
&= 4^n\frac{(n+1)(n+2)}{2}.
\end{aligned}$$

5.3.2 Evaluation of inverse Z-transform using partial fractions method

While finding the inverse $Z-$transform, resolve $\frac{X(z)}{z}$ into partial fractions which will result in the correct procedure for finding the same.

Example 5.6. Find $Z^{-1}\left[\frac{z\left(z^2-z+2\right)}{(z+1)(z-1)^2}\right]$. [May 2010]

Solution. Let $X(z) = \frac{z\left(z^2-z+2\right)}{(z+1)(z-1)^2}$.

$$\frac{X(z)}{z} = \frac{z^2-z+2}{(z+1)(z-1)^2}.$$

$$\begin{aligned}
\text{Let } \frac{z^2-z+2}{(z+1)(z-1)^2} &= \frac{A}{z+1}+\frac{B}{z-1}+\frac{C}{(z-1)^2} \\
&= \frac{A(z-1)^2+B(z+1)(z-1)+C(z+1)}{(z+1)(z-1)^2}
\end{aligned}$$

$$\therefore z^2-z+2 = A(z-1)^2+B(z+1)(z-1)+C(z+1).$$

When $z=-1, 1+1+2 = A(-1-1)^2$

$$4A = 4$$

$$\boxed{A=1.}$$

When $z=1, C(1+1) = 1-1+2$

$$2C = 2$$

$$\boxed{C=1.}$$

Equating the coefficients of z^2 we get

$$A+B=1$$
$$1+B=1$$
$$B=0.$$

$$\therefore \frac{z^2-z+2}{(z+1)(z-1)^2}=\frac{1}{z+1}+\frac{1}{(z-1)^2}$$
$$\frac{X(z)}{z}=\frac{1}{z+1}+\frac{1}{(z-1)^2}$$
$$\therefore X(z)=\frac{z}{z+1}+\frac{z}{(z-1)^2}.$$
$$Z^{-1}[X(z)]=Z^{-1}\left[\frac{z}{z+1}\right]+Z^{-1}\left[\frac{z}{(z-1)^2}\right]$$
$$=(-1)^n+n.$$

Example 5.7. Find the inverse $Z-$transform of $\dfrac{10z}{z^2-3z+2}$. [Dec 2009]

Solution. Let $X(z)=\dfrac{10z}{z^2-3z+2}$.

$$\frac{X(z)}{z}=\frac{10}{(z-1)(z-2)}.$$

Let $\dfrac{10}{(z-1)(z-2)}=\dfrac{A}{z-1}+\dfrac{B}{z-2}$

$$10=A(z-2)+B(z-1).$$

When $z=1, -A=10$

$$\therefore A=-10.$$

When $z=2, B=10.$

$$\frac{X(z)}{z}=-\frac{10}{z-1}+\frac{10}{z-2}.$$
$$X(z)=-10\frac{z}{z-1}+10\frac{z}{z-2}.$$
$$Z^{-1}[X(z)]=-10Z^{-1}\left[\frac{z}{z-1}\right]+10Z^{-1}\left[\frac{z}{z-2}\right]$$
$$=-10\times 1+10\times 2^n$$
$$=10\left[2^n-1\right].$$

Example 5.8. Find the inverse $Z-$transform of $\dfrac{z^2+z}{(z-1)(z^2+1)}$ using partial fractions. [Nov 2014]

Solution. Let $X(z)=\dfrac{z^2+z}{(z-1)(z^2+1)}$.

$$\frac{X(z)}{z}=\frac{z+1}{(z-1)(z^2+1)}.$$

Let $\dfrac{z+1}{(z-1)(z^2+1)} = \dfrac{A}{z-1} + \dfrac{Bz+c}{z^2+1}$.

$\therefore z+1 = A(z^2+1) + (Bz+c)(z-1)$

When $z=1$, $2A = 2 \Rightarrow A = 1$

Equating the coefficients of z^2 we get

$A + B = 0 \Rightarrow B = -A \Rightarrow B = -1.$

Equating the constant terms we get

$A - C = 1 \Rightarrow C = A - 1 = 0 \Rightarrow C = 0.$

$$\therefore \frac{X(z)}{z} = \frac{1}{z-1} - \frac{z}{z^2+1}.$$

$$X(z) = \frac{z}{z-1} - \frac{z^2}{z^2+1}.$$

$$Z^{-1}[X(z)] = Z^{-1}\left[\frac{z}{z-1}\right] - Z^{-1}\left[\frac{z^2}{z^2+1}\right] = 1 - \cos\frac{n\pi}{2}.$$

5.3.3 Evaluation of inverse Z-transform by Residue method

By inversion integral method, the inverse $Z-$transform of $X(z)$ is given by $x(n) = \dfrac{1}{2\pi i}\displaystyle\int_c X(z)z^{n-1}dz$, where c is the closed contour containing all the isolated singularities of $X(z)$.

