# Probability and Statistics

Midsem Solutions

August 2023

## Section I (6 Marks)

## Problem 1

Consider a random variable $X$ with the following pdf. Find the mean and variance for $X$.

$$f_X(x) = \begin{cases} 0.5\lambda e^{-\lambda x}, & \text{if } x \geq 0 \\ 0.5\lambda e^{\lambda x}, & \text{if } x < 0 \end{cases}$$

## Solution 1

### Mean calculation

To find the mean (expected value) of $X$, we use the following formula:

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} x f_X(x) dx$$

$$\mathbb{E}[X] = \int_{-\infty}^{0} x \cdot 0.5\lambda e^{\lambda x} dx + \int_{0}^{\infty} x \cdot 0.5\lambda e^{-\lambda x} dx$$

Consider,

$$I = \int_{-\infty}^{0} x \cdot 0.5\lambda e^{\lambda x} dx$$

Putting $t = -x$ we get, $dt = -dx,\ t \to \infty$ as $x \to -\infty$ and $t = 0$ at $x = 0$. So,

$$I = \int_{\infty}^{0} t \cdot 0.5\lambda e^{-\lambda t} dt = -\int_{0}^{\infty} t \cdot 0.5\lambda e^{-\lambda t} dt = -\int_{0}^{\infty} x \cdot 0.5\lambda e^{-\lambda x} dx$$

Hence, we get the value of $\mathbb{E}[X]$ as follows:

$$\mathbb{E}[X] = \int_{0}^{\infty} x \cdot 0.5\lambda e^{-\lambda x} dx - \int_{0}^{\infty} x \cdot 0.5\lambda e^{-\lambda x} dx = 0$$

## Variance calculation

* Calculate $\mathbb{E}[X^2]$:

$$\mathbb{E}[X^2] = \int_{-\infty}^{0} x^2 \cdot 0.5\lambda e^{\lambda x} dx + \int_{0}^{\infty} x^2 \cdot 0.5\lambda e^{-\lambda x} dx$$

Again consider,

$$I = \int_{-\infty}^{0} x^2 \cdot 0.5\lambda e^{\lambda x} dx$$

Putting $t = -x$ we get, $dt = -dx,\ t \to \infty$ as $x \to -\infty$ and $t = 0$ at $x = 0$. So,

$$I = -\int_{\infty}^{0} t^2 \cdot 0.5\lambda e^{-\lambda t} dt = \int_{0}^{\infty} t^2 \cdot 0.5\lambda e^{-\lambda t} dt = \int_{0}^{\infty} x^2 \cdot 0.5\lambda e^{-\lambda x} dx$$

Now, for the first integral, use integration by parts:

$$\int_{-\infty}^{0} x^2 e^{\lambda x} dx = \left[\frac{x^2}{\lambda} e^{\lambda x} - \frac{2x}{\lambda^2} e^{\lambda x} + \frac{2}{\lambda^3} e^{\lambda x}\right]_{-\infty}^{0} = \frac{2}{\lambda^3} - 0 = \frac{2}{\lambda^3}$$

Now, we can substitute these results back into our calculation of $\mathbb{E}[X^2]$:

$$\mathbb{E}[X^2] = \frac{\lambda}{2}\left(\frac{2}{\lambda^3}\right) + \frac{\lambda}{2}\left(\frac{2}{\lambda^3}\right) = \frac{2}{\lambda^2}$$

* Calculate the variance:

$$\begin{aligned} \mathrm{Var}[X] &= \mathbb{E}[X^2] - (\mathbb{E}[X])^2 \\ &= \frac{2}{\lambda^2} - (0)^2 = \frac{2}{\lambda^2} \end{aligned}$$

Hence, the mean and the variance of the random variable $X$ are:

$$\mathbb{E}[X] = 0 \text{ and } \mathrm{Var}[X] = \frac{2}{\lambda^2}$$

# Grading Criteria

* Showing $\mathbb{E}[X] = 0$ gets 3 marks
* Showing $Var[X] = \frac{2}{\lambda^2}$ gets 3 marks

## Question 2

Let X be a Uniform U[0, 1] random variable. Let Y = $e^{2X}$. Find PDF and CDF of Y.

