PH 534 Autumn 2024-2025 – Quantum information and computation

Practice problems Set 3
(Dated: October 2024)

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1. LOCC transformations for pure bipartite states

a) In the following set of transformations, the Hilbert space is $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ and $\dim(\mathcal{H}_A) = \dim(\mathcal{H}_B) = 6$. Please state, what deterministic LOCC transformations (one-way, two-way, no-way) are possible between the states $|\phi\rangle$ and $|\psi\rangle$, when

- $|\phi\rangle = \frac{1}{\sqrt{3}}(|00\rangle + |11\rangle + |55\rangle)$ and $|\psi\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$
- $|\phi\rangle = \frac{1}{\sqrt{2}}|00\rangle + \sqrt{\frac{2}{5}}|11\rangle + \frac{1}{\sqrt{10}}|55\rangle$ and $|\psi\rangle = \sqrt{\frac{3}{5}}|00\rangle - \frac{1}{\sqrt{5}}|10\rangle + \frac{1}{\sqrt{5}}|22\rangle$
- $|\phi\rangle = \frac{1}{2}(|00\rangle + |11\rangle + |22\rangle + |33\rangle)$ and all $|\psi\rangle \in \mathcal{H}$.

b) Roughly how many singlets are needed to construct 100 copies of the state,

$$|\phi\rangle = \mathcal{N}(|12\rangle - 3|03\rangle - |04\rangle + 2|34\rangle),$$

under the LOCC protocol? (Here, $\mathcal{N}$ is the normalization constant)

c) Define the majorization relation on pair of vectors. Given two vectors:

$$\mathbf{a} = \{a_1, a_2, \dots, a_N\} \text{ and } \mathbf{b} = \{b_1, b_2, \dots, b_N\},$$

we can define $\mathbf{a} \otimes \mathbf{b} = \{a_1 b_1, a_1 b_2, a_1 b_3, \dots, a_N b_N\}$, which is similar to forming diagonal matrices with entries $\mathbf{a}$ and $\mathbf{b}$ and taking the tensor product of the two. Now, prove that if $\mathbf{a} \prec \mathbf{c}$ and $\mathbf{b} \prec \mathbf{d}$ then $\mathbf{a} \otimes \mathbf{b} \prec \mathbf{c} \otimes \mathbf{d}$. Interpret this mathematical result physically in terms of LOCC on pure bipartite quantum systems.

d) Show that for the state $|\phi\rangle = \sqrt{\frac{2}{5}}(|00\rangle + |11\rangle) + \frac{1}{\sqrt{10}}(|22\rangle + |33\rangle)$ we cannot have a deterministic transformation under LOCC to the state $|\psi\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{2}(|11\rangle + |22\rangle)$, but the transformation, $|\phi\rangle \otimes |\chi\rangle \longrightarrow |\psi\rangle \otimes |\chi\rangle$, *is possible* under LOCC, for the state $|\chi\rangle = \sqrt{\frac{3}{5}}|00\rangle + \sqrt{\frac{2}{5}}|11\rangle$.

2. Entanglement witness, mixed state entanglement and the PPT criterion

a) Write the state $\rho_{AB} = \frac{1}{2}|\phi^+\rangle\langle\phi^+| + \frac{2}{5}|\psi^-\rangle\langle\psi^-| + \frac{1}{10}|01\rangle\langle 01|$ out as a matrix, and determine if it is entangled or not, where $|\phi^+\rangle$ and $|\psi^-\rangle$ are two of the Bell states.

b) Define the *Werner state* for two qubits as $\rho_{AB}(p) = (1-p)\frac{\mathbb{I}}{4} + p|\psi^-\rangle\langle\psi^-|$, obtained as a convex combination of the normalized maximally entangled singlet and the maximally mixed identity state. For what range of p is ρ_{AB} a valid quantum state, and for where is it entangled?

c) Construct a witness W that detects entanglement by considering a maximally entangled pure state $|\Psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$, using the following approach. Represent the state $|\Psi\rangle$ using its Schmidt decomposition, and define a Hermitian operator,

$$W = \mathbb{I} - \alpha\, |\Psi\rangle\langle\Psi|.$$

Here $\alpha > 0$ is some number, which we shall try to fix such that W is an entanglement witness. Now, explain why imposing the defining property of entanglement witness for pure product states will mean it also holds for all separable mixed states $\sigma_{AB} \in \mathcal{D}_{sep}$.

d) Show that W is an entanglement witness for $\alpha = d = \dim(\mathcal{H})$, and that its associated hyperplane actually touches the set of separable state at the product state points $|\varphi^*\rangle \otimes |\varphi\rangle$. Provide an example of a state ρ_{AB} for which W detects entanglement.

e) Consider a bipartite, pure quantum state $|\psi\rangle_{AB}$. Write down $|\psi\rangle\langle\psi|^{T_B}$ using a Schmidt decomposition $|\psi\rangle_{AB} = \sum_{k=1}^{M} \sqrt{\lambda_k}\, |k\rangle_A \otimes |k\rangle_B$.

Hence, show that $|\psi\rangle\langle\psi|^{T_B}$ has $M^2 - M$ entangled eigenvectors, with eigenvalues of the form $\pm\alpha$, and the remaining M eigenvectors (with non-zero eigenvalues) are product states. (Perhaps start with the case $M = 2$ and generalize).

f) Find an entanglement witness arising from the negative partial transpose condition.

3. Choi-Jamiolkowski isomorphism

a) Show if the following single system operators are positive but not completely positive or completely positive:

i) $\mathcal{E}(\rho) = \mathrm{Tr}[\rho].\mathbb{I} - \rho$ and ii) $\mathcal{E}(\rho) = \mathrm{Tr}[\rho].\mathbb{I} - \rho^T$

b) Prove that every single system CPTP operator $\mathcal{E}(\rho)$ is associated with an (unnormalized) bipartite quantum state $\mathcal{J}(\mathcal{E}) = (\mathcal{E} \otimes \mathbb{I})|\mathrm{vec}(\mathbb{I})\rangle\langle\mathrm{vec}(\mathbb{I})|$.

Show that if $\mathcal{J}(\mathcal{E})$ is a pure state, then $\mathcal{E}(\rho)$ is a unitary operator.

c) Show that for any valid entanglement witness $W = \mathcal{J}(\mathcal{E})$, it is necessary to show that $\langle\phi|\, \mathcal{J}(\mathcal{E})|\phi\rangle \geq 0$, where $|\phi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ and $|\phi\rangle \in \mathcal{D}_{sep}$.

4. Computation of entanglement

Use Mathematica or Python (or any other language) to do the computation. You are free to use in-built functions and packages to perform partial trace/ transposition.

a) Write the matrix form of the *Werner state* for two qubits:

$$\rho_{AB}(p) = (1-p)\frac{\mathbb{I}}{4} + p|\psi^-\rangle\langle\psi^-|.$$

b) Numerically estimate the quantum mutual information, negativity, logarithmic negativity, entanglement of formation and concurrence for the Werner state, ρ_{AB}.

c) Plot all the functions in the same figure as a function of the parameter p, excluding the part where the density matrix is not a valid quantum state.