By Cauchy's residue theorem,

$x(n)$ =sum of the residues of $X(z)z^{n-1}$ at the isolated singularities.

Method of evaluation of the residues

1. If $z=a$ is a simple pole, then $R(a) = \lim\limits_{z\to a}(z-a)X(z)z^{n-1}$.

2. If $z=a$ is a pole of order m, then $R(a) = \lim\limits_{z\to a}\dfrac{1}{(m-1)!}\dfrac{d^{m-1}}{dz^{m-1}}\left\{(z-a)^m X(z)z^{n-1}\right\}$.

Example 5.9. Using inversion integral method, find the inverse $Z-$transform of $\dfrac{z}{(z-1)(z-2)}$.

Solution. Let $X(z) = \dfrac{z}{(z-1)(z-2)}$.

$z = 1$ and $z = 2$ are simple poles .

Let $R(1)$ and $R(2)$ be the residues of $X(z)z^{n-1}$ at $z=1$ and $z=2$ respectively.

$$\begin{aligned}
\text{Now } R(1) &= \lim_{z\to 1}(z-1)X(z)z^{n-1}\\
&= \lim_{z\to 1}(z-1)\frac{z}{(z-1)(z-2)}z^{n-1}\\
&= \lim_{z\to 1}\frac{z}{z-2}z^{n-1}\\
&= \lim_{z\to 1}\frac{z^n}{z-2} = \frac{1}{-1} = -1.\\
R(2) &= \lim_{z\to 2}(z-2)X(z)z^{n-1}\\
&= \lim_{z\to 2}(z-2)\frac{z}{(z-1)(z-2)}z^{n-1}\\
&= \lim_{z\to 2}\frac{z^n}{z-1} = 2^n.
\end{aligned}$$

By the inversion integral method,

$Z^{-1}[X(z)] = x(n) =$ sum of the Residues

$= R(1) + R(2)$

$= -1 + 2^n = 2^n - 1.$

Example 5.10. Find the inverse $Z-$transform of $\dfrac{z}{(z+1)^2}$. [Dec. 2013]

Solution. Let $X(z) = \dfrac{z}{(z+1)^2}$.

$z=-1$ is a pole of order 2.

Residue of $X(z)z^{n-1}$ at $z=-1$ is

$$\begin{aligned}
R(-1) &= \lim_{z\to -1}\frac{1}{1!}\frac{d}{dz}\left\{(z+1)^2X(z)z^{n-1}\right\}\\
&= \lim_{z\to -1}\frac{d}{dz}\left\{\cancel{(z+1)^2}\times\frac{z}{\cancel{(z+1)^2}}\times z^{n-1}\right\}\\
&= \lim_{z\to -1}\left\{\frac{d}{dz}(z^n)\right\}\\
&= \lim_{z\to -1}\left\{nz^{n-1}\right\}\\
&= n(-1)^{n-1}.
\end{aligned}$$

By inversion integral method

$Z^{-1}[X(z)] = x(n) =$ sum of the Residues

$= R(-1) = n(-1)^{n-1}.$

Example 5.11. Find $Z^{-1}\left[\dfrac{2z^2+4z}{(z-2)^3}\right]$ using residue theorem.

Solution. Let $X(z) = \dfrac{2z^2+4z}{(z-2)^3}$.

$z = 2$ is a pole of order 3.