## Solution 2

Given: $X$ is a Uniform $U[0,1]$ random variable and $Y = e^{2X}$. So the CDF of X is given by :-

$$F_X(x) = \begin{cases} 0, & \text{if } x < 0 \\ x, & \text{if } 0 \leq x \leq 1 \\ 1, & \text{if } x > 1 \end{cases}$$

And PDF by :-

$$f_X(x) = \begin{cases} 1, & \text{if } 0 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

Now,

$$0 \leq X \leq 1$$
$$0 \leq 2X \leq 2$$
$$e^0 \leq e^{2X} \leq e^2$$
$$1 \leq Y \leq e^2$$

So,

$$\begin{aligned} &F_Y(y) = P(Y \leq y) \\ &= P(e^{2X} \leq y) \\ &= P(2X \leq \ln(y)) \\ &= P(X \leq \ln(\sqrt{y})) \\ &= \ln(\sqrt{y}) \qquad (Since\ P(X <= x) = x) \end{aligned}$$

So, the CDF of Y is given by :-

$$F_Y(y) = \begin{cases} 0, & \text{if } y < 1 \\ \ln(\sqrt{y}), & \text{if } 1 \leq y \leq e^2 \\ 1, & \text{if } y > e^2 \end{cases}$$

Also, we know that:-

$$
\begin{aligned}
f_Y(y) &= \frac{d}{dy} F_Y(y) \\
&= \frac{d}{dy} \ln(\sqrt{y}) \\
&= \frac{1}{\sqrt{y}} * \frac{1}{2\sqrt{y}} \\
&= \frac{1}{2y}
\end{aligned}
$$

So, The PDF of Y is given by :-

$$
f_Y(y) = \begin{cases} \frac{1}{2y}, & \text{if } 1 \leq y \leq e^2 \\ 0, & \text{otherwise} \end{cases}
$$

**Grading Scheme (6 marks) :**

1) 3 marks for PDF
2) 3 marks for CDF
3) Partial marks for intermediate steps

# Question 3

The joint probability mass function of the discrete random variables $X$ and $Y$ is given by $p_{X,Y}(x, y) = \frac{1}{2^{x+y}}$, where $x = 1, 2, \ldots$ and $y = 1, 2, \ldots$.

**(a)** Find the expressions for the marginal PMFs $p_X(x)$ and $p_Y(y)$.

**(b)** Find $\mathbb{E}[XY]$ and determine if $X$ and $Y$ are independent.

# Solution 3

(a) The marginal PMFs $p_X$ and $p_Y$ can be obtained from the joint PMF as follows:

$$
p_X(x) = \sum_y p_{XY}(x, y) \quad \text{and} \quad p_Y(y) = \sum_x p_{XY}(x, y).
$$

Therefore,

$$
p_X(x) = \sum_{y \in R} p_{XY}(x, y) = \sum_{y \in N} \frac{1}{2^{x+y}} = \frac{1}{2^x} \sum_{y \in N} \frac{1}{2^y} = \frac{1}{2^x} \cdot \frac{1/2}{1 - 1/2} = \frac{1}{2^x} \cdot \frac{1/2}{1/2} = \frac{1}{2^x}
$$

$$
p_Y(y) = \sum_{x \in R} p_{XY}(x, y) = \sum_{x \in N} \frac{1}{2^{x+y}} = \frac{1}{2^y} \sum_{x \in N} \frac{1}{2^x} = \frac{1}{2^y} \cdot \frac{1/2}{1 - 1/2} = \frac{1}{2^y} \cdot \frac{1/2}{1/2} = \frac{1}{2^y}
$$

**1.5 Marks** - Calculation of $p_X(x)$
**1.5 Marks** - Calculation of $p_Y(y)$

(b) Two random variables $X$ and $Y$ are independent if, for all $(x, y) \in \mathbb{R}^2$, the following is true:

$$p_{XY}(x, y) = p_X(x) \cdot p_Y(y).$$

$$p_{XY}(x, y) = \frac{1}{2^{x+y}} = \frac{1}{2^x} \cdot \frac{1}{2^y} = p_X(x) \cdot p_Y(y)$$

Thus, $X$ and $Y$ are independent random variables.

