1. Definition of the Internal Carotid Artery (ICA)

The centerline of the ICA was extracted from medical imaging data using V-Modeler (an in-house software developed in Oshima Lab). After the 3d thinning algorithm, a fifth-order spline interpolation was applied to generate a smooth and continuous representation of the vessel centerline.

The ICA centerline can be expressed as an ordered sequence of points:

$$\boldsymbol{C}_{ICA} = \{\boldsymbol{p}_i \mid i = 1, 2, \dots, N\} \tag{1}$$

where each point:

$$\boldsymbol{p}_i = (x_i, y_i, z_i)^{\top} = \begin{bmatrix} x_i \\ y_i \\ z_i \end{bmatrix} \tag{2}$$

represents the three-dimensional coordinates of the centerline.

In this study, the cumulative arc length s is measured along the centerline starting from the bifurcation between the ICA and the middle cerebral artery (MCA) and ending at the bifurcation between the ICA and the common carotid artery (CCA). Along the centerline, the curvature κ and cumulative arc length s_i are defined respectively as:

$$\kappa_i = \kappa(\boldsymbol{p}_i) = \kappa(s_i), \;\; s_i = s(\boldsymbol{p}_i) \tag{3}$$

where $\kappa(\boldsymbol{p}_i)$ denotes the curvature at point $\boldsymbol{p}_i$, which is equivalently expressed as $\kappa(s_i)$ along the arc-length parameterization. $s(\boldsymbol{p}_i)$ represents the cumulative arc length from the starting point of the centerline to the position of $\boldsymbol{p}_i$.

2. Definition of the ICA Siphon

From an anatomical perspective, the ICA siphon is located between the C4 and C6 segments of the internal carotid artery, as illustrated in *Figure 1*. This region is characterized by two distinct curvatures: the superior bend (B1) and the inferior bend (B2).

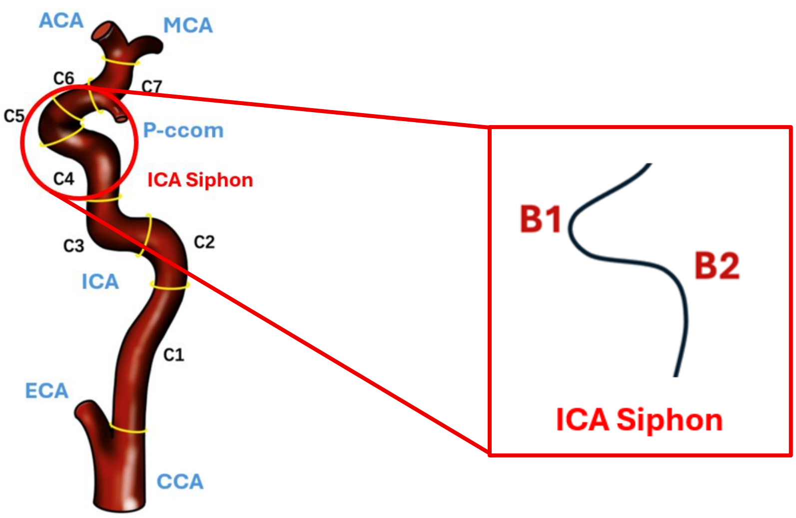

Figure 1. Definition of the ICA and Siphon

The maximum curvature of the B1 bend is generally located approximately $20–40mm$ downstream from the starting portion of the ICA segment (the ICA/MCA bifurcation). In most cases, the maximum curvature at B1 corresponds to the global maximum curvature of the entire ICA centerline. The next local curvature peak downstream of B1 is defined as the B2 curvature peak.

In this study, the arc-length positions of the B1 and B2 curvature peaks are denoted as s_{B1} and s_{B2}, respectively, while their corresponding curvature values are denoted as $\kappa_{B1}$and κ_{B2}, as shown in *Figure 2*.

