Su2025MATH4991

Chapter 8 - Crosscurrents

Geometry, Astronomy, and the Decline of Hellenistic Innovation

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Cultural Invocation

- **Civilization:** Hellenistic Greece to Early Byzantium
- **Time Period:** 3rd century BCE to 6th century CE
- **Figures:** Eratosthenes, Aristarchus, Hipparchus, Ptolemy, Heron, Pappus

The age following Archimedes was not one of decline but diffusion — a crossroads of astronomy, engineering, trigonometry, and practical geometry. What began as geometry for ideal forms evolved into geometry for measurement, navigation, and ultimately, survival.

Map of the Tetrarchy in 293, showing the dioceses and the four tetrarchs' zones of influence.

Problem–Solution Cycles

Problem 1: Can Geometry Measure the Earth?

Concept Preview

Can a shadow reveal the size of the planet? Eratosthenes used sunlight, wells, and a bold assumption to stretch geometry from local observation to global insight.

Topic: Eratosthenes and the Size of the Planet

1. **Assumption:** Syene and Alexandria lie on the same **meridian** (a north–south line).
2. **Observation in Syene:** At noon on the summer solstice, the **sun shines straight down a well**.
3. **Observation in Alexandria:** At the same time, a vertical stick **casts a shadow**.
4. **Measurement:** The shadow angle is **1/50 of a circle** (roughly 7.2 degrees).
5. **Distance between cities: 5,000 stades**.
6. **Conclusion:** Earth's circumference is:

$$50 \times 5{,}000 = 250{,}000 \text{ stades}$$

Crosscurrents

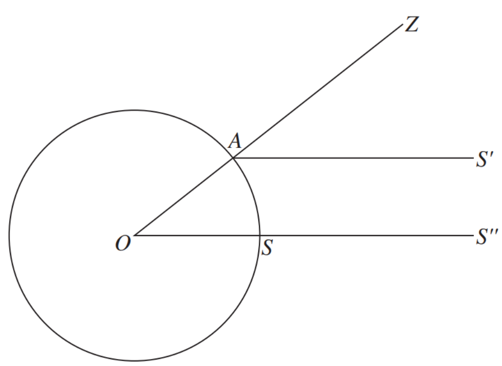


FIG. 8.1

Figure 1: Eratosthenes' method: equal angles at Earth's center and surface.

Meta-Realization: *Geometry becomes geodesy. With just a shadow, distance, and angle, the Earth becomes legible. This is measurement on a planetary scale — without leaving the ground.*

Problem 2: Can You Measure the Cosmos With Angles?

Concept Preview

Aristarchus turned his gaze upward. He wanted to know: how far is the Sun? And how big is it — compared to the Moon?

Topic: Aristarchus and the Distances to the Sun and Moon

1. **Key Moment:** When the **Moon is half-lit**, the triangle Earth–Moon–Sun is right-angled.
2. **Measurement:** $\angle MES \approx 87^\circ$
3. **Method:** Using triangle geometry:
   - The Sun is about **19× farther** than the Moon.
   - From eclipse data: the Sun is also about **19× wider**.

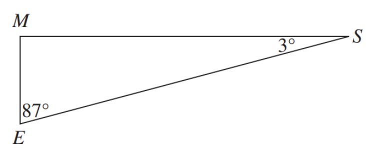


FIG. 8.2

Figure 2: Angle between Sun and Moon at half-illumination (angle MES).

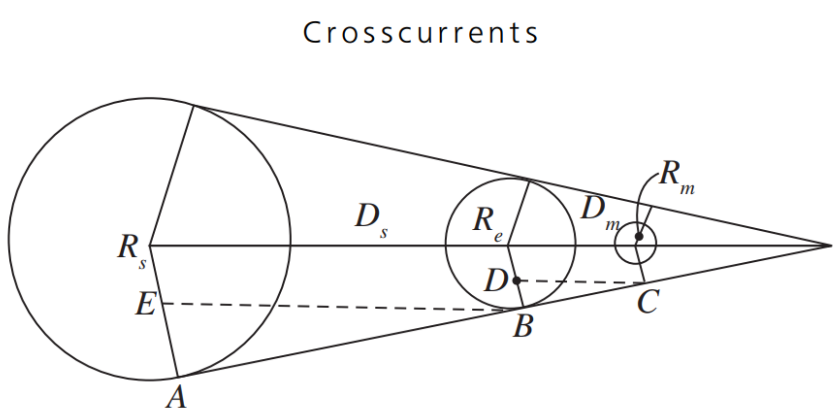


FIG. 8.3

Figure 3: Using eclipses and triangles to find size ratios.