If random variables $X$ and $Y$ are independent, then we have:

$$\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$$

$$\mathbb{E}[X] = \sum_{x\in\mathbb{N}} x \cdot p_X(x) = \sum_{x\in\mathbb{N}} \frac{x}{2^x} = 2$$

$$\mathbb{E}[Y] = \sum_{y\in\mathbb{N}} y \cdot p_Y(y) = \sum_{y\in\mathbb{N}} \frac{y}{2^y} = 2$$

Solving for the sum of Arithmetico-Geometric Series:

$$\begin{aligned}
S &= \frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \frac{4}{2^4} + \dots \\
\frac{S}{2} &= \frac{1}{2^2} + \frac{2}{2^3} + \frac{3}{2^4} + \dots \\
S - \frac{S}{2} &= \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \dots = \frac{1/2}{1 - 1/2} = 1 \\
S &= 2
\end{aligned}$$

Therefore,

$$\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y] = 2 \times 2 = 4$$

**1.5 Marks** - Proving the Independence of Random Variables X and Y
**1.5 Marks** - Calculation of $\mathbb{E}[XY]$

## Question 4

The joint pdf of random variables $X$ and $Y$ is given by $f_{X,Y}(x, y) = \lambda e^{-\lambda x + y}$, where $x \geq 0$, $y \geq 0$, and $\lambda > 0$.

**(a)** Find the expressions for the marginal pdfs $f_X(x)$ and $f_Y(y)$.
**(b)** Find the joint cdf $F_{X,Y}(x, y)$. Are $X$ and $Y$ independent ?

## Solution 4

a)

$$f_X(x) = \int_y f(x,y)\,dy$$

$$f_Y(y) = \int_x f(x,y)\,dx$$

$$f_X(x) = \int_y \lambda \cdot e^{-\lambda x - y}\,dy$$

$$= -\left(\lambda e^{-\lambda x}\right) e^{-y}\Big|_0^\infty$$

$$= \lambda e^{-\lambda x}$$

Similarly,

$$f_Y(y) = e^{-y}$$

b)

$$F(x,y) = P(X \le x, Y \le y)$$

$$= \int_0^x \int_0^y f(x,y)\,dy\,dx$$

$$= \int_0^x \int_0^y \lambda \cdot e^{-\lambda x - y}\,dy\,dx$$

$$= (1 - e^{-y}) \int_0^x \lambda \cdot e^{-\lambda x}\,dx$$

$$= (1 - e^{-y})(1 - e^{-\lambda x})$$

Two variables are considered independent if,

$$f_{XY}(x,y) = f_X(x) \cdot f_Y(y)$$

$$f_X(x) \cdot f_Y(y) = \lambda \cdot e^{-\lambda x} \cdot e^{-y}$$

$$= \lambda \cdot e^{-\lambda x - y}$$

$$= f_{XY}(x,y)$$

Thus they are independent.

**Marks Distribution:**

a) Marginal pdf for X - 1.5 Marks
Marginal pdf for Y - 1.5 Marks
b) Joint CDF - 1.5 Marks
Independence - 1.5 Mark

## Question 5

Let X, Y and Z be independent exponential random vaiables with parameters $\lambda_1$, $\lambda_2$ and $\lambda_3$. Let W = min(X,Y,Z). Find the PDF and CDF of W.

## Solution 5

The exponential distribution has a PDF and CDF given by:

$$f_X(x) = \lambda_1 e^{-\lambda_1 x}$$

$$F_X(x) = 1 - e^{-\lambda_1 x} for x \geq 0$$

and similarly for Y and Z.
The CDF of W, denoted as $F_W(w)$, can be expressed as the probability that atleast one of the random variable (X, Y, and Z) is less than or equal to w or probability that all three random variables (X, Y, and Z) are greater than w. Since X, Y, and Z are independent, we can write:

$$F_W(w) = 1 - P(min(X,Y,Z) > w) = 1 - P(X > w) * P(Y > w) * P(Z > w)$$

Using the exponential distribution CDF for each random variable:

$$F_W(w) = 1 - (1 - (1 - e^{-\lambda_1 w})) * (1 - (1 - e^{-\lambda_2 w})) * (1 - (1 - e^{-\lambda_3 w}))$$
$$F_W(w) = 1 - e^{-(\lambda_1 + \lambda_2 + \lambda_3) w}$$

To find the PDF of W, we can differentiate the CDF with respect to w:

$$f_W(w) = \frac{d(1 - e^{-(\lambda_1 + \lambda_2 + \lambda_3) w})}{dw}$$
$$f_W(w) = (\lambda_1 + \lambda_2 + \lambda_3) e^{-(\lambda_1 + \lambda_2 + \lambda_3) w}$$

## Grading Criteria:

* 3 marks for derivation of CDF.
* 3 marks for derivation of PDF.
* Partial marks for intermediate steps like PDF/CDF of X,Y,Z.

