Quiz 1

Introduction to Quantum Information and Computation

30th January, 2024

Total Points: 20 Time: 45 mins

Instructions: Questions 1, 2, and 3 are compulsory. Out of questions 4 and 5 you can answer any one.

Notation:

All log values are taken with the base 2

$\mathbb{I}_n$ stands for the n by n identity matrix.

Some frequently used states:

$|0\rangle = \begin{bmatrix}1\\0\end{bmatrix}, |1\rangle = \begin{bmatrix}0\\1\end{bmatrix},$

$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle), |-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle),$

$|+i\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle), |-i\rangle = \frac{1}{\sqrt{2}}(|0\rangle - i|1\rangle)$

Pauli Matrices:

$\sigma_0 = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}, \sigma_x = \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}, \sigma_y = \begin{bmatrix}0 & -i\\i & 0\end{bmatrix}, \sigma_z = \begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}.$

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1. [2.5 + 2.5 points] Recall the discussion on how to distinguish between 2 quantum states.

   a Given two states $\rho_0 = |+\rangle\langle+|$ and $\rho_1 = |-\rangle\langle-|$, can we distinguish between them? If so, how?

   b What if we had the states $\rho_2 = |+\rangle\langle+|$ and $\rho_3 = |-i\rangle\langle-i|$. Can we distinguish between them? If so, how?

2. [2 + 2 points] Alice and Bob share a bipartite state $\rho_{AB} = \Sigma_{i,j}\rho_{i,j}|i\rangle\langle i| \otimes |j\rangle\langle j|$. The partial trace of ρ_{AB} with respect to Bob's basis is given by $\rho_A = tr_B[\rho_{AB}] = \Sigma_j(\mathbb{I} \otimes \langle j|)\rho_{AB}(\mathbb{I} \otimes |j\rangle)$, which is the state corresponding to Alice's subsystem (reduced density operator).

   a Show that ρ_A is a valid density operator (by showing that it follows Hermiticity, Positivity, Normalization).

   b Say $\rho_{AB} = |\phi\rangle\langle\phi|$ where $|\phi\rangle = \frac{1}{\sqrt{2}}(|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle)$. Find the reduced density operator ρ_A.

3. [2 + 1 + 3 points] The von Neumann entropy characterizes the amount of 'uncertainty' of some quantum state ρ, and is given by

$$S(\rho) = -tr[\rho\ log_2(\rho)]$$

   Note that by continuity arguments, we can define $0log0 = 0$.

   Say you had the states $\rho_1 = \begin{bmatrix}\frac{1}{2} & 0\\0 & \frac{1}{2}\end{bmatrix}$ and $\rho_2 = \begin{bmatrix}\frac{3}{4} & 0\\0 & \frac{1}{4}\end{bmatrix}$.

   a Intuitively, which of the two states above has more entropy? Verify this by calculating the von Neumann entropies of both ρ_1 and ρ_2. You can use: $log_2(\frac{3}{4}) = -0.415$.

b Prove that the von Neumann entropy of a pure state is 0.

c Prove that the von Neumann entropy remains the invariant under any unitary transormation. That is, show that the von Neumann entropy of a state ρ is the same, even after some unitary U is applied to it ($S(\rho) = S(U\rho U^\dagger)$).

Hint: Recall how to find the function of an operator. If an operator A has the spectral decomposition $A = \Sigma_i \lambda_i |\psi_i\rangle \langle\psi_i|$, then $f(A) = \Sigma_i f(\lambda_i) |\psi_i\rangle \langle\psi_i|$.

4. [2 + 3 points] **Answer any one question out of questions 4 and 5**

Fidelity is one of the widely used metric to compare closeness of two quantum states. A higher fidelity value means the states are closer, and the fidelity between the same states is 1, while fidelity between orthogonal states is 0.

The fidelity between the two states ρ and σ is given by,

$$F(\rho,\sigma) = (tr[\sqrt{\rho^{1/2}\sigma\rho^{1/2}}])^2$$

Property of Fidelity: $F(\rho,\sigma) = F(\sigma,\rho)$,
Fidelity is symmetric, even though it seems asymmetric at the first glance.
You are allowed to take this property for granted.

a Find the fidelity between the states $|\psi_1\rangle = |+\rangle$ and $|\psi_2\rangle = |+i\rangle$
$F(|\psi_1\rangle\langle\psi_1|, |\psi_2\rangle\langle\psi_2|)$

b Given that $\rho = |\psi_\rho\rangle\langle\psi_\rho|$ is a pure state,
prove that $F(\sigma,\rho) = \langle\psi_\rho|\sigma|\psi_\rho\rangle$ for any arbitrary state σ

5. [1 + 1 + 3 points] **Answer any one question out of questions 4 and 5**

For any single qubit quantum state, its density operator ρ can be written as

$$\rho = \frac{1}{2}(\mathbb{I}_2 + \vec{r}.\vec{\sigma})$$

where $\vec{\sigma}$ is a vector of the Pauli matrices, $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ and $\vec{r} = (r_x, r_y, r_z) \in \mathbb{R}^3$ such that $|\vec{r}| \le 1$. $\vec{r}$ is the parameter that gives us the position of the state in the Bloch Sphere.

a Find the values of $\vec{r}$ for which we get the density operators $\rho_0 = |0\rangle\langle 0|$, $\rho_{+i} = |+i\rangle\langle +i|$.

b Locate the above states on the Bloch Sphere.

c Prove that states are pure states if and only if they lie on the surface of the sphere ($|\vec{r}| = 1$).

Hint: You can use the following properties: $\sigma_i^2 = \mathbb{I}_2$ for any $i \in \{0, x, y, z\}$. $\sigma_x\sigma_y = i\sigma_z$ $\sigma_y\sigma_z = i\sigma_s$ $\sigma_z\sigma_x = i\sigma_y$