CS7.302: Computer Graphics

Final Exam on Feb 26, 2024. Total: **100 points**
(Answer *any 5 out of 6* questions)

1 Question 1

$p(\omega) = k \cos\theta e^{\phi}$
$p(\omega) = \sin\theta p(\theta, \phi)$
(As given in paper, since this is confusing, we have been lenient in grading)

1.1 (a)

PDF should be normalized over the integration domain (upper hemisphere), thus:

$$\int_{\Omega} p(\omega) d\omega = 1$$

Using $d\omega = \sin\theta d\theta d\phi$,

$$\begin{aligned}
\int_0^{\pi/2} \int_0^{2\pi} p(\omega) \sin\theta d\phi d\theta &= 1 \\
\int_0^{\pi/2} \int_0^{2\pi} k \cos\theta e^{\phi} \sin\theta d\phi d\theta &= 1 \\
k \int_0^{\pi/2} \cos\theta \sin\theta d\theta \int_0^{2\pi} e^{\phi} d\phi &= 1 \\
k\frac{1}{2} \int_0^{\pi/2} \sin 2\theta d\theta \int_0^{2\pi} e^{\phi} d\phi &= 1 \qquad \text{(Identity)} \\
k\frac{1}{2}[-\frac{1}{2}\cos 2\theta]_0^{\pi/2} \int_0^{2\pi} e^{\phi} d\phi &= 1 \qquad \text{(Integrate)} \\
-k\frac{1}{4}[\cos\pi - \cos 0] \int_0^{2\pi} e^{\phi} d\phi &= 1 \\
-k\frac{1}{4}[-1-1] \int_0^{2\pi} e^{\phi} d\phi &= 1 \\
k\frac{1}{2} \int_0^{2\pi} e^{\phi} d\phi &= 1 \\
k\frac{1}{2}[e^{\phi}]_0^{2\pi} &= 1 \\
k\frac{1}{2}[e^{2\pi} - e^0] &= 1 \\
k &= \frac{2}{e^{2\pi} - 1}
\end{aligned}$$

1.2 (b)

$$p(\omega) = \frac{2}{e^{2\pi} - 1} \cos\theta e^{\phi} \tag{1}$$

Using the identity given in question:

$$\begin{aligned}
p(\omega) &= \sin\theta p(\theta, \phi) \\
p(\theta, \phi) &= \frac{2}{e^{2\pi} - 1} \cos\theta e^{\phi} \frac{1}{\sin\theta} \\
p(\theta, \phi) &= \frac{2}{e^{2\pi} - 1} \cot\theta e^{\phi}
\end{aligned}$$

Marginalizing ϕ to get $p(\theta)$:

$$\begin{aligned}
p(\theta) &= \int_0^{2\pi} p(\theta, \phi) d\phi \\
p(\theta) &= \int_0^{2\pi} \frac{2}{e^{2\pi} - 1} \cot\theta e^{\phi} d\phi \\
p(\theta) &= \frac{2}{e^{2\pi} - 1} \cot\theta \int_0^{2\pi} e^{\phi} d\phi \\
p(\theta) &= \frac{2}{e^{2\pi} - 1} \cot\theta [e^{\phi}]_0^{2\pi} \\
p(\theta) &= \frac{2}{e^{2\pi} - 1} \cot\theta [e^{2\pi} - 1] \\
p(\theta) &= 2\cot\theta
\end{aligned}$$

Marginalizing θ to get $p(\phi)$:

$$\begin{aligned}
p(\phi) &= \int_0^{\pi/2} p(\theta, \phi) d\theta \\
p(\phi) &= \int_0^{\pi/2} \frac{2}{e^{2\pi} - 1} \cot\theta e^{\phi} d\theta \\
p(\phi) &= \frac{2}{e^{2\pi} - 1} e^{\phi} \int_0^{\pi/2} \cot\theta d\theta \\
p(\phi) &= \frac{2}{e^{2\pi} - 1} e^{\phi} \ln|\sin\theta|_0^{\pi/2}
\end{aligned}$$

Not possible to go after this since you get infinities.
Marks are given generously, as long as you have written the procedure after this:

- Calculate CDF by integrating PDF.
- Invert the CDF to sample.