Meta-Realization: *Angles become telescopes. Aristarchus couldn't measure the Sun directly. Instead, he used logic and geometry to peer across space.*

Problem 3: How Do You Build Trigonometry Without Sines?

Concept Preview

No calculators. No sine or cosine. Just a circle — and the Greeks' sharp eyes.

Topic: Chords and Circle Geometry

1. **Definition:** A **chord** is the straight line connecting two points on a circle.
2. **Insight:** For a central angle θ:
$$\text{chord} = 2R\sin\left(\frac{\theta}{2}\right)$$
3. **Hipparchus:** Created a **table of chords**.
4. **Menelaus:** Extended chord geometry to **spherical triangles**.

Crosscurrents

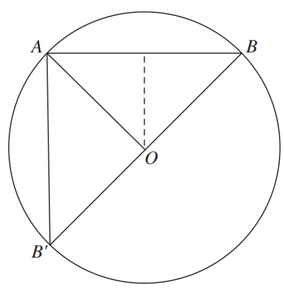


FIG. 8.4

Figure 4: Chord AB in a circle: precursor to the sine function.

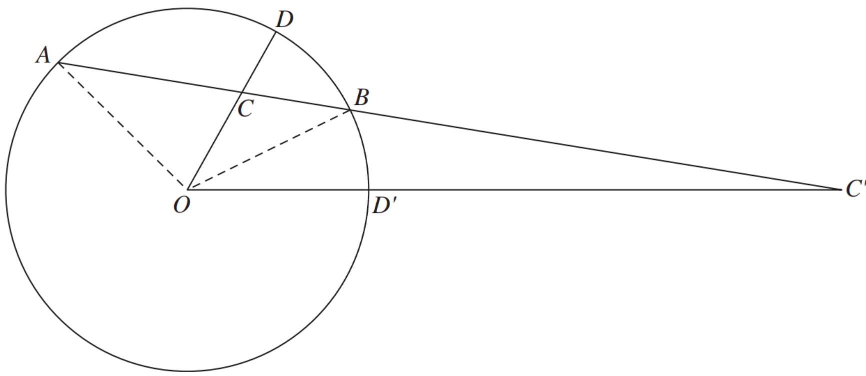


FIG. 8.5

Figure 5: Menelaus's geometric lemma using intersecting chords and radii.

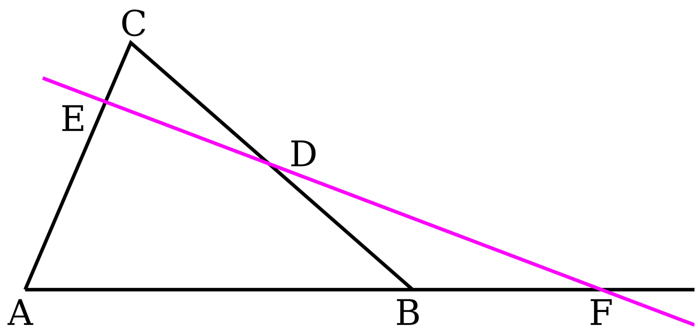


Figure 6: Menelaus's Theorem: A transversal cuts triangle ABC at points D, E, F. The identity holds: $\frac{AD}{DB} \cdot \frac{BE}{EC} \cdot \frac{CF}{FA} = 1$.

Meta-Realization: *Trigonometry was born from astronomy. The stars demanded accuracy, and geometry delivered — even before modern notation existed.*

Problem 4: What Geometric Trick Builds a Whole Trigonometry Table?

Concept Preview

Ptolemy wanted better data for astronomy. Without algebra or functions, he had to build his trigonometry tables using just geometry.

