QUANTUM FIELD THEORY

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1 Lie Groups

A Lie group is a group whose elements g depend in a continuous and differentiable way on a set of real parameters θ. Therefore a Lie group is at the same time a group and a differentiable manifold. W.l.o.g. the coordinates θ can be chosen such that the identity element e of the group corresponds to $\theta = 0$, that means $g(0) = e$.
A (linear) *representation* R of a group is an operation that assigns to each group element a linear operator D_R

$$g \mapsto D_R(g) \tag{1}$$

The properties are:

(i): $D_R(e) = 1$, where 1 is the identity operator

(ii): $D_R(g_1)D_R(g_2) = D_R(g_1 g_2)$, so that the mapping preserves the group structure

The linear space on which the operators D_R act is called the *basis* of the representation R. E.g. taking as group $SO(3)$ and as base space the spatial vectors $\mathbf{v}$, an element $g \in SO(3)$ can be interpreted physically as a rotation in three-dimensional space. A representation is *reducible*, if it has a subspace invariant under the transformation, i.e. elements from the subspace dont́ leave it if $D_R s$ act on them. The representation is *completely reducible*, if for all g and a certain choice of basis the matrices can be written in block diagonal form (quadratic matrices on the diagonal). That means that the basis vectors split into subsets that dont́ mix. It means that the completely reducible representation can be written as the direct sum of *irreducible* representations, representation without not-mixing subspaces. *Equivalent* representations are achieved by change of basis.

There is the *Lie algebra* that is independent on whatever representation is taken. For infinitesimal (assumption of smoothness) D_R (rotations) we can write:

$$\boxed{D_R(\theta) = 1 + i\theta T_R + \mathcal{O}(\theta^2)} \tag{2}$$

with

$$\boxed{T_R = -i\frac{\partial D_R}{\partial \theta}\bigg|_{\theta=0}} \tag{3}$$

The T_R are called the *generators* of the group in the representation R. With appropriate choice of parametrization far from the identity the group element can always be represented by

$$D_R(g(\theta)) = e^{i\theta T_R} \tag{4}$$

The factor i is chosen so that with hermitean generators the matrices D_R are unitary, so that the representation is called *unitary*.
With the property (ii):

$$e^{i\alpha T_R} e^{\beta T_R} = e^{i\delta T_R} \tag{5}$$

with some $\delta(\alpha, \beta)$. Expanding the logarithm of the relation to second order, we get:

$$\alpha^a \beta^b \left[T_R^a, T_R^b\right] = i\gamma_c(\alpha, \beta) T_R^c \tag{6}$$

with $\gamma_c(\alpha, \beta) \propto (\delta_c - \alpha_c - \beta_c)$. Since this is true for all α and β, γ_c must be linear in these, and takes the general form $\gamma_c = \alpha_a \beta_b f_c^{ab}$ with some constants f_c^{ab}. We have derived the Lie-algebra of the group:

$$\boxed{\left[T^a, T^b\right] = i f_c^{ab} T^c} \tag{7}$$

It can be shown that the equation is valid to any order. The *structure constants* f_c^{ab} are independent of the representation and obey the identities:

$$\boxed{f_C^{AB} = -f_C^{BA}} \tag{8}$$

$$\boxed{f_D^{AB} f_E^{CD} + f_D^{CA} f_E^{BD} + f_D^{BC} f_E^{AD} = 0} \tag{9}$$

Thus the structure constants define the Lie algebra and the problem of finding all possible matrix representations of the group amounts to the algebraic problem of finding all possible solutions T_R^a of the equation.
A group is called *abelian* if all its elements commute, otherwise it is *non-abelian*. Therefore

for an abelian group the structure constants vanish. All irreducible representations of an abelian Lie group are one-dimensional.
Another important quantity of a Lie group are the *Casimir operators*. These are constructed from such T that commute with all other T of the group. In each irreducible representation the Casimir operators are proportional to the identity matrix and the proportionality constant labels the representation.
E.g. the angular momentum algebra is $\left[J^i, J^j\right] = i\epsilon^{ijk}J^k$ and the Casimir operator is J^2. It commutes with all angular momentum components and, on an irreducible representation, J^2 is equal to $j(j+1)$ times the identity matrix, with $j = 0, \frac{1}{2}, 1, ...$
Spatial rotations are a compact manifold, and hence are called a *compact* group. The Lorentz and Poincare Lie groupy are non-compact manifolds. Such groups have no non-trivial unitary finite-dimensional representations. So an *infinite-dimensional representation* is needed to get hermitean operators, which only can be identified with physical observables.

2 The Lorentz Group

The Lorentz group is defined as the group that acts in the vector representation (e.g. coordinate transformations) as:

$$x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu \tag{10}$$

which leave the quantity

$$\eta_{\mu\nu}x^\mu x^\nu = t^2 - x^2 - y^2 - z^2 \tag{11}$$

unchanged. The transformation of the coordinates is due to the observer with another reference frame x'^μ which moves inertially $\mathbf{v}$ relative to the reference frame x^μ. The physical laws and c have to be Lorentz scalars. On the contrary an acceleration nean leaving of one's inertial frame and produces a different physical state. However also the acceleration can be accounted for in this formalism, but is most properly dealt with in the ART. The Lorentz transform is the orthogonal group $O(3,1)$, with 3 referring to the negative spatial coordinates and 1 to the positive time coordinate.
This condition can be rewritten as

$$\eta = \Lambda^T \eta \Lambda \tag{12}$$

This implies that $\det \Lambda = \pm 1$ both satisfy the condition. The transformation with the plus sign can always be written as the product of a transformation with the minus sign and a odd sign reversal transformation, e.g. a parity transformation or a single axis reflection. The plus sign transformation is the *proper Lorentz transformation*. It is denoted $SO(3,1)$. The condition also implies that $(\Lambda^0{}_0)^2 \geq 1$. The transformation with the plus sign is called *orthochronous*, the other *non-orthochronous*. The non-orthochrounous transformation can be related to the orthochronous one similar as above. So we reduce the degrees of freedom

to $SO(3,1)$ with $\Lambda^0{}_0 \geq 1$.
Considering a infinitesimal transformation (proportional to the parameters):

$$\Lambda^{\mu}{}_{\nu} = \delta^{\mu}_{\nu} + \omega^{\mu}{}_{\nu} \tag{13}$$

the group condition yields:

$$\omega_{\mu\nu} = -\omega_{\nu\mu} \tag{14}$$

which means six independent parameters for an antisymmetric 4x4 matrix. The spatial rotations group $SO(3)$ leaves the t axis invariant and gives the three rotational angles as parameters. The three transformations which leave $t^2 - x_i$ invariant are called *boost* along the respective axis. This yields the three parameters of the $\mathbf{v}$ vector. Since $\mathbf{v}$ is in a non-compact interval, the Lorentz group is a *non-compact* group.
To six parameters correspond six generators. We arrange them again in a 4x4 antisymmetric matrix $J^{\mu\nu}$. A general element is therefore written as:

$$\Lambda = e^{-\frac{1}{2}\omega_{\mu\nu}J^{\mu\nu}} \tag{15}$$

avoiding double-counting. All physical quantities can be classified accordingly to their transformation properties under the Lorentz group into the categories:

1. *scalar*, which doesn't change under transformation
2. *contravariant four-vector* V^{μ}, which transforms as

$$V^{\mu} \to \Lambda^{\mu}{}_{\nu} V^{\nu} \tag{16}$$

3. *covariant four-vector* V_{μ} transforms as

$$V_{\mu} \to \Lambda_{\mu}{}^{\nu} V^{\nu} \tag{17}$$

with $\Lambda_{\mu}{}^{\nu} = \eta_{\mu\rho}\eta^{\nu\sigma}\Lambda^{\rho}{}_{\sigma}$. In case of four-vector representation each generator $J^{\mu\nu}$ is the 4x4 matrix

$$(J^{\mu\nu})^{\rho}{}_{\sigma} = i(\eta^{\mu\rho}\delta^{\nu}_{\rho} - \eta^{\nu\rho}\delta^{\mu}_{\sigma}) \tag{18}$$

This representation is irreducible, since a generic Lorentz transformation mixes all 4 components of a four-vector.
Computing the commutator:

$$[J^{\mu\nu}, J^{\rho\sigma}] = i(\eta^{\nu\rho}J^{\mu\sigma} - \eta^{\mu\rho}J^{\nu\sigma} - \eta^{\nu\sigma}J^{\mu\rho} + \eta^{\mu\sigma}J^{\nu\rho}) \tag{19}$$

this is the Lie algebra of $SO(3,1)$.
The six components of $J^{\mu\nu}$ can be arranged into two three-vectors:

$$J^i = \frac{1}{2}\epsilon^{ijk}J^{jk}, K^i = J^{i0} \tag{20}$$

With this the commutator splits into three commutators $\left[J^i, J^j\right]$, $\left[J^i, K^j\right]$, $\left[K^i, K^j\right]$, which is now the Lie algebra of the Lorentz group. The first one is equivalent to the commutator of the angular momentum, the second shows that **K** is a spatial vector. We also split the parameters ω into 2 three-vectors:

$$\Theta^i = (1/2)\epsilon^{ijk}\omega^{jk}, \quad \eta^i = \omega^{i0} \tag{21}$$

so that the Lorentz transformation becomes:

$$\Lambda = exp(-i\mathbf{\Theta} \cdot \boldsymbol{J} + i\boldsymbol{\eta} \cdot \boldsymbol{K}) \tag{22}$$

The signs are chosen such that the first term is a counterclockwise rotation of a point P w.r.t. a fixed reference frame and the second is a boost by **v** on a particle at rest, which results in a particle with velocity $+\boldsymbol{v}$.

2.1 Tensor representation

Since the four-vector is the irreducible *fundamental representation*, of dimension **4**, the tensor representation is just derived as a tensor product $4 \otimes 4$ from it. A four-tensor transforms like

$$T^{\mu\nu} \to \Lambda^{\mu}{}_{\mu'}\Lambda^{\nu}{}_{\nu'}T^{\mu'\nu'} \tag{23}$$

This representation has 16 components, however it is reducible. Lorentz transform doesn't mix symmetric and antisymmetric parts of a tensor. Hence it is split into six-dimensional antisymmetric representation $A^{\mu\nu} = (1/2)(T^{\mu\nu} - T^{\nu\mu})$ and a 10-dimensional symmetric representation $S^{\mu\nu} = (1/2)(T^{\mu\nu} + T^{\nu\mu})$. The Lorentz group also conserves the trace of a symmetric tensor, in particular a traceless tensor remains traceless. Therefore the 10 dimensional representation decomposes into the **9** dimensional irreducible *symmetrical traceless tensor* representation $S^{\mu\nu} - (1/4)\eta^{\mu\nu}S$ and a **1** dimensional scalar representation S. In one line:

$$\mathbf{4} \otimes \mathbf{4} = \mathbf{1} \otimes \mathbf{6} \otimes \mathbf{9} \tag{24}$$

Here we deal with tensors which have two indices. However tensors of arbitrary number of indices can be reduced by the same procedure: remove all traces and then symmetrizing or anti-symmetrizing over all pairs of indices.

2.2 Spinorial representation

We consider the subgroup $SO(3)$, i.e. spatial rotations. A kindred group is the SU(2) group and it has the same Lie algebra: $\left[J^i, J^j\right] = i\epsilon^{ijk}J^k$, that means they are infinitesimally similar. The representations of this group, the spinors, change sign under rotation by 2π. So, to satisfy the SU(2) group, the solutions of the Lie algebra above only with half-integer spin may be taken. To satisfy SO(3) the solutions of the Lie algebra with integer spin are suitable. The sign difference doesn't play in physics a role, since their representations are

used only as squares in wavefunction, which is experimental evidence. So in fact spinorial representation for SU(2) with half-integer spin must be included too, since it represents a realization that nature can and will take, although if it is not a realization of SO(3) roup. Experimental evidence is that SU(2) is the more fundamental representation for spin. So we have the possibilities of the solutions of Lie algebra with spin values $0, \frac{1}{2}, 1, \frac{3}{2}, ...$ A spin j representation has dimension 2j+1.
The representation with $j = 1/2$ has dimension 2, is a SU(2) representation and the generators are:

$$J^i = \frac{\sigma^i}{2} \tag{25}$$

where σ^i are the Pauli matrices. They satisfy the algebraic identity:

$$\sigma^i \sigma^j = \delta^{ij} + i\epsilon^{ijk}\sigma^k \tag{26}$$

The spin 1/2 representation, the *spinorial representation* is the fundamental one for SU(2), since all other representations can be constructed as tensor products:

$$\mathbf{\frac{1}{2}} \otimes \mathbf{\frac{1}{2}} = \mathbf{0} \otimes \mathbf{1} \tag{27}$$

The state with j=1 has dimension 2j+1 three (triplett), while the j=0 scalar state has dimension one (singlett).

2.3 Spinors in four-dimensional space

We want to keep spinors, but how to redefine the Lorentz algebra to mathematically include these? We define:

$$\boldsymbol{J}^{\pm} = \frac{\boldsymbol{J} \pm i\boldsymbol{K}}{2} \tag{28}$$

with this the Lorentz algebra becomes two copies of angular momentum relation for $\boldsymbol{J}^{\pm}$. We have rewritten the Lorentz algebra as algebra of SU(2)xSU(2). Note that this group is not any more the Lorentz one. Note however that SU(2)xSU(2) is the *universal covering group* of SO(4), which is obtained by taking the time purely imaginary, similar to the fact that SU(2) is the universal covering group of SO(3).
This representations are complex and include the all tensor representations of the Lorentz group and all spinorial representations. Observe that though $\boldsymbol{J}^{\pm}$ have as elements of SU(2) half integer (or zero) values, all integer spin values can also be recovered, since $J = J^+ + J^-$. So the representations are labeled by half integers (j_-, j_+), are of $(2j_- + 1)(2j_+ + 1)$ dimensions, with all possible spin values j in integer steps between $|j_+ - j_-|$ and $j_+ + j_-$. Following some of the easier cases.
$(\mathbf{0}, \mathbf{0})$, no spin, dimension one, therefore scalar representation.
$(\mathbf{\frac{1}{2}}, \mathbf{0})$ and $(\mathbf{0}, \mathbf{\frac{1}{2}})$. Dimension two and spin 1/2 (spinorial). Denote by $(\psi_L)_\alpha$ with $\alpha = 1, 2$

a spinor in $(1/2, 0)$ and by $(\psi_R)_\alpha$ a spinor in $(0, 1/2)$. $\psi_{L/R}$ is called the *left- and right-handed Weyl spinor*. Now determine **J** and **K**.
Consider (1/2, 0). Then

$$J = \frac{\boldsymbol{\sigma}}{2}, \boldsymbol{K} = i\frac{\boldsymbol{\sigma}}{2} \tag{29}$$

so **K** is not hermitean, since the Lorentz group is non-compact and has only the representation in finite dimensions where the non-compact generators (K) are represented trivially, i.e. $K^i = 0$. So the left-handed Weyl spinor transforms under the Lorentz transform as:

$$\psi_L \to \Lambda_L \psi_L = exp((-i\boldsymbol{\theta} - \boldsymbol{\eta}) \cdot \frac{\boldsymbol{\sigma}}{2})\psi_L \tag{30}$$

Consider (0, 1/2). We find

$$J = \frac{\boldsymbol{\sigma}}{2}, \boldsymbol{K} = -i\frac{\boldsymbol{\sigma}}{2} \tag{31}$$

and

$$\psi_R \to \Lambda_R \psi_R = exp((-i\boldsymbol{\theta} + \boldsymbol{\eta}) \cdot \frac{\boldsymbol{\sigma}}{2})\psi_R \tag{32}$$

One can show that

$$\sigma^2 \Lambda_L^* \sigma^2 = \Lambda_R \tag{33}$$

Therefore, if ψ_L is left-handed, then $\sigma^2 \xi_L^*$ is righthanded.
We define operation of *charge conjugation* on Weyl spinors as

$$\psi_R = \psi_L^c = i\sigma^2 \xi_L^* \tag{34}$$

$$\psi_L = \psi_R^c = -i\sigma^2 \xi_R^* \tag{35}$$

$(\mathbf{\frac{1}{2}}, \mathbf{\frac{1}{2}})$. This representation has complex dimension four and $j = 0, 1$. It is a complex four-vector representation. A generic element is $((\psi_L)_k, (\xi_R)_k)$ with two independent Weyl spinorsa and $k = 1, 2$. We want to relate these complex quantities to complex four-vectors by defining the matrices:

$$\sigma^\mu = (1, \sigma^i), \quad \bar{\sigma}^\mu = (1, -\sigma^i) \tag{36}$$

Then the following quantities are contravariant four-vector representations:

$$\xi_R^\dagger \sigma^\mu \psi_R \tag{37}$$

$$\xi_L^\dagger \bar{\sigma}^\mu \psi_L \tag{38}$$

Since $\Lambda^\mu{}_\nu$ is real, one can impose a reality condition $V_\mu = V_\mu^*$. This condition is then true for all frames. Hence we obtain the real four-vector representation.
$(\mathbf{1}, \mathbf{0})$ and $(\mathbf{0}, \mathbf{1})$. These correspond to *self-dual and anti-self-dual* antisymmetric tensors and each have complex dimension three, i.e. real dimension six.

