Su2024MATH4991Lecture2Mk3

Hellenic Foundations

Mr. Harley Caham Combest

May 2025

Chapter 4 - Hellenic Traditions

I. Cultural Invocation

- **Civilization:** Hellenic Greece
- **Time Period:** 600 BCE – 300 BCE
- **Place:** Ionia, Athens, Croton, Tarentum
- **Social Roles:** Philosophers, Geometers, Astronomer-Mystics, Rationalists

Opening Statement: *Welcome to the Heroic Age — when Greeks, newly awakened to abstraction, asked not just how many, but why. Gone are the ropes and trade ledgers of Egypt and Babylon. Here begins a new kind of math — born in argument, tempered in paradox, and driven by the desire to know.*

II. Problem–Solution Cycles

Problem 1: How do you divide a line in the most balanced way possible?

Topic: The Golden Ratio

Figures: Fig. 4.1, Fig. 4.2, Fig. 4.3, Fig. 4.10

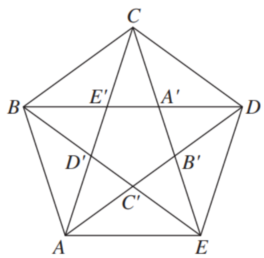
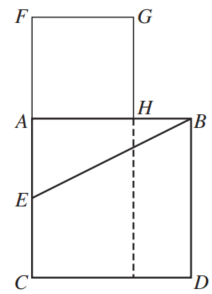


FIG. 4.1

FIG. 4.2



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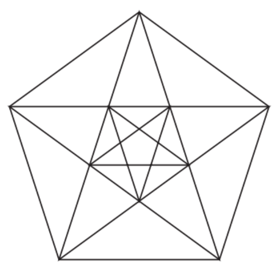


FIG. 4.10

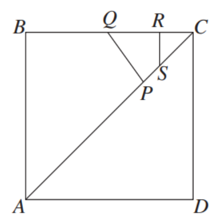


FIG. 4.11

The Greeks weren't looking for halves — they were looking for harmony. They discovered a special way to divide a line such that:

$$\frac{\text{long part}}{\text{short part}} = \frac{\text{whole}}{\text{long part}}$$

This self-replicating division became known as the **golden section**, or golden ratio ($\phi \approx 1.618$).

Fig. 4.1 shows the pentagram, where intersecting diagonals naturally divide one another in the golden ratio.

Fig. 4.2 demonstrates the recursive structure of the ratio — each subdivision replicates the same proportion infinitely.

Fig. 4.3 shows how to construct this division using a square and circle alone.

Fig. 4.10 reveals golden ratios embedded inside a regular pentagon.

The Greeks, especially the Pythagoreans, saw this ratio not as a number, but as a cosmic principle — a visual echo of balance.

Problem 2: Can you measure the area of a crescent shape?

Topic: Squaring the Lune

Figures: Fig. 4.5, Fig. 4.6, Fig. 4.7, Fig. 4.8

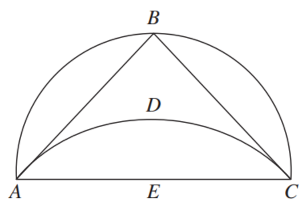


FIG. 4.5

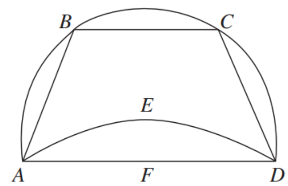
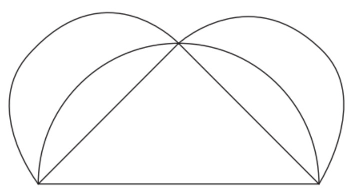


FIG. 4.6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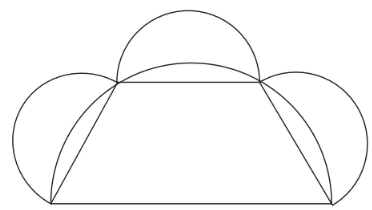


FIG. 4.8

The circle was sacred — but unmeasurable. Enter Hippocrates of Chios.

