Quiz 1 - Solutions

Question 1

We can prove the existence of x like this.

Let $x = \frac{1}{3}(w + (-v))$. Then
$v + 3\left(\frac{1}{3}w + \frac{1}{3}(-v)\right) = v + w + (-v) = (v + (-v)) + w = 0 + w = w$, so that x exists.

Since $x - \frac{1}{3}(w + (-v)) = 0$, we have that x is the inverse of $-\frac{1}{3}(w + (-v))$.

Since scalar multiplication and vector addition are defined in V, then $-\frac{1}{3}(w + (-v)) \in V$.

Now to prove the uniqueness of this x, we assume that

$$\begin{aligned} v + 3x &= w \\ v + 3x' &= w \end{aligned}$$

where, x and $x^{'}$ belong to V.

Subtracting the second equation from the first, we get

$$\begin{aligned}
&3x - 3x' = 0 \\
&3\left(x - x'\right) = 0 \text{ (distributive law)} \\
&\frac{1}{3}(3\left(x - x'\right) = \frac{1}{3}0 \text{ (multiplication by a constant preserves equality)} \\
&\left(\frac{1}{3}3\right)\left(x - x'\right) = 0 \text{ (associativity of scalar multiplication)} \\
&1\left(x - x'\right) = 0 \text{ (arithmetic)} \\
&x - x' = 0 \text{ (identity rule for scalar multiplication)} \\
&x = x' \text{ (add } x' \text{ to both sides; use additive inverses to cancel.)}
\end{aligned}$$

Since every vector in V has a unique inverse, x must be unique.

Question 2

Question 2.1)

Since A is invertible, we know that A^{-1} exists. Therefore, multiplying A^{-1} on both sides, we get,

$$\begin{aligned}
A^{-1}(AB) &= (A^{-1})0 \\
(A^{-1}A)B &= 0 \text{ (by commutativity of matrix multiplication)} \\
IB &= 0 \\
B &= 0
\end{aligned}$$

Question 2.2)

Since A is not invertible, $AX = 0$ must have a non-trivial solution v. Let B be the matrix all of whose columns are equal to v. Then $B \neq 0$, but $AB = 0$.

Question C)

Question C.1)

Need to prove both forward and backward implication.

Reverse implication:

$W_1 \cup W_2$ is a subspace iff either $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$.

WLOG, let us assume $W_2 \subseteq W_1$. Now, we can say that W_1 is a subspace, as $W_1 \cup W_2 = W_1$. Similarly, if we assume $W_1 \subseteq W_2$, W_2 is a subspace as $W_1 \cup W_2 = W_2$. Hence, in either case, we have shown that $W_1 \cup W_2$ is a subspace.

Forward implication:

Let us assume $W_1 \cup W_2$ is a subspace. Now, let $W_1 \nsubseteq W_2$ and $W_2 \nsubseteq W_1$.

$$\begin{aligned} &\therefore \exists\, w_1 \in W_1 \backslash W_2, \text{such that } w_1 \notin W_2 \\ &\implies w_1 \in W_1 \cup W_2 \\ \\ &\therefore \exists\, w_2 \in W_2 \backslash W_1, \text{such that } w_2 \notin W_1 \\ &\implies w_2 \in W_1 \cup W_2 \end{aligned}$$

Now, since $W_1 \cup W_2$ is assumed as a subspace, all its elements are closed under addition. Therefore, $w_1 + w_2 \in W_1 \cup W_2$.

From this, we can deduce that, either $w_1 + w_2 \in W_1$ or $w_1 + w_2 \in W_2$.

WLOG, let us take $w_1 + w_2 \in W_1$. Then, we can write see that,

$$\begin{gathered} w_1 \in W_1 \\ w_1 + w_2 \in W_1 (\text{assumed WLOG}) \\ \therefore w_2 = (w_1 + w_2) - w_1 \in W_1 (\text{closure of vectors under addition}) \end{gathered}$$

However, this contradicts our original assumption that $w_2 \notin W_1$.