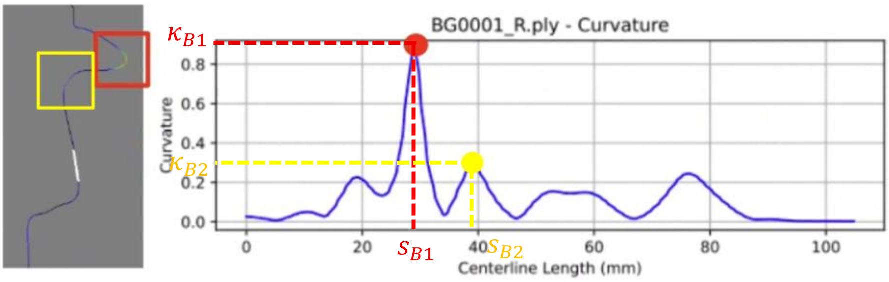

Figure 2. The identification of the two major curvature peaks, B1 and B2, within the ICA siphon region.

After determining the positions of the maximum curvature points corresponding to the B1 and B2 bends, the curvature distribution along the centerline was further analyzed to identify the adjacent local minima associated with each peak. These local minima were used to define the boundaries of the siphon region, as illustrated in *Figure 3*.

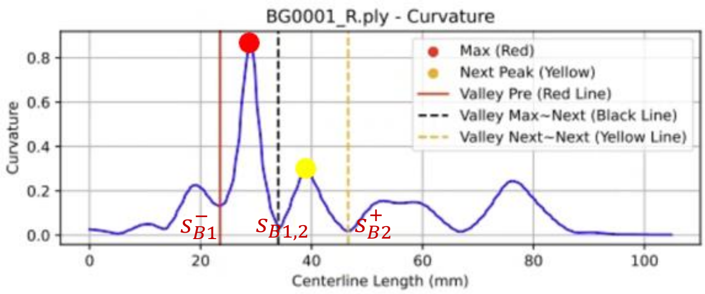

Figure 3. Identification of the curvature valleys defining the proximal ($\mathrm{s}_{\mathrm{B1}}^{-}$), *distal* ($\mathrm{s}_{\mathrm{B2}}^{+}$), *and intermediate* ($\mathrm{s}_{\mathrm{B1,2}}$) *boundaries of the ICA siphon.*

The red line in *Figure 3* represents the curvature valley located proximal to the B1 peak (proximal start), denoted as s_{B1}^{-}, while the yellow line indicates the curvature valley near the B2 peak (distal end), denoted as s_{B2}^{+}. The black line corresponds to the curvature valley between the two peaks, denoted as $s_{B1,2}$, which serves as the boundary separating the two bends.

The corresponding set of centerline points contained within this region is represented as:

$$\boldsymbol{C}_{Siphon} = \{\boldsymbol{p}_i \in \boldsymbol{C}_{ICA} \mid s_{B1}^{-} \leq s_i \leq s_{B2}^{+}\} \tag{4}$$

Within the siphon region, the centerline points corresponding to the two major bends are defined as:

$$\boldsymbol{C}_{B1} = \{\boldsymbol{p}_i \in \boldsymbol{C}_{Siphon} \mid s_{B1}^{-} \leq s_i \leq s_{B12}\} \tag{5}$$

$$\boldsymbol{C}_{B2} = \{\boldsymbol{p}_i \in \boldsymbol{C}_{Siphon} \mid s_{B12} \leq s_i \leq s_{B2}^{+}\} \tag{6}$$

3. Extraction of Geometric Parameters of the ICA Siphon

Curvature and torsion are fundamental quantities for describing the geometric properties of a spatial curve. However, in this study, they are not used as indicators of the siphon's geometric features. This is because curvature often exhibits nonlinear variation, making direct comparison between different cases difficult. Furthermore, the calculation of torsion is highly sensitive to local noise and numerical fluctuations, which can reduce its stability. The ICA siphon shape typically consists of two prominent bends (B1 and B2). Therefore, this study employs a geometry-based fitting method to describe its overall shape.

The detail is illustrated in *Figure 4*. In step (c), for each bending region, a segment of the centerline that represents the characteristic circular arc of the bend is first selected. In step (d), a best-fit plane is obtained by applying the least-squares method to the selected point set, which approximately fits the spatial distribution of these points. In step (e), the centerline points of the segment are orthogonally projected onto the fitted plane, and a circular arc is further fitted on the plane to obtain the representative geometric feature of the bend. Finally, in step (f), the radii of the two fitted circles, CR_{B1} and CR_{B2}, as well as the inter-plane angle βbetween the two fitted planes, are determined. These parameters respectively represent the degree of curvature of each bend and the degree of non-coplanarity between the two bends in three-dimensional space.

