Module - I Part 2

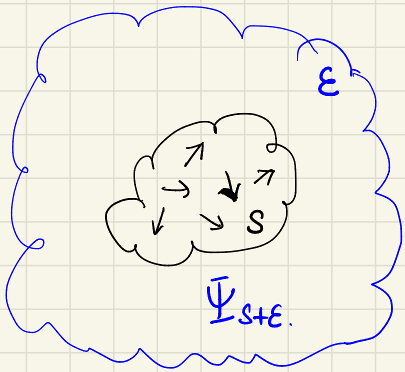
Foundations

Module 1

PH 534 QIC

② OPEN SYSTEMS

What is an open system?



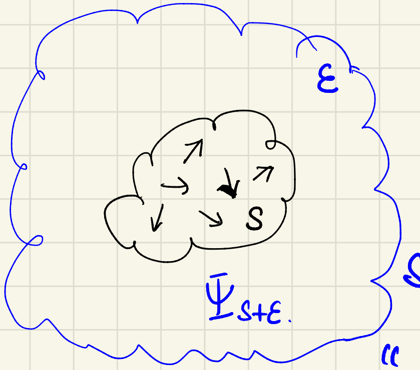
Note that the state of the larger system $S+\mathcal{E}$ is still given by $\Psi_{S+\mathcal{E}}$ which follows the earlier axioms or postulates.

Sometimes our system of interest S isolated. Rather it interacts or is embedded inside a larger space. Say, there exists an environment $\mathcal{E}$, such that $S+\mathcal{E}$ is the larger space. The system in S is no longer given by $|\psi\rangle$ but rather probabilities (p_i) of states $(|\psi_i\rangle)$ — the density matrix $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$

As such, we need to define a new set of rules to describe the unisolated or "open" system given by the density matrix ρ. These rules should be consistent with all axioms pertaining to the larger Hilbert space.

Why is this necessary?

- No quantum physical system of interest is truly isolated. To understand the interactions and dynamics "open system" description is necessary.
- To allow for robust error correction in these systems.
- Allow connection between quantum and statistical physics.



$$\underbrace{\Psi_{S+\mathcal{E}}}_{} \longrightarrow \underbrace{\rho_S}_{}$$

contains all the knowledge of the larger $S+\mathcal{E}$ Hilbert space

"Closed"

contains the knowledge of only the system S

"Open"

- Why does ρ_S only contain partial information or some ignorance / probability ??

An intuitive answer is because a lot of the information in $S+\mathcal{E}$ could be in the correlations between S and $\mathcal{E}$, which cannot be captured while defining ρ_S. On the other hand, if no correlation exists between S and $\mathcal{E}$, then ρ_S contains all the information about S i.e.,

$$\rho_S = |\psi\rangle\langle\psi|_S \quad \text{(note there is no probability any more)}$$

So, one can connect $|\psi\rangle$ with complete information about a quantum system or a "pure state" and $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$, with probabilistic or "mixed" state of a system.

Moreover, the notion of ρ can also be connected to imperfections in preparing a state as discussed later.

A) THE DENSITY MATRIX

Also called the "density operator" — is an equivalent description of quantum states that take into account the fact that the relevant system is not isolated or closed.

But let us begin our discourse with the alternate approach — ignorance or ambiguity of the quantum state, before we reconcile everything with the notion of open systems.

i) Imperfect state preparation — ensemble of states

Imagine a quantum experiment that outputs the quantum state $|\psi\rangle$ — but each time there is a certain error, say due to fluctuations in the laser or some magnetic field.

Say, we actually end up with the state $|\psi_i\rangle$ with probability p_i — an ensemble of pure states $\{p_i, |\psi_i\rangle\}$ Need not be orthonormal

Expectation value of some observable A : $\underbrace{\langle\psi_i|A|\psi_i\rangle}_{\text{from previous discussion}}$

What I mean here is that instead of just $\langle\psi|A|\psi\rangle$, you have to deal with an ensemble of $\{|\psi_i\rangle\}$ with some probability $\{p_i\}$

But now there is also the ensemble $\{p_i, |\psi_i\rangle\}$

$$\therefore \langle A\rangle = \sum_i p_i \langle\psi_i|A|\psi_i\rangle$$

$$= \sum_{i,j} p_i \langle\psi_i|A|j\rangle\langle j|\psi_i\rangle$$

$\{|j\rangle\}$ is some complete orthonormal basis

$$= \sum_{j,i} p_i \langle j|A|\psi_i\rangle\langle\psi_i|j\rangle$$

$$= \sum_j \langle j|A \underbrace{\sum_i p_i|\psi_i\rangle\langle\psi_i|}_{\rho}|j\rangle$$

$$\langle A\rangle = \mathrm{Tr}(A\rho)$$

The density matrix of the ensemble $\{p_i, |\psi_i\rangle\}$

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$$

It is to easy to see that the density matrix of a quantum state $|\psi\rangle$ is nothing but $|\psi\rangle\langle\psi|$.

* So many texts refer to ρ as the state of a system.

* A state $|\psi\rangle$ $(\rho = |\psi\rangle\langle\psi|)$ is called a "pure state", as opposed to an ensemble $\{p_i, |\psi_i\rangle\}$ $(\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|)$ which is called a "mixed state", implying it is a classical mixture of pure quantum states.

Operationally, $\rho = \sum_i p_i \rho_i$, for the ensemble $\{p_i, \rho_i\}$ where ρ_i can be either pure or mixed. *

ii) Properties of the density matrix ρ

- It is Hermitian — $\rho^\dagger = \sum_i p_i^* (|\psi_i\rangle\langle\psi_i|)^\dagger$

$$= \sum_i p_i |\psi_i\rangle\langle\psi_i| \quad (\because p_i \in \mathbb{R}, \forall i)$$

$$= \rho$$

- It has unit trace — $\mathrm{Tr}(\rho) = \mathrm{Tr}\left(\sum_i p_i |\psi_i\rangle\langle\psi_i|\right)$

$$= \sum_i p_i \underbrace{\mathrm{Tr}(|\psi_i\rangle\langle\psi_i|)^*}_{=1} = 1$$

(as $\sum_i p_i = 1$ by definition)

Exercise: Prove that $\mathrm{Tr}(|\psi\rangle\langle\psi|) = 1$ in words.

Exercise: Show that the one can define $\rho = \sum_i p_i \rho_i$, where ρ_i is a set of mixed state density matrices i.e., $\rho_i \neq |\psi_i\rangle\langle\psi_i|$.

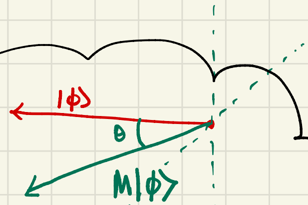
- φ is positive semidefinite : Any Hermitian linear operator $M \in \mathcal{L}(\mathcal{H})$ is positive semidefinite if $\langle\phi|M|\phi\rangle \geq 0$, $\forall\, |\phi\rangle \in \mathcal{H}$.

Exercise : Prove that $\langle\phi|M|\phi\rangle \in \mathbb{R}$ if M is Hermitian.

Now, $\langle\phi|\varphi|\phi\rangle = \sum_i p_i \langle\phi|\psi_i\rangle\langle\psi_i|\phi\rangle$

$= \sum_i p_i |\langle\phi|\psi_i\rangle|^2 \geq 0$

, as $p_i \geq 0\ \forall\, i$



$\cos\theta = \langle\phi|M|\phi\rangle \geq 0$

Both $|\phi\rangle$ and $M|\phi\rangle$ live on the same side in some heuristic sense. M does not flip the state $|\phi\rangle$.

If we diagonalize φ and write its eigendecomposition (or its spectral decomposition), then by extension of the above arguments, we need all its eigenvalues to be positive.

$\therefore\ \varphi = \sum_i p_i |\psi_i\rangle\langle\psi_i|$

$= \sum_i \lambda_i |e_i\rangle\langle e_i|$; where

$\{\lambda_i, |e_i\rangle\}$ is the eigenvalues and eigenstates of φ.

