Ch5 Flashcards

Harley Caham Combest

Fa2025 2025-10-24 MATH5353

Chapter 5 — Direct and Semidirect Products, and Abelian Groups

This chapter develops two complementary themes: (1) assembling groups from known pieces via direct and semidirect products, and (2) classifying and recognizing structure, especially for finitely generated abelian groups and for groups of small order.

- **Building groups by products.** The chapter begins with (external/internal) *direct products* $G_1 \times \cdots \times G_n$, emphasizing axis embeddings, projection maps, behavior of orders and centers, and the commutativity across different factors. This gives a controlled way to construct larger groups from known ones.

- **Classification of finitely generated abelian groups.** The Fundamental Theorem is presented in both *invariant-factor* and *elementary-divisor* forms:

  $$G \cong \mathbb{Z}^r \times \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_s} \quad \text{with} \quad n_s \mid \cdots \mid n_1,$$

  and equivalently as a product of p-primary cyclic components. Practical conversions between the two forms are outlined, enabling complete listings of all abelian groups of a given order.

- **Small-order landscape.** Using the abelian classification together with standard nonabelian families, the text compiles a table of groups of small order (presentations, quick invariants, and examples such as dihedral, quaternionic, and other semidirect constructions).

- **Recognizing internal direct products.** A recognition criterion is proved: if $H, K \trianglelefteq G$, $H \cap K = 1$, and $G = HK$, then $G \cong H \times K$; elements of H commute with those of K, and each $g \in G$ has a unique decomposition $g = hk$ with $h \in H$, $k \in K$.

- **Semidirect products and controlled nonabelian extensions.** Relaxing normality to one factor yields $H \rtimes_\varphi K$ via an action $\varphi : K \to \mathrm{Aut}(H)$. A matching internal recognition theorem shows that whenever $H \triangleleft G$, $G = HK$, and $H \cap K = 1$, the group is a semidirect product. This framework systematically produces many nonabelian groups and underpins several order-specific classifications by analyzing possible actions into $\mathrm{Aut}(H)$.

Why it matters. Chapter 5 provides

1. constructive tools (direct/semidirect products) to *manufacture* groups,
2. definitive structure theorems to *classify* all finitely generated abelian groups, and
3. recognition criteria to *detect* internal splittings.

These methods are foundational for later work on extensions, nilpotent/solvable groups, and representation theory.

5.1 Direct Products

Definition. For groups $G_1, \dots, G_n$, the (external) direct product $G = G_1 \times \cdots \times G_n$ is the set of n-tuples with componentwise multiplication. Then G is a group with identity $(1, \dots, 1)$ and $(g_1, \dots, g_n)^{-1} = (g_1^{-1}, \dots, g_n^{-1})$. If all G_i are finite, $|G| = \prod_i |G_i|$. Each G_i embeds as the "i-th axis" subgroup $\{(1, \dots, 1, g_i, 1, \dots, 1)\} \leq G$, and the coordinate projections $\pi_i : G \to G_i$ are surjective homomorphisms with kernels $\prod_{j \neq i} G_j$. Elements supported in different factors commute.

Basic consequences.

- $Z(G_1 \times \cdots \times G_n) = Z(G_1) \times \cdots \times Z(G_n)$; hence the product is abelian iff each factor is abelian.
- Reordering factors yields an isomorphic product.
- For $x_i \in G_i$, $\mathrm{ord}(x_1 \cdots x_n) = \mathrm{lcm}_i\{\mathrm{ord}(x_i)\}$ when components commute as above.

5.2 Fundamental Theorems for Finitely Generated Abelian Groups

Invariant-factor form. Every finitely generated abelian group G is isomorphic to

$$G \cong \mathbb{Z}^r \times \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_s}, \qquad r \geq 0,\ 2 \leq n_s \mid n_{s-1} \mid \cdots \mid n_1,$$

with r the free rank (Betti number) and $(n_1, \ldots, n_s)$ the *invariant factors*, unique up to isomorphism. Finite abelian groups are exactly those with $r = 0$; their order is $\prod_j n_j$.

