Chapter 15 – Analysis, Synthesis, the Infinite, and Numbers

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Su2025 MATH4991 Lecture Notes – Mk1

I. Cultural Invocation

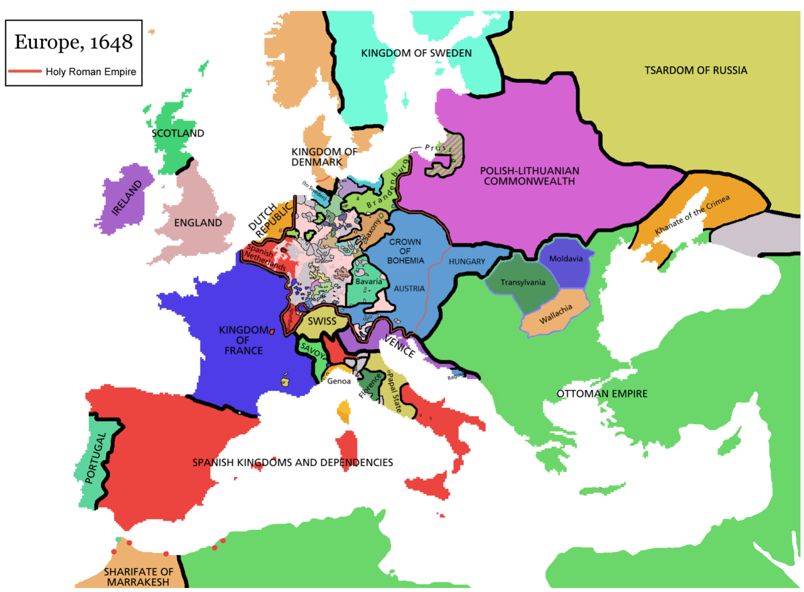
- **Civilization:** Seventeenth-Century Western Europe
- **Time Period:** c. 1630–1670
- **Figures:** Galileo Galilei, Bonaventura Cavalieri, Evangelista Torricelli, René Descartes, Pierre de Fermat, Blaise Pascal, Christiaan Huygens

Mathematics crossed a threshold — from mechanical calculation to conceptual reckoning.

The tools of Kepler and Stevin gave way to infinitesimals, analytic planes, and questions that pierced metaphysical veils:

What is a curve? What is the infinite? Can a part be equal to the whole?

This was an age of paradox, of thresholds, of men who saw the future through the infinitesimal eye.



Europe, 1648

II. Faces of the Era

Galileo Galilei (1564–1642).

- Articulated projectile motion via parabolic decomposition.
- Touched the infinite through physics, not abstraction.
- Misidentified the catenary — but intuited infinitesimal higher orders.

Bonaventura Cavalieri (1598–1647).

- Formalized the method of indivisibles; proto-integral reasoning.
- Asserted that lines compose area, areas compose volume — atomic geometry.
- Proposed Cavalieri's Principle (area/volume equivalence via sections).

Evangelista Torricelli (1608–1647).

- Extended Cavalieri's indivisibles to tangents, arcs, and spirals.
- Proved infinite area can generate finite volume (Gabriel's Horn).
- Sketched early logarithmic graphs.

René Descartes (1596–1650).

- Created analytic geometry: number fused with shape.
- Saw equations as curves, curves as equations.
- Rejected mechanical (non-algebraic) curves as "inexact."

Pierre de Fermat (1601–1665).

- Discovered methods of maxima, minima, and tangents — early differential reasoning.
- Operated through symbolic induction and infinite descent.
- Knew the fundamental theorem of calculus in all but name.

Blaise Pascal (1623–1662).

- Crafted Pascal's Hexagon Theorem and the logic of infinity.
- Proved tangents to conics geometrically.
- Nearly discovered integration from the cycloid.

Christiaan Huygens (1629–1695).

- Defined involutes and evolutes, curvature and radius.
- Made the cycloid a tautochrone; found its arc length.
- Reconciled time, geometry, and motion.