2.4 Scalar field

The field representations are infinite dimensional. We require that by Lorentz coordinate transformation the field transforms to a new field as a function of new coordinates. The simplest possibility is that of a scalar field:

$$\phi'(x') = \phi(x) \tag{39}$$

the moving observer at x' sees the same field at point P as the observer in the reference frame x looking at point P, as for any other Lorentz observer. The field is Lorentz invariant. So by looking at $\delta\phi = \phi'(x') - \phi(x)$ we evaluate the single degree of freedom, how the field at point x changes for an observer, because $\phi(x)$ is kept fixed and ϕ' is scalar, which depends in one-dimensional way on the four dimensional coordinates, something like $\phi'(|x'^{\mu}|)$. For the scalar field $\delta\phi = 0$, which corresponds to the fact that it is the trivial representation with the generators $J^{\mu\nu} = 0$. A four-vector field would have four degrees of freedom but $\delta\phi$ is still finite dimensional.
When we consider

$$\delta_0\phi = \phi'(x) - \phi(x) \tag{40}$$

we vary the point P over the whole spacetime, so this task is infinite-dimensional. Imagine a field $\delta_0\phi(x)$, then this field has infinite dimensions of continuation of the field from infinitely many observers at each spacetime point x' to x, though $\phi(x)$ is kept fixed.
So we are looking at a infinite-dimensional representation of the Lorentz group: the field $\delta_0\phi(x)$. We expand the field to first order in dx:

$$\delta_o\phi = -\delta x^{\rho}\partial_{\rho}\phi(x) \tag{41}$$

We know that $\delta x^{\rho} = \omega^{\rho}{}_{\sigma}x^{\sigma} = -\frac{i}{2}\omega_{\mu\nu}(J^{\mu\nu})^{\rho}{}_{\sigma}x^{\sigma}$. We use this to write:

$$\delta_0\phi = -\frac{i}{2}\omega_{\mu\nu}L^{\mu\nu}\phi \tag{42}$$

with the definition

$$L^{\mu\nu} = -(J^{\mu\nu})^{\rho}{}_{\sigma}x^{\sigma}\partial_{\rho} = i(x^{\mu}\partial^{\nu} - x^{\nu}\partial^{\mu}) \tag{43}$$

If one haves a complex scalar field $\phi(x)$, one can impose the Lorentz invariant reality condition $\phi(x) = \phi^*(x)$, which once imposed is then true in any frame.

2.5 Weyl field

A left-handed Weyl field is defined as:

$$\psi_L(x) \to \psi'_L(x') = \Lambda_L\psi_L(x) \tag{44}$$

as

$$x^{\mu} \to x'^{\mu} = \Lambda^{\mu}{}_{\nu}x^{\nu} \tag{45}$$

In the classical theory it is a filed of commuting complex numbers. The representation of the Lorentz generators on ψ_L can be found with:

$$\delta_0\psi_L = (\Lambda_L - 1)\psi_L(x) - \delta x^\rho \partial_\rho \psi_L(x) \tag{46}$$

Writing $\Lambda_L = e^{-\frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}}$ with S not yet defined and using the result for the free field (angular momentum L) we get:

$$\delta_0\psi_L = -\frac{i}{2}\omega_{\mu\nu}J^{\mu\nu}\psi_L \tag{47}$$

with

$$J^{\mu\nu} = L^{\mu\nu} + S^{\mu\nu} \tag{48}$$

We use the result for Weyl spinors to identify:

$$S^i = \frac{1}{2}\epsilon^{ijk}S^{jk} = \frac{\sigma^i}{2}, \qquad S^{i0} = i\frac{\sigma^i}{2} \tag{49}$$

which is the spin. The angular momentum has always the same form, independent of the representation, while the spin varies.

3 Parity Transformation

3.1 Dirac field

We know that in nature the weak interaction violates the parity. This is because the right and left-handed spinors enter the theory in a fundamentally different way. However the scale of the weak interactions is $O(100)\ GeV$ much higher than the scale of strong and electromagnetic interactions. At sufficiently low energies the parity holds.

Consider a parity transformation $(t, \boldsymbol{x}) \to (t, -\boldsymbol{x})$. Under this operation the boost generators transform as true vectors and change sign, $\boldsymbol{K} \to -\boldsymbol{K}$, since the parity reverses the velocity $\mathbf{v}$ of the boost. However $\boldsymbol{J}$ is a pseudo-vector that doesn't change. However the $J^i_\pm$ transform into each other, i.e. the representation (j_-, j_+) is transformed into (j_+, j_-). Therefore this representation of the Lorentz group is not a basis for a representation of the parity transformation. In particular this holds for ψ_L and ψ_R *separately*. Therefore we need the field:

$$\Psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} \tag{50}$$

which is the *Dirac field*. It transforms under a Lorentz transformation as:

$$\Psi \to \Lambda_D \Psi \tag{51}$$

with

$$\Lambda_D = \begin{pmatrix} \Lambda_L & 0 \\ 0 & \Lambda_R \end{pmatrix} \tag{52}$$

and under the parity transform:

$$\begin{pmatrix} \psi_L(x) \\ \psi_R(x) \end{pmatrix} \rightarrow \begin{pmatrix} \psi_R(x') \\ \psi_L(x') \end{pmatrix} \tag{53}$$

and therefore

$$\Psi(x) \rightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Psi(x') \tag{54}$$

We can define charge conjugation for the Dirac field using the one for the Weyl fields:

$$\Psi^c = \begin{pmatrix} -i\sigma^2\psi_R^* \\ i\sigma^2\psi_L^* \end{pmatrix} = -i \begin{pmatrix} 0 & \sigma^2 \\ -\sigma^2 & 0 \end{pmatrix} \Psi^* \tag{55}$$

with

$$(\Psi^c)^c = \Psi \tag{56}$$

Note that the coordinates are unchanged under charge conjugation. The Weyl fields are the more fundamental objects as the Dirac field.

3.2 Majorana field

A *Majorana field* is a Dirac field with the degrees of freedom of a single Weyl field:

$$\Psi_M = \begin{pmatrix} \psi_L \\ i\sigma^2\psi_L^* \end{pmatrix} \tag{57}$$

the Majorana field is invariant under charge conjugation. On Majorana field a Lorentz invariant reality condition can be imposed, similar to the four-vector representation. For the Dirac spinor the situation is different and the reality condition is not Lorentz invariant. So one can see the Majorana field as the real Dirac field.

3.3 Vector field

A contravariant vector field transforms simply as:

$$V^\mu(x) \rightarrow V'^\mu(x') = \Lambda^\mu{}_\nu V^\nu(x) \tag{58}$$

as x transforms with the Lorentz transform. It has a spin-0 and a spin-1 component. Since the vector field belongs to the (1/2, 1/2) representation, it has $j_- = j_+$ and therefore satisfies parity. We will see that for the electromagnetic gauge the gauge invariance eliminates the spin-0 component and the state with $(j = 1, J_z = 0)$ with z being the propagation direction. Tensor fields are defined similarly.

4 The Poincaré Group

We require invariance under space-time translations. The generic element element of the *translation group* is:

$$exp(-iP^{\mu}a_{\mu}) \tag{59}$$

with a_{μ} is the translation vector. The components of the four-momentum operator P^{μ} are the generators. The Lorentz group plus translation group is the *Poincaré group*, also called *Lorentz inhomogeneous group ISO(3,1)*. Since the translations commute, we have:

$$[P^{\mu}, P^{\nu}] = 0 \tag{60}$$

Additionally the momentum is a vector under rotations and the energy is the scalar under rotations:

$$\left[J^i, P^j\right] = i\epsilon^{ijk}P^k \tag{61}$$

$$\left[J^i, P^0\right] = 0 \tag{62}$$

The unique Lorentz covariant generalization is:

$$[P^{\mu}, J^{\rho\sigma}] = i(\eta^{\mu\rho}P^{\sigma} - \eta^{\mu\sigma}P^{\rho}) \tag{63}$$

Together with the Lorentz algebra these equations define the Poincaré algebra with all the conservation properties known from the classical physics.

4.1 Fields

We require that all fields independent of their Lorentz transformation properties behave as scalars under space-time translations. Then under a *translation* the field transforms like:

$$\phi'(x') = \phi(x) \tag{64}$$

We have under the infinitesimal translation $x^{\mu} \to x'^{\mu} = x^{\mu} + \epsilon^{\mu}$ to first order:

$$\delta_0\phi = -\epsilon^{\mu}\partial_{\mu}\phi(x) \tag{65}$$

We also know that:

$$\phi'(x' - \epsilon) = e^{i\epsilon_{\mu}P^{\mu}}\phi(x) \tag{66}$$

and therefore

$$\delta_o\phi = i\epsilon_{\mu}P^{\mu}\phi(x) \tag{67}$$

We see that the momentum operator is represented as:

$$P^{\mu} = i\partial^{\mu} \tag{68}$$

Observe that:

$$H = P^0 = i\partial^0 = i\partial_0 = i\frac{\partial}{\partial x^0}, \qquad P^i = i\partial^i = -i\partial_i = -i\frac{\partial}{\partial x^i} \tag{69}$$

using that by $\eta = (1, -1, -1, -1)$ to switch from the covariant to contra-covarinat vectors the time component stays unchanged. So we have the generators J for rotation, K for boost, P^i for space translation and H for time translation.

4.2 One-particle states

The mysterious question is: how the particle emerges from the field quantization? So represent the Poincaré group in the basis of the Hilbert space of a free particle. Denote the state of a free particle with $|\boldsymbol{p}, s\rangle$ where $\mathbf{p}$ is the momenum and s are all the other quantum numbers. Since $\mathbf{p}$ is a continuous and unbounded variable, this base-space is infinite-dimensional. A theorem by Wigner states that on this Hilbert space any symmetry transformation can be represented by a unitary operator. Therefore in this base space a Poincaré transformation is represented by a unitary matrix and the generators J, K, P, H are hermitean. We look for the Casimir operators that label a representation. The first is $P_\mu P^\mu$, on a one-particle state it has the value m, where m is the *mass* of the particle. The other is given by $W_\mu W^\mu$, where W^μ is the *Pauli-Lubanski four-vector*:

$$W^\mu = -\frac{1}{2}\epsilon^{\mu\nu\rho\sigma} J_{\nu\rho} P_\sigma \tag{70}$$

Since $W_\mu W^\mu$ is Lorentz-invariant, we can compute it in the frame that we prefer.
If we deal with a massive particle $m \neq 0$, it is convenient to choose the rest frame:

$$W^\mu = (-m/2)\epsilon^{\mu\nu\rho 0} J_{\nu\rho} = (m/2)\epsilon^{0\mu\nu\rho 0} J_{\nu\rho} \tag{71}$$

and

$$W^i = mJ^i, \qquad W^0 = 0 \tag{72}$$

Therefore

$$-W_\mu W^\mu = m^2 j(j+1) \tag{73}$$

We restrict our consideration to m real and positive. Therefore the representation is labeled by the mass m and by the spin j. For a massive particle we can bring the momentum into the form $P^\mu = (m, 0, 0, 0)$, which is the rest frame. This form still leaves the freedom of spatial rotations. The group of transformations which leaves invariant a chosen P^μ is called the *little group*. Since we want to include spinor representations, in our case the little group is $SU(2)$ with all integer and half-integer spin values. This means that the massive particle of spin j has (2j+1) degrees of freedom.

For a massless particle $m = 0$ there is no rest frame, but one can reduce P^μ to the form

$P^\mu = (\omega, 0, 0, \omega)$. The little group is the set of all transformations that leave this vector unchanged. One can see immediately that this is the set of rotations in the (x, y) plane. This is the $SO(2)$ group generated by $J^3 = J^z$. As for any abelian group, the irreducible representations of $SO(3)$ are one-dimensional. J^3 is the generator and the labels are h, the eigenvalues of J^3, which is called the *helicity*. It represents the angular momentum in the direction of propagation. The helicity is quantized. The proof is based on the fact that the universal covering of the Lorentz group is $SL(2, C)$, this is a double covering, therefore any Lorentz rotation by 4π is the same as the identity matrix *(see Weinberg(1995), pages 86-90)*. Therefore we conclude that massless particles has only one degree of freedom, characterized by its helicity h. Therefore it can be represented by a $U(1)$ group:

$$U(\theta) = exp(-i\hbar\theta) \tag{74}$$

From the view of the Poincaré group only, the particles with helicity $+h$ and $-h$ are two different species since they belong to different representations. However we observe that h changes sign under parity, since it is a pseudoscalar:

$$h = \hat{\mathbf{p}}_{\boldsymbol{z}} \cdot \boldsymbol{J} \tag{75}$$

If a interaction conserves parity, then to each particle with $+h$ must be another particle with $-h$ and they both must enter the theory in a symmetric way. Since electromagnetic interaction conserves parity, the particles with $\pm h$ are given the same name, the *left- and right-handed photon*. Neutrino doesn't conserve parity, so both states are given different names: the neutrino and anti-neutrino.

5 Equations of Motion

The quantum mechanics is obtained from the first quantization of the fields, so the wavefunction is only a collective approximation to field dynamics with a large number of excitations. A QFT field refers to the photon wavefunction same as a gauge potential refers to the electric field. The Klein-Gordon equation and the Dirac equation approximate to the Schrödinger equation is the non-relativistic limit. However in presence of a magnetic field one should rather take for the electrons the Pauli equation, which accounts for interaction with the spin. Even in the first quantization one meets the anti-particles with negative energy solutions.

