PH 534 Autumn 2024-2025 – Quantum information and computation

Practice problems

(Dated: August 2024)

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1. Consider the vectors: $|0\rangle = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$, $|1\rangle = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$, and $|2\rangle = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$.

   Show that you can create an orthonormal basis, i.e. $\langle i|j\rangle = \delta_{ij}$, using these vectors.

2. Given a vector space, the choice of a basis is not unique. Consider the bases $B = \{u_1, u_2, u_3\}$ and $\tilde{B} = \{\tilde{u}_1, \tilde{u}_2, \tilde{u}_3\}$ for $V = \mathbb{R}^3$, where

$$u_1 = \begin{pmatrix} -3 \\ 0 \\ -3 \end{pmatrix}, u_2 = \begin{pmatrix} -3 \\ 2 \\ -1 \end{pmatrix}, u_3 = \begin{pmatrix} 1 \\ 6 \\ -1 \end{pmatrix}, \tag{1}$$

$$\tilde{u}_1 = \begin{pmatrix} -6 \\ -6 \\ 0 \end{pmatrix}, \tilde{u}_2 = \begin{pmatrix} -2 \\ -6 \\ 4 \end{pmatrix}, \tilde{u}_3 = \begin{pmatrix} -2 \\ -3 \\ 7 \end{pmatrix}. \tag{2}$$

   (a) Find the matrix that does this basis transformation.

   (b) Express the vector $\begin{pmatrix} -5 \\ 8 \\ -5 \end{pmatrix}$ in terms of both the bases, B and $\tilde{B}$.

3. As with Hermitian operators, an anti-Hermitian operator is defined as, $B^\dagger := -B$. Prove that the eigenvalues of non-degenerate (all eigenvalues are distinct), anti-Hermitian operator B are purely imaginary.

4. Consider the state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ where $|\alpha|^2 + |\beta|^2 = 1$.

   (a) Find the expectation value of the observable $\sigma_x, \sigma_y, \sigma_z$.

   (b) What are the expectation values for $\sigma_x, \sigma_y, \sigma_z$ for $\alpha = 1/\sqrt{2}$ and $\beta = 1/\sqrt{2}$.

(c) When $\alpha = \cos\theta$ and $\beta = \sin\theta e^{i\phi}$, find the expectation value for the observable $\hat{O} = \sin\alpha\cos\beta\sigma_x + \sin\alpha\sin\beta\sigma_y + \cos\alpha\sigma_z$.

5. Consider the Pauli matrices σ_x, σ_y, and σ_z. Refer to the Notes to study similarity transformations that change basis.

   (a) Write the Pauli matrices in the basis of σ_x.

   (b) Write the Pauli matrices in the basis of σ_y.

6. Consider a two-qubit system in the state $|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$.

   (a) Compute the expectation value of $\sigma_z \otimes \sigma_z$, $\sigma_x \otimes \sigma_x$ and $\sigma_y \otimes \sigma_x$.

   (b) Compute $\rho = |\psi\rangle\langle\psi|$ and take a partial trace over the second qubit. Is the resulting matrix pure or mixed?

7. Consider the following three-qubit states

   (a) $|\text{GHZ}\rangle_{123} = \frac{|000\rangle+|111\rangle}{\sqrt{2}}$.

   (b) $|\text{W}\rangle_{123} = \dfrac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle)$.

   Find partial trace $\text{Tr}_1\{\, |GHZ\rangle\, \langle GHZ|\, \}$, $\text{Tr}_{12}\{\, |GHZ\rangle\, \langle GHZ|\, \}$, $\text{Tr}_1\{\, |W\rangle\, \langle W|\, \}$ and $\text{Tr}_{12}\{\, |W\rangle\, \langle W|\, \}$.

8. We know that the evolution of a pure state $|\psi\rangle$ is given by

$$\frac{d\,|\psi\rangle}{dt} = -iH\,|\psi\rangle\,, \tag{3}$$

   where H is the system Hamiltonian. Using this, derive an evolution equation for a density matrix given by $\rho = \sum_i |\psi_i\rangle\,\langle\psi_i|$.

9. For a state evolving under a time-dependent Hamiltonian $H(t) = \omega(t)\sigma_z$,

   (a) Suppose $\omega(t) = A$ is a constant for all times, determine the time evolution operator $U(t)$ for a time t.