# Section II (10 Marks)

## Problem 1

Let Y = a$X^2$ + b where X is a continuous random variable. Derive the expression for CDF and pdf of Y in terms of pdf of X

## Solution 1

Let $F_Y, f_Y$ be the CDF and pdf of Y and $F_X, f_X$ be the CDF and pdf of X respectively. We have,

$$\begin{aligned}
F_Y(y) &= P(Y \leq y) \\
&= P(aX^2 + b \leq y) \\
&= P(aX^2 \leq y - b) \\
Case: a > 0 \\
&= P(X^2 \leq \frac{y-b}{a}) \\
&= P(|X| \leq \sqrt{\frac{y-b}{a}}) \text{ for } y > b \\
&= P\left(-\sqrt{\frac{y-b}{a}} \leq X \leq \sqrt{\frac{y-b}{a}}\right) \\
&= F_X\left(\sqrt{\frac{y-b}{a}}\right) - F_X\left(-\sqrt{\frac{y-b}{a}}\right) \\
&= \int_{-\sqrt{\frac{y-b}{a}}}^{\sqrt{\frac{y-b}{a}}} f_X(x)dx
\end{aligned}$$

$Case : a < 0$

$$
\begin{aligned}
&= P(X^2 \geq \frac{y-b}{a}) \\
&= P(|X| \geq \sqrt{\frac{y-b}{a}}) \text{ for } y < b \\
&= P\left(X \leq -\sqrt{\frac{y-b}{a}} or X \geq \sqrt{\frac{y-b}{a}}\right) \\
&= P\left(X \leq -\sqrt{\frac{y-b}{a}}\right) + P\left(X \geq \sqrt{\frac{y-b}{a}}\right) \\
&= F_X\left(-\sqrt{\frac{y-b}{a}}\right) + 1 - F_X\left(\sqrt{\frac{y-b}{a}}\right) \\
&= 1 - \int_{-\sqrt{\frac{y-b}{a}}}^{\sqrt{\frac{y-b}{a}}} f_X(x)dx
\end{aligned}
$$

$$
\therefore F_Y(y) = \begin{cases} \int_{-\sqrt{\frac{y-b}{a}}}^{\sqrt{\frac{y-b}{a}}} f_X(x)dx, & \text{if } y > b \text{ and } a > 0 \\ 1 - \int_{-\sqrt{\frac{y-b}{a}}}^{\sqrt{\frac{y-b}{a}}} f_X(x)dx, & \text{if } y < b \text{ and } a < 0 \\ 0, & \text{otherwise} \end{cases}
$$

Now,

$$
f_Y(y) = \frac{d}{dy} F_Y(y)
$$

$$
\frac{d}{dy} \int_{-\sqrt{\frac{y-b}{a}}}^{\sqrt{\frac{y-b}{a}}} f_X(x)dx = \frac{1}{2\sqrt{a(y-b)}} \left(f_X\left(\sqrt{\frac{y-b}{a}}\right) + f_X\left(-\sqrt{\frac{y-b}{a}}\right)\right)
$$

Therefore-

$$
f_Y(y) = \begin{cases} \frac{1}{2\sqrt{a(y-b)}} \left(f_X\left(\sqrt{\frac{y-b}{a}}\right) + f_X\left(-\sqrt{\frac{y-b}{a}}\right)\right), & \text{if } y > b \text{ and } a > 0 \\ -\frac{1}{2\sqrt{a(y-b)}} \left(f_X\left(\sqrt{\frac{y-b}{a}}\right) + f_X\left(-\sqrt{\frac{y-b}{a}}\right)\right), & \text{if } y < b \text{ and } a < 0 \\ 0, & \text{otherwise} \end{cases}
$$

## Grading Criteria

* CDF $F_Y$ in terms of $f_X$ - 5 marks
* PDF $f_Y$ in terms of $f_X$ - 5 marks

Other approaches, if correct, will also be considered on case to case basis.

## Problem 2

Let X be a uniform random variable with support [a,b]. Let Y be a Poisson random variable with parameter $\lambda$. Find the mean and variance for each.