5.1 Noether theorem in Lagrangian formalism

If $\mathcal{L}$ changes only by a total derivative of some $\mathcal{F}$, it leaves S invariant. A *symmetry* is the field transformation, with which the Lagrangian changes only by a total derivative. For a continuous symmetry you can write to first order:

$$\phi \to \phi + \epsilon\delta\phi := \phi + \epsilon\mathcal{X}(\phi) \tag{76}$$

One can observe without use the Equations of Motion (”off-shell”), that with this the Lagrangian varies to first order by a total derivative of some $\mathcal{F}$:

$$\mathcal{L} \to \mathcal{L} + \epsilon\delta\mathcal{L} := \mathcal{L} + \epsilon\partial_\mu \mathcal{F}^\mu \tag{77}$$

Under a general transformation the Lagrangian varies as:

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\phi}\delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta(\partial_\mu\phi) = \frac{\partial\mathcal{L}}{\partial\phi}\delta\phi + \partial_\mu\Big(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi\Big) - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi \tag{78}$$

with the use of the product rule. We see that the term $\partial_\mu\Big(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi\Big)$ is unphysical since it always vanishes. If the transformation $\delta\phi$ is a symmetry $\mathcal{X}$ then the term which is left over must also be a total derivative of some quantity j which is the *Noether current*:

$$\frac{\partial\mathcal{L}}{\partial\phi}\delta\phi - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi = \Big(\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\Big)\mathcal{X} := -\partial_\mu j^\mu \tag{79}$$

We see that ”on-shell” $\partial_\mu j^\mu = 0$. There is a time-conserved quantity related to it which is conserved ”on-shell”, the *Noether charge* Q:

$$Q := \int d^3x j^0 \tag{80}$$

$$\dot{Q} = \int d^3x \frac{\partial}{\partial t} j^0 = 0 \tag{81}$$

since $\partial_\mu j^\mu = 0$. In finite volume this quantity also satisfies local charge conservation. To have a symmetry we require:

$$\partial_\mu\Big(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\mathcal{X}\Big) + \Big(\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\Big)\mathcal{X} \equiv \partial_\mu\mathcal{F}^\mu \tag{82}$$

Therefore we see the relation:

$$\partial_\mu j^\mu + \partial_\mu\mathcal{F}^\mu = \partial_\mu\Big(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\mathcal{X}\Big) \tag{83}$$

And so:

$$j^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\mathcal{X} - \mathcal{F}^\mu \tag{84}$$

If the Lagrangian is invariant under the symmetry, $\partial_\mu\mathcal{F}^\mu = 0$. Therefore $\mathcal{F}$ decouples from the equations and can be set zero! Alltogether we see that every continuous symmetry gives rise to a Noether current and a Noether charge. Note: for an infinitesimal symmetry

$$q^i \to q^i + \delta q^i,\ p_j \to p_j + \delta p_j \tag{85}$$

with p defined as $p^j = \frac{\partial\mathcal{L}}{\partial\dot{q}^j}$ the Noether charge is:

$$Q = \delta q^j p_j - \mathcal{F} \tag{86}$$

5.2 Canonical Transformation - Hamiltonian formalism

A transformation of the canonical coordinates $p \to P$, $q \to Q$ is canonical, if the Poisson brackets satisfy:

$$\{P_i, Q_j\}_{p,q} = \delta_{ij} \tag{87}$$

and all others are equal 0. A canonical transformation leaves the Hamiltonian equations of motion invariant:

$$\dot{q} = \frac{\partial H}{\partial p},\ \dot{p} = -\frac{\partial H}{\partial q} \tag{88}$$

A canonical transformation in infinitesimal form is:

$$q^i \to q^i + \delta q^i,\ p_j \to p_j + \delta p_j \tag{89}$$

with

$$\delta q^i = \left\{q^i, Q\right\},\ \delta p_j = \{p_j, Q\} \tag{90}$$

with $Q(\boldsymbol{q}, \boldsymbol{p})$ is the *generator* of the transformation. Any function $G(\boldsymbol{q}, \boldsymbol{p})$ transforms as:

$$G \to G + \delta G,\ \delta G = \{G, K\} \tag{91}$$

If a canonical transformation leaves the Hamiltonian invariant:

$$\delta H = \{H, Q\} = 0 \tag{92}$$

it is a continuous symmetry in Noether's sense. This implies also the time-invariance of the generator Q by the Heisenberg equation:

$$\frac{dA}{dt} = \{A, H\} + \frac{\partial A}{\partial t} \tag{93}$$

5.3 Real scalar field

We are now able to construct Poincaré invariant actions. In this case it is:

$$S = \frac{1}{2}\int d^4x(\partial_\mu\phi\partial^\mu\phi - m^2\phi^2) \tag{94}$$

The Euler-Lagrange equation gives the free *Klein-Gordon equation*:

$$(\Box + m^2)\phi = 0 \tag{95}$$

Its solution are plane waves $e^{\pm ipx}$ with p satisfying the relativistic dispersion relation $p^2 = m^2$. Therefore we recognize the parameter m in the action as the *mass* of the particle. For a real field we must have a real superposition of plane waves:

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3\sqrt{2E_{\boldsymbol{p}}}}\left(a_{\boldsymbol{p}}e^{-ipx} + a^*_{\boldsymbol{p}}e^{+ipx}\right) \tag{96}$$

The solution is evaluated on the positive p^0, we are interested in positive energies $E_{\boldsymbol{p}} = p^0 = +\sqrt{\boldsymbol{p}^2 + m^2}$.
The momentum conjugate to ϕ, the Hamiltonian density and the energy-momentum tensor are:

$$\Pi_\phi = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)} = \partial_0 \phi, \tag{97}$$

$$\mathcal{H} = \Pi_\phi \partial_0 \phi - \mathcal{L} = \frac{1}{2}\{\Pi_\phi^2 + (\nabla\phi)^2 + m^2\phi^2\} \tag{98}$$

$$\Theta^{\mu\nu} = \partial^\mu \phi \partial^\nu \phi - \eta^{\mu\nu}\mathcal{L} \tag{99}$$

with $\Theta^{00} = \mathcal{H}$.
With Noether theorem we find out, that $H = \int d^3x \Theta^{00}$ is the conserved charge related to the invariance under time translation. M is the conserved charge related to the spacial rotations of Lorentz invariance, with :

$$M^{ij} = \frac{i}{2}\int d^3x \{\phi L^{ij}(\partial_0\phi) - (\partial_0\phi)L^{ij}\phi\} \tag{100}$$

We define the scalar product(not positive definite) that is conserved on solutions of the Klein-Gordon equation:

$$\langle \phi_1 | \phi_2 \rangle = \frac{i}{2}\int d^3x \phi_1 \overleftrightarrow{\partial}_0 \phi_2 \tag{101}$$

with $f\overleftrightarrow{\partial_\mu} g = f\partial_\mu g - (\partial_\mu f)g$.
We recognize that:

$$M^{ij} = \langle \phi | L^{ij} | \phi \rangle \tag{102}$$

This holds generally as a correspondence between the representration of the generators of the Lie algebra as operators acting on fields (e.g. L^{ij}) and the value of the corresponding charges on a given solution of the equations of motion. One more example:

$$P^\mu = \langle \phi | i\partial^\mu | \phi \rangle \tag{103}$$

The generalization of the action to the self-interacting "free" field is:

$$S = \int d^4x \Big(\frac{1}{2}\partial_\mu \phi \partial^\mu \phi - V(\phi)\Big) \tag{104}$$

The quadratic part of the potential is then the mass term and higher order parts $O(\phi^3)$, $O(\phi^4)$, ... are the self-interactions.

5.4 Complex scalar field

We consider

$$\phi = (\phi_1 + i\phi_2)/\sqrt{2} \tag{105}$$

The action is:

$$S = \int d^4x(\partial_\mu\phi^*\partial^\mu\phi - m^2\phi^*\phi) \tag{106}$$

This field also satisfies the KG-equation and can therefore be expanded in modes. We observe the global $U(1)$ symmetry of the action under:

$$\phi(x) \to e^{i\Theta}\phi(x), \qquad \phi^*(x) \to e^{-i\Theta}\phi^*(x) \tag{107}$$

The U(1) charge is found from the respective Noether current. On the complex field the conserved scalar product is:

$$\left\langle \phi \middle| \phi' \right\rangle = i\int d^3x\phi^*\overleftrightarrow{\partial}_0\phi' \tag{108}$$

We observe that:

$$\mathcal{Q}_{U(1)} = i\int d^3x\phi^*\overleftrightarrow{\partial}_0\phi = \langle\phi|\phi\rangle \tag{109}$$

and the corresponding generator of $U(1)$ transformation is the identity operator.

5.5 Weyl spinor

We consider a left handed Weyl spinor field $\psi_L(x)$. Then the Lorentz invariant kinetic term is:

$$\mathcal{L}_L = i\psi_L^\dagger\bar{\sigma}^\mu\partial_\mu\psi_L(=0) \tag{110}$$

since on a classical solution $\bar{\sigma}^\mu\partial_\mu\psi_L = 0$. The equation of motion is:

$$(\partial_0 - \sigma^i\partial_i)\psi_L = 0 \tag{111}$$

This implies the massless Klein-Gordon equation:

$$\Box\psi_L = 0 \tag{112}$$

But there is more to the equation of motion. Consider a plane-wave solution:

$$\psi_L(x) = u_L e^{-ipx} \tag{113}$$

where u_L is a constant spinor. Then the EqMo gives:

$$\frac{\boldsymbol{p}\cdot\boldsymbol{\sigma}}{E}u_L = -u_L \tag{114}$$

and the Klein-Gordon equation gives:

$$E = |\boldsymbol{p}| \tag{115}$$

So we get:

$$(\hat{\boldsymbol{p}} \cdot \boldsymbol{J}) u_L = -\frac{1}{2} u_L \tag{116}$$

Therefore a left-handed massless Weyl spinor has *helicity* $h = -\frac{1}{2}$. The energy-momentum tensor is:

$$\theta^{\mu\nu} = i\psi_L^\dagger \bar{\sigma}^\mu \partial^\nu \psi_L \tag{117}$$

Tis Lagrangian has a $U(1)$ symmetry and the conserved charge is:

$$\mathcal{Q}_{U(1)} = \int d^3x \psi_L^\dagger \psi_L \tag{118}$$

There is another Lagrangian possible, which gives the same equations of motion:

$$\mathcal{L}_L' = \mathcal{L}_L - \frac{i}{2}\partial_\mu(\psi_L^\dagger \bar{\sigma}^\mu \psi_L) = \frac{i}{2}\psi_L^\dagger \bar{\sigma}^\mu \overleftrightarrow{\partial_\mu} \psi_L \tag{119}$$

The conserved currents are different, but the charges are the same. For the right-handed Weyl spinor the Lagrangian is:

$$\mathcal{L}_R = i\psi_R^\dagger \sigma^\mu \partial_\mu \psi_R \tag{120}$$

The equation of motion is:

$$\sigma^\mu \partial_\mu \psi_R = (\partial_0 + \sigma^i \partial_i)\psi_R = 0 \tag{121}$$

The right-handed Weyl spinor has helicity $+1/2$. A massive particle can also be described with a single Weyl spinor.

5.6 Dirac spinor

Having both the Weyl spinors at the disposal, there are more Lorentz-scalars possible:

$$\psi_L^\dagger \psi_R, \qquad \psi_R^\dagger \psi_L \tag{122}$$

Also real Lorentz scalars can be constructed from them:

$$\psi_L^\dagger \psi_R + \psi_R^\dagger \psi_L, \qquad i(\psi_L^\dagger \psi_R - \psi_R^\dagger \psi_L) \tag{123}$$

Under parity transformation, the first one is a scalar and the second one is a pseudo-scalar. We set up the Dirac Lagrangian:

$$\mathcal{L}_D = i\psi_L^\dagger \bar{\sigma}^\mu \partial_\mu \psi_L + i\psi_R^\dagger \sigma^\mu \partial_\mu \psi_R - m(\psi_L^\dagger \psi_R + \psi_R^\dagger \psi_L) \tag{124}$$

This Lagrangian is invariant under parity transformation, since $\psi_R \leftrightarrow \psi_R$ and $\partial_i \leftrightarrow -\partial_i$. The equations of motion are:

$$\bar{\sigma}^\mu i\partial_\mu \psi_L = m\psi_R \tag{125}$$
$$\sigma^\mu i\partial_\mu \psi_R = m\psi_L \tag{126}$$

This is the *Dirac equation*. The Weyl-spinors are no longer helicity eigenstates. Applying $\sigma^\mu i\partial_\mu$ on both sides of the first equation, one gets:

$$-\sigma^\mu \bar{\sigma}^\nu \partial_\mu \partial_\nu \psi_L = m^2 \psi_L \tag{127}$$

Since $\partial_\mu \partial_\nu$ is symmetric, we can replace $\sigma^\mu \bar{\sigma}\nu$ with $(1/2)(\sigma^\mu \bar{\sigma}^\nu + \sigma^\nu \bar{\sigma}^\mu)$. Using

$$\sigma^\mu \bar{\sigma}^\nu + \sigma^\nu \bar{\sigma}^\nu = 2\eta^{\mu\nu} \tag{128}$$

we then find:

$$(\Box + m^2)\psi_L = 0 \tag{129}$$
$$(\Box + m^2)\psi_R = 0 \tag{130}$$

Therefore the Dirac equation implies the massive Klein-Gordon equation and the parameter m is a mass term. We take the Dirac field:

$$\Psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} \tag{131}$$

this is the *chiral representation*. We define the 4x4 γ matrices in the *chiral representation*:

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} \tag{132}$$

which is

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \tag{133}$$

We see that the γ matrices satisfy the *Clifford-algebra*:

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} \tag{134}$$

In terms of the Dirac spinor the Dirac equation becomes:

$$(i\gamma^\mu \partial_\mu - m)\Psi = 0 \tag{135}$$

With the definition:

$$\bar{\Psi} = \Psi^\dagger \gamma^0 \tag{136}$$

and the *Dirac slash notation* $\not{A} = \gamma^\mu A_\mu$ the Dirac Lagrangian becomes:

$$\mathcal{L} = \bar{\Psi}(i\not{\partial} - m)\Psi \tag{137}$$

One also defines $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$. In the chiral representation it is:

$$\gamma^5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \tag{138}$$

The *projector on Weyl spinors* acting on the Dirac field is:

$$\frac{1-\gamma^5}{2}\Psi = \begin{pmatrix} \psi_L \\ 0 \end{pmatrix}, \qquad \frac{1+\gamma^5}{2}\Psi = \begin{pmatrix} 0 \\ \psi_R \end{pmatrix} \tag{139}$$

Though the neutrino can be described as a massless left-handed Weyl spinor, it is convenient to use the Dirac ν spinor:

$$\nu = \begin{pmatrix} \nu_L \\ 0 \end{pmatrix} \tag{140}$$

The chiral representation is good for the relativistic limit: the mass is neglected and the Weyl spinors are helicity eigenstates. Another useful representation called the *standard representation*, which is used in the non-relativistic limit can be obtained using the invariance of the *Clifford algebra* under $\gamma^\mu \to U\gamma^\mu U^\dagger$ wuth U unitary. The representation is obtained using:

$$U = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \tag{141}$$

$$\Psi = \frac{1}{\sqrt{2}}\begin{pmatrix} \psi_R + \psi_L \\ \psi_R - \psi_L \end{pmatrix} \tag{142}$$

The γ matrices become:

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \gamma^0 = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \tag{143}$$

$$\gamma^5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{144}$$

The the equations of motion stays the same form in terms of the redefined Dirac fields and γ matrices. The general solution of the massive Dirac equation is a superposition of plane waves of positive energy:

$$\Phi(x) = u(p)e^{-ipx} \tag{145}$$

and the negative energy solution:

$$\Phi(x) = v(p)e^{ipx} \tag{146}$$

5.7 Gauge field

The electromagnetic field is described by a four-vector A_μ, the gauge potential. The field strength tensor is defined as:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{147}$$

The Lagrangian of the free electromagnetic field is:

$$L = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \tag{148}$$

The equations of motion derived from this Lagrangian are:

$$\partial_\mu F^{\mu\nu} = 0 \tag{149}$$

We furthermore define:

$$\tilde{F}^{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma} \tag{150}$$

and see that also:

$$\partial_\mu \tilde{F}^{\mu\nu} = 0 \tag{151}$$

These two equations comprise the four free Maxwell equations. We observe that the electromagnetic Lagrangian is symmetric under:

$$A_\mu \to A_\mu - \partial_\mu \theta(x) \tag{152}$$

which is a local symmetry. We therefore have the choice of a gauge to eliminate the redundant degrees of freedom. Observe that this symmetry is trivial in the sense that it doesn't give rise to a Noether current. One choice is the *radiation gauge*:

$$A_0 = 0, \qquad \boldsymbol{\nabla} \cdot \boldsymbol{A} = 0 \tag{153}$$

which implies the Lorentz gauge:

$$\partial_\mu A^\mu = 0 \tag{154}$$

The equations of motion in the *radiation gauge* become:

$$\Box A^\nu = 0 \tag{155}$$

This is the massless Klein-Gordon equation. Its plane wave solutions are:

$$A_\mu(x) = \epsilon_\mu(k) e^{-ikx} + c.c. \tag{156}$$

These plane waves have two perpendicular to k degrees of freedom. We see that *radiation gauge* reproduces the degrees of freedom correctly. Unfortunately it is not Lorentz covariant, so there is something missing. With the *Lorentz gauge* its the other way round. We compute the energy-momentum tensor:

$$\theta^{\mu\nu} = -F^{\mu\rho} \partial^\nu A_\rho + \frac{1}{4} \eta^{\mu\nu} F^2 \tag{157}$$

This form is not gauge invariant, since it explicitly depends on A_μ. Under a gauge transformation it changes as:

$$\theta^{\mu\nu} \to \theta^{\mu\nu} + F^{\mu\rho}\partial^\nu\partial_\rho\theta = \theta^{\mu\nu} + \partial_\rho(F^{\mu\rho}\partial^\nu\theta) \tag{158}$$

However the conserved charges are gauge invariant:

$$P^\nu \to P^\nu + \int d^3x \partial_i(F^{0i}\partial^\nu\theta) \tag{159}$$

since the second term integrates to zero. We can modify the Energy-momentum tensor to be gauge-invariant by adding to it a term $\partial_\rho(F^{\mu\rho}A^\nu)$. Then it becomes:

$$T^{\mu\nu} = F^{\mu\rho}F_\rho{}^\nu + \frac{1}{4}\eta^{\mu\nu}F^2 \tag{160}$$

In this form it has convenient properties, too. For example:

$$E = \int d^3x T^{00} \tag{161}$$

$$P^i = \int d^3x T^{0i} \tag{162}$$

We see that the Energy-momentum tensor, and generally in QFT the currents are not unique and therefore not observabe. What is observable are the charges.

