Su2024MATH4991 – Chapter 7 Mk4

Apollonius of Perga: Conics, Curves, and Cosmic Geometry

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I. Cultural Invocation

- **Civilization**: Hellenistic Greece (Alexandria, Pergamum)
- **Time Period**: ˜262–190 BCE
- **Roles**: Geometers, Astronomers, Mathematician-Scribes

Opening Statement:
Welcome to the mathematical high renaissance of antiquity. Apollonius, heir to Euclid and Archimedes, took the simple idea of slicing a cone — and birthed the geometry that would guide Newton to the stars. This is not mathematics for artisans. This is mathematics as metaphysics.

II. Problem–Solution Cycles

Problem 1: Can a single cone give rise to all curves?

Topic: Unified Theory of Conic Sections

- **Context**: Before Apollonius, mathematicians believed each conic type (ellipse, parabola, hyperbola) required a different kind of cone.
- **Breakthrough**: Apollonius showed that by slicing a single double-napped cone at different angles, one could generate all three types.
- **Figure Insight**:

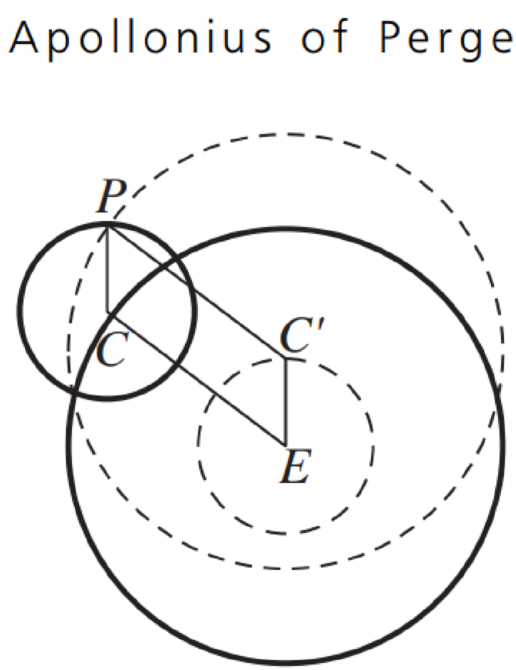


FIG. 7.1

FIG. 7.1 — This diagram shows a planet tracing an epicycle (small circle) whose center moves along a deferent (large circle). Though drawn later, it reflects Apollonius's core insight: nested, rotating motion can be reduced to pure geometry.

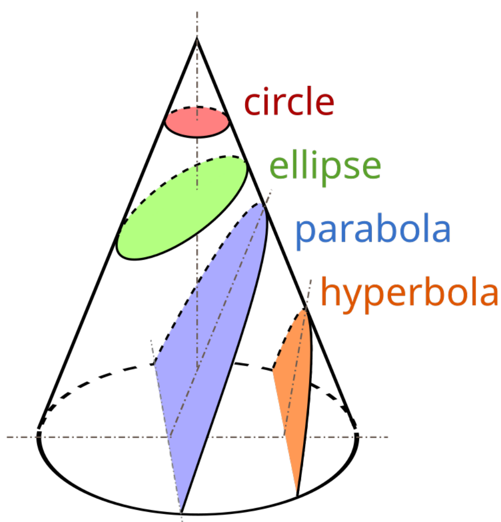
Meta-Realization: A single shape — the cone — hides a multitude of curves. This teaches students the power of unity within variation.

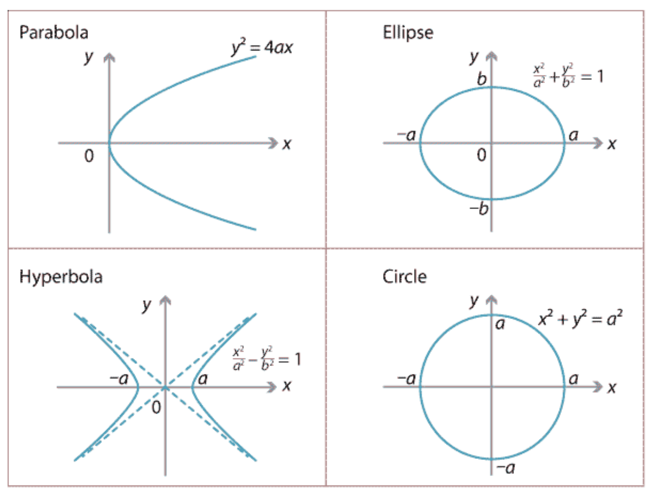
Problem 2: What's the curve whose name means "deficiency"?