Then φ is positive semidefinite if $\lambda_i \geq 0$, $\forall\, i$. This holds true for all positive semidefinite matrices or operators.

- Measurements : Again we begin with a set of operators $\{M_k\}$ with outcomes $\{m_k\}$. Let φ correspond to some ensemble $\{p_i, |\psi_i\rangle\}$. Please note that $\sum_k M_k^\dagger M_k = \mathbb{I}$.

$$\underbrace{p(m_k|p_i)}_{\text{probability of outcome } m_k \text{, subject to the state being } |\psi_i\rangle \text{ with probability } p_i} = \langle\psi_i|M_k^\dagger M_k|\psi_i\rangle$$

So, the probability of getting the outcome m_k is

$$p(m_k) = \sum_i p(m_k|p_i)\, p_i$$

$$= \sum_i p_i \langle \psi_i | M_k^\dagger M_k | \psi_i \rangle$$

$$\mathrm{Tr}(M_k^\dagger M_k \rho) \leftarrow = \mathrm{Tr}\left(p_i M_k^\dagger M_k |\psi_i\rangle\langle\psi_i| \right)$$

Similarly, for the post measurement state :-

$$|\psi_i^k\rangle = M_k|\psi_i\rangle / \sqrt{p(m_k|p_i)}$$

$$\therefore \rho_k = \sum_i p_i |\psi_i^k\rangle\langle\psi_i^k|$$

$$= \sum_i p_i \; M_k|\psi_i^k\rangle\langle\psi_i^k| M_k^\dagger / p(m_k|p_i)$$

$$= \frac{M_k \rho M_k^\dagger}{\mathrm{Tr}(M_k^\dagger M_k \rho)} .$$

- Quantum evolution : for closed system the evolution is straight forward,

$$\rho \xrightarrow{U} \sum_i p_i \; U|\psi_i\rangle\langle\psi_i| U^\dagger = U \rho U^\dagger$$

In general, any quantum evolution is described a linear operator, $\rho \rightarrow \mathcal{E}(\rho)$, such that $\rho(t) = \mathcal{E}\rho(0)$ is also a valid density matrix.

Composite system — Again similar to the axioms of quantum mechanics, the Hilbert space is a tensor product and since density matrices are linear operators,

$$\rho \rightarrow \rho_1 \otimes \rho_2 \otimes \rho_3 \otimes \cdots \otimes \rho_N .$$

iii) The density matrix of a qubit

Let us consider an operational definition of a density matrix in the qubit Hilbert space. Therefore, ρ for a qubit is a 2×2 Hermitian matrix that satisfies $\mathrm{eigs}(\rho) \geq 0$ and $\mathrm{Tr}(\rho) = 1$.

$$\rho = \begin{pmatrix} a & c \\ c^* & b \end{pmatrix}; \quad a + b = 1$$

Now we know that all linear operators in the qubit space is spanned by the Pauli matrices and the identity:

$$\mathbb{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; \; \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \; \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}; \; \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

So any Hermitian ρ can be written as

$$\rho = \alpha\left(\mathbb{I} + x\sigma_x + y\sigma_y + z\sigma_z \right)$$

Now, $\mathrm{Tr}(\sigma_i)_{i=x,y,z} = 0$, therefore $\alpha = 1/2$ and we have

$$\rho = 1/2\left(\mathbb{I} + \vec{r}\cdot\vec{\sigma} \right)$$

where $\vec{\sigma} = \{\sigma_x, \sigma_y, \sigma_z\}$ and $\vec{r} = \{x, y, z\}$. For positivity condition, $\mathrm{eigs}(\rho) \geq 0$,

$$\mathrm{eigs}(\rho) = 1/2\left(1 \pm \sqrt{x^2+y^2+z^2} \right) \geq 0.$$

$\therefore$ For $\mathrm{eigs}(\rho) \geq 0$, we need $\vec{r}\cdot\vec{r} = x^2+y^2+z^2 \leq 1$.

Therefore, the length of the Bloch vector $\vec{r}$ is less than or equal to unity.

For $|\vec{r}| = 1$, eigs$(\rho) = 1, 0$ and $\rho = |\psi\rangle\langle\psi|$ and is therefore pure.

Importantly, if we consider a Bloch sphere, all vectors that lie on the surface of the sphere are pure quantum states.

Exercise : i) Find out where does the state $\rho = \frac{1}{2}\mathbb{I}$ lies on the Bloch sphere.

ii) Using the ensemble definition of ρ, show that $\mathrm{Tr}(\rho^2) \leq 1$. When is $\mathrm{Tr}(\rho^2) = 1$?

iv) The classical notion of ensembles — coherent v/s incoherent

In a previous section we discussed the importance of the superposition by looking at the state

$$|+\rangle = \frac{1}{\sqrt{2}}\{|0\rangle + |1\rangle\} \text{ in the } \sigma_z \text{ basis}$$

While the state gives $|0\rangle$ and $|1\rangle$ with probability $\frac{1}{2}$ when measured in the σ_z basis, there is always a preferred basis (σ_x in this case), where we get $|+\rangle$ with probability 1. However, for the state

$$\rho = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1|, \text{ again in } \sigma_z \text{ basis}$$

measurement in any basis will always give probability $\frac{1}{2}$.

$$\langle\psi(\theta,\phi)|\rho|\psi(\theta,\phi)\rangle = \frac{1}{2}$$

This distinguishes the coherent superposition in $|+\rangle$ with the incoherent mixing in ρ.

V) Composite quantum systems and the partial trace

We finally make the connection between density operators and composite quantum systems.

For example consider a bipartite system AB, with the Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$. A density matrix ρ_{AB} in this space is a positive semidefinite Hermitian matrix, with unit trace.

For two qubits: $\rho_{AB} = \rho_A \otimes \rho_B$ (product state)

where $\rho_A = \frac{1}{2}(\mathbb{I} + \vec{a}\cdot\vec{\sigma})$ and $\rho_B = \frac{1}{2}(\mathbb{I} + \vec{b}\cdot\vec{\sigma})$

Examples: $\rho_{AB} = |0\rangle\langle 0|_A \otimes |+\rangle\langle +|_B$ (pure state)

$\rho_{AB} = \mathbb{I}/2 \otimes \mathbb{I}/2$ (maximally mixed)

$\rho_{AB} = |\psi^-\rangle\langle\psi^-|$, where $|\psi^-\rangle = \frac{1}{\sqrt{2}}\{|01\rangle - |10\rangle\}$ (singlet state)

How do we find the state of the subsystem A? We do a partial trace over the rest of the system.

If $L(\mathcal{H})$ is the Hilbert space of all linear operators in $\mathcal{H}$, then partial trace is a linear map, such that $Tr_B : L(\mathcal{H}_A \otimes \mathcal{H}_B) \rightarrow L(\mathcal{H}_A)$. Here, we restrict to the state space of density operators.

$$\rho_A = Tr_B(\rho_{AB})$$

We trace out everything that is NOT A

$$= \sum_k (\mathbb{1}_A \otimes \langle k|_B)\, \rho_{AB}\, (|k\rangle_B \otimes \mathbb{1}_A)$$

where $\{|k\rangle_B\}$ is any orthonormal basis in $\mathcal{H}_B$.

If $\rho_{AB} = \rho_A \otimes \rho_B$; $Tr_B(\rho_{AB}) = \rho_A \otimes \underbrace{Tr(\rho_B)}_{=1}$

(Partial trace is trivial for product states)

Similarly, we can trace out A and we get,

$$\rho_B = Tr_A(\rho_{AB})$$ } Again, trace out everything that is not B

Let us consider the state,

$$\rho_{AB} = |\psi^-\rangle\langle\psi^-|$$
$$= \frac{1}{2}\{|01\rangle\langle 01| - |01\rangle\langle 10| - |10\rangle\langle 01| + |10\rangle\langle 10|\}$$

Using the basis : $\{|0\rangle_B, |1\rangle_B\}$ we get

$$\rho_A = Tr_B(|\psi^-\rangle\langle\psi^-|) = \mathbb{1}_A \otimes \langle 0|_B \rho_{AB}|0\rangle_B \otimes \mathbb{1}_A + \mathbb{1}_A \otimes \langle 1|_B \rho_{AB}|1\rangle_B \otimes \mathbb{1}_A$$

$$= \frac{1}{2}\{\mathbb{1}_A \otimes \langle 0|10\rangle\langle 10|0\rangle \otimes \mathbb{1}_A + \mathbb{1}_A \otimes \langle 1|01\rangle\langle 01|1\rangle \otimes \mathbb{1}_A\}$$

$$\therefore \quad \rho_A = \frac{1}{2}\{|1\rangle\langle 1| + |0\rangle\langle 0|\} = \underbrace{\frac{1}{2}\mathbb{I}_A}$$

reduced state on A for a pure singlet $|\psi^-\rangle_{AB}$

Exercise : Find the reduced ρ_B.

