Assignment 2, PNS monsoon 2023

TAs

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Question 1

Find stationary distribution π for the Markov chain with the following transition matrix. State if π is unique. If not, state all stationary distribution with justification.

$$P = \begin{bmatrix} 0.5 & 0.5 & 0 \\ 0.5 & 0.5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Solution 1

A stationary distribution satisfies $\pi P = \pi$, or $P^T \pi^T = \pi^T$. Now clearly P has rank ≤ 3, however, our π is also not independent! As a matter of fact, P has rank 1 and our π has 2 independent parameters. Concretely, note that π belongs to the null space of $P^T - I$.

$$P^T - I = \begin{bmatrix} -0.5 & 0.5 & 0 \\ 0.5 & -0.5 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

This matrix can reduced to (via R2 = R2+R1)

$$P^T - I = \begin{bmatrix} -0.5 & 0.5 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

This has rank = 1 (number of independent rows or columns). Let us write π as [a,b,c]. Clearly solving the above equation $(P^T - I)\pi^T = 0$, we get c be anything and a=b. But we also know a+b+c = 1, so $\pi = [\frac{1-c}{2}, \frac{1-c}{2}, c]$. Hence not a unique solution.

Marking Scheme

- 3.5 marks for solving for π

- 1.5 marks for stating if π is unique with justification.
- Partial marks will be awarded for correct procedure.

Question 2

Consider a bivariate normal vector $X = (X_1, X_2)^T$ with mean vector $(2, 4)^T$ and covariance matrix 6I, where I is the identity matrix. From the density of vector X derive marginal distribution of $f_{X_1}(x_1)$ and conditional distribution $f_{X_1|X_2}(x_1|x_2)$

Solution 2

We know that the density function for a bivariate normal vector with covariance Σ, mean μ, and dimension p is given by -

$$f(x) = \left(\frac{1}{2\pi}\right)^{p/2} |\Sigma|^{-1/2} \exp\left\{\frac{-1}{2}(\mathbf{x}-\mu)^T\Sigma^{-1}(\mathbf{x}-\mu)\right\}$$

For bivariate p = 2, and substituting Σ and μ into above equation we get joint distribution -

$$f(x_1, x_2) = \frac{1}{12\pi} \exp\left\{\frac{-1}{12}\left((x_1-2)^2 + (x_2-4)^2\right)\right\}$$

where we used the fact that $|\Sigma| = 36$. This is very similar to standard normal distribution. As a matter of fact, x_1 and x_2 are independent. One way to see the covariance matrix is I (multiple to be precise). Another brute way is, we can organize the distribution as -

$$f(x_1, x_2) = \frac{1}{\sqrt{12\pi}} \exp\left\{\frac{-1}{12}(x_1-2)^2\right\} \frac{1}{\sqrt{12\pi}} \exp\left\{\frac{-1}{12}(x_2-4)^2)\right\}$$

When we marginalize the marginalized co-ordinate will sum to 1 (measure of entire space = 1).

$$\begin{aligned}
f(x_1) &= \frac{1}{\sqrt{12\pi}} \exp\left\{\frac{-1}{12}(x_1-2)^2\right\} \left[\int_{-\infty}^{\infty} \frac{1}{\sqrt{12\pi}} \exp\left\{\frac{-1}{12}(x_2-4)^2)\right\} dx_2\right] \\
&= \frac{1}{\sqrt{12\pi}} \exp\left\{\frac{-1}{12}(x_1-2)^2\right\} \times 1 \\
&= N(2, 6)
\end{aligned}$$

Integral is nothing but the area under the density function, which by definition is 1. Both terms are, hence, nothing but $N(2, 6)$, and clearly, their product is the joint distribution which implies their independence. Hence $f_{X_1}(x_1)$ is N(2, 6).

Also, we know that X and Y are independent random variables, from conditioning definition $f_{X,Y}(x, y) = f_X(x)f_{Y|X}(y|x)$, but $f(x, y) = f_X(x)f_Y(y)$, which gives $f_Y(y) = f_{Y|X}(y|x)$. Hence $f_{X_1|X_2}(x_1|x_2)$ is simply $f_{X_1}(x_1)$ which is N(2,6).