Residue of $X(z)z^{n-1}$ at $z = 2$ is

$$
\begin{aligned}
R(2) &= \frac{1}{2!}\lim_{z\to 2}\frac{d^2}{dz^2}\left\{(z-2)^3 X(z) z^{n-1}\right\} \\
&= \frac{1}{2}\lim_{z\to 2}\frac{d^2}{dz^2}\left\{(z-2)^3 \times \frac{2z^2+4z}{(z-2)^3} \times z^{n-1}\right\} \\
&= \frac{1}{2}\lim_{z\to 2}\frac{d^2}{dz^2}\left\{(2z^2+4z)z^{n-1}\right\} \\
&= \frac{1}{2}\lim_{z\to 2}\frac{d^2}{dz^2}\left\{2z^{n+1}+4z^n\right\} \\
&= \frac{1}{2}\lim_{z\to 2}\frac{d}{dz}\left\{2(n+1)z^n+4nz^{n-1}\right\} \\
&= \frac{1}{2}\lim_{z\to 2}\left\{2n(n+1)z^{n-1}+4n(n-1)z^{n-2}\right\} \\
&= \frac{1}{2}\left\{2n(n+1)2^{n-1}+4n(n-1)2^{n-2}\right\} \\
&= \frac{1}{2}\left\{n(n+1)2^n+2^2n(n-1)2^{n-2}\right\} \\
&= \frac{1}{2}\left[n(n+1)2^n+n(n-1)2^n\right] \\
&= \frac{1}{2}\times 2^n \times n(n+1+n-1) \\
&= \frac{1}{2}\times 2^n \times n \times 2n = n^2 \times 2^n.
\end{aligned}
$$

By the residue method

$Z^{-1}[X(z)] = x(n) =$ sum of the Residues $= R(2) = n^2 \times 2^n$.

Example 5.12. Find the inverse $Z-$transform of $\dfrac{2z}{(z-1)(z^2+1)}$ by inversion integral method.

Solution. Let $X(z) = \dfrac{2z}{(z-1)(z^2+1)}$.

The poles are given by

$$
\begin{aligned}
(z-1)(z^2+1) &= 0 \\
(z-1)(z+i)(z-i) &= 0 \\
z = 1, z = i, z &= -i.
\end{aligned}
$$

All the poles are simple poles.

Let $R(1), R(i)$ and $R(-i)$ be the residues of $X(z)z^{n-1}$ at $z=1, z=i$ and $z=-i$ respectively.

$$\begin{aligned}
R(1) &= \lim_{z\to 1}(z-1)X(z)z^{n-1}\\
&= \lim_{z\to 1}(z-1)\frac{2z}{(z-1)(z^2+1)}z^{n-1}\\
&= \lim_{z\to 1}\frac{2z^n}{z^2+1} = \frac{2}{2} = 1.\\
R(i) &= \lim_{z\to i}(z-i)X(z)z^{n-1}\\
&= \lim_{z\to i}(z-i)\frac{2z}{(z-1)(z-i)(z+i)}z^{n-1}\\
&= \lim_{z\to i}\frac{2z^n}{(z-1)(z+i)}\\
&= \frac{2i^n}{(i-1)2i} = \frac{i^{n-1}}{i-1}\\
R(-i) &= \lim_{z\to -i}(z+i)X(z)z^{n-1}\\
&= \lim_{z\to -i}(z+i)\frac{2z}{(z-1)(z+i)(z-i)}z^{n-1}\\
&= \lim_{z\to -i}\frac{2z^n}{(z-1)(z-i)}\\
&= \frac{2(-i)^n}{(-i-1)(-2i)} = \frac{(-i)^{n+1}}{1+i}.
\end{aligned}$$

By the inversion integral method,

$Z^{-1}[X(z)] = x(n) =$ sum of the Residues

$= R(1) + R(i) + R(-i)$

$$= 1 + \frac{i^{n-1}}{i-1} + \frac{(-i)^{n+1}}{1+i}.$$

Example 5.13. Find $Z^{-1}\left[\dfrac{z}{z^2-2z+2}\right]$ by residue method.

Solution. Let $X(z) = \dfrac{z}{z^2-2z+2}$.

The poles are given by

$$\begin{aligned}
z^2-2z+2 = 0 &\Rightarrow (z-1)^2+2-1 = 0\\
(z-1)^2+1 &= 0\\
(z-1)^2 &= -1\\
z-1 &= \pm i\\
z &= 1\pm i.
\end{aligned}$$

$1+i$ and $1-i$ are simple poles.

Let $R(1+i)$ and $R(1-i)$ be the residues of $X(z)z^{n-1}$ at $1+i$ and $1-i$ respectively.

$$\begin{aligned}
R(1+i) &= \lim_{z\to(1+i)} (z-1-i)X(z)z^{n-1} \\
&= \lim_{z\to 1+i} (z-1-i)\frac{z}{(z-1-i)(z-1+i)}z^{n-1} \\
&= \lim_{z\to 1+i} \frac{z^n}{(z-1+i)} \\
&= \frac{(1+i)^n}{1+i-1+i} = \frac{(1+i)^n}{2i} \\
R(1-i) &= \lim_{z\to(1-i)} (z-1+i)X(z)z^{n-1} \\
&= \lim_{z\to 1-i} (z-1+i)\frac{z}{(z-1-i)(z-1+i)}z^{n-1} \\
&= \lim_{z\to 1-i} \frac{z^n}{(z-1-i)} \\
&= \frac{(1-i)^n}{1-i-1-i} = \frac{(1-i)^n}{-2i}
\end{aligned}$$