   (b) find the expectation value of σ_x at time t.

   (c) Compute the expectation value of σ_z for this state as a function of time.

   (d) Compute the state $|\psi(t)\rangle$ if the initial state is $|\psi(0)\rangle = |0\rangle$.

10. Consider the Hadamard gate H, which allows for the following transformation:

$$H|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle); \quad H|1\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

   (a) Find the matrix form of H in the $\{|0\rangle, |1\rangle\}$ basis.

   (b) Find the sum of outerproduct form of H, i.e. $H = \sum_{i,j} h_{i,j}|i\rangle\langle j|$.

   (c) Show that H is unitary and preserves the norm of any arbitrary state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$.

11. Imagine a Stern-Gerlach setup, however with an inbuilt "demon" that sorts $|0\rangle$ and $|+\rangle$ with probability 2/3 and 1/3 respectively.

   (a) Write down the density matrix (ρ) for this system.

   (b) Find the eigenvalues of ρ and prove that it is a valid quantum state.

   (c) Is this a pure state or a mixed state?

   (d) Find the expectation value of the operator σ_z for this state.

12. For $|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$, find out the range of p for which the sets of density matrices: $\rho = p|\psi\rangle\langle\psi| + (1-p)\mathbb{I}_2/4$ are valid density matrices. $\mathbb{I}_2$ is the identity operator in the two-dimensional Hilbert space.

13. An operator of the form $\rho = \frac{1}{2}(\mathbb{I}_2 + \vec{r}.\vec{\sigma})$ gives the position of a point in a unit ball. Here, $\mathbb{I}_2$ is the identity operator, $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ and $\vec{r} = (r_x, r_y, r_z)$, $|\vec{r}| \leq 1$.

   (a) Show that this is a valid density operator.

   (b) Find out where eigenstates of σ_x, σ_y and σ_z will lie on the sphere, sketch them.

   (c) Check that the surface of the sphere is formed by all the pure states.

   (d) Identify the position of a state $\rho = \cos^2\delta|0\rangle\langle 0| + \sin^2\delta|1\rangle\langle 1|$. (express answer in polar coordinates)

14. Let T be a linear operator acting on a space of complex $d \times d$ matrices such that $T(A) \rightarrow A^T$, where A^T is the transpose of A. Consider a state ρ such that $r = 1, \theta = \pi/4, \phi = \pi/4$. Where on the Bloch sphere does $T(\rho)$ lie? Is $T(\rho)$ a valid density matrix?

15. Consider a quantum system described by a two-dimensional Hilbert space. The Positive Operator-Valued Measure (POVM) elements are given by:

$$E_1 = (1-p)\mathbb{I}_2/2 + p|0\rangle\langle 0|,$$

$$E_2 = (1-p)\mathbb{I}_2/2 + p|1\rangle\langle 1|,$$

where p is a real parameter with $0 \leq p \leq 1$, and $\mathbb{I}_2$ is the identity operator in the two-dimensional Hilbert space.

   (a) Show that the POVM elements E_1 and E_2 can be written as projectors in a higher-dimensional Hilbert space.

   (b) Identify the higher-dimensional space and construct the explicit projectors corresponding to E_1 and E_2.

16. Consider the two measurement schemes given below.

   (a) The first scheme $\Pi = \{\Pi_0, \Pi_1\}$, where

$$\Pi_0 = |0\rangle\langle 0|; \qquad \Pi_1 = |1\rangle\langle 1|,$$

   that measures the quantum state $\sigma = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$.

      i. Show that Π obeys the conditions required to define a measurement.

      ii. What is the probability of obtaining a measurement outcome 0? What about 1?

      iii. What is the probability of obtaining the following outputs: (i) (0, 0), (ii) (0, 1), (iii) (1, 0), (iv) (1, 1) in a sequential measurement?

