Chapter 22 – Analysis

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Su2025 MATH4991 Lecture Notes – Mk1

I. Cultural Invocation

- **Civilization:** 19th-century Europe — post-Newtonian, post-Eulerian, striving for mathematical rigor amid physical discovery.
- **Time Span:** Early 1800s to early 1900s
- **Epochal Axes:** Continuity, Convergence, Integration, Real Number Construction, Function Theory
- **Figures:** Cauchy, Riemann, Weierstrass, Dedekind, Cantor, Heine, Dirichlet, Bolzano, Hermite, Liouville, Poincaré

Analysis is the mathematics of change — but in the 19th century, it became the mathematics of control.

After centuries of intuitive manipulation — Newtonian fluents, Eulerian expansions — cracks appeared. Infinitesimals had no grounding. Convergence was misunderstood. Continuity was presumed.

Rigor had to be rebuilt.

- Cauchy redefined limits and derivatives through clarity, not speed.
- Weierstrass formalized ε–δ language and banished intuition.
- Riemann, from intuition again, created integrals, surfaces, and the zeta function.
- Dedekind cut the rationals to construct the reals.
- Cantor made infinity mathematical — and countability a property.

Each contribution was both surgical and philosophical:

- Analysis was not just about solving problems — it was about defining the conditions under which solutions mean anything.

- Where algebra built structures, analysis secured the foundations beneath them.

It was not a linear refinement. It was a reconstruction — brick by logical brick — of what "real" even means.

II. Big Pictures

Analysis was not simply extended in the 19th century — it was redefined.

Calculus had worked. But its tools were heuristic, its foundations unclear.

The 1800s did not abandon its power — they rebuilt its logic.

Here are the major arcs that shaped this reconstruction:

1. **Cauchy's Reformation** - Defined limits and continuity with rigor. - Introduced power series and formal convergence. - Founded rigorous function theory and complex analysis.

2. **Weierstrass and the ε–δ Revolution**

- Banished infinitesimals from analysis.

- Defined limits, continuity, and differentiability with formal logic.

- Made pathological functions central — continuous nowhere-differentiable examples.

3. **Riemann's Intuition**

- Reclaimed intuition with depth: integration, surfaces, analytic continuation.

- Introduced the Riemann integral, Riemann surfaces, and the Riemann zeta function.

- Bridged physics, geometry, and function theory.

4. **Construction of the Real Numbers**

- Dedekind cuts and Cantor sequences made the real line logically complete.

- Méray, Heine, Bolzano contributed to convergence, series, and irrationality.

- Arithmetic replaced geometry as the foundation of analysis.

5. **Cantor's Set Theory and Infinity**

- Countability, uncountability, and transfinite numbers.

- Function spaces became sets with structure — not just rules.

- Real analysis became inseparable from set theory.

6. **National Currents**

- *Germany*: Rigor, completeness, function theory (Weierstrass, Dirichlet, Dedekind, Cantor).

- *England*: Physics-led analysis (Stokes, Maxwell, Airy).

- *France*: Analytical continuity and transcendence (Cauchy, Hermite, Poincaré).

Together these arcs did more than restore clarity. They redrew what analysis was — and what it was for.

From motion to logic. From curves to completeness. Analysis ceased to describe the world, and began to define its mathematics.

III. Epochal Outline — Analysis Through Iteration

Analysis matured through confrontation.

Each generation found flaws in the last — and redefined what it meant to be correct, convergent, or complete.

Here are eight major phases in the transformation of analysis:

1. **Pre-Rigorous Calculus** *(Newton, Euler)*

- Derivatives and integrals treated as operations on infinitesimals.
- Formal expansions used freely, often diverging or misapplied.
- Power without precision — results were right, reasoning uncertain.

2. **Cauchyan Function Theory**

- Limits, derivatives, and continuity defined via sequences and convergence.
- The derivative becomes a local property defined by a limit.
- Complex analysis begins in earnest.