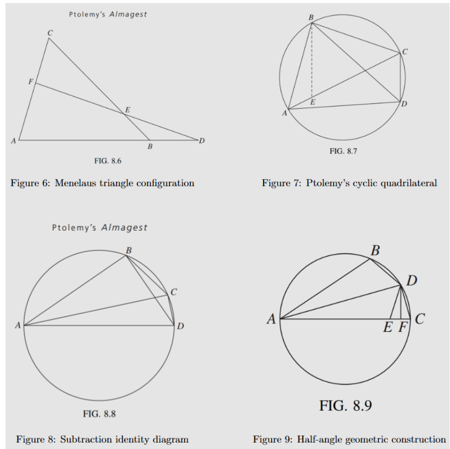
Topic: Ptolemy's Theorem and Table of Chords

1. He worked with cyclic quadrilaterals (4 points on a circle).
2. Using a theorem now named after him:

$$AC \cdot BD = AB \cdot CD + AD \cdot BC$$

3. From this he derived:
   - $\sin(\alpha \pm \beta)$ and $\cos(\alpha \pm \beta)$ identities
   - Recursive chord approximations using halving and difference
4. He built a full table of chords from 0.5° to 180°.

arc°	chord			sixtieths			
$\frac{1}{2}$	0	31	25	0	1	2	50
1	1	2	50	0	1	2	50
$1\frac{1}{2}$	1	34	15	0	1	2	50
⋮	⋮	⋮	⋮		⋮	⋮	⋮
109	97	41	38	0	0	36	23
$109\frac{1}{2}$	97	59	49	0	0	36	9
110	98	17	54	0	0	35	56
$110\frac{1}{2}$	98	35	52	0	0	35	42
111	98	53	43	0	0	35	29
$111\frac{1}{2}$	99	11	27	0	0	35	15
112	99	29	5	0	0	35	1
$112\frac{1}{2}$	99	46	35	0	0	34	48
113	100	3	59	0	0	34	34
⋮	⋮	⋮	⋮		⋮	⋮	⋮
179	119	59	44	0	0	0	25
$179\frac{1}{2}$	119	59	56	0	0	0	9
180	120	0	0	0	0	0	0



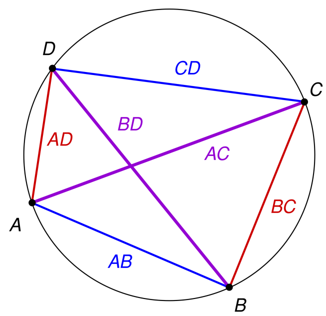


Figure 7: Ptolemy's Theorem: In cyclic quadrilateral ABCD, the diagonals and sides relate via $AC \cdot BD = AB \cdot CD + AD \cdot BC$.

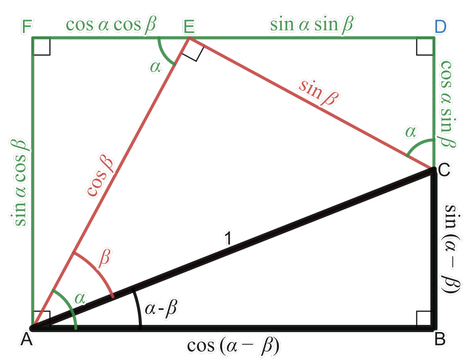


Figure 8: Geometric basis for $\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$. Angles α and β are labeled on intersecting arcs and chords.

Meta-Realization: *A clever geometric identity becomes a factory for trigonometry — centuries before sine tables had names.*

Problem 5: How Do You Model Weird Planetary Motion Using Only Circles?

Concept Preview

Planets don't move smoothly across the sky — they pause, reverse, loop. The Greeks hated irregularity. So Ptolemy did something bold.

Topic: Ptolemy's Epicycles and the Equant

1. Modeled planets using:
   - **Deferent**: large circle around Earth
   - **Epicycle**: smaller circle whose center moves on deferent
2. Introduced the **equant** — a point from which the center of the epicycle appears to move uniformly.
3. This violated Greek ideals of uniform circular motion — but matched observations.

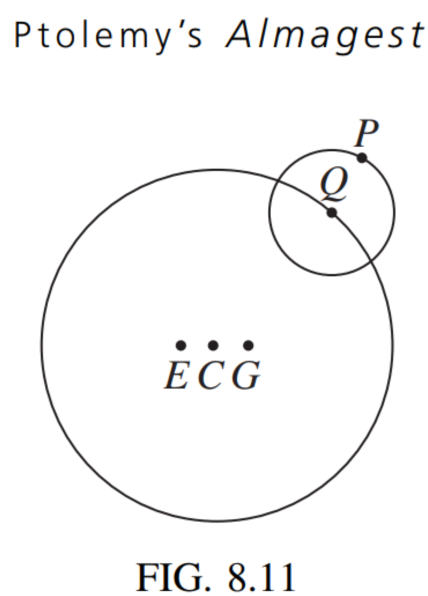


Figure 9: Ptolemy's epicycle + equant construction: a bold departure from geometric purity.

