# Solutions

## Quiz-2 Discrete Structures

## Q1-a

1. **(a)** The order of a permutation $f$ is the least positive integer $n$ such that $f^n = I$, $I$ being the identity permutation. Here, $f^n = \underbrace{f \cdot f \cdots f}_{n\,times}$ and $\cdot$ is the function composition.

   Prove that the order of an $r$-cycle is $r$.

### Solution

(i) Let $f = (a_1\ a_2 \cdots a_r)$ be an $r$-cycle.

Then, $f(a_1) = a_2$.

$f^2(a_1) = f(f(a_1)) = f(a_2) = a_3, \ldots$

(12)

...ilarly, $f^r(a_i) = a_i$, for all $i$.

Hence, $f^r = I$.

Therefore, the order of $f$ is $r$. $\square$

## Q1-b

**(b)** It is well-known that the order of a permutation $f$ can be also expressed as the $lcm$ (least common multiple) of the lengths of its disjoint cycles. Using this, find the order of the permutation

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 2 & 1 \end{pmatrix}$$

### Solution

Part 2:

Part: (b)

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 2 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 3 & 5 & 2 & 4 \\ 3 & 5 & 1 & 4 & 2 \end{pmatrix}$$

$$= (1\ 3\ 5) \cdot (2\ 4)$$

Now, the order of the given permutation $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 2 & 1 \end{pmatrix}$ is the least common multiple (l.c.m.) of the order of permutation $(1\ 3\ 5)$ and the order of the permutation $(2\ 4)$

= l.c.m. of 3 and 2

= lcm (3, 2)

= 6.

# Question 2-a

2. **(a)** Prove that any semigroup $[S, \circ]$ which has a left identity $e$ such that $e \circ x = x, \forall x \in S$ and a left inverse $x^{-1}$ such that $x^{-1} \circ x = e, \forall x \in S$, then $S$ is a group, where $\circ$ is the binary composition.

## Solution

group.

Proof. Given that $[S, \cdot]$ is a semigroup, so

(i) closure holds i.e, $\forall x, y \in S$, $x \cdot y \in S$, an

(ii) Associativity holds i.e, $\forall x, y, z \in S$,

$(x \cdot y) \cdot z = x \cdot (y \cdot z)$.

(iii) Given $e$ is the left identity i.e, $e \cdot x = x, \forall x \in S$

(iv) Given $x^{-1}$ is the left inverse i.e, $x^{-1} \cdot x = e, \forall x \in$

R.T.P. $[S, \cdot]$ is a group

, by (i), closure holds

. by (ii), associativity holds.

$\therefore$ R.T.P.

(a) [right identity]

i.e, $\forall x \in S$, $x \cdot e = x$ ; and

(b) [right inverse]

i.e, $x \cdot x^{-1} = e$.

(a) Now, $e \cdot e = e$

$\Rightarrow (x^{-1} \cdot x) \cdot (x^{-1} \cdot x) = x^{-1} \cdot x$

$\Rightarrow x^{-1} \cdot (x \cdot x^{-1}) \cdot x = x^{-1} \cdot x$

$\Rightarrow (x \cdot x^{-1}) \cdot x = x$ , by left cancellation law

$\Rightarrow x \cdot (x^{-1} \cdot x) = x$

$\Rightarrow x \cdot e = x$

$\therefore$ e is the right identity in $[S, \cdot]$

(2)

(b) $x(x^{-1}x)x^{-1} = x \cdot e \cdot x^{-1} = x x^{-1}$

$\Rightarrow x(x^{-1} \cdot x) \cdot x^{-1} = x \cdot x^{-1}$

$\Rightarrow (x^{-1}x) \cdot x^{-1} = x^{-1}$, by left cancellation law.

$\Rightarrow x^{-1}(x \cdot x^{-1}) = x^{-1} \cdot e$

$\Rightarrow x \cdot x^{-1} = e$ ;

$\therefore$ $x^{-1}$ is also the right inverse of x in $[S, \cdot]$.

Hence $[S, \cdot]$ is a group.

$\Delta$

## Question 2-b

**(b)** If $a$ is a fixed element of the group $[G, \cdot]$, then show that the set

$$N(a) = \{x \in G | x \cdot a = a \cdot x\}$$

is a subgroup of $G$.

### Solution

Solution:-

Given $N(a) = \{x \in G \mid xa = ax\}$.

R.T.P. $N(a)$ is a subgroup of $G$

i.e, $\forall x, y \in N(a)$, $xy^{-1} \in N(a)$.

Since $x, y \in G$, $x^{-1}$ and $y^{-1} \in G$.

Let $x, y \in N(a)$.

Then $xa = ax \cdots (1)$, and

$ya = ay \cdots (2)$.

from (2):

$$y^{-1}(ya)y^{-1} = y^{-1}(ay)y^{-1}$$

or, $(y^{-1}y)(ay^{-1}) = (y^{-1}a)(yy^{-1})$

or, $e(ay^{-1}) = (y^{-1}a)(e)$, where $e$ is the identity in $G$.

or, $y^{-1}a = ay^{-1}$

$\therefore$ $y^{-1} \in N(a)$.

Consider, $(xy^{-1})a$.

Now, 
$$\begin{aligned}(xy^{-1})a &= x(y^{-1}a) \\ &= x(ay^{-1}) \\ &= (xa)y^{-1} \\ &= (ax)y^{-1}, \text{ from (1)} \\ &= a(xy^{-1})\end{aligned}$$

$\Rightarrow xy^{-1} \in N(a)$.

$\therefore$ $N(a)$ is a subgroup of $G$. $\Delta$.

## Question 2-c

**(c)** If $H$ be a subgroup of a group $\langle G, \circ \rangle$ and $h \in H$, then show that $h \circ H = H$.

### Solution

i.e, $Hh \subseteq H$, and $H \subseteq Hh$.

(i) ~~Let~~ $h' \in H$. Then $h.h' \in hH$, by def$^n$.
Since $H$ is a subgroup, so
$h \in H, h' \in H \Rightarrow h.h' \in H$

$\therefore$ every element of $hH$ is also an element of $H$. Hence $hH \subseteq H$ ---(1)

Again, let $h' \in H$.
Then, $h' = (h h^{-1}) h'$
$= h(h^{-1}h') \in hH$,
Since $h^{-1} \in H, h' \in H \Rightarrow h^{-1}h' \in H$.

$\therefore H \subseteq hH$ --(2)

From (1) & (2), we conclude that
$\boxed{hH = H}$.