PH 534 Autumn 2024-2025 – Quantum information and computation

Practice problems Set 2

(Dated: September 2024)

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1. Please answer the following questions:

   (a) Consider the state, $\rho = \frac{1}{2}\,\mathbb{I}_2$ and the operator, $\mathcal{E} = |0\rangle\langle 0|$. Ensure $\mathcal{E}$ preserves trace and show that $S(\mathcal{E}(\rho)) < S(\rho)$, where $S(\rho)$ is the von Neumann entropy.

   (b) Write down two unique purifications for the state $\rho = \frac{1}{2}\,\mathbb{I}_2$.

   (c) Write down the un-normalised state, $|vec(\mathbb{I}_2)\rangle$ and the operator, $|vec(\mathbb{I}_2)\rangle\langle vec(\mathbb{I}_2)|$.

2. Your friend and you want to exchange classical bits of information using a newly bought optical fiber setup that allows you noiselessly encode, store and send qubits. Say, you want to send two bits of information (labeled $i = 0, 1, 2, 3$) and the states you intend to send to your friend are $\rho_i = \frac{1}{2}(\mathbb{I}_2 + \sigma_i)$, where $\sigma_i \in \{\mathbb{I}_2, \sigma_x, \sigma_y, \sigma_z\}$ are the identity operator and the usual Pauli matrices. Say, you encode the information with equal probability.

   (a) What is the register (a set of distinguishable quantum states) you use to keep track of your classical bits?

   (b) What is the joint state or density matrix ρ_{AB} shared by you and your friend?

   (c) Say, the dimension of ρ_{AB} is $d_A \times d_B$, where d_A and d_B are the dimensions of your and your friends Hilbert space, respectively. What is the value of d_A and d_B?

   (d) What is the maximum information your friend can retrieve or access? Can they retrieve two bits of information?

3. The physics lab has recently purchased a quantum entanglement machine, that can input two qubits and generate a maximally entangled singlet, $|\psi^-\rangle$. However, once in a while with a probability p it creates a highly mixed state, given by the normalized two-qubit identity matrix, $\mathbb{I}_4/4$.

(a) Write down the state ρ_{AB} produced by the quantum entanglement machine?

(b) What is the quantum mutual information of these states as a function of p?

(c) Show that the conditional entropy, defined as $H(A|B) = S(\rho_{AB}) - S(\rho_B)$, can actually be negative for some values of p (or not).

4. Consider an atomic system with two distinct energy levels, with energies given by $E_0 = \log_e 0.1$ and $E_1 = \log_e 0.2$. Let's say, these energy levels are represented by the orthonormal states $|e_0\rangle$ and $|e_1\rangle$, respectively.

   (a) Write the energy operator $\hat{E}$ in a spectral form.

   (b) Assume that a laser shines upon the isolated atom and creates a coherent state, $|\psi\rangle$ with equal superposition between it's energy eigen states. Calculate the expectation value of the energy, $\langle \hat{E} \rangle$, for this state.

   (c) Now the atom interacts with it's environment at temperature, $T = 1/\beta$, and ends up in a thermal state, given by the density matrix,

$$\rho = \frac{1}{Z} \sum_i e^{-\beta E_i} \, |e_i\rangle\langle e_i|,$$

   where, $Z = \sum_i e^{-\beta E_i}$ is the partition function. For $\beta = 1$, calculate $\langle \hat{E} \rangle$.

5. Consider the following communication protocol using qubits: Information is encoded in qubits $|0\rangle$ and $|+\rangle$, with probabilities λ and $1 - \lambda$.

   Show that there is (or there isn't) a non-zero value of λ for which the von-Neumann entropy is equal to the Shannon entropy.