Marking Scheme

- 1 mark for writing the complete density function $\phi(\mathbf{x})$
- 2 marks for maginalized PDF $f_{X_1}(x_1)$
- 2 marks for conditional PDF $f_{X_1|X_2}(x_1|x_2)$
- Partial marks will be awarded for correct procedure

Question 3

Let $\mathbf{Z}$ denote an n length standard normal vector and let B be an $n \times n$ matrix. Consider $Y = BX + b$ for vector b of appropriate dimension. Derive an expression for the PDF $f_Y(y)$.

Solution 3

A random vector $\mathbf{Z}$ is called as a standard normal vector if its components Z_i are independent and standard normal.

$$\mathbf{Z} = \begin{bmatrix} Z_1 \\ Z_2 \\ . \\ . \\ . \\ Z_n \end{bmatrix}$$

where Z_i 's are i.i.d. and $Z_i \sim N(0,1)$. Then, we have

$$\begin{aligned}
f_{\mathbf{Z}}(\mathbf{z}) &= f_{Z_1,Z_2,\ldots,Z_n}(z_1,z_2,\ldots,z_n) \\
&= \prod_{i=1}^{n} f_{Z_i}(z_i) \\
&= \frac{1}{(2\pi)^{\frac{n}{2}}} \exp\left\{-\frac{1}{2}\sum_{i=1}^{n} z_i^2\right\} \\
&= \frac{1}{(2\pi)^{\frac{n}{2}}} \exp\left\{-\frac{1}{2}\mathbf{z}^T\mathbf{z}\right\}.
\end{aligned}$$

For a standard normal random vector $\mathbf{Z}$, where Z_i 's are i.i.d. and $Z_i \sim N(0,1)$, the PDF is given by

$$f_{\mathbf{Z}}(\mathbf{z}) = \frac{1}{(2\pi)^{\frac{n}{2}}} \exp\left\{-\frac{1}{2}\mathbf{z}^T\mathbf{z}\right\}.$$

Now we know that the pdf of Y from pdf of X when $Y = g(X)$ (where g is monotone, continuous, differentiable), is $f_Y(y) = f_X(h(y))\left|\frac{dh}{dy}(y)\right|$ where h is the inverse function of g.
When $\mathbf{Y} = G(\mathbf{X})$ where $G : \mathbb{R}^n \to \mathbb{R}^n$, continuous invertible with continuous partial derivatives. Let H denote its inverse. Then

$$f_{\mathbf{Y}}(\mathbf{y}) = f_{\mathbf{X}}(H(\mathbf{y}))|J|$$

where J is the determinant of the Jacobian matrix.
When $\mathbf{Y} = \mathbf{BZ} + \mathbf{b}$, since B is invertible, we can write

$$Z = B^{-1}(Y - b).$$

where J is the Jacobian of H defined by

$$J = \det \begin{bmatrix} \frac{\partial H_1}{\partial y_1} & \frac{\partial H_1}{\partial y_2} & \cdots & \frac{\partial H_1}{\partial y_n} \\ \frac{\partial H_2}{\partial y_1} & \frac{\partial H_2}{\partial y_2} & \cdots & \frac{\partial H_2}{\partial y_n} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial H_n}{\partial y_1} & \frac{\partial H_n}{\partial y_2} & \cdots & \frac{\partial H_n}{\partial y_n} \end{bmatrix}$$

We can see that, $\frac{\partial H_i}{\partial y_j}$ is nothing but the (i,j) element of the inverse matrix of B. (Since differentiation drops the constant term b)

$$J = \det \begin{bmatrix} B_{11}^{-1} & B_{12}^{-1} & \ldots & B_{1n}^{-1} \\ B_{21}^{-1} & B_{22}^{-1} & \ldots & B_{2n}^{-1} \\ \vdots & \vdots & \vdots & \vdots \\ B_{n1}^{-1} & B_{n2}^{-1} & \ldots & B_{nn}^{-1} \end{bmatrix}$$

$$J = \det\left(B^{-1}\right) = \frac{1}{\det(B)}.$$

Thus, we conclude that

$$f_{\mathbf{Y}}(\mathbf{y}) = \frac{1}{|\det(B)|} f_{\mathbf{Z}}\left(B^{-1}(\mathbf{y}-\mathbf{b})\right)$$

$$f_{\mathbf{Y}}(\mathbf{y}) = \frac{1}{(2\pi)^{\frac{n}{2}}|\det(B)|}\exp\left\{-\frac{1}{2}\left(B^{-1}(\mathbf{y}-\mathbf{b})\right)^{T}\left(B^{-1}(\mathbf{y}-\mathbf{b})\right)\right\}$$