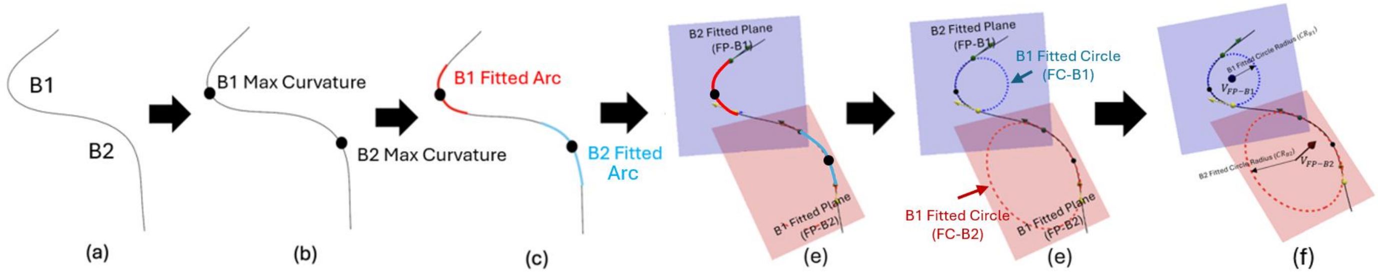

Figure 4. Stepwise procedure for geometric fitting of the ICA siphon: (a–f) illustrate the selection of bending segments, plane and circle fitting, and extraction of curvature radii and inter-plane angle.

3.1. Definition of the Fitted Arc

To perform circular fitting for each bending segment, it is first necessary to determine the portion of the centerline that represents the characteristic geometry of the vessel. In this study, the range of the fitted arc was defined based on a curvature threshold α $(0 < \alpha < 1)$.

Taking the B1 bend as an example, the start and end positions of the fitted arc were determined by searching along both upstream and downstream directions from the point of maximum curvature κ_{B1}. The continuous region where the curvature values satisfied $\kappa_i > \alpha\kappa_{B1}$ was defined as the fitted arc segment.

The cumulative arc lengths corresponding to the first and last points of this region were denoted as $s_{B1}^{start}$and s_{B1}^{end}, respectively. The fitted arc segment $\boldsymbol{C}_{B1\_fit}$ was then defined as:

$$\boldsymbol{C}_{B1\_fit} = \{\boldsymbol{p}_i \in \boldsymbol{C}_{B1} \mid s_{B1}^{start} \le s_i \le s_{B1}^{end}\} \tag{7}$$

- α: threshold parameter $(0 < \alpha < 1)$, which controls the length of the fitting range.

3.2. Definition of the Fitting Plane

To construct the fitted plane for each bending segment, the set of points forming the fitted arc C_{B1}^{fit} is used. This set can be expressed as:

$$\boldsymbol{C}_{B1\_fit} = \{\boldsymbol{p}_j^{B1} \mid j = 1, 2, \dots, N_{B1}\} \tag{8}$$

where each point:

$$\boldsymbol{p}_j^{B1} = (x_j, y_j, z_j)^{\mathsf{T}} = \begin{bmatrix} x_j \\ y_j \\ z_j \end{bmatrix} \tag{9}$$

An arbitrary plane in space can be represented in the general form:

$$ax + by + cz + d = 0 \tag{10}$$

where the vector $\mathbf{n} = (a, b, c)^{\mathsf{T}} \neq \mathbf{0}$ denotes the plane's normal vector.

The centroid of the selected point set is given by:

$$\boldsymbol{\mu}_{B1} = (\bar{x}, \bar{y}, \bar{z})^T = \frac{1}{N_{B1}} \sum_{j=1}^{N_{B1}} \begin{pmatrix} x_j \\ y_j \\ z_j \end{pmatrix} \tag{11}$$

Because the centroid lies on the fitted plane, substituting it into the plane equation gives:

$$a\bar{x} + b\bar{y} + c\bar{z} + d = 0 \tag{12}$$

which yields $d = -(\bar{x} + b\bar{y} + c\bar{z})$. Thus, the plane equation becomes:

$$a(x - \bar{x}) + b(y - \bar{y}) + c(z - \bar{z}) = 0 \tag{13}$$

The signed distance from point $\boldsymbol{p}_j^{B1}$ to the plane is defined as:

$$d_j = a(x_j - \bar{x}) + b(y_j - \bar{y}) + c(z_j - \bar{z}) \tag{14}$$

The total squared error is given by:

$$Q_{B1}(\mathrm{a}, \mathrm{b}, \mathrm{c}) = \sum_{j=1}^{N_{B1}} d_j^2 = \sum_{j=1}^{N_{B1}} \left[a(x_j - \bar{x}) + b(y_j - \bar{y}) + c(z_j - \bar{z})\right]^2 \tag{15}$$