Fig. 4.5 shows a crescent (or **lune**) bounded by a semicircle and a triangle. Hippocrates proved the area of the lune equals the area of the triangle — a radical breakthrough: a curved figure made measurable.

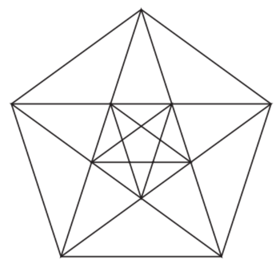
Figs. 4.6–4.7 extend this to other configurations. In *Fig. 4.7*, two lunes together equal the area of a right triangle.

Fig. 4.8 demonstrates a failed leap: trying to patch together many lunes to square the full circle.

These early steps laid groundwork for later concepts of integration and analysis — long before calculus, the Greeks were trying to measure what bends.

Problem 3: Can the diagonal of a square be measured exactly?

Topic: Incommensurability
Figure: Fig. 4.11



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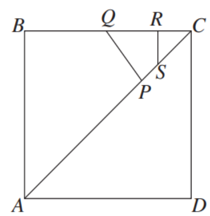


FIG. 4.11

The Pythagoreans believed all lengths could be expressed as ratios of whole numbers. Then came the diagonal of the square.

Fig. 4.11 shows a recursive proof by contradiction — assuming the diagonal is rational leads to an infinite descent. This proved that $\sqrt{2}$ is irrational, or as the Greeks called it, **incommensurable** with the unit length.

This discovery shattered the belief that "all is number" and forced Greek thinkers to distinguish between discrete and continuous — number vs. magnitude.

Problem 4: Can you do algebra using only geometry?

Topic: Geometric Algebra

Figures: Fig. 4.12, Fig. 4.13, Fig. 4.14, Fig. 4.15, Fig. 4.16, Fig. 4.17

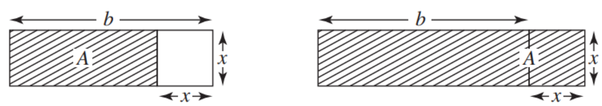
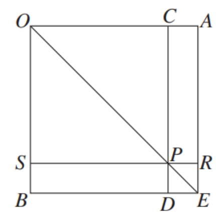


FIG. 4.12



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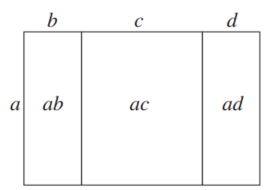

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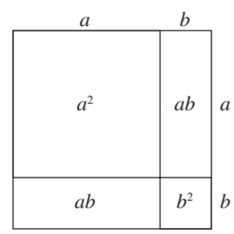
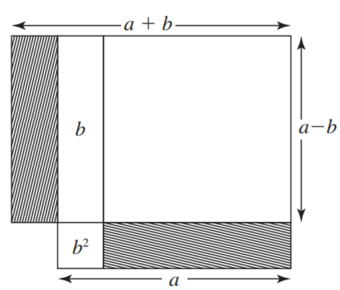


FIG. 4.15



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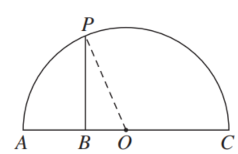

FIG. 4.17

Before symbolic algebra, the Greeks used diagrams to express equations.

Fig. 4.15 shows $(a+b)^2$ as a large square composed of four parts:

$$a^2 \quad \text{(top-left)}, \quad b^2 \quad \text{(bottom-right)}, \quad \text{and two } ab \text{ rectangles}$$

Fig. 4.16 illustrates the identity $a^2 - b^2 = (a + b)(a - b)$ by rearranging a large square minus a smaller square into a rectangle.

Fig. 4.17 uses a semicircle and a right triangle to geometrically construct $\sqrt{ab}$.

Figs. 4.12–4.14 provide additional area-based reasoning for quadratics and proportions.

This was not a workaround — it was a robust form of reasoning. Their algebra was **spatial**, not symbolic.