2 Question 1

$p(\omega) = k \cos\theta e^{\phi}$
$\sin\theta p(\omega) = p(\theta, \phi)$
(The correct formula. Again, since this is confusing, we have been lenient in grading)

2.1 (a)

Same as previous.

2.2 (b)

$$p(\omega) = \frac{2}{e^{2\pi} - 1} \cos\theta e^{\phi}$$

Using the identity given in question:

$$\sin\theta p(\omega) = p(\theta, \phi)$$
$$p(\theta, \phi) = \frac{2}{e^{2\pi} - 1} \cos\theta e^{\phi} \sin\theta$$

Marginalizing ϕ to get $p(\theta)$:

$$p(\theta) = \int_0^{2\pi} p(\theta, \phi) d\phi$$
$$p(\theta) = \int_0^{2\pi} \frac{2}{e^{2\pi} - 1} \cos\theta e^{\phi} \sin\theta d\phi$$
$$p(\theta) = \frac{2}{e^{2\pi} - 1} \cos\theta e^{\phi} \sin\theta \int_0^{2\pi} e^{\phi} d\phi$$
$$p(\theta) = \frac{2}{e^{2\pi} - 1} \cos\theta e^{\phi} \sin\theta [e^{\phi}]_0^{2\pi}$$
$$p(\theta) = \frac{2}{e^{2\pi} - 1} \cos\theta e^{\phi} \sin\theta [e^{2\pi} - 1]$$
$$p(\theta) = 2 \cos\theta e^{\phi} \sin\theta$$
$$p(\theta) = \sin 2\theta$$

Marginalizing θ to get $p(\phi)$:

$$p(\phi) = \int_0^{\pi/2} p(\theta, \phi) d\theta$$
$$p(\phi) = \int_0^{\pi/2} \frac{2}{e^{2\pi} - 1} \cos\theta e^{\phi} \sin\theta d\theta$$
$$p(\phi) = \frac{1}{e^{2\pi} - 1} e^{\phi} \int_0^{\pi/2} 2 \cos\theta \sin\theta d\theta$$
$$p(\phi) = \frac{1}{e^{2\pi} - 1} e^{\phi} \int_0^{\pi/2} \sin 2\theta d\theta$$
$$p(\phi) = \frac{1}{e^{2\pi} - 1} e^{\phi} [-\frac{1}{2} \cos 2\theta]_0^{\pi/2}$$

$$p(\phi) = \frac{1}{e^{2\pi} - 1} e^{\phi} (-\frac{1}{2})[\cos \pi - \cos 0]$$
$$p(\phi) = \frac{e^{\phi}}{e^{2\pi} - 1}$$

2.3 (c)

$$P(\theta) = \int_0^{\theta} p(\theta') d\theta'$$
$$P(\theta) = \int_0^{\theta} \sin 2\theta' d\theta'$$
$$P(\theta) = [-\frac{1}{2} \cos 2\theta']_0^{\theta}$$
$$P(\theta) = [-\frac{1}{2}(\cos 2\theta - \cos 0)]$$
$$P(\theta) = \frac{1 - \cos 2\theta}{2}$$

$$P(\phi) = \int_0^{\phi} p(\phi') d\phi'$$
$$P(\phi) = \int_0^{\phi} \frac{e^{\phi}}{e^{2\pi} - 1} d\phi'$$
$$P(\phi) = \frac{1}{e^{2\pi} - 1} [e^{\phi}]_0^{\phi}$$
$$P(\phi) = \frac{e^{\phi} - 1}{e^{2\pi} - 1}$$

2.4 (d)

To sample θ:

$$\xi_1 = P(\theta)$$
$$\xi_1 = \frac{1 - \cos 2\theta}{2}$$
$$2\xi_1 = 1 - \cos 2\theta$$
$$\cos 2\theta = 1 - 2\xi_1$$
$$2\theta = \cos^{-1}(1 - 2\xi_1)$$
$$\theta = \frac{\cos^{-1}(1 - 2\xi_1)}{2}$$

To sample ϕ:

$$\begin{aligned}
\xi_2 &= P(\phi) \\
\xi_2 &= \frac{e^{\phi} - 1}{e^{2\pi} - 1} \\
(e^{2\pi} - 1)\xi_2 &= e^{\phi} - 1 \\
(e^{2\pi} - 1)\xi_2 + 1 &= e^{\phi} \\
\phi = \ln\left[(e^{2\pi} - 1)\xi_2 + 1\right]
\end{aligned}$$

