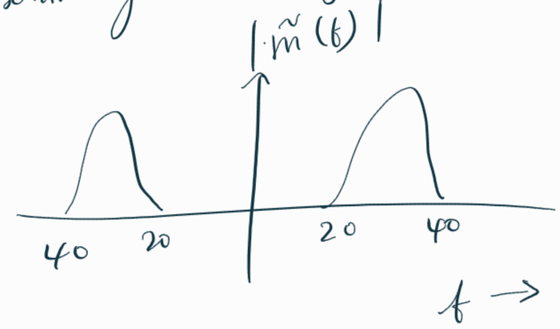
Mid 1 Solutions (Info & Comm 2023)

Q1.

(a) False:

Most signals relevant in engineering are band limited. So they can be sampled and reconstructed.

(b) False: Essentially the signal's spectrum looks like this.



This is a baseband signal with highest freq component = 40 kHz

Hence its bandwidth is 40 kHz.

(c) False: Note that $\mathcal{F}$ is not closed under complements. That is, if $A \in \mathcal{F}$, this means A contains stars each of which has mass $> C$. But this means $A^C = \Omega \setminus A$ contains stars which have mass $< C$ also (assuming that Ω has such stars, which is a reasonable assumption for this question).

(d) False:

$G(x)$ is not even a function. Observe the last three entries. The value of $G(x)$ is multiply defined for $x \geq 100 + 10^{-100}$.

(e) False: Such examples do exist, if we go purely by definition. For example, let A be any event whose probability is 0.
& B be any event disjoint from A (for ex: $B=\phi$ works).

Thus, $P(A\cap B) \leq P(B) = 0 = P(A).P(B)$.

Thus A & B are independent also.

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2. We know that the pdf of a Gaussian RV X with mean μ & variance σ^2 is as follows.

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\left[\frac{(x-\mu)^2}{2\sigma^2}\right]}, \quad \forall x \in \mathbb{R}.$$

Now, it is given that the mean $\mu = 15$.
Further it is also given that

$$\mathbb{E}[X^2] = 10\, Var(X) = 10\left(\mathbb{E}[X^2] - \mu^2\right)$$

$$= 10\left(\mathbb{E}[X^2] - 225\right)$$

$$\mathbb{E}[X^2] = 10\,\mathbb{E}[X^2] - 2250$$

Thus, $9\,\mathbb{E}[X^2] = 2250$.

$$\Rightarrow \mathbb{E}[X^2] = 2250/9 = 250$$

Thus $Var(X) = \sigma^2 = \mathbb{E}[X^2]/10 = 25$.

Thus the required pdf of the Gaussian is

$$f_X(x) = \frac{1}{5\sqrt{2\pi}} e^{-\left[\frac{(x-15)^2}{50}\right]}, \quad \forall x \in \mathbb{R}.$$

3.

(a) The sampled signal is given by

$$x_s(t) = \sum_{k \in \mathbb{Z}} x(kT)\,\delta(t-kT)$$

Observation 1:

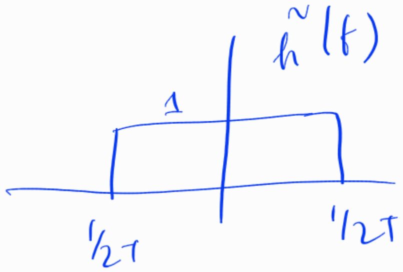
When this is passed through a filter with impulse response $h(t)$, we get output

$$\boxed{y(t) = x_s(t) * h(t) = \sum_{k \in \mathbb{Z}} x(kT)\, h(t-kT).}$$

Now, the ideal LPF with cut-off freq at $\frac{1}{2T}$ has spectrum

$$\tilde{h}(f) = \begin{cases} 1, & \forall f \text{ such that } |f| \le \frac{1}{2T} \\ 0 & \text{otherwise} \end{cases}$$

(b)



[Note: perhaps the student mentions the height as T or as some constant. That is ok]

Thus, $\tilde{h}(f) = \text{rect}(f/\tfrac{1}{T})$

$= \text{rect}(fT)$

& this has the IFT (by using the fact proved in class)

that $\boxed{h(t) = \frac{1}{T}\,\text{sinc}(t/T)}$ → observation ②

Using the two observations, we see that O/p of passing $x_s(t)$ through the Ideal LPF gives