Problem 5: How do you double the volume of a cube using only geometry?

Topic: Conic Sections and the Cube Root of 2

Figures: Fig. 4.19, Fig. 4.20

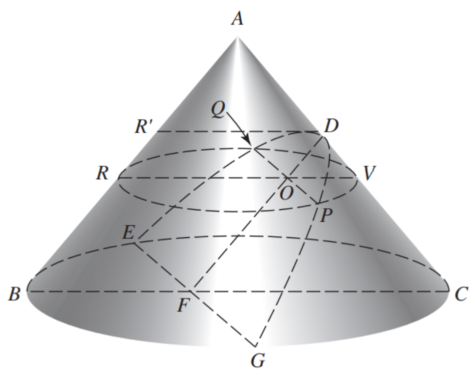

FIG. 4.19

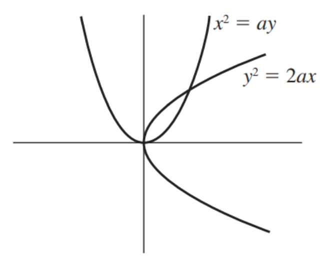

FIG. 4.20

The Delian oracle asked the Athenians to double a cubical altar. They thought they could just double the edge length — but that would octuple the volume.

The real challenge: find x such that

$$x^3 = 2a^3$$

Straightedge and compass couldn't solve it. So Menaechmus — a student of Plato's academy — invented a new tool: **conic sections**.

Fig. 4.19 shows the generation of a parabola from slicing a cone.

Fig. 4.20 shows the breakthrough: intersecting a parabola and hyperbola produces a point that solves the problem.

This wasn't just a clever fix — it marked the invention of a new kind of curve, one that would later underpin analytic geometry and the idea of coordinates.

III. Civilizational Logic Summary

- **Core Principles:** Proof, Harmony, Ontology, Proportion, Crisis
- **Mathematical Identity:** *This was a civilization of Reason.*
- **Conception of Truth:** Not "what works," but "what must be."

IV. Closing Dialectic

Summary Statement:

They didn't just solve problems — they defined what it meant for a problem to be solved.

Exit Prompt:

You are a student of Plato. You must prove a truth of geometry with no ruler, no calculator, and no tolerance for contradiction. What would you choose to prove — and how would you begin?

Chapter 5 - Euclid

I. Cultural Invocation

- **Civilization:** Hellenistic Egypt under the Ptolemies
- **Time Period:** c. 300 BCE
- **Place:** Alexandria — Library and Museum
- **Social Roles:** Philosopher-scholars, Geometers, Platonic inheritors

Opening Statement: *Here in the bright halls of the Alexandrian Museum, geometry is no longer a craft — it is the mind's cathedral. Euclid stands at the intersection of Plato and Aristotle, compiling not just the truths of geometry, but the logic by which truth itself may be pursued. No longer will we rely on ritual or intuition. Here, at last, there is no royal road.*

II. Problem–Solution Cycles

Problem 1: How can you divide a shape with total fairness?

You're given a trapezoid — not just any shape, but one bounded by points a, b, q, and d. The upper side ab is shorter than the base qd, and both sides extend upward to meet at point e, forming triangle aed.

Your task:

Draw a line parallel to the bases that cuts the trapezoid abqd exactly in half — dividing its area into two equal parts.

This isn't guesswork. It's geometric reasoning.

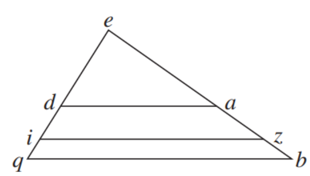

FIG. 5.1

Fig. 5.1 shows this setup. A line zi is drawn from point z on segment ae to point i on segment ed, parallel to both bases.

The key insight: A line parallel to the bases creates a smaller, similar trapezoid. But only one such line — placed precisely — will divide the total area evenly.

