Linear Algebra (UG1, Spring 2023)

Midsem [20 marks]; Time: 90 mins (+45 mins)

April 29, 2023

Notations are from class lectures unless stated otherwise. Each step of the proof should be clear. Appropriate reasoning for your claims are must.

Question A [9 marks]

1. Suppose $V_1, V_2, \ldots, V_m$ are subspaces of a vector space V defined over the field $\mathbf{F}$. Prove that $V_1 + V_2 + \ldots + V_m$ is the smallest subspace of V containing $V_1, V_2, \ldots, V_m$. [3 marks]

2. Suppose the set of vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m$ is linearly in dependent in the vector space V over field $\mathbf{F}$. Prove that if the set of vectors $\vec{v}_1 + \vec{w}, \vec{v}_2 + \vec{w}, \ldots, \vec{v}_m + \vec{w}$ is linearly dependent in V, then $\vec{w}$ is spanned by the set of linearly dependent vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m$. [3 marks]

3. Prove that a vector space V defined over a field $\mathbf{F}$ is infinite-dimensional if and only if there is a sequence $\vec{v}_1, \vec{v}_2, \ldots$ of vectors in V such that $\vec{v}_1, \ldots, \vec{v}_m$ is linearly in dependent for every positive integer m. [3 marks]

Question B [6 marks]

Let $M_{2\times 2}(\mathbb{R})$ be the vector space of 2×2 matrices defined over the field $\mathbb{R}$ of real numbers. If $T : M_{2\times 2}(\mathbb{R}) \to \mathbb{R}$ is the trace map $T = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = a + d$, i.e., T is the sum of the diagonal entries of a square matrix. Then,

- Show that T is a linear transformation. [1.5 marks]
- Find the nullity and the rank of T. [3 marks]
- State the nullity-rank theorem. Verify whether the theorem holds for T or not. [1.5 marks]

Question C [5 marks]

Using notations from the previous question, consider the map $P : \mathbb{R}^4 \to M_{2\times 2}(\mathbb{R})$ given by $P(x_1, x_2, x_3, x_4) = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}$.

- Show that the map P is a linear transformation. [2 marks]
- Find the null space and the range of P. [1 marks]
- Does the inverse of P exist? If yes, what is the inverse of P? If no, then why not? [2 marks]