IIIT-H, Monsoon Semester 24, Final Exam: 22nd November 2024

Introduction to Quantum Field Theory (SC1.421)

Full Marks 60, Duration: 180 min

There are 6 questions. The question paper consists of two sides of a sheet.

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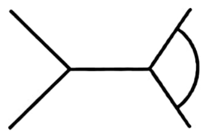
1. The Hamiltonian of a real scalar field is
$$H = \int \frac{d^3k}{(2\pi)^3} E_k \left(a_k^\dagger a_k\right),$$
where $a_k^\dagger$ and a_k are raising and lowering operators such that $[a_k, a_{k'}^\dagger] = (2\pi)^3\delta^3(k-k')$, and $[a_k, a_{k'}] = 0 = [a_k^\dagger, a_{k'}^\dagger]$.
   (a) Show that $[H, a_k] = -E_k a_k$ [4]
   (b) Show that $H(a_k|E\rangle) = (E - E_k)(a_k|E\rangle)$. What is the physical interpretation of the result. [3]
   (c) Prove that $\langle E|H|E\rangle \geq 0$. [3]

2. Consider the following Feynman diagram of a two-particle to two-particle scattering in a scalar field theory.



   (a) If the interaction term in the Lagrangian (density) that generates the diagram is $-\frac{\lambda}{n!}\phi^n$ then what is the value of n? [2]
   (b) Write down the amplitude (*i.e.*, matrix element $i\mathcal{M}$) for the diagram using Feynman rules. *Momentum and their directions must be labeled.* [6]

3. Without any specific representation, and only using the properties of γ matrices
   (a) Show that $\frac{1}{2}(1-\gamma_5)\frac{1}{2}(1-\gamma_5) = \frac{1}{2}(1-\gamma_5)$ and $\frac{1}{2}(1-\gamma_5)\frac{1}{2}(1+\gamma_5) = 0$ [2]
   (b) Explicitly show that $\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^5) = 0$. *Hint: You may use anti-commutation of γ matrices.* [3]
   (c) Calculate the trace $\mathrm{Tr}(\gamma^\mu \not{p}_1 \gamma^\nu (1-\gamma_5)\not{p}_2)$, where $\not{p} = \gamma_\mu p^\mu$. [5]

4. The plane-wave solutions to the Dirac equation are: $u^s(\vec{p}) = \begin{pmatrix} \sqrt{p\cdot\sigma}\xi^s \\ \sqrt{p\cdot\bar{\sigma}}\xi^s \end{pmatrix}$, where $\sigma^\mu = (1, \vec{\sigma})$ and $\bar{\sigma}^\mu = (1, -\vec{\sigma})$ and ξ^s, with $s = 1,2$, is a basis of orthonormal two-component spinors, satisfying $(\xi^r)^\dagger \cdot \xi^s = \delta^{rs}$. Show that
   (a) $u^r(\vec{p})^\dagger \cdot u^s(\vec{p}) = 2p_0\delta^{rs}$
   (b) $\bar{u}^r(\vec{p}) \cdot u^s(\vec{p}) = 2m\delta^{rs}$
   (c) $\sum_{s=1}^{2} u^s(\vec{p})\bar{u}^s(\vec{p}) = \not{p} + m$

   You may need the identity $(p.\sigma)(p.\bar{\sigma}) = p^2 = m^2$. [2+2+2]

5. Under Lorentz Transformation,
$$\psi^\alpha(x) \to S[\Lambda]^\alpha_\beta \psi^\beta(\Lambda^{-1}x),$$
where $\Lambda = \exp\left(\frac{1}{2}\Omega_{\rho\sigma}M^{\rho\sigma}\right)$, and $S[\Lambda] = \exp\left(\frac{1}{2}\Omega_{\rho\sigma}S^{\rho\sigma}\right)$.
The generators of the Lorentz Algebra satisfy,
$$[M^{\rho\sigma}, M^{\tau\nu}] = g^{\sigma\tau}M^{\rho\nu} - g^{\rho\tau}M^{\sigma\nu} + g^{\rho\nu}M^{\sigma\tau} - g^{\sigma\nu}M^{\rho\tau} \quad (1)$$

   Consider $S^{\rho\sigma} = \frac{1}{4}[\gamma^\rho, \gamma^\sigma]$.
   (a) Compute the commutation relation, $[S^{\mu\nu}, \gamma^\rho]$.

(b) Use the above relations to show that the $S^{\mu\nu}$ matrices associated with the spinor representation, satisfy the Lorentz algebra commutation relations, Eq. (1).