Why does partial trace work ?

From the axioms of QM, we know that any pure state in a composite system can be written in a joint basis such that if $\{|i\rangle_A\}$ is a basis in $\mathcal{H}_A$ and $\{|j\rangle_B\}$ is a basis in $\mathcal{H}_B$, then $\{|i\rangle_A \otimes |j\rangle_B\}$ forms a basis in $\mathcal{H}_A \otimes \mathcal{H}_B$

$$\therefore \quad |\psi\rangle_{AB} = \sum_{ij} \alpha_{ij}\, |i\rangle_A \otimes |j\rangle_B = \sum_{ij} \alpha_{ij} |i\rangle_A |j\rangle_B$$

If we wish to find the expectation value of observable M_A that acts of $\mathcal{H}_A$:

$$\langle M_A \rangle = \langle\psi|_{AB}(M_A \otimes \mathbb{I}_B)|\psi\rangle_{AB}$$

$$= \sum_{i,j} \alpha_{ij}^{*} \langle j|_B \langle i|_A (M_A \otimes \mathbb{I}_B) \sum_{i',j'} \alpha_{i'j'} |i'\rangle_A |j'\rangle_B$$

$$= \sum_{i,j,i',j'} \alpha_{ij}^{*} \alpha_{i'j'} \langle i|_A M_A |i'\rangle_A \otimes \underbrace{\langle j|_B \mathbb{I}_B |j'\rangle_B}_{\delta_{jj'}}$$

$$= \sum_{i,i',j} \alpha_{ij}^{*} \alpha_{i'j} \langle i|_A M_A |i'\rangle_A$$

$$= \mathrm{Tr}(M_A \rho_A) \Big\} \quad \rho_A = \mathrm{Tr}_B(|\psi\rangle\langle\psi|_{AB})$$

$$\mathrm{Tr}_B(|\psi\rangle\langle\psi|_{AB}) = \sum_k \langle k|_B \sum_{i,j,i',j'} \alpha_{ij}^{*} \alpha_{i'j'} |i'\rangle_A |j'\rangle_B \langle i|_A \langle j|_B |k\rangle$$

$$= \sum \alpha_{ij}^{*} \alpha_{i'j'} |i'\rangle\langle i|_A \otimes \langle k_B|j'\rangle_B \langle j|k\rangle_B$$

$$= \sum_{i,i',j} \alpha_{ij}^{*} \alpha_{i'j} |i'\rangle\langle i|_A = \rho_A$$

$$\therefore \mathrm{Tr}(M_A \rho_A) = \sum_k \langle k|_A M_A \rho_A |k_A\rangle$$

$$= \sum_{i,i',j,k} \langle k|_A \alpha_{ij}^{*} \alpha_{i'j} M_A |i'\rangle\langle \underbrace{i|k}_{\delta_{ik}}\rangle_A$$

$$= \sum_{i,i',j} \alpha_{ij}^{*} \alpha_{i'j} \langle i|M_A|i'\rangle$$

So, performing a measurement or calculating the expectation value of a subsystem in a composite is equivalent to doing the measurement directly on the reduced state.

Show that, $\mathrm{Tr}[(M_A \otimes \mathbb{I}_B)\rho_{AB}] = \mathrm{Tr}_A[M_A \rho_A]$.

vi) Purification of a mixed quantum state

All quantum states fall under what is often termed as the "Church of the Larger Hilbert space," which implies that all states are part of a larger quantum system that is pure and deterministic. For mixed states, this quite simple :

Again, you can also show this without taking any orthonormal set of states for ρ

Since ρ is a Hermitian operator, it has a spectral decomposition

$$\rho = \sum_i \lambda_k \underbrace{|k\rangle\langle k|}_{\text{orthonormal}} ; \quad \underbrace{\lambda_k \geq 0 \ \forall\, k}_{\rho \geq 0} ; \quad \underbrace{\sum_k \lambda_k = 1}_{\mathrm{Tr}(\rho) = 1}$$

Purification of ρ implies that there is a composite pure state $|\psi\rangle_{AB}$, such that $\rho_A \equiv \mathrm{Tr}\left(|\psi\rangle\langle\psi|_{AB}\right)$

$$|\psi\rangle_{AB} = \sum_k \sqrt{\lambda_k}\, |k\rangle_A \otimes |e_k\rangle_B$$

So, every mixed state ρ can always be viewed as arising from a global pure $|\psi\rangle_{AB}$, and the mixedness now arises due to the correlations in the state $|\psi\rangle_{AB}$, between A and B.

So this connects the randomness in ensemble preparation and the ambiguity to nonclassical correlations that existed because ρ_A is part of a larger quantum system.

Purifications are not unique :-

$$|\phi\rangle_{AB} = \left(\mathbb{1}_A \otimes \mathcal{U}_B\right) |\psi\rangle_{AB}$$

also gives you; $\rho_A = \mathrm{Tr}_B\left(|\phi\rangle\langle\phi|_{AB}\right)$, $\because\ \mathcal{U}_B^{\dagger}\mathcal{U}_B = \mathbb{I}$

vii) Pure state decompositions of a quantum state

For an n-dimensional quantum system A, the state ρ_A is given by :

$$\rho_A = \sum_{i=1}^{n} \lambda_i |i\rangle\langle i|$$

where $\{\lambda_i, |i\rangle\}$ is the eigendecomposition of ρ_A. Let there also exist some other pure state decomposition, such that

$$\rho_A = \sum_{k=1}^{m} p_k |\phi_k\rangle\langle\phi_k|$$

Then, we have : $\sqrt{p_k}\,|\phi_k\rangle = \sum_{i=1}^{n} u_{ki}\sqrt{\lambda_i}\,|i\rangle$

where u_{ki} are the elements of some unitary U.

Proof: $$|\psi\rangle_{AB} = \sum_i \sqrt{\lambda_i}\,|i\rangle_A \otimes |i\rangle_B$$

and $$|\phi\rangle_{AC} = \sum_k \sqrt{p_k}\,|\phi_k\rangle_A \otimes |k\rangle_C$$

Now; $|\phi\rangle_{AC} = (\mathbb{1} \otimes U)|\psi\rangle_{AB}$. Using $\{|i\rangle_A\}$

$$(\mathbb{1} \otimes {}_C\langle k|)|\phi\rangle_{AC} = \mathbb{1}_A \otimes {}_C\langle k|U|\psi\rangle_{AB}$$

$$\Rightarrow \sqrt{p_k}\,|\phi_k\rangle_A = \sum_i u_{ki}\sqrt{\lambda_i}\,|i\rangle_A$$

where $u_{ki} = {}_C\langle k|U|i\rangle_B$

Note : Here, $\{|i\rangle\}$ is the eigen decomposition of ρ_A but the second decomposition is arbitrary, so the above relation will hold for all pure state decompositions.

viii) The Schmidt decomposition

Any bipartite pure state $|\psi\rangle$ can be written as:

$$|\psi\rangle_{AB} = \sum_{i=1}^{r} \lambda_i \, |i\rangle_A \otimes |i\rangle_B$$

where $\{|i\rangle_A\}$ and $\{|i\rangle_B\}$ are a set of orthonormal basis in A and B. λ_i's are the Schmidt coefficients and 'r' is the Schmidt rank.

