Project Euler - 108

Supporting math for problem 108

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21st of July 2022

Introduction

We want to find a function $\Psi(n)$ such that it returns the number of distinct solutions for the Diophantine equation $\frac{1}{x} + \frac{1}{y} = \frac{1}{n}$. Note that since we are looking for distinct solutions we can assume without loss of generality that $x \leq y$ and note that obviously $n < x$.

Solution

Note that we can rewrite the Diophantine equation such that $n = \frac{xy}{x+y}$ i.e. $nx + ny = xy$. This is not helpful at the moment, however we can further rewrite the equation: $xy - nx - ny = 0$. This is more familiar, this looks a lot like something squared but we are missing a term, we get: $xy - nx - ny + n^2 = n^2$ i.e. $(x-n)(y-n) = n^2$. This means that $\Psi(n)$ models the number of distinct factorizations of n^2 into two numbers. It turns out that there is a simple formula for this. If $m = \prod_{p|n} p^{a_p}$ then $d(m) = \lceil \frac{1}{2} \prod_{p|m} (a_p + 1) \rceil$ where $d(m)$ is the number of ways to write m as a product of two numbers. We then get that

$$\Psi(n) = \lceil \frac{1}{2} \prod_{p|n} (2a_p + 1) \rceil$$

We can now simply brute force through iterations of n until we get $\Psi(n) > 1000$.