Probability Theory - Overview

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What is probability?

Fundamentally related to the frequencies of repeated events

- Frequentist

Fundamentally related to our own certainty and uncertainty of events - Bayesians.

<https://courses.edx.org/asset-v1:UCSanDiegoX+DSE210x+3T2017+type@asset+block@HistoryOfProbAndStat.pdf>

Sample Space

The sample space , denoted by *S*, is the collection of all possible outcomes of a random experiment.

Outcome

An outcome is a result of a random experiment.

Event

An event is a set of outcomes of an experiment (a subset of the sample space) to which a probability is assigned.

Counting

(CLASSICAL DEFINITION OF PROBABILITY)

$$P(\text{Event}) = \frac{\#\ \text{Favorable Outcomes}}{\#\ \text{Possible Outcomes}}$$

What are our favorable outcomes?

What are the possible outcomes?

Axioms of Probability

$$(1)\ P(A) \geq 0 \text{ for all } A \subset S$$
$$(2)\ P(S) = 1$$
$$(3)\ \text{If } A \cap B = \emptyset,$$
$$\text{then } P(A \cup B) = P(A) + P(B)$$

Independent Events

The probability one event occurs in no way affects the probability of the other event occurring. Eg: Owning a dog and growing your own herb garden.

Dependent Events

When two events are said to be dependent, the probability of one event occurring influences the likelihood of the other event. Eg: Robbing a bank and going to jail.

Mutual Exclusivity

Two events are said to be mutually exclusive if they can't occur at the same time. For a given sample space, it is either one or the other but not both. Eg: Head or Tail.

Out of 400 students, 270 do Computer Science, 300 do English and 50 do Business studies. All those doing Computer Science do English, 20 take Computer Science and Business studies and 35 take English and Business studies. Calculate the probability that a pupil drawn at random will take:

- English, but not Business studies or Computer Science
- English or Business studies but not Computer Science

Conditional probability

The conditional probability of an event A, given an event B with $\mathbf{P}(B) > 0$, is defined by

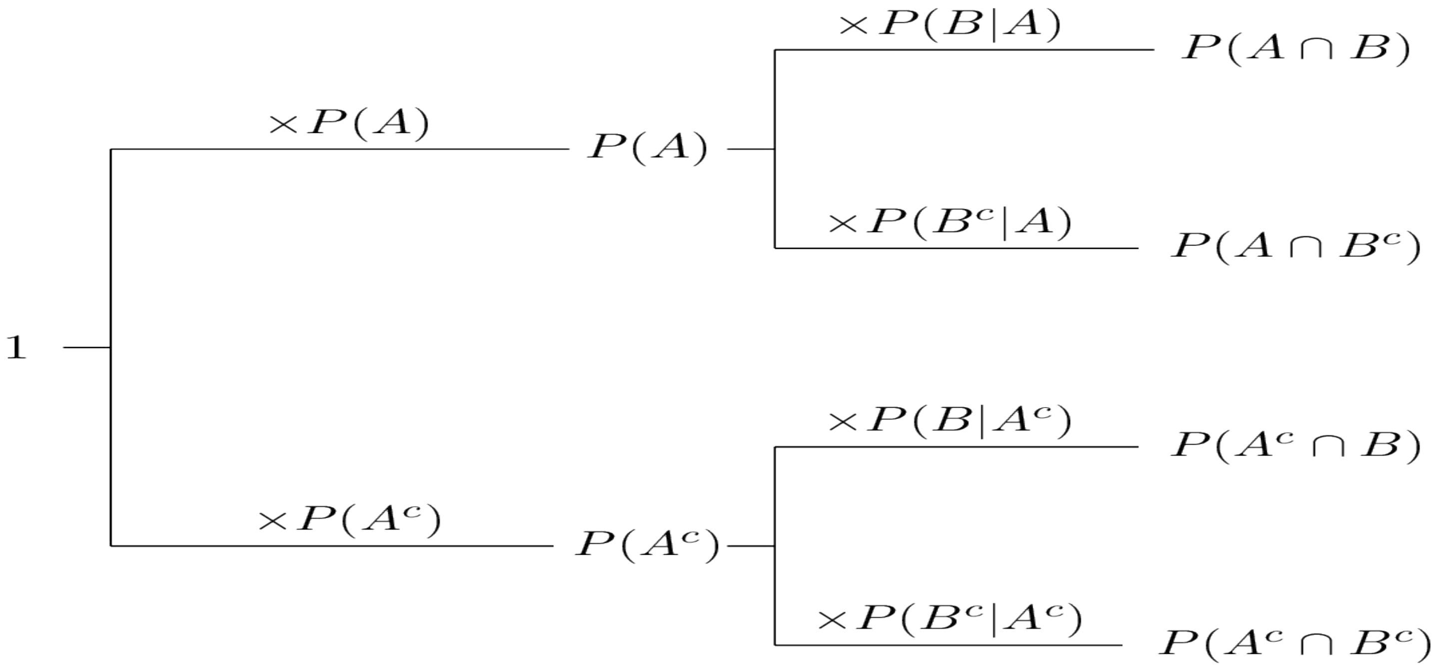
$$\mathbf{P}(A \mid B) = \frac{\mathbf{P}(A \cap B)}{\mathbf{P}(B)},$$