Importantly: $\rho_A = \sum_i \lambda_i^2 |i\rangle\langle i|_A$; $\rho_B = \sum_i \lambda_i^2 |i\rangle\langle i|_B$ have identical eigenvalues. Also; $\sum_i \lambda_i^2 = 1$.

$\{|i\rangle_A\}$ and $\{|i\rangle_B\}$ are the eigen basis for ρ_A and ρ_B.

Proof: $|\psi\rangle_{AB} = \sum_{i,j} a_{ij} \, |i\rangle_A \otimes |j\rangle_B$, where $\{|i\rangle_A\}$ and $\{|j\rangle_B\}$ are both orthonormal basis for A and B.

$|\psi\rangle_{AB} = \sum_{i,j} a_{ij} |i\rangle_A |j\rangle_B = \sum_i |i\rangle_A \otimes |\tilde{i}\rangle_B$, where $\{|\tilde{i}\rangle_B = \sum_j a_{ij} |j\rangle_B\}$. Now; in general, $\rho_A = \lambda_i^2 |i\rangle\langle i|_A$

Also from; $\rho_A = \mathrm{Tr}_B(|\psi\rangle\langle\psi|_{AB}) = \mathrm{Tr}_B\Big(\sum_{i,j} |i\rangle\langle j|_A \otimes |\tilde{i}\rangle\langle \tilde{j}|_B\Big)$

$$= \sum_{i,j} |i\rangle\langle j|_A \otimes \sum_k \langle k|\tilde{i}\rangle\langle\tilde{j}|k\rangle_B$$

Compare with

$$= \sum_{i,j} |i\rangle\langle j|_A \otimes \sum_k \langle \tilde{j}|k\rangle\langle k|\tilde{i}\rangle_B$$

$\rho_A = \sum_i \lambda_i^2 |i\rangle\langle i|_A \longleftarrow$

$$= \sum_{i,j} \underbrace{\langle \tilde{j}|\tilde{i}\rangle_B}_{\lambda_i^2 \delta_{ij}} |i\rangle\langle j|_A \quad \Big\} \because \sum_k |k\rangle\langle k| = \mathbb{I}$$

$\therefore \{\lambda_i |\tilde{i}\rangle_B\}$ forms an orthonormal basis in B, which gives

$$|\psi\rangle_{AB} = \lambda_i \, |i\rangle_A \otimes |\tilde{i}\rangle_B$$

ix) The ambiguity of an ensemble

A linear combination of two density matrices is also a density matrix (ensemble interpretation):

$$\rho = p\,\rho_1 + (1-p)\,\rho_2, \text{ where } 0 \leq p \leq 1.$$

Exercise: Check that ρ satisfies the criteria for a density matrix if ρ_1 and ρ_2 do.

So, density matrices are a convex subset of the real vector space of Hermitian operators.

Importantly, no PURE quantum state, i.e., $\rho = |\psi\rangle\langle\psi|$ can be expressed as a convex combination of density matrices.

Let us now consider the density matrix:

$$\rho_1 = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1| = \frac{1}{2}\mathbb{I}_2$$

where $\frac{1}{2}\mathbb{I}_2$ is the maximally mixed state (identity matrix).

But consider the states: $|\pm\rangle = \frac{1}{\sqrt{2}}\{|0\rangle \pm |1\rangle\}$

Now; $$\rho_2 = \frac{1}{2}|+\rangle\langle +| + \frac{1}{2}|-\rangle\langle -| = \frac{1}{2}\mathbb{I}_2$$

So, $\rho_1 = \rho_2$ regardless of the underlying ensemble states, which is weird. Let us say that we have a cat, which is either alive ($|0\rangle$) or dead ($|1\rangle$) with probability $\frac{1}{2}$. But, now if the cat is in the more complicated Schrodinger state ($|\pm\rangle$), with superpositions, but we still get the same density matrix.

There is no operation that can distinguish the two and this feature has no classical analogue.

B) QUANTUM OPERATIONS

The question we now ask is how can we study the dynamics of open quantum systems. We know that in a closed system, all dynamics in the Hilbert space is governed by linear operators that are unitary. However, we now know that for realistic settings a quantum system is open and represented by density matrices and all dynamics must account for interaction with the external environment, which give rise to noise.

In open systems, all dynamics are governed by quantum operators, which are a set of linear operators defined on the Hilbert space. The quantum operator formalism can account for both unitary and noisy evolution, as well as measurements.

A quantum operator is a linear map, $\varphi \to \sigma = \mathcal{E}(\varphi)$, which is convex (preserves probabilities):

$$\mathcal{E}(p\varphi_0 + (1-p)\varphi_1) = p\mathcal{E}(\varphi_0) + (1-p)\mathcal{E}(\varphi_1)$$

for all states φ_0, φ_1 and for all $0 \leq p \leq 1$. For a deterministic process, the map $\mathcal{E}$ always returns a valid quantum state $\mathcal{E}(\varphi)$, which means $\mathrm{Tr}[\mathcal{E}(\varphi)] = 1$, $\mathrm{eigs}[\mathcal{E}(\varphi)] \geq 0$ and $\mathcal{E}(\varphi)^\dagger = \mathcal{E}(\varphi), \forall \varphi$.

A formal definition of a quantum operation from a quantum system A to a quantum system B is any linear mapping:

$$\mathcal{E}: \mathcal{L}(\mathcal{H}_A) \to \mathcal{L}(\mathcal{H}_B), \ \forall \varphi \in \mathcal{L}(\mathcal{H}_A)^{\#}$$

There are a few different approaches to understand quantum operators, first from a purely operational view and second from a more physical, intuitive notion. Let us begin with the operational point of view.

The space of linear operators is typically denoted by $\mathcal{L}(\mathcal{H})$ or $\mathcal{B}(\mathcal{H})$.

i) Operator sum representation — Kraus operators

We begin with "an operator sum" representation of a quantum operator (or also called a quantum channel), where $\mathcal{E}$ is defined as:

$$\mathcal{E}(\varphi) = \sum_i A_i \varphi A_i^{\dagger} ; \; \forall \sum_i A_i^{\dagger} A_i = \mathbb{1}$$

and $\mathcal{E}$ maps a quantum state φ to another $\sigma = \mathcal{E}(\varphi)$.

A quantum operator $\mathcal{E}(\varphi)$ must preserve trace and be positive. In fact, $\mathcal{E}(\varphi)$ must be completely positive i.e, for all

$$\mathcal{E}_A(\varphi_A) \geq 0, \forall \varphi_A \in \mathcal{H}_A \quad (\text{positive semidefinite})$$

it must also satisfy: $(\mathcal{E}_A \otimes \mathbb{I}_B)\varphi_{AB} \geq 0, \forall \varphi_{AB} \in \mathcal{H}_A \otimes \mathcal{H}_B$.

All completely positive, trace preserving (CPTP) maps, $\mathcal{E}$, have a simple description is the operator sum representation, called the Kraus operators, $\{A_i\}$.

ii) Completely positive maps that preserve the trace

Quantum operators $\mathcal{E}(\varphi)$ as defined by the Kraus operators $\{A_i\}$ is given by $\mathcal{E}(\varphi) = \sum A_i \varphi A_i^{\dagger}$, where $\sum_i A_i^{\dagger} A_i = \mathbb{I}$.
It is straightforward to show that $\mathcal{E}(\varphi)$ is completely positive.

Let, M be positive: $\langle \phi | M | \phi \rangle \geq 0$, $M \in \mathcal{L}(\mathcal{H}_A \otimes \mathcal{H}_B)$, $\forall \, |\phi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$

$(A_i^{\dagger} \otimes \mathbb{I})|\phi\rangle = |\psi_i\rangle$; $\therefore \langle \phi | (A_i \otimes \mathbb{I}) M (A_i^{\dagger} \otimes \mathbb{I}) |\phi\rangle = \langle \psi_i | M | \psi_i \rangle \geq 0$

which implies; $\sum_i \langle \psi_i | M | \psi_i \rangle \geq 0 \Rightarrow \sum_i \langle \phi | (A_i \otimes \mathbb{I}) M (A_i^{\dagger} \otimes \mathbb{I}) | \phi \rangle \geq 0$

$\therefore \; \langle \phi | (\mathcal{E} \otimes \mathbb{I}) M | \phi \rangle \geq 0$, for all positive M and therefore all φ.