By the inversion integral method,

$Z^{-1}[X(z)] = x(n) = \text{sum of the Residues}$

$$= R(1+i) + R(1-i)$$

$$= \frac{(1+i)^n}{2i} - \frac{(1-i)^n}{-2i} = \frac{1}{2i}\left[(1+i)^n - (1-i)^n\right].$$

Let $1+i = r(\cos\theta + i\sin\theta)$.

$r\cos\theta = 1, r\sin\theta = 1.$

$r^2 = 1+1 = 2$

$r = \sqrt{2}.$

Substituting $r = \sqrt{2}$ we get,

$$\cos\theta = \frac{1}{\sqrt{2}}, \sin\theta = \frac{1}{\sqrt{2}}$$

$$\therefore \theta = \frac{\pi}{4}.$$

Hence ,$1+i = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

$$\begin{aligned}
(1+i)^n &= (\sqrt{2})^n\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)^n \\
&= (\sqrt{2})^n\left(\cos\frac{n\pi}{4} + i\sin\frac{n\pi}{4}\right).
\end{aligned}$$

Changing i into $-i$ we get

$$(1-i)^n = (\sqrt{2})^n\left(\cos\frac{n\pi}{4} - i\sin\frac{n\pi}{4}\right)$$

$$\begin{aligned}
(1+i)^n - (1-i)^n &= (\sqrt{2})^n\left[\cos\frac{n\pi}{4} + i\sin\frac{n\pi}{4} - \cos\frac{n\pi}{4} + i\sin\frac{n\pi}{4}\right]. \\
&= (\sqrt{2})^n \times 2i\sin\frac{n\pi}{4}
\end{aligned}$$

$$\therefore Z^{-1}[X(z)] = \frac{1}{2i}(\sqrt{2})^n \times 2i\sin\frac{n\pi}{4} = 2^{\frac{n}{2}}\sin\frac{n\pi}{4}.$$

5.3.4 Solution of difference equations using $Z-$transform

Example 5.14. Using $Z-$transform, solve the difference equation $y_{n+2} + 2y_{n+1} + y_n = n$ given that $y_0 = 0 = y_1$. [Dec 2013]

Solution. The given difference equation is

$y_{n+2} + 2y_{n+1} + y_n = n.$

Taking Z − transform on both sides we get

$Z[y_{n+2}] + 2Z[y_{n+1}] + Z[y_n] = Z[n]$

$z^2\left[Z[y_n] - y_0 - \frac{y_1}{z}\right] + 2z[Z[y_n] - y_0] + Z[y_n] = \frac{z}{(z-1)^2}$

$z^2[Z[y_n] - 0 - 0] + 2z[Z[y_n] - 0] + Z[y_n] = \frac{z}{(z-1)^2}$

$Z[y_n]\left[z^2 + 2z + 1\right] = \frac{z}{(z-1)^2}$

$Z[y_n] = \frac{z}{(z-1)^2(z^2+2z+1)} = \frac{z}{(z-1)^2(z+1)^2}$

$\Rightarrow y_n = Z^{-1}\left[\frac{z}{(z-1)^2(z+1)^2}\right].$

Let $Y(z) = \frac{z}{(z-1)^2(z+1)^2}$.
The poles are given by
$(z-1)^2(z+1)^2 = 0.$
$z = 1, \ -1$ which are of order 2.
Let $R(1)$ and $R(-1)$ be the Residues of $Y(z)z^{n-1}$ at $z = 1$ and $z = -1$ respectively.

$$\begin{aligned}
R(1) &= \frac{1}{1!}\lim_{z\to 1}\frac{d}{dz}\left[(z-1)^2 Y(z) z^{n-1}\right] \\
&= \lim_{z\to 1}\frac{d}{dz}\left[(z-1)^2\frac{z}{(z-1)^2(z+1)^2}z^{n-1}\right] \\
&= \lim_{z\to 1}\frac{d}{dz}\left[\frac{z^n}{(z+1)^2}\right] \\
&= \lim_{z\to 1}\left[\frac{(z+1)^2 \times n \times z^{n-1} - z^n \times 2(z+1)}{(z+1)^4}\right] \\
&= \frac{4n1^{n-1} - 1^n \times 4}{16} = \frac{4n-4}{16} \\
R(1) &= \frac{n-1}{4}.
\end{aligned}$$