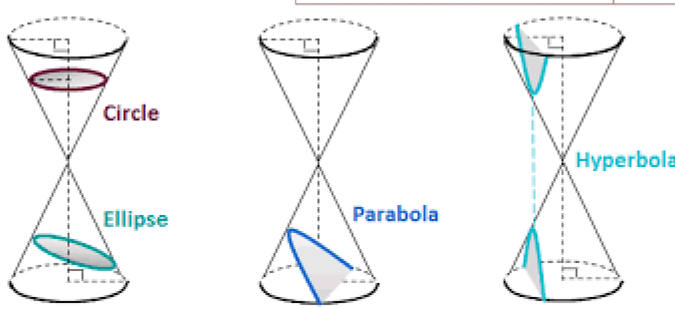
Topic: Naming the Conics

- **Historical Language**: The names *ellipse*, *parabola*, and *hyperbola* come from earlier algebraic descriptions:
  - Ellipse: $y^2 < lx$ (a "deficiency")
  - Parabola: $y^2 = lx$ (a "just-right" match)
  - Hyperbola: $y^2 > lx$ (an "excess")
- **Pedagogical Note**: The names are not just vocabulary — they express the relationship between squared distances and linear terms.

Clarification: *Ontology* — here — means "what kind of thing something really is." Apollonius wasn't just naming shapes — he was classifying mathematical realities.







Problem 3: How do you define a curve without an equation?

Topic: Geometry Without Coordinates

- **Method**: Apollonius used a cone and slicing planes to define curves, long before coordinates were invented.
- **Goal**: Turn a 3D cutting process into a 2D rule — using only geometry.

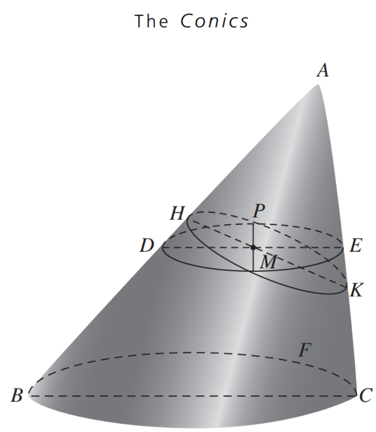


FIG. 7.2

FIG. 7.2 — A cone sliced by a plane. The result is a conic section: ellipse, parabola, or hyperbola depending on the plane's angle.

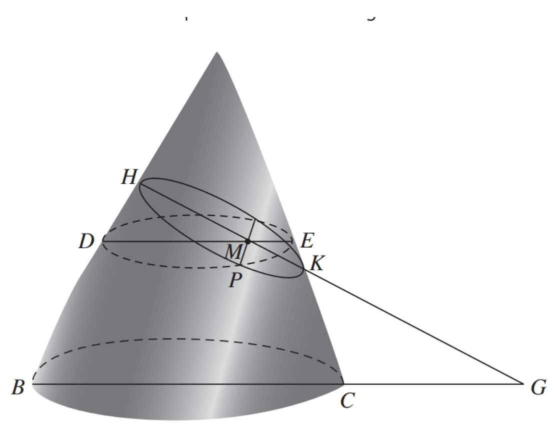


FIG. 7.3

FIG. 7.3 — Geometric construction from Book I, Prop. 13 of the Conics. It shows how to derive properties of conics directly from cone geometry.

Meta-Realization: Apollonius didn't need coordinates — he extracted laws from the shapes themselves.

Problem 4: Can a line "touch" a curve without calculus?

Topic: Tangents and Conjugate Diameters

- **Challenge**: Today we use derivatives to find tangents. But how did Apollonius do it without algebra or limits?
- **Greek Insight**: Use conjugate diameters — pairs of lines that define symmetry in ellipses and hyperbolas.
- **Key Idea**: A line tangent to a conic is the only one that doesn't cross it and just "kisses" it at one point.