Gilles de Roberval (1602–1675).

- Developed method of tangents via decomposition of motion.
- Analyzed curves like the cycloid without algebraic coordinates.
- Precursor to vector-based reasoning in kinematics and geometry.

Philippe de La Hire (1640–1718).

- Extended Descartes' coordinate approach into three dimensions.
- Constructed surfaces using equations in three variables.
- Provided a geometric bridge from conics to analytic solids.

Girard Desargues (1591–1661).

- Founded projective geometry; introduced poles, polars, and perspectivity.
- Showed how conic sections could be unified under projection.
- His ideas were rediscovered and canonized by Pascal and La Hire.

Marin Mersenne (1588–1648).

- Central figure in the European mathematical network — the "Mersenne Circle."
- Facilitated correspondence between Descartes, Fermat, Roberval, Torricelli, and others.
- Issued mathematical challenges that catalyzed key developments in tangents and quadrature.
- Though not a creator of theorems, he served as the intellectual hub of the era's mathematical ferment.

Jean de Beaugrand (1584–1640).

- Advocate of Desargues' geometry and one of its earliest public defenders.
- Known more for polemics and commentary than for original theory.
- His writings served to publicize projective approaches in an era of skepticism.

Frans van Schooten (1615–1660).

- Editor and disseminator of Descartes' *La Géométrie*.
- His annotated Latin editions made Cartesian geometry widely accessible.
- Trained a generation of Dutch mathematicians who carried analytic methods forward.

Johann Hudde (1628–1704).

- Developed algebraic techniques for finding maxima and minima.
- Independently discovered methods parallel to Fermat's tangents.
- Influenced Newton through his correspondence and algebraic manuscripts.

René-François de Sluse (1622–1685).

- Created general rules for tangents to polynomial curves.
- His formulas anticipated the derivative concept in symbolic form.
- Corresponded with Barrow and Newton, forming a bridge between eras.

Isaac Barrow (1630–1677).

- Newton's teacher and the first to clearly state the inverse relation of differentiation and integration.
- Developed a rigorous geometric approach to infinitesimal methods.
- Stood at the edge of full calculus, shaping its final conceptual form.

III. Works of the Era

Galileo Galilei — *Dialogo sopra i due massimi sistemi del mondo* (1632).

- Framed dynamics through geometric thought, using infinitesimal comparisons.
- Argued for higher-order infinitesimals in projectile motion.
- Initiated discussions on one-to-one correspondences in infinite sets.

Bonaventura Cavalieri — *Geometria indivisibilibus continuorum nova quadam ratione promota* (1635).

- Developed the method of indivisibles: lines as area, areas as volume.
- Proposed Cavalieri's Theorem: solids with equal cross-sectional areas have equal volumes.
- Prefigured integral formulas: $\int_0^a x^n dx = \frac{a^{n+1}}{n+1}$ (rhetorically, geometrically).

Evangelista Torricelli — *De Dimensione Parabolae* (1644).

- Applied indivisibles to parabolas, cycloids, and tangents.
- Provided early quadrature of the cycloid, both via exhaustion and indivisibles.
- Demonstrated finite volume from infinite area (solid of revolution of a hyperbola).

René Descartes — *La Géométrie* (1637).

- Founded analytic geometry; fused algebra and Euclidean construction.
- Introduced coordinate-based classification of curves via polynomial degree.
- Rejected non-algebraic ("mechanical") curves — privileging exact constructibility.

Pierre de Fermat — *Methodus ad disquirendam maximam et minimam* (posth. 1679).

- Used difference quotients to find extrema and tangents — an early differential method.
- Discovered rules equivalent to the power rule in calculus.
- Introduced coordinate analysis of loci and symbolic recursion ("infinite descent").

Blaise Pascal — *Lettres de A. Dettonville* (1658–59), *Traité du triangle arithmétique* (1665).

