Phase lines

Horseshoe crab blood (which is bright blue since it is based on copper rather than iron) contains cells that respond to bacteria at about one part in a trillion. Starting in 1970 it has been used to detect bacterial infections in humans. A start-up in Maryland is considering creating a horseshoe crab farm, and seeks advice about how large a herd they should create in order to sustain the level of harvesting that would make a profit, namely, 250 crabs per month.

Here's a model. The unknown a is proportional to the size of the farm. This proportionality constant is the maximum population in the absence of harvesting (measured in kilocrabs). The larger the farm, the larger the maximum population with no harvesting. Crabs multiply fast in an unconstrained environment, at the rate of 1 month^{-1}. So the logistic growth rate is $(1 - y/a)$, and the logistic equation is

$$\dot{y} = (1 - (y/a))y = y - y^2/a\,.$$

If we now include desired harvest rate we get

$$\dot{y} = -\frac{1}{4} + y - \frac{y^2}{a}\,.$$

Our job is to determine the behavior of the crab population for various values of a.

(a) Just as an experiment, sketch the phase line for $a = 2$. Be sure to include all critical points; say whether they are stable or unstable (or neither), and include the direction of travel of solutions. Then sketch representative graphs of the population evolution, one of each type. (There are five types.)

(b) It looks as if the farm could be made smaller and still produce the desired harvest. To see how small a can be made, it will be useful for you invoke the Phase Lines Mathlet. Select the appropriate autonomous equation from it, and set $a = 2$. You can create solutions by clicking on the graphing window, and visualize the phase line by selecting the [Phase Line] button. What color is used to denote stable equilibria? What color is used to denote unstable equilibria? Do these images corroborate your computations?

(c) Now vary a on the tool. What seems to be the minimum value of a resulting in a stable positive population of crabs? If you roll the cursor over the window, a readout of the coordinates appears below the graph. Use this to estimate the that minimum stable population.

(d) You can visualize all these phase lines simultaneously using the [Bifurcation diagram] option. Select it, and then vary a. You can see the various phase lines being formed and changing as a varies. The pair of critical points for large a merge at a certain value of a; and for smaller values of a the population crashes.

(e) What is the relation between a and y describing the curve you see on the bifurcation diagram? Write it in the form (a function of a and y) $= 0$.

(f) Finally, determine by computation the position of the semi-stable critical points and compare your answers with what you measured earlier and with what you see on the Mathlet.