Meta-Realization: *Sometimes, models are messy. But if they predict the sky — they win.*

Problem 6: Can You Measure Sight Itself?

Concept Preview

Light reflects at equal angles — why? Heron figured it out long before physics had equations.

Topic: Heron's Optics and the Shortest Path

1. Heron asked: why does light "bounce" off a mirror with equal angles?
2. His answer: the path from source to eye via mirror is the **shortest possible path**.
3. The proof is purely geometric — using reflection and triangles.

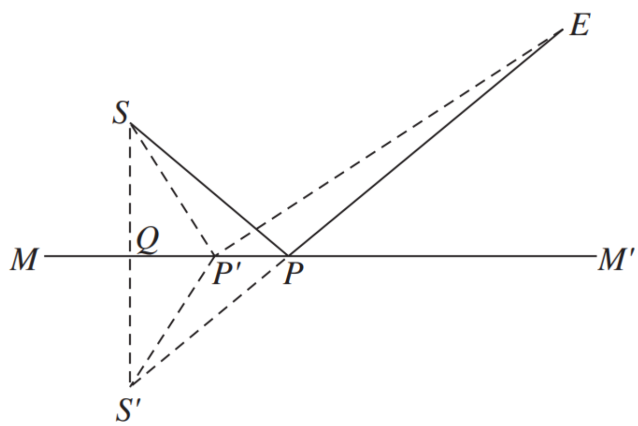


FIG. 8.12

Figure 10: Light's path from object to eye via mirror: the shortest route.

Meta-Realization: *Light obeys geometry — because nature takes the shortest route.*

Problem 7: Can You Trisect an Angle Without Breaking the Rules?

Concept Preview

The Greeks proved you can't trisect any angle using only a compass and straightedge. But what if you added a new tool — like a conic?

Topic: Pappus and Trisection by Conics

1. Classical restriction: Compass + straightedge only.
2. Pappus broke this rule — by introducing **hyperbolas**.
3. Result: An exact trisection for any angle, constructed geometrically.

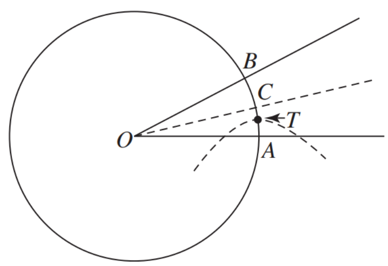


FIG. 8.13

Figure 11: Pappus's hyperbola-based trisection.

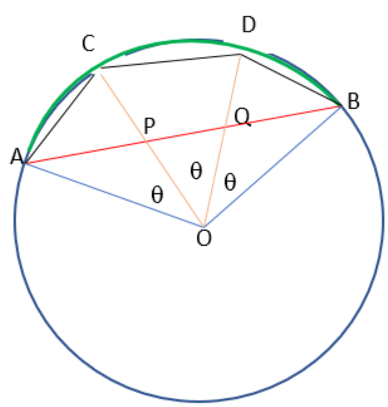


Figure 12: Pappus's trisection method: angle divided into three equal arcs using a conic intersection.

Meta-Realization: *Mathematics evolves by accepting its own limits — then breaking them with new ideas.*

Visual Interlude: The Three Classical Problems

Before we move further, let's step back. The Greeks faced three legendary construction challenges — problems so deceptively simple that entire centuries chased them with only a straightedge and compass. These problems became icons of the limits of classical geometry.

1. Trisecting an Angle

The challenge: Given an arbitrary angle, divide it into three equal parts using only classical tools.

What's the catch? Some angles (like 90 degrees) are easy, but the general problem is impossible with just compass and straightedge.

Continuity:

- **Chapter 6:** Archimedes used both the spiral and a neusis (a movable straightedge) to achieve trisection by physical construction.
- **Chapter 8:** Pappus brought in conic sections — showing that with a hyperbola, even the impossible could become solvable.