(c) Write down the form of the rotation matrix under spinor representation. Writing the rotation parameters as $\Omega_{\rho\sigma} = -\epsilon_{\rho\sigma k}\varphi^k$ (meaning $\Omega_{12} = -\varphi^3$) work out the value of the spinor representation $S[\Lambda]$ under a rotation by 2π about the z axis. [3+4+4]

6. The Lagrangian of the spinors along with the interaction of the vector field is:

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi + e\bar{\psi}\gamma^\mu\psi A_\mu.$$

Consider the process $e^-(p_1)\ e^+(p_2) \to e^-(p_3)\ e^+(p_4)$, also known as Bhabha Scattering. For simplification, you may work in the ultra-relativistic limit $E_{cm} >> m_e$, where the mass of the electron can be neglected ($m_e = 0$).

(a) Draw the leading order Feynman diagrams for the desired process, with appropriate labeling.

(b) Use the Feynman rules to write down the total amplitude $\mathcal{M}$. (Be sure, you have the correct relative sign between these diagrams.)

(c) Square the amplitude, average over the initial particle's spin, and sum over the final particle spins. Use the completeness relations, $\sum_{s=1}^{2} u^s(\vec{p})\bar{u}^s(\vec{p}) = \not{p} + m_e$, $\sum_{s=1}^{2} v^s(\vec{p})\bar{v}^s(\vec{p}) = \not{p} - m_e$, and evaluate the traces using the trace formula.

(d) Introduce the Mandelstam variables, s, t, and u, in terms of p_1, p_2, p_3, p_4. Show that the final result is

$$\frac{1}{4}|\mathcal{M}|^2 = 2e^4\left[\frac{u^2+t^2}{s^2} + \frac{2u^2}{st} + \frac{u^2+s^2}{t^2}\right]$$

. Express all 4-momentum vectors in the center of mass frame of reference, in terms of a suitably chosen set of variables such as E and θ. Considering $m_e = 0$, what is the value for $s+t+u$?

(e) Use crossing symmetry to compute $\frac{1}{4}|\mathcal{M}|^2_{e^-e^-\to e^-e^-}$ for the process $e^-(p)\ e^-(k) \to e^-(p')\ e^-(k')$ and show

$$\frac{1}{4}|\mathcal{M}|^2_{e^-e^-\to e^-e^-} = 2e^4\left[\frac{s^2+t^2}{u^2} + \frac{2s^2}{ut} + \frac{u^2+s^2}{t^2}\right]$$

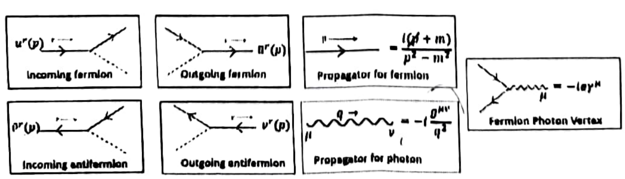
[2+3+5+3+2]

Some formulas you may need

- The Lagrangian of the spinors along with the interaction of the vector field is:

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi + e\bar{\psi}\gamma^\mu\psi A_\mu.$$

The relevant Feynman rules are listed below:



- In the Weyl representation the γ matrices are

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$

where, 1 is a 2×2 identity matrix and σ^i are 2×2 Pauli matrices.

- The Pauli Matrices satisfy $\{\sigma_i, \sigma_j\} = 2\delta_{ij}$, $\sigma^i\sigma^j = i\epsilon^{ijk}\sigma_k$ and are denoted by:

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

- Trace formulas

$\mathrm{Tr}[\mathbb{I}] = 4$, $\mathrm{Tr}[\text{any odd number of } \gamma\text{'s}] = 0$, $\mathrm{Tr}[\gamma^\mu\gamma^\nu] = 4g^{\mu\nu}$

$\mathrm{Tr}[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma] = 4(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho})$,

$\mathrm{Tr}[\not{p}_2\gamma^\mu\not{p}_1\gamma_\nu\not{p}_3\gamma_\mu\not{p}_4\gamma^\nu] = -32(p_1\cdot p_4)(p_2\cdot p_3)$, $\mathrm{Tr}[\not{p}_2\gamma^\mu\not{p}_4\gamma_\nu\not{p}_3\gamma_\mu\not{p}_1\gamma^\nu] = -32(p_1\cdot p_4)(p_2\cdot p_3)$

$\mu = \sigma \quad \nu = \mu \quad \rho = \rho \quad \sigma = \nu$