5.7.1 Coupling to matter

We consider the action:

$$S = -\int d^4x \left(\frac{1}{4}F^2 + j^\mu A_\mu\right) \tag{163}$$

This action is gauge invariant, if we additionally require:

$$\partial_\nu j^\nu = 0 \tag{164}$$

The equations of motion from this action are:

$$\partial_\mu F^{\mu\nu} = j^\nu \tag{165}$$

A very general method of writing a gauge invariant action is the following. We start from a theory with a global $U(1)$ invariance. Consider as a simple case the Dirac action:

$$\mathcal{L}_D = \bar{\Psi}(i\gamma^\mu\partial_\mu - m)\Psi \tag{166}$$

Therefore one considers the global transformation:

$$\Psi \to e^{iq\theta}\Psi \tag{167}$$

At the same time the action of the free electromagnetic field invariant under a local gauge:

$$A_\mu \to A_\mu - \partial_\mu \theta \tag{168}$$

with $\theta = \theta(x)$. Consider now the local $U(1)$ transformation for the Dirac action, i.e. $\theta \to \theta(x)$. One defines the *covariant derivative* of Ψ, which transforms in the same way as Ψ:

$$D_\mu \Psi = (\partial_\mu + iqA_\mu)\Psi \tag{169}$$

Under the *local* $U(1)$ plus the electromagnetic gauge transformation we find:

$$D_\mu \Psi \to e^{iq\theta(x)} D_\mu \Psi \tag{170}$$

i.e. the covariant derivative transforms uder $\theta(x)$ the same as Ψ. Therefore one obtains the locally $U(1)$ invariant action by replacing the regualr derivatives by covariant derivatives:

$$\mathcal{L}_D = \bar{\Psi}(i\gamma^\mu D_\mu - m)\Psi \tag{171}$$

This procedure is called local gauging of the $U(1)$ symmetry of Ψ by *minimally coupling* the Dirac field to a *gauge field* A_μ through the covariant derivative. Other couplings are possible, for example through higher electromagnetic moments. This is introduced by *non-minimal coupling*. More explicitly we find:

$$\mathcal{L}_D = \bar{\Psi}(i\gamma^\mu \partial_\mu - m)\Psi - qA_\mu \bar{\Psi}\gamma^\mu \Psi \tag{172}$$

Now we consider the implied **global** symmetry of the electrodynamics:

$$\begin{aligned} \Psi &\to e^{iq\theta}\Psi \\ A_\mu &\to A_\mu \end{aligned} \tag{173}$$

This gives rise to a conserved Noether current:

$$j^\mu_V = \bar{\Psi}\gamma^\mu \Psi \tag{174}$$

We see that the covariant derivative simply couples the gauge field to a conserved current. The respective conserved charge is:

$$Q = \int d^3x \bar{\Psi}\gamma^0 \Psi = \int d^3x \Psi^\dagger \Psi \tag{175}$$

and it has the meaning of electrical charge in units of e. The electrodynamic equations of motion are:

$$(i\gamma^\mu D_\mu - m)\Psi = 0 \tag{176}$$

Another example is the complex scalar field which symmetry is again $U(1)$ and can be minimally coupled to a gauge field A_μ:

$$\begin{aligned}\mathcal{L} &= (D_\mu\phi)^* D^\mu\phi - m^2\phi^*\phi \qquad (177)\\ &= \partial_\mu\phi\partial^\mu\phi^* + iqA^\mu(\phi\partial_\mu\phi^* - \phi^*\partial_\mu\phi) + q^2|\phi|^2 A_\mu A^\mu - m^2\phi^*\phi \qquad (178)\end{aligned}$$

with $D_\mu\phi = (\partial_\mu + iqA_\mu)\phi$. We recognise the conserved current of the complex Klein-Gordon theory coupled to the gauge field. We also see the third term which is important in the Higgs mechanism and the superconductivity. Trying to couple a real field to an electroomagnetic field one would find $q = 0$, so a real scalar field describes particles that are neutral under electromagnetism.

6 Field Quantization - Particles

6.1 Scalar field

By quantization one finds how the field produces particles with all observed properties. At the core of the canonical quantization are the relations:

$$\Big[\phi(t^*, \boldsymbol{x}), \Pi(t^*, \boldsymbol{y})\Big] = i\delta^{(3)}(\boldsymbol{x} - \boldsymbol{y}) \qquad (179)$$

$$\Big[\phi(t^*, \boldsymbol{x}), \phi(t^*, \boldsymbol{y})\Big] = 0 \qquad (180)$$

$$\Big[\Pi(t^*, \boldsymbol{x}), \Pi(t^*, \boldsymbol{y})\Big] = 0 \qquad (181)$$

And the real field is promoted to a hermitean operator by taking the Fourier transform and promoting the coefficients $a_{\boldsymbol{p}}$, $a^*_{\boldsymbol{p}}$ to operators $a_{\boldsymbol{p}}$, $a^\dagger_{\boldsymbol{p}}$. The commutation relation are then incorporated in the momentum space by:

$$\Big[a_{\boldsymbol{p}}, a^\dagger_{\boldsymbol{q}}\Big] = (2\pi)^3\delta^{(3)}(\boldsymbol{p} - \boldsymbol{q}) \qquad (182)$$

$$\Big[a_{\boldsymbol{p}}, a_{\boldsymbol{q}}\Big] = 0 \qquad (183)$$

$$\Big[a^\dagger_{\boldsymbol{p}}, a^\dagger_{\boldsymbol{q}}\Big] = 0 \qquad (184)$$

One takes a box of finite dimension L^3 and hence introduces a *infrared cutoff* to heal divergencies due to infinite box sizes, or equivalently from small momenta. Then:

$$\int d^3p \to \Big(\frac{2\pi}{L}\Big)^3 \sum_{\boldsymbol{n}} \qquad (185)$$

One recognizes the analogy with the qm harmonic oscillator, identifies the *construction* and *destruction operators* and introduces the respective *Fock space* of excitations. The vacuum energy is the zero-point energy:

$$E_{vac} = \frac{1}{2} V \int \frac{d^3p}{(2\pi)^3} E_p \tag{186}$$

with the energy density:

$$\rho_{vac} = \frac{1}{V} E_{vac} = \frac{1}{2} \int \frac{d^3p}{(2\pi)^3} E_p \tag{187}$$

For large $\boldsymbol{p}$ the integral diverges. One introduces the *ultraviolett cutoff* $|\boldsymbol{p}| < \Lambda$. Then the vacuum energy density diverges as:

$$\rho_{vac} \sim \int^{\Lambda} p^3 dp \sim \Lambda^4 \tag{188}$$

One shifts the potential and sets the zero-point energy zero. Then the Hamiltonian is:

$$H = \int \frac{d^3p}{(2\pi)^3} E_{\boldsymbol{p}} a^\dagger_{\boldsymbol{p}} a_{\boldsymbol{p}} \tag{189}$$

Alternatively one can derive this Hamiltonian from the classical one in two steps, by promoting the field to an operator and applying the *normal ordering*:

$$H = \frac{1}{2} \int d^3x : \Pi^2 + (\nabla\phi)^2 + m^2\phi^2 : \tag{190}$$

This holds generally. From this one can find the energies of the state which is the sum of the energies of the contributing excitations, i.e. particles. One finds the quantum momentum operator:

$$P^i = \int d^3x : \theta^{0i} := \int d^3x : \partial_0\phi\partial^i\phi : \tag{191}$$

Which, observing that the terms quadratic in the destruction operator vanish, gives:

$$P^i = \int \frac{d^3p}{(2\pi)^3} p^i a^\dagger_{\boldsymbol{p}} a_{\boldsymbol{p}} \tag{192}$$

The momentum of the state is again the sum of the individual momenta. We have found the *bosons* $(2E_{\boldsymbol{p}})^{1/2} a^\dagger_{\boldsymbol{p}}$.

6.2 Normalization

$$|\vec{p}\rangle := \sqrt{2\omega_p} a^\dagger \tag{193}$$

6.3 Complex scalar field - the Higgs boson

We have for the field:

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3\sqrt{2E_{\boldsymbol{p}}}}(a_{\boldsymbol{p}}e^{-ipx} + b^\dagger_{\boldsymbol{p}}e^{ipx}) \tag{194}$$

and its complex conjugate becomes its hermitean conjugate:

$$\phi^\dagger(x) = \int \frac{d^3p}{(2\pi)^3\sqrt{2E_{\boldsymbol{p}}}}(a^\dagger_{\boldsymbol{p}}e^{ipx} + b_{\boldsymbol{p}}e^{-ipx}) \tag{195}$$

With canonical commutation relations becoming:

$$\left[a_{\boldsymbol{p}}, a^\dagger_{\boldsymbol{q}}\right] = \left[b_{\boldsymbol{p}}, b^\dagger_{\boldsymbol{q}}\right] = (2\pi)^3\delta^{(3)}(\boldsymbol{p} - \boldsymbol{q}) \tag{196}$$

All **other** commutators are zero. The Hamiltonian and the momentum operator become:

$$H = \int \frac{d^3p}{(2\pi)^3}E_{\boldsymbol{p}}(a^\dagger_{\boldsymbol{p}}a_{\boldsymbol{p}} + b^\dagger_{\boldsymbol{p}}b_{\boldsymbol{p}}) \tag{197}$$

$$P^i = \int \frac{d^3p}{(2\pi)^3}p^i(a^\dagger_{\boldsymbol{p}}a_{\boldsymbol{p}} + b^\dagger_{\boldsymbol{p}}b_{\boldsymbol{p}}) \tag{198}$$

We see that the quanta of the complex scalar field are the *particle and anti-particle* $(2E_{\boldsymbol{p}})^{1/2}a^\dagger_{\boldsymbol{p}}$ and $(2E_{\boldsymbol{p}})^{1/2}b^\dagger_{\boldsymbol{p}}$. We also quantize the $U(1)$ charge:

$$\mathcal{Q}_{U(1)} = \int \frac{d^3p}{(2\pi)^3}(a^\dagger_{\boldsymbol{p}}a_{\boldsymbol{p}} - b^\dagger_{\boldsymbol{p}}b_{\boldsymbol{p}}) \tag{199}$$

Since the operator $a^\dagger a$ is the *number operator* of the harmonical oscillator, we observe that the particle has the charge 1 and its anti-particle has the charge -1. So in case of the real scalar field, due to the reality condition, the particle is its own anti-particle with the same charge.

6.4 Spin 1/2 field - the fermion

We start with the chiral representation Lagrangian:

$$\mathcal{L} = \bar{\Psi}(i\gamma^\mu\partial_\mu - m)\Psi \tag{200}$$

The conjugate momentum is:

$$\Pi_\Psi = i\bar{\Psi}\gamma^0 = i\Psi^\dagger \tag{201}$$

For the Dirac field we need the *anti-commutator* instead of the commutator, because of the spin-statistics theorem and because the commutator in this case gives inplausible results. So we have the canonical quantization with the anti-commutator:

$$\left\{\Psi_a(\boldsymbol{x},t),\Psi_b^\dagger(\boldsymbol{y},t)\right\} = \delta^{(3)}(\boldsymbol{x}-\boldsymbol{y})\delta_{ab} \tag{202}$$

with a, b the Dirac indices. We expand the free field:

$$\Psi(x) = \int \frac{d^3p}{(2\pi)^3\sqrt{2E_{\boldsymbol{p}}}} \sum_{s=1,2} (a_{\boldsymbol{p},s}u^s(p)e^{-ipx} + b_{\boldsymbol{p},s}v^s(p)e^{ipx}) \tag{203}$$

We construct also the hermitean conjugate like in the complex scalar case:

$$\bar{\Psi}(x) = \int \frac{d^3p}{(2\pi)^3\sqrt{2E_{\boldsymbol{p}}}} \sum_{s=1,2} (b_{\boldsymbol{p},s}\bar{v}^s(p)e^{-ipx} + a^\dagger_{\boldsymbol{p},s}\bar{u}^s(p)e^{ipx}) \tag{204}$$

So we see already that from the two-component structure of the Dirac field follows that the fermions also have anti-particles $(2E_{\boldsymbol{p}})^{1/2}b^\dagger_{\boldsymbol{p},s}$. The commutator relation becomes:

$$\left\{a^r_{\boldsymbol{p}},a^{s\dagger}_{\boldsymbol{q}}\right\} = \left\{b^r_{\boldsymbol{p}},b^{s\dagger}_{\boldsymbol{q}}\right\} = (2\pi)^3\delta^{(3)}(\boldsymbol{p}-\boldsymbol{q})\delta^{rs} \tag{205}$$

with all other commutators equal to zero. The classical Hamiltonian is:

$$H = \int d^3x\bar{\Psi}(-i\gamma\cdot\boldsymbol{\nabla}+m)\Psi \tag{206}$$

In this case the normal ordering is abtained *adding a minus sign by each exchange of a creation and annihilation operator.* We get:

$$H = \int \frac{d^3p}{(2\pi)^3} \sum_{s=1,2} E_{\boldsymbol{p}}(a^\dagger_{\boldsymbol{p},s}a_{\boldsymbol{p},s} + b^\dagger_{\boldsymbol{p},s}b_{\boldsymbol{p},s}) \tag{207}$$

At this point using the commutator would result in an unbounded negative energy! The momentum operator is:

$$\boldsymbol{P} = \int \frac{d^3p}{(2\pi)^3} \sum_{s=1,2} \boldsymbol{p}(a^\dagger_{\boldsymbol{p},s}a_{\boldsymbol{p},s} + b^\dagger_{\boldsymbol{p},s}b_{\boldsymbol{p},s}) \tag{208}$$

So the situation is the same to the complex scalar field, but with an additional degree of freedom, the spin. The angular momentum is a Noether charge, the spin part becomes:

$$\boldsymbol{S} = \frac{1}{2}\int d^3x\Psi^\dagger\boldsymbol{\Sigma}\Psi \tag{209}$$

with

$$\Sigma^i = \begin{pmatrix} \sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix} \tag{210}$$

We find that for $a^\dagger_{\boldsymbol{p},s}$ the state with $s = 1$ has $J_z = +1/2$ and the state with $s = 2$ has $J_z = -1/2$. For the state $b^\dagger_{\boldsymbol{p},s}$ the situation is reversed. Performing a boost in z drection the $a^\dagger_{\boldsymbol{p},s=1}$ state gets a *helicity* $= +1/2$. The conserved charge becomes:

$$\mathcal{Q}_{U(1)} = \int \frac{d^3p}{(2\pi)^3} \sum_{s=1,2} (a^\dagger_{\boldsymbol{p},s} a_{\boldsymbol{p},s} - b^\dagger_{\boldsymbol{p},s} b_{\boldsymbol{p},s}) \tag{211}$$

6.5 Gauge field - gauge bosons*

We proceed in the radiation gauge. We find from the masseless Klein-Gordon equation of motion the most general plane wave solution:

$$\boldsymbol{A} = \int \frac{d^3p}{(2\pi)^3 \sqrt{2\omega_{\boldsymbol{p}}}} \sum_{\lambda=1,2} \left\{ \boldsymbol{\epsilon}(\boldsymbol{p}, \lambda) a_{\boldsymbol{p},\lambda} e^{-ipx} + c.c. \right\} \tag{212}$$