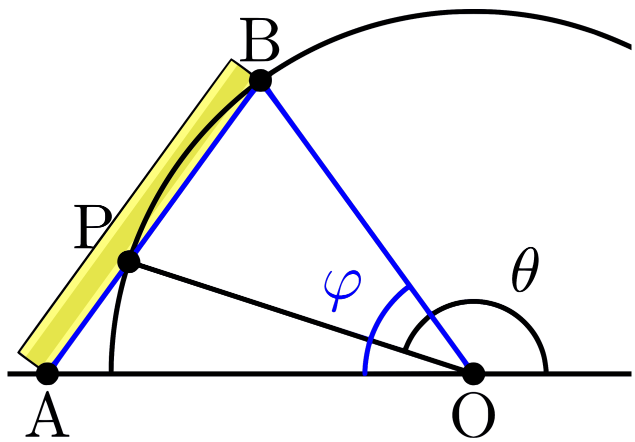


Figure 13: Angle trisection: impossible with straightedge + compass alone, solvable with conics.

2. Doubling the Cube (Delian Problem)

The challenge: Given a cube, construct another cube with exactly twice its volume.

What's the catch? This requires constructing the length $\sqrt[3]{2}$ — not possible with compass and straightedge alone.

Continuity:

- This problem was not solved classically, but Menaechmus used intersecting conics — a method later formalized by Apollonius (Chapter 7).
- While Archimedes (Chapter 6) didn't tackle this problem directly, his volume work and solid slicing techniques resonate with the spirit of the challenge.

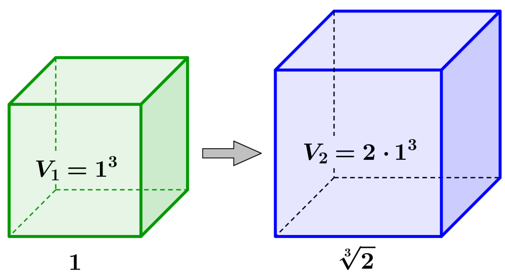


Figure 14: Doubling the cube means solving for a side of length $\sqrt[3]{2}$.

3. Squaring the Circle

The challenge: Construct a square with the same area as a given circle.

What's the catch? This would require constructing $\sqrt{\pi}$ — impossible since π is transcendental.

Continuity:

- **Chapter 6:** Archimedes approximated π geometrically with astonishing accuracy and even used his spiral to square the circle in a non-classical way.
- Later centuries would prove the classical construction impossible — but the Greek insight laid the foundation for reasoning about π long before calculus.

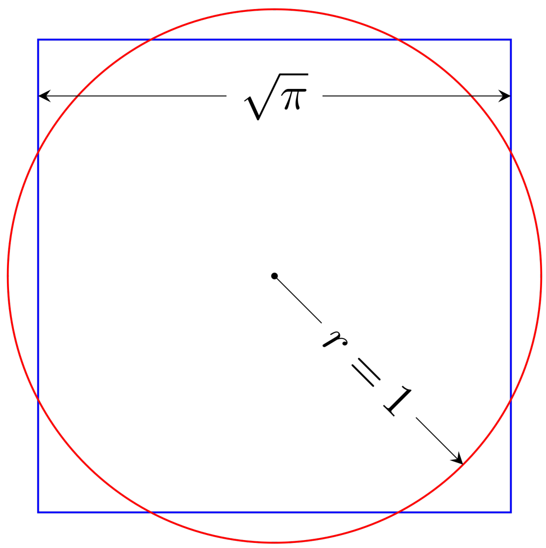


Figure 15: No classical construction can yield a square equal in area to a given circle.

Meta-Realization: **The Greeks discovered not everything can be constructed — and that realization itself was revolutionary.**

Problem 8: What's the Definition of a Curve, Really?

Concept Preview

Can a curve be defined by a rule, not a shape? Pappus showed that proportion — not pictures — could generate geometry.