3. **Weierstrassian Rigidity**

- Analysis becomes fully formal — ε–δ definitions rule.
- Continuity, differentiability, and convergence are divorced from intuition.
- Pathological functions (continuous but nowhere differentiable) are accepted, even embraced.

4. **Riemannian Integration and Surfaces**

- Defines integration as a limiting process over partitions.
- Introduces multi-valued functions, surfaces, and analytic continuation.
- Analysis becomes geometric again — but abstractly.

5. **Construction of the Reals** *(Dedekind, Cantor, Méray)*

- The real number line is constructed from rational data.
- Continuity becomes a property of completeness — not intuition.
- The irrational and transcendental gain rigorous meaning.

6. Set Theory Enters *(Cantor)*

- Infinite sets are classified: countable vs. uncountable.
- Functions are reconceived as set-theoretic mappings.
- Transfinite numbers extend arithmetic beyond the finite.

7. Analysis and Physics *(Stokes, Kelvin, Maxwell)*

- Analysis is applied to wave equations, heat, electromagnetism.
- Fourier series, boundary problems, and differential equations take center stage.
- Physical meaning guides the construction of formal tools.

8. French Continuity and Transcendence *(Hermite, Poincaré)*

- Hermite proves transcendence (e.g. e).
- Poincaré explores differential equations and chaos.
- Analysis moves toward dynamical systems and qualitative behavior.

IV. Iterative Approaches to a Core Problem

A central question in analysis:

What does it mean for a function to be continuous?

Each school of analysis answered this differently — and in doing so, revealed new possibilities, new dangers, and new definitions of mathematical truth.

1. Pre-Rigorous Intuition *(Euler)*

- A function is continuous if it can be drawn without lifting the pen.
- Continuity is a geometric notion — obvious when seen.
- Discontinuities are rare, visible, and usually removable.

2. Cauchy's Reformulation

- A function is continuous if the limit exists and equals the value at the point.
- Formulated using sequences: continuity as limit-preserving.
- Still grounded in geometric intuition — but formalized.

3. Weierstrass's ε–δ Definition

- Continuity is a local condition defined without reference to motion or graph.
- For every $\varepsilon > 0$, there exists a $\delta > 0 \ldots$
- This framework admits strange functions: continuous, nowhere differentiable.

4. Riemann's Perspective

- Continuity is examined in the context of integration and complex analysis.
- Some discontinuous functions are still integrable — the nature of discontinuity matters.
- Continuity becomes one aspect of analytic structure.

5. Cantor and the Set-Theoretic View

- Continuity is about the behavior of preimages of open sets.
- Introduces ideas of pointwise vs. uniform convergence.
- Distinguishes between countable and uncountable sets of discontinuities.

6. Poincaré and Beyond

- Continuity is no longer a guarantee of predictability.
- Qualitative behavior — not just limit behavior — becomes central.
- Dynamical systems expose sensitivity within continuous flows.

From visualization to limit. From limit to formal logic. From logic to abstraction. From abstraction to chaos.

Continuity survived each shift — but its meaning never stayed fixed.

V. Closing Dialectic

Analysis is where mathematics learned to doubt itself — and from doubt, to rebuild.

It asked questions no longer about values, but about definitions:

> What is a limit? What is a function? What is a number? What is infinity?

It took the intuitive tools of calculus — motion, tangent, area — and subjected them to discipline.

- Where Euler calculated, Weierstrass proved.
- Where Newton saw fluxions, Cauchy demanded limits.
- Where geometry once guided, set theory now governed.

The real line became a construction. Continuity became a condition. Infinity became a landscape with coordinates.

But with this rigor came abstraction: - Functions no longer had graphs. - Convergence no longer implied insight. - Continuity no longer guaranteed comprehension.

Analysis did not collapse under this weight. It ascended.

And in that ascent, it gave modern mathematics its tone:

> No result without rigor. No structure without definition. No truth without foundation.

This was analysis — not the end of intuition, but the beginning of a much greater degree of certainty.