By minimizing $Q(a, b, c)$ under the constraint $a^2 + b^2 + c^2 = 1$, and using the Lagrange multiplier method, a unique normal vector can be obtained as $\boldsymbol{n}_{B1} = (a_{B1}, b_{B1}, c_{B1})^T$, from which the fitting plane equation is determined as:

$$\Pi_{B1}: \quad \boldsymbol{n}_{B1} \cdot (\mathbf{x} - \boldsymbol{\mu}_{B1}) = 0 \tag{16}$$

where $\mathbf{x}$ is the coordinate vector of any point on the plane.

In addition, to ensure that the point of maximum curvature used for subsequent arc fitting lies on the fitting plane, the fitting plane Π_{B1} must pass through the point corresponding to the maximum curvature.

3.3. Point Projection onto the Fitted Plane

After obtaining the fitting plane Π_{B1}, the centerline points of the fitted arc $\boldsymbol{C}_{B1\_fit}$ were orthogonally projected onto the plane to extract the geometric characteristics of the curved segment and to perform circular fitting within the plane.

For each point $\boldsymbol{p}_j^{B1}$ belonging to $\boldsymbol{C}_{B1\_fit}$, a vector $\mathbf{v}_j$ from the centroid $\boldsymbol{\mu}_{B1}$ of the fitted arc to that point is defined as:

$$\mathbf{v}_j = \boldsymbol{p}_j^{B1} - \boldsymbol{\mu}_{B1} \tag{17}$$

This vector can be decomposed into two components:

- Normal components along the plane's normal direction: $\mathbf{v}_{j,\perp} = (\mathbf{v}_j \cdot \mathbf{n}_{B1})\mathbf{n}_{B1}$.
- The in-plane component parallel to the plane: $\mathbf{v}_{j,\parallel} = \mathbf{v}_j - (\mathbf{v}_j \cdot \mathbf{n}_{B1})\mathbf{n}_{B1}$。

By removing the normal component, the projected point $\boldsymbol{q}_j$ on the fitted plane can be expressed as the three-dimensional coordinates of the point after being orthogonally projected onto the plane:

$$\boldsymbol{q}_j = \boldsymbol{p}_j^{B1} - (\mathbf{v}_j \cdot \mathbf{n}_{B1})\mathbf{n}_{B1} \tag{18}$$

Substituting *Equation (17)* into *Equation (18)*, the explicit form becomes:

$$\boldsymbol{q}_j = \boldsymbol{p}_j^{B1} - \left[\left(\boldsymbol{p}_j^{B1} - \boldsymbol{\mu}_{B1}\right) \cdot \mathbf{n}_{B1}\right]\mathbf{n}_{B1} \tag{19}$$

To describe the positions of these projection points on the plane, a local 2D orthogonal coordinate system $\{\mathbf{e}_1, \mathbf{e}_2\}$ is established on the fitting plane, satisfying:

$$\mathbf{e}_1 \perp \mathbf{n}_{B1}, \qquad \mathbf{e}_2 \perp \mathbf{n}_{B1}, \qquad \mathbf{e}_2 \perp \mathbf{e}_1 \tag{20}$$

In this coordinate system, the two-dimensional coordinates of the projected points are given by:

$$u_j = \left(\boldsymbol{q}_j - \boldsymbol{\mu}_{B1}\right) \cdot \mathbf{e}_1, \qquad v_j = \left(\boldsymbol{q}_j - \boldsymbol{\mu}_{B1}\right) \cdot \mathbf{e}_2 \tag{21}$$

3.4. Definition of Fitted Circles and Shape Parameters

To determine the parameters of the fitted circle, the least-squares method was applied to the projected points (u_j, v_j) obtained in the local 2D coordinate system defined on the fitted plane.

For each projected point, the circular relationship is given by:

$$\left(u_j - a\right)^2 + \left(v_j - b\right)^2 = r^2 \tag{22}$$

where (a, b) represents the coordinates of the circle's center, and r denotes the radius.