- Solved arc length, area, and center of gravity problems for the cycloid.
- Formalized binomial expansions and recursive triangle logic.
- Unified combinatorics and geometry under principles of symmetry and infinity.

Christiaan Huygens — *Horologium Oscillatorium* (1673).

- Defined curvature, radius of curvature, and evolutes/involutes.
- Proved the cycloid to be a tautochrone and rectified its arc.
- Combined infinitesimal reasoning with mechanical applications (pendulum design).

Philippe de La Hire — *Nouveaux Éléments des Sections Coniques* (1679).

- Extended projective geometry and conic analysis with algebraic tools.
- Introduced one of the first analytic surface equations in three variables.
- Bridged classical conic theory with the emerging methods of coordinate geometry.

Isaac Barrow — *Lectiones Mathematicae* (1670), *Geometrical Lectures* (1674).

- Provided a geometric foundation for calculus through the inverse relation of integration and differentiation.
- Developed rigorous tangent and area constructions without limits.
- His lectures deeply influenced Newton and formed a conceptual bridge from geometry to calculus.

IV. Historical Overview

The mid-seventeenth century was an age of conceptual upheaval. Tools became theory. Curves became equations. Infinity became a topic not of awe, but of action. The thinkers of this era stood at the boundary between geometry and analysis, between static magnitude and dynamic form.

1. Indivisibles Supersede Exhaustion.

- Classical geometry favored the method of exhaustion — rigorous but limited.
- Cavalieri introduced indivisibles: treating area as composed of line elements, volume as composed of planes.
- The result was a functional shift — from finite approximations to infinitesimal decompositions.

2. Geometry Enters Algebraic Space.

- Descartes transformed curves into equations, and equations into spatial loci.
- Problems of ancient construction were now solvable by polynomial classification.
- Coordinates became the grammar through which geometry spoke algebra.
- Descartes revolutionized geometry by treating curves as algebraic equations.
- However, early use of coordinates was partial: many diagrams and solutions avoided negative values.
- The full implications of the Cartesian plane — with both axes extending in both directions — took time to manifest in practice.
- Graphs remained quadrant-bound; symmetry and signed distances were not yet universally accepted.

3. Tangents, Maxima, and the Rise of the Differential.

- Fermat devised methods to find tangents and extrema via symbolic perturbation.
- These foreshadowed derivatives — limits without formal limit theory.
- Hudde and Sluse added procedural rules; tangents became calculable objects.

4. The Infinite Begins to Fragment.

- Galileo confronted paradox: as many squares as natural numbers — yet fewer?
- Pascal and Cavalieri spoke of infinite sums, but with rhetorical caution.
- Huygens, Torricelli, and others handled infinite curves and volumes pragmatically, if not formally.

5. Integration Emerges Without Name.

- Cycloids, parabolas, spirals — all invited new quadratures.
- Torricelli used indivisibles and geometry to find volumes from infinitesimal arcs.
- Pascal and Fermat each touched the essence of integration — summation over continuous form.

6. The Center of Gravity Shifts.

- The early seventeenth century belonged to France — Descartes, Pascal, and Roberval stood at the mathematical vanguard.
- But by the 1670s, the French dominance had waned. Pascal was dead, Roberval's methods unformalized, and Descartes' metaphysics had outpaced his math.
- The Netherlands rose in their place: van Schooten published annotated editions of Descartes, Hudde extended Fermat's methods, and Huygens combined geometry with mechanics.
- In England, Barrow began a geometric calculus; his student Newton would soon synthesize it all.
- Even as Cartesian rationalism held court in salons and pulpits, the actual center of mathematical innovation was moving — from Paris to Leiden, from Port-Royal to Cambridge.
- The Dutch intellectual network centered around Leyden — van Schooten's editions, Hudde's algebra, and Huygens' geometry defined its peak.
- But the *Rampjaar* of 1672 shattered this center. With the death of de Witt and political instability, Leyden's mathematical community declined.
- In its place rose the Royal Society of London — founded in 1660, chartered in 1662.
- Where Leyden thrived on correspondence and Cartesian rationalism, the Royal Society privileged experiment, publication, and mechanical philosophy.
- The institutional gravity had shifted — and with it, the next chapter of mathematics would be written in Cambridge.

