Finite Difference Method (FDM)

: A numerical method of solving partial differential equations

- 1-dimensional Poisson equation

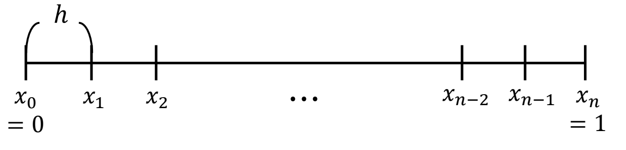
$$\begin{cases} -u''(x) = f(x)\,, & 0 < x < 1 \\ u(0) = u(1) = 0 \; : & boundary\ conditions \end{cases}$$

Exact solution : $u(x) = \sin \pi x \quad \rightarrow \quad f(x) = \pi^2 \sin \pi x$

- Finite difference method

GOAL : Find the approximations to the solution $u(x)$ at $x = x_j\,, j = 0,1,\cdots,n$

1. Discretize the domain using uniform mesh $\boldsymbol{h = \frac{1}{n}}$.



We have $n+1$ nodes : $x_j = x_0 + jh = jh \quad (j = 0, 1, \cdots, n)$

2. Discretize the derivative.

The derivatives are approximated using Taylor expansion : $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$.

- The first order derivative $u'(x)$

$$\begin{aligned} u(x+h) &= u(x) + u'(x)h + \frac{u''(x)}{2} h^2 + \cdots \\ &= u(x) + u'(x)h + O(h^2) \rightarrow 0 \;\; as \;\; h \rightarrow 0 \end{aligned}$$

$$\Rightarrow \quad u'(x) \approx \frac{u(x+h) - u(x)}{h}$$: forward difference approximation

- The first order derivative $u'(x)$

$$u(x-h) = u(x) - u'(x)h + O(h^2)$$

$$\Rightarrow \quad u'(x) \approx \frac{u(x) - u(x-h)}{h}$$: backward difference approximation

- The second order derivative $u''(x)$

$$u''(x) \overset{\text{forward}}{\approx} \frac{u'(x+h)-u'(x)}{h} \overset{\text{backward}}{\approx} \frac{1}{h}\left(\frac{u(x+h)-u(x)}{h} - \frac{u(x)-u(x-h)}{h}\right)$$

$$\approx \frac{u(x+h)-2u(x)+u(x-h)}{h^2}$$: centered difference approximation

$$\left(\begin{array}{ll} -\dfrac{u(x_j+h)-2u(x_j)+u(x_j-h)}{h^2} = f(x_j) & j = 1,\cdots,n-1 \\ u(x_0) = u(x_n) = 0 & \end{array}\right)$$

3. Set up the algebraic system.

Let u_j be an approximation to $u(x_j)$ and $f_j = f(x_j)$ for $j = 0,\cdots,n$.

$$\frac{u_{j+1}-2u_j+u_{j-1}}{h^2} = -f_j \qquad j = 1,\cdots,n-1$$

$$j=1: \quad \frac{\cancelto{0}{u_0}-2u_1+u_2}{h^2} = -f_1$$

$$j=2: \quad \frac{u_1-2u_2+u_3}{h^2} = -f_2$$

$$\vdots$$

$$j=n-1: \quad \frac{u_{n-2}-2u_{n-1}+\cancelto{0}{u_n}}{h^2} = -f_{n-1}$$

$\Rightarrow$

$u_0 = u_n = 0,\ f_j$: knowns

$(n-1)$ unknowns : $u_1,\cdots,u_{n-1}$

$(n-1)$ equations

Construct the linear system!

$$\underbrace{\frac{1}{h^2}\begin{bmatrix} -2 & 1 & 0 & \cdots & & & \\ 1 & -2 & 1 & 0 & & & \\ 0 & 1 & -2 & 1 & & & \\ & & & & \ddots & & \\ & & & & 1 & -2 & 1 \\ & & & & 0 & 1 & -2 \end{bmatrix}}_{\boldsymbol{A}} \underbrace{\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{n-2} \\ u_{n-1} \end{bmatrix}}_{\boldsymbol{u}} = \underbrace{\begin{bmatrix} -f_1 \\ -f_2 \\ \vdots \\ -f_{n-2} \\ -f_{n-1} \end{bmatrix}}_{\boldsymbol{b}}$$

