The Distance Between Two Vectors

Fold

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Sometimes we will want to calculate the distance between two vectors or points. We will derive some special properties of distance in Euclidean n-space thusly. Given some vectors $\vec{u}, \vec{v} \in \mathbb{R}^n$, we denote the distance between those two points in the following manner.

Definition: Let $\vec{u}, \vec{v} \in \mathbb{R}^n$. Then the **Distance** between $\vec{u}$ and $\vec{v}$ is $d(\vec{u}, \vec{v}) = \|\vec{u} - \vec{v}\| = \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2 \dots (u_n - v_n)^2}$.

We will now look at some properties of the distance between points in $\mathbb{R}^n$.

Theorem 1 (Symmetry Property of Distance): If $\vec{u}, \vec{v} \in \mathbb{R}^n$ then $d(\vec{u}, \vec{v}) = d(\vec{v}, \vec{u})$.

- **Proof:** We note that $d(\vec{u}, \vec{v}) = ||\vec{u} - \vec{v}|| = \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2 \dots (u_n - v_n)^2}$ and that $d(\vec{v}, \vec{u}) = ||\vec{v} - \vec{u}|| = \sqrt{(v_1 - u_1)^2 + (v_2 - u_2)^2 \dots (v_n - u_n)^2}$. To show these are equal, we must only show that $(u_i - v_i)^2 = (v_i - u_i)^2$ for $1 \leq i \leq n$ and $i \in \mathbb{N}$.
- Notice that $(u_i - v_i)^2 = u_i^2 - 2u_iv_i + v_i^2 = v_i^2 - 2u_iv_i + 2u_i^2 = (v_i - u_i)^2$. It therefore follows that $d(\vec{u}, \vec{v}) = d(\vec{v}, \vec{u})$ as the value underneath the square roots is equal. ■

Theorem 2 (Non-Negativity of Distances): If $\vec{u}, \vec{v} \in \mathbb{R}^n$ then $d(\vec{u}, \vec{v}) \geq 0$.

- **Proof:** Since $d(\vec{u}, \vec{v}) = ||\vec{u} - \vec{v}|| = \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2 \dots (u_n - v_n)^2}$ and $(u_i - v_i)^2 \geq 0$ for all $1 \leq i \leq n$, $i \in \mathbb{N}$ then clearly $d(\vec{u}, \vec{v}) \geq 0$. ■

Theorem 3 (The Triangle Inequality of Distances): If $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^n$ then $d(\vec{u}, \vec{v}) \leq d(\vec{u}, \vec{w}) + d(\vec{w}, \vec{v})$.

- **Proof:** We will begin by operating on the lefthand side:

$$\begin{aligned} d(\vec{u}, \vec{v}) &= ||\vec{u} - \vec{v}|| \\ d(\vec{u}, \vec{v}) &= ||\vec{u} - \vec{w} + \vec{w} - \vec{v}|| \\ d(\vec{u}, \vec{v}) &= ||(\vec{u} - \vec{w}) + (\vec{w} - \vec{v})|| \\ d(\vec{u}, \vec{v}) &\leq ||(\vec{u} - \vec{w})|| + ||(\vec{w} - \vec{v})|| \\ d(\vec{u}, \vec{v}) &\leq d(\vec{u}, \vec{w}) + d(\vec{w}, \vec{v}) \quad \blacksquare \end{aligned} \tag{1}$$

Example 1

Determine the Euclidean distance between $\vec{u} = (2, 3, 4, 2)$ and $\vec{v} = (1, -2, 1, 3)$.

Applying the formula given above we get that:

$$\begin{aligned} d(\vec{u}, \vec{v}) = ||\vec{u} - \vec{v}|| &= \sqrt{(2-1)^2 + (3+2)^2 + (4-1)^2 + (2-3)^2} \\ d(\vec{u}, \vec{v}) = ||\vec{u} - \vec{v}|| &= \sqrt{1 + 25 + 9 + 1} \\ d(\vec{u}, \vec{v}) = ||\vec{u} - \vec{v}|| &= \sqrt{36} \\ d(\vec{u}, \vec{v}) = ||\vec{u} - \vec{v}|| &= 6 \end{aligned} \tag{2}$$

Therefore $d(\vec{u}, \vec{v}) = 6$.