V. Problem–Solution Cycle

Problem 1: Galileo's Higher-Order Infinitesimal (c. 1632). Statement. Show how an object on a rotating Earth remains attached to the surface despite tangential motion.
Solution.

- Consider a small arc swept by Earth's rotation through angle θ.
- Tangential motion is represented by segment PQ; vertical fall by segment PS.
- Galileo argues that PS is an infinitesimal of higher order than PQ.
- Hence, even a minuscule gravitational tendency is sufficient to retain surface contact.

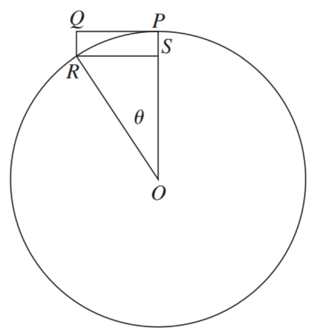


FIG. 15.1

FIG. 15.1: Galileo's infinitesimal motion argument.

Problem 2: Cavalieri's Equal Area via Indivisibles (c. 1635). Statement. Prove that two triangles within a parallelogram are equal in area using indivisibles.
Solution.

- Divide the parallelogram $AFDC$ into triangles via diagonal CF.
- Use horizontal line segments BM and HE at equal heights.
- One-to-one correspondence of line segments shows the triangles are area-equal.

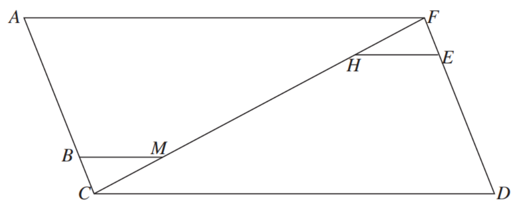


FIG. 15.2

FIG. 15.2: Cavalieri's triangle equivalence by indivisibles.

Problem 3: Transforming a Parabola into a Spiral (c. 1635). Statement. Map the parabola $x^2 = ay$ onto an Archimedean spiral using polar transformation.
Solution.

- Consider ordinates of the parabola as radial segments from fixed point O.
- Twist the curve into spiral form via $x = r, y = r\theta$.
- Resulting curve: $r = a\theta$, showing equivalence of area via polar geometry.

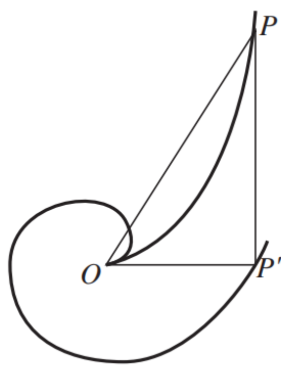


FIG. 15.3

FIG. 15.3: Cavalieri's spiral-parabola transformation.

Problem 4: Solving a Quadratic via Circle Construction (c. 1637). Statement. Solve $z^2 = az + b^2$ geometrically, without algebra.
Solution.

- Draw segment $LM = b$, construct perpendicular $NL = a/2$.
- Form a circle centered at N with radius $a/2$.
- The line MN intersects the circle at point O; then $OM = z$.

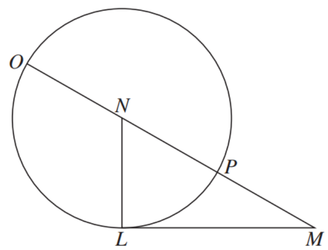


FIG. 15.4

FIG. 15.4: Descartes' geometric root extraction.

Problem 5: The Cartesian Trident from Pappus (c. 1637). Statement. Derive a cubic locus from five lines using Pappus' proportionality rule.
Solution.

