Chapter 16 – British Techniques and Continental Methods

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I. Cultural Invocation

- **Civilization:** Late Seventeenth to Early Eighteenth Century Europe
- **Time Period:** c. 1650–1725
- **Epochs:** English Restoration, Dutch Decline, Newtonian Synthesis
- **Figures:** John Wallis, James Gregory, Isaac Barrow, Isaac Newton, Abraham De Moivre, Gottfried Wilhelm Leibniz

The infinite ceased to be taboo. The legacy of Descartes passed into the hands of Wallis, Barrow, and Newton—where geometry met flux, and proof met power. As the Continent soared with curves, Britain grounded itself with rigor: proportions became series, indivisibles became fluxions, and the spiral of history bent toward calculus. *Who may claim invention? What is the rigor of infinitesimals? Can geometry survive under infinite descent?* This was the age where mathematics became empire.

Geopolitical Map of World, 1700AD

II. Faces of the Era

John Wallis (1616–1703).

- Introduced the symbol ∞; expanded Cavalieri's indivisibles arithmetically.
- Developed Wallis's product for π and early integral formulas.
- Advanced analytic geometry and infinite series in England.

Isaac Barrow (1630–1677).

- First to articulate the inverse relation between integration and differentiation.
- Taught Newton and edited early lectures on geometric calculus.
- Reconciled geometry with early infinitesimal reasoning.

James Gregory (1638–1675).

- Anticipated Taylor series and found exact expressions for arctangent integrals.
- Studied quadratures of curves; contributed to early convergence theory.
- Introduced series expansions for inverse trigonometric functions.

Isaac Newton (1642–1727).

- Invented fluxional calculus and infinite series for fractional exponents.
- Formulated the laws of motion and universal gravitation in the *Principia*.
- Developed the binomial theorem and the method of fluxions.

Gottfried Wilhelm Leibniz (1646–1716).

- Independently developed differential and integral calculus.
- Created the notation dx, $\int$, and symbolic form of the derivative.
- Founded the analytic style that dominated Continental mathematics.

Jacques (James) Bernoulli (1654–1705).

- Founded the theory of probability in *Ars Conjectandi*.
- Introduced Bernoulli numbers and the Law of Large Numbers.
- Investigated the logarithmic spiral and early combinatorics.

Johann (Jean) Bernoulli (1667–1748).

- Prolific expositor and extender of Leibnizian calculus.
- Advanced differential equations, brachistochrones, and exponential calculus.
- Mentored Euler and disseminated analysis across Europe.

Abraham De Moivre (1667–1754).

- Developed analytic trigonometry using complex numbers.
- Authored *The Doctrine of Chances*, foundational to probability theory.
- Anticipated the normal distribution and Stirling's approximation.

Brook Taylor (1685–1731).

- Formulated the Taylor series for function expansion.
- Studied curvature, oscillations, and perspective geometry.
- Bridged Newtonian fluxions and modern analytic mechanics.

Alexis Claude Clairaut (1713–1765).

- Founded solid analytic geometry and studied space curves.
- Extended curvature and potential theory to three dimensions.
- Applied calculus to gravitational theory and the figure of the Earth.

III. Works of the Era

John Wallis — *Arithmetica Infinitorum* (1655).

- Replaced Cavalieri's indivisibles with arithmetical series.
- Extended integration to general powers of x using interpolation.
- Introduced infinite products and early conceptions of area under curves.

John Wallis — *Treatise of Algebra, Both Historical and Practical* (1685).

- Merged symbolic manipulation with historical development.
- Asserted English priority in algebraic methods over Descartes.
- Provided foundational reference for Newton and other English mathematicians.

Isaac Newton — *De analysi per aequationes numero terminorum infinitas* (1669 / 1711).

- Introduced Newton's general method of infinite series expansion.
- Showed that curves could be expressed and integrated via power series.
- Marked the beginning of Newton's analytic calculus.

Isaac Newton — *Philosophiae Naturalis Principia Mathematica* (1687).

- Unified physics through geometric derivation of universal gravitation.
- Formalized laws of motion and mathematical astronomy.
- Contained hidden uses of calculus under geometric language.