Note: In general a quantum operator or map need not preserve the trace. Think about measurements, where $\mathcal{E}(\varphi) = E^{\dagger} \varphi E > 0$, which is a probability. Nielsen and Chuang has an interesting discussion in Section 8.2.3 and 8.2.4. For our purposes we restrict ourselves to CPTP maps and treat probabilistic measurements separately.

It is easy to show that Kraus operators also preserve the trace

$$\mathrm{Tr}\left[\mathcal{E}(\varphi)\right] = \mathrm{Tr}\sum_i A_i \varphi A_i^{+} = \sum_i \mathrm{Tr}(A_i \varphi A_i^{+})$$

$$= \sum_i \mathrm{Tr}(A_i^{+} A_i \varphi)$$

cyclicity of trace operation

$$= \mathrm{Tr}\Big(\underbrace{\sum_i A_i^{+} A_i}_{\sum_i A_i^{+}A_i = \mathbb{I}} \varphi\Big) = \mathrm{Tr}(\varphi) = 1$$

iii) Combining two quantum operations gives you a quantum operation

Suppose, $\mathcal{E}_1 = \{A_i\}$ takes you from quantum system A to system B and $\mathcal{E}_2 = \{B_i\}$ takes you from B to system C.

$$\mathcal{E}_2 \cdot \mathcal{E}_1(\varphi) = \mathcal{E}_2(\mathcal{E}_1(\varphi)) = \mathcal{E}_2 \sum_i A_i \varphi A_i^{+}$$

$$= \sum_i \mathcal{E}_2 A_i \varphi A_i^{+} = \sum_{i,j} B_j A_i \varphi A_i^{+} B_j^{+}$$

Now, $\mathcal{E}_3 = \{ C_{ij} = B_j A_i \}$ is a quantum operator if

$$\sum_{i,j} C_{ij}^{+} C_{ij} = \sum_{i,j} A_i^{+} B_j^{+} B_j A_i = \sum_i A_i^{+} \underbrace{\sum_j B_j^{+} B_j}_{\mathbb{1}} A_i$$

$$= \sum_i A_i^{+} A_i = \mathbb{1}$$

Therefore, $\mathcal{E}_3$ maps states from system A to system C.

iv) Environment and quantum operations

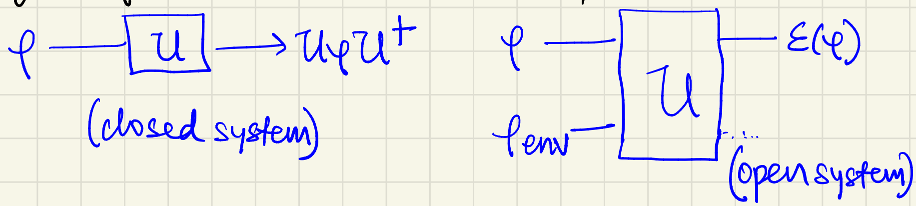
In the "operator sum" representation we defined quantum operations in an axiomatic manner. But the set of Kraus operators can also have a more intuitive and physical picture coming from the interaction of the system with an external environment.

Stinespring dilation and the "church of the larger Hilbert space"

A quantum operation $\mathcal{E}: B(\mathcal{H}_A) \to B(\mathcal{H}_B)$, but there exists an external environment E, such that the map can be written in terms of a unitary on quantum system A and E.

$$\mathcal{E}(\rho) = \mathrm{Tr}_E\left(\mathcal{U}(\rho \otimes |\eta\rangle\langle\eta|_E)\mathcal{U}^\dagger\right)$$

This is a natural way to describe the dynamics of an open system, where the system and its environment are part of a larger Hilbert space, where the joint system is closed and evolves under a unitary operator. But the system by itself evolves under a CPTP map.



All quantum operations can be reduced to unitaries and adding ancillas (isometries) and doing partial traces.

v) The Choi representation of a quantum operation

Some preliminaries: Vectorization of a matrix

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \longrightarrow |\mathrm{vec}(A)\rangle = \begin{pmatrix} a_{11} \\ a_{12} \\ a_{21} \\ a_{22} \end{pmatrix}$$

If X and Y are two matrices :-

$$XA \to |\mathrm{vec}(XA)\rangle = (X \otimes \mathbb{1})|\mathrm{vec}(A)\rangle$$

$$AY \to |\mathrm{vec}(AY)\rangle = (\mathbb{1} \otimes Y^T)|\mathrm{vec}(A)\rangle$$

$$(X \otimes Y)|\mathrm{vec}(A)\rangle = |\mathrm{vec}(XAY^T)\rangle$$

"Choi-Jamiołkowski representation" for a quantum operator $\mathcal{E}$ is given by:

$$J(\mathcal{E}) := (\mathcal{E} \otimes \mathbb{1})\, |vec(\mathbb{1})\rangle\langle vec(\mathbb{1})|$$

where $|vec(\mathbb{1})\rangle = \sum_k |k\rangle|k\rangle$ (not normalized)

Example: Consider the Kraus operators $\{|0\rangle\langle 0|, |1\rangle\langle 1|\}$

$|vec(\mathbb{1})\rangle = |00\rangle + |11\rangle$ as $\mathbb{1} = 1|0\rangle\langle 0| + 1|1\rangle\langle 1|$

$$|vec(\mathbb{1})\rangle\langle vec(\mathbb{1})| = |00\rangle\langle 00| + |00\rangle\langle 11| + |11\rangle\langle 00| + |11\rangle\langle 11|$$

$$= |0\rangle\langle 0| \otimes |0\rangle\langle 0| + |0\rangle\langle 1| \otimes |0\rangle\langle 1| + |1\rangle\langle 0| \otimes |1\rangle\langle 0| + |1\rangle\langle 1| \otimes |1\rangle\langle 1|$$

$$\Rightarrow J(\mathcal{E}) = (\mathcal{E} \otimes \mathbb{1})\, |vec(\mathbb{1})\rangle\langle vec(\mathbb{1})|$$

$$= \mathcal{E}(|0\rangle\langle 0|) \otimes |0\rangle\langle 0| + \mathcal{E}\underbrace{(|0\rangle\langle 1|)}_{=0} \otimes |0\rangle\langle 1| + \underbrace{\cdots}_{=0} + \mathcal{E}(|1\rangle\langle 1|) \otimes |1\rangle\langle 1|$$

(the Kraus operators kill the $|0\rangle\langle 1|$ and $|1\rangle\langle 0|$ terms)

$$J(\mathcal{E}) = |0\rangle\langle 0| \otimes |0\rangle\langle 0| + |1\rangle\langle 1| \otimes |1\rangle\langle 1|$$

The Choi representation becomes really useful when the map $\mathcal{E}$ is known but we do not know the Kraus operators. We do not know the quantum operators representing an explicit map !!

Example:- $\mathcal{E}(X) = \frac{1}{3}\left(Tr(X)\,\mathbb{1} + X^T\right)$, for any matrix M

$$J(\mathcal{E}) = (\mathcal{E} \otimes \mathbb{1})\left(\sum_{ij} |i\rangle\langle j| \otimes |i\rangle\langle j|\right)$$

$$= \sum_{ij} \mathcal{E}(|i\rangle\langle j|) \otimes |i\rangle\langle j|$$

$$= \frac{1}{3} \sum_{ij} \left(tr(|i\rangle\langle j|)\mathbb{1} + |j\rangle\langle i|\right) \otimes |i\rangle\langle j|$$

$$J(\mathcal{E}) = \frac{1}{3}\sum_{ij}\left(\mathrm{tr}(|i\rangle\langle j|\mathbb{1} + |j\rangle\langle i|\right)\otimes |i\rangle\langle j|$$

$$= \frac{1}{3}\left\{\sum_{ij}\delta_{ij}\,\mathbb{1}\otimes|j\rangle\langle i| + \sum_{i,j}|ji\rangle\langle ij|\right\}$$

$$= \frac{1}{3}\left(\mathbb{1} + SWAP\right)$$

Swap operator (S_W) is a unitary transformation that $|i,j\rangle \longrightarrow |j,i\rangle$ in the computational basis.