$$R(-1) = \frac{1}{1!} \lim_{z \to -1} \frac{d}{dz} \left[(z-1)^2 Y(z) z^{n-1} \right]$$
$$= \lim_{z \to -1} \frac{d}{dz} \left[(z-1)^2 \frac{z}{(z-1)^2 (z+1)^2} z^{n-1} \right]$$
$$= \lim_{z \to -1} \frac{d}{dz} \left[\frac{z^n}{(z-1)^2} \right]$$
$$= \lim_{z \to -1} \left[\frac{(z-1)^2 \times n \times z^{n-1} - z^n \times 2(z-1)}{(z-1)^4} \right]$$
$$= \frac{4n(-1)^{n-1} + 4(-1)^{n-1}}{16} = 4(-1)^{n-1} \frac{n+1}{16}$$
$$R(-1) = \frac{n+1}{4} (-1)^{n-1}.$$

By the inversion integral method,

$$y_n = Z^{-1}[Y(z)] = \text{Sum of the residues}$$
$$= \frac{n-1}{4} + \frac{n+1}{4}(-1)^{n-1}$$
$$= \frac{1}{4}\left[n - 1 + (n+1)(-1)^{n-1} \right].$$

Example 5.15. Solve $y_{n+2} + 6y_{n+1} + 9y_n = 2^n$ given that $y_0 = y_1 = 0$.

[Dec 2012, Dec 2009]

Solution. The given difference equation is

$y_{n+2} + 6y_{n+1} + 9y_n = 2^n$.

Taking $Z-$transform on both sides we get

$$Z[y_{n+2}] + 6Z[y_{n+1}] + 9Z[y_n] = Z[2^n]$$
$$z^2 \left[Z[y_n] - y_0 - \frac{y_1}{z} \right] + 6z\left[Z[y_n] - y_0 \right] + 9Z[y_n] = \frac{z}{z-2}$$
$$z^2 [Z[y_n]] + 6z[Z[y_n]] + 9Z[y_n] = \frac{z}{z-2}$$
$$Z[y_n]\left[z^2 + 6z + 9 \right] = \frac{z}{z-2}$$
$$Z[y_n](z+3)^2 = \frac{z}{z-2}$$
$$Z[y_n] = \frac{z}{(z-2)(z+3)^2}$$
$$y_n = Z^{-1}\left[\frac{z}{(z-2)(z+3)^2} \right].$$

Let $Y(z) = \dfrac{z}{(z-2)(z+3)^2}$.

The poles are given by $z = 2, z = -3$.

$z = 2$ is a simple pole and $z = -3$ is a pole of order 2.

Residue of $Y(z)z^{n-1}$ at $z=2$ is

$$\begin{aligned}R(2) &= \lim_{z\to 2}(z-2)Y(z)z^{n-1}\\ &= \lim_{z\to 2}(z-2)\frac{z}{(z-2)(z+3)^2}z^{n-1}\\ &= \lim_{z\to 2}\frac{z^n}{(z+3)^2}\\ &= \frac{2^n}{25}.\end{aligned}$$

Residue of $Y(z)z^{n-1}$ at $z=-3$ is

$$\begin{aligned}R(-3) &= \frac{1}{1!}\lim_{z\to -3}\frac{d}{dz}\left[(z+3)^2Y(z)z^{n-1}\right]\\ &= \lim_{z\to -3}\frac{d}{dz}\left[(z+3)^2\frac{z}{(z-2)(z+3)^2}z^{n-1}\right]\\ &= \lim_{z\to -3}\frac{d}{dz}\left[\frac{z^n}{(z-2)}\right]\\ &= \lim_{z\to -3}\left[\frac{(z-2)\times n\times z^{n-1}-z^n\times 1}{(z-2)^2}\right]\\ &= \frac{-5n(-3)^{n-1}-(-3)^n}{25}\\ R(-3) &= \frac{(3-5n)(-3)^{n-1}}{25}.\end{aligned}$$

By residue theorem

$$\begin{aligned}y_n &= Z^{-1}[Y(z)]\\ &= \text{Sum of the residues}\\ &= R(2)+R(-3)\\ &= \frac{2^n}{25}+\frac{3-5n}{25}(-3)^{n-1}.\end{aligned}$$

Example 5.16. Solve $y(k+2)+y(k)=1$, $y(0)=y(1)=0$. [Jun 2012]