The sum of squared errors is then defined as:

$$S(a,b,r) = \sum_{j=1}^{N_{B1}} \left[\left(u_j - a\right)^2 + \left(v_j - b\right)^2 - r^2 \right]^2 \tag{23}$$

By minimizing $S(a,b,r)$, the optimal parameters (a,b,r) of the fitted circle can be obtained.

The fitted circle radii CR_{B1} and CR_{B2} represent the curvature intensity of the bending segments B_1 and B_2, respectively. To quantify the degree of spatial non-coplanarity between the two bending segments, the normal vectors of their corresponding fitted planes are defined as $\mathbf{n}_{B1}$ and $\mathbf{n}_{B2}$, and the angle between them is defined as:

$$\beta = cos^{-1}\left(\frac{\mathbf{n}_{B1} \cdot \mathbf{n}_{B2}}{|\mathbf{n}_{B1}||\mathbf{n}_{B2}|}\right) \tag{24}$$

4. Selection of the Optimal Fitted Arc

As described in Section 3.1, the fitting range of the circular arc is determined by the threshold parameter α. To further obtain a fitted arc that best represents the shape of the curved segment, this study calculated combinations of different α values and selected the optimal arc based on the principle of minimizing fitting error.

Taking the B1 as an example, the error function between the curvature of the fitted circle and the original curvature distribution is defined as:

$$Error_{B1} = \left|\frac{1}{CR_{B1}} - \kappa_{B1}\right| \tag{25}$$

The error represents the deviation between the equivalent curvature of the fitted circle and the mean curvature of the original curve.

For each centerline, α was varied within the range of 0.1 to 0.9, and the corresponding error function was computed to compare the distribution of fitting errors. The α value that yielded the smallest error was selected as the optimal threshold parameter, denoted as α_{opt}. Therefore, the start and end points of the optimal fitted arc, denoted as $s_{B1\_opt}^{start}$ and $s_{B1\_opt}^{end}$, these positions represent the points along the centerline where the curvature values first and last satisfy $\kappa_i > \alpha_{opt} \cdot \kappa_{B1}$.

Accordingly, the set of points forming the optimal fitted arc can be expressed as:

$$\boldsymbol{C}_{B1\_fit}^{opt} = \left\{\boldsymbol{p}_i \in \boldsymbol{C}_{B1} \,\middle|\, s_{B1\_opt}^{start} \leq s_i \leq s_{B1\_opt}^{\mathrm{end}}\right\} \tag{26}$$

Through this method, the optimal fitted arcs for both B1 and B2 segments were determined for each centerline. Subsequently, the corresponding geometric parameters were extracted using the procedures described in the previous sections, providing the basis for quantitative analysis in the following study.