- A fair 4-sided die is rolled twice and we assume all 16 possible outcomes are equally likely. Let X and Y be the result of the 1st and the 2nd roll respectively. Determine the conditional probability P(A I B), where

$$A = \{\max(X,Y) = 4\} \qquad\qquad B = \{\min(X,Y) = 2\}$$

- If an aircraft is present in a certain area, a radar detects it and generates an alarm signal with probability 0.99. If an aircraft is not present. the radar generates a (false) alarm, with probability 0.10. We assume that an aircraft is present with probability 0.05. What is the probability of no aircraft presence and a false alarm? What is the probability of aircraft presence and no detection?

Chain Rule



$\times P(B|A)$
$P(A \cap B)$
$\times P(A)$
$P(A)$
$\times P(B^c|A)$
$P(A \cap B^c)$
1
$\times P(B|A^c)$
$P(A^c \cap B)$
$\times P(A^c)$
$P(A^c)$
$\times P(B^c|A^c)$
$P(A^c \cap B^c)$

Generalization

Chain rule for conditional probability:

$$P(A1 \cap A2 \cap \cdots \cap An) = P(A1)P(A2|A1)P(A3|A2,A1) \cdots P(An|An-1,An-2,\cdots,A1)$$

Problem

In a factory there are 100 units of a certain product, 5 of which are defective. We pick three units from the 100 units at random. What is the probability that none of them are defective?

Total probability theorem

Let $A_1, \ldots, A_n$ be disjoint events that form a partition of the sample space (each possible outcome is included in exactly one of the events $A_1, \ldots, A_n$) and assume that $\mathbf{P}(A_i) > 0$, for all i. Then, for any event B, we have

$$\begin{aligned}\mathbf{P}(B) &= \mathbf{P}(A_1 \cap B) + \cdots + \mathbf{P}(A_n \cap B) \\ &= \mathbf{P}(A_1)\mathbf{P}(B \mid A_1) + \cdots + \mathbf{P}(A_n)\mathbf{P}(B \mid A_n).\end{aligned}$$

Problem

In a certain country 60% of registered voters are Republicans ,30% are Democrats and 10% are Independents.When those voters were asked about increasing military spending 40% of Republicans opposed it , 65% of the Democrats opposed it and 55% of the Independents opposed it . What is the probability that a randomly selected voter in this county opposes increased military spending?

Bayes' Theorem

- For any two events A and B, where $P(A) \neq 0$, we have

$$P(B|A) = \frac{P(A|B)P(B)}{P(A)}.$$

- If $B_1, B_2, B_3, \cdots$ form a partition of the sample space S, and A is any event with $P(A) \neq 0$, we have

$$P(B_j|A) = \frac{P(A|B_j)P(B_j)}{\sum_i P(A|B_i)P(B_i)}.$$

Problems

- A factory production line is manufacturing bolts using three machines, A, B and C. Of the total output, machine A is responsible for 25%, machine B for 35% and machine C for the rest. It is known from previous experience with the machines that 5% of the output from machine A, 4% from machine B and 2% from machine C are defective. A bolt is chosen at random from the production line and found to be defective. What is the probability that it came from machine A?