- Four vertical lines evenly spaced at intervals a, plus one horizontal line.
- Use point P such that the product of distances satisfies the relation:

$$(a + x)(a - x)(2a - x) = axy$$

- Result is the cubic curve known as the "Cartesian trident" or "parabola of Newton."

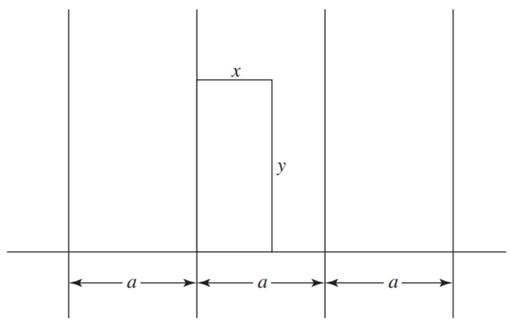


FIG. 15.5

FIG. 15.5: Construction of a Pappus cubic curve.

Problem 6: Spiral Length by Tangent Rectification (Torricelli, c. 1644). Statement. Rectify the arc of a logarithmic spiral using elementary geometry.
Solution.

- Construct a logarithmic spiral centered at O with point P on the spiral.
- Draw the tangent line PT at point P.
- Torricelli showed that the total arc length from $\theta = 0$ to P equals the fixed segment PT.
- This marks one of the first exact rectifications of a transcendental curve.

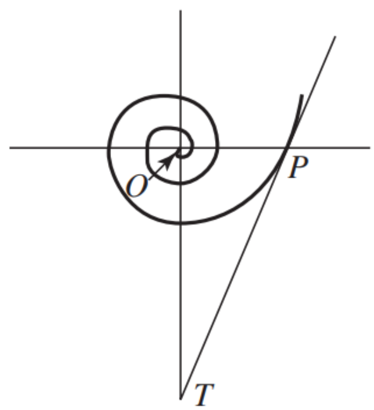


FIG. 15.6

FIG. 15.6: Rectifying the logarithmic spiral.

Problem 7: Tangent to a Curve via Indirect Differencing (Fermat, c. 1630s). Statement. Find the tangent to a curve $y = f(x)$ without using calculus.
Solution.

- Choose a point P on the curve and nearby point P' at $x + E$.
- Compute $\frac{f(x+E)-f(x)}{E}$ and set $E = 0$.
- Subtangent $TQ = c$ is determined by similar triangles.
- The process anticipates the derivative by analyzing limiting slopes.

During the very years in which Fermat was developing his analytic geometry, he also discovered how to apply his neighborhood process to find the tangent to an algebraic curve of the form $y = f(x)$. If P is a point on the curve $y = f(x)$ at which the tangent is desired, and if the coordinates of P are (a, b), then a neighboring point on the curve with coordinates $x - a + E$, $y = f(a + E)$ will lie so close to the tangent that one can think of it as approximately on the tangent, as well as on the curve. If, therefore, the subtangent at the point P is $TQ = c$ (Fig. 15.7), the triangles TPQ and $TP'Q'$ can be taken as being virtually similar. Hence, one has the proportion

$$\frac{b}{c} = \frac{f(a+E)}{c+E}.$$

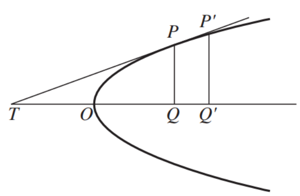


FIG. 15.7

FIG. 15.7: Fermat's tangent method using small increments.

Problem 8: Quadrature of Power Functions (Fermat, c. 1630s). Statement. Determine the area under the curve $y = x^n$ using geometric summation.
Solution.

- Subdivide the interval from $x = 0$ to $x = a$ into exponentially spaced rectangles.
- Approximate the area with a geometric series of rectangle areas.
- Letting the spacing ratio tend to 1 yields:

$$\int_0^a x^n dx = \frac{a^{n+1}}{n+1}$$

- Fermat generalized this for all rational exponents (except $n = -1$).