Primary (elementary-divisor) form. If $|G| = n = \prod_i p_i^{\alpha_i}$, then

$$G \cong \bigoplus_i A_i, \quad |A_i| = p_i^{\alpha_i}, \quad A_i \cong \mathbb{Z}_{p_i^{\beta_{i1}}} \times \cdots \times \mathbb{Z}_{p_i^{\beta_{it_i}}},$$

with $\beta_{i1} \geq \cdots \geq \beta_{it_i} \geq 1$ and $\sum_j \beta_{ij} = \alpha_i$. These p-power cyclic moduli are the *elementary divisors*. The two forms are equivalent and unique. The number of isomorphism types of abelian groups of order $n = \prod_i p_i^{\alpha_i}$ is $\prod_i \mathsf{p}(\alpha_i)$, where $\mathsf{p}(\cdot)$ is the partition function.

Practical conversions.

- From invariant factors to elementary divisors: factor each n_j into prime powers and regroup by primes.
- From elementary divisors to invariant factors: for each prime p, sort the p-powers in nonincreasing order into a column; pad shorter columns with 1's, then multiply across rows to get $(n_1, \ldots, n_s)$ with divisibility $n_s \mid \cdots \mid n_1$.

5.3 Table of Groups of Small Order (highlights)

Using the above, one lists all abelian types for small n and cites standard nonabelian families (e.g., dihedral D_{2n}, quaternion Q_{2^m}, semidirects like $\mathbb{Z}_3 \rtimes \mathbb{Z}_4$, Frobenius group of order 20, etc.) together with presentations.

5.4 Recognizing Internal Direct Products

Commutators. $[x, y] = x^{-1}y^{-1}xy$; the subgroup G' generated by commutators is characteristic, and G/G' is the largest abelian quotient.

Criterion (Internal Direct Product). If $H, K \trianglelefteq G$ and $H \cap K = 1$, then $HK \cong H \times K$; moreover every element of H commutes with every element of K, and each $g \in HK$ decomposes uniquely as hk with $h \in H$, $k \in K$.

5.5 Semidirect Products

Construction. Given a homomorphism $\varphi : K \to \mathrm{Aut}(H)$, define $H \rtimes_\varphi K$ on the set $H \times K$ by

$$(h_1, k_1) \cdot (h_2, k_2) = (\, h_1 \cdot \varphi(k_1)(h_2),\ k_1k_2\,).$$

Then $H \lhd H \rtimes K$, $H \cap K = 1$, and $khk^{-1} = \varphi(k)(h)$. The product is direct iff φ is trivial (equivalently $K \lhd H \rtimes K$).

Recognition (Internal Semidirect Product). If G has subgroups $H \lhd G$ and $K \leq G$ with $H \cap K = 1$ and $G = HK$, then $G \cong H \rtimes_\varphi K$ where φ is conjugation of K on H.

Standard classifications via semidirect products (samples).

- *Order pq ($p < q$ primes).* If $p \nmid (q-1)$ then $G \cong \mathbb{Z}_{pq}$; if $p \mid (q-1)$ there are exactly two types: cyclic and a unique nonabelian semidirect $\mathbb{Z}_q \rtimes \mathbb{Z}_p$.
- *Order* 12*.* Five types: three abelian ($\mathbb{Z}_{12}$, $\mathbb{Z}_6 \times \mathbb{Z}_2$, $\mathbb{Z}_3 \times \mathbb{Z}_2 \times \mathbb{Z}_2$) and two core nonabelian families obtained as semidirects (e.g. D_{12}, A_4, or $\mathbb{Z}_3 \rtimes \mathbb{Z}_4$ depending on the action).
- p^3 *(odd p).* Exactly three types: two abelian ($\mathbb{Z}_{p^3}$, $\mathbb{Z}_{p^2} \times \mathbb{Z}_p$, $\mathbb{Z}_p^3$ gives three abelian total) and two nonabelian: the Heisenberg-type $(\mathbb{Z}_p^2) \rtimes \mathbb{Z}_p$ (exponent p) and the $\mathbb{Z}_{p^2} \rtimes \mathbb{Z}_p$ type (contains elements of order p^2).