7 Perturbation Theory and Feynman Diagrams

7.1 S-matrix

The amplitude for the process in which the initial Schrödinger picture state $|a\rangle$ evolves into the final state $|b\rangle$ is given by:

$$\langle b| e^{-iH(T_f - T_i)} |a\rangle \tag{213}$$

with H the second quantized Hamiltonian of field theory. In the limit $T_f - T_i \to \infty$ the evolution operator $e^{-iH(T_f - T_i)}$ is called the *S-matrix*. This matrix is an unitary operator, $S^\dagger S = SS^\dagger = 1$. This expresses the conservation of probability:

$$\sum_n |\langle n| S |a\rangle|^2 = 1 \tag{214}$$

In QFT the *Heisenberg representation* is most useful, since in QFT the operators are just the fields and from the point of view of Lorentz covariance must depend on x and t. The conversion from the Schrödinger picture is the following:

$$|a\rangle (t) \to |a_H, t\rangle = e^{iHt} |a\rangle (t) \tag{215}$$

$$A \to A_H(t) = e^{iHt} A e^{-iHt} \tag{216}$$

The states in the Heisenberg picture are time independent, the label t is carried as a parameter, to say that the state $|a_H, t^\star\rangle$is aneigenvector of a time dependent operator at $A(t^\star)$. Therefore the matrix element becomes in the Heisenberg picture

$$\langle a| S |b\rangle = \langle b, T_f | a, T_i\rangle \tag{217}$$

A general S-matrix element for a N-particle state can be calculated from the *N-particle Greens function*:

$$\langle 0| T\phi(x_1)\phi(x_2)...\phi(x_n) |0\rangle \tag{218}$$

this is stated by the LSZ reduction formula. In other words the connected matrix element is the *residue* in the sence of complex analysis of the Fourier transformed time-ordered correlators (Green's functions):

$$\begin{aligned} &\prod_{i=1}^{m}\int d^4x_i e^{-ik_ix_i}\prod_{j=1}^{n}\int d^4y_j e^{+ip_jy_j}\times \\ &\langle 0| T\{\phi(x_1)...\phi(x_m)\phi(y_1)...\phi(y_n)\} |0\rangle \\ = \Big(\prod_{i=1}^{m}\frac{i\sqrt{Z}}{k_i^2-m^2}\Big)&\Big(\prod_{j=1}^{n}\frac{i\sqrt{Z}}{p_j^2-m^2}\Big)\langle \boldsymbol{p}_1...\boldsymbol{p}_n| iT |\boldsymbol{k}_1...\boldsymbol{k}_m\rangle \end{aligned} \tag{219}$$

First of all the left hand side is computed *off-shell*, i.e. without using any relation between p_0^2 and $\boldsymbol{p}^2$. In the limit, in which we send the particles *on-shell*, i.e. $p^2 - m^2 = 0$, the left-hand side develops poles of the form $1/(k_i^2 - m^2)$ for each incoming and $1/(p_j^2 - m^2)$ for each outgoing particle. These factors cancel the same factors on the right hand side and we remain with finite quantities.
Therefore we are first of all interested in an approximation to the N-point Green's function.

7.2 Perturbative expansion in Interaction/Dirac picture

7.2.1 Setting up the interaction picture

At the classical level the interacting field $\phi(x)$ has a complicated equations of motion, so it cannot be decomposed into plane wave, i.e creation and annihilation operators. Therefore we need to relate it perturbatively to a free field ϕ_I in interaction picture. This happens in the following way:
Observe that ϕ_I can be expanded into plane waves and evolves with the free Hamiltonian:

$$\phi_I(t,x) = e^{iH_0(t-t_0)}\phi_I(t_0,x)e^{-iH_0(t-t_0)} \tag{220}$$

The interacting field evolves in the same manner with the full Hamiltonian H. Observe the relation to the *Schrödinger picture* for the states:

$$|\psi(t)\rangle_I = e^{iH_0(t-t_0)} |\psi(t)\rangle_S \tag{221}$$

and for the operators:

$$O_I(t) = e^{iH_0(t-t_0)} O_S e^{-iH_0(t-t_0)} \tag{222}$$

Set $\phi(t,x)$ and $\phi_I(t,x)$ equal at some time t_0, then we get:

$$\phi(t,x) = e^{iH\tau} e^{-iH_0\tau} \phi_I(t,x) e^{iH_0\tau} e^{-iH\tau} \tag{223}$$

with $\tau = t - t_0$.

7.2.2 Finding the time evolution operator

Define the time evolution operator

$$U(t,t_0) \equiv e^{iH_0\tau} e^{-iH\tau} \tag{224}$$

Define the interaction picture Hamiltonian H_I:

$$H_I(t) = e^{iH_0\tau} H_{int} e^{-iH_0\tau} \tag{225}$$

Observe by easy computation that $U(t,t_0)$ evolution with time is governed by a Schrödinger-like equation:

$$i\frac{\partial}{\partial t} U(t,t_0) = H_I(t) U(t,t_0) \tag{226}$$

This is equivalent to:

$$U(t,t_0) = -i \int_{t_0}^{t} H_I(t') U(t',t_0) dt' + U(t_0,t_0) \tag{227}$$

Solving this differential equation iteratively, with appropriate boundary conditions, one gets:

$$U(t,t_0) = Texp[-i \int_{t_0}^{t} dt' H_I(t')] \tag{228}$$

with the time ordering meaning that all terms in the Taylor expansion must be ordered:

$$U(t,t_0) = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_1 ... \int_{t_0}^{t} dt_n T H_I(t_1)..H_I(t_n) \tag{229}$$

From the last equation one can read all the properties of the time evolution operator. We observe from the second order expansion that $U(t_1,t_0)U(t_0,t_2) = U(t_1,t_2)$ and $U(t_1,t_2)\dagger = U(t_1,t_2)^{-1} = U(t_2,t_1)$.

7.2.3 Green's function

Back to the Green's function:

$$\langle 0| \phi(x_1)\phi(x_2)...\phi(x_n) |0\rangle \tag{230}$$

where the x_i are **time-ordered**. We first rewrite it in terms of the free field ϕ_I:

$$\langle 0| U^\dagger(t_1,t_0)\phi_I(x_1)U(t_1,t_0)U^\dagger(t_2,t_0)\phi_I(x_2)U(t_2,t_0)... \tag{231}$$

Using the identities one gets:

$$\langle 0| U^\dagger(t_1,t_0)\phi_I(x_1)U(t_1,t_2)\phi_I(x_2)U(t_2,t_3)...U(t_{n-1},t_n)\phi_I(x_n)U(t_n,t_0) |0\rangle \tag{232}$$

Then we can introduce a new very large variable $t >> t_i$. We rewrite with use of the identities:

$$U(t_n,t_0) = U(t_n,-t)U(-t,t_0) \tag{233}$$

and

$$U^\dagger(t_1,t_0) = U^\dagger(t,t_0)U(t,t_1) \tag{234}$$

and write:

$$\langle 0| U^\dagger(t,t_0)\{U(t,t_1)\phi_I(x_1)...\phi_I(x_n)U(t_n,-t)\}U(-t,t_0) |0\rangle \tag{235}$$

The term in the bracket is time ordered including the time ordering of the expansions of U. Therefore it can be written as:

$$\begin{aligned}
\{...\} =& T\{\phi_I(x_1)...\phi_I(x_n)U(t,t_1)...U(t_n,-t)\} && (236)\\
=& T\{\phi_I(x_1)...\phi_I(x_n)U(t,-t)\} && (237)\\
=& T\{\phi_I(x_1)...\phi_I(x_n)exp(-i\int_{-t}^{t} dt' H_I(t'))\} && (238)
\end{aligned}$$

We arrive at the equation:

$$\langle 0| T\phi(x_1)\phi(x_2)...\phi(x_n) |0\rangle = \langle 0| U^\dagger(t,t_0)T\phi_I(x_1)...\phi_I(x_n)exp(-i\int_{-t}^{t} dt' H_I(t'))U(-t,t_0) |0\rangle \tag{239}$$

Now we deal with the boundaries. t_o is arbitrary. Therefore we choose $t_0 = -t$ and send $t \to -\infty$. Then:

$$U(-t,t_0) = 1 \tag{240}$$

$$U^\dagger(t,t_0) \to U^\dagger(\infty,-\infty) \tag{241}$$

$\langle 0| U^\dagger(\infty,-\infty)$ is the hermitean conjugate of $U(\infty,-\infty)|0\rangle$, which is the state obtained by evolving the vacuum state from $-\infty$ to ∞. Ii is clear, that this must be again the vacuum. However in quantum mechanics the states are degenerate up to a phase:

$$U(\infty,-\infty)|0\rangle = e^{i\alpha}|0\rangle \tag{242}$$

Now we calculate the phase factor by simply multiplying this equation with $\langle 0|$. We obtain:

$$e^{i\alpha} = \langle 0| Texp(-i\int_{-\infty}^{+\infty} dt' H_I(t'))\,|0\rangle \tag{243}$$

We need the hermitean cojugate of it, $e^{-i\alpha}$, which is just the inverse of it. Since H_I is only time dependent, we can write the integral with dx^4. So eventially we we arrive at the formula for the N-point Greens function:

$$\langle 0| T\phi(x_1)\phi(x_2)...\phi(x_n)\,|0\rangle = \frac{\langle 0| T\phi_I(x_1)...\phi_I(x_n)exp(-i\int dx^4 H_I)\,|0\rangle}{\langle 0| Texp(-i\int dx^4 H_I)\,|0\rangle} \tag{244}$$

Some comments on it can be made. First of all we have expressed the Greens function in terms of the free field. H_I is expressed very simply in terms of the free field, since the functional dependence of H_I on ϕ_I is the same as of H_{int} on ϕ. For example: $H_{int} = (\lambda/4!)\phi^4 \to H_I(t) = (\lambda/4!)\phi_I^4$. If we want to calculate the expression perturbatively, we expand the exponential in powers of H_I and left with the products of the free fields. Since they can be expressed in terms of creation and annihilation operators, it is easily done, but fastly becomes combersome as a brute-force technique, therefore in the next chapter the more feasible techhnique based on Feynman graphs.

7.3 The Feynman propagator

We start with the simpler case of two free fields, the *Feynman propagator*, which is defined as:

$$\langle 0| T\phi_I(x)\phi_I(y)\,|0\rangle \tag{245}$$

*Since from now we have expressed the full field perturbatively as interaction picture field ϕ_I, we will never use it again, so we drop the subscript **I**.* We first separate the field into the creation and annihilation parts:

$$\phi(x) = \phi^+(X) + \phi^-(x) \tag{246}$$

where

$$\phi^+(x) = \int \frac{d^3p}{(2\pi)^3\sqrt{2E_p}} a_p e^{-ipx} \tag{247}$$

$$\phi^-(x) = \int \frac{d^3p}{(2\pi)^3\sqrt{2E_p}} a_p^\dagger e^{+ipx} \tag{248}$$

Consider first the case $x^0 > y^0$.Then

$$T\phi(x)\phi(y) = \phi^+(x)\phi^+(y) + \phi^+(x)\phi^-(y) + \phi^-(x)\phi^+(y) + \phi^-(x)\phi^-(y) \tag{249}$$
$$= \phi^+(x)\phi^+(y) + \phi^-(y)\phi^+(x) + \phi^-(x)\phi^+(y) + \phi^-(x)\phi^-(y) + \left[\phi^+(x), \phi^-(y)\right] \tag{250}$$
$$=: \phi(x)\phi(y) : + \left[\phi^+(x), \phi^-(y)\right] \tag{251}$$

with columns denoting the *normal ordering*, i.e. all creation operators appear on the left side and all annihilation operators on the right side.
For $y^0 > x^0$ we get $T\phi(x)\phi(y) =: \phi(x)\phi(y) : +[\phi^+(y), \phi^-(x)]$. Therefore for both cases:

$$T\phi(x)\phi(y) =: \phi(x)\phi(y) : + D(x-y) \tag{252}$$

where $D(x-y)$ is the *contraction*, a combination of opposed Feynman propagators and theta-functions. Observe that the expectation value of a normal ordered term is 0. We also observe that $\langle 0| D(x-y) |0\rangle = D(x-y) \langle 0|0\rangle = D(x-y)$, since the commutator of a_p and $a_p^\dagger$ is a c-number. Therefore we get:

$$\langle 0| T\phi_I(x)\phi_I(y) |0\rangle = D(x-y) \tag{253}$$

for the Feynman propagator. Computing the commutators and rewriting as a four-space integral we find for the Feynman propagator:

$$D(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\epsilon} e^{-ip(x-y)} \tag{254}$$

where $\epsilon \to 0^+$.
In momentum space the Feynman propagator gets even simpler:

$$\tilde{D}(p) = F(D(x)) = \frac{i}{p^2 - m^2 + i\epsilon} \tag{255}$$

So the Feynman propagator is just the Green's function of the operator $\Box + m^2$:

$$(\Box_x + m^2)D(x-y) = -i\delta^{(4)}(x-y) \tag{256}$$

We now introduce the *Wick's theorem* as the procedure to rewrite a general Greens function. E.g.:

$$T(\phi_1\phi_2\phi_3\phi_4) =: \phi_1\phi_2\phi_3\phi_4 : + D_{12} : \phi_3\phi_4 : + D_{13} : \phi_2\phi_4 : \tag{257}$$
$$+ D_{14} : \phi_2\phi_3 : + D_{23} : \phi_1\phi_4 : + D_{24} : \phi_1\phi_3 : \tag{258}$$
$$+ D_{34} : \phi_1\phi_2 : + D_{12}D_{34} + D_{13}D_{24} + D_{14}D_{23} \tag{259}$$

With a vacuum expectation value we get:

$$\langle 0| T(\phi_1\phi_2\phi_3\phi_4) |0\rangle = D_{12}D_{34} + D_{13}D_{24} + D_{14}D_{23} \tag{260}$$

At this point the product of Feynman propagators $D_{12}D_{34}$ can be physically interpreted as the amplitude for the process of propagation of a particle from space point x_1 to space point x_2 and another particle from x_3 to x_4. The sum is the sum over all diagrams. When we expand the exponential in powers of H_I, the diagrams become less trivial.

7.4 Vacuum bubbles, vertices, external legs

The theory for 2 ingoing and 2 outgoing particles is $H_I = (\lambda/4!)\phi^4$.
At zeroth order in λ, i.e. $\lambda = 0$ the exponent disapppears. For this theory is $Z = 1+O(\lambda^2)$, so we set $Z = 1$. We observe that the right hand side produces only two poles which cannot cancel the four poles on the left hand side as one goes on mass shell. Therefore the non-interacting diagrams from the supergroup of *disconnected diagrams*, don't contribute to the scattering amplitude, as expected.
To the first order in λ the exponential becomes simply a factor $\Big(-i\frac{\lambda}{4!}\Big)$ and, for a moment not considering the denominator, we get:

$$\begin{aligned}
&\Big(\prod_{i=1}^{2}\frac{i}{p_i^2-m^2}\Big)\Big(\prod_{j=1}^{2}\frac{i}{k_i^2-m^2}\Big)\langle \boldsymbol{p}_1\boldsymbol{p}_2| iT |\boldsymbol{k}_1\boldsymbol{k}_2\rangle \\
&= \int d^4x_1 d^4x_2 d^4x_3 d^4x_4 exp\{i(p_1x_1+p_2x_2-k_1x_3-k_2x_4)\} \\
&\times\Big(-i\frac{\lambda}{4!}\Big)\int d^4x\, \langle 0| T\{\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\phi^4(x)\} |0\rangle
\end{aligned} \tag{261}$$

Notice that point x is integrated over. This is an *internal point*, i.e. the vortex of the interaction. The points x_i are the *external points*, i.e. the ends of the *external legs* which produce factors that cancel the divergencies in the LSZ formula.
The graphs divide into disconnected graphs as above which don't contribute and 4 connected graphs, which are obtained by *contracting* each of the $\phi(x_i)$ with $\phi(x)$. There are 4! possible contractions of this type (think about it). This is why this term is included in H_I and it cancels:

$$\begin{aligned}
&\int d^4x d^4x_1 d^4x_2 d^4x_3 d^4x_4 exp\{i(p_1x_1+p_2x_2-k_1x_3-k_2x_4)\} \\
&\times(-i\lambda)D(x_1-x)D(x_2-x)D(x_3-x)D(x_4-x)
\end{aligned} \tag{262}$$

Rewriting the integration in a new variable $y_i = x_i - x$ and performing the integration, we get:

$$\begin{aligned}&(-i\lambda)\tilde{D}(p_1)\tilde{D}(p_2)\tilde{D}(k_1)\tilde{D}(k_2)\int d^4xe^{i(p_1+p_2-k_1-k_2)x}\\ &= (-i\lambda)(2\pi)^4\delta^{(4)}(p_1+p_2-k_1-k_2)\Big(\prod_{i=1}^{2}\frac{i}{p_i^2-m^2}\Big)\Big(\prod_{j=1}^{2}\frac{i}{k_j^2-m^2}\Big)\end{aligned} \tag{263}$$

With $\tilde{D}$ is the Feynman propagator in the momentum space. Therefore we get:

$$\langle \boldsymbol{p}_1\boldsymbol{p}_2|\, iT\, |\boldsymbol{k}_1\boldsymbol{k}_2\rangle = (-i\lambda)(2\pi)^4\delta^{(4)}(p_1+p_2-k_1-k_2) \tag{264}$$

Now we consider the denominator. It gives only the *vacuum-to-vacuum graphs* i.e. loops with only internal points. However, we postponned to this point that all graphs from the nominator, the connected and disconnected ones, also contain all possible vacuum-to-vacuum diagrams. This situation is called *partially connected graph* + the disconnected vacuum part. We observe that all Feynman graphs with vacuum parts factorize into a progression of vacuum diagrams times the partially connected graph. The progression looks like:

$$\infty + \frac{1}{2!}\infty\infty + \frac{1}{3!}\infty\infty\infty + ... \tag{265}$$

which is the perturbative expansion of the denominator, where the factors account for the number (not symmetry!) of double-loops.

$$\langle 0|\, Texp(-i\int dx^4 H_I)\, |0\rangle = \prod_i e^{V_i} = e^{\sum_i V_i} \tag{266}$$

where V_i is the value of the disconnected vacuum diagram. Therefore the vacuum contributions from the nominator and the denominator cancel each other. Therefore from now on we don't have to consider the denominator or *vacuum bubbles* in the nominator. We generally only need to consider the connected graphs. The denominator is also known as the *partition function*, which is directly related to the vacuum energy.

