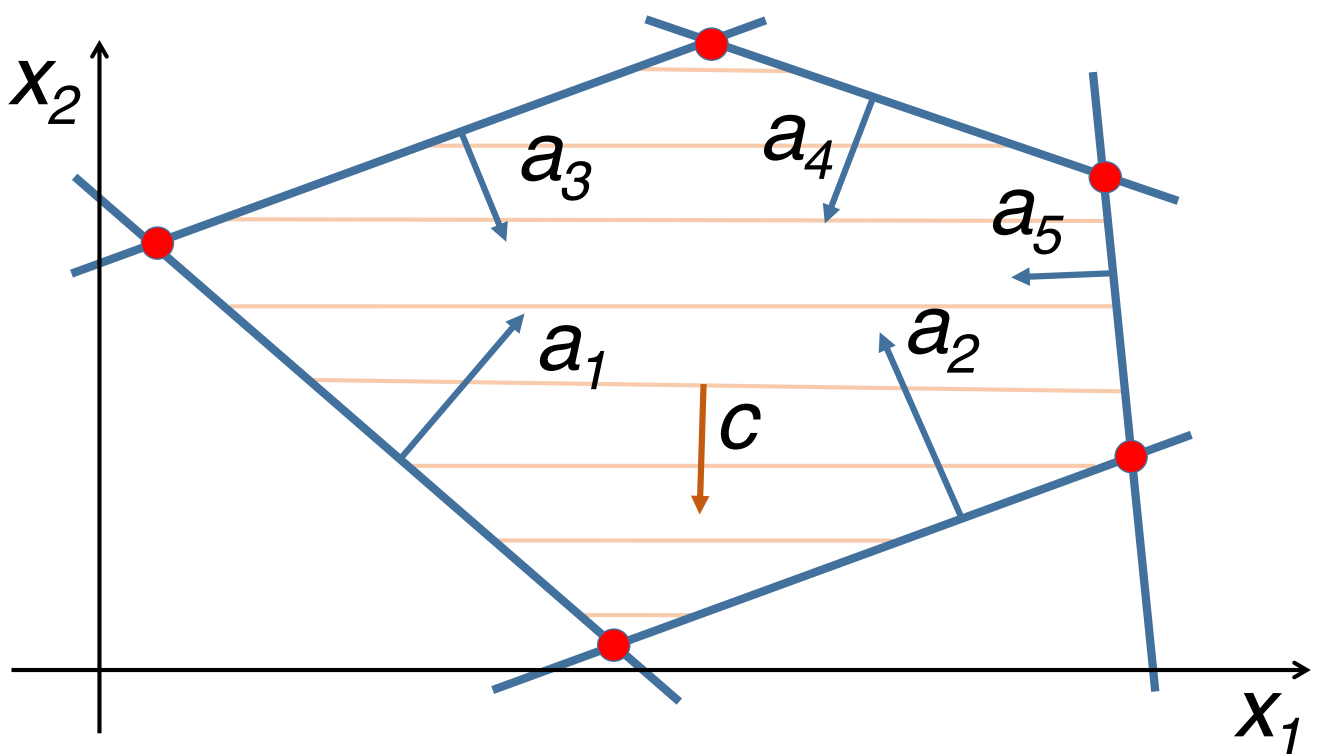


What is LP

Generally speaking, all problems with linear objective and linear equalities/inequalities constraints could be considered as Linear Programming. However, there are some widely accepted formulations.

$$\begin{aligned} \min_{x \in \mathbb{R}^n} & c^\top x \\ \text{s.t. } & Ax \leq b \end{aligned} \quad (\text{LP.Basic})$$



for some vectors $c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$ and matrix $A \in \mathbb{R}^{m \times n}$. Where the inequalities are interpreted component-wise.

Standard form

This form seems to be the most intuitive and geometric in terms of visualization. Let us have vectors $c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$ and matrix $A \in \mathbb{R}^{m \times n}$.

$$\begin{aligned} \min_{x \in \mathbb{R}^n} & c^\top x \\ \text{s.t. } & Ax = b \\ & x_i \geq 0, \quad i = 1, \dots, n \end{aligned} \quad (\text{LP.Standard})$$

Canonical form

$$\begin{aligned} \min_{x \in \mathbb{R}^n} c^\top x \\ \text{s.t. } Ax \leq b \\ x_i \geq 0, i = 1, \dots, n \end{aligned} \quad (\text{LP. Canonical})$$