5.1: Exercise 14. Let $G = A_1 \times A_2 \times \cdots \times A_n$, and for each i let B_i be a normal subgroup of A_i. Prove that $B_1 \times B_2 \times \cdots \times B_n \trianglelefteq G$ and that

$$(A_1 \times A_2 \times \cdots \times A_n)/(B_1 \times B_2 \times \cdots \times B_n) \cong (A_1/B_1) \times (A_2/B_2) \times \cdots \times (A_n/B_n).$$

As General Proposition: If each $B_i \trianglelefteq A_i$, then $\prod_{i=1}^n B_i \trianglelefteq \prod_{i=1}^n A_i$ and the quotient by $\prod_i B_i$ is naturally isomorphic to $\prod_i (A_i/B_i)$.
As Conditional Proposition: Let $G = \prod_{i=1}^n A_i$ and $B_i \trianglelefteq A_i$ for all i. Then $B := \prod_{i=1}^n B_i \trianglelefteq G$ and $G/B \cong \prod_{i=1}^n (A_i/B_i)$.

. . . .

Intuition. Conjugation in a direct product is coordinatewise, so normality of each B_i in A_i forces normality of $\prod_i B_i$ in G. For the quotient, map $(a_1, \ldots, a_n)$ to $(a_1B_1, \ldots, a_nB_n)$; its kernel is exactly $\prod_i B_i$ and it is onto, so the First Isomorphism Theorem gives $G/\prod_i B_i \cong \prod_i(A_i/B_i)$.

. . . .

Proof.

Step 1 (Set-up). Write $G = \prod_{i=1}^n A_i$ and $B = \prod_{i=1}^n B_i$ with $B_i \trianglelefteq A_i$ for each i.

Step 2 (Normality via coordinatewise conjugation). For $a = (a_1, \ldots, a_n) \in G$ and $b = (b_1, \ldots, b_n) \in B$,

$$aba^{-1} = (a_1b_1a_1^{-1}, \ldots, a_nb_na_n^{-1}),$$

and since $a_ib_ia_i^{-1} \in B_i$ for each i, we have $aba^{-1} \in B$; hence $B \trianglelefteq G$.

Step 3 (Define the comparison map). Define $\varphi : G \to \prod_{i=1}^n(A_i/B_i)$ by

$$\varphi(a_1, \ldots, a_n) = (a_1B_1, \ldots, a_nB_n).$$

This is a homomorphism because multiplication is coordinatewise on both domain and codomain.

Step 4 (Kernel).

$$\ker \varphi = \{(a_1, \ldots, a_n) : a_i \in B_i \ \forall i\} = \prod_{i=1}^n B_i = B.$$

Step 5 (Surjectivity). Each projection $A_i \to A_i/B_i$ is onto, so φ is onto $\prod_i(A_i/B_i)$.

Step 6 (Apply First Isomorphism Theorem). By the First Isomorphism Theorem,

$$G/B \cong \prod_{i=1}^n(A_i/B_i),$$

which is the desired natural isomorphism.

5.2: Exercise 4(a). In each of parts (a) to (d) determine which pairs of abelian groups listed are isomorphic (here $\{a_1, a_2, \ldots, a_k\}$ denotes $\mathrm{Z}_{a_1} \times \mathrm{Z}_{a_2} \times \cdots \times \mathrm{Z}_{a_k}$).
(a) $\{4, 9\},\ \{6, 6\},\ \{8, 3\},\ \{9, 4\},\ \{6, 4\},\ \{64\}$.

As General Proposition: Two finite abelian groups are isomorphic iff their p-primary decompositions agree for every prime p (equivalently, they have the same multiset of invariant factors/elementary divisors).
As Conditional Proposition: Among the six groups above, the only isomorphic pair is $\{4, 9\} \cong \{9, 4\}$. The other four are pairwise non-isomorphic.

. . . .