7.5 Internal lines

we consider as an example the three phi theory, here three lines meet at each vertex. Since we want $2 \rightarrow 2$ and only connected graphs, we automatically will have an internal line. In this theory the leading order is $O(\lambda^2)$, since in the first order we have for the Greens function:

$$\langle 0|\, T\{\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\phi^3\}\, |0\rangle \tag{267}$$

where an odd order of fields (7) cannot be contracted and the vacuum expectation value is zero. So to the second order we have:

$$\int d^4x_1 d^4x_2 d^4x_3 d^4x_4 exp\{i(p_1x_1 + p_2x_2 - k_1x_3 - k_2x_4)\}$$
$$\times\frac{1}{2!}\Big(-i\frac{\lambda}{3!}\Big)^2\int d^4x\int d^4y\,\langle 0|\,T\{\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\phi^3(x)\phi^3(y)\}\,|0\rangle \tag{268}$$

The new aspect is that in order to contract all fields and obtain a connected diagram, we have to contract one field $\phi(x)$ with one $\phi(y)$, where x and y are the positions of the two vertices. We notice that the order of expansion of the exponential also gives a combinatorial factor, here $\frac{1}{2!}$. However we can interchange x and y which cancels this factor.
So the final result for a picked diagram is:

$$\int d^4x_1 d^4x_2 d^4x_3 d^4x_4 exp\{i(p_1x_1 + p_2x_2 - k_1x_3 - k_2x_4)\}$$
$$\times\int d^4x\int d^4y(-i\lambda)^2 D(x_1 - x)D(x_2 - x)D(x - y)D(y - x_3)D(y - x_4)$$
$$= (-i\lambda)^2(2\pi)^4\delta^{(4)}(p_1 + p_2 - k_1 - k_2)\tilde{D}(p_1)\tilde{D}(p_2)\tilde{D}(p_1 + p_2)\tilde{D}(k_1)\tilde{D}(k_2) \tag{269}$$

We recognise the internal line with the momentum $p_1 + p_2$.

We generalize the results we get to all Feynman graphs in the scalar theory:

1 Only connected graphs contribute

2 For connected graphs, the propagators associated to the external legs cancel exactly the pole factors in the LSZ formula. Therefore one can omit the factors from the beginning on. This is often expressed as consideing graphs with *external legs amputated*

3 Each interaction vertex gives a factor $-i$ times the coupling constant.

4 There is an integration over a vertex x which gives a Dirac delta imposing the overall energy-momentum conservation. Each vertex must be considered separately.

5 To each internal line associates the momentum space Feynman propagator with the momentum given by the momentum conservation.

6 One has to account for all realizations of the process, therefore there is a combinatorial factor from the number of equivalent contractions, the factor $\frac{1}{n!}$ from the expansion of the exponential to the order n and numerical factors associated to the definition of the coupling constant such as $1/4!$ in $\frac{\lambda}{4!}\phi^4$ theory.

7.6 Internal loops

Consider again the four phi theory to the second order, i.e. with two vertices. Automatically there have to be an internal loop, accounting for two internal lines. The important point here is that the internal momentum is not completely fixed. Hence in the end we face an integration over the momentum k of one of the two inner lines:

$$
\begin{aligned}
1/2(-i\lambda)^2\tilde{D}(p_1)\tilde{D}(p_2)\tilde{D}(k_1)\tilde{D}(k_2)(2\pi)^4\delta^{(4)}(p_1+p_2-k_1-k_2)\\
\times\int\frac{d^4k}{(2\pi)^4}\tilde{D}(k)\tilde{D}(p_1+p_2-k)
\end{aligned}
\tag{270}
$$

So we add the rule:

7 associate a propagator to each internal line in a loop, use momentum conservation at the vertices to reduce the number of independent momenta, and integrate over the remaining unfixed momenta with the measure $d^4k/(2\pi)^4$.

We define

$$
A(p)=\frac{(-i\lambda)^2}{2}\int\frac{d^4k}{(2\pi)^4}\frac{i}{k^2-m^2+i\epsilon}\frac{i}{(p-k)^2-m^2+i\epsilon}
\tag{271}
$$

where k is the momentum of one internal line and p is the momentum of the other internal lines.

The $2\to 2$ scattering amplitude at one loop level is then (M is the matrix element omitting Dirac delta and $(2\pi)^4$ for simplicity):

$$
iM_{2\to 2}=-i\lambda+A(p_1+p_2)+A(p_1-k_1)+A(p_1-k_2)
\tag{272}
$$

accounting for the three processes. Notice that the integral diverges at large k. This is called the *UV-divergence*. One proceeds as following. For simplicity set $p=0$ (here it is sufficient for extracting the divergent part, since the divergence comes from the region where $(p-k)^2\to k^2$ as $k\to\infty$). According to $+i\epsilon$ the pole at $k^0>0$ is below the real axis and te pole at $k^0<0$ is above. Therefore one can perform a *Wick rotation* counterclockwise from the real to the imaginary axis. We get:

$$
A(0)=i\frac{\lambda^2}{2}\int\frac{d^4k}{(2\pi)^4}\frac{1}{(k^2+m^2)^2}
\tag{273}
$$

where $k^2=(k^0)^2+\boldsymbol{k}^2$. We introduce a cutoff for the integration over the momentum:

$$
\begin{aligned}
A(0)&=i\frac{\lambda^2}{2}\int\frac{d^4k}{(2\pi)^4}\frac{1}{k^4}+finite\quad parts\\
&i\frac{\lambda^2}{2}\frac{1}{(2\pi)^4}(2\pi^2)\int^{\Lambda}\frac{dk}{k}+finite\quad parts
\end{aligned}
\tag{274}
$$

$$
=i\lambda^2\frac{1}{16\pi^2}log\Lambda+finite\quad parts
\tag{275}
$$

In this graph the divergent part is independent of p, therefore the three possible graphs contribute the same to the divergence. As a result, at one loop, the $2 \to 2$ scattering amplitude in $\lambda\phi^4$ is:

$$iM_{2\to 2} = -i\lambda + i\lambda^2(\beta_0 log\Lambda + finite \quad parts) \tag{276}$$

with

$$\beta_0 = \frac{3}{16\pi^2} \tag{277}$$

7.7 Corrections to the Feynman propagator - tadpoles and two-loops

Consider the first application of the above - the corrections of $D(x-y) = \langle 0| T\{\phi(x)\phi(y)\,|0\rangle$ in the ϕ^4 theory at order λ. There are another example of divergence, the *tadpole graphs*. Using the Feynman rules and Wicks rotation one arrives at the imaginary quantity:

$$-iB = \frac{-i\lambda}{2}\int \frac{d^4k}{(2\pi)^4}\frac{1}{k^2+m^2} = -i\frac{\lambda}{32\pi^2}\Big(\Lambda^2 - m^2 log\frac{\Lambda^2+m^2}{m^2}\Big) \tag{278}$$

so this new term associated to a tadpole has a quadratic and a logarithmic divergence, both coming from the large k, so thiis are again UV divergencies. Observe that the tadpole loop graph with the legs amputated is independent of the external momentum p. Actually the tadpole graphs can be even resummed. Including the external legs the result for the two-point function is:

$$\begin{aligned} &\tilde{D}(p) + \tilde{D}(p)(-iB)\tilde{D}(p) + \tilde{D}(p)(-iB)\tilde{D}(p)(-iB)\tilde{D}(p) + ... \\ &= \tilde{D}(p)\Big(1 + (-iB\tilde{D}(p)) + (-iB\tilde{D}(p))^2 + ...\Big) \\ &= \tilde{D}(p)\frac{1}{1+iB\tilde{D}(p)} = \frac{1}{p^2-m^2}\left(\frac{1}{1-\frac{B}{p^2-m^2}}\right) = \frac{i}{p^2-m^2-B} \end{aligned} \tag{279}$$

So the net effect is to shift the mass from m^2 to $m^2 + B(\Lambda)$. This effect is important for the renormalization theory and will be soon generalized.
At $O(\lambda^2)$ there are further contributions to the two-point function. One possibility is the two-loop correction. The Feynman graph gives:

$$i\frac{\lambda^2}{6}\int \frac{d^4k_2}{(2\pi)^4}\frac{d^4k_2}{(2\pi)^4}\frac{1}{\{(p-k_1-k_2)^2-m\}(k_1^2-m^2)(k_2^2-m^2)} \tag{280}$$

This integral is quite complicated and turns out to be:

$$\lambda^2p^2\Big(log\frac{\Lambda^2}{p^2} + C\Big) \tag{281}$$

The two loop geometric series can also be resummed to:

$$\frac{i}{E(\Lambda,p^2)p^2 - m^2 - B(\Lambda)} \tag{282}$$

with $E(\Lambda,p^2) = 1 - \lambda^2(c_1 log\frac{\Lambda^2}{p^2} + c_2)$, where c_i are some constants.
So in the end we are facing the fact that disconnected and trivial diagrams doesn't matter. The loops and the two-loop corrections to the 2-point Feynman propagator, can be resummed. But there are more possible diagrams to it. To organize all possible diagrams, we define *one-particle irreducible diagrams* . These are diagrams with just one non-trivial event that is integrated over. We further define the *sum of all possible one-particle irreducible diagrams = 1PI*. The value of this sum will be of the form: $-iM^2(p^2)$, that in QED is called the *self-energy*. Irrespective of the exact value of $1PI$, which was in the previous cases A and B and can depend on p and Λ, we generally state that all corrections can be organized as a geometric series in $1PI$ diagrams. We resum in this way the 2-point Greens function and get the value of the *Dyson resummed propagator*, reinstating $i\epsilon$:

$$D(p^2) = \frac{i}{p^2 - (m^2 + M^2(p^2)) + i\epsilon} \tag{283}$$

Now we can expand M in λ. We extract the first analytic pole in $D(p^2)$ and call it m^2 with now $m \neq m_0$ i.e. in general a shift. Then we get:

$$D(p^2) = \frac{iZ}{p^2 - m^2 + i\epsilon} + terms\ regular\ at\ m^2 \tag{284}$$

Z is by definition the residue of $D(p^2)$ at its first analytic pole m^2. We get beautifully two important quantities - the *wave function renormalization Z* and the full physical mass m of a free particle in an interacting theory i.e. the pole mass of the particle as a result of self-interactions. The wave function renormalization is defined as:

$$\langle vacuum|\, \phi(0)\, |\boldsymbol{p} = 0\rangle \tag{285}$$

It is also called the field strength. in the free theory $Z = 1$ in the interacting theory however $Z < 1$, because the field $\phi(0)$ creates not just one particle as in the free theory, and so this overlap with one-particle state is smaller.

So the complete set of rules becomes:

1 Only connected graphs contribute

2 For connected graphs, the propagators associated to the external legs cancel exactly the pole factors in the LSZ formula. Therefore one can omit the factors from the beginning on. This is often expressed as considering graphs with *external legs amputated*

3 Each interaction vertex gives a factor $-i$ times the coupling constant.

4 There is an integration over a vertex x which gives a Dirac delta imposing the overall energy-momentum conservation. Each vertex must be considered separately.

5 To each internal line associates the momentum space Feynman propagator with the momentum given by the momentum conservation. The propagator experiences corrections, i.e. is Dyson resummed to given order in λ.

6 One has to count all realizations of the process, i.e. all possible interaction processes. Therefor there are combinatorial factors adding, i.e. a combinatorial factor from the number of equivalent contractions, the factor $\frac{1}{n!}$ from the expansion of the exponential to the order n and numerical factors associated to the definition of the coupling constant such as 1/4! in $\frac{\lambda}{4!}\phi^4$ theory. Also the symmetry group of the loops contribute, a single loop has 2 realizations, two loops 2*2*2 realizations. In addition k similar vertices have $k!$ permutations. The resulting factor contributes to the strength of the interaction, i.e. is an overall factor.

7 associate a propagator to each internal line in a loop, use momentum conservation at the vertices to reduce the number of independent momenta, and integrate over the remaining unfixed momenta with the measure $d^4k/(2\pi)^4$.

7.8 Fermion propagator

Wick's theorem can be generalized to Fermionic fields. We define

$$T\{\Psi(x)\bar{\Psi}(y)\} = \begin{cases} \Psi(x)\bar{\Psi}(y) & x^0 > y^0 \\ -\bar{\Psi}(y)\Psi(x) & x^0 < y^0 \end{cases}$$

The Feynman propagator for the Dirac field is

$$S(x-y) = \langle 0| T\{\Psi(x)\bar{\Psi}(y)\} |0\rangle \tag{286}$$

Its Fourier transform, the momentum space propagator is:

$$\tilde{S}(p) = \frac{i(\gamma^\mu p_\mu + m)}{p^2 - m^2 + i\epsilon} \tag{287}$$

this can be rewritten to the simple form:

$$\tilde{S}(p) = \frac{i}{\gamma^\mu p_\mu - m + i\epsilon} \tag{288}$$

7.9 Photon propagator

By definition:

$$D_{\mu\nu}(x-y) = \langle 0| \, T\{A_\mu(x)A_\nu(y)\} \, |0\rangle \tag{289}$$

Using the *covariant quantization* of the gauge fields the photon propagator is computed similarly to the massless scalar field. The propagator is then:

$$\tilde{D}_{\mu nu}(k) = \frac{-i}{k^2 + i\epsilon}\eta_{\mu\nu} \tag{290}$$

with Minkowski metric γ^μ. The only difference is the different sign of the time component. In QED the interaction Hamiltonian is:

$$H_{int} = eA_\mu \bar{\Psi}\gamma^\mu \Psi \tag{291}$$

The field Ψ in terms of the creation/annihilation operators creates a positron and destroys an electron, the field $\bar{\Psi}$ creates an electron and destroys an positron. The gauge field in the covariant quantization can create and destroy a photon. Therefore this Hamiltonian has two fermion and one photon line, and conserves charge: e.g. $e^- \to e^-\gamma$, $e^- \to e^-\gamma$ or $e^+e^- \to \gamma$. All possibilities are summarized by associating the factor

$$-\, ie\gamma^\mu \tag{292}$$

to the interaction vertex. γ^μ is the Minkowski metric. To external legs are associated the factors e^{ipx}, e^{-iky}. They represent the initial and final particles, give eventually the overall energy-momentum conservation and the Fourier transform the propagators into momentum space. There are more factors associated to the external legs.