- In my town, it's rainy one third of the days. Given that it is rainy, there will be heavy traffic with probability 1/2, and given that it is not rainy, there will be heavy traffic with probability 1/4. If it's rainy and there is heavy traffic, I arrive late for work with probability 1/2. On the other hand, the probability of being late is reduced to 1/8 if it is not rainy and there is no heavy traffic. In other situations (rainy and no traffic, not rainy and traffic) the probability of being late is 0.25. You pick a random day.
    - Given that I arrived late at work, what is the probability that it rained that day?

- 1% of women at age forty who participate in routine screening have breast cancer. 90% of women with breast cancer will get positive mammographies. 8% of women without breast cancer will also get positive mammographies. A woman in this age group had a positive mammography in a routine screening. What is the probability that she actually has breast cancer?

- The blue M&M was introduced in 1995. Before then, the color mix in a bag of plain M&Ms was (30% Brown, 20% Yellow, 20% Red, 10% Green, 10% Orange, 10% Tan). Afterward it was (24% Blue , 20% Green, 16% Orange, 14% Yellow, 13% Red, 13% Brown).

  A friend of mine has two bags of M&Ms, and he tells me that one is from 1994 and one from 1996. He won't tell me which is which, but he gives me one M&M from each bag. One is yellow and one is green. What is the probability that the yellow M&M came from the 1994 bag?

Joint, Marginal and conditional probabilities

Joint probabilities for rain and wind:

	no wind	some wind	strong wind	storm
no rain	0.1	0.2	0.05	0.01
light rain	0.05	0.1	0.15	0.04
heavy rain	0.05	0.1	0.1	0.05

- P (no wind)?
- P (light rain)?
- P (some wind | heavy rain)?
- P (light rain | storm)?

Probability with Counting

- Terminologies
    - Sampling - choosing an element from a set
    - With / without replacement - whether we put back an element after each draw
    - Ordered / unordered - If ordering matters i.e. (a_1, b_1) is not the same as (b_1, a_1)

Four possibilities:

- Ordered sampling with replacement
- Ordered sampling without replacement
- Unordered sampling without replacement
- Unordered sampling with replacement

- Ordered sampling with replacement

A local telephone number is a 7-digit sequence, but the first digit has to be different from 0 or 1 .

How many distinct telephone numbers are there?

- Ordered sampling without replacement - Permutation

Count the number of words that consist of four distinct letters. This is the problem of counting the number of 4-permutations of the 26 letters in the alphabet.

- Unordered sampling without replacement - Combination

    - Bernoulli Trials - a random experiment with two possible outcomes “success” and “failure”

Suppose that I have a coin for which $P(H)=p$ and $P(T)=1-p$. I toss the coin 5 times.

What is the probability that the outcome is $THHHH$?

What is the probability that I will observe exactly three heads and two tails?

- Unordered sampling with replacement

Ten passengers get on an airport shuttle at the airport. The shuttle has a route that includes 5 hotels, and each passenger gets off the shuttle at his/her hotel. The driver records how many passengers leave the shuttle at each hotel. How many different possibilities exist?

Summary

ordered sampling with replacement	n^k
ordered sampling without replacement	$P_k^n = \frac{n!}{(n-k)!}$
unordered sampling without replacement	$\binom{n}{k} = \frac{n!}{k!(n-k)!}$
unordered sampling with replacement	$\binom{n+k-1}{k}$

Partitions

Definition. The **number of distinguishable permutations** of n objects, of which:

- n_1 are of one type
- n_2 are of a second type
- ... and ...
- n_k are of the last type

and $n = n_1 + n_2 + ... + n_k$ is given by:

$$\binom{n}{n_1 n_2 n_3 \ldots n_k} = \frac{n!}{n_1!n_2!n_3!\ldots n_k!}$$

Problem

How many different words (letter sequences) can be obtained by rearranging the letters in the word “TATTOO”?

Extras

- What is a random variable.
- What is a distribution.
- What are mostly used distributions.
- What is PDF, PMF, CDF.
- What is mean, variance , covariance.

- Best book
  - Introduction to Probability by Dimitri Bertsekas and John Tsitsiklis (First three chapters)

- Best resources
  - https://projects.iq.harvard.edu/stat110
  - https://www.edx.org/course/introduction-probability-science-mitx-6-041x-2

References

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