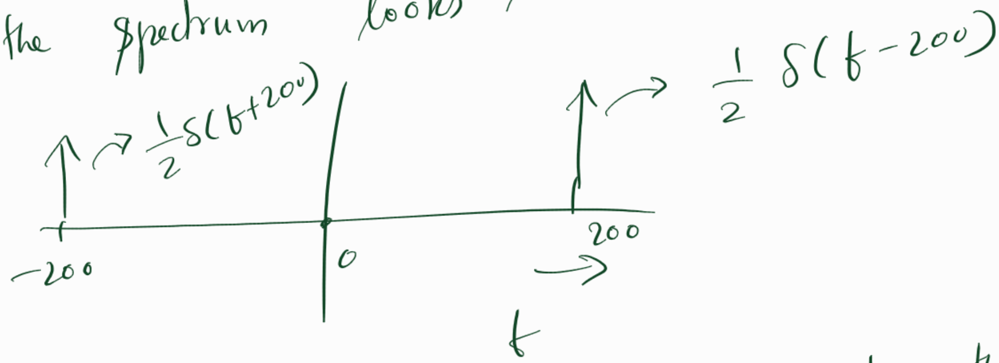
$$y(t) = \frac{1}{T} \sum_{k \in \mathbb{Z}} x(kT)\, \text{sinc}\left(t/T - k\right)$$

Note: If the ideal LPF was assumed as $T\, \text{rect}(t/T)$, then the O/P would have been $y(t) = \sum_{k \in \mathbb{Z}} x(kT)\, \text{sinc}(t/T - k)$

(b) The Nyquist sampling rate of $x(t) = \cos 2\pi \times 200 \times t$ is 400 samples/second (Correct unit if student writes 400 Hz But don't cut marks).

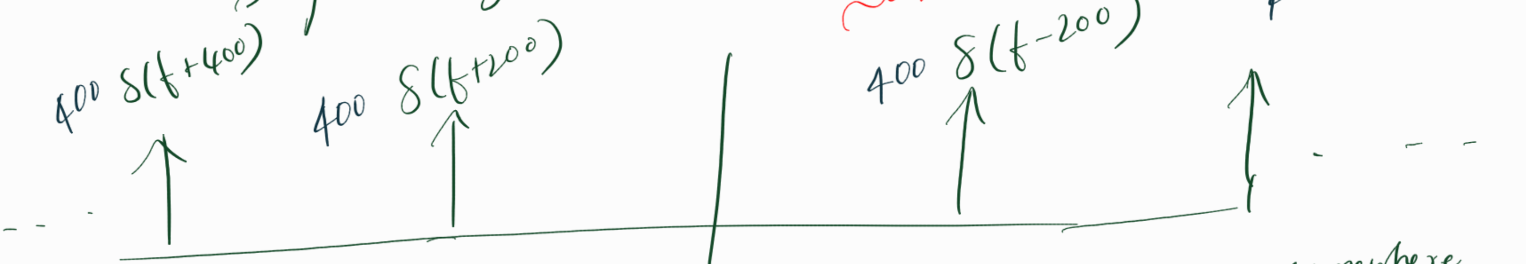
as the highest freq component is 200 Hz.