Now, $S_W^2 = \mathbb{1}$ (double swap gives the same state), and therefore has eigenvalues ± 1. For two qubits, the symmetric subspace $(+1)$ is spanned by the vectors: $|00\rangle$, $|11\rangle$ and $|\psi^+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$, whereas the anti-symmetric subspace has the vectors $|\psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$.

So, the symmetric projector: $\Pi_+ = |00\rangle\langle 00| + |11\rangle\langle 11| + |\psi^+\rangle\langle\psi^+|$

and the antisymmetric: $\Pi_- = |\psi^-\rangle\langle\psi^-|$

Now, $\Pi_+ + \Pi_- = 1$ (complete basis state)

Also, $S_W = \Pi_+ - \Pi_-$; $\therefore \frac{1}{2}(\mathbb{1} + SWAP) = \Pi_+$

$$J(\mathcal{E}) = \frac{2}{3}\Pi_+ = \frac{2}{3}\left(|00\rangle\langle 00| + |11\rangle\langle 11| + |\psi^+\rangle\langle\psi^+|\right)$$

$$= \sum_i |vec(A_i)\rangle\langle vec(A_i)|$$; where $\{A_i\}$ are the Kraus operators for $\mathcal{E}$.

$\therefore \frac{2}{3}|00\rangle = |vec(A_1)\rangle$; $A_1 = \sqrt{\frac{2}{3}}\,|0\rangle\langle 0|$

$\frac{2}{3}|11\rangle = |vec(A_2)\rangle$; $A_2 = \sqrt{\frac{2}{3}}\,|1\rangle\langle 1|$

$\frac{2}{3}|\psi^+\rangle = |vec(A_3)\rangle$; $A_3 = \sqrt{\frac{2}{3}}\begin{pmatrix} 0 & 1/\sqrt{2} \\ 1/\sqrt{2} & 0 \end{pmatrix} = \frac{1}{\sqrt{3}}\sigma_x$

$$\sum_i A_i^\dagger A_i = A_1^\dagger A_1 + A_2^\dagger A_2 + A_3^\dagger A_3 = \begin{pmatrix} 2/3 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 2/3 \end{pmatrix} + \begin{pmatrix} 1/3 & 0 \\ 0 & 1/3 \end{pmatrix} = \mathbb{1}$$

So, $\{A_i\}$ forms the set of Kraus operators.

vi) Non uniqueness of Kraus operators

We know that the Choi representation give us,

$$J(\mathcal{E}) = \sum_k |\mathrm{vec}(A_k)\rangle\langle \mathrm{vec}(A_k)|$$

where $\{A_k\}$ are the Kraus operators for the CPTP map $\mathcal{E}$. But $J(\mathcal{E})$ is a positive matrix and has an eigen decomposition.

$$J(\mathcal{E}) = \sum_k \lambda_k |e_k\rangle\langle e_k| \,; \quad \lambda_k \geq 0$$

From Section B (vii) of this notes, we know that other pure state decomposition can be obtained using the eigen decomposition, via a unitary, i.e.,

$$J(\mathcal{E}) = \sum_k \lambda_k |e_k\rangle\langle e_k| = \sum_i p_i |\phi_i\rangle\langle\phi_i|$$

where $\sqrt{p_i}|\phi_i\rangle = \sum_i u_{ik}\sqrt{\lambda_k}|e_k\rangle$ for some element u_{ik} of a unitary matrix. Both these decompositions will give Kraus operators corresponding to the same quantum operator $\mathcal{E}$.

$$\sqrt{\lambda_k}|e_k\rangle = \sqrt{\lambda_k}|\mathrm{vec}(E_k)\rangle, \therefore \{A_k\} = \{\sqrt{\lambda_k}E_k\}$$

$$\sqrt{p_i}|\phi_i\rangle = \sqrt{p_i}|\mathrm{vec}(F_i)\rangle; \therefore \{B_k\} = \{\sqrt{p_i}F_i\}$$

So, overall any two Kraus decompositions of the quantum channel / operator $\mathcal{E}$ is connected by a unitary :

$$B_i = \sum_k u_{ik} A_k$$

vii) Examples of quantum operators

- Unitary and isometry dynamics – Let us consider the case where $\mathcal{E} = \{U\}$, which satisfies the relation $\sum_i A_i^\dagger A_i = U^\dagger U = \mathbb{1}$. For unitary operators, $UU^\dagger = \mathbb{1}$ but for isometries (operators that map a state to a larger Hilbert space)[#] we restrict ourselves only to $U^\dagger U = \mathbb{1}$. Both unitary and isometry are valid quantum operations. For example, the isometry on a single qubit is given by

$$U|0\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + |2\rangle\right); \quad U|1\rangle = \frac{1}{\sqrt{2}}\left(|1\rangle + |3\rangle\right).$$

The isometry maps states in the Hilbert space $\{|0\rangle, |1\rangle\}$ to the larger Hilbert space $\{|0\rangle, |1\rangle, |2\rangle, |3\rangle\}$.

The notion that the dimension of the input can be different from the dimension of the output makes quantum operations a very potent tool in quantum information theory.

Now we see how does the unitary operator act on the qubit density matrix in the Bloch sphere picture. Let a single qubit be given by: $\varphi = \frac{1}{2}(\mathbb{I} + \vec{r}\cdot\vec{\sigma})$

When acted upon by a unitary: $\mathcal{E}(\varphi) = U\varphi U^\dagger = \frac{1}{2}(\mathbb{I} + \vec{s}\cdot\vec{\sigma})$, which gives us the relation that

$$U(\vec{r}\cdot\vec{\sigma})U^\dagger = \vec{s}\cdot\vec{\sigma}$$

Now, let us calculate, $(\vec{s}\cdot\vec{\sigma})(\vec{s}\cdot\vec{\sigma}) = \sum_{ij} s_i s_j \sigma_i \sigma_j$

$\sigma_i\sigma_j + \sigma_j\sigma_i = 0$, for $i \neq j$ and $\sigma_i^2 = \mathbb{1}$, $\forall \sigma_i \in \{\sigma_x, \sigma_y, \sigma_z\}$ $\Big\{ \; = \sum_{ij} s_i s_j \delta_{ij} \mathbb{1} = s^2 \mathbb{1}$

Also, $(\vec{s}\cdot\vec{\sigma})(\vec{s}\cdot\vec{\sigma}) = U(\vec{r}\cdot\vec{\sigma})U^\dagger U(\vec{r}\cdot\vec{\sigma})U^\dagger$

$$= U(\vec{r}\cdot\vec{\sigma})(\vec{r}\cdot\vec{\sigma})U^\dagger$$

$$= U r^2 \mathbb{1} U^\dagger = r^2 \mathbb{1}$$

$$\therefore \; s^2 \mathbb{1} = r^2 \mathbb{1}$$

This means the lengths of the vector $\vec{r}$ and $\vec{s}$ are equal and the Bloch vectors must be related by some orthogonal rotation. Thus, unitary transformations only rotate the Bloch vector.

- **The dephasing map** – Let us look at the CPTP map given by the Kraus operators, $A_0 = |0\rangle\langle 0|$ and $A_1 = |1\rangle\langle 1|$ acting on a single qubit.