- A factor $\epsilon^*_\mu(k)$ for each final photon with momentum k and polarization ϵ_μ.
- A factor $\epsilon_\mu(k)$ for each initial photon with polarization ϵ_μ.
- A factor $u^s(p)$ for each initial electron with momentum p and spin s
- A factor $v^s(p)$ for each final positron with momentum p and spin s
- A factor $\bar{u}^s(p)$ for each final electron with momentum p and spin s
- A factor $\bar{v}^s(p)$ for each initial positron with momentum p and spin s

In one word, to each initial particle is associated its wave-function, and to each final particle is associated the complex conjugate of the wave function (or the Dirac adjoint).
Finally to each closed fermionic loop there is an **additional minus sign** due to the anticommuting nature of fermions.

8 Renormalization

The basic ingredient is a cutoff Λ. As long as it is taken finite, the theory depends in all its constituents on Λ, e.g. the 4ϕ theory:

$$\mathcal{L} = \frac{1}{2}(\partial\phi(\Lambda))^2 - \frac{1}{2}m^2(\Lambda)\phi^2(\Lambda) - \frac{\lambda(\Lambda)}{4!}\phi^4(\Lambda) \tag{293}$$

we call this Λ dependent quantities the *bare field, bare coupling, bare mass* and denote it with a subscript zero. Recall the one-loop correction to the Feynman propagator at $\mathcal{O}(\lambda)$:

$$\langle 0| T\{\phi_0(x,\Lambda)\phi_0(y,\Lambda)\} |0\rangle = \mathcal{F}\bigg(D_F(p) = \frac{i}{p^2 - m_0^2(\Lambda) - B(\Lambda)}\bigg) \tag{294}$$

where, in comparison to $\mathcal{O}(\lambda = 0)$ the bare mass m_0 is modified by a factor B which is divergent with Λ^2. The idea is, that we doesn't observe the m_0 and B, but the renormalized, i.e. interacting mass m_R. We choose the parameters m_0 and B to cancel the divergencies of each other and it it leaves us with a finite value of m_R equal to the measured value. At the two-loop level $\mathcal{O}(\lambda^2)$ the situation is similar, there is an additional divergence coming from $E(\Lambda)$:

$$D_F(p) = \frac{i}{E(\Lambda,p^2)p^2 - m_0^2(\Lambda) - b(\Lambda)} \tag{295}$$

We define the mass as the position of the pole of the propagator:

$$\{E(\Lambda,p^2)p^2 - m_0^2(\Lambda) - b(\Lambda)\}_{|p^2=m_R^2} = 0 \tag{296}$$

This is the first condition to eliminate the two divergencies and is called the *mass renormalization*. We expand the two-loop corrected correlation function near the pole:

$$\int d^4x e^{ipx} \langle 0| T\{\phi_0(x,\Lambda)\phi_0(0,\Lambda)\} |0\rangle_{connected} = \frac{iZ}{p^2 - m_R^2} + ... \tag{297}$$

where

$$Z\Big(\lambda_0(\Lambda), \frac{\Lambda}{m_R}\Big) = \bigg(\Big(\frac{d}{d(p^2)}A(\Lambda,p^2)p^2\Big)_{|p^2=m_R^2}\bigg)^{-1} \tag{298}$$

Now define the *renormalized field* ϕ_R by:

$$\phi_0(x,\Lambda) = Z^{1/2}\Big(\lambda_0(\Lambda), \frac{\Lambda}{m_R}\Big)\phi_R(x) \tag{299}$$

By definition ϕ_R is independent of the cutoff and this fixes the dependence of ϕ_0 on Λ. This is the second condition. The factor $Z^{1/2}$ is called the *wave function renormalization*. We see from the expansion above that the factor Z is the same as in the LSZ formula of a free field with mass m_R. So Z disappears from the LSZ formula, if one replaces $\phi_0 \to \phi_R$.

Thus, after *mass and wave function renormalization*, the on-shell two-point function is finite.
Now we deal with the four-point function. At one-loop there are two kinds of divergence: the one associated to unfixed internal momenta, which is logarithmic, and tadpoles on external legs. The divergencies due to tadpoles on a leg are automatically cured by mass and wave-function renormalization of the two-point function, since it concerns only the two-point subgraph! So lets consider the logarithmic divergence. To renormalize the divergence one should make clear, what are the observable quantities. The scattering amplitude λ_R is measurable in the scattering experiment, it is the *renormalized coupling*. We consider a scattering amplitude in the limit of $p_1 = p_2 = k_1 = k_2 = (m_R, 0)$, i.e. zero spatial momentum. We rewrite the previous result for a $2 \to 2$ scattering amplitude to order λ_0^2 as:

$$i\mathcal{M}_{2\to 2}(\boldsymbol{p}_i = \boldsymbol{k}_i = 0) = -i\lambda_R = -i\lambda_0(\Lambda)\Big(1 - \lambda_0(\Lambda)\Big(\beta_0 log\frac{\Lambda}{m_R} + finite\ parts\Big)\Big) \quad (300)$$

So the relevant dimensional scale is provided by m_R. By setting λ_R equal to the experimental value, we can choose λ_0 freely. Once set, the result of a finite λ_R comes out in any other limit of kinematic parameters. To any order genuinely other divergencies can only occure in irreducible (1PI) graphs. Any other graph on the other hand can be reduced to subgraphs of a lower order and their diverencies are automatically cured. With each renormalization at a higher and higher order the theory looses its predictive power. If one needs infinitely many experimental values for λ_R for infinitely many orders, the theory is called a *non-renormalizable theory*. Consider also as an anticipation of the chapter *Running of coupling constant* the scattering amplitude in the different kinematic regime where the *energy transfer* or *CM energy* $q^2 := (p_1^2 + p_2^2) \gg m_R^2$ is large and is potentially the leading dimensional scale. The calculation gives a different value for the amplitude:

$$i\mathcal{M}_{2\to 2}(q^2) = -i\lambda_0(\Lambda)\Big(1 - \lambda_0(\Lambda)\Big(\frac{\beta_0}{2} log\frac{\Lambda^2}{q^2} + finite\ parts\Big)\Big) + \mathcal{O}(\lambda_0^3) \quad (301)$$

Writing in terms of the renormalized coupling constant one gets:

$$i\mathcal{M}_{2\to 2}(q^2) = -\lambda_R\Big(1 + \lambda_R\frac{\beta_0}{2} log\frac{q^2}{m_R^2}\Big) + \mathcal{O}(\lambda_R^3) \quad (302)$$

which is not the same result as in case of low energy. First, the transition amplitude is now dependent on q, i.e. on the energy scale. One cannot get rid of this dependence since it is physical as will be seen later and this is known as the *running of the coupling constant*. Second, the coupling constant λ_R is not the observable quantity any more. We rather have to introduce the *effective coupling constant* as:

$$i\mathcal{M}_{2\to 2}(q^2) = -i\lambda_{eff}(E) \quad (303)$$

$$\lambda_{eff}(E) = \lambda_R + \lambda_R^2 \beta_0 \log \frac{E}{m_R} + \mathcal{O}(\lambda_R^3) \tag{304}$$

The same formula hold for $E << m_R$. So we see that

$$i\mathcal{M}_{2\to 2} = -i\lambda_R \tag{305}$$

is only the special situation with $E = 0$. We will see that the parameter β_0 is the zero point of the *beta function* and the sign of the parameter is positive for QED and negative for an *asymptotically free theory* like QCD.

There is a general statement about renormalizability of a theory. Consider for example a theory $\lambda\phi^n$ with $n > 3$ integer, in four space-time dimensions. Recall that each loop gives a 4th power in momentum integration d^4k and a factor of $\frac{1}{\mathcal{O}(p^2)}$ of the type $\frac{1}{((k-p)^2-m^2)}$ for each propagator on an internal line. We define the *superficial degree of divergence*:

$$D = 4L - 2N_i \tag{306}$$

where L are the loops and N_I are the internal lines. One can also show:

$$L = N_i - V + 1 \tag{307}$$

where V are the vertices in a graph. There is also a relation for n, the number of lines at each vertex:

$$2N_i + N_e = nV \tag{308}$$

where N_e is the number of external lines and the factor 2 reflects the fact that one internal line connects two vertices. Combined, the expressions give:

$$D = (n-4)V + 4 - N_e \tag{309}$$

If $D \geq 0$ one expects that the diagram is divergent, unless some numerical cancelations in the leading term occur.$D = 0$ corresponds to the factor $\int d^4k/k^4$, i.e. a logarithmic divergence. If $D \leq 0$, the diagramm might have divergencies, which are of a finite number. The condition of renormalizability is therefore a finite number of Greens functions with $D \geq 0$. For the $\lambda\phi^4$ we have:

$$D = 4 - N_e \tag{310}$$

That means, the divergent parts are the graphs with no external legs, i.e. are vacuum bubbles which diverge as Λ^4, see the *cosmological constant problem*, the two-point function graphs, which diverge as Λ^2 and the four-point function that diverges as $log\Lambda$. Therefore the theory is renormalizable. The theories with $n < 4$ are also renormalizable. However the theories with $n \geq 4$ are non-renormalizable. These theories are however perfectly suitable as low-energy theories. we generalize to a general theory with the dimensional argumentation: the field ϕ has dimension of mass, since the action is dimensionless. The kinetic term is $\sim \int d^4x(\partial\phi)^2$ and $\partial \sim 1/length = mass$. Requiring that $\int d^4x\lambda_n\phi^n$ is dimensionless, we see that the coupling λ_n has dimension of $(mass)^{4-n}$. So the criterion $n \geq 4$ means that:

> *Terms in the Lagrangian whose coefficients have either a positive mass dimension or are dimensionless are renormalizable. Terms with negative mass dimension are non-renormalizable.*

9 Running of the coupling constant

In this secton one generalizes the method of mass and wavefunction renormalization to those high energy situations, where the masses can be neglected, and explains, why the coupling constant is energy dependent. Consider a generic n-point function. A general *renormalized* n-point function Γ_R will depend on the momenta p_i or in a simple kinematic situation just on one invariant q^2, on a generic renormalized coupling g_R (the generalization to more than one couplings is straight forward) and on the *mass scale* μ which defines the renormalization:

$$\Gamma_R = \Gamma_R(p_i; g_R, \mu) \tag{311}$$

In the previous section we have chosen $\mu = m_R$ as leading scale and defined it as the position of the pole of the propagator as should be expected of the physical mass: $p^2 = m_R^2$. Then we obtained the field by choosing it such (Z) that its residue in the LSZ formula is $+i$. Then the amplitude was finally obtained in a kinematic situation, where m_R is again the leading scale: we used a zero momentun situation with $q = 4m_R^2$ the CM energy. However we have seen that also other choices are possible, like $q^2 = \mu^2 >> m_R$ in a relativistic situation. So in the high energy limit the energy is the leading scale and the mass dependence can be neglected. We have seen also that since Z is dimensionless, in the high energy limit it can only depend on Λ and μ through Λ/μ. The general high energy relation holds:

$$\Gamma_R(p_i; g_R, \mu) = Z^{-n/2}\Gamma_0(p_i; g_0(\Lambda), \Lambda) \quad Z = Z\Big(g_0(\Lambda), \frac{\Lambda}{\mu}\Big) \tag{312}$$

Since Γ_R is independent of Λ the following relation is true:

$$\Lambda\frac{d\Gamma_R}{d\Lambda} = 0 \tag{313}$$

With this we obtain:

$$\Big(\Lambda\frac{\partial}{\partial\Lambda} + \beta(g_0)\frac{\partial}{\partial g_0} - n\eta(g_0)\Big)\Gamma_0(p_i; g_0, \Lambda) = 0 \tag{314}$$

where

$$\beta(g_0) = \Lambda \frac{dg_0}{d\Lambda} = \frac{dg_0}{d\log(\Lambda)} \tag{315}$$

$$\eta(g_0) = \frac{1}{2}\Lambda \frac{d}{d\Lambda} \log Z \tag{316}$$

The equation is called the *renormalization group equation* and is solved by method of characteristics. $\beta(g_0)$ is called the *beta function*. In general the renormalization of the coupling has the form:

$$g_R = g_0 - \beta_0 g_0^2 \log \Lambda + \mathcal{O}(g_0^3) \tag{317}$$

Therefore:

$$\beta(g_0) = \beta_0 g_0^2 + \mathcal{O}(g_0^3) \tag{318}$$

This shows that there is always a zero of the beta function at $g_0 = 0$, and it is possible to remove the cutoff while at the same time sendin $g_0(\Lambda) \to 0$. In other words, given a regularized theory, i.e. with a cutoff in momentum space or on a space-time lattice, one finds the limit $\Lambda \to \infty$ tuning the bare couplings towards a zero of the beta function. This method is particularly useful in statistical physics.

There is another way to look at it. Lets look instead on:

$$\Gamma_0(p_i; g_0(\Lambda), \Lambda) = Z^{n/2}\Gamma_R(p_i; g_R, \mu) \tag{319}$$

then one uses that Γ_0 is independent on the energy scale μ and finds:

$$0 = \mu \frac{d\Gamma_0}{d\mu} = \Big(\mu \frac{\partial}{\partial \mu} + \beta(g_R)\frac{\partial}{\partial g_R} + n\gamma(g_R)\Big)\Gamma_R(p_i; g_R, \mu) \tag{320}$$

with

$$\beta(g_R) = \mu \frac{dg_R}{d\mu} \tag{321}$$

$$\gamma(g_R) = \frac{1}{2}\mu \frac{d}{d\mu} \log Z \tag{322}$$

which is the *Callan-Symanzik equation*. This way of looking is more useful in high energy physics. The beta function is now the more physical dependence of the renormalized coupling on the enery scale. If Γ_R has mass dimension M, then for dimensional reasons:

$$\Gamma_R(p_i; g_R, \mu) = \mu^M F\Big(g_R, \frac{p_i}{\mu}\Big) \tag{323}$$

with F a *dimensionless* function. Solving the Callan-Symanzik equation with the method of characteristics, one introduces the *dilatation parameter* u, which is later set to zero. The solution is:

$$\Gamma_R(p_i; g_r, \frac{\mu}{u}) = Z_{eff}^{-n/2}(u)\Gamma_R(p_i; g_{eff}(u), \mu) \tag{324}$$

with $g_{eff}(u)$ is defined as the solution of:

$$u\frac{dg_{eff}}{du} = \beta(g_{eff}(u)) \tag{325}$$

with the initial condition $g_{eff}(1) = g_R$. And Z_{eff} is defined as the solution to:

$$\frac{1}{2}u\frac{d}{du}\log Z_{eff} = -\gamma(g_{eff}(u)) \tag{326}$$

with the initial condition $Z_{eff}(1) = 1$. We observe also:

$$\Gamma_R\left(p_i; g_R, \frac{\mu}{u}\right) = \frac{1}{u^M}\Gamma_R(up_i; g_R, \mu) \tag{327}$$

With this we find the solution:

$$\Gamma_R(up_i; g_r, \mu) = u^M Z_{eff}^{-n/2}(u)\Gamma_R(p_i; g_{eff}(u), \mu) \tag{328}$$

This can be written in the integral form as:

$$\Gamma_R(up_i; g_R, \mu) = u^M exp\left\{n\int_0^{\log u}\gamma(g_{eff})(u')d\log u'\right\}\Gamma_R(p_i; g_{eff}(u), \mu) \tag{329}$$

We are interested in the quantity $\Gamma_R(p_i; g_{eff}(u), \mu)$ independent on the mass renormalized coupling. We see that apart from an overall factor we get an additional dimension γ which is called *anomalous dimension* and is the answer to the cutoff Λ which is sent to infinity and replaced by the factor μ, an additional mass-scale. In the end it gives the effective field-strength renormalization Z_{eff}.
We have computed the result for the internal loops at one-loop and have found:

$$\lambda_R = \lambda_0 + \lambda_0^2\frac{3}{16\pi^2}\log\Lambda \tag{330}$$

The calculation at one loop is simple, since there is no wave-function renormalization. We see that:

$$\beta(\lambda_0) = \beta_0\lambda_0^2 + \mathcal{O}(\lambda_0^3), \qquad \beta_0 = \frac{3}{16\pi^2} \tag{331}$$

Considering just one-loop we integrate:

$$E\frac{d}{dE}\lambda_{eff} = \beta_0\lambda_{eff}^2 \tag{332}$$

over the energy with the initial condition $\lambda_{eff}(E = \mu) = \lambda_*$ and find for λ_{eff}:

$$\lambda_{eff}(E) = \frac{\lambda_*}{1 - \beta_0\lambda_*\log(E/\mu)} \tag{333}$$

Comparing to the previous result

$$\lambda_{eff}(E) = \lambda_R + \lambda_R^2 \beta_0 \log \frac{E}{m_R} + \mathcal{O}(\lambda_R^3) \tag{334}$$

we see that we just got a resummation of the geometric series. So the result is the same but now extended to all orders in $\lambda_R = \lambda_*$. One can also expand the logarithm to get approximate results:

$$\lambda_{eff}(E) = \lambda_* \left(1 + \sum_{n=1}^{\infty} c_n(E)\lambda_*\right)^n \tag{335}$$

with

$$c_n(E) = \left(\beta_0 \log \frac{E}{\mu}\right)^n \tag{336}$$

We have used only the one-loop beta function for this result. However the corrections are negligible in case $\log(E/\mu) >> 1$. Since in this case also $\lambda_{eff}(E) << 1$, we conclude that the resummed version converges only if $\beta_0 < 0$, i.e. in the asymptotically free case. In QED on the other hand the running coupling increases in the UV ans the series diverges, so one better consider only few leading terms and not too high energies. However the running is very slow, since it is logarithmic. Therefore long before the theory enters the strong coupling regime, one enters the electroweak scale where one has to consult the Standard Model.