So the spectrum looks like



If we sample exactly at 400 samples/second, then the sampled signal has spectrum

This is 400 because of aliasing



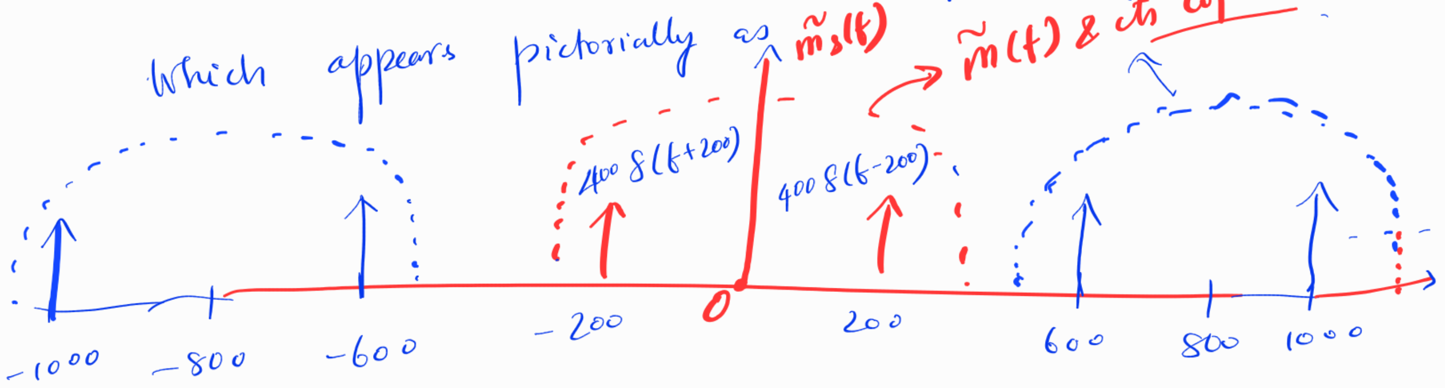
Now, passing this through a LPF with cutoff frequency somewhere between (200, 400) is good for reconstruction. However, in real world scenarios, sampling/LPF/sinusoids/etc. may not be ideal, so we go for sampling rate slightly higher than Nyquist.

(c) If we sample at rate of $\frac{1}{T} = 800$ samples per second, the sampling period $T = \frac{1}{800}$ s. The spectrum of

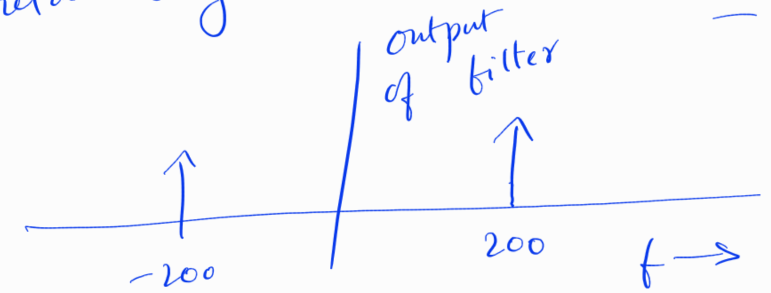
The sample signal is essentially given as

$$\tilde{x}_s(f) = \frac{1}{T} x(f) * \sum_{k \in \mathbb{Z}} \delta(f - k/T)$$

Which appears pictorially as



→ Passing this through ideal LPF with cut-off 400 Hz we retain only the central part above —



Thus giving the original signal (except for some scaling constant).

Gvn $p = 0.2$; $n = 100$.

4. (a) The appropriate prob distribution would be the binomial distribution. Let N = no of flips.

$$P(N=k) = \binom{n}{k} p^k (1-p)^{n-k} \rightarrow (1)$$

To prove that this is a valid PMF, we need to check that (a) $P(N=k) \geq 0$ (which is true as all quantities in RHS of (1) are ≥ 0).

& (b) $\sum_{k=0}^{n} P(N=k) = 1$.

The proof of (b) follows by writing

$$1 = (p + (1-p))^n = \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k}$$

(b) Expected no of flips.

$$\sum_{k=0}^{n} k \cdot \binom{n}{k} p^k (1-p)^{n-k}$$

$$= \sum_{k=1}^{n} k \binom{n}{k} p^k (1-p)^{n-k}$$

$$= \sum_{k'=0}^{n-1} n \binom{n-1}{k'} p^{k'+1} (1-p)^{n-1-k'}$$

$$= np \times \left[\sum_{k'=0}^{n-1} \binom{n-1}{k'} p^{k'} (1-p)^{(n-1)-k'} \right]$$

$$\boxed{\text{using } \binom{n-1}{k-1} = \frac{k}{n}\binom{n}{k} \text{ \& } k' = k-1}$$

$$= np \times 1 = np. = 100 \times 0.2 = 20$$ [this number is needed]

(c) $$P(N \geq 71) = \sum_{k=71}^{100} \binom{100}{k} \left(\frac{1}{5}\right)^k \times \left(\frac{4}{5}\right)^{100-k}$$

$$= \frac{1}{5^{100}} \sum_{k=71}^{100} \binom{100}{k} 4^{100-k}.$$