Discrete Structures Spring 2022 IIITH: Midterm Exam

Discrete Structures Instructors

20th December 2022 10:30pm-12noon

1. Total Duration: 90min — Total Marks: 90
2. Please submit question paper with answer sheet. Please only write your roll number and seat number on it. Nothing else. Failing to do so will cost you 10 points.
3. This is a closed book exam. You are not permitted to seek assistance from any individual or refer to any reading material during the exam.
4. Proofs should be clear and concise, with each step (judgement) enumerated on a separate line. Points will be deducted for proofs not adhering to this format. Unless asked explicitly, you do not need to write your proofs in fitch style.

Roll Number: ________________ Seat Number: ________________

1. (30 points)
   1. Using the generating function, prove that the number of distinct binary trees with n nodes is $\frac{1}{n+1}{}^{2n}C_n$, for all $n \geq 0$.
   2. Consider the following recurrence relation for the numeric function

$$b = (b_0, b_1, b_2, \cdots, b_r, \cdots)$$

:

$$
\begin{aligned}
b_0 &= 1 \\
b_1 &= 2 \\
b_2 &= 3 \\
b_r &= b_{r-1} + b_{r-2} - b_{r-3} + 4, \text{ for all } r \geq 3
\end{aligned}
$$

Using the generating function, derive b_r.

3. State the *Comparison Tests of Convergence* of an infinite series. Using this, test for convergence or divergence of the series $\sum_{n=1}^{\infty} u_n$, where $u_n = \sqrt{n^4+1} - \sqrt{n^4-1}$.

[10 + 12 + (2 + 6) = 30]

2. (30 points) Consider the set of propositional formulas built using propositional variables V and propositional constructors NOT, AND, OR and $IMPLIES$. Let p, q, r be variables. Examples of propositional formulas are:

1. $AND(p, q)$
2. $NOT(OR(IMPLIES(p, q), r))$

Consider the ordering $\prec$ between propositional formulas. $A \prec B$ iff formula A is an immediate subformula of formula B.

(a) (6 points) Give three examples of members of the $\prec$ relation.

Solution:

1. $p \prec p \wedge q$
2. $\neg p \prec (\neg p \rightarrow q)$
3. $p \vee q \prec (p \vee q) \wedge r$

Obviously, other examples possible.

(b) (10 points) Define the $\prec$ relation as a first order logic formula with equality.

Solution: $x \prec y \stackrel{\text{def}}{=}$

$$
\begin{aligned}
y = NOT(x) \vee \\
\exists z \ \ y = AND(x, z) \ \vee \ y = AND(z, x) \vee \\
y = OR(x, z) \ \vee \ y = OR(z, x) \vee \\
y = IMPLIES(x, z) \ \vee \ y = IMPLIES(z, x)
\end{aligned}
$$

(c) (14 points) Argue that $\prec$ is well-founded (i.e., it admits no infinite descending sequences). Be brief.

> **Solution:**
>
> 1. Assume $\prec$ is not well-founded.
> 2. Then there is an infinite sequence $s = \ldots x_3 \prec x_2 \prec x_1 \prec x_0$.
> 3. Now, note that if $x \prec y$, x has fewer symbols than y.
> 4. Consider now the sequence c derived from s obtained by taking the number of symbols in each x_i
> 5. If s is infinite then c is infinite as well.
> 6. But c can not be an infinite descending sequence, it can not go below 0.
> 7. Hence s can not be infinitely long.
> 8. Hence $\prec$ is well-founded.

3. (30 points) Consider the set $X = \mathbf{B}^n$ where $\mathbf{B} = \{0, 1\}$ is the set of booleans and n is a natural number greater than 0. A subset S of X is called a *box* if $S = X_1 \times \ldots \times X_n$ and $X_i \subseteq \mathbf{B}$, $1 \leq i \leq n$.

   In each of the questions below, either give a justification (via a proof) or provide a counter-example.

   (a) (4 points) Express the proposition $x \in S$ in terms of the components x_i and X_i.

   > **Solution:** $x = S$ iff $x_i \in X_i$ for each $i : [1..n]$

   (b) (6 points) Let $x \in \mathbf{B}^n$. Is $\{x\}$ a box? Justify your answer.

   > **Solution:** Let $x = x_1 \ldots x_n$. Then $\{x\}$ is a box because x may be written as $x = \{x_1\} \times \ldots \times \{x_n\}$.

   (c) (10 points) Is the union of two boxes a box? Justify your answer.

   > **Solution:** Let $S' = S \cup \{x\}$, where S is a box. Clearly $\{x\}$ is a box. We claim that S' is not a box.
   > Counter-example: take $n = 2$ Let $S = \{01\} = \{0\} \times \{1\}$. Let $x = 10$. Then $S' = \{01, 10\}$. S' is not a box because it can not be written as a product $S' = X_1 \times X_2$.

1. X_i is nonempty for each $i \in \{0, 1\}$
2. X_i can not both be singletons: If so, S' would have only one element.
3. X_i can not both be B : If so, S' would have four elements.
4. X_1 is either $\{0\}$ or $\{1\}$ and $X_2 = \mathbf{B}$. In the first case, this forces every element of S' start with 0, which is clearly not the case with S'. The second case is similar.
5. A similar argument works for X_2 being either $\{0\}$ or $\{1\}$.

(d) (10 points) Is the intersection of two boxes a box? Justify your answer.

Solution: Let $S = P \cap Q$ where P and Q are boxes. Let $P = P_1 \times \ldots \times P_n$ and $Q = Q_1 \times \ldots \times Q_n$. Let $S' = P_1 \cap Q_1 \times \ldots \times P_n \cap Q_n$. We claim $S = S'$ Proof:

1. First we prove that $S' \subseteq S$
2. Let $x \in S'$. Then $x_i \in P_i \cap Q_i$, for each $i : [1..n]$.
3. Then $x_i \in P_i$ and $x_i \in Q_i$ for each $i : [1..n]$.
4. This implies that $x \in P$ and $x \in Q$
5. Hence $x \in P \cap Q$.
6. This proves that $S' \subseteq S$.
7. Now we prove that $S \subseteq S'$.
8. Now assume, $x \in S$, i,e., $x \in P \cap Q$
9. Hence $x \in P$ and $x \in Q$
10. Hence $x_i \in P_i$ for each $i : [1..n]$ and $x_i \in Q_i$ for each $i : [1..n]$.
11. Hence $x_i \in P_i \cap Q_i$ for each $i : [1..n]$.
12. Hence $x \in S'$.
13. This proves that $S \subseteq S'$
14. Hence we conclude that $P \cap Q = S$
15. Hence the intersection of two boxes is a box.