International Institute of Information Technology, Hyderabad

(Deemed to be University)

Introduction to Quantum Information and Computation (H1)

(Spring-2024)

Mid Sem Exam

Maximum Duration: 90 Minutes

Maximum Marks: 100

General guidelines:

1. This is a closed book exam where the use of calculators is not prohibited.
2. One A4 size cheat sheet is allowed.
3. First four questions (Q1-Q4) are to be solved in the exam hall and rest (Q5-Q8) are take home and should be submitted on 4th March.

Good Luck!

Question (1) [9 Marks] Let $|\psi\rangle$ be a bipartite state in the Hilbert Space $\mathcal{H}_A \otimes \mathcal{H}_B$. The Schmidt number (or Schmidt Rank) of $|\psi\rangle$ over $\mathcal{H}_A \otimes \mathcal{H}_B$ is the number of non-zero coefficients in the Schmidt decomposition of the state. For $f : \{0,1\}^2 \to \{0,1\}$, let the state $|\psi_f\rangle$ be defined as follows:

$$|\psi_f\rangle_{AB} := \frac{1}{2} \sum_{a,b \in \{0,1\}} (-1)^{f(a,b)} |a\rangle_A \otimes |b\rangle_B .$$

Find the Schmidt rank of the state $|\psi_f\rangle$ if f is taken as the:

(a) **[3 Marks]** AND operation

(b) **[3 Marks]** OR operation

(c) **[3 Marks]** XOR operation

Question (2) [16 Marks] Let us consider two ensembles $E_1 = \{p, |\psi_1\rangle\langle\psi_1|;\ (1-p), |\psi_2\rangle\langle\psi_2|\}$ an $E_2 = \{q, |\phi_1\rangle\langle\phi_1|;\ (1-q), |\phi_2\rangle\langle\phi_2|\}$ of qutrit states. Let $p = 1/2$, $q = 1/3$, and

$$|\psi_1\rangle = \frac{1}{\sqrt{2}}\left[|0\rangle - |2\rangle\right]; \quad |\phi_1\rangle = \frac{1}{\sqrt{2}}\left[|0\rangle + |2\rangle\right]; \quad |\psi_2\rangle = |1\rangle = |\phi_2\rangle, \text{ where}$$

$$|0\rangle = \begin{pmatrix}1\\0\\0\end{pmatrix}; \quad |1\rangle = \begin{pmatrix}0\\1\\0\end{pmatrix}; \quad |2\rangle = \begin{pmatrix}0\\0\\1\end{pmatrix}.$$

(a) **[4 Marks]** Write down the density matrices ρ_1 and ρ_2 corresponding to the ensembles E_1 anc E_2, respectively.

(b) **[4 Marks]** For two arbitrary quantum states σ_1 and σ_2 (density matrices), write down th expression for the maximum success probability in the discrimination of these states.

(c) **[4 Marks]** For states ρ_1 and ρ_2 in question 1(a), what is the value of the maximum succes probability in the discrimination of these states.

(d) **[4 Marks]** Write down the measurement that gives maximum success probability in discrim inating ρ_1 and ρ_2.

Question (3) [13 Marks] Quantum Fourier transform of a state $|j\rangle$, where $\{|j\rangle\}_{j=0}^{N-1}$ forms a orthonormal basis, is defined by the transformation

$$|j\rangle \mapsto U_{\text{QFT}}|j\rangle = \frac{1}{\sqrt{N}}\sum_{k=0}^{N-1} e^{-\frac{2\pi ijk}{N}}|k\rangle,$$

where $i^2 = -1$. Let $N = 2^n$, i.e., the basis $\{|j\rangle\}_{j=0}^{N-1}$ spans a Hilbert space of n qubits.

(a) **[3 Marks]** Show that U_{QFT} is a unitary matrix. Also, show that the matrix form of U_{QFT} i: given by

$$U_{\text{QFT}} = \frac{1}{\sqrt{2^n}}\sum_{j,k=0}^{2^n-1} e^{-\frac{2\pi ijk}{2^n}}|k\rangle\langle j|.$$