Now, let us take $w_1 + w_2 \in W_2$. Then, we can write see that,

$$\begin{gathered} w_2 \in W_2 \\ w_1 + w_2 \in W_2 (\text{assumed WLOG}) \\ \therefore w_1 = (w_1 + w_2) - w_2 \in W_2 (\text{closure of vectors under addition}) \end{gathered}$$

However, this contradicts our original assumption that $w_1 \notin W_2$.

In either case, we have ended up with a contradiction.

$\therefore$, either $W_1 \subset W_2$ or $W_2 \subset W_1$ for $W_1 \cup W_2$ to be a subspace.

Question C.2)

Let us represent the system of equations as a matrix $\mathbf{A}$. Let the solutions to this system of equations be represented as b.

$$\mathbf{A} = \begin{bmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \end{bmatrix}, x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, b = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

Hence, we can say,

$$Ax = b$$

Now, we make use of the fact that a matrix that represents a system of equations, has the same solutions as the row reduced form of the matrix, which also represents a set of equations. Hence, we get the matrix $\mathbf{A}$ into it's RREF, to make it easier to find the solutions to the system of equations. If R represents the RREF of matrix A then, we can say $Rx = b$ has the same solutions.

Step 1: Subtract half of the first row from the second row to get a zero in the (2,1) position.

$$\begin{bmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \end{bmatrix} \xrightarrow{R_2 - \frac{1}{2}R_1 \to R_2} \begin{bmatrix} 1 & 1/2 & 1/3 \\ 0 & 1/12 & 1/12 \\ 1/3 & 1/4 & 1/5 \end{bmatrix}$$

Step 2: Subtract one-third of the first row from the third row to get a zero in the (3,1) position.

$$\begin{bmatrix} 1 & 1/2 & 1/3 \\ 0 & 1/12 & 1/12 \\ 1/3 & 1/4 & 1/5 \end{bmatrix} \xrightarrow{R_3 - \frac{1}{3}R_1 \to R_3} \begin{bmatrix} 1 & 1/2 & 1/3 \\ 0 & 1/12 & 1/12 \\ 0 & 1/12 & 2/45 \end{bmatrix}$$

Step 3: Multiply the second row by 12 to get a leading 1 in the (2,2) position.

$$\begin{bmatrix} 1 & 1/2 & 1/3 \\ 0 & 1/12 & 1/12 \\ 0 & 1/12 & 2/45 \end{bmatrix} \xrightarrow{12R_2 \to R_2} \begin{bmatrix} 1 & 1/2 & 1/3 \\ 0 & 1 & 1 \\ 0 & 1/12 & 2/45 \end{bmatrix}$$

Step 4: Subtract one-twelfth of the second row from the third row to get a zero in the (3,2) position.

$$\begin{bmatrix} 1 & 1/2 & 1/3 \\ 0 & 1 & 1 \\ 0 & 1/12 & 2/45 \end{bmatrix} \xrightarrow{R_3 - \frac{1}{12}R_2 \to R_3} \begin{bmatrix} 1 & 1/2 & 1/3 \\ 0 & 1 & 1 \\ 0 & 0 & 1/15 \end{bmatrix}$$

Step 5: Multiply the third row by 15 to get a leading 1 in the (3,3) position.

$$\begin{bmatrix} 1 & 1/2 & 1/3 \\ 0 & 1 & 1 \\ 0 & 0 & 1/15 \end{bmatrix} \xrightarrow{15R_3 \to R_3} \begin{bmatrix} 1 & 1/2 & 1/3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Step 6: Subtract one-half of the second row from the first row to get a zero in the (1,2) position.

$$\begin{bmatrix} 1 & 1/2 & 1/3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_1 - \frac{1}{2}R_2 \to R_1} \begin{bmatrix} 1 & 0 & -1/6 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Step 7: Add one-sixth of the third row to the first row to get a zero in the (1,3) position.

$$\begin{bmatrix} 1 & 0 & -1/6 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_1 + \frac{1}{6}R_3 \to R_1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Step 8: Subtract the third row from the second row to get a zero in the (2,3) position.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_2 - R_3 \to R_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

We observe that this reduces to an identity matrix, and hence, we can now see that

$$\begin{aligned} x_1 &= 0 \\ x_2 &= 0 \\ x_3 &= 0 \end{aligned}$$

Hence, the trivial solution $x = 0$ is the only solution that exists.