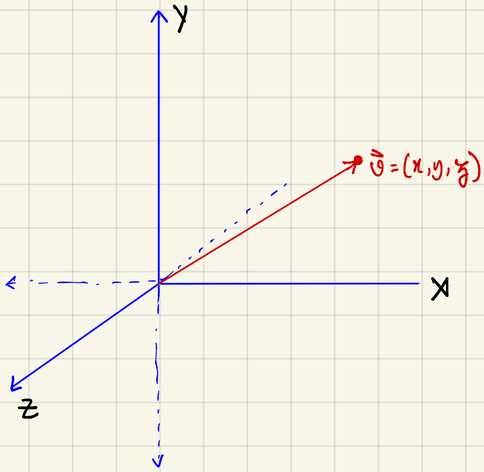
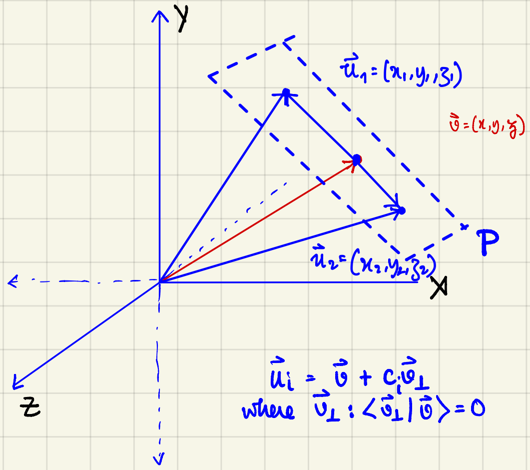
- Understanding hyperplanes and entanglement witness



Consider the real vector space of 3D vectors. A vector $\vec{v}$ can be written in terms of the basis vectors $\hat{x}$, $\hat{y}$ and $\hat{z}$ such that $\vec{v} = (x,y,z) = x\hat{x} + y\hat{y} + z\hat{z}$.

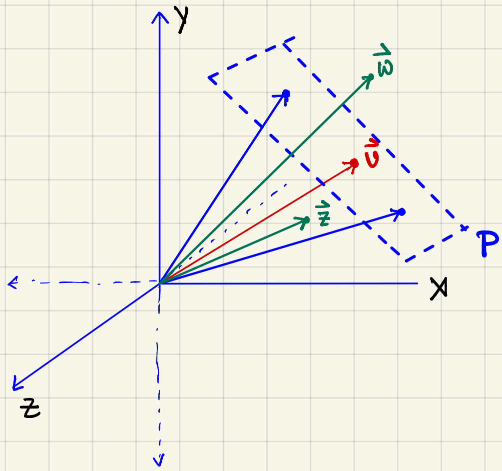
A hyperplane is a subspace of the vector space with co-dimension 1. For the 3D vector space above the hyperplane is nothing but a 2D plane in the 3D space. Note that the 2D plane extends to infinity unlike the figure.



$\vec{u}_i = \vec{v} + c_i \vec{v}_\perp$
where $\vec{v}_\perp : \langle \vec{v}_\perp | \vec{v} \rangle = 0$

Let us consider the hyperplane P defined by the vector $\vec{v}$ as shown in the figure to the right.

$P = \{ \vec{u} : \langle \vec{u}, \vec{v} \rangle = |\vec{v}|^2 = c \}$. As such all vectors on the hyperplane (vector subspace) P have a fixed inner product with $\vec{v}$.



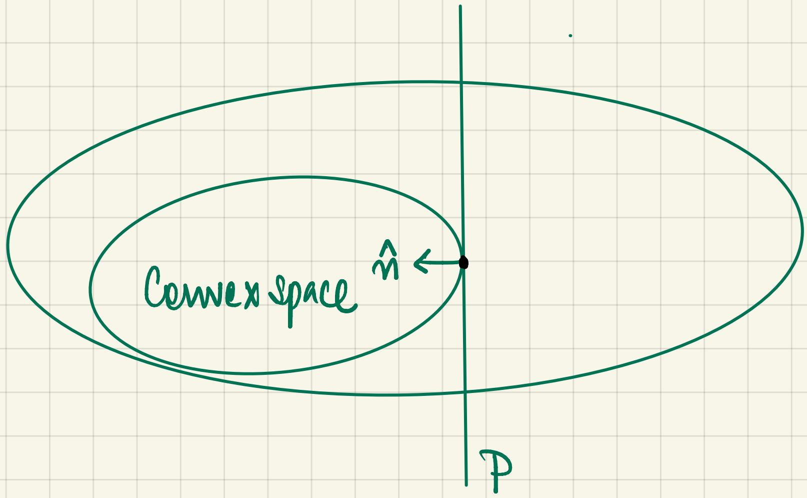
Now consider the points $\vec{w}$ and $\vec{z}$ that lie on either side of the hyperplane P defined by vector $\vec{v}$.

For the vectors $\vec{w}$ and $\vec{z}$, we have

$$\langle \vec{w}, \vec{v} \rangle > c \quad \text{and} \quad \langle \vec{z}, \vec{v} \rangle < c$$

Therefore, the hyperplane divides the entire vector space (that is not part of the subspace defined by P) into two regions.

- Convex sets — For a convex set such as the separable states a hyperplane can separate the entire set from the rest of the vector space.



Hyperplane P defined by the unit vector $\hat{n}$ such that all points $\vec{v}$ on the subspace is $\langle \vec{v}, \hat{n} \rangle = 0$.

The hyperplane can completely separate a convex set C, such that $\langle y, \hat{n} \rangle \geq 0$ $\forall\, y \in C$

The notion of hyperplanes can now be extended to vector space of density matrices and observables consisting of Hermitian linear operators.

Since, the set of separable state is convex, one can always find an observable or Hermitian operator to define a hyperplane that separates the set of separable state from the rest.

Such a Hermitian operator is called an entanglement witness W, with the hyperplane being defined by all vectors φ_{AB} such that $Tr[W \varphi_{AB}] = 0$.

For separable states σ_{AB}, $Tr[W \sigma_{AB}] \geq 0$ and $Tr[W \varphi_{AB}] < 0$ implies the state is entangled.

Entanglement witnesses play a critical role in quantum information protocols. Since witnesses are observables, they can readily verify and certify entanglement in laboratories, where it may not be possible to explicitly determine the quantum state to calculate entanglement.