Solution. The given difference equation is

$$y(k+2)+y(k)=1.$$

Taking $Z-$ transform both sides we get

$$\begin{aligned}&Z\left[y(k+2)\right]+Z\left[y(k)\right]=Z\left[1\right]\\ &z^2\left[Z\left[y(k)\right]-y(0)-\frac{y(1)}{z}\right]+Z\left[y(k)\right]=\frac{z}{z-1}\\ &z^2Z\left[y(k)\right]+Z\left[y(k)\right]=\frac{z}{z-1}\\ &Z[y(k)]\left(z^2+1\right)=\frac{z}{z-1}\\ &Z[y(k)]=\frac{z}{(z-1)\left(z^2+1\right)}.\end{aligned}$$

Let $Y(z) = \dfrac{z}{(z-1)(z^2+1)}$.

The poles are given by

$(z-1)\left(z^2+1\right) = 0$

$z = 1, z^2 = -1.$

$z = \pm i.$

The poles $z = 1$, $z = i$, $z = -i$ which are simple.

Residue of $Y(z)z^{n-1}$ at $z = 1$ is

$$\begin{aligned}
R(1) &= \lim_{z\to 1}(z-1)Y(z)z^{n-1} \\
&= \lim_{z\to 1}(z-1)\frac{z}{(z-1)(z^2+1)}z^{n-1} \\
&= \lim_{z\to 1}\frac{z^n}{z^2+1} = \frac{1^n}{2} = \frac{1}{2}.
\end{aligned}$$

Residue of $Y(z)z^{n-1}$ at $z = i$ is

$$\begin{aligned}
R(i) &= \lim_{z\to i}(z-i)Y(z)z^{n-1} \\
&= \lim_{z\to i}(z-i)\frac{z}{(z-1)(z-i)(z+i)}z^{n-1} \\
&= \lim_{z\to i}\frac{z^n}{(z-1)(z+i)} \\
&= \frac{i^n}{(i-1)\times 2i} \\
&= \frac{1}{2}\frac{i^n}{-1-i} = -\frac{1}{2}\frac{i^n}{1+i} = -\frac{1}{2}\frac{(1-i)i^n}{2} = -\frac{1}{4}(1-i)i^n.
\end{aligned}$$

Residue of $Y(z)z^{n-1}$ at $z = -i$ is

$$\begin{aligned}
R(i) &= \lim_{z\to -i}(z+i)Y(z)z^{n-1} \\
&= \lim_{z\to -i}(z+i)\frac{z}{(z-1)(z-i)(z+i)}z^{n-1} \\
&= \lim_{z\to -i}\frac{z^n}{(z-1)(z-i)} \\
&= \frac{(-i)^n}{(-i-1)\times(-2i)} = \frac{1}{2}\frac{i^n}{-1+i} = -\frac{1}{2}\frac{i^n}{1-i} = -\frac{1}{2}\frac{(1+i)i^n}{2} = -\frac{1}{4}(1+i)i^n.
\end{aligned}$$

By residue theorem

$$\begin{aligned}
y(k) &= \text{Sum of the residues} \\
&= \frac{1}{2} - \frac{1}{4}(1-i)i^n - \frac{1}{4}(1+i)i^n \\
&= \frac{1}{2} - \frac{1}{4}i^n(1-i+1+i) = \frac{1}{2} - \frac{1}{4}i^n \times 2 = \frac{1}{2} - \frac{1}{2}i^n \\
y(k) &= \frac{1}{2}\left(1-i^n\right).
\end{aligned}$$

Example 5.17. Solve the difference equation $y_{n+3} - 3y_{n+1} + 2y_n = 0$, given that $y_0 = 4, y_1 = 0, y_2 = 8$.

[Dec 2012, May 2011]

Solution. The given difference equation is

$y_{n+3} - 3y_{n+1} + 2y_n = 0.$

Taking $Z-$transform on both sides we get

$$Z[y_{n+3}] - 3Z[y_{n+1}] + 2Z[y_n] = 0$$
$$z^3\left[Z[y_n] - y_0 - \frac{y_1}{z} - \frac{y_2}{z^2}\right] - 3z[Z[y_n] - y_0] + 2Z[y_n] = 0$$
$$z^3\left[Z[y_n] - 4 - \frac{8}{z^2}\right] - 3z[Z[y_n] - 4] + 2Z[y_n] = 0$$
$$z^3 Z[y_n] - 4z^3 - 8z - 3zZ[y_n] + 12z + 2Z[y_n] = 0$$
$$Z[y_n]\left[z^3 - 3z + 2\right] = 4z^3 - 4z$$
$$Z[y_n] = \frac{4z^3 - 4z}{z^3 - 3z + 2}$$
$$= \frac{4z\left(z^2 - 1\right)}{(z+1)(z+2)(z-1)}$$
$$= \frac{4z(z+1)(z-1)}{(z+1)(z+2)(z-1)}$$
$$Z[y_n] = \frac{4z}{z+2}$$
$$\therefore y_n = Z^{-1}\left[\frac{4z}{z+2}\right] = 4Z^{-1}\left[\frac{z}{z+2}\right] = 4(-2)^n$$

which is the required solution.