Real world problems

Diet problem

Imagine, that you have to construct a diet plan from some set of products: 🍌🍰🍗🥚🐟. Each of the products has its own vector of nutrients. Thus, all the food information could be processed through the matrix W . Let also assume, that we have the vector of requirements for each of nutrients $r \in \mathbb{R}^n$. We need to find the cheapest configuration of the diet, which meets all the requirements:

$$\begin{aligned} \min_{x \in \mathbb{R}^p} c^\top x \\ \text{s.t. } Wx \geq r \\ x_i \geq 0, i = 1, \dots, p \end{aligned}$$



$$W \in \mathbb{R}^{n \times p},$$

Requirements

$$r \in \mathbb{R}^n$$

Proteins
Carbs
Fats
Calories
Vitamin D

$c \in \mathbb{R}^p$ - cost per 100 g

$$\begin{aligned} \min_{x \in \mathbb{R}^p} c^\top x \\ Wx \geq r \end{aligned}$$

How to retrieve LP

Basic transformations

Inequality to equality by increasing the dimension of the problem by m .

$$Ax \leq b \leftrightarrow \begin{cases} Ax + z = b \\ z \geq 0 \end{cases}$$

unsigned variables to nonnegative variables.

$$x \leftrightarrow \begin{cases} x = x_+ - x_- \\ x_+ \geq 0 \\ x_- \geq 0 \end{cases}$$

Chebyshev approximation problem

$$\min_{x \in \mathbb{R}^n} \|Ax - b\|_\infty \leftrightarrow \min_{x \in \mathbb{R}^n} \max_i |a_i^\top x - b_i|$$

$$\begin{aligned} & \min_{t \in \mathbb{R}, x \in \mathbb{R}^n} t \\ \text{s.t. } & a_i^\top x - b_i \leq t, \quad i = 1, \dots, n \\ & -a_i^\top x + b_i \leq t, \quad i = 1, \dots, n \end{aligned}$$

l_1 approximation problem

$$\min_{x \in \mathbb{R}^n} \|Ax - b\|_1 \leftrightarrow \min_{x \in \mathbb{R}^n} \sum_{i=1}^n |a_i^\top x - b_i|$$

$$\begin{aligned} & \min_{t \in \mathbb{R}^n, x \in \mathbb{R}^n} \mathbf{1}^\top t \\ \text{s.t. } & a_i^\top x - b_i \leq t_i, \quad i = 1, \dots, n \\ & -a_i^\top x + b_i \leq t_i, \quad i = 1, \dots, n \end{aligned}$$

Idea of simplex algorithm

- The Simplex Algorithm walks along the edges of the polytope, at every corner choosing the edge that decreases $c^\top x$ most
- This either terminates at a corner, or leads to an unconstrained edge ($-\infty$ optimum)

