1 A real number x is *badly approximable* if there is some $c > 0$ such that for every rational p/q we have

$$\left| x - \frac{p}{q} \right| > \frac{c}{q^2}.$$

A somewhat weaker condition (let's say α-*badly approximable*) is when q^2 is replaced by q^α for some $\alpha > 2$.

For $c > 0$ we say $x \in \mathbb{R}$ is (α, c)**-bad** if for every rational p/q we have

$$\left| x - \frac{p}{q} \right| > \frac{c}{q^\alpha}.$$

Then x is α**-bad** (α-badly approximable) if it is (α, c)-bad for some $c > 0$.

For $c > 0$ we say $x \in \mathbb{R}$ is (α, c)**-good** if there is a rational p/q such that

$$\left| x - \frac{p}{q} \right| < \frac{c}{q^\alpha}.$$

We say $x \in \mathbb{R}$ is α**-good** if it is (α, c)-good for all $c > 0$.

Clearly, if x is α-good then it must not be α-bad, since α-good uses a strict inequality, which is stronger than the negation of α-bad. On the other hand, if x is not α-good, then it must not be (α, c)-good for some $c > 0$. In which case, $c < c/2$ so x is $(\alpha, c/2)$-bad—hence α-bad.

We may therefore conclude that a real number is α-good if and only if it is not α-bad.

Show that there are lots of α-badly approximable numbers and lots of not-α-badly approximable numbers (for example, both sets are uncountable).

Proof. For $c > 0$ define the set

$$G_c = \bigcup_{p/q \in \mathbb{Q}} B_{c/q^\alpha}(p/q) = \{x \in \mathbb{R} : x \text{ is } (\alpha, c)\text{-good}\},$$

(where representatives $p/q \in \mathbb{Q}$ are chosen such that $\gcd(p, q) = 1$) and its complement

$$N_c = \mathbb{R} \setminus G_c = \{x \in \mathbb{R} : x \text{ is } (\alpha, c)\text{-bad}\}.$$

Then define

$$G = \bigcap_{n \in \mathbb{N}} G_{1/n} \quad \text{and} \quad N = \mathbb{R} \setminus G = \bigcup_{n \in \mathbb{N}} N_{1/n}.$$

We claim that G is precisely the set of α-good real numbers and, by extension, N is the set of α-bad real numbers. Note that if $x \in \mathbb{R}$ is α-good, then it must be $(\alpha, 1/n)$-good for every $n \in \mathbb{N}$, i.e., $x \in G$. On the other hand, if $x \in G$ and $c > 0$ then there is some $n \in \mathbb{N}$ such that $1/n < c$. In which case, x is $(\alpha, 1/n)$-good, implying it is (α, c)-good—hence α-good.

Note that G_c is an open "neighborhood" of $\mathbb{Q}$ since it is the union of open balls—one for each rational. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, G_c is also dense in $\mathbb{R}$. It follows that N_c is nowhere dense as a closed set with dense complement, therefore N is meager. Then G is comeager, so it cannot be meager. Since countable sets are meager, we conclude that G is uncountable.

Let λ denote the Lebesgue measure on $\mathbb{R}$. For any $c > 0$ we have $G \subseteq G_c$ so

$$\lambda(G) \leq \lambda(G_c) \leq \sum_{p/q \in \mathbb{Q}} \lambda(B_{c/q^\alpha}(p/q)) = \sum_{p/q \in \mathbb{Q}} \frac{2c}{q^\alpha}.$$

Again, we are assuming $\gcd(p, q) = 1$ so

$$\sum_{p/q \in \mathbb{Q}} \frac{2c}{q^\alpha} = \sum_{n \in \mathbb{N}} \sum_{\substack{k \in \mathbb{Z} \\ \gcd(n,k)=1}} \frac{1}{n^\alpha} = \sum_{n \in \mathbb{N}} \frac{2\varphi(n)}{n^\alpha},$$

where φ denote Euler's totient function, i.e., $\varphi(n)$ is the number of positive integers less than or equal to n that are relatively prime to n. Note that $\varphi(n) \leq n$ for all $n \in \mathbb{N}$, (with equality only if n is prime). Then

$$\sum_{n \in \mathbb{N}} \frac{\varphi(n)}{n^\alpha} \leq \sum_{n \in \mathbb{N}} \frac{n}{n^\alpha} = \sum_{n \in \mathbb{N}} \frac{1}{n^{\alpha-1}}.$$

This is precisely the sum of the p-series for $p = \alpha - 1$. With $\alpha > 2$ we have $p > 1$ so the sum is $\zeta(\alpha - 1)$, i.e., the evaluation of the Riemann zeta function at $\alpha - 1$. In particular, we conclude that $\lambda(G) \leq \zeta(\alpha - 1) < \infty$. Therefore $\lambda(N) = \infty$, implying N is uncountable. $\square$

2

(a) Prove the **Uniform boundedness principle**:

Let X be a complete metric space and let $\mathcal{F}$ be a subset of $C(X)$ such that for each $x \in X$, the set

$$\mathcal{F}_x = \{f(x) : f \in \mathcal{F}\}$$

is bounded. Then there is a nonempty open set U of X on which the functions in $\mathcal{F}$ are *uniformly bounded*, i.e. there is an M such that $|f(x)| \leq M$ for all $x \in U$ and $f \in \mathcal{F}$.

Hint: Choose $A_N = \{x \in X : |f(x)| \leq N \text{ for all } f \in \mathcal{F}\}$.

Proof. Given $x \in X$ we know by assumption that $\mathcal{F}_x \subseteq [-N, N]$ for some $N \in \mathbb{N}$. In other words, $|f(x)| \leq N$ for all $f \in \mathcal{F}$, hence $X = \bigcup_{N \in \mathbb{N}} A_N$. Assuming $X = \operatorname{int} X$ is nonempty, the Baire category theorem implies it cannot be meager. Therefore, some A_N must not be nowhere dense, i.e., $\operatorname{int} \overline{A_N} \neq \varnothing$. We can write

$$A_N = \bigcap_{f \in \mathcal{F}} f^{-1}([-N, N]),$$

which means A_N is closed. Then

$$U = \operatorname{int} A_N = \operatorname{int} \overline{A_N} \neq \varnothing$$

is an open set of X such that $|f(x)| \leq N$ for all $x \in U$ and $f \in \mathcal{F}$. $\square$

(b) Suppose now that X is a Banach space and the functions in $\mathcal{F}$ are linear. Show that there is an M such that for every $x \in X$ and $f \in \mathcal{F}$, $|f(x)| \leq M\|x\|$.

Proof. Let U and M be as in part (a). Choose $z \in U$ and $\varepsilon > 0$ such that $B_\varepsilon(z) \subseteq U$. Then for any $f \in \mathcal{F}$ and $u \in B_\varepsilon(0)$ we have $z + u \in U$ so

$$|f(u)| = |f(-z + z + u)| = |-f(z) + f(z + u)| \leq |f(z)| + |f(z + u)| \leq 2M.$$

In other words, $B_\varepsilon(0) \subseteq A_{2M}$. Given $x \in X$ choose