Given an equation, $x_1 + x_2 + x_3 + \dots + x_r = n$

Sum = n units → We need to divide those units in r parts, so we put separators between them

$$\underbrace{0000 \dots 000}_{n \text{ (units)}} \qquad \underbrace{|\,|\,| \dots |\,|\,|}_{(r-1) \text{ (separators)}}$$

ex. $a+b=4$

	a	b
00001	4	0
00010	3	1
00100	2	2
01000	1	3
10000	0	4

No. of non-negative solutions will be equal to number of ways of arranging these $(n+r-1)$ objects (which are identical)

$$= \frac{(n+r-1)!}{n!\,(r-1)!} = \boxed{{}^{n+r-1}C_{r-1}}$$

Suppose given $x_1 \geq 3$, $x_2 \geq 4$, ... etc., put $x_1 = 3 + k_1$, $x_2 = 4 + k_2$, ... etc. So the eq^n simplifies to

$$k_1 + k_2 + \dots + x_r = n - 3 - 4 - \dots = k'$$

No. of non-negative sol^{n}s of this eq^n = $\boxed{{}^{k'+r-1}C_{r-1}}$

= no. of sol^{n}s of prev. eq^n as to every k_i corresponds an x_i.

Suppose given, $a_1 x_1 + a_2 x_2 + \dots + a_r x_r = n$

Number of non-negative sol^{n}s of this eq^n

= Number of ways of getting (i.e. the coefficient of) x^n in eq^n

$$(x^0 + x^{a_1} + x^{2a_1} + \dots)(x^0 + x^{a_2} + x^{2a_2} + \dots) \dots (x^0 + x^{a_r} + x^{2a_r} \dots)$$

i.e. $\boxed{\prod_{i=1}^{r} \left(\sum_{k=0}^{\infty} x^{ka_i} \right)}$

because the expansion gives (for a rough estimate)

$$\dots + x^{2a_1+a_2+3a_3+\dots+6a_r} + x^{a_1+2a_2+4a_3+\dots+2a_r} + \dots$$

Suppose we have 2 sol^{n}s $(2,1,3,\dots,6)$ & $(1,2,4,\dots,2)$

$$\Rightarrow \dots + x^n + x^n + \dots = \dots + 2x^n + \dots$$

So, in the coefficient we get 2 as we ~~here~~ assumed that there are these 2 sol^{n}s in given locality.
So, coeff (of x^n) = no. of sets of sol^{n}s. of $\sum_{i=1} x_i = n$

⇒ no. of non-negative sol^n = coeff of x^n in

$$(1-x^{a_1})^{-1}(1-x^{a_2})^{-1}(1-x^{a_3})^{-1}\dots(1-x^{a_r})^{-1}$$

☆ Coeff of x^r in $(1-x^k)^{-n}$

$= \text{Coeff of } y^{r/k} \text{ in } (1-y)^{-n} = \boxed{{}^{(n+\frac{r}{k}-1)}C_{(r/k)}}$

i.e. Coeff of x^r in $(1-x)^{-n} = {}^{n+r-1}C_r$ (repeated selections)

Working method

$x+y+z=3$ number in right + signs in the LHS

no. of non-negative integral sol^n = ${}^{5}C_{2}$ signs in LHS

no. of +ve integral sol^n, i.e. $x, y, z \geq 1$

$\underset{1}{x}+\underset{1}{y}+\underset{1}{z}=3 \Rightarrow x+y+z=0 \Rightarrow {}^{2}C_{2}$

Aliter, multinomial method, no: of non-negative integral sol^n

$= \text{coeff of } x^3 \text{ in } (1+x+x^2+\ldots)(1+x+x^2+\ldots)(1+x+x^2+\ldots)$

$= \text{coeff of } x^3 \text{ in } (1-x)^{-3}$

$= {}^{3+3-1}C_{3} = {}^{5}C_{2} \; (= {}^{5}C_{3})$

no. of +ve integral sol^n

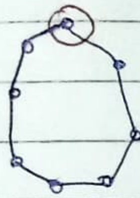
$= \text{coeff of } x^3 \text{ in } (x+x^2+\ldots)^3$

$= \text{coeff of } x^3 \text{ in } x^3(1-x)^{-3}$

$= \text{coeff of } x^0 \text{ in } (1-x)^{-3}$

$= {}^{3+0-1}C_{0}$

2. Making r-sided polygon by joining the vertices of n-sided convex polygon such that no side of the r-sided polygon is ~~eq~~ common to any side of n-sided polygon.