G. F. A. de L'Hospital — *Analyse des infiniment petits* (1696).

- First textbook on differential calculus using Leibniz's notation.
- Contained l'Hospital's Rule, originating from Johann Bernoulli.
- Made calculus accessible to the broader European mathematical community.

Isaac Newton — *De Quadratura Curvarum* (1704).

- Developed integration techniques explicitly tied to fluxions.
- Introduced the notion of the "ultimate ratio" — a precursor to limits.
- Signaled Newton's response to the rising popularity of Leibniz's methods.

Jacques Bernoulli — *Ars Conjectandi* (1713).

- Founded the modern theory of probability.
- Introduced Bernoulli numbers and the Law of Large Numbers.
- Unified combinatorics, expectation, and mathematical rigor.

Isaac Newton — *Methodus Fluxionum et Serierum Infinitorum* (1736).

- Posthumous English publication of Newton's method of fluxions.
- Developed rules for differentiation and their inverse (integration).
- Presented Newtonian calculus more directly than the *Principia*.

Abraham De Moivre — *Miscellanea Analytica* (1730).

- Applied complex numbers to trigonometric expansions.
- Advanced probability theory and approximations (precursor to the normal distribution).
- Introduced what would become De Moivre's Theorem.

Colin Maclaurin — *Treatise of Fluxions* (1742).

- Most rigorous defense of Newtonian fluxions against Berkeley's critique.
- Grounded calculus in classical geometric logic.
- Preserved Newton's framework during the rise of Continental analysis.

Note: Leibniz's Foundational Papers on Calculus
Although not included among the full treatises listed above, the following works by Gottfried Wilhelm Leibniz merit special recognition as the earliest published expositions of differential and integral calculus:

- *Nova Methodus pro Maximis et Minimis, itemque Tangentibus...* (*Acta Eruditorum*, 1684)
- *De Geometria Recondita et Analysi Indivisibilium atque Infinitorum* (*Acta Eruditorum*, 1686)

These journal papers (in The Acts of the Erudite [1st Scientific Journal of German-Speaking Europe]) introduced the d notation, the integral symbol $\int$, and the philosophical framing of calculus as a symbolic, general method. Though brief, they seeded a revolution in mathematical practice and notation across the Continent.

IV. Historical Overview

The birth of calculus was not a singular moment but a divided unfolding—one British, geometric, and guarded; the other Continental, symbolic, and bold.

1. England's Geometric Conservatism.

- Newton's calculus remained veiled in geometry, driven by fluxions, ratios, and limits concealed under classical form.
- Barrow's rigor and Wallis's infinite products shaped a method of reasoning that valued continuity with Euclid over symbolic convenience.
- British thinkers distrusted algebraic abstraction—preferring the solidity of geometric construction.

2. The Rise of Symbolic Calculus in Europe.

- Leibniz introduced dx, dy, and $\int$ as autonomous symbols of change and summation.
- Johann Bernoulli translated these into usable algorithms—turning calculus into a system.
- L'Hospital's textbook formalized these tools and ensured their spread from Paris to Basel.

3. Probability and Analysis Merge.

- Jacques Bernoulli's *Ars Conjectandi* unified combinatorics, expectation, and inductive reasoning.
- De Moivre extended this into continuous forms—paving the path to Gaussian distributions and statistical calculus.

4. Controversy over Foundations.

- Berkeley's *The Analyst* attacked the logical coherence of fluxions as "ghosts of departed quantities."
- Maclaurin responded with classical rigor—attempting to ground Newton's method in geometric first principles.
- Meanwhile, Continental analysts moved forward pragmatically, refining convergence and approximation through series and limits.

5. The Divide Solidifies.

- By 1750, the Continental school—Leibniz, the Bernoullis, Euler—had overtaken British methods in clarity and computational power.
- British loyalty to Newton's fluxional method, though noble, delayed full engagement with modern analysis for nearly a century.