## Solution 2

PDF of X is given by $f(x) = \frac{1}{b-a}$, where x $\in$ [a,b]. Mean of X is given by -

$$\begin{aligned}
\mu = E[X] &= \int_a^b x f(x) dx \\
&= \int_a^b x \cdot \frac{1}{b-a} dx \\
&= \frac{1}{b-a} \int_a^b x dx \\
&= \frac{1}{b-a} \left[ \frac{x^2}{2} \right]_a^b \\
&= \frac{1}{2(b-a)} (b^2 - a^2) \\
&= \frac{b+a}{2}
\end{aligned}$$

The variance of X is given by -

$$
\begin{aligned}
Var[X] &= E[(X-\mu)^2] \\
&= \int_a^b (x-\mu)^2 f(x)dx \\
&= \int_a^b (x - \frac{b+a}{2})^2 \cdot \frac{1}{b-a} dx \\
&= \frac{1}{b-a} \int_a^b (x^2 - bx - ax + \frac{b^2+2ab+a^2}{4}) dx \\
&== \frac{1}{b-a} \left[ \frac{x^3}{3} - \frac{bx^2}{2} - \frac{ax^2}{2} + \frac{b^2x}{4} + \frac{2abx}{4} + \frac{a^2x}{4} \right]_a^b \\
&== \frac{1}{12(b-a)} (b^4 - a^4 - 6b^3a + 6a^3b) = \frac{(b-a)^2}{12}
\end{aligned}
$$

Or another way is by $Var[X] = E[X^2] - E[X]^2$

$$
\begin{aligned}
E[x^2] &= \int_a^b x^2 f(x)dx \\
&= \int_a^b x^2 \cdot \frac{1}{b-a} dx \\
&= \frac{1}{b-a} \int_a^b x^2 dx \\
&= \frac{1}{b-a} \left[ \frac{x^3}{3} \right]_a^b \\
&= \frac{1}{3(b-a)} (b^3 - a^3) \\
&= \frac{(b^2+ab+a^2)}{3} \\
Var[X] &= E[x^2] - E[x]^2 \\
&= \frac{(b^2+ab+a^2)}{3} - \left( \frac{b+a}{2} \right)^2 \\
&= \frac{4(b^2+ab+a^2) - 3(b^2+2ab+a^2)}{12} \\
&= \frac{(b-a)^2}{12}
\end{aligned}
$$

Now for Y, the mean is as follows -

$$
\begin{aligned}
E[X] &= \sum_{x=0}^{\infty} x \cdot P(X = x) \\
&= \sum_{x=0}^{\infty} x \cdot \frac{\lambda^x e^{-\lambda}}{x!} \\
&= \sum_{x=1}^{\infty} x \cdot \frac{\lambda^x e^{-\lambda}}{x!} \\
&= \lambda \sum_{x=1}^{\infty} \frac{\lambda^{x-1} e^{-\lambda}}{(x-1)!} \\
&= \lambda e^{-\lambda} \sum_{y=0}^{\infty} \frac{\lambda^y}{y!} \\
&= \lambda e^{-\lambda} e^{\lambda} \\
&= \lambda
\end{aligned}
$$

And calculation for variance is as follows -

$$
\begin{aligned}
Var[X] &= E[X^2] - E[X]^2 \\
\mathrm{E}[X\,(X-1)] &= \sum_{x=0}^{\infty} x\,(x-1) \cdot \frac{\lambda^x\, e^{-\lambda}}{x!} \\
&= \sum_{x=2}^{\infty} x\,(x-1) \cdot \frac{\lambda^x\, e^{-\lambda}}{x!} \\
&= e^{-\lambda} \cdot \sum_{x=2}^{\infty} x\,(x-1) \cdot \frac{\lambda^x}{x \cdot (x-1) \cdot (x-2)!} \\
&= \lambda^2 \cdot e^{-\lambda} \cdot \sum_{x=2}^{\infty} \frac{\lambda^{x-2}}{(x-2)!} \\
&= \lambda^2 e^{-\lambda} e^{\lambda} \\
&= \lambda^2
\end{aligned}
$$

Giving $E[X^2] = \lambda^2 + \lambda$, using linearity of expectations. Hence $Var[X] = \lambda$

# Grading Criteria

* Right calculation for the mean of X - 2.5 marks
* Right calculation for the variance of X - 2.5 marks
* Right calculation for the mean of Y - 2.5 marks
* Right calculation for the variance of Y - 2.5 marks