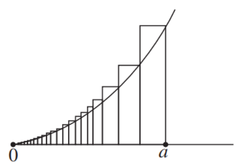


FIG. 15.8

$$\frac{a^{n+1}(1-E)}{1-E^{n+1}} \quad \text{or} \quad \frac{a^{n+1}}{1+E+E^2+\cdots+E^n}.$$

As E tends toward 1—that is, as the rectangles become narrower—the sum of the areas of the rectangles approaches the area under the curve. On letting $E = 1$ in the previous formula for the sum of the rectangles, we obtain $(a^{n+1})/(n+1)$, the desired area under the curve $y = x^n$ from $x = 0$ to $x = a$. To show that this also holds for rational fractional values, p/q, let $n = p/q$. The sum of the geometric progression then is

$$a^{(p+q)/q}\left(\frac{1-E^q}{1-E^{p+q}}\right) = a^{(p+q)/q}\left(\frac{1+E+E+\cdots+E^{q\ \ 1}}{1+E+E^2+\cdots+E^{p+q\ \ 1}}\right),$$

and, when $E = 1$, this becomes

$$\frac{q}{p+q}a^{(p+q)/q}.$$

FIG. 15.8: Fermat's proto-integral via rectangles.

Problem 9: Tangent to the Cycloid via Composition of Motions (Roberval, c. 1630s).
Statement. Construct the tangent to a cycloid using physical motion.
Solution.

- Let point P on a rolling circle trace the cycloid.
- Decompose motion into translation PS and rotation PR.
- Bisect the angle between these vectors to obtain the tangent direction PT.
- Roberval's method interpreted tangents kinematically — motion over form.

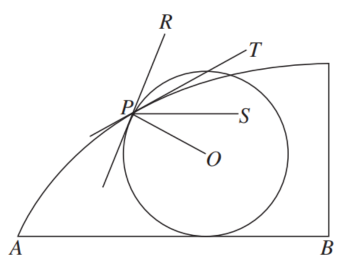


FIG. 15.9

FIG. 15.9: Tangent via vector decomposition.

Problem 10: Harmonic Configurations on a Conic (Desargues, c. 1639). Statement. Relate poles and diagonals using projective geometry.
Solution.

- Construct quadrangle $ABCD$ inscribed in a conic.
- Determine diagonal points E, F, G from intersections of opposite sides.
- The line through any two diagonal points is the polar of the third.
- This reveals harmonic and duality relationships central to early projective theory.

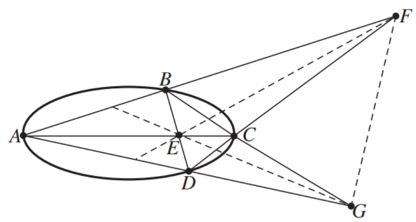


FIG. 15.10

FIG. 15.10: Desargues' pole-polar projective theorem.

Problem 11: Pascal's Mystic Hexagon (c. 1640). Statement. Demonstrate that opposite sides of a hexagon inscribed in a conic intersect in collinear points.
Solution.

- Let A, B, C, D, E, F be points on a conic.
- Extend sides AB and DE, BC and EF, CD and FA to form points P, Q, R.
- Pascal's Theorem states: points P, Q, R lie on a single straight line.
- This was one of the earliest and most powerful results of projective geometry.