$$f_{\mathbf{Y}}(\mathbf{y}) = \frac{1}{(2\pi)^{\frac{n}{2}}|\det(B)|}\exp\left\{-\frac{1}{2}(\mathbf{y}-\mathbf{b})^{T} B^{-T}B^{-1}(\mathbf{y}-\mathbf{b})\right\}$$

$$f_{\mathbf{Y}}(\mathbf{y}) = \frac{1}{(2\pi)^{\frac{n}{2}}|\det(B)|}\exp\left\{-\frac{1}{2}(\mathbf{y}-\mathbf{b})^{T}\left(BB^{T}\right)^{-1}(\mathbf{y}-\mathbf{b})\right\}$$

For $\mathbf{Y} = B\mathbf{Z} + \mathbf{b}$, we have
$E[\mathbf{Y}] = \mathbf{b}$ and $\Sigma = C_{\mathbf{Y}} = BB^{T}$.

$$f_{\mathbf{Y}}(\mathbf{y}) = \frac{1}{(2\pi)^{n/2}\sqrt{\det\left(C_{\mathbf{Y}}\right)}}\exp\left\{-\frac{1}{2}(\mathbf{y}-\mathbf{b})^{T}C_{Y}^{-1}(\mathbf{y}-\mathbf{b})\right\}$$

Marking Scheme

- 2 marks for PDF of standard normal vector Z.
- 2 marks for derivation the det(J).
- 3 marks for derivation of PDF of a general function of Z.
- 3 marks for calculating the PDF of Y.

Question 4

(Question too long, please read from paper please)

Solution 4

$$\begin{aligned}
F_{ii} =& P(\text{ Coming back to state i having stated in state i}) \\
&= P(\text{Coming back after 1 step}) \\
&+ P(\text{Coming back after 2 steps and not before}) \\
&+ P(\text{Coming back after 3 steps and not before}) \\
&\vdots \\
&= \sum_{n=1}^{\infty} P(\text{coming back exactly after n steps and not before}) \\
&= \sum_{n=1}^{\infty} f_{ii}^n
\end{aligned}$$

For P_1:-
For state 3, we can come back to state 3 for the first time after 1 step with probability 0.5. But, if we leave state 3 at anytime, we can never come back as no other states come back to state 3. Thus all other f_{33}^n are 0, except for n=1 where it is 0.5. Thus, $F_{33} = 0.5$.
If we are in state 1 or state 2, we will come back to the same state after 2 steps. Thus,

$$f_{jj}^n = 0 \text{ for } n \text{ in natural numbers except 2 and } f_{jj}^n = 1 \text{ for } n = 2 \text{ for } j = 1, 2$$

Thus clearly, $F_{11}, F_{22} = 1$. Thus, states 1 and 2 are recurrent, while state 3 is transient
For P_2:-
Consider the case of starting in state 1. The probability of coming back to state 1 after exactly 1 step is clearly 0.5. The probability of coming back to state 1 after exactly n steps for $n > 1$ is given by:-

$$\begin{aligned}
f_{11}^n =& P(\text{going to state 2 from state 1}) \times \\
& P(\text{staying in state 2 for n-2 steps}) \times \\
& P(\text{going to state 1 from state 2})) \\
& = p_{12} \times p_{22}^{(n-2)} \times p_{21} \\
& = 0.25(0.5)^{(n-2)}
\end{aligned}$$

Summing this over all n, we get

$$\begin{aligned}
\sum_{n=1}^{\infty} f_{ii}^{n} =& 0.5 + \sum_{n=2}^{\infty} f_{ii}^{n} \\
&= 0.5 + \sum_{n=2}^{\infty} 0.25(0.5)^(n-2) \\
&= 0.5 + 0.25 \sum_{n=0}^{\infty} (0.5)^(n) \\
&= 0.5 + 0.25 \times 2 \;\; \text{(summing the geometric progression)} \\
&= 1 = F_{11}
\end{aligned}$$

Thus, state 1 is recurrent. For similar reasoning, state 2 is also recurrent. For reasoning similar to P_1, state 3 is also transisient

Marking Scheme

- 4 marks for the proof.
- 3 + 3 marks for deducing transient and recurrent states for P1 and P2