Selecting one point, ${}^nC_1$ ——— (I)
Let the other $(r-1)$ pts. be a_i, $i \in [1, r-1] \cap \mathbb{N}$

$$\underset{}{x_1}\ \underset{a_1}{|}\ x_2\ \underset{a_2}{|}\ x_3\ \underset{a_3}{|}\ x_4 \cdots x_{r-1}\ \underset{a_{r-1}}{|}\ x_r$$

x_i are unselected pts. b/w a_{i-1} and a_i
$\Sigma x_i = x_1 + x_2 + x_3 + \dots + x_{r-1} = n - r$
ATQ, $x_i \geq 1$ (No side is common to polygons)

$\Rightarrow x_1 + x_2 + x_3 + \dots + x_{r-1} = n - 2r$
no. of sol^{n}s of this eq^n $\equiv$ no. of ways of selecting polygons
$= {}^{n-2r+r-1}C_{r-1} = {}^{n-r-1}C_{r-1}$ ——— (II)

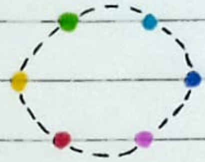
Doing ${}^nC_1$, we repeat sets of vertices of polygon
e.g. x_1 selected $\rightarrow (x_1, x_2, x_3)$ then x_2 selected $\rightarrow$ (x_1, x_2, x_3) & since these represent same polygons, we have to divide by no. of times they are repeated.

$\Rightarrow$ Total no. of ways $= \frac{1}{r} \times {}^nC_1 \times {}^{n-r-1}C_{r-1}$

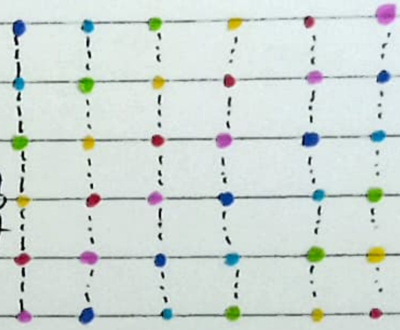
$$\boxed{= \left(\frac{n}{r}\right) {}^{n-r-1}C_{r-1}}$$

CIRCULAR PERMUTATIONS

Consider an arrangement of blue, cyan, green, yellow, red and magneta beads in a circle.



For this particular arrangement of six beads, there are six ways to list the arrangement in counterclockwise order depending on whether we start the list with the blue, cyan, green, yellow, red, or magneta bead.



Conversely, each of these six linear arrangements of these six beads corresponds to six linear arrangements.
i.e.

1 circular arrangement corresponds to 6 linear arrangements
or, generally, for n things,

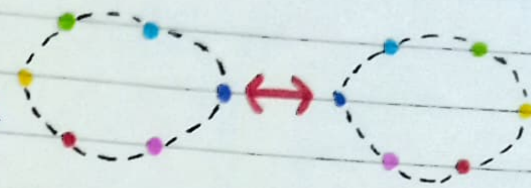
1 circular arrangement corresponds to n linear arrangements

Total linear arrangements $= n!$

$\therefore$ Total Circular permutations $= \dfrac{n!}{n}$

$$\boxed{= (n-1)!}$$

In BRACELETS OR NECKLACES, clockwise or counterclockwise does not matter, as rotation of one about any axis on this plane yield the other.



So, a pair of prev arrangements is counted once.
Hence, total arrangements $= \boxed{\frac{1}{2}(n-1)!}$