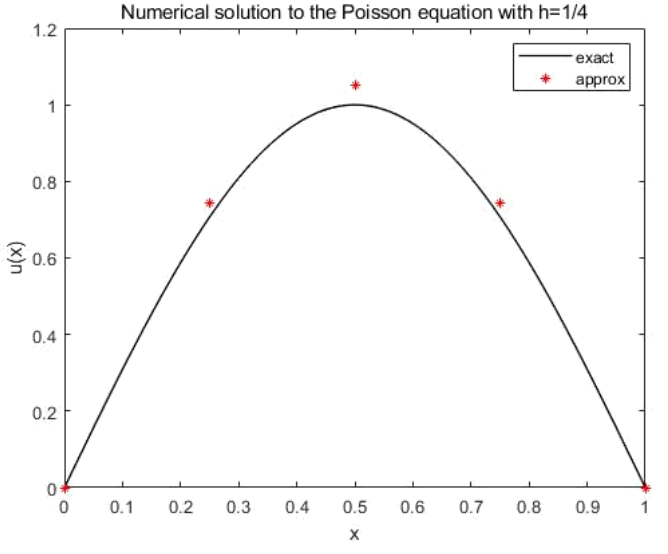
(tridiagonal, symmetric, sparse)

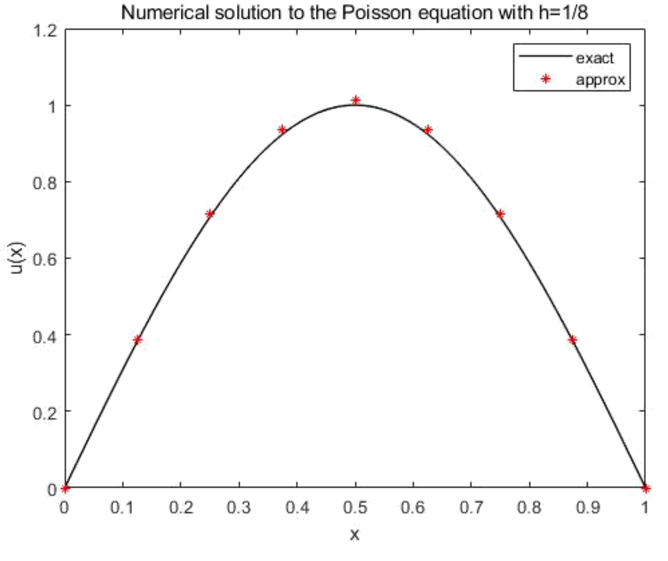
$$\therefore\ u = A^{-1}b$$

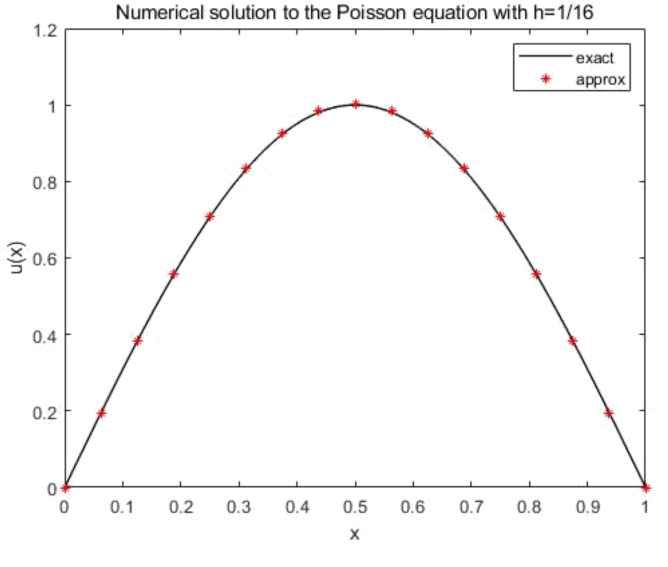
Solve the linear system $Au = b$ $\Leftrightarrow$ Find the approximations to $u(x_0),\cdots,u(x_n)$

- Numerical results

h	$1/4$	$1/8$	$1/16$
u_0	0	0	0
u_1	0.7446	0.3876	0.1957
u_2	1.0530	0.7163	0.3839
u_3	0.7446	0.9358	0.5574
u_4	0	1.0130	0.7094
u_5		0.9358	0.8341
u_6		0.7163	0.9269
u_7		0.3876	0.9839
u_8		0	1.0032
u_9			0.9839
u_{10}			0.9269
u_{11}			0.8341
u_{12}			0.7094
u_{13}			0.5574
u_{14}			0.3839
u_{15}			0.1957
u_{16}			0