(b) **[4 Marks]** Let $j = j_1 j_2 \cdots j_n$ be the bit representation of integer j, show that

$$U_{\text{QFT}}|j\rangle = \left[\left(\frac{|0\rangle + e^{2\pi i 0.j_n}|1\rangle}{\sqrt{2}}\right) \otimes \left(\frac{|0\rangle + e^{2\pi i 0.j_{n-1}j_n}|1\rangle}{\sqrt{2}}\right) \otimes \cdots \otimes \left(\frac{|0\rangle + e^{2\pi i 0.j_1 j_2\cdots j_n}|1\rangle}{\sqrt{2}}\right)\right]$$

(c) **[6 Marks]** Let us consider a quantum state for $n = 3$

$$|\psi\rangle := \frac{1}{2}\sum_{j=0}^{7}\cos(\pi j/4)\,|j\rangle.$$

Show that the state is normalized. Further, find the state $U_{\text{QFT}}|\psi\rangle$. Write your answer in the basis $\{|a\rangle \otimes |b\rangle \otimes |c\rangle\}_{a,b,c=0}^{1}$.

Question (4) [12 Marks] Let X, Y, Z be the Pauli matrices and $\vec{\sigma} = (X, Y, Z)^T$ be the vector of Pauli matrices. Define for $\theta \in [0, 2\pi]$

$$R_{\hat{n}}(\theta) := e^{-\frac{i\theta}{2}\hat{n}\cdot\vec{\sigma}},$$

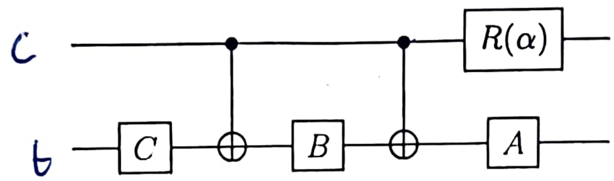
where $\hat{n} = (n_x, n_y, n_z)$ is a three dimensional unit vector.

(a) **[4 Marks]** Prove that $XYX = -Y$ and further show that

$$XR_y(\theta)X = R_y(-\theta).$$

(b) **[3 Marks]** Let C_U be a controlled unitary operation on two qubits for some qubit unitary U. Let c denotes the control and t denotes the target, then write down the action of C_U on the state $|x\rangle_c \otimes |y\rangle_t$, where $x, y \in \{0, 1\}$.

(c) **[5 Marks]** For an arbitrary qubit unitary matrix U prove that there exists 2×2 unitary matrices A, B, C with the property that $ABC = \mathbb{I}$ and a real number α such that the circuit below implements the gate C_U:



where $R(\alpha) = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\alpha} \end{pmatrix}$.

Take home questions

Question (5) [8 Marks] Let us consider a function $f : \{0, 1\}^n \mapsto \{0, 1\}^n$. Consider two types of oracles U_f and V_f such that on the n qubit query register Q and the n qubit response register R they act respectively as

$$U_f\left(|x\rangle_Q \otimes |y\rangle_R\right) := |x\rangle_Q \otimes |y \oplus f(x)\rangle_R;$$
$$V_f\left(|x\rangle_Q \otimes |y\rangle_R\right) := (-1)^{y\cdot f(x)}\,|x\rangle_Q \otimes |y\rangle_R,$$

where $x, y \in \{0, 1\}^n$. If $x = x_1 \cdots x_n$ and $y = y_1 \cdots y_n$, then $x \cdot y = \oplus_{i=1}^n x_i y_i$, where $\oplus$ is addition modulo 2. U_f is called normal oracle and V_f is called the phase oracle.

(a) **[3 Marks]** For $x, y \in \{0, 1\}^n$ show that

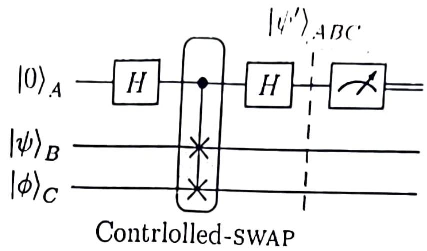
$$\sum_{z\in\{0,1\}^n} (-1)^{(x\oplus y)\cdot z} = 2^n \delta(x, y).$$

(b) [5 Marks] Show that for any $x, y \in \{0,1\}^n$

$$V_f(|x\rangle_Q \otimes |y\rangle_R) = \left(\mathbb{I}_Q \otimes H^{\otimes n}\right) U_f \left(\mathbb{I}_Q \otimes H^{\otimes n}\right) (|x\rangle_Q \otimes |y\rangle_R),$$

where H is the Hadamrd operator. That is prove that $V_f = \left(\mathbb{I}_Q \otimes H^{\otimes n}\right) U_f \left(\mathbb{I}_Q \otimes H^{\otimes n}\right)$.

