Quiz - 2 solⁿ

Q.1

(a) Assume A, B & C be 3 collections of RV containing X, Y & Z respectively. So each collection is a non-empty collection of RU with 1 RV in each.

$\because \quad I(A;B|C) \geq 0$

$\because \quad H(A|C) - H(A|B,C) \geq 0$

(2)

$\because \quad H(A|C) \geq H(A|B,C)$

Replacing collections by the RV in the respective collection:

$\because \quad H(X|Z) \geq H(X|Y,Z)$

(2)

Also, we are given that equality holds if A & B are independent given C which here means X & Y are independent given Z. So, here equality holds if X & Y are independent given Z.

(don't cut marks)

Commⁿ Setup interpretation: Not Necessary

Signal observed by two people

$$\begin{tikzcd} & X \\ Z \arrow[ur, dash, "n_1"] \arrow[dr, dash, "n_2"'] & \\ & Y \end{tikzcd}$$

independent
n_1 & n_2 → noise

here $\underline{\underline{H(X|Z) = H(X|Y,Z)}}$

(b)

④ $H(X_1, \ldots, X_n \mid Y) = \sum_{y \in supp(P_Y)} P_Y(y)\, H(X_1, \ldots, X_n \mid Y = y)$

where $H(X_1, \ldots, X_n \mid Y=y) = -\sum_{(x_1, \ldots, x_n) \in supp(P_{X_1 \ldots X_n \mid Y=y})} P_{X_1, \ldots, X_n \mid Y=y}(x_1, \ldots, x_n) \log P_{X_1 \ldots X_n \mid Y=y}(x_1 \ldots x_n)$

⑤ Chain Rule states that

$$H(X_1, \ldots, X_n \mid Y) = H(X_1 \mid Y) + H(X_2 \mid X_1, Y) + H(X_3 \mid X_1, X_2, Y) + \cdots + H(X_n \mid X_1, X_2, \ldots X_{n-1}, Y)$$

To prove this,

$$H(X, Y/Z) = \sum_{z} P(Z=z)\, H(X, Y / Z=z)$$

$$= -\sum_{x,y,z} P(X=x, Y=y / Z=z) P(Z=z) \log P(X=x, Y=y / Z=z)$$

$$= -\sum_{x,y} P(X=x / Z=z)\, p(Z=z) \log P(X=x / Z=z)$$

$$- \sum_{x,y,z} p(Y=y \mid X=x, Z=z) P(X=x, Z=z) \log P(Y=y \mid X=x, Z=z)$$

$$= H(X/Z) + H(Y \mid X, Z)$$

$X \& Y$

we showed this for two variables. it will be similarly extended to n variables as

$$P(X_1=x_1, X_2=x_2, \ldots, X_n=x_n \mid Y=y) = P(X_1=x_1 \mid Y=y)\, P(X_2=x_2 \mid X_1=x_1, Y=y)$$

$$\ldots\ P(X_n=x_n \mid X_1=x_1, \ldots, X_{n-1}=x_{n-1}, Y=y)$$

which comes from breaking the joint probability sequentialy.

① → For $H(X_1, \ldots X_n \mid Y) = \sum_{i=1}^{m} H(X_i \mid Y)$, it

holds when all X_i are independent of all other X_j given Y. This comes from the understanding that

$$H(X_i \mid X_1, \ldots, X_{i-1}, Y) = H(X_i \mid Y)$$

if X_i is independen of $X_1, \ldots x_{i-1}$ given Y.

Q.2

$$X \sim Ber(0.5)$$

(2)

$$\Rightarrow P(X=0) = P(X=1) = 0.5$$

$$\Rightarrow P_{Y|X}(0|0) = P_{Y|X}(1|1) = 1-p$$

$$P_{Y|X}(\epsilon|0) = P_{Y|X}(\epsilon|1) = p$$

(1)

$$P_Y(0) = \sum_x P_{Y|X}(0|x) P_X(x)$$

$$= \frac{1-p}{2}$$

$$P_Y(1) = \frac{1-p}{2}$$

$$P_Y(\epsilon) = p$$

$$\because H(Y) = -p \log p - \frac{(1-p)}{\cancel{2}} \log\left(\frac{1-p}{2}\right) \times \cancel{2}$$

$$= -p \log p - (1-p)\log(1-p) + (1-p)$$

$$H(Y|X) = \sum_x P_X(x)\, H(Y|X=x)$$

④

$H(Y|X=0) = H_2(P) = -P\log P - (1-P)\log(1-P)$

$H(Y|X=1) = H_2(P)$

$\because \quad H(Y|X) = H_2(P)$

$\because \quad I(X;Y) = \cancel{H_2(P)} + (1-P) - \cancel{H_2(P)}$

$$\boxed{I(X;Y) = 1-P}$$

Q.3

Since X is binomial RV,

$$P(X=i) = {}^{n}C_i \, p^i (1-p)^{n-i}$$

here $n=4$ & $p=0.1$

$1\frac{1}{2}$

$\therefore$

x	$P_X(x)$
0	$9^4 \times 10^{-4}$
1	$4 \times 9^3 \times 10^{-4}$
2	$6 \times 9^2 \times 10^{-4}$
3	$4 \times 9 \times 10^{-4}$
4	10^{-4}

To get avg length short, we assign shorter codewords to values of X which have higher probability

$2\frac{1}{2}$

$\therefore$

value of X	codeword
'0'	0
'1'	10
'2'	110
'3'	1110
'4'	1111