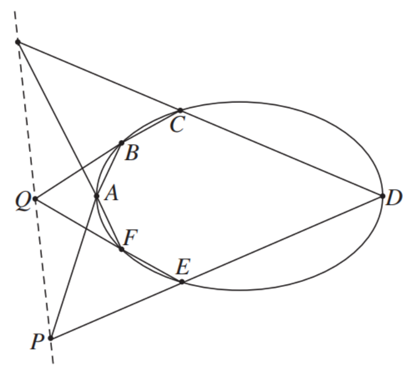
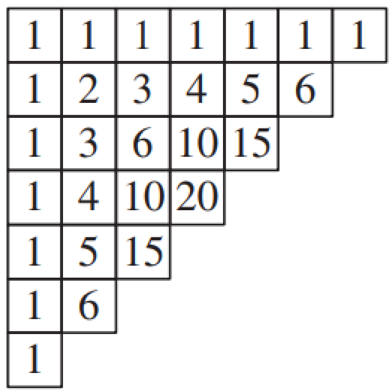


FIG. 15.11

FIG. 15.11: Pascal's Hexagon Theorem — collinearity from conics.

Problem 12: Recursive Harmony in Pascal's Triangle (c. 1654). Statement. Reveal hidden proportionality laws within Pascal's arithmetic triangle.
Solution.

- Arrange integers in a triangle where each entry is the sum of the two above it.
- Observe that for any base diagonal, the ratio of adjacent entries relates to their vertical positions.
- These patterns reflect the binomial coefficients and guided Pascal's probability work.
- The triangle encodes combinatorial symmetry and recursive structure.



1	1	1	1	1	1	1
1	2	3	4	5	6	
1	3	6	10	15		
1	4	10	20			
1	5	15				
1	6					
1						

FIG. 15.12

FIG. 15.12: Pascal's Triangle and its proportional diagonals.

Problem 13: First Steps into Solid Analytic Geometry (Lahire, c. 1679). Statement. Construct a surface defined by a three-variable equation from geometric configuration.
Solution.

- Set up a coordinate system with axis OB and plane OBA.
- Let point P lie such that perpendicular PB exceeds OB by fixed distance a.
- The locus of such points satisfies:

$$a^2 + 2ax + x^2 = y^2 + v^2$$

- This is the equation of a cone in three dimensions — an early instance of surface plotting.

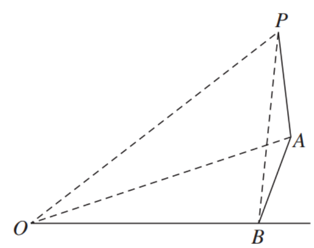


FIG. 15.13

FIG. 15.13: Lahire's early cone via three-coordinate geometry.

Problem 14: Cycloidal Oscillation and Isochronism (Huygens, c. 1673). Statement. Design a pendulum path with equal-time oscillations for all amplitudes.
Solution.

- Suspend a pendulum from point P between two inverted cycloidal cheeks PQ and PR.
- Let the bob swing along arc QSR, itself a cycloid identical in shape to the cheeks.
- The result: the pendulum's period is independent of amplitude — it is a tautochrone.
- Huygens used this to improve clock precision via geometric constraint.

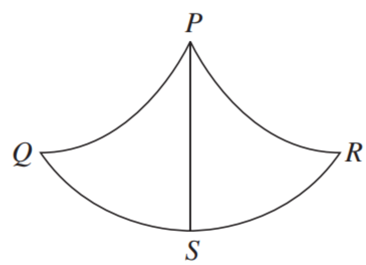


FIG. 15.14

FIG. 15.14: Huygens' cycloidal pendulum — the tautochrone.

Problem 15: Radius of Curvature and the Evolute (Huygens, c. 1673). Statement. Determine the center of curvature at a point on a curve using normals.
Solution.

- Take a smooth curve c_i and two nearby points P and Q.
- Construct normals at each point; let them intersect at I.
- As $Q \to P$, the point $I \to O$, the center of curvature.
- The path traced by O as P moves is the evolute c_e — the envelope of normals.

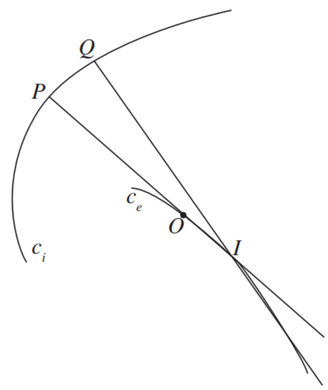


FIG. 15.15

FIG. 15.15: Evolute as locus of curvature centers.