Topic: The Pappus Problem

1. Given several fixed lines, define a curve such that:

   Product of distances to some lines = Product of distances to others

2. This defines conics — and higher-order algebraic curves.
3. This approach anticipates analytic geometry.

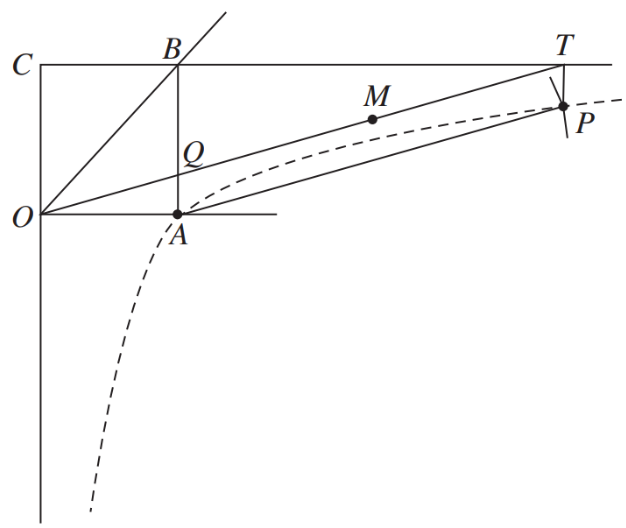


FIG. 8.14

Figure 16: Locus defined by distance products: The Pappus Problem.

Meta-Realization: *The curve isn't drawn — it's deduced. Proportion becomes place.*

Problem 9: Can You See Multiple Types of Average at Once?

Concept Preview

We learn arithmetic, geometric, and harmonic means separately. Pappus showed they could be visualized — all in one diagram.

Topic: Pappus's Means in One Diagram

1. Draw a semicircle on diameter AC.
2. Drop perpendicular DB and construct segments:
   - DO = Arithmetic Mean of AB, BC
   - DB = Geometric Mean
   - DF = Harmonic Mean
3. All means emerge from one configuration.

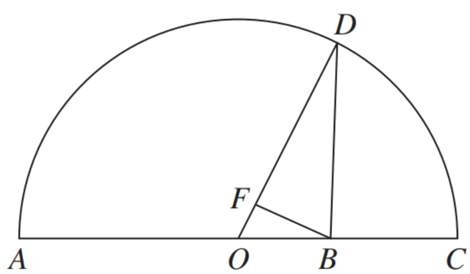


FIG. 8.15

Figure 17: Visualizing the arithmetic, geometric, and harmonic means.

Meta-Realization: *Geometry doesn't just prove — it visualizes relationships we usually treat as abstract.*

Problem 10: Why Hexagons?

Concept Preview

Why do bees make honeycombs with hexagons? Because no other shape tiles space so efficiently with so much area.

Topic: Isoperimetry and Pappus's Bee Theorem

1. Question: What shape with a fixed perimeter encloses the most area?
2. Pappus (and earlier Zenodorus) showed: among regular tilings, the **hexagon wins**.
3. Nature seems to "know" this — bees instinctively build hexagonal cells.

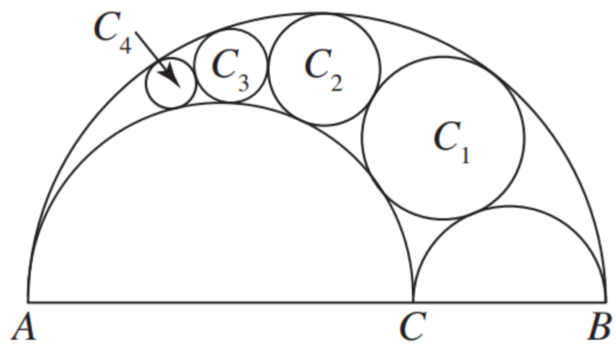


FIG. 8.17

Figure 18: Bees choose hexagons: the best area-to-edge ratio in a tiling.

Meta-Realization: *Nature optimizes — and sometimes, it beats us to the proof.*

IV. Closing Dialectic

Summary

These thinkers didn't just do geometry — they built machines for thinking. Tables, tools, techniques — some abstract, some brutally applied. Their ideas crossed borders and languages:

- Greek **chord tables** became Indian **sine tables**.
- Ptolemy's **planetary models** influenced Islamic astronomy and later **Copernican systems**.
- In parallel, Chinese scholars developed precise **numerical methods, calendars**, and **approximation algorithms** — often centuries ahead of their time.
- They all laid groundwork for:
  - China's enduring **computational ingenuity**,
  - India's **decimal revolution**,
  - Islam's **algebraic synthesis**,
  - and Europe's **rediscovery of mathematical reason along the same lines** in the Renaissance.

This chapter closes the curtain on ancient geometry — but leaves the stage set for its transformation in new languages, new tools, and new visions of the universe.

Exit Prompt

> **You are Heron.** No textbook. No computer. Just mirrors, ropes, pulleys, and mind. *What will you measure — and how will you prove it?*