- Vector calculus

  - Gradient : the direction in which $v(\mathbf{x})$ increases most quickly from the point $\mathbf{x}$.

$$\nabla v(\mathbf{x}) = \left(\frac{\partial v}{\partial x_1}, \cdots, \frac{\partial v}{\partial x_n}\right)^T \qquad \mathbf{x} = (x_1, \cdots, x_n)$$

v : scalar-valued function

  - Divergence : the extent to which a vector field at $\mathbf{x}$ spreads out.

$$\nabla \cdot \mathbf{V} = \frac{\partial V_1}{\partial x_1} + \cdots + \frac{\partial V_n}{\partial x_n} \qquad \mathbf{V} = \left(V_1(\mathbf{x}), \cdots, V_n(\mathbf{x})\right)$$

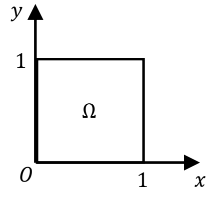
: vector-valued function

  - Laplacian/Laplace operator : the divergence of gradient of v (diffusion).

$$\nabla^2 v = \nabla \cdot \nabla v = \left(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n}\right) \cdot \left(\frac{\partial v}{\partial x_1}, \cdots, \frac{\partial v}{\partial x_n}\right)^T = \frac{\partial^2 v}{\partial x_1^2} + \cdots + \frac{\partial^2 v}{\partial x_n^2}$$

- 2-dimensional Poisson equation

$$\begin{cases} -\nabla^2 u = f & in \quad \Omega = (0,1)^2 \\ \quad u = 0 & on \quad \Gamma \coloneqq \partial\Omega \end{cases}$$



Example) Calculate $f(x,y)$, assuming the exact solution is $u(x,y) = \sin \pi x \, sin\, \pi y$.
Let us check if $u(x,y)$ satisfies the boundary condition.

$$\begin{aligned} u(x,0) &= \sin \pi x \sin 0 = 0 \quad \text{on } (0,1) \times \{y=0\} \\ u(1,y) &= \sin \pi \sin \pi y = 0 \quad \text{on } \{x=1\} \times (0,1) \\ u(x,1) &= \sin \pi x \sin \pi = 0 \quad \text{on } (0,1) \times \{y=1\} \\ u(0,y) &= \sin 0 \sin \pi y = 0 \quad \text{on } \{x=0\} \times (0,1) \end{aligned}$$

Boundary value problem (BVP) is a problem of finding a solution $u(x,y)$ that satisfies the boundary condition for given $f(x,y)$. In general, we don't know the exact solution, but we compute the right-hand side f using a well-known solution for problem settings.

$$\frac{\partial u}{\partial x} = \pi \cos \pi x \sin \pi y \quad \rightarrow \quad \frac{\partial^2 u}{\partial x^2} = -\pi^2 \sin \pi x \sin \pi y$$

$$\frac{\partial u}{\partial y} = \pi \sin \pi x \cos \pi y \quad \rightarrow \quad \frac{\partial^2 u}{\partial y^2} = -\pi^2 \sin \pi x \sin \pi y$$

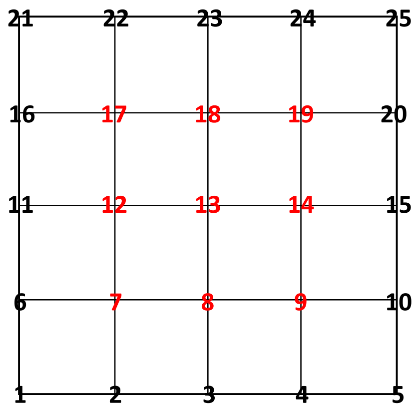
$$\Rightarrow \; f(x,y) = -\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = 2\pi^2 \sin \pi x \sin \pi y$$

- Finite difference method

GOAL : Find the approximations to the solution $u(x,y)$ at $x = x_j,\ y = y_j$.

1. Discretize the domain using uniform mesh $h = \frac{1}{4}$.

Number the nodes of two-dimensional array of size 5x5 in one direction.
Arrange the x, y coordinates of 25 nodes in order of index.