3 Question 2

3.1 (a)

$$\langle I \rangle = \frac{|D|}{N} \sum_{i=1}^{N} f(X_i)$$

$$\langle I \rangle = \frac{2\pi}{N} \sum_{i=1}^{N} f(X_i)$$

3.2 (b)

$$\langle I \rangle = \frac{1}{N} \sum_{i=1}^{N} \frac{f(X_i)}{p(X_i)}$$

3.3 (c)

We need to prove that on average, MC estimator computes the right answer.
Taking the average, or expected value of the MC estimator:

$$\begin{aligned}
E[\langle I \rangle] &= E\left[\frac{1}{N} \sum_{i=1}^{N} \frac{f(X_i)}{p(X_i)}\right] \\
&= \frac{1}{N} E\left[\sum_{i=1}^{N} \frac{f(X_i)}{p(X_i)}\right] \qquad \text{(Since } E[aX] = aE[X]\text{)} \\
&= \frac{1}{N} \sum_{i=1}^{N} E\left[\frac{f(X_i)}{p(X_i)}\right] \qquad \text{(Since } E[X_1 + X_2] = E[X_1] + E[X_2]\text{)} \\
&= \frac{1}{N} \sum_{i=1}^{N} \int \frac{f(x)}{p(x)} \cdot p(x) dx \qquad \text{(Since } E[X] = \int x \cdot p(x) dx\text{)} \\
&= \frac{1}{N} \sum_{i=1}^{N} \int f(x) dx \qquad \text{(Cancelling PDF)}
\end{aligned}$$

Thus,

$$E[\langle I \rangle] = \int f(x)dx$$

4 Question 3

4.1 (a)

Nearest neighbour was $(0.5, 0.25)$ with the value 18. Marks have also been awarded partially or fully if you made different assumptions about the uv coordinates (for example, setting $v' = (1 - v)$) and written this assumption.

4.2 (b)

$$\begin{aligned}
c_u &= \frac{(0.5 - 0.4) * 17 + (0.4 - 0.25) * 18}{0.25} = 17.6 \\
c_l &= \frac{(0.5 - 0.4) * 22 + (0.4 - 0.25) * 23}{0.25} = 22.6 \\
c &= \frac{(0.25 - 0.24) * 22.6 + (0.24 - 0.00) * 17.6}{0.25} = 17.8
\end{aligned}$$

Note: 2.5 Marks have been cut if the division by 0.25 is missing.

4.3 (c)

4.3.1 Finding out the uv coordinates (5 Marks)

$$\begin{aligned}
\frac{y}{x} &= \frac{r \sin u \sin v}{r \cos u \sin v} \\
&= \tan(u) \\
u &= \arctan\left(\frac{y}{x}\right) = \frac{\pi}{4}
\end{aligned}$$

$$\begin{aligned}
v &= \arccos\left(\frac{z}{r}\right) \\
&= \arccos\left(\frac{\sqrt{2}}{2}\right) \\
&= \arccos\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}
\end{aligned}$$

We need to normalize u, v to the range of $(0, 1)$, i.e $uv = \frac{(\pi/4, \pi/4)}{(2\pi, \pi)} = (0.125, 0.25)$.
Note: 2.5 marks have been cut if the normalization is missing.

4.3.2 Bi-linear interpolation (5 marks)

$$c = \frac{16 * (0.25 - 0.125) + 17 * (0.125 - 0.00)}{0.25} = 16.5$$

Full marks are given even if you have used $(\frac{\pi}{4}, \frac{\pi}{4})$. The correct answer in that case is 8.44.
Note: 2.5 Marks have been cut if the division by 0.25 is missing.