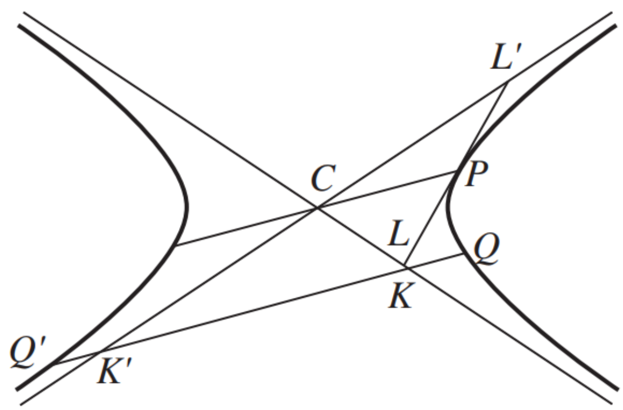


FIG. 7.4

FIG. 7.4 — Any tangent to a conic touches it at a single point and intersects the asymptotes symmetrically.

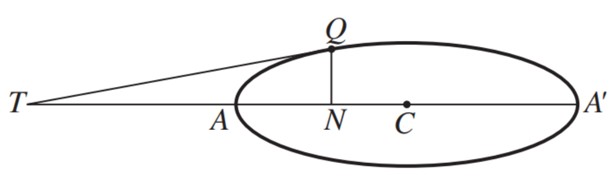


FIG. 7.5

FIG. 7.5 — Construction of a tangent using harmonic division. Apollonius shows how to find tangents using only compass and straightedge.

Meta-Realization: Even without calculus, the Greeks had precise ways to find tangents — using ratios, reflections, and geometry alone.

Problem 5: What is the locus of a triangle's hidden center?

Topic: The Problem of Apollonius

- **Geometric Task**: Given three objects — each either a point, line, or circle — can you construct a circle tangent to all three?
- **Range of Possibilities**: There are 10 distinct cases, depending on the combinations of objects. The most famous is the three-circle case (CCC), which can yield up to 8 distinct solution circles.
- **Pedagogical Framing**: This is an early example of a deep "locus" problem — identifying all points (here, centers of circles) satisfying geometric conditions.

Historical Note: This problem does not appear in the surviving books of Apollonius' *Conics*. It is known through a summary in **Pappus's Mathematical Collection**, where it is attributed to Apollonius's now-lost work *Tangencies*. Pappus reports that Apollonius solved all 10 combinations of point, line, and circle.

Modern Connections:

- **Viète** rediscovered and expanded the problem in the 1600s.
- **Descartes** gave an algebraic solution (Descartes' Circle Theorem).
- **Soddy** (1936) gave an elegant poetic solution in Nature.
- **Eppstein** (2001) and others have reconstructed solutions using both classical and computational methods.

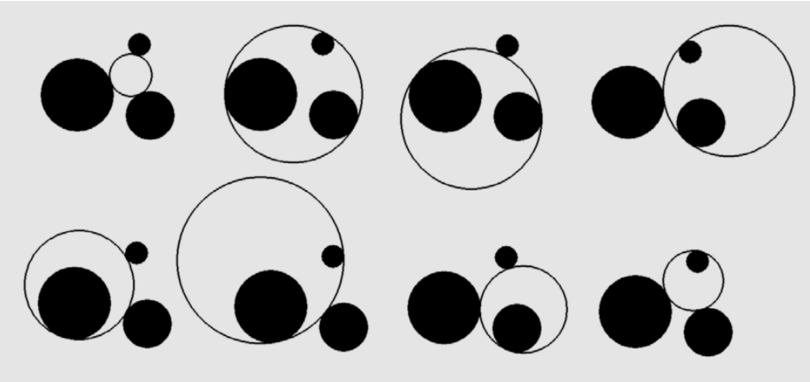


Figure 1. Eight solution circles to Apollonius' problem.

FIG. R1 — The most general case of the Problem of Apollonius: finding circles tangent to three given circles. Depending on the arrangement, up to eight distinct solution circles (shown here) may exist. Image adapted from Gisch and Ribando (2004).

Meta-Realization: Even without coordinate geometry, ancient mathematicians could pose — and in some form solve — problems that still challenge and inspire mathematicians and computer scientists today.

Problem 6: What is the shape of the planets' path?