   (b) Consider a different measurement $\mathcal{M} = \{M_0, M_1, M_2\}$, where

$$M_0 = \sqrt{\frac{2\cos\theta}{1+\cos\theta}}\,|0\rangle\langle 0|; \quad M_1 = \sqrt{\frac{1}{1+\cos\theta}}\,|\psi_+\rangle\langle\psi_+|; \quad M_2 = \sqrt{\frac{1}{1+\cos\theta}}\,|\psi_-\rangle\langle\psi_-|,$$

   and $|\psi_\pm\rangle = \sin(\theta/2)\,|0\rangle \mp \cos(\theta/2)\,|1\rangle$, that measures the same state σ as above.

      i. Show that $\mathcal{M}$ obeys the conditions required to define a measurement.

      ii. What is the probability of obtaining a measurement outcome 0? What about 1?

iii. What is the probability of obtaining the following outputs: (i) (0, 0), (ii) (0, 1), (iii) (1, 0), (iv) (1, 1) in a sequential measurement?

17. Check whether the operator forms a POVM measurement or not: $\mathcal{M}_i = p\,|\psi_i\rangle\,\langle\psi_i| + (1-p)\mathbb{I}_4/4$, where $|\psi_i\rangle$ are the four Bell states given by $|\psi_{1,2}\rangle = \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle)$ and $|\psi_{3,4}\rangle = \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle)$. $\mathbb{I}_4$ is the identity matrix in the two-qubit Hilbert space.

18. The exponential of a scalar λ times a matrix A (a representation of a linear operator, as we will learn later) is defined as $e^{\lambda A} = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} A^k$. Note here that $0!$ is defined as 1 and $A^0 = \mathbb{1}$, the identity matrix with the same dimensionality as A.

    (a) Consider two matrices X and Y. Expand $(X+Y)^2$.

    (b) Assume for a matrix P that $P^2 = P$. Evaluate $e^{\lambda P}$, where λ is a real scalar.

    (c) Prove that: $e^{-i\theta\sigma_x/2} = \cos(\theta/2)\mathbb{I}_2 - i\sin(\theta/2)\sigma_x$, where $\mathbb{I}_2$ is the identity matrix.

    (d) Given a matrix B, the matrix logarithm of B is defined as the matrix A such that $e^A = B$. Prove that $\log$ is a non-unique function (i.e., for every B, there are several A that satisfy the matrix formula given).

    (e) Prove that for two matrices, if $AB = BA$, then $\log(AB) = \log(A) + \log(B)$.

19. Unitary operators are defined by the relation $UU^\dagger = U^\dagger U := \mathbb{I}$. Show that the eigenvalues of unitary operators are complex numbers with unit amplitude. (hint: Let S be a matrix that diagonalizes U. Show that S diagonalizes $U^\dagger$ and use this information).

20. Consider the Bloch sphere representation:

    (a) Consider two normalized states $|\psi_{1,2}\rangle = \cos\left(\frac{\theta_{1,2}}{2}\right)|0\rangle + e^{i\phi_{1,2}}\sin\left(\frac{\theta_{1,2}}{2}\right)|1\rangle$. Find the condition on the Bloch angles for $|\psi_1\rangle + |\psi_2\rangle$ to be a normalized state as well.

    (b) Consider $|\psi_0\rangle = |0\rangle$, $|\psi_{1,2}\rangle = -\frac{1}{2}\,|0\rangle \pm \frac{\sqrt{3}}{2}\,|1\rangle$. Denote these three quantum states on the Bloch sphere.

    (c) Represent the quantum state $\rho = \begin{pmatrix} 1/3 & 0 \\ 0 & 2/3 \end{pmatrix}$ on the Bloch Sphere. What about the state $\rho = \begin{pmatrix} \frac{1}{3} & \frac{\sqrt{2}}{3}e^{-i\pi/4} \\ \frac{\sqrt{2}}{3}e^{i\pi/4} & \frac{2}{3} \end{pmatrix}$?

(d) Devise two separate experiments to produce the two states in the problem above. (For additional amusement, try to come up with a "dial" that goes from 0 to 1 with the following property: If the dial is set at 0, the machine produces the pure state and at 1 it produces the mixed state.).

(e) Consider the state $\rho = \frac{p}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + (1-p)\begin{pmatrix} \frac{1}{3} & \frac{\sqrt{2}}{3}e^{-i\pi/4} \\ \frac{\sqrt{2}}{3}e^{i\pi/4} & \frac{2}{3} \end{pmatrix}$, $0 \leq p \leq 1$. Prove that this is a good density matrix.

(f) For the state in the previous problem, find where this state is on the Bloch sphere as a function of p.