5 Question 4

5.1 (a)

$$\begin{aligned}
I &= \int_0^2 (x+2)^2 dx \\
&= \int_0^2 (x^2 + 4x + 4) dx \\
&= \frac{x^3}{3} + 2x^2 + 4x \Big|_0^2 \\
&= \frac{56}{3}
\end{aligned}$$

5.2 (b)

$$\begin{aligned}
f(x_1) &= (1.16 + 2)^2 = 9.98 \\
f(x_2) &= (1.98 + 2)^2 = 18.84 \\
f(x_3) &= (0.54 + 2)^2 = 6.45 \\
f(x_4) &= (1.26 + 2)^2 = 10.26
\end{aligned}$$

$$\begin{aligned}
\langle I \rangle &= \frac{1}{4} \sum_{i=0}^{4} \frac{f(x_i)}{p(x_i)} \\
&= \frac{1}{4} \cdot \frac{9.98 + 18.84 + 6.45 + 10.26}{\frac{1}{2}} \\
&= 21.45
\end{aligned}$$

5.3 (c)

$$\begin{aligned}\langle I \rangle &= \frac{1}{4}\sum_{i=0}^{4}\frac{f(x_i)}{p(x_i)} \\ &= \frac{1}{4}\sum_{i=0}^{4}\frac{(x+2)^2}{\frac{3}{56}\cdot(x+2)^2} \\ &= \frac{56}{3}\end{aligned}$$

Full marks have also been awarded to those who have done the complete calculations correctly.

5.4 (d)

The convergence graph was asked to show the value of the MC estimate as the number of samples is increased. If you have done this and also mentioned that (c) converges faster, then full marks have been awarded.

6 Question 5

See derivation in: Ray Tracing in One Weekend. 2 Marks for stating $t \geq 0$ and 2 marks for stating discriminant $\Delta \geq 0$.

7 Question 6

7.1 (a)

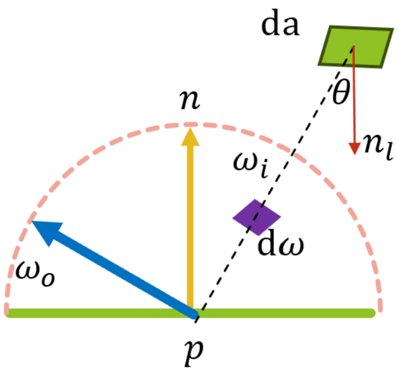

Figure 1: Solid angle.

The area element dA can be arbitrarily oriented (given by it's normal n_l), and it subtends a solid angle $d\omega$.

Assume a perpendicular area element $dA^{\perp}$ with sides dl. Then $dA^{\perp} = dl \cdot dl$.

Assume the sides of the solid angle element are $d\omega_l$. Then $d\omega = d\omega_l \cdot d\omega_l$.

By similar triangles:

$$\frac{d\omega_l}{2} = \frac{dl}{2r}$$
$$d\omega_l = \frac{dl}{r}$$

Thus,

$$d\omega = d\omega_l \cdot d\omega_l$$
$$d\omega = \frac{dl}{r} \cdot \frac{dl}{r}$$
$$d\omega = \frac{dA^{\perp}}{r^2}$$
$$d\omega = \frac{dA \cos\theta_l}{r^2}$$

7.2 (b)

$$L_o(x,\omega_o) = \int_{\Omega} f(x,\omega_o,\omega_i) L_i(x,\omega_i) \cos\theta d\omega_i$$
$$L_o(x,\omega_o) = \int_{A} f(x,\omega_o,\omega_i) L_i(x,\omega_i) \cos\theta J_T dA \qquad \text{(Change of variables, introduces a jacobian)}$$
$$L_o(x,\omega_o) = \int_{A} f(x,\omega_o,\omega_i) L_i(x,\omega_i) V(x,\omega_i) \cos\theta J_T dA \qquad \text{(We also explicitely add visibility)}$$
$$L_o(x,\omega_o) = \int_{A} f(x,\omega_o,\omega_i) L_i(x,\omega_i) V(x,\omega_i) \cos\theta \frac{\cos\theta_l}{r^2} dA \qquad \text{(Jacobian is given from above)}$$

7.3 (c)

Since PDF is uniform over area light, $p_a = \frac{1}{|A|}$, where $|A|$ is the area of the light.

The MC estimator is:

$$\langle L_o(x,\omega_o)\rangle = \frac{1}{N}\sum_{i=1}^{N} \frac{f(x,\omega_o,\omega_i) L_i(x,\omega_i) V(x,\omega_i) \cos\theta \frac{\cos\theta_l}{r^2}}{p_a}$$
$$\langle L_o(x,\omega_o)\rangle = \frac{|A|}{N}\sum_{i=1}^{N} f(x,\omega_o,\omega_i) L_i(x,\omega_i) V(x,\omega_i) \cos\theta \frac{\cos\theta_l}{r^2}$$