Example 5.18. Solve $y_{n+2} + 4y_{n+1} + 3y_n = 2^n$ with $y_0 = 0, y_1 = 1$. [Dec 2010]

Solution. From the previous problem we get

$$Z[y_n](z+3)(z+1) = \frac{z}{z-2} + z$$
$$= z\left[\frac{1}{z-2} + 1\right]$$
$$= z\left[\frac{1 + z - 2}{z-2}\right]$$
$$= z\frac{(z-1)}{z-2}$$
$$Z[y_n] = \frac{z(z-1)}{(z+1)(z-2)(z+3)}$$
$$y_n = Z^{-1}\left[\frac{z(z-1)}{(z+1)(z-2)(z+3)}\right].$$

Let $Y(z) = \dfrac{z(z-1)}{(z+1)(z-2)(z+3)}$

The poles are given by $z = -1$, $z = 2$, $z = -3$, all of them are simple.

Residue of $Y(z)z^{n-1}$ at $z = 2$ is

$$\begin{aligned}
R(-1) &= \lim_{z\to -1}(z+1)Y(z)z^{n-1} \\
&= \lim_{z\to -1}(z+1)\frac{z(z-1)}{(z+1)(z-2)(z+3)}z^{n-1} \\
&= \frac{(-1)(-2)(-1)^{n-1}}{(-3)2} \\
&= \frac{2(-1)^{n-1}}{-3\times 2} = -\frac{1}{3}(-1)^{n-1}.
\end{aligned}$$

Residue of $Y(z)z^{n-1}$ at $z = 2$ is

$$\begin{aligned}
R(2) &= \lim_{z\to 2}(z-2)Y(z)z^{n-1} \\
&= \lim_{z\to 2}\frac{(z-2)z(z-1)}{(z+1)(z-2)(z+3)}z^{n-1} \\
&= \frac{2}{3\times 5}(2)^{n-1} = \frac{2^n}{15}.
\end{aligned}$$

Residue of $Y(z)z^{n-1}$ at $z = -3$ is

$$\begin{aligned}
R(-3) &= \lim_{z\to -3}(z+3)Y(z)z^{n-1} \\
&= \lim_{z\to -3}\frac{(z+3)z(z-1)}{(z+1)(z-2)(z+3)}z^{n-1} \\
&= \lim_{z\to -3}\frac{z(z-1)}{(z+1)(z-2)}z^{n-1} \\
&= \frac{(-3)(-4)}{(-2)(-5)}(-3)^{n-1} \\
&= -2\frac{(-3)^n}{5}.
\end{aligned}$$

By the method of residues, the solution is given by

$$\begin{aligned}
y_n &= \text{Sum of the residues} \\
&= R(-1) + R(2) + R(-3) \\
&= -\frac{1}{3}(-1)^{n-1} + \frac{1}{15}2^n - \frac{2}{5}(-3)^n.
\end{aligned}$$

Example 5.19. Solve $U_{n+2} - 2U_{n+1} + U_n = 2^n$ given that $U_0 = 2, U_1 = 1$. [May 2010]

