PH 534 Autumn 2024-2025 – Quantum information and computation

Practice problems Set 1b

(Dated: August 28, 2024)

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21. Statistical vs Quantum Randomness: Imagine a Stern-Gerlach setup, however with an inbuilt "demon" that sorts $|0\rangle$ and $|+\rangle$ with probability 2/3 and 1/3 respectively.

    (a) Write down the density matrix (ρ) for this system.

    (b) Find the eigenvalues of ρ and prove that it is a valid quantum state.

    (c) Find the expectation value of the operator σ_z for this state.

    (d) Compute the purification of the above single qubit state ρ.

    (e) Perform a Schmidt decomposition of this purified state.

    (f) Verify that a partial trace over the second subsystem gives you the state ρ that you started with!

22. Compute purifications for the following single qubit states:

    (a) $\rho_1 = \frac{1}{2}\mathbb{I}_2$, where $\mathbb{I}_2$ is a 2×2 identity matrix. Is the purification unique?

    (b) $\rho_2 = \frac{1}{2}(|0\rangle\langle 0| + |+\rangle\langle +|)$, where $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$.

    (c) $\rho_3 = \frac{2}{5}\rho_1 + \frac{3}{5}\rho_2$. This is an example of a mixture of a mixture!

23. Find the Schmidt decomposition for the state,

    (a) $|\psi_{AB}\rangle = \frac{1}{\sqrt{2}}(|0\rangle_A |0\rangle_B + |+\rangle_A |1\rangle_B)$.

    (b) What is the Schmidt rank for the above state? What do you infer about the presence of entanglement?

    (c) Prove that the Schmidt rank of a separable pure state is always 1.

24. Choi-Jamiołkowski isomorphism: Isomorphisms between Maps and States

    Consider the following maps on a qubit state ρ:

    (a) Transpose map, i.e., $\Phi(\rho) = \rho^T$.

(b) Thermal map, i.e., $\Phi(\rho) = \tau = p\,|0\rangle\langle 0| + (1-p)\,|1\rangle\langle 1|$ for some value of p, $0.5 \leq p \leq 1$.

(c) Write the Choi state for the map Φ given in (a) and (b), and find the Kraus operators.

(d) Find which of these maps are CPTP. Prove your assertion by checking the Choi state and the Kraus operators.

25. Partial trace and expectation values

(a) Consider the state $|\psi\rangle_{AB} = \cos\theta\,|00\rangle + \sin\theta\,|11\rangle$ in the Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$. Find the correlation matrix $\chi_{AB} = \rho_{AB} - \rho_A \otimes \rho_B$, where $\rho_{AB} = |\psi\rangle\langle\psi|_{AB}$, and ρ_A and ρ_B are the reduced states.

(b) Measure the expectation value of σ_z in the Hilbert space $\mathcal{H}_A$ using the full state $|\psi\rangle_{AB}$ and the reduced state ρ_A.