It's easy to check that $A_0^\dagger A_0 + A_1^\dagger A_1 = |0\rangle\langle 0|0\rangle\langle 0| + |1\rangle\langle 1|1\rangle\langle 1| = \mathbb{1}$

Again, let the single qubit density matrix be given by:

$$\rho = \frac{1}{2}(\mathbb{I} + \vec{\sigma}.\vec{r}) = \frac{1}{2}\begin{pmatrix} 1+z & x-iy \\ x+iy & 1-z \end{pmatrix}$$

So, now
$$\begin{aligned}
\mathcal{E}(\rho) &= A_0 \rho A_0^\dagger + A_1 \rho A_1^\dagger \\
&= \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1+z & x-iy \\ x+iy & 1-z \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1+z & x-iy \\ x+iy & 1-z \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \\
&= \frac{1}{2}\begin{pmatrix} 1+z & x-iy \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 & 0 \\ x+iy & 1-z \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \\
&= \frac{1}{2}\begin{pmatrix} 1+z & 0 \\ 0 & 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 & 0 \\ 0 & 1-z \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1+z & 0 \\ 0 & 1-z \end{pmatrix}
\end{aligned}$$

So, the dephasing map in σ_z-basis, completely kills off the off diagonal components or the coherences and returns simply a classical mixture of the two eigenstates of σ_z.

- **The fully depolarizing channel** — If we consider a state σ in the Hilbert space, a completely depolarizing channel maps all states, ρ, to the state σ, i.e.,

$$\mathcal{E}(\rho) = \sigma \quad \forall \text{ all } \rho$$

The Kraus operators for the map $\mathcal{E}$ is given by the eigen decomposition of σ. So, if $\sigma = \sum_i \lambda_i |e_i\rangle\langle e_i|$, then Kraus operators are:

$$A_{ij} = \sqrt{\lambda_i}\, |e_i\rangle\langle j|,$$ where $|j\rangle$ is the computational basis

So, how does the Kraus operators give us the fully depolarizing map :

$$\mathcal{E}(\varphi) = \sum_{ij} A_{ij}\, \varphi\, A_{ij}^{+}$$

$$= \sum_{ij} \sqrt{\lambda_i}\, |e_i\rangle\langle j|\varphi|j\rangle\langle e_i|\, \sqrt{\lambda_i}$$

$$= \sum_i \lambda_i |e_i\rangle \underbrace{\sum_j \langle j|\varphi|j\rangle}_{Tr(\varphi)=1} \langle e_i|$$

$$= \sum_i \lambda_i |e_i\rangle\langle e_i| = \sigma$$

Also; $\sum_{ij} A_{ij}^{+} A_{ij} = \sum_{ij} \sqrt{\lambda_i}\, |e_i\rangle\langle j|j\rangle\langle e_i|\, \sqrt{\lambda_i}$

$$= \sum_i |e_i\rangle\langle e_i| \underbrace{\sum_j \lambda_i}_{} = 1 \text{ (sum of eigenvalues)}$$

$$= \mathbb{I} \text{ (completeness)}$$

- Partial trace — Discarding a quantum state

So this map gives us : $\mathcal{E}(\varphi_{AB}) = tr_B(\varphi_{AB}) = \varphi_A$

The Kraus operators for this map are simply : $\{A_i = \mathbb{1}_A \otimes \langle i|_B\}$, which gives us, $\sum_i A_i^{+} A_i = \sum_i \mathbb{1}_A \otimes |i\rangle\langle i|_B = \mathbb{1}_{AB}$

and, $\sum_i \mathbb{1}_A \otimes {}_B\langle i|\, \varphi_{AB}\, |i\rangle_B = tr_B(\varphi_{AB})$ (by definition!)

- Preparing a quantum state

Takes a 1D state $|0\rangle$ to a quantum state $\varphi = \underbrace{\sum_i \lambda_i |e_i\rangle\langle e_i|}_{\text{eigen decomposition}}$

So, Kraus operators are : $\{A_i = \sqrt{\lambda_i}\, |e_i\rangle\langle 0|\}$

$$\mathcal{E}(|0\rangle) = \sum_i \lambda_i |e_i\rangle\underbrace{\langle 0|0\rangle}_{=1}\underbrace{\langle 0|0\rangle}_{=1}\langle e_i| = \sum_i \lambda_i |e_i\rangle\langle e_i| = \varphi.$$

Also, $\sum_i A_i^{+} A_i = \sum_i \lambda_i |0\rangle\langle e_i|e_i\rangle\langle 0| = (\sum_i \underbrace{\lambda_i}_{=1})|0\rangle\langle 0| = \underbrace{\mathbb{1}}_{}$

$|0\rangle\langle 0|$ is also the identity

- The bit flip and phase flip channels

The bit flip channel flips the state $|0\rangle$ to $|1\rangle$ (and $|1\rangle$ to $|0\rangle$) with probability $1-p$, i.e., $\mathcal{E}(|0\rangle) = |1\rangle$ and $\mathcal{E}(|1\rangle) = |0\rangle$.

The Kraus operators for $\mathcal{E}$ are given by:

$$A_0 = \begin{pmatrix} \sqrt{p} & 0 \\ 0 & \sqrt{p} \end{pmatrix} = \sqrt{p}\,\mathbb{I} \quad \text{and} \quad A_1 = \begin{pmatrix} 0 & \sqrt{1-p} \\ \sqrt{1-p} & 0 \end{pmatrix} = \sqrt{1-p}\,\sigma_x$$

So let us consider the qubit density matrix: $\varphi = \frac{1}{2}\begin{pmatrix} 1+z & x-iy \\ x+iy & 1-z \end{pmatrix}$

$$\mathcal{E}(\varphi) = \frac{1}{2}\begin{pmatrix} \sqrt{p} & 0 \\ 0 & \sqrt{p} \end{pmatrix}\begin{pmatrix} 1+z & x-iy \\ x+iy & 1-z \end{pmatrix}\begin{pmatrix} \sqrt{p} & 0 \\ 0 & \sqrt{p} \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 & \sqrt{1-p} \\ \sqrt{1-p} & 0 \end{pmatrix}\begin{pmatrix} 1+z & x-iy \\ x+iy & 1-z \end{pmatrix}\begin{pmatrix} 0 & \sqrt{1-p} \\ \sqrt{1-p} & 0 \end{pmatrix}$$

$$= \frac{p}{2}\begin{pmatrix} 1+z & x-iy \\ x+iy & 1-z \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{(1-p)}{2}\begin{pmatrix} x+iy & 1-z \\ 1+z & x-iy \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$= \frac{p}{2}\begin{pmatrix} 1+z & x-iy \\ x+iy & 1-z \end{pmatrix} + \frac{(1-p)}{2}\begin{pmatrix} 1-z & x+iy \\ x-iy & 1+z \end{pmatrix}$$

$$= p\,\varphi + (1-p)\,\hat{\varphi} \quad // \text{ spin flipped state}$$

The above state can also be rewritten as:

$$\mathcal{E}(\varphi) = \tfrac{1}{2}\left(\mathbb{I} + x\sigma_x - y(1-2p)\sigma_y - z(1-2p)\sigma_z\right)$$

So, the bit flip channel keeps the x axis in the Bloch sphere unchanged but contracts the z-y plane by a factor $(1-2p)$.

The phase flip channel changes the relative phase of a vector by a factor $e^{i\pi}$ or is simply a sign flip channel, i.e.,

$$\alpha|0\rangle + \beta e^{i\phi}|1\rangle \rightarrow \alpha|0\rangle - \beta e^{i\phi}|1\rangle \quad (\because e^{i\pi} = -1)$$

The Kraus operators for the phase or sign flip channel with probability $1-p$, is given by :

$$A_0 = \sqrt{p}\, \mathbb{I} = \begin{pmatrix} \sqrt{p} & 0 \\ 0 & \sqrt{p} \end{pmatrix} \text{ and } A_1 = \sqrt{1-p}\, \sigma_z = \begin{pmatrix} \sqrt{1-p} & 0 \\ 0 & -\sqrt{1-p} \end{pmatrix}$$

$$\mathcal{E}(\rho) = \frac{p}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1+z & x-iy \\ x+iy & 1-z \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{(1-p)}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1+z & x-iy \\ x+iy & 1-z \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= p\rho + \frac{(1-p)}{2}\begin{pmatrix} 1+z & x-iy \\ -(x+iy) & -(1-z) \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= p\rho + \frac{(1-p)}{2}\begin{pmatrix} 1+z & -(x-iy) \\ -(x+iy) & 1-z \end{pmatrix} = p\rho + (1-p)\check{\rho} \quad \text{// phase flipped}$$

$$= \frac{1}{2}\left(\mathbb{I} + z\sigma_z + (2p-1)x\sigma_x + (2p-1)y\sigma_y \right)$$

So, the phase flip channel keeps the z-axis unchanged but contracts the y-x plane by a factor $(1-2p)$ in the Bloch sphere (ball).