10 Path Integral

Path integral formulation of quantum mechanics leads to the same QFT results as the Hilbert space formalism. It corresponds to the well known duality between the wave-function formalism and operator formalism in quantum mechanics. We non-trivially rewrite the transition amplitude as a path integral:

$$\langle q_F, t_F | q_I, t_I \rangle = \int_{q(t_I)=q_I}^{q(t_F)=q_F} \mathcal{D}q(t)\mathcal{D}p(t) exp\left\{ i \int_{t_I}^{t_F} dt(p\dot{q} - H(p,q)) \right\} \tag{337}$$

In case $H = p^2/2 + V(q)$, the path integral is:

$$\langle q_F, t_F | q_I, t_I \rangle = \int_{q(t_I)=q_I}^{q(t_F)=q_F} \mathcal{D}q(t) exp\left\{ \frac{i}{\hbar} S \right\} \tag{338}$$

Where integration over p was absorbed into the integral measure $\mathcal{D}q(t)$. S is the classical action

$$S = \int_{T_i}^{T_f} dt L(q,\dot{q}), \qquad L = \frac{1}{2}\dot{q}^2 - V(q) \tag{339}$$

So the transition amplitude is the sum over all paths $q(t)$ which satisfy the initial and final boundary conditions. The particla takes all paths simultaneously and the quantum amplitude is obtained by summing up all trajectories. One easily recovers the classical limit: only the trajectories, which have stationary action, i.e. action that changes little under small path deformation contribute in the limit $\hbar \to 0$.

$$\frac{\partial}{\partial q} S = 0 \tag{340}$$

All other trajectories oscillate largely and therefore cancel each other. The correlation functions become:

$$\langle q_F, t_F | \, \hat{q}(t) \, | q_I, t_I \rangle = \int_{q(t_I)=q_I}^{q(t_F)=q_F} \mathcal{D}q(t) q(t) exp \left\{ \frac{i}{\hbar} S \right\} \tag{341}$$

$$\langle q_F, t_F | \, T\{\hat{q}(t_1)\hat{q}(t_2)\} \, | q_I, t_I \rangle = \int_{q(t_I)=q_I}^{q(t_F)=q_F} \mathcal{D}q(t) q(t_1) q(t_2) exp \left\{ \frac{i}{\hbar} S \right\} \tag{342}$$

So the time-ordering appears automatically.

10.1 Free scalar theory

From now on we again drop $\hbar$. For the vacuum correlation function (Green's function) in the free scalar theory one gets:

$$\langle 0 | \, T\hat{\phi}(x_1)...\hat{\phi}(x_n) \, | 0 \rangle = \frac{\int \mathcal{D}\phi \; \phi(x_1)...\phi(x_n) e^{iS}}{\int \mathcal{D}\phi e^{iS}} \tag{343}$$

Recall the analogous procedure in QFT1, where we found the vacuum denominator, observing that in the limit $t_0 = -t$, $t \to \infty$ we are left with the term $U(\infty, -\infty) \, |0\rangle$ which is a evolution of vacuum and should again give the vacuum. We then observed that vacuum state is defined up to a phase $e^{i\alpha}$ and this phase is the denominator. Here the denominator is obtained more handsomely: you take the prescription for the transition amplitudes, but now with coordinates q replaced by the field ϕ. Then if one chooses the vacuum as the initial and final state, $\phi_i(x) = \phi_f(x) = 0$ and integrate over the complete time dimension $-\infty$ to ∞, one gets for the vacuum -to-vacuum transition amplitude:

$$\langle 0, t = \infty | 0, t = -\infty \rangle = \int \mathcal{D}\phi e^{iS} \tag{344}$$

where the integral is performed over all field configurations that vanish at $t \to \pm\infty$. The denominator can be considered as a simple normalization factor and may be missing, but is implied, in the following sections.
The path integral is defined on a regularized theory with IR und UV cutoff. The IR divergence is cured by a finite V, the UV diverence can be cured by discretizing the space-time.

Thus we are left with the 3-dimensional Quantum Mechanics with finite degrees of freedom, where the time can be taken continuously by Wick rotation. The way to compute the path integral non-perturbatively is **Lattice Field Theory**. Otherwise, taking the space continuously, one has to evaluate the path integral perturbatively and introduce the renormalization.

Consider the Greens correlation functions defined as above and the action of the free scalar field:

$$S = \frac{1}{2}\int d^4x \left(\partial^\mu\phi\partial_\mu\phi - m^2\phi^2\right) \tag{345}$$

To improve the convergence one can introduce a small convergence factor $e^{-\epsilon\int d^4x\phi^2/2}$ and take the limit $\epsilon \to 0^+$. One therefore considers:

$$G(x_1, ..., x_n) = \int \mathcal{D}\phi\phi(x_1)...\phi(x_n)exp\left\{\frac{i}{2}\int d^4x[\partial^\mu\phi\partial_\mu\phi - (m^2 - i\epsilon)\phi^2]\right\} \tag{346}$$

and the convergence factor amounts to the replacement $m^2 \to m^2 - i\epsilon$, exactly as in QFT1. We next introduce:

$$W[J] = \int \mathcal{D}\phi exp\left\{\frac{i}{2}\int d^4x[\partial^\mu\phi\partial_\mu\phi - (m^2 - i\epsilon)\phi^2] + \int d^4x\phi(x)J(x)\right\} \tag{347}$$

$W[J]$ is a functional of the field J and performing the *functional derivatives* w.r.t. J, one obtains the respective Greens functions.

A short comment on functional theory. Analogous to defining in QM the state as the value of the functional/wave-function of a field $q(t)$:

$$|q_0, t\rangle = \psi(q(t)) = \delta(q(t) - q_0(t)) \tag{348}$$

A pure state is a "collaps" of the continuous-valued wave-function to a delta function which returns a number. The same, one can see the QFT states and scalar quantities as the wavefunctional of the QFT field:

$$|\phi_0(x,t)\rangle = \Psi[\phi(t)] = \delta[\phi(x,t) - \phi_0(x,t)] \tag{349}$$

The *functional derivative* is defined as:

$$\frac{\delta}{\delta J}J(x) = \delta^{(4)}(x - y) \tag{350}$$

plus the standard rules for the derivative of composite functions and by the rule that we can carry it inside the integral sign:

$$\frac{\delta}{\delta J(y)}\int d^4x J(x)\phi(x) = \phi(y) \tag{351}$$

Therefore it is straightforward to see:

$$G(x_1,...x_n) = \left(\frac{\delta}{\delta J(x_1)}...\frac{\delta}{\delta J(x_n)}W[J]\right)_{|J=0} \tag{352}$$

For this reason $W[J]$ is called the *generating functional* of the Greens functions.
We now compute $W[J]$ explicitly to be able to evaluate the Greens function. First of all we write in momentum space using:

$$\begin{aligned}&\frac{i}{2}\int d^4x\left(\partial^\mu\phi\partial_\mu\phi-(m^2-i\epsilon)\phi^2\right)+\int d^4x\phi(x)J(x) \\ &=\frac{1}{2}\int\frac{d^4p}{(2\pi)^4}\left(\tilde{\phi}(-p)i(p^2-m^2+i\epsilon)\tilde{\phi}(p)+\tilde{J}(-p)\tilde{\phi}(p)\right)\end{aligned} \tag{353}$$

We recognize a quadratic form in $\tilde{\phi}$ and the integral is a Gaussian integral. One discretizes the integral by working in a finite space-time volume. Therefore the four-momenta are discrete and the integrals are repaced by sums. The integration over all possible functions $\phi(x)$ can be written as an integration over all possible Fourier modes $\tilde{\phi}(x)$:

$$\mathcal{D}\phi = \prod_p d\tilde{\phi}(p) \tag{354}$$

Proportionality factors are irrelevant, since they cancel in the denominator of the Greens function. This equation can be taken as the definition of the integral measure. There is a useful identity for the multivariate Gaussian:

$$\begin{aligned}&\int_{-\infty}^{+\infty}\prod_{i=1}^{N}dy_i\exp\left[-\frac{1}{2}\sum_{i,j=1}^{N}y_iA_{ij}y_j+\sum_{i=1}^{N}y_iz_i\right] \\ &=(2\pi)^{N/2}(\det A)^{-1/2}\exp\left[+\frac{1}{2}\sum_{i,j=1}^{N}z_i(A^{-1})_{ij}z_j\right]\end{aligned} \tag{355}$$

with A_{ij} an invertible semi-definite matrix. Coming back to the continuum notation, one immediately recovers the Feynman propagator with:

$$W[J] = W[0]\exp\left\{\frac{1}{2}\int\frac{d^4p}{(2\pi)^4}\tilde{J}(-p)\tilde{D}(p)\tilde{J}(p)\right\} \tag{356}$$

where

$$W[0] = \int \mathcal{D}\phi e^{iS} \tag{357}$$

and

$$\tilde{D}(p) = \frac{i}{p^2 - m^2 + i\epsilon} \tag{358}$$

Therefore the propagator is simply the inverse of the kinetic term. We write $W[J]$ in the coordinate space:

$$W[J] = W[0] \exp\left\{\frac{1}{2}\int d^4x d^4y J(x) D(x-y) J(y)\right\} \tag{359}$$

Now we can find all the Green's functions of the free scalar theory. For the two-point function one finds:

$$\frac{G(x_1, x_2)}{\int \mathcal{D}\phi e^{iS}} = D(x_1 - x_2) \tag{360}$$

Taking multiple derivatives, one obtains all higher-point Green's functions. For the four-point Green's function one finds:

$$\frac{G(x_1, ..., x_4)}{W[0]} = D_{12}D_{34} + D_{13}D_{24} + D_{14}D_{23} \tag{361}$$

By introducing:

$$Z[J] = \log W[J] \tag{362}$$

one has in free theory:

$$Z[J] = Z[0] + \frac{1}{2}\int d^4 d^4y J(x) D(x-y) J(y) \tag{363}$$

and therefore $Z[J]$ is the generating functional of the *connected* Green's functions, which is in the free case of course only the two-point function. The interactions are introduces perturbatively by a term $V(\phi) = \lambda\phi^4/4!$. The path integral becomes:

$$\begin{gathered} \int \mathcal{D}\phi \phi(x_1) ... \phi(x_n) \\ \times \exp\left\{ i \int d^4x \left[\frac{1}{2}(\partial^\mu\phi\partial_\mu\phi - (m^2 - i\epsilon)\phi^2) - \frac{\lambda}{4!}\phi^4\right]\right\} \end{gathered} \tag{364}$$

It is non-Gaussian and can be evaluated only perturbatively in λ. Using $Z[J]$ one again gets the *connected* diagrams. By taking an interacting action and evaluating the path integral perturatively, one recovers again the whole set of Feynman rules.

10.2 Euclidean formulation

The perturbative expansion reproduces the expansion in terms of Feynman graphs, but it is not the end of the story. The greatest virtue of the path integral is its intrinsical non-perturbativeness and it allows to non-perturbatively *define* the field theory and to compute the non-perturbative effects. To this end one ensures the convergence of the path integral in another way: define the Euclidean time $t_E = it$ such that now the metric becomes $(1, 1, 1, 1,)$. Then $d^4x = -i(d^4x)_E$ and $\frac{\partial}{\partial t} = i\frac{\partial}{\partial t_E}$. We find also: $S = iS_E$:

$$\begin{aligned} S &= \int d^4x \left[\frac{1}{2}(\partial^\mu\phi\partial_\mu\phi - m^2\phi^2) - V(\phi)\right] && (365) \\ &= \int d^4x \left[\frac{1}{2}(\partial_t\phi)^2 - \frac{1}{2}(\partial_i\phi)^2 - \frac{1}{2}m^2\phi^2 - V(\phi)\right] \\ &= -i\int (d^4x)_E \left[-\frac{1}{2}(\partial_{t_E}\phi)^2 - \frac{1}{2}(\partial_i\phi)^2 - \frac{1}{2}m^2\phi^2 - V(\phi)\right] && (366) \\ &= iS_E \end{aligned}$$

So the Euclidean action is:

$$S_E = \int d^4x \left[\frac{1}{2}\partial_\mu\phi\partial_\mu\phi + \frac{1}{2}m^2\phi^2 + V(\phi)\right] \tag{367}$$

where $(d^4x)_E$ is from now on implied. Now all scalar products are calculated with the Euklidean metric, i.e. four real dinesions arre implied. The potential can be shifted arbitrary to achieve that the Euklidean action is positive definite and the factor:

$$e^{iS} = e^{-S_E} \tag{368}$$

ensures convergence of the integration over large fluctuations, since paths with a large action are exponentially suppressed. We find also:

$$\begin{aligned} G_E(\phi(x_1), ..., \phi(x_n)) &= \int \mathcal{D}\phi\phi(x_1)...\phi(x_n)e^{-S} && (369) \\ \tilde{D}_E(p) &= \frac{1}{p^2 + m^2} \end{aligned}$$

where $p^2 = p_0^2 + \mathbf{p}^2$. And each vertex carries a factor $-\lambda$ instead of $-i\lambda$. The Euclidean Green's functions are afterwards mapped back to the Minkowski space by an analytic continuation. The path integral in the Euclidean space is solved through a discretization on a grid or by a Monte-Carlo simulation.

10.3 Critical phenomena

Looking at the *correlation function*:

$$\frac{\int \mathcal{D}\phi\phi(x_1)...\phi(x_n)e^{-S}}{\int \mathcal{D}\phi e^{-S}} \tag{370}$$

one recognizes a statistical average of a classical system in a four dimensional space. The two-point function in a massive theory is:

$$\langle\phi(x)|\phi(0)\rangle = \int \frac{d^4p}{(2\pi)^4}\frac{e^{ipx}}{p^2+m^2} \tag{371}$$

Consider a system on a lattice with the lattice side a s.t. $x = na$. Consider also $m|x| >> 1$. Then

$$\langle\phi(x)|\phi(0)\rangle \sim e^{-amn} = e^{-n/\chi} \tag{372}$$

defining the *correlation length* χ. By taking the continuum limit while keeping m fixed, the correlation length diverges. One has to adjust the couplings to obtain finite quantities. Since the divergent correlation length is characteristic for critical systems, the removal of the cutoff in Quantum Field Theory is equivalent to tuning the statistical system towards a critical point. Consider the easy example of the Ising model. Below the critical temperature there is a spontaneous magnitization, so that the spins correlate even at infinite distance, the correlation length is infinite. Therefore the system is the same on all scales. Take a manetized condensed matter sample. Obviously since we are dealing with infinitely many degrees of freedom we have to take the situation on a scale, where we can average largely. Then physics beyond the atomic spacing cannot be described with the macroscopic Hamiltonian only, one has to beyond the model and explore the alternatives.

References

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