Question (6) [11 Marks] Let $|0\rangle_A$, $|\psi\rangle_B$ and $|\phi\rangle_C$ be three normalized qubit states. They are undergoing the transformation according to the following circuit:



(a) [5 Marks] Write the tripartite state $|\psi'\rangle_{ABC}$.

(b) [3 Marks] Let us perform the computational basis measurement on the qubit A. Write down the probability of getting 0 as p_0 and 1 as p_1. Compute p_0 and p_1.

(c) [3 Marks] Suppose you perform the above circuit N times using N copies of everything. And suppose you get 0 outcome m times. Write the fidelity between the states $|\psi\rangle_A$ and $|\phi\rangle_B$ in terms of m and N.

Question (7) [18 Marks] Let us consider a function $f : \{0,1\}^n \mapsto \{0,1\}^n$. It is promised that there is some unknown $d \in \{0,1\}^n$ such that for all $x, y \in \{0,1\}^n$

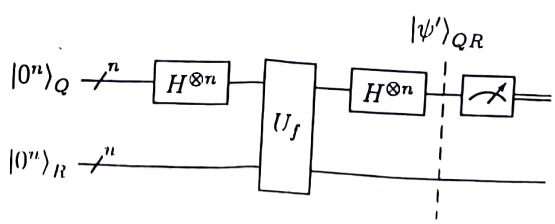
$$f(x) = f(y) \text{ if and only if } x \oplus y \in \{0^n, d\}.$$

The function f can be accessed as a black box (oracle) via the unitary U_f such that on the n qubit query register Q and the n qubit response register R we have

$$U_f\left(|x\rangle_Q \otimes |y\rangle_R\right) := |x\rangle_Q \otimes |y \oplus f(x)\rangle_R.$$

(a) [3 Marks] Prove that the function f is one-to-one when $d = 0^n$ and two-to-one otherwise.

(b) [5 Marks] Find the state $|\psi'\rangle_{QR}$ in the circuit below.



(c) **[5 Marks]** Perform an n qubit computational basis measurement on the first set of qubits Q in the state $|\psi'\rangle_{QR}$. Prove that the probability of getting outcome $j = j_1 \cdots j_n$ is given by

$$p(j) = \left\| \frac{1}{2^n} \sum_{z \in \text{range}(f)} \left(1 + (-1)^{j \cdot d}\right) |z\rangle \right\|^2 .$$

(d) **[2 Marks]** Show that $p(j)$ is nonzero only if $j \cdot d = 0$. This means that if you repeat the above procedure N times you get N values x_i such that $x_i \cdot d = 0$ for all $i = 1, \cdots, N$. Solving these equations classically one can determine d.

(e) **[3 Marks]** The above algorithm to determine d uses single query to f. In how many queries to f one can determine d classically naively?

Question (8) [13 Marks] Say Alice and Bob share an arbitrary pure state $|\Psi\rangle_{AB}$. Bob wants to make a measurement given by projection operators $\{Q_y^B\}_y$, but his measurement apparatus is faulty nevertheless he can still perform any unitary operations on his subsystem. Alice on the other hand, can apply any measurement she wants. Prove that there exists a set of projectors $\{P_y^A\}_y$ on Alice's side such that if projector P_y^A ends up being applied, then both Alice and Bob can implement unitaries U_y^A and V_y^B, respectively, such that ??

$$|\Psi'\rangle_{AB} = (\mathbb{I}_A \otimes Q_y^A)\,|\Psi\rangle_{AB} = (U_y^A \otimes V_y^B)(P_y^A \otimes \mathbb{I}_B)\,|\Psi\rangle_{AB}$$

Note how $|\Psi'\rangle_{AB}$ is unnormalized, so it implicitly encodes the probability of outcome y. Through this we can see that Alice can help Bob using a measurement on her subsystem, after which she gets some measurement outcome y. She then classically sends this value y to Bob, and both of them correct the state to $|\Psi'\rangle_{AB}$ using local unitaries U_y^A and V_y^B.

Hint 1: For Alice to replicate the measurement that Bob needs, the probabilities of the different outcomes of $\{P_y^A\}_y$ and $\{Q_y^B\}_y$ should be identical.

Hint 2: Use Schmidt decomposition to work in the Schmidt basis of $|\Psi\rangle_{AB}$.