< Node generation >

$$
\begin{array}{c}
j \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ \\ 21 \\ 22 \\ 23 \\ 24 \\ 25
\end{array}
\begin{array}{c}
\begin{array}{cc} x_j & y_j \end{array} \\
\begin{bmatrix}
0 & 0 \\
0.25 & 0 \\
0.50 & 0 \\
0.75 & 0 \\
1 & 0 \\
\vdots & \vdots \\
0 & 1 \\
0.25 & 1 \\
0.50 & 1 \\
0.75 & 1 \\
1 & 1
\end{bmatrix}
\end{array}
$$

Separate the indices of nodes into two parts to impose the boundary condition.

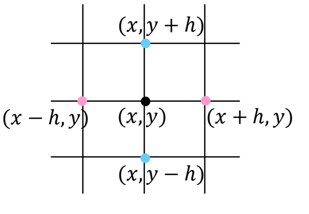
$B = \{1, 2, 3, 4, 5, 6, 10, 11, 15, 16, 20, 21, 22, 23, 24, 25\}$: 16 nodes

$I = \{7, 8, 9, 12, 13, 14, 17, 18, 19\}$: 9 nodes

2. Discretize the derivatives using centered difference approximations.

$$\frac{\partial^2 u}{\partial x^2} \approx \frac{u(x-h,y) - 2u(x,y) + u(x+h,y)}{h^2}$$

$$\frac{\partial^2 u}{\partial y^2} \approx \frac{u(x,y-h) - 2u(x,y) + u(x,y+h)}{h^2}$$



5 points!

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \approx \frac{u(x-h,y) + u(x,y-h) - 4u(x,y) + u(x+h,y) + u(x,y+h)}{h^2}$$

$$\left(\begin{array}{l}
\dfrac{\overset{\text{left}}{u(x_j-h,y_j)} + \overset{\text{down}}{u(x_j,y_j-h)} - \overset{\text{current}}{4u(x_j,y_j)} + \overset{\text{right}}{u(x_j+h,y_j)} + \overset{\text{up}}{u(x_j,y_j+h)}}{h^2} = -f(x_j,y_j), \quad j \in I \\[2ex]
\text{Boundary condition} : u = 0 \text{ on } \Gamma \quad \Rightarrow \quad u(x_j,y_j) = 0, \quad j \in B
\end{array}\right)$$

$\Rightarrow$ There are only 9 unknowns, the approximations at interior nodes.

3. Set up the linear system.

Let u_j be an approximation to $u(x_j, y_j)$ and $f_j = f(x_j, y_j)$ for $j \in I$.

$$j = 7 \ : \ \frac{\cancelto{0}{u_6} + \cancelto{0}{u_2} - 4u_7 + u_8 + u_{12}}{h^2} = -f_7$$

$$j = 8 \ : \ \frac{u_7 + \cancelto{0}{u_3} - 4u_8 + u_9 + u_{13}}{h^2} = -f_8$$

$$j = 9 \ : \ \frac{u_8 + \cancelto{0}{u_4} - 4u_9 + \cancelto{0}{u_{10}} + u_{14}}{h^2} = -f_9$$

$$j = 12 \ : \ \frac{u \quad + u \quad - 4u \quad + u \quad + u}{h^2} = -f_{12}$$

$$j = 13 \ : \ \frac{u \quad + u \quad - 4u \quad + u \quad + u}{h^2} = -f_{13}$$

$$j = 14 \ : \ \frac{u \quad + u \quad - 4u \quad + u \quad + u}{h^2} = -f_{14}$$

$$j = 17 \ : \ \frac{u \quad + u \quad - 4u \quad + u \quad + u}{h^2} = -f_{17}$$

$$j = 18 \ : \ \frac{u \quad + u \quad - 4u \quad + u \quad + u}{h^2} = -f_{18}$$

$$j = 19 \ : \ \frac{u \quad + u \quad - 4u \quad + u \quad + u}{h^2} = -f_{19}$$

$\Rightarrow$

$u_B, \ f_j$: knowns

9 unknowns : $u_7, \cdots, u_{19}$

9 equations

Construct the linear system!

