# Probability and Statistics

Assignment 1 Solutions

October 27, 2023

## Question 1

Show that convergence in mean square implies convergence in probability.

## Solution 1

We can apply the Markov inequality to a generic term of the sequence $\{(X_n - X)^2\}$

$$P\left((X_n - X)^2 \geq \epsilon^2\right) \leq \frac{E\left[(X_n - X)^2\right]}{\epsilon^2} \tag{1}$$

For any strictly positive real number $\epsilon$. Taking the square root of both sides of the left-hand inequality, we obtain:

$$P(|X_n - X| \geq \epsilon) \leq \frac{E\left[(X_n - X)^2\right]}{\epsilon^2} \tag{2}$$

Taking limits on both sides, we get:

$$\lim_{n\to\infty} P(|X_n - X| \geq \epsilon) \leq \lim_{n\to\infty} \frac{E\left[(X_n - X)^2\right]}{\epsilon^2} \leq \frac{\lim_{n\to\infty} E\left[(X_n - X)^2\right]}{\epsilon^2} = 0 \tag{3}$$

Since,

$$\lim_{n\to\infty} E[(X_n - X)^2] = 0 \tag{4}$$

And by the definition of probability,

$$P(|X_n - X| \geq \epsilon) \geq 0 \tag{5}$$

Then it must be that also:

$$\lim_{n\to\infty} P(|X_n - X| \geq \epsilon) = 0 \tag{6}$$

### Marking Scheme

- 3 marks for applying Markov inequality and taking the square root.
- 3 marks for calculations in the next steps.

## Question 2

Consider a sequence of random variables $\{X_n,\, n = 1, 2, 3..\}$ such that

$$X_n = \begin{cases} \frac{-1}{n^2} \text{ with probability } 0.3 \\ \frac{1}{n^2} \text{ with probability } 0.7 \end{cases}$$

Show that $X_n$ converges to 0 almost surely.

## Solution 2

By the Theorem above, it suffices to show that

$$\sum_{n=1}^{\infty} P\left(|X_n| > \epsilon\right) < \infty.$$

Note that, we have

$$|X_n| = \frac{1}{n^2}$$

.
Thus, $|X_n| > \epsilon$ if and only if $n^2 < 1/\epsilon$.
Thus, we conclude

$$\sum_{n=1}^{\infty} P\left(|X_n| > \epsilon\right) \leq \sum_{n=1}^{\left\lfloor \frac{1}{\sqrt{\epsilon}} \right\rfloor} P\left(|X_n| > \epsilon\right)$$

We have $P(|X_n| > \epsilon) = 0.7 + 0.3 = 1$.

$$\sum_{n=1}^{\infty} P\left(|X_n| > \epsilon\right) = \left\lfloor \frac{1}{\sqrt{\epsilon}} \right\rfloor$$

$$\sum_{n=1}^{\infty} P\left(|X_n| > \epsilon\right) < \infty$$

### Marking Scheme

- 2 mark for finding the limit of n.
- 2 mark for simplifying the limits of summation.
- 2 marks for calculating and showing $X_n$ converges.

## Question 3

Let $X_1, X_2, X_3, \cdots$ be a sequence of random variables such that

$$X_n \sim Binomial\left(n, \frac{\lambda}{n}\right), \qquad \text{for } n \in \mathbb{N}, n > \lambda,$$

where $\lambda > 0$ is a constant. Show that $X_n$ converges in distribution to $Poisson(\lambda)$

### Solution 3

We will use the following theorem to prove this question:

Consider the sequence $X_1, X_2, X_n, \cdots$ and the random variable $X$. Assume that $X$ and $X_n$ (for all $n$) are non-negative and inter-valued, i.e.

$$\begin{aligned} &R_X \subset \{0, 1, 2, \cdots\}, \\ &R_{X_n} \subset \{0, 1, 2, \cdots\}, \qquad \text{for } n = 1, 2, 3, \cdots. \end{aligned}$$

Then $X_n \xrightarrow{d} X$ if and only if

$$\lim_{n\to\infty} P_{X_n}(k) = P_X(k), \qquad \text{for } k = 0, 1, 2, \cdots.$$

Now, We have

$$\begin{aligned} \lim_{n\to\infty} P_{X_n}(k) &= \lim_{n\to\infty} \binom{n}{k} \left(\frac{\lambda}{n}\right)^k \left(1 - \frac{\lambda}{n}\right)^{n-k} \\ &= \lambda^k \lim_{n\to\infty} \frac{n!}{k!(n-k)!} \left(\frac{1}{n^k}\right) \left(1 - \frac{\lambda}{n}\right)^{n-k} \\ &= \frac{\lambda^k}{k!} \cdot \lim_{n\to\infty} \left( \left[\frac{n(n-1)(n-2)...(n-k+1)}{n^k}\right] \left[\left(1 - \frac{\lambda}{n}\right)^n\right] \left[\left(1 - \frac{\lambda}{n}\right)^{-k}\right] \right). \end{aligned}$$