To find the correct position of point z, the Greeks used the condition:

$$ze^2 = \frac{1}{2}(eb^2 + ea^2)$$

Here:

- ze is the distance from point z to the apex e,

- ea and eb are the distances from e to the ends of the upper base,
- The equation ensures the area of triangle ezi equals half the area of trapezoid $abqd$.

This geometric condition isn't derived from modern calculus — it's built from Euclidean reasoning and proportional logic.

What it teaches:

- Area can be reasoned about geometrically — not just measured.
- Visual symmetry often hides numerical balance.
- Construction problems in Greek mathematics reflect deeper ideas of fairness and precision.

This problem comes from Euclid's lost work *Division of Figures*, and represents one of the earliest known examples of geometric area bisection based on squared lengths.

Problem 2: What if a figure contains its own solution?

Topic: Deductive Geometry

Figure: Fig. 5.2

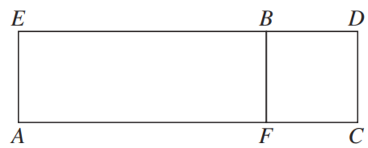

FIG. 5.2

Some diagrams don't need numbers to speak. They contain relationships within them — areas that constrain each other, shapes whose positions imply conclusions.

Fig. 5.2 shows a rectangle divided by a vertical segment BF, forming two parts. If the area of one part is known, the shape itself contains enough logic to deduce the other.

This type of problem is drawn from Euclid's *Data*, which focused on what follows from what — not through equations, but through spatial reasoning.

Problem 3: Can you build a perfect triangle with no measurements?

Topic: Classical Construction

Figure: Fig. 5.3

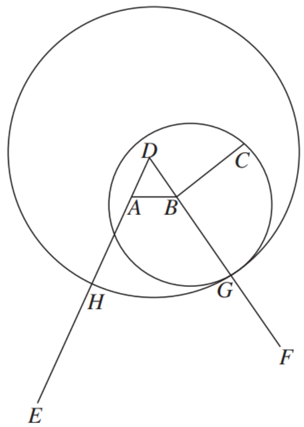

FIG. 5.3

Imagine you're given only a segment. No ruler. No numbers. Can you construct an equilateral triangle?

Fig. 5.3 shows Euclid's solution from Proposition I.1. He draws two circles — one centered at each endpoint of the segment — and their intersection becomes the third vertex of the triangle.

Each side is defined by construction, not by measurement. This is the foundation of Euclid's *Elements* — building truth from definitions and postulates.

Problem 4: How do you prove the Pythagorean Theorem without algebra?

Topic: Geometric Proof

Figure: Fig. 5.4

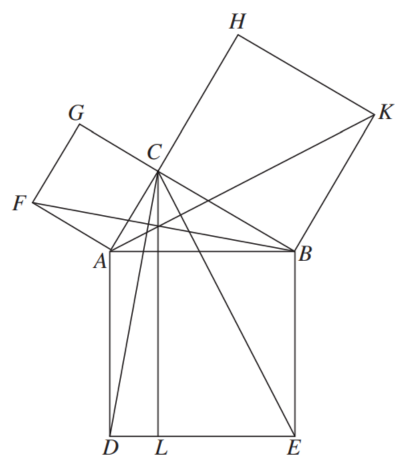

FIG. 5.4

Fig. 5.4 contains no numbers — only squares drawn on the sides of a right triangle. It's one of the most elegant visual proofs in mathematics.

The two smaller squares are dissected and rearranged. Their pieces fit exactly into the larger square. Thus:

$$\text{area on side } a + \text{area on side } b = \text{area on hypotenuse } c$$

This is Euclid's Proposition I.47. It shows that geometry can establish certainty, not just approximate results.

Problem 5: How do you show that multiplication distributes — without multiplying anything?

Topic: Geometric Algebra

Figure: Fig. 5.5

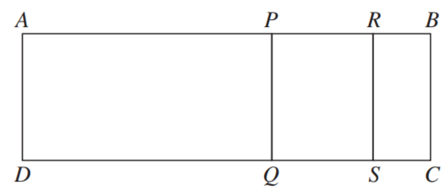

FIG. 5.5

You have a rectangle. Divide one side into two parts — call them b and c. Now draw lines to form two inner rectangles inside the whole.