	Section	Symbol	Type	Description
1	1	$\boldsymbol{C}_{ICA}$	Set	Set of points representing the ICA centerline
2		$\boldsymbol{p}_i = (x_i, y_i, z_i)^{\top} = \begin{bmatrix} x_i \\ y_i \\ z_i \end{bmatrix}$	Vector	Coordinate vector of the i-th point
3		$\kappa_i = \kappa(\boldsymbol{p}_i) = \kappa(s_i)$	Scalar	Curvature value at point $\boldsymbol{p}_i$ equivalently expressed as $\kappa(s_i)$ along the arc-length parameterization.
4		$s_i = s(\boldsymbol{p}_i)$	Scalar	Cumulative arc length corresponding to point $\boldsymbol{p}_i$
5	2	s_{B1}, s_{B2}	Scalar	Arc-length positions of the curvature peaks B1 and B2 along the ICA centerline.
6		κ_{B1}, κ_{B2}	Scalar	Curvature values corresponding to the B1 and B2 curvature peaks.
7		$\mathrm{s}_{\mathrm{B1}}^{-}$	Scalar	Arc length corresponding to the curvature valley proximal to the B1 peak (start of the siphon).
8		$\mathrm{s}_{\mathrm{B2}}^{+}$	Scalar	Arc length corresponding to the curvature valley distal to the B2 peak (end of the siphon).
9		$\mathrm{s}_{\mathrm{B1,2}}$	Scalar	Arc length corresponding to the curvature valley between B1 and B2 (boundary separating the two bends).
10		$\boldsymbol{C}_{Siphon}$	Set	Set of points within the ICA siphon, defined for $s_{B1}^{-} \le s_i \le s_{B2}^{+}$
11		$\boldsymbol{C}_{B1}$	Set	Set of points corresponding to the B1 bend, defined for $s_{B1}^{-} \le s_i \le s_{B12}$
12		$\boldsymbol{C}_{B2}$	Set	Set of points corresponding to the B2 bend, defined for $s_{B12} \le s_i \le s_{B2}^{+}$
13	3.1	α	Scalar	Threshold parameter $(0 < \alpha < 1)$ controlling the curvature-based fitting range.
14		s_{B1}^{start}	Scalar	Cumulative arc length at the first point where the curvature κ_i satisfies $\kappa_i > \alpha\kappa_{B1}$.
15		s_{B1}^{end}	Scalar	Cumulative arc length at the last point where the curvature κ_i satisfies $\kappa_i > \alpha\kappa_{B1}$.
16		$\boldsymbol{C}_{B1\_fit}$	Set	Set of points between s_{B1}^{start} and s_{B1}^{end} that meet the curvature condition $\kappa_i > \alpha\kappa_{B1}$.
17	3.2	$\boldsymbol{p}_j^{B1}$	Vector	Coordinate vector of the j-th point belonging to the set $\boldsymbol{C}_{B1\_fit}$.
18		$\boldsymbol{\mu}_{B1}$	Vector	Centroid of all points in the set $\boldsymbol{C}_{B1\_fit}$.
19		d_j	Scalar	Signed distance from point $\boldsymbol{p}_j^{B1}$ to the fitting plane
20		$Q_{B1}(a, b, c)$	Scalar	Sum of squared distances (error function) used in plane fitting. a, b and c are the components of the normal vector $\boldsymbol{n}_{B1}$
21		$\boldsymbol{n}_{B1}$	Vector	Normalized normal vector of the fitted plane obtained by minimizing Q_{B1}.
22		$\mathbf{x}$	Vector	Coordinate vector of an arbitrary point on the plane
23		Π_{B1}	Plane	$\boldsymbol{C}_{B1\_fit}$ Fitted plane
24	3.3	$\mathbf{v}_j$	Vector	Vector from the centroid $\boldsymbol{\mu}_{B1}$ to point $\boldsymbol{p}_j^{B1}$.
25		$\mathbf{v}_{j,\perp}$	Vector	Normal component of $\mathbf{v}_j$ along the plane normal $\boldsymbol{n}_{B1}$.

26		$\mathbf{v}_{j,\parallel}$	Vector	In-plane component of $\mathbf{v}_j$ parallel to the fitted plane.
27		$\boldsymbol{q}_j$	Vector	Three-dimensional coordinates of the point after being orthogonally projected onto the fitted plane
28		$\mathbf{e}_1, \mathbf{e}_2$	Vector	Local orthogonal basis vectors on the fitted plane
29		u_j, v_j	Scalar	2D coordinates of the projected point $\boldsymbol{q}_j$ in the plane coordinate system.
30	3.4	$S(a, b, r)$	Scalar	Sum of squared errors between projected points and the fitted circle, used to determine optimal circle parameters. Here, a and b are the coordinates of the circle center, and r is the circle radius.
31		CR_{B1}	Scalar	Radius of the fitted circle corresponding to the B1 bend.
32		CR_{B2}	Scalar	Radius of the fitted circle corresponding to the B2 bend.
33		β	Scalar	Angle between the fitted planes of B1 and B2, representing the degree of spatial non-coplanarity between the two bends.
34	4	$Error_{B1}$	Scalar	Represents the deviation between the equivalent curvature of the fitted circle and the curvature of the original curve
35		α_{opt}	Scalar	The optimal threshold parameter obtained by varying α within the range of 0.1–0.9 and selecting the value that minimizes the fitting error.
36		$s_{B1\_opt}^{start}, s_{B1\_opt}^{end}$	Scalar	The cumulative arc-length positions of the start and end points of the optimal fitted arc, corresponding to the points along the centerline where the curvature first and last satisfies $\kappa_i > \alpha_{opt} \cdot \kappa_{B1}$.
37		$\boldsymbol{C}_{B1\_fit}^{opt}$	set	Set of points forming the optimal fitted arc.