Intuition. First split by order: possible orders are 36 ($\{4,9\},\{6,6\}$), 24 ($\{8,3\},\{6,4\}$), and 64 ($\{64\}$). Within a fixed order, compare p-parts: cyclic vs. noncyclic 2-parts or 3-parts force non-isomorphism.

. . . .

Proof.

Step 1 (Orders).

$$|\{4,9\}| = |\{6,6\}| = 36, \quad |\{8,3\}| = |\{6,4\}| = 24, \quad |\{64\}| = 64.$$

Groups of different orders cannot be isomorphic, so compare within $\{36\}$- and $\{24\}$-blocks.

Step 2 (36-block).

$$\{4,9\} \cong \mathrm{Z}_4 \times \mathrm{Z}_9 \cong \mathrm{Z}_{36} \quad \text{(coprime factors } 4,9\text{)},$$

so it is cyclic. Also $\{9,4\}$ is the same group. But

$$\{6,6\} \cong \mathrm{Z}_6 \times \mathrm{Z}_6 \cong (\mathrm{Z}_2 \times \mathrm{Z}_3) \times (\mathrm{Z}_2 \times \mathrm{Z}_3) \cong \mathrm{Z}_2^2 \times \mathrm{Z}_3^2,$$

which is not cyclic (its 2-part is Z_2^2). Hence $\{6,6\} \ncong \{4,9\}$ and $\{6,6\} \ncong \{9,4\}$, while $\{4,9\} \cong \{9,4\}$.

Step 3 (24-block).

$$\{8,3\} \cong \mathrm{Z}_8 \times \mathrm{Z}_3 \cong \mathrm{Z}_{24} \quad \text{(coprime } 8,3;\ \text{cyclic)},$$

whereas

$$\{6,4\} \cong \mathrm{Z}_6 \times \mathrm{Z}_4 \cong (\mathrm{Z}_2 \times \mathrm{Z}_3) \times \mathrm{Z}_4 \cong (\mathrm{Z}_4 \times \mathrm{Z}_3) \times \mathrm{Z}_2 \cong \mathrm{Z}_{12} \times \mathrm{Z}_2,$$

which is not cyclic. Hence $\{8,3\} \ncong \{6,4\}$.

Step 4 (64-singleton). $\{64\} \cong \mathrm{Z}_{64}$ has order 64, so it cannot be isomorphic to any of the others.

Conclusion. The only isomorphic pair is $\boxed{\{4,9\} \cong \{9,4\}}$. All others are non-isomorphic.

Additional: Exercise 1. Let G be a finite group. Prove that G is abelian if and only if all of its Sylow subgroups are normal and abelian.

As General Proposition: A finite group is abelian $\iff$ each of its Sylow subgroups is normal and abelian.
As Conditional Proposition: Let $|G| = \prod_{i=1}^{k} p_i^{a_i}$. Then G is abelian if and only if for every i the Sylow p_i-subgroup P_i is normal in G and P_i is abelian.

. . . .

Intuition. The forward direction is immediate: subgroups of an abelian group are abelian and normal. For the converse, if all Sylow subgroups P_i are normal and abelian, then different P_i's commute (their commutator lies in $P_i \cap P_j = 1$ by coprime orders), and G is the internal direct product $P_1 \cdots P_k \cong P_1 \times \cdots \times P_k$, hence abelian.

. . . .

Proof.

($\Rightarrow$) If G is abelian then its Sylow subgroups are normal and abelian. Any subgroup of an abelian group is abelian. In an abelian group every subgroup is normal. Hence each Sylow subgroup of G is both normal and abelian.

($\Leftarrow$) If all Sylow subgroups are normal and abelian then G is abelian. Let $P_i \in \mathrm{Syl}_{p_i}(G)$ be the Sylow p_i-subgroups, assumed normal and abelian.

Step 1 (Pairwise intersections are trivial). For $i \neq j$, $|P_i \cap P_j|$ divides both $|P_i| = p_i^{a_i}$ and $|P_j| = p_j^{a_j}$; since $(p_i, p_j) = 1$, we have $P_i \cap P_j = \{e\}$.