There is another operation called the bit-phase flip operator with Kraus operators : $A_0 = \sqrt{p}\, \mathbb{I}$ and $A_1 = \sqrt{1-p}\, \sigma_y$. We know that, $\sigma_y = i\sigma_x\sigma_z$, therefore the above operation does the phase flip followed by the bit flip.

Exercise : Show the action of the bit-phase flip channel on the Bloch sphere.

- The amplitude damping channel

A much more physical operation closely connected to the description of energy dissipation or loss of energy in the quantum system. The amplitude damping channel mimics such physical processes including spontaneous emission of a photon, loss of energy due to environmental interaction etc.

The amplitude damping channel can be represented by the set of Kraus operators $\{A_k\}$:

$$A_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{pmatrix} \text{ and } A_1 = \begin{pmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{pmatrix}$$

where γ is the rate at which dissipation or damping takes place. This satisfies the condition for Kraus operators

$$A_0^+ A_0 + A_1^+ A_1 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ \sqrt{\gamma} & 0 \end{pmatrix}\begin{pmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1-\gamma \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & \gamma \end{pmatrix} = \mathbb{1}$$

Let us consider the state: $\varphi = \frac{1}{2}\begin{pmatrix} 1+z & x-iy \\ x+iy & 1-z \end{pmatrix}$

$$\mathcal{E}(\varphi) = \frac{1}{2}\left\{\begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{pmatrix}\begin{pmatrix} 1+z & x-iy \\ x+iy & 1-z \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{pmatrix} + \begin{pmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1+z & x-iy \\ x+iy & 1-z \end{pmatrix}\begin{pmatrix} 0 & 0 \\ \sqrt{\gamma} & 0 \end{pmatrix}\right\}$$

$$= \frac{1}{2}\left\{\begin{pmatrix} 1+z & x-iy \\ \sqrt{1-\gamma}(x+iy) & \sqrt{1-\gamma}(1-z) \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{pmatrix} + \begin{pmatrix} \sqrt{\gamma}(x+iy) & \sqrt{\gamma}(1-z) \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ \sqrt{\gamma} & 0 \end{pmatrix}\right\}$$

$$= \frac{1}{2}\left\{\begin{pmatrix} 1+z & \sqrt{1-\gamma}(x-iy) \\ \sqrt{1-\gamma}(x+iy) & (1-\gamma)(1-z) \end{pmatrix} + \begin{pmatrix} \gamma(1-z) & 0 \\ 0 & 0 \end{pmatrix}\right\}$$

$$= \frac{1}{2}\begin{pmatrix} 1+z+\gamma(1-z) & \sqrt{1-\gamma}(x-iy) \\ \sqrt{1-\gamma}(x+iy) & (1-\gamma)(1-z) \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} \varphi_{00} + \gamma\varphi_{11} & \sqrt{1-\gamma}\,\varphi_{01} \\ \sqrt{1-\gamma}\,\varphi_{10} & (1-\gamma)\varphi_{11} \end{pmatrix}$$

So, if we take $\varphi = \begin{pmatrix} \varphi_{00} & \varphi_{01} \\ \varphi_{10} & \varphi_{11} \end{pmatrix}$ only the state $|0\rangle\langle 0|$ remains invariant. This implies that $|0\rangle\langle 0|$ is the lowest energy or zero temp. state. #

\# For generalized amplitude damping to some finite temperature state we use the following set of Kraus operators:

$A_0 = \sqrt{p}\begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{pmatrix}$, $A_1 = \sqrt{p}\begin{pmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{pmatrix}$, $A_2 = \sqrt{1-p}\begin{pmatrix} \sqrt{1-\gamma} & 0 \\ 0 & 1 \end{pmatrix}$, $A_3 = \sqrt{1-p}\begin{pmatrix} 0 & 0 \\ \sqrt{\gamma} & 0 \end{pmatrix}$, where the invariant state is $\varphi_\infty = p|0\rangle\langle 0| + (1-p)|1\rangle\langle 1|$

vi) The Choi-Jamiołkowski isomorphism

Here we provide a simple introduction to how the CJ isomorphism comes. Consider any Hermitian operator acting J acting on the Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$. J being Hermitian has an eigendecomposition:

$$
\begin{aligned}
J &= \sum_k \lambda_k |e_k\rangle\langle e_k| \\
&= \sum_k |\mathrm{vec}(A_k)\rangle\langle \mathrm{vec}(B_k)| \\
&= \sum_k A_k \otimes \mathbb{1} \, |\mathrm{vec}(\mathbb{1})\rangle\langle \mathrm{vec}(\mathbb{1})| \, B_k^\dagger \otimes \mathbb{1} \\
&= (\mathcal{E} \otimes \mathbb{1}) |\mathrm{vec}(\mathbb{1})\rangle\langle \mathrm{vec}(\mathbb{1})|
\end{aligned}
$$

as λ_k is real but not necessarily positive

where $\mathcal{E}(X) = \sum_k A_k X B_k^\dagger$, which maps an object in B to A, i.e., $\mathcal{E}: B(\mathcal{H}_B) \to B(\mathcal{H}_A)$. $\{A_k\}$ and $\{B_k\}$ are different as J is not positive.

CJ isomorphism: For every superoperator* $\mathcal{E}: B(\mathcal{H}_B) \to B(\mathcal{H}_A)$ we can find a Choi operator $J(\mathcal{E}) \in B(\mathcal{H}_A \otimes \mathcal{H}_B)$ given by

$$J(\mathcal{E}) = (\mathcal{E} \otimes \mathbb{1}) |\mathrm{vec}(\mathbb{1})\rangle\langle \mathrm{vec}(\mathbb{1})|.$$

Alternately, for every operator $J(\mathcal{E}) \in B(\mathcal{H}_A \otimes \mathcal{H}_B)$ acting on a bipartite system we can use the singular value decomposition to write J in the form of

$$J = \sum_k |\mathrm{vec}(A_k)\rangle\langle \mathrm{vec}(B_k)|$$

for some matrices $\{A_k\}$ and $\{B_k\}$, which can be used to define the superoperator $\mathcal{E}(X) = \sum_k A_k X B_k^\dagger$.

* It is typical to called maps that take a linear operator to another as a superoperator, in contrast to operators that act on a vector.

If $\mathcal{J}$ is Hermitian, then $\mathcal{E}(X)^\dagger = \mathcal{E}(X)$ if $X = X^\dagger$. $\mathcal{E}(X)$ are called Hermiticity preserving maps.

Importantly, if $\mathcal{J}$ is a positive operator then

$$\mathcal{J} = \sum_k \lambda_k |e_k\rangle\langle e_k| = \sum_k |\mathrm{vec}(A_k)\rangle\langle \mathrm{vec}(A_k)| \left.\right\} \text{as } \lambda\text{'s are now positive}$$

$$= \sum_k A_k \otimes \mathbb{1} \, |\mathrm{vec}(\mathbb{1})\rangle\langle \mathrm{vec}(\mathbb{1})| \, A_k^\dagger \otimes \mathbb{1}$$

$$= (\mathcal{E} \otimes \mathbb{1}) |\mathrm{vec}(\mathbb{1})\rangle\langle \mathrm{vec}(\mathbb{1})|$$

where $\mathcal{E}(\varphi) = \sum_k A_k \varphi A_k^\dagger$ is a completely positive map and $\{A_k\}$ are the corresponding Kraus operators. So, this sets up an interesting channel-state duality:

The set of unnormalized bipartite quantum states in $\mathcal{H}_A \otimes \mathcal{H}_B$ is isomorphic to the set of completely positive maps from states in $\mathcal{H}_B$ to states in $\mathcal{H}_A$.