EC5.406 - Signal Detection and Estimation Theory
Final Exam

Date: 27th November, 2024
Instructor: Santosh Nannuru

Maximum marks: 50
Exam duration: 180 minutes

Instructions:

a) There are *6 questions* for a total of *50 marks*.

b) Mention any additional assumptions you make that is not given in the question.

c) Clearly show the steps used to arrive at the solutions.

d) You are allowed one handwritten cheat sheet (at most A4 size).

1. **[10 marks]**
It is known that the data $x[n]$ follows a Gaussian distribution. Based on some preliminary analysis, Siva claims that the data distribution is in fact i.i.d. Gaussian, $\mathcal{N}(\mu_1, \sigma^2)$. Based on some further analysis, Madhuri claims that the data is indeed i.i.d. Gaussian with variance σ^2, but the mean of the Gaussian is higher than μ_1. Given the data samples $x[n], n = 0, 1, \dots, N-1$, you are tasked with deciding whether Siva or Madhuri are correct in their analysis. The two hypothesis are:

$$\text{Siva is correct}: x[n] \sim \mathcal{N}(\mu_1, \sigma^2)$$
$$\text{Madhuri is correct}: x[n] \sim \mathcal{N}(\mu_2, \sigma^2),\ \mu_2 > \mu_1$$

The parameters μ_1 and σ^2 are known but μ_2 is unknown.

(a) [5] Does a uniformly most powerful (UMP) test exist for this composite hypothesis testing problem for a given false alarm rate of α? If yes, find it.

(b) [3] Find the relation between probability of detection, P_D, and probability of false alarm, $P_{FA} = \alpha$. How does the unknown parameter μ_2 affect the ROC curve? Explain with diagram.

(c) [2] Find the generalized likelihood ratio test (GLRT) detector for this problem.

2. **[10 marks]**
The observed data $x[n]$ has the following distribution conditioned on the parameter θ,

$$p(x[n]|\theta) = \begin{cases} \exp\left[-(x[n]-\theta)\right], & x[n] > \theta\,, \\ 0, & x[n] < \theta\,, \end{cases}$$

while the parameter θ has the following PDF,

$$p(\theta) = \begin{cases} \exp\left[-\theta\right], & \theta > 0\,, \\ 0, & \theta < 0\,. \end{cases}$$

(a) [4] Given one sample of observation data, $x[0]$, find and sketch the exact posterior distribution $p(\theta|x[0])$. What is the MMSE estimator of θ?

(b) [4] If N samples of data $x[n]$ are available, find and sketch the exact posterior distribution $p(\theta|\underline{x})$. What is the MAP estimator of θ?

(c) [2] If θ is assumed to be deterministic (rather than random), and the expression $p(x[n]|\theta)$ above is treated as the likelihood function $p(x[n];\theta)$, can we apply the CRLB analysis? If yes, find the Fisher information.

3. **[12 marks]**
A drone is surveying for presence of ice on planet Z. It is programmed to transmit a specific signal to its base station if it detects presence of ice on the planet, otherwise it does not transmit anything. Thus, the receiver at the base station has to decide between the following options:

$$\text{No signal from the drone}: \; x[n] = w_0[n]\,,$$
$$\text{Signal transmitted by the drone}: \; x[n] = s[n] + w_1[n].$$

The receiver knows the signal $s[n]$ that will be transmitted by the drone. The statistics of the noise $w_0[n]$ and $w_1[n]$ are also known to the receiver and it employs the Neyman-Pearson detection framework.

(a) [4] If $w_0[n] = w_1[n] \sim \mathcal{N}(0, \sigma^2)$ are i.i.d. Gaussian noises, derive the detector structure given N observations.

(b) [5] Let $w_0[n] \sim \mathcal{N}(0, \sigma^2)$ and $w_1[n] \sim \mathcal{N}(0, 2\sigma^2)$ be i.i.d. Gaussian noises. For a single observation, $N = 1$ and $s[0] = A > 0$, derive the simplified form for the N-P detector. Clearly identify and label the decision regions.

(c) [3] For false alarm rate of α, find expression for the probability of detection, P_D.

4. [6 marks]
The detection performance of a certain binary detector depends on the parameter $\beta > 0$. Specifically, the relation between probability of detection, P_D, and probability of false alarm, $P_{FA} = \alpha$, is given by

$$P_D = Q[Q^{-1}(\alpha) - \beta],$$

where, $Q()$ is the right tail probability of standard normal Gaussian distribution.

(a) [3] Plot and compare the ROC curves for $\beta_1 < \beta_2 < \beta_3$. Explain your answers.

(b) [3] If instead, the relation is given as,

$$P_D = Q[Q^{-1}(\alpha) - \beta] - Q[Q^{-1}(\alpha) + \beta],$$

what is the constraint on the parameter β such that this detector performs better than the random detector when $\alpha = \frac{1}{2}$?

5. [8 marks]
For binary hypothesis testing, derive the minimum Bayes Risk detector. Furthermore, as a special case, derive the minimum probability of error detector.

6. [4 marks]
Are the following statements true or false? Give justification.

(a) The MVU estimator always exists as long as the CRLB can be computed. False

(b) If model is linear, the BLUE is efficient. False

(c) When the noise is white Gaussian, deterministic signal detector performance only depends on signal energy and not on the signal shape. True

(d) When the noise and the signal are both white Gaussian, the random signal detector threshold does not depend on the statistics of the signal. False.