The Choi-Jamiolkowski isomorphism

Before we begin, there is an important distinction that one often encounters while discussing operators that live in the relevant Hilbert space of operators. This is the distinction between superoperators or linear maps and operators in the Hilbert space. Going further back, if our states $|\psi\rangle_{AB}$ live in the Hilbert space $\mathcal{H} = H_A \otimes H_B$, our operators live in the space $\mathcal{B}(\mathcal{H})$, with inner product defined by the Hilbert-Schmidt norm i.e. $Tr[AB^\dagger]$ for $A, B \in \mathcal{B}(\mathcal{H})$.

Superoperators are operators that linearly map states and other operators in the Hilbert space (also called channels).

Example: $\mathcal{E}(\varphi) \to \sigma$, where $\sigma = \sum_i A_i \varphi A_i^\dagger$

is a superoperator that maps density matrices.

On the other hand, density matrices and observables are termed as operators.

Example: $\langle \sigma_x \rangle = Tr[\sigma_x \varphi]$, where both σ_x and φ are operators in the Hilbert space.

Choi-Jamiolkowski isomorphism provides a natural way to connect observables or operators in the Hilbert space $B(\mathcal{H})$ to superoperators acting on the subsystem $B(\mathcal{H}_A)$.

For instance, every unnormalized density matrix $\mathcal{J}(\mathcal{E})$ in $B(\mathcal{H}_A \otimes \mathcal{H}_B)$ is related to a completely positive quantum map or channel in the space $B(\mathcal{H}_A)$. This is called the channel-state duality.

- **Mapping Hermitian operators to Hermiticity preserving channels**

Here we provide a simple introduction to how the CJ isomorphism comes. Consider any Hermitian operator acting J acting on the Hilbert space $B(\mathcal{H}_A \otimes \mathcal{H}_B)$. J being Hermitian has an eigendecomposition:

$$
\begin{aligned}
J &= \sum_k \lambda_k |e_k\rangle\langle e_k| \\
&= \sum_k |\mathrm{vec}(A_k)\rangle\langle \mathrm{vec}(B_k)| \\
&= \sum_k A_k \otimes \mathbb{1} \, |\mathrm{vec}(\mathbb{1})\rangle\langle \mathrm{vec}(\mathbb{1})| \, B_k^\dagger \otimes \mathbb{1} \\
&= (\mathcal{E} \otimes \mathbb{1}) |\mathrm{vec}(\mathbb{1})\rangle\langle \mathrm{vec}(\mathbb{1})|
\end{aligned}
$$

as λ_k is real but not necessarily positive

where $\mathcal{E}(X) = \sum_k A_k X B_k^\dagger$, which maps an object in B to A, i.e., $\mathcal{E}: B(\mathcal{H}_B) \to B(\mathcal{H}_A)$. $\{A_k\}$ and $\{B_k\}$ are different as J is not positive.

CJ isomorphism: For every superoperator $\mathcal{E}: B(\mathcal{H}_B) \to B(\mathcal{H}_A)$ we can find a Choi operator $J(\mathcal{E}) \in B(\mathcal{H}_A \otimes \mathcal{H}_B)$ given by

$$
J(\mathcal{E}) = (\mathcal{E} \otimes \mathbb{1}) |\mathrm{vec}(\mathbb{1})\rangle\langle \mathrm{vec}(\mathbb{1})|.
$$

Alternately, for every operator $J(\mathcal{E}) \in B(\mathcal{H}_A \otimes \mathcal{H}_B)$ acting on a bipartite system we can use the singular value decomposition to write J in the form of

$$
J = \sum_k |\mathrm{vec}(A_k)\rangle\langle \mathrm{vec}(B_k)|
$$

for some matrices $\{A_k\}$ and $\{B_k\}$, which can be used to define the superoperator $\mathcal{E}(X) = \sum_k A_k X B_k^\dagger$.

If $\mathcal{J}$ is Hermitian, then $\mathcal{E}(X)^\dagger = \mathcal{E}(X)$ if $X = X^\dagger$. $\mathcal{E}(X)$ are called Hermiticity preserving maps.

• Channel-state duality

Importantly, if $\mathcal{J}$ is a positive operator then

$$\mathcal{J} = \sum_k \lambda_k |e_k\rangle\langle e_k| = \sum_k |\mathrm{vec}(A_k)\rangle\langle \mathrm{vec}(A_k)| \quad \Big\} \text{ as } \lambda\text{'s are now positive}$$

$$= \sum_k A_k \otimes \mathbb{1} \, |\mathrm{vec}(\mathbb{1})\rangle\langle \mathrm{vec}(\mathbb{1})| \, A_k^\dagger \otimes \mathbb{1}$$

$$= (\mathcal{E} \otimes \mathbb{1}) |\mathrm{vec}(\mathbb{1})\rangle\langle \mathrm{vec}(\mathbb{1})|$$

where $\mathcal{E}(\varphi) = \sum_k A_k \varphi A_k^\dagger$ is a completely positive map and $\{A_k\}$ are the corresponding Kraus operators. So, this sets up an interesting channel-state duality:

The set of unnormalized bipartite quantum states in $\mathcal{H}_A \otimes \mathcal{H}_B$ is isomorphic to the set of completely positive maps from states in $\mathcal{H}_B$ to states in $\mathcal{H}_A$.

Namely given any completely positive map $\mathcal{E}: B(\mathcal{H}_B) \to B(\mathcal{H}_A)$, it's Choi operator is a positive bipartite operator $\mathcal{J}(\mathcal{E}) \geq 0$ and so can be viewed as unnormalized bipartite state in the Hilbert space $B(\mathcal{H}_A \otimes \mathcal{H}_B)$. Conversely given any bipartite operator $\mathcal{J}(\mathcal{E}) \geq 0$ on $B(\mathcal{H}_A \otimes \mathcal{H}_B)$, its corresponding superoperator $\mathcal{E}$ is a completely positive map.