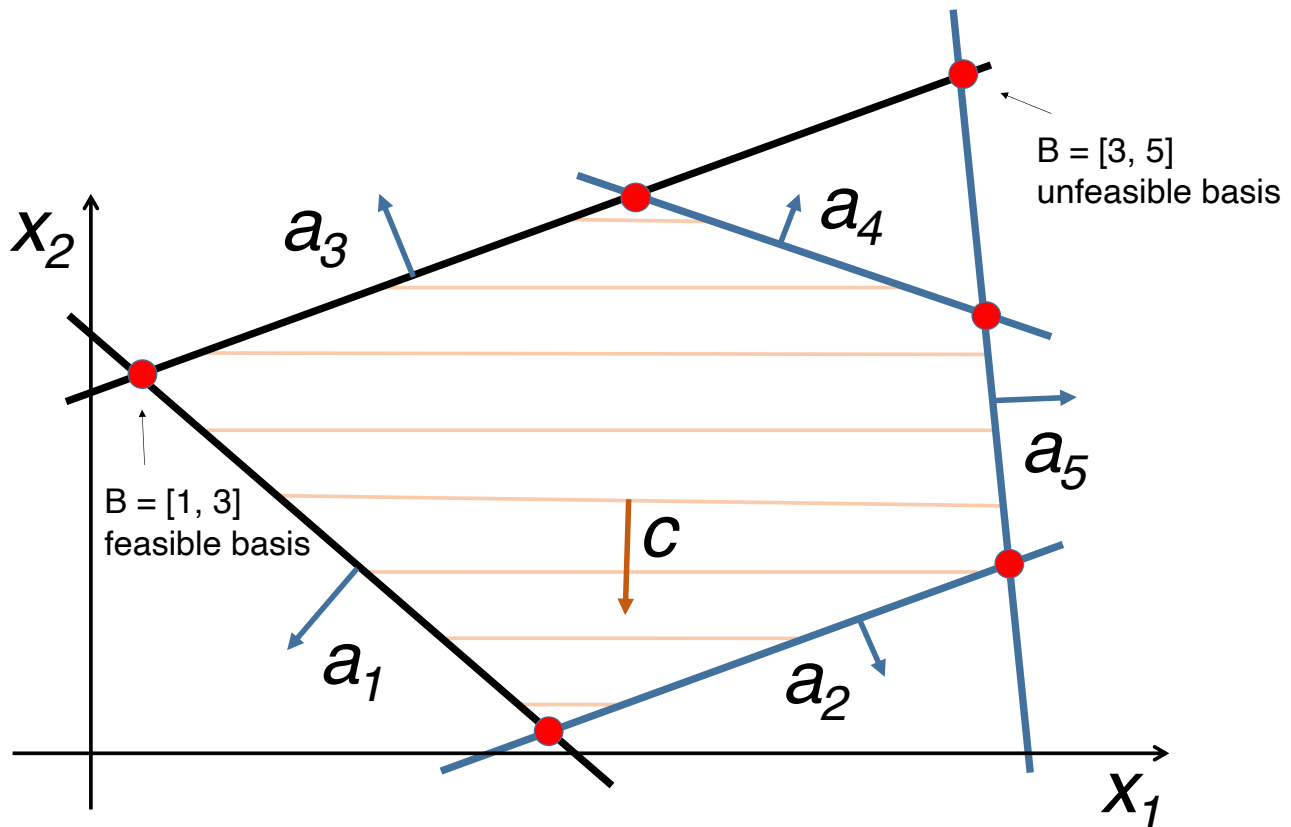
We will illustrate simplex algorithm for the simple inequality form of LP:

$$\begin{aligned} & \min_{x \in \mathbb{R}^n} c^\top x \\ \text{s.t. } & Ax \leq b \end{aligned} \quad (\text{LP.Inequality})$$

Definition: a **basis** B is a subset of n (integer) numbers between 1 and m , so that $\text{rank} A_B = n$. Note, that we can associate submatrix A_B and corresponding right-hand side b_B with the basis B . Also, we can derive a point of intersection of all these hyperplanes from basis: $x_B = A_B^{-1}b_B$.

If $Ax_B \leq b$, then basis B is **feasible**.

A basis B is optimal if x_B is an optimum of the LP. Inequality.



Since we have a basis, we can decompose our objective vector c in this basis and find the scalar coefficients λ_B :

$$\lambda_B^\top A_B = c^\top \leftrightarrow \lambda_B^\top = c^\top A_B^{-1}$$

Main lemma

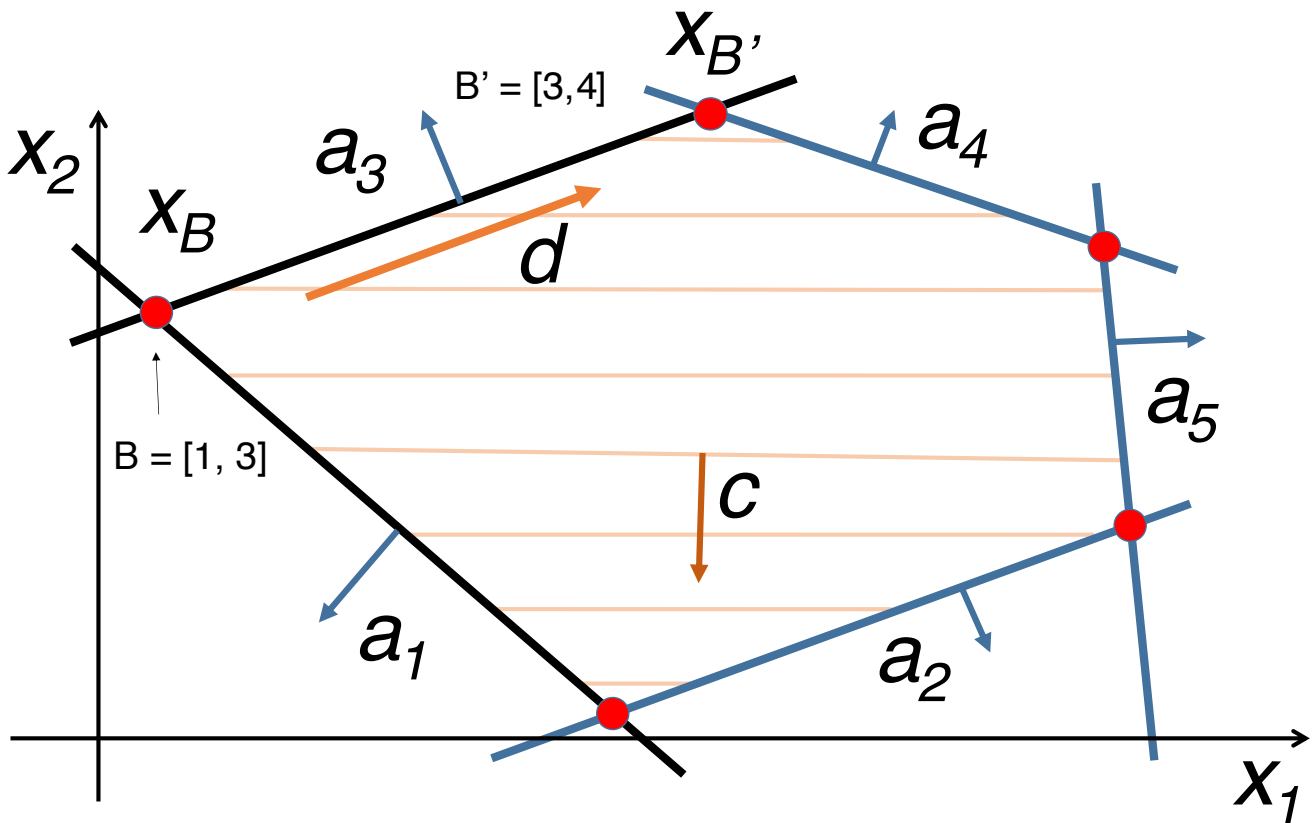
If all components of λ_B are non-positive and B is feasible, then B is optimal.

Proof:

$$\begin{aligned}
\exists x^* : Ax^* &\leq b, c^\top x^* < c^\top x_B \\
A_B x^* &\leq b_B \\
\lambda_B^\top A_B x^* &\geq \lambda_B^\top b_B \\
c^\top x^* &\geq \lambda_B^\top A_B x_B \\
c^\top x^* &\geq c^\top x_B
\end{aligned}$$

Changing basis

Suppose, some of the coefficients of λ_B are positive. Then we need to go through the edge of the polytope to the new vertex (i.e., switch the basis)



$$x_{B'} = x_B + \mu d = A_{B'}^{-1} b_{B'}$$

Finding an initial basic feasible solution

Let us consider LP.Canonical.

$$\begin{aligned}
&\min c^\top x \\
&x \in \mathbb{R}^n \\
&\text{s.t. } Ax = b \\
&x_i \geq 0, i = 1, \dots, n
\end{aligned}$$

The proposed algorithm requires an initial basic feasible solution and corresponding basis. To compute this solution and basis, we start by multiplying by -1 any row i of $Ax = b$ such that $b_i < 0$. This ensures that $b \geq 0$. We then introduce artificial variables $z \in \mathbb{R}^m$ and consider the following LP:

$$\begin{aligned} & \min_{x \in \mathbb{R}^n, z \in \mathbb{R}^m} 1^\top z \\ & \text{s.t. } Ax + Iz = b \\ & \quad x_i, z_j \geq 0, \quad i = 1, \dots, n \quad j = 1, \dots, m \end{aligned} \quad (\text{LP.Phase 1})$$

which can be written in canonical form $\min\{\tilde{c}^\top \tilde{x} \mid \tilde{A}\tilde{x} = \tilde{b}, \tilde{x} \geq 0\}$ by setting

$$\tilde{x} = \begin{bmatrix} x \\ z \end{bmatrix}, \quad \tilde{A} = [A \ I], \quad \tilde{b} = b, \quad \tilde{c} = \begin{bmatrix} 0_n \\ 1_m \end{bmatrix}$$

An initial basis for LP.Phase 1 is $\tilde{A}_B = I$, $\tilde{A}_N = A$ with corresponding basic feasible solution $\tilde{x}_N = 0$, $\tilde{x}_B = \tilde{A}_B^{-1}\tilde{b} = \tilde{b} \geq 0$. We can therefore run the simplex method on LP.Phase 1, which will converge to an optimum \tilde{x}^* . $\tilde{x} = (\tilde{x}_N \ \tilde{x}_B)$. There are several possible outcomes:

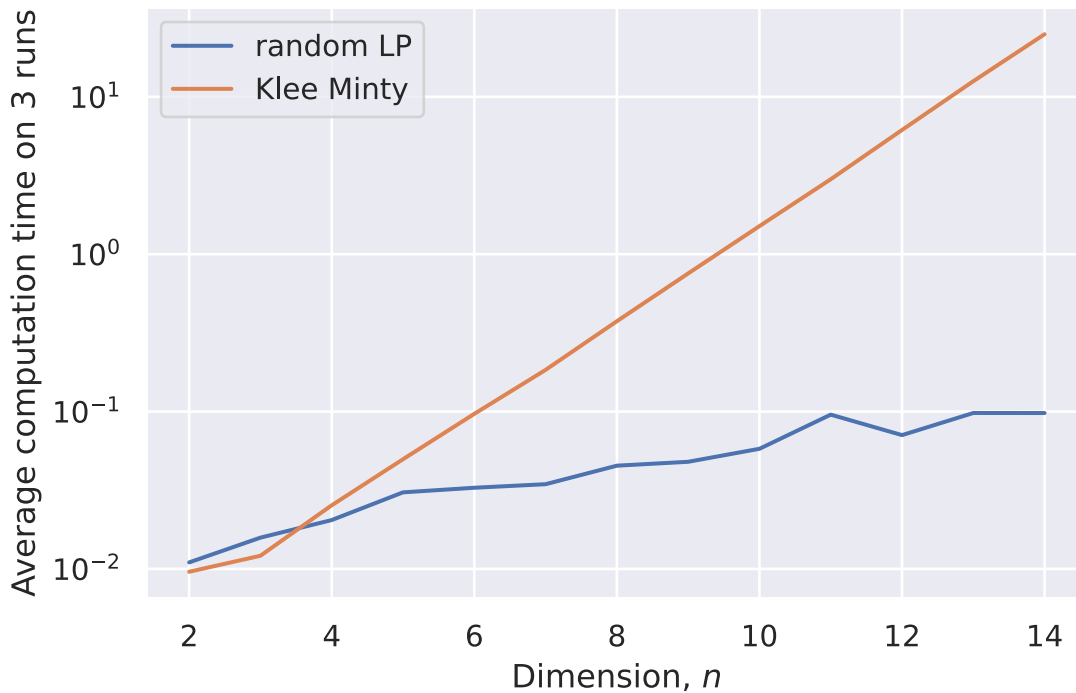
- $\tilde{c}^\top \tilde{x} > 0$
 . Original primal is infeasible.
- $\tilde{c}^\top \tilde{x} = 0 \rightarrow 1^\top z^* = 0$
 . The obtained solution is a start point for the original problem (probably with slight modification).

Convergence

Klee Minty example

In the following problem simplex algorithm needs to check $2^n - 1$ vertexes with $x_0 = 0$.

$$\begin{aligned} & \max_{x \in \mathbb{R}^n} 2^{n-1}x_1 + 2^{n-2}x_2 + \dots + 2x_{n-1} + x_n \\ & \text{s.t. } x_1 \leq 5 \\ & \quad 4x_1 + x_2 \leq 25 \\ & \quad 8x_1 + 4x_2 + x_3 \leq 125 \\ & \quad \dots \\ & \quad 2^n x_1 + 2^{n-1}x_2 + 2^{n-2}x_3 + \dots + x_n \leq 5^n \quad x \geq 0 \end{aligned}$$



Strong duality

There are four possibilities:

- Both the primal and the dual are infeasible.
- The primal is infeasible and the dual is unbounded.
- The primal is unbounded and the dual is infeasible.
- Both the primal and the dual are feasible and their optimal values are equal.

Summary

- A wide variety of applications could be formulated as the linear programming.
- Simplex algorithm is simple, but could work exponentially long.
- Khachiyan's ellipsoid method is the first to be proved running at polynomial complexity for LPs. However, it is usually slower than simplex in real problems.
- Interior point methods are the last word in this area. However, good implementations of simplex-based methods and interior point methods are similar for routine applications of linear programming.

Code

Materials

- [Linear Programming](#). in V. Lempitsky optimization course.
- [Simplex method](#). in V. Lempitsky optimization course.
- [Overview of different LP solvers](#)
- [TED talks watching optimization](#)
- [Overview of ellipsoid method](#)
- [Comprehensive overview of linear programming](#)
- [Converting LP to a standard form](#)