Project Euler - 110

Supporting math for problem 110

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Introduction

This problem is almost exactly the same as problem number 108, the reader should therefore familiarize themselves with that solution before reading this one. The key equation to solving this problem is

$$\Psi(n) = \lceil \frac{1}{2} \prod_{p|n} (2a_p + 1) \rceil$$

Solution

We need to make a few key observations. First of all, we want to minimize n for a large $\Psi(n)$. Notice that the product will always have odd terms but never 1 as a term. Now suppose we have a number $n = p_1^{a_1} \cdot p_2^{a_2} \cdot \ldots \cdot p_m^{a_m}$. Now, suppose that $1 = a_1 = a_2 = \ldots = a_m$ then we would have that $\Psi(n) = \lceil \frac{1}{2} \cdot 3^m \rceil$. This means that $\Psi(n)$ is big when n has a lot of unique prime factors. Now let us just focus on the product. Now suppose that for $p_1 = 2$ that $a_1 = 2$ but everything else remains the same, what does this mean for our product? Essentially, it means that if we keep a constant number of prime factors (not distinct) we "lose" two three-s. So our product is divided by 9 and multiplied by 5. Now it is clear that we are looking for some combination of $a_1, a_2, \ldots, a_m$ and $p_1, p_2, \ldots, p_m$ such that $\Psi(n) > 4.000.000$. Let us make a few more observations, note that whatever our combination of $a_1, a_2, \ldots, a_m$ turns out to be, we want the highest powers on the smallest primes, and we want our primes to be consecutive since we want to minimize n and whatever primes we choose does not influence $\Psi(n)$ since it only depends on $a_1, a_2, \ldots, a_m$. Note also that it is very improbable that we would have for some i that $a_i = 5$ since that would mean that at minimum we would have a factor of n be $2^5 = 32$ and at that point it is "cheaper" to use another prime instead of having a high power of 2. We can now check all combinations of a, b, c, d where

a, b, c, d are the number of primes to the power of 1,2,3,4 respectively. Then we can just take the minimum of all those kinds of numbers and that will return our target number.