Note that for a fixed k, we have

$$\begin{aligned} &\lim_{n\to\infty} \frac{n(n-1)(n-2)...(n-k+1)}{n^k} = 1, \\ &\lim_{n\to\infty} \left(1 - \frac{\lambda}{n}\right)^{-k} = 1, \\ &\lim_{n\to\infty} \left(1 - \frac{\lambda}{n}\right)^n = e^{-\lambda}. \end{aligned}$$

Thus, we conclude

$$\lim_{n\to\infty} P_{X_n}(k) = \frac{e^{-\lambda}\lambda^k}{k!}.$$

**Marking Scheme**

- 1 mark for writing the distributions of Poisson and Binomial
- 1.5 marks for mentioning the above theorem
- 2 mark for simplifying the expression of binomial
- 1.5 marks for writing the limits of the final 3 terms

# Q4

1. Sample X from U[0,1], s.t $F_X(x) = 1 - e^{-\sqrt{x}}$.
2. Say you have samples of X, how will you sample U[0,1]?

# Solution 4

1. We can sample X via sample u from U[0,1] and pass it into inverse CDF. $F_X^{-1}(u) = \ln^2(1-u)$. For example, say 0.2 is sample from U[0,1], than $F_X^{-1}(0.2) = \ln^2(1-0.2) \approx 0.5$ is a sample from X.
2. Consider the following claim, Let $F_X$ be the CDF of random variable X. Then $F_X(X)$ is a random variable with uniform distribution.

$$\begin{aligned} P(F_X(X) \leq u) &= P(X \leq F_X^{-1}(u)) \\ &= F(F_X^{-1}(u)) \\ &= u \end{aligned}$$

Here we used the fact that $F_X$ is a bijective function in the first step! One can note that the range of $F_X(X)$ is [0,1] and the CDF of $F_X(X)$ is uniform. Now if we are given samples from X we can find the corresponding sample from U[0,1] by taking an image of x under $F_X$.

# Question 5

Suppose $X_n$, n = 1, 2, 3, ... are i.i.d uniform U[0,1] and let $Y_n = \min(X_1, ..., X_n$. Show that $Y_n$ converges to 0 in probability (4 marks). Does it also converge in almost sure sense? Justify your answer (2 marks).

## Solution 5

$$F_{X_n}(x) = \begin{cases} 0, & x < 0 \\ x, & 0 \leq x \leq 1 \\ 1, & x > 1 \end{cases}$$

Also, $R_{Y_n} = [0, 1]$
For $0 \leq y \leq 1$,

$$\begin{aligned}
F_{Y_n}(y) &= P(Y_n \leq y) \\
&= 1 - P(Y_n > y) \\
&= 1 - P(X_1 > y, X_2 > y, ..., P(X_n > y) \\
&= 1 - P(X_1 > y)P(X_2 > y)...P(X_n > y) \text{ (since } X_i\text{'s are independent)} \\
&= 1 - (1 - F_{X_1}(y))(1 - F_{X_2}(y))...(1 - F_{X_n}(y)) \\
&= 1 - (1 - y)^n
\end{aligned}$$

Therefore,

$$F_{Y_n}(y) = \begin{cases} 0, & y < 0 \\ 1 - (1-y)^n, & 0 \leq y \leq 1 \\ 1, & y > 1 \end{cases} \qquad \text{(1 mark)}$$

In particular, note that $Y_n$ is continuous. To show $Y_n \xrightarrow{p} 0$, we need to show that

$$\lim_{n \to \infty} P(|Y_n| \geq \epsilon) = 0 \qquad \text{for all } \epsilon > 0$$

Since $Y_n \geq 0$, it suffices to show that

$$\lim_{n \to \infty} P(Y_n \geq \epsilon) = 0 \qquad \text{for all } \epsilon > 0$$

For $\epsilon \in (0, 1), we have$

$$\begin{aligned}
P(Y_n \geq \epsilon) &= 1 - P(Y_n < \epsilon) \\
&= 1 - P(Y_n \leq \epsilon) \qquad \text{(since } Y_n \text{ is a continuous random variable)} \\
&= 1 - F_{Y_n}(\epsilon) \\
&= (1 - \epsilon)^n \qquad \text{(2 marks)}
\end{aligned}$$

Therefore,

$$\begin{aligned}
\lim_{n \to \infty} P(|Y_n| \geq \epsilon) &= \lim_{n \to \infty} (1 - y)^\epsilon \\
&= 0 \qquad \text{for all } \epsilon \in (0, 1] \qquad \text{(1 mark)}
\end{aligned}$$

$Y_n$ converges in the almost sure sense. (1 mark)
To prove that $Y_n \xrightarrow{a.s} 0$, we need to prove

$$\sum_{n=1}^{\infty} P(|Y_n| > \epsilon) < \infty$$

From above we get,

$$\begin{aligned}
\sum_{n=1}^{\infty} P(|Y_n| > \infty) &= \sum_{n=1}^{\infty} (1 - \epsilon)^n \\
&= \frac{1 - \epsilon}{\epsilon} < \infty \qquad \text{(geometric series)} \qquad \text{(1 mark)}
\end{aligned}$$