End-Semester Examination

(MA6.102) Probability and Random Processes, Monsoon 2023

29th November, 2023

Max. Duration: 3 Hours

Special Instruction: The solution to each question should begin on a new page.

Question 1. (a) (2 Marks) For three events A, B, C, prove that

$$P(A|B) = P(A|B \cap C)P(C|B) + P(A|B \cap C^{\mathsf{c}})P(C^{\mathsf{c}}|B).$$

Hint: Conditional total probability theorem.

(b) (3 Marks) For two events A and B, is it true that $P(A|A \cup B) \geq P(A|B)$? If not, give a counterexample.

Question 2 (5 Marks)**.** For an event A, let $\mathbb{I}_A$ denote the indicator random variable of A, i.e., $\mathbb{I}_A(\omega) = 1$ if $\omega \in A$ and $\mathbb{I}_A(\omega) = 0$ if $\omega \in A^{\mathsf{c}}$. For any two events A and B, show that the following are equivalent.

- The events A and B are independent.
- The random variables $\mathbb{I}_A$ and $\mathbb{I}_B$ are independent.

Question 3 (5 Marks)**.** Let X be a discrete random variable that is uniformly distributed over $\{a, a+1, \ldots, b-1, b\}$, where a and b are integers with $a < 0 < b$. Let $Y = \max\{0, X\}$ and $Z = \min\{0, X\}$. Find the PMFs P_Y and P_Z.

Question 4 (5 Marks)**.** A permutation on the numbers in $[1:n]$ can be represented as a function $\pi : [1:n] \to [1:n]$, where $\pi(i)$ is the position of i in the ordering given by the permutation. A fixed point of a permutation $\pi : [1:n] \to [1:n]$ is a value x for which $\pi(x) = x$. Let X be number of fixed points of a permutation chosen uniformly at random from all permutations. Find $\mathbb{E}[X]$.
Hint: Express X as a sum of indicator random variables.

Question 5 (5 Marks)**.** For any two random variables X and Y, Cauchy-Schwarz inequality states that

$$|\mathbb{E}[XY]| \leq \sqrt{\mathbb{E}[X^2]\mathbb{E}[Y^2]}$$

with equality if and only if $X = \alpha Y$, for some constant $\alpha \in \mathbb{R}$. Prove this and use it to show that $|\rho(X,Y)| \leq 1$, where $\rho(X,Y)$ is the correlation coefficient of X and Y given by

$$\rho(X,Y) = \frac{\mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]}{\sqrt{\text{var}(X)\text{var}(Y)}}.$$

Question 6. (a) (2 Marks) Show that X and Y are independent continuous random variables if and only if their joint PDF f_{XY} factorizes as the product of the functions of the single variables x and y alone, i.e., $f_{XY}(x,y) = g(x)h(y)$, for all x, y.

(b) (3 Marks) Let X and Y be independent exponential random variables with parameter λ. Find the joint PDF of

$$Z = X + Y \text{ and } W = \frac{X}{Y}$$

and show that they are independent.

Question 7. Let X and Y be two random variables with the associated MGFs $M_X(s)$ and $M_Y(s)$, respectively. Let Z be a random variable MGF

$$M_Z(s) = M_X(s)^2 M_Y(s)^3.$$

Find $\mathbb{E}[Z]$ and var(Z) in terms of $\mathbb{E}[X]$, var(X), $\mathbb{E}[Y]$, and var(Y).

Question 8 (5 Marks)**.** Consider two sequences of random variables $X_1, X_2, \ldots$ and $Y_1, Y_2, \ldots$ which converge in probability to x and y, respectively. That is, for every $\epsilon > 0$, we have

$$\lim_{n\to\infty} P(|X_n - x| \geq \epsilon) = 0,$$
$$\lim_{n\to\infty} P(|Y_n - y| \geq \epsilon) = 0.$$

Prove that the sequence $X_1Y_1, X_2Y_2, \ldots$ converges in probability to xy.
Hint: Show that $\lim_{n\to\infty} P(|(X_n - x)(Y_n - y)| \geq \epsilon) = 0$.

Question 9 (5 Marks)**.** Let $X_1, Y_1, X_2, Y_2, \ldots$ are independent random variables and uniformly distributed over the interval $[0, 1]$, and let

$$W = \frac{\sum_{i=1}^{16} X_i - \sum_{i=1}^{16} Y_i}{16}.$$

Find an approximate value to the quantity $P(|W - \mathbb{E}[W]| < 0.001)$ in terms of the CDF of standard Gaussian random variable $\mathcal{N}(0, 1)$.

Question 10 (5 Marks)**.** Let $g : \mathbb{R} \to \mathbb{R}$ be a periodic function with period T, i.e., $g(t + T) = g(t)$, for all $t \in \mathbb{R}$. Consider the random process

$$X(t) = g(t + U), \text{ for all } t \in \mathbb{R},$$

where U is a random variable uniformly distributed over the interval $[0, T]$. Is $X(t)$ a wide-sense stationary (WSS) process?