$A \ : 9 \times 9 \ \ \& \ \ u, b : 9 \times 1$

$$\frac{1}{h^2}\begin{bmatrix} & & & & & & & & \\ & & & & & & & & \\ & & & & & & & & \\ & & & & & & & & \\ & & & & & & & & \\ & & & & & & & & \\ & & & & & & & & \\ & & & & & & & & \\ & & & & & & & & \end{bmatrix} \begin{bmatrix} u_7 \\ u_8 \\ u_9 \\ u_{12} \\ u_{13} \\ u_{14} \\ u_{17} \\ u_{18} \\ u_{19} \end{bmatrix} = \begin{bmatrix} -f_7 \\ -f_8 \\ -f_9 \\ -f_{12} \\ -f_{13} \\ -f_{14} \\ -f_{17} \\ -f_{18} \\ -f_{19} \end{bmatrix}$$

$\boldsymbol{A}$ (symmetric, sparse, 5 diagonals, nine 3x3 blocks) $\quad \boldsymbol{u} \quad \boldsymbol{b}$

$$\therefore \ u = A^{-1} b$$

Solve the linear system $Au = b$ $\Leftrightarrow$ Find the approximations to $u(x_1, y_1), \cdots, u(x_{25}, y_{25})$

3. Set up the linear system.

Let u_j be an approximation to $u(x_j, y_j)$ and $f_j = f(x_j, y_j)$ for $j \in I$.

$$j = 7 \ : \ \frac{\overset{0}{\cancel{u_6}} + \overset{0}{\cancel{u_2}} - 4u_7 + u_8 + u_{12}}{h^2} = -f_7$$

$$j = 8 \ : \ \frac{u_7 + \overset{0}{\cancel{u_3}} - 4u_8 + u_9 + u_{13}}{h^2} = -f_8$$

$$j = 9 \ : \ \frac{u_8 + \overset{0}{\cancel{u_4}} - 4u_9 + \overset{0}{\cancel{u_{10}}} + u_{14}}{h^2} = -f_9$$

$$j = 12 \ : \ \frac{u_{11} + u_7 - 4u_{12} + u_{13} + u_{17}}{h^2} = -f_{12}$$

$$j = 13 \ : \ \frac{u_{12} + u_8 - 4u_{13} + u_{14} + u_{18}}{h^2} = -f_{13}$$

$$j = 14 \ : \ \frac{u_{13} + u_9 - 4u_{14} + u_{15} + u_{19}}{h^2} = -f_{14}$$

$$j = 17 \ : \ \frac{u_{16} + u_{12} - 4u_{17} + u_{18} + u_{22}}{h^2} = -f_{17}$$

$$j = 18 \ : \ \frac{u_{17} + u_{13} - 4u_{18} + u_{19} + u_{23}}{h^2} = -f_{18}$$

$$j = 19 \ : \ \frac{u_{18} + u_{14} - 4u_{19} + u_{20} + u_{24}}{h^2} = -f_{19}$$

$\Rightarrow$

u_B, f_j : knowns

9 unknowns : $u_7, \cdots, u_{19}$

9 equations

Construct the linear system!

A : 9×9 & u, b : 9×1

$$\frac{1}{h^2}\left[\begin{array}{ccc:ccc:ccc}
-4 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & -4 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & -4 & 0 & 0 & 1 & 0 & 0 & 0 \\
\hdashline
1 & 0 & 0 & -4 & 1 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 1 & -4 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 & -4 & 0 & 0 & 1 \\
\hdashline
0 & 0 & 0 & 1 & 0 & 0 & -4 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & -4 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & -4
\end{array}\right]
\left[\begin{array}{c}
u_7 \\ u_8 \\ u_9 \\ \hdashline u_{12} \\ u_{13} \\ u_{14} \\ \hdashline u_{17} \\ u_{18} \\ u_{19}
\end{array}\right]
=
\left[\begin{array}{c}
-f_7 \\ -f_8 \\ -f_9 \\ -f_{12} \\ -f_{13} \\ -f_{14} \\ -f_{17} \\ -f_{18} \\ -f_{19}
\end{array}\right]$$

$\boldsymbol{A}$ $\qquad\qquad$ $\boldsymbol{u}$ $\qquad\qquad$ $\boldsymbol{b}$

(symmetric, sparse, 5 diagonals, nine 3x3 blocks)

$$\therefore \ u = A^{-1}b$$

Solve the linear system $Au = b$ $\Leftrightarrow$ Find the approximations to $u(x_1, y_1), \cdots, u(x_{25}, y_{25})$

- Numerical results