Step 2 (Different Sylow subgroups commute). Because $P_i, P_j \trianglelefteq G$, the commutator subgroup $[P_i, P_j] \leq P_i \cap P_j = \{e\}$, so P_i and P_j centralize each other. Hence every element of P_i commutes with every element of P_j.

Step 3 (Product is a subgroup of full order). The product

$$H := P_1 P_2 \cdots P_k$$

is a subgroup (by normality of each P_i) and, using Step 1 and induction,

$$|H| = \prod_{i=1}^{k} |P_i| = \prod_{i=1}^{k} p_i^{a_i} = |G|.$$

Thus $H = G$.

Step 4 (Internal direct product). By Steps 1–2, the multiplication map

$$P_1 \times \cdots \times P_k \longrightarrow G, \qquad (x_1, \ldots, x_k) \mapsto x_1 \cdots x_k$$

is an injective homomorphism with image G, hence an isomorphism. Therefore

$$G \;\cong\; P_1 \times \cdots \times P_k.$$

Each P_i is abelian, so their direct product is abelian. Hence G is abelian. □

Additional: Exercise 2. Let N, H be groups.

(a) Suppose $\varphi_1, \varphi_2 : H \to \mathrm{Aut}(N)$ are homomorphisms and there exist $\psi \in \mathrm{Aut}(N)$ and an isomorphism $\sigma : H \to H$ such that

$$\psi\,\varphi_1(h)\,\psi^{-1} \;=\; \varphi_2(\sigma(h)) \quad \text{for all } h \in H.$$

Prove that $N \rtimes_{\varphi_1} H \;\cong\; N \rtimes_{\varphi_2} H$.

(b) Show that there are exactly four groups of order 28 up to isomorphism. (Hint: use part (a) and Sylow; you may use $\mathrm{Aut}(\mathrm{Z}/7) \cong C_6$.)

As General Proposition: (a) Semidirect products $N \rtimes_{\varphi_1} H$ and $N \rtimes_{\varphi_2} H$ are isomorphic whenever φ_1 and φ_2 are related by pre/post-composition with automorphisms of H and N as above. (b) Up to isomorphism there are exactly four groups of order 28:

$$C_{28}, \qquad C_{14} \times C_2, \qquad D_{14}, \qquad \mathrm{Dic}_7 \text{ (dicyclic of order 28)}.$$

As Conditional Proposition: (a) With the given $\psi \in \mathrm{Aut}(N)$ and $\sigma \in \mathrm{Aut}(H)$, $N \rtimes_{\varphi_1} H \cong N \rtimes_{\varphi_2} H$. (b) Every group G of order $28 = 2^2 \cdot 7$ is isomorphic to exactly one of the four groups listed above.

. . . .

Intuition. (a) View a semidirect product as $N\times H$ with twisted multiplication $(n,h)(m,k) = (n\cdot\varphi(h)(m),\, hk)$. If two actions differ by rebasing N via ψ and relabeling H via σ, the coordinate change $(n,h)\mapsto(\psi(n),\sigma(h))$ transports one product law to the other. (b) By Sylow, the Sylow-7 subgroup $P\cong C_7$ is normal (either unique or characteristic inside a normal C_{14}). Then $G\cong C_7\rtimes H$ with $|H|=4$ and action $H\to\mathrm{Aut}(C_7)\cong C_6$. The only possible images have order dividing $\gcd(4,6)=2$: either trivial or the unique order-2 subgroup $\{\pm1\}$ (inversion). Taking $H\cong C_4$ or V_4 gives exactly two abelian and two nonabelian outcomes.

. . . .

Proof.

Part (a).

Step 1 (Definitions). On the set $N\times H$ define

$$(n,h)\cdot_{\varphi_i}(m,k) \;=\; \big(n\,\varphi_i(h)(m),\, hk\big) \qquad (i=1,2).$$

Let $F: N\times H\to N\times H$ be $F(n,h) := (\psi(n),\, \sigma(h))$.