VI. Decline and Disruption

The breakthroughs of the seventeenth century were immense, but they were not seamless. The thinkers of this era approached the infinite with daring and ingenuity — yet lacked the rigor to seal their claims beyond doubt. Behind the advances in tangents, quadrature, and coordinates, unresolved tensions accumulated.

1. Lack of Formal Limit Theory.

- Fermat, Pascal, and Torricelli relied on intuitive reasoning involving infinitesimals.
- No clear concept of convergence, epsilon-delta rigor, or error bounds existed.
- Critics began to question whether such reasoning was sound or merely symbolic sleight of hand.

2. Infinitesimals Under Attack.

- Indivisibles were seen by some as metaphysical absurdities.
- Jesuit mathematicians resisted Cavalieri's methods as violating Aristotelian continuity.
- The Church's philosophical orthodoxy clashed with atomic or infinitesimal geometry.

3. Algebraic Elitism.

- Descartes dismissed mechanical or non-algebraic curves as "inexact."
- His analytic geometry excluded spirals, cycloids, and transcendental forms.
- This sidelined some of the most fruitful geometrical discoveries of the era.

4. Fragmentation of Techniques.

- Torricelli, Fermat, Roberval, and Pascal all developed methods in parallel, with little synthesis.
- No unified notation, method, or discipline of "calculus" yet existed.
- Concepts like tangents, areas, and curvature lacked shared language — a tower of Babel before Newton and Leibniz.

5. Clockmakers Without Clocks.

- Huygens' cycloidal pendulum and isochronous motion offered profound implications for mechanics and timekeeping.
- Yet, without a fully developed physics of force, inertia, or energy, the full framework for motion remained incomplete.
- The tools were forged — but the system they implied was still in waiting.

Historical Sidebar: The Cannibalism of the de Witts (1672). *"Anno 1672 was het rampjaar: het volk was redeloos, de regering radeloos, en het land reddeloos."*

In August of 1672 — the Dutch *Rampjaar* or "Year of Disaster" — a mob in The Hague lynched and mutilated Johan de Witt, Grand Pensionary of Holland, along with his brother Cornelis. Blamed for weakening Dutch defenses and accused of plotting against the House of Orange, they were ambushed, hacked to death, and — as multiple eyewitnesses reported — their bodies were cannibalized in public.

The event marked the symbolic collapse of the Cartesian republican elite. De Witt had supported the mathematical and philosophical circles of van Schooten, Hudde, and other Dutch Cartesians. Their secular, rationalist orientation was tied to a political regime now annihilated by populist violence.

> **Legacy:** The mob consumed the body of a mathematician's patron. The Dutch Golden Age turned in on itself.

VII. Closing Dialectic

The seventeenth century was not the age of calculus — it was the age that made calculus inevitable.

Cavalieri's indivisibles challenged the exhaustion method, asking whether a line could be a sum of points, a surface a sum of lines. Descartes refused to engage non-algebraic curves, declaring the mechanical "inexact." Pascal balanced conics with cycloids; Fermat whispered derivatives without limits. They had no epsilon, no delta, no completeness axiom — only will, intuition, and symbol.

And yet, their tools worked.

What the ancients had forbidden — the use of the infinite — these thinkers touched and wielded with impunity. Where the Greeks sought certainty through abstention, these moderns pursued power through approximation. And in their paradoxes, they laid both the foundation and the fault lines of modern mathematics.

The dialectic remains:

> Can one use the infinite without incoherence?
> Can intuition build a system before justification arrives?
> Can geometry survive when its curves outgrow construction?

These were not merely technical questions. They were philosophical thresholds — and it would take Newton, Leibniz, and a century of reckoning to cross them.