Solution. The given difference equation is

$$U_{n+2} - 2U_{n+1} + U_n = 2^n.$$

Taking $Z-$transform on both sides we get

$$\begin{aligned}
Z\left[U_{n+2}\right]-2Z\left[U_{n+1}\right]+Z\left[U_n\right]&=Z\left[2^n\right]\\
z^2\left[Z\left[U_n\right]-U_0-\frac{U_1}{z}\right]-2z\left[Z\left[U_n\right]-U_0\right]+Z\left[U_n\right]&=\frac{z}{z-2}\\
z^2\left[Z\left[U_n\right]-2-\frac{1}{z}\right]-2z\left[Z\left[U_n\right]-2\right]+Z\left[U_n\right]&=\frac{z}{z-2}\\
z^2Z\left[u_n\right]-2z^2-z-2zZ\left[U_n\right]+4z+Z\left[U_n\right]&=\frac{z}{z-2}\\
Z\left[U_n\right]\left[z^2-2z+1\right]-2z^2+3z&=\frac{z}{z-2}\\
Z\left[U_n\right](z-1)^2&=\frac{z}{z-2}+2z^2-3z\\
&=z\left[\frac{1}{z-2}+2z-3\right]\\
&=z\left[\frac{1+(2z-3)(z-2)}{z-2}\right]\\
&=z\left[\frac{1+2z^2-4z-3z+6}{z-2}\right]\\
&=\left[\frac{z\left(2z^2-7z+7\right)}{z-2}\right]\\
Z\left[U_n\right]&=\left[\frac{z\left(2z^2-7z+7\right)}{(z-2)(z-1)^2}\right]\\
U_n&=Z^{-1}\left[\frac{z\left(2z^2-7z+7\right)}{(z-2)(z-1)^2}\right].
\end{aligned}$$

Let $U(z)=\dfrac{z\left(2z^2-7z+7\right)}{(z-2)(z-1)^2}$.

The poles are given by $z=2$,and $z=1$.

$z=2$ is a simple pole and $z=1$ is a pole of order 2.

Residue of $U(z)z^{n-1}$ at $z=2$ is

$$\begin{aligned}
R(2)&=\lim_{z\to 2}(z-2)U(z)z^{n-1}\\
&=\lim_{z\to 2}(z-2)\frac{z\left(2z^2-7z+7\right)}{(z-2)(z-1)^2}z^{n-1}\\
&=\frac{2(8-14+7)}{1}2^{n-1}\\
&=2^n.
\end{aligned}$$

Residue of $U(z)z^{n-1}$ at $z=1$ is

$$
\begin{aligned}
R(1) &= \frac{1}{1!}\lim_{z\to 1}\frac{d}{dz}\left[(z-1)^2U(z)z^{n-1}\right] \\
&= \lim_{z\to 1}\frac{d}{dz}\left\{(z-1)^2\frac{z\left(2z^2-7z+7\right)}{(z-2)(z-1)^2}z^{n-1}\right\} \\
&= \lim_{z\to 1}\frac{d}{dz}\left[\frac{2z^{n+2}-7z^{n+1}+7z^n}{z-2}\right] \\
&= \lim_{z\to 1}\left[\frac{(z-2)\left(2(n+2)z^{n+1}-7(n+1)z^n+7nz^{n-1}\right)-\left(2z^{n+2}-7z^{n+1}+7z^n\right)\times 1}{(z-2)^2}\right] \\
&= \frac{-(2n+4-\cancel{7n}-7+\cancel{7n})-(2-7+7)}{1} \\
&= -2n+3-2 \\
&= 1-2n.
\end{aligned}
$$

By residue method, the solution is given by

$$
\begin{aligned}
U_n &= \text{Sum of the residues} \\
&= R(2)+R(1) \\
&= 2^n+1-2n.
\end{aligned}
$$

5.4 Assignment V[Z-Transform and difference equations]

1. Find the Z-transform of $\cos n\theta$ and $\sin n\theta$. Hence deduce the Z-transform of $a^n\cos n\theta$ and $a^n\sin n\theta$.

2. Find $Z\left(na^n\sin n\theta\right)$.

3. Find $z^{-1}\left(\dfrac{z\left(z^2-z+2\right)}{(z+1)(z-1)^2}\right)$ by using method of partial fraction.

4. Find $Z^{-1}\left[\dfrac{z\left(z^2-z+2\right)}{(z+1)(z-1)^2}\right]$ and $Z^{-1}\left[\dfrac{z}{(z-1)(z-2)}\right]$

5. Using convolution theorem, find the $Z^{-1}\left[\dfrac{z^2}{(z-4)(z-3)}\right]$

6. Using convolution theorem find the inverse Z-transform of $\dfrac{12z^2}{(3z-1)(4z+1)}$

7. Find the inverse Z-transform of $\dfrac{z(z+1)}{(z-1)^3}$ by residue method.

8. Derive the difference equation from $y_n=(A+B_n)(-3)^n$.

9. Solve the difference equation using Z-transform $y_{(n+3)} - 3y_{(n+2)} + 2y_{(n)} = 0$ given that $y_0 = 4, y_1 = 0, y_2 = 8$.

10. Solve $y_{(n+2)} + 3y_{(n+1)} + 9y_{(n)} = 2^n$ given that $y_0 = y_1 = 0$.