Step 2 (Homomorphism check). For all $n,m\in N$, $h,k\in H$,

$$\begin{aligned}
F\big((n,h)\cdot_{\varphi_1}(m,k)\big) &= F\big(n\,\varphi_1(h)(m),\, hk\big) = (\psi(n\,\varphi_1(h)(m)),\, \sigma(hk))\\
&= (\psi(n)\cdot\psi\varphi_1(h)\psi^{-1}(\psi(m)),\, \sigma(h)\sigma(k))\\
&= (\psi(n)\cdot\varphi_2(\sigma(h))(\psi(m)),\, \sigma(h)\sigma(k))\\
&= F(n,h)\cdot_{\varphi_2}F(m,k),
\end{aligned}$$

so $F:(N\times H,\cdot_{\varphi_1})\to(N\times H,\cdot_{\varphi_2})$ is a homomorphism. Since ψ,σ are bijective, F is bijective with inverse $(n,h)\mapsto(\psi^{-1}(n),\sigma^{-1}(h))$. Hence F is an isomorphism, proving $N\rtimes_{\varphi_1}H\cong N\rtimes_{\varphi_2}H$.

Part (b).

Step 1 (Normal 7-Sylow and reduction to semidirects). Let G have order $28=4\cdot7$. By Sylow,

$$n_7\equiv1 \pmod 7,\quad n_7\mid 4\;\Rightarrow\; n_7\in\{1,4\}.$$

If $n_7=1$ then the Sylow-7 subgroup $P\cong C_7$ is normal. If $n_7=4$, then G has a normal cyclic subgroup $\langle r\rangle\cong C_{14}$ (the rotation subgroup in the dihedral/ dicyclic cases), and its unique subgroup of order 7, $P=\langle r^2\rangle\cong C_7$, is characteristic in $\langle r\rangle$ and hence normal in G. Thus in all cases $P\trianglelefteq G$, and

$$G\;\cong\;C_7\rtimes_\varphi H \qquad \text{for some } H\le G,\ |H|=4,\ \varphi: H\to\mathrm{Aut}(C_7)\cong C_6.$$

Step 2 (Possible actions). Since $|H|=4$ and $|\mathrm{Aut}(C_7)|=6$, the image $\varphi(H)$ has order dividing 2. Hence either:

- *Trivial action* ($\varphi = 1$): $G \cong C_7 \times H$.
- *Inversion action* (φ onto the unique order-2 subgroup $\{\pm 1\} \leq C_6$): a nontrivial semidirect.

By part (a), actions that differ by automorphisms of H or C_7 yield isomorphic semidirect products; since C_6 has a unique subgroup of order 2, there is only one nontrivial action type for each isomorphism type of H.

Step 3 (Take $H \cong C_4$ or $H \cong V_4$).

- $H \cong C_4$:
  - Trivial action: $C_7 \times C_4 \cong C_{28}$.
  - Nontrivial action (generator acts by $x \mapsto x^{-1}$): the dicyclic group Dic_7 of order 28 with presentation

$$\langle a, x \mid a^{14} = 1,\ x^4 = 1,\ x^2 = a^7,\ xax^{-1} = a^{-1} \rangle$$

    (equivalently $C_7 \rtimes C_4$ with kernel of the action generated by x^2).
- $H \cong V_4$:
  - Trivial action: $C_7 \times V_4 \cong C_{14} \times C_2$.
  - Nontrivial action (each nonidentity in V_4 acts by inversion, image $\cong C_2$): the dihedral group D_{14} of order 28.

Step 4 (No further isomorphisms). The four groups are pairwise nonisomorphic: the abelian ones are distinct by invariant factors ($C_{28} \not\cong C_{14} \times C_2$); among nonabelian groups, D_{14} has an index-2 cyclic subgroup of order 14 whose quotient by it is C_2, whereas Dic_7 has a cyclic quotient of order 4 by its normal C_7 and contains an element of order 4 whose square lies in C_7—properties not shared by D_{14}. Hence exactly four isomorphism classes occur.