6. Geometry, Revived but Redefined.

- Analytic geometry expanded into three dimensions with Clairaut and Hermann.
- Newton's solid curves were reinterpreted as algebraic loci; surfaces began to be classified symbolically.
- Geometry had not died—it had transformed into algebraic structure.

Note: Newtonian vs. Leibnizian Calculus — A Comparative Overview

Newtonian Calculus	**Leibnizian Calculus**
Fluxions and fluents: quantities changing over time	*Differentials and integrals*: infinitesimal increments and summation
Grounded in geometry and kinematics	Grounded in symbolic analysis
Notation: $\dot{x}$ (fluxion), $\ddot{x}$ (second fluxion)	Notation: dx, dy, $\int$ (integral)
Defined change as a ratio of vanishing quantities ("ultimate ratios")	Treated dx and dy as actual algebraic entities manipulable in equations
First developed in *De Analysi* and *Principia*; method often concealed in geometric form	First published in 1684–1686 in *Acta Eruditorum*; method explicitly symbol-based
Developed independently in England; spread slowly due to lack of clear exposition	Disseminated quickly through Europe via L'Hospital, Bernoulli, and Euler
Cautious about symbolic generalization; prioritized rigor via classical geometry	Embraced generalization; prioritized versatility and computation

Despite these differences, both systems captured the same underlying truths. They approached limits, tangents, and areas from different angles—but converged in power. The calculus was not born twice. It was revealed in two dialects of genius.

V. Problem–Solution Cycle

Problem 1: Subtangent by Infinitesimal Triangle (Barrow, 1670s). Statement. Given Barrow's triangle $\triangle MRN$ along a curve with known ordinate m, derive the subtangent t using triangle similarity.
Solution.

- Let a and e be the vertical and horizontal legs of triangle MRN, with m the ordinate at M.
- By similarity: $\frac{a}{e} = \frac{m}{t} \Rightarrow t = \frac{e \cdot m}{a}$.
- This is an early geometric precursor to the concept of the derivative.

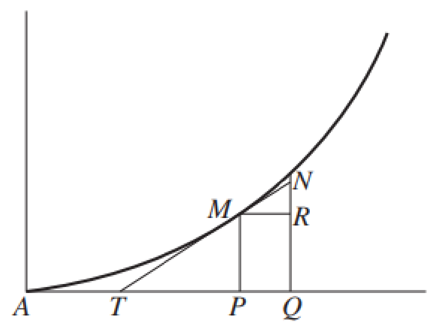

FIG. 16.1

FIG. 16.1: Barrow's subtangent triangle via infinitesimals.

Problem 2: Deriving Gregory's Series for $\arctan x$ (Gregory, 1671). Statement. Expand $\frac{1}{1+x^2}$ as a power series and integrate to obtain Gregory's series.
Solution.

- Divide: $\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots$
- Integrate term-by-term:

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

- Converges for $|x| \leq 1$; anticipates later Taylor expansions.

Problem 3: Newton's Ultimate Ratio of Powers (Newton, 1671). Statement. Use binomial expansion on $(x+o)^n - x^n$ and find the ultimate ratio as $o \to 0$.
Solution.

- Expand: $(x+o)^n = x^n + nx^{n-1}o + \cdots$
- Subtract x^n, divide by o, let $o \to 0$:

$$\lim_{o\to 0} \frac{(x+o)^n - x^n}{o} = nx^{n-1}$$

- A geometric justification of the derivative.

Problem 4: Newton's Polygon and the Folium (Newton, 1676). Statement. Analyze the folium $x^3 + y^3 = 3axy$ using Newton's lattice diagram.
Solution.

- Organize terms by total degree into diagonal lattice segments.
- Each segment provides a local approximation (branch) of the curve.
- This anticipates Newton's polygon method for singularities.