Fig. 5.5 shows this layout. Visually, it becomes obvious:

$$a(b + c) = ab + ac$$

To the Greeks, this was algebra — done with area. The logic of multiplication and distribution was grounded in shape, not symbols.

Problem 6: How can a square reveal a deeper identity?

Topic: Difference of Squares

Figure: Fig. 5.6

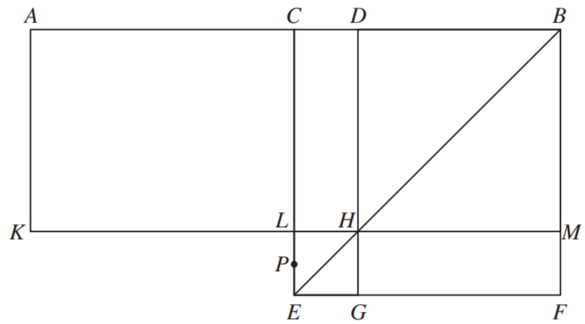

FIG. 5.6

You start with a large square. Remove a smaller one from its corner. What remains?

Fig. 5.6 shows how the leftover region can be rearranged into a rectangle — whose sides are $a + b$ and $a - b$. From this, the Greeks saw:

$$a^2 - b^2 = (a + b)(a - b)$$

This wasn't derived — it was seen. Geometry made abstract relationships visible.

Problem 7: What shape repeats itself forever — and always feels balanced?

Topic: Golden Rectangles and Spirals

Figures: Fig. 5.7, Fig. 5.8

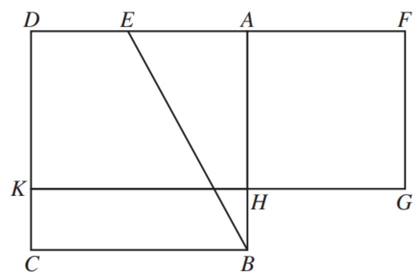

FIG. 5.7

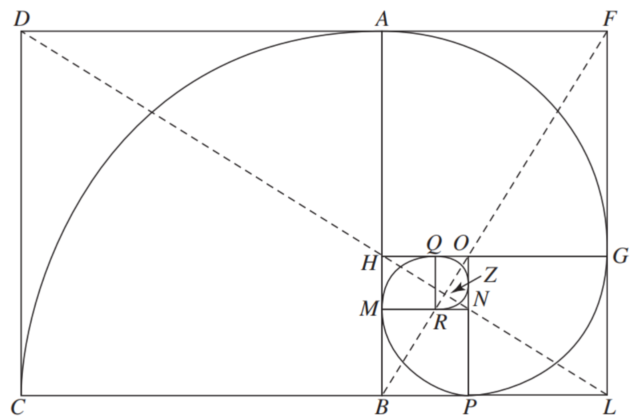

FIG. 5.8

Remove a square from a golden rectangle. What remains is another golden rectangle.

Fig. 5.7 shows this self-replicating process: a rectangle whose side lengths stay in golden ratio no matter how many times you repeat the subtraction.

Fig. 5.8 completes the thought — each golden rectangle spirals into the next, forming a shape that converges toward a point but never quite lands. This is an ancient glimpse into the modern concept of a limit.

The Greeks called such shapes *gnomons* — pieces that preserve the whole.

III. Civilizational Logic Summary

- **Core Principles:** Axiom, Proof, Construction, Continuity, Exhaustion
- **Mathematical Identity:** *This was a civilization of structure.*
- **Conception of Truth:** That which follows necessarily from first principles.

IV. Closing Dialectic

Summary Statement:
They didn't just do geometry. They built the scaffolding of knowledge itself.

Exit Prompt:
You are Euclid. Prove that any triangle's interior angles equal two right angles — using only lines, logic, and prior propositions.