Topic: Epicycles and Eccentrics

- **Ancient Astronomy**: Earlier astronomers like Eudoxus used nested spheres. Apollonius introduced epicycles — circles on circles — to model planetary motion more precisely.
- **Key Insight**: You can describe complex, looping paths using only circular motion.

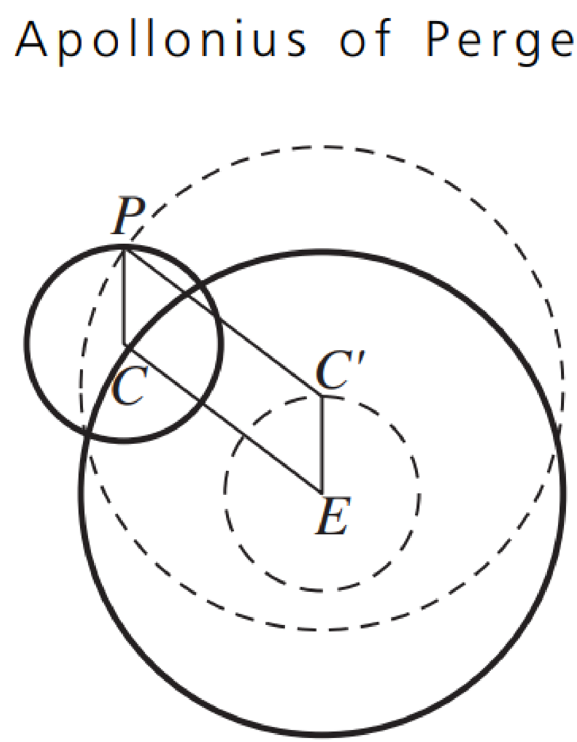


FIG. 7.1

FIG. 7.1 Revisited — A visual model of epicyclic motion: a planet moves around a small circle whose center moves along a larger circle.

Meta-Realization: What looks chaotic (like planetary retrograde motion) can emerge from simple, elegant geometry.

Problem 7: Can you construct the normal to a curve without algebra?

Topic: Normals and Evolutes

- **Definition Reminder**: A normal is a line that is perpendicular to the tangent at a point on a curve.
- **Challenge**: Apollonius found normals by finding shortest or longest distances from a point to the curve — before analytic methods existed.
- **Bonus**: He effectively discovered the evolute — the curve tracing the centers of curvature of another curve.

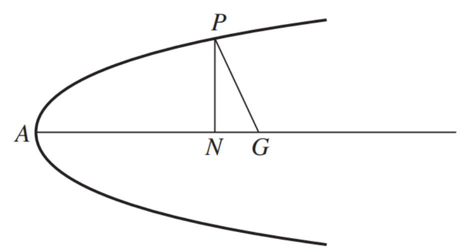


FIG. 7.6

FIG. 7.6 — Apollonius uses minimum-length reasoning to define the normal to a parabola — an early precursor to the derivative.

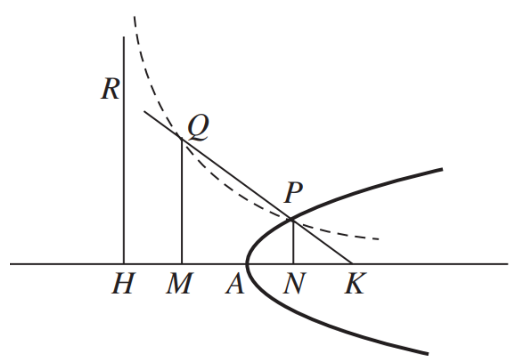


FIG. 7.7

FIG. 7.7 — Geometric construction of a normal from an external point using an auxiliary hyperbola.

Meta-Realization: Without symbols or calculus, Apollonius still traced the core ideas of curvature and optimization.

III. Civilizational Logic Summary

- **Core Principles**: Geometry as ontology, locus as solution, reasoning without coordinates.
- **Mathematical Identity**: Civilization of ideal forms and logical precision.
- **Conception of Truth**: That which is shown in space and follows necessarily from form.

IV. Closing Dialectic

Summary Statement:
Apollonius did not merely study curves. He turned geometry into a cosmos.

Exit Prompt:
You are Apollonius. No symbols. No variables. Only cone, plane, and reason. Construct a line that touches a curve — and prove that no other can do so.