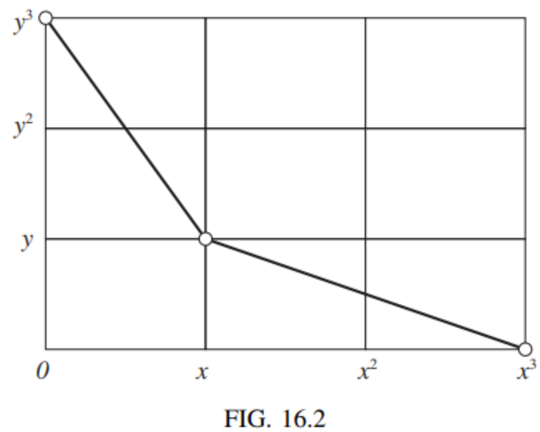

FIG. 16.2: Newton's diagram for the folium of Descartes.

Problem 5: Triangle-Arc Ratio and $\frac{d\theta}{dx}$ (Pascal–Leibniz, 1680s). Statement. Use a circle diagram to show that $\frac{AD}{DI} = \frac{EE}{RR} \approx \frac{d\theta}{dx}$.
Solution.

- Treat EE as arc, RR as projection; triangle ADI is infinitesimal.
- Ratio approximates:
$$\frac{1}{\sin\theta} = \frac{d\theta}{dx}$$
- Visual bridge from geometry to differential identities.

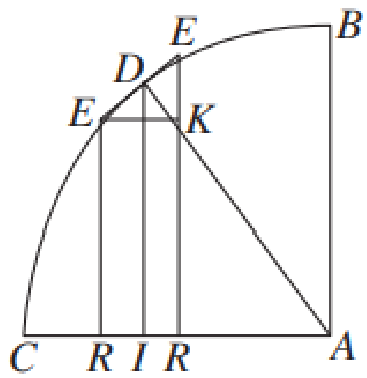

FIG. 16.3

FIG. 16.3: Pascal's triangle of differential ratios.

Problem 6: Leibniz's Product Rule from Differentials (Leibniz, 1684). Statement. Derive $d(xy) = x\,dy + y\,dx$ by expanding $(x + dx)(y + dy)$.
Solution.

- Expand: $(x + dx)(y + dy) - xy = x\,dy + y\,dx + dx\,dy$
- Discard $dx\,dy$ as second-order.
- Final rule: $d(xy) = x\,dy + y\,dx$

Problem 7: Integral of x^x via Series (Bernoulli, 1690s). Statement. Express x^x as a power series and integrate $\int_0^1 x^x dx$.
Solution.

- Use: $x^x = e^{x \ln x} = \sum \frac{(x \ln x)^n}{n!}$
- Integrate term-by-term:
$$\int_0^1 x^x dx = 1 - \frac{1}{2^2} + \frac{1}{3^3} - \cdots$$
- Early use of exponentials and infinite series.

Problem 8: Permutations from Conditional Probability (De Moivre, 1711). Statement. Derive the number of 2-letter permutations using probability logic.
Solution.

- $P(\text{first letter}) = 1/6$, $P(\text{second}) = 1/5$
- Product $= 1/30 \Rightarrow 30$ permutations
- General form:
$$P(n, r) = \frac{n!}{(n-r)!}$$

VII. Closing Dialectic

The calculus was not invented. It was uncovered—twice.

Newton, in solitude, conjured fluxions from motion and geometry. Leibniz, in correspondence, spun differentials from language and logic. They did not merely compute—they redefined what it meant to know change.

What united them was not method, but destiny: A universe in motion required a mathematics that could speak its grammar.

- Newton fused Euclid and motion, offering a calculus of force.
- Leibniz fused logic and infinity, offering a calculus of thought.
- The Bernoullis transformed that calculus into technique.
- De Moivre and Bernoulli seeded probability and the shape of chance.
- Maclaurin and Clairaut defended structure at the edge of rigor.

And yet, neither fluxions nor differentials escaped unscathed. The infinite still lacked foundation. The derivative still hid its definition. The integral still demanded formal precision.
The dialectic remains:

- Can motion be mathematized without first understanding limit?
- Can symbolism yield truth when its foundations remain implicit?
- Can two systems born in rivalry give birth to one enduring legacy?

Calculus did not begin with certainty. It began with power. And it would take a century of correction, translation, and confrontation before its fire would be tempered into clarity.

From fluxions to functions. From rivalry to reason.
This was the birth of the modern.