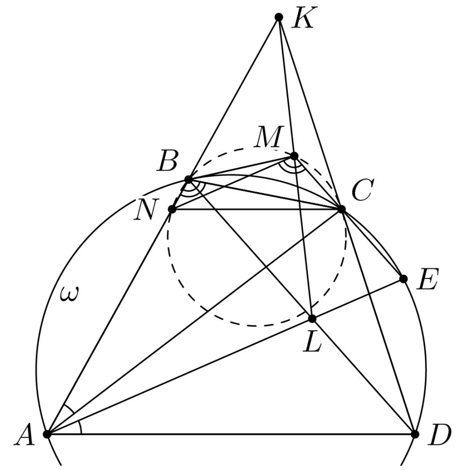
XXI Silk Road Mathematical Competition
March 2022

Attention! *We ask you not to* ***disclose*** *these problems and not to discuss them publicly (especially through Internet) before May 25, 2022.*

Solutions and marking schemes

Problem №1. Convex quadrilateral $ABCD$ is inscribed in circle ω. Rays AB and DC intersect at K. L is chosen on the diagonal BD so that $\angle BAC = \angle DAL$. M is chosen on the segment KL so that $CM \parallel BD$. Prove that the line BM touches ω. *(Kungozhin M.)*

First solution. Let N be a point on the line AK so that $MN \parallel AL$. Since $\dfrac{CM}{DL} = \dfrac{NM}{AL} = \dfrac{KM}{KL}$ and $\angle CMN = \angle DLA$, it follows that K is a center of homothety which sends $\triangle CMN$ into similar $\triangle DLA$. On the other hand, $\triangle DLA$ is similar to $\triangle CBA$ because $\angle DAL = \angle BAC$ and $\angle ADL = \angle ACB$. Consequently, $\angle(BN, BC) = \angle(MN, MC)$, and thus points N, B, M, C are cyclic. Therefore, $\angle CBM = \angle CNM = \angle CAB$ from which it follows that BM touches ω.



Second solution. For this solution we need the following theorem.

Pascal's theorem. Points A, B, C, D, E, F (not necessarily in this order) lie on some circle. Then the intersections of the lines AB and DE, BC and EF, CD and FA lie on a straight line.

Back to the problem. Let E be the second intersection of the line AL and ω. Define ℓ_b as a tangent line to ω at B. Let's apply Pascal's theorem on points B, B_1, A, E, C, D (here B_1 coincides with B) and pairs of lines (BB_1, EC), (B_1A, CD), (AE, DB). These pairs of lines coincide with pairs of lines (ℓ_b, EC), (BA, CD), (AE, DB). Then from Pascal's theorem, the straight line connecting $K = BA \cap CD$ and $L = AE \cap DB$ will also contain $\ell_b \cap EC$. Since $M = EC \cap KL$, it follows that BM touches ω, as desired.

Marking scheme.

1. Unfinished bashing
(using coordinates, complex numbers, vectors, trigonometry, etc.): . . . . **0 points**
2. Proof of the similarity $\triangle ADL \sim \triangle ACB$: . . . . **1 points**
3. Proof of the similarity $\triangle CMN \sim \triangle DLA$: . . . . **3 points**
4. It is proven that the points N, B, M, C are cyclic: . . . . **2 points**
5. It is proven that the points M, C, E lie on a straight line (doesn't add up with 2): . . . . **1 points**
6. Application of Pascal's theorem without any details (depending on which collection of points to apply): **0 points**
7. Application of Pascal's theorem on wrong collection of points: . . . . **0 points**
8. Application of Pascal's theorem on the right collection of points without details on the pairs of lines: . . **2 points are deducted**
9. For the non-consideration of all possible configurations of points no points are deducted.

Problem №2. Distinct positive integers A and B are given. Prove that there exist infinitely many positive integers that can be represented both as $x_1^2 + Ay_1^2$ for some positive coprime integers x_1 and y_1, and as $x_2^2 + By_2^2$ for some positive coprime integers x_2 and y_2. *(Golovanov A.S.)*

Solution. Without loss of generality $A > B$.

Choose an arbitrary prime $p > 2$ and let's find x_1 and x_2 so that

$$x_1^2 + A(2p)^2 = x_2^2 + B(2p)^2,$$

Hence, $x_2^2 - x_1^2 = 4Cp^2$ where $C = A - B$. Set $x_1 = Cp^2 - 1$ and $x_2 = Cp^2 + 1$. If x_1 and x_2 are both odd, then they are both coprime with $y = 2p$, and we have $x_1^2 + Ay^2 = x_2^2 + By^2$. If they are both even , then $\frac{x_1}{2}$ and $\frac{x_2}{2}$ are both coprime with $y = p$, and we have $\left(\frac{x_1}{2}\right)^2 + Ay^2 = \left(\frac{x_2}{2}\right)^2 + By^2$.

The number to which we have found two such forms will be not less than p^2, thus proving there are infinitely many such numbers.

Marking scheme.

1. It is proven that if $x_1 = Cp^2 - 1$ and $x_2 = Cp^2 + 1$ are both odd, then the number $x_1^2 + Ay^2 = x_2^2 + By^2$ satisfies the problem conditions, where $y = 2p$: .... **5 points**.
2. It is proven that if $x_1 = Cp^2 - 1$ and $x_2 = Cp^2 + 1$ are both even, then the number $\left(\frac{x_1}{2}\right)^2 + Ay^2 = \left(\frac{x_2}{2}\right)^2 + By^2$ satisfies the problem conditions, where $y = p$: .... **5 points**
3. It is proven that infinitely many such numbers exist (only if 1. or 2. is present): .... **1 point**
4. Points for 1. and 2. do not add up.
5. If 1. and 2. are both present: .... **plus 1 point**

Problem №3. In an infinite sequence $\{\alpha\}$, $\{\alpha^2\}$, $\{\alpha^3\}$, ... there are only finitely many distinct values. Show that α is an integer. ($\{x\}$ denotes the fractional part of x, i.e. $\{x\} = x - [x]$, where $[x]$ is the greatest integer not greater than x.) *(Golovanov A.S.)*

Solution. *Step 1.* We show that there is a positive integer l s.t. α^l is rational. Say the sequence is of length $k - 1$. For any positive integer n we the sequence $\{\alpha^{nk}\}$, $\{\alpha^{nk+1}\}$, ..., $\{\alpha^{nk+k-1}\}$ contains two equal elements. Hence, there are infinitely many pairs i, j, $0 < i - j < k$, such that $\{\alpha^i\} = \{\alpha^j\}$, i.e. $\alpha^j(\alpha^{i-j} - 1)$ is an integer. Since there are finitely many possible values of $i - j$, at least one of them occurs infinitely often. So we can find m such that $\alpha^j(\alpha^m - 1)$ is an integer for infinitely many j. We can divide two such numbers to get α^l is rational for some positive integer l.

Step 2. Conclusion. Now if α^l is not an integer, say, is equal to $\frac{a}{b}$ for $b > 1$, $\gcd(a, b) = 1$, then $\{\alpha^{ln}\}$ is an irreducible fraction with denominator b^n. This is true for any n so we get infinitely many values, contradiction. So α^l is an integer. If α is irrational, then $\alpha^{nl+1} = \alpha^{nl} \cdot \alpha$ is irrational for any natural n and have distinct fractional parts (Indeed, $\alpha^{il+1} - \alpha^{jl+1} = \alpha(\alpha^{il} - \alpha^{jl})$ cannot be an integer). This contradicts finiteness as well. Thus α is rational, and similar to above we can conclude it is an integer.

Marking scheme.

1. Proof of step 1: .... **4 points**
2. Attempt to construct m such that $\alpha^j(\alpha^m - 1)$ is an integer for two different j's: .... **1 point**
3. Found i and j such that $\{\alpha^i\} = \{\alpha^j\}$ and $|i - j|$ is bounded above by some constant which might depend on k (and not on i or j): .... **2 points**
4. Proof of step 2, i.e. that if α^l is rational for some l, then α is an integer: .... **2 points**
5. Proof of the fact that α is either irrational or an integer (or equivalent): .... **1 points**
6. Point 1 does not add up with points 2 or 3, point 4 does not add up with point 5.

Problem №4. In a language, an alphabet with 25 letters is used; *words* are exactly all sequences of (not necessarily different) letters of length 17. Two ends of a paper strip are glued so that the strip forms a ring; the strip bears a sequence of 5^{18} letters. Say that a word is *singular* if one can cut out a piece bearing exactly that word from the strip, but one cannot cut out two such non-overlapping pieces. It is known that one can cut out 5^{16} non-overlapping pieces each containing the same word. Determine the largest possible number of singular words. *(Bogdanov I.)*

Answer. $2 \cdot 5^{17}$.

Solution. Let the alphabet consist of letters $a_1, a_2, \dots, a_{25}$. By a *piece* we always mean a piece of the strip containing exactly 17 consecutive letters; different pieces may contain the same word. Say that a piece is *singular* if the word it contains is such.

We start with constructing an example containing $N = 2 \cdot 5^{17}$ singular words. Define a word $W = a_1a_2 \dots a_{17}$; this will be the word having $k = 5^{16}$ non-overlapping copies on the strip. There exist exactly $25^8 = k$ possible 8-letter sequences, consisting of letters $a_{18}, a_{19}, \dots, a_{25}$; put them onto the strip in an arbitrary order, separating each two sequences by an instance of W. Each segment of the strip containing one 8-sequence mentioned above (and no other letters) will be referred to as a *part*. Notice that the strip contains exactly $(8 + 17)k = 5^{18}$ letters.

Clearly, the obtained strip contains k non-overlapping copies of W. Now we show that any piece containing a whole part is singular — moreover, that the word it contains is met on no other piece. Since a part can be situated in a piece at 10 different positions (starting from the 1-st, from the 2-nd, …, or from the 10-th letter of a piece), we will get that there are at least $10 \cdot 5^{16} = N$ singular words.

Consider an arbitrary piece p containing a word P. Either this piece contains a unique nonempty prefix which coincides with some suffix of W, or there is no such prefix — only in this case we will say that such prefix is empty. Let b be the length of the defined prefix. Define similarly a suffix of P which coincides with a prefix of W, and denote its length by e. Notice that the defined prefix and suffix do not overlap whenever $P \neq W$ (if $P = W$, we have $b = e = 17$).

If the piece contains no whole part, then $\max\{b, e\} > 9$. If the piece contains a part, then $b + e = 9$ and $0 \leq b, e \leq 9$. Thus, piece p contains a part if and only if $\max\{b, e\} \leq 9$, and in this case the position of the part at P (and hence the position of p at the strip) is uniquely determined. Therefore, in this case P is met only on piece p, so this piece is singular. We have proven that the constructed example works.

It remains to prove that the number of singular words cannot exceed N. Enumerate the positions in the strip successively by $1, 2, \dots, 5^{18}$ (the numeration is cyclic modulo 5^{18}). Let p_i denote the piece starting at position i, and let P_i be the word on that piece. Let $n_1, \dots, n_k$ be positions such that the pieces p_{n_1}, p_{n_2}, …, p_{n_k} are pairwise disjoint and contain the same word W (from the problem statement). Clearly, those pieces are not singular.

For $i = 1, 2, \dots, 8$ and $1 \leq s \leq k$, we say that a piece p_{n_s+i} is a *rank i follower*, while p_{n_s-i} is a *rank i predecessor*. All these pieces (followers and predecessors) are distinct; moreover, followers of a fixed rank are pairwise disjoint, and the same holds for predecessors. We will show that *among* $8 \cdot 5^{16}$ *followers of all ranks, at most* 5^{16} *pieces are singular* (we will call this statement a *quoted claim* in the future); by symmetry, the same bound holds for predecessors. This will yield that there are at least $5^{16} + 7 \cdot 5^{16} + 7 \cdot 5^{16} = 3 \cdot 5^{17}$ non-singular pieces, which implies the desired bound.

Thus, we are left to prove the claim quoted above. For any rank i follower p_{n_s+i} define its *tail* as its suffix of length i (the tail consists of all letters which do not lie in p_{n_s}; we regard a tail as a sequence of letters). We show by induction on $m = 0, 1, \dots, 8$ that for every sequence U consisting of $(8 - m)$ letters, there are no more than 25^m followers whose tails contain U as a prefix. The desired claim is obtained by setting $m = 8$.

The base case $m = 0$ is obvious: if a follower with tail U is singular, then there is only one such follower. Let us perform the inductive step. If there is no singular follower whose tail is U, then every singular follower's tail starting with U starts in fact with some word of the form Ua_i. For every $i = 1, 2, \dots, 25$, there are at most 25^{m-1} such followers, by the inductive hypothesis. So the total number of such followers does not exceed $25 \cdot 25^{m-1} = 25^m$, as desired.

Finally, if there is a singular follower P_{n_s+8-m} whose tail is U, then such follower is unique. Therefore, all followers of larger ranks whose tails start with U correspond to the same copy p_{n_s} of W. Then the number of such followers (including P_{n_s+8-m} itself) is at most $m+1 \leq 25^m$, as desired again. The claim, and the bound, are proven.

Remark. We present a shorter (yet more ideological) proof of the quoted claim on the number of singular followers. Say that a singular follower's tail T is *minimal* if none of its proper prefixes is a singular follower's tail. In particular, no minimal tail can be a proper prefix of other minimal tail.

For every minimal tail T let us write down all 8-letter sequences starting with T; if the length of T is d, then the number of such sequences is 25^{8-d}. No sequence could be written down twice; therefore, if there are M minimal tails of lengths $d_1, \dots, d_M$, then

$$\sum_{i=1}^{M} 25^{8-d_i} \leq 25^8.$$

On the other hand, each singular follower's tail has a prefix which is a minimal tail. For a minimal tail T of length d, there are at most $9-d$ singular followers whose tails start with T — at most one per tail's length. Therefore, the number of singular followers does not exceed

$$\sum_{i=1}^{M} (9-d_i) \leq \sum_{i=1}^{M} 25^{8-d_i} \leq 25^8,$$

since $9 - d \leq 25^{8-d}$ for all $d = 1, 2, \dots, 8$.

Marking scheme.

1. An example with $2 \cdot 5^{17}$ singular words: .... **2 points**
2. Proof of the example's correctness: .... **1 point**
3. Proof of the fact that the answer is not greater than $2 \cdot 5^{17}$: .... **4 points**
4. Formulation of the quoted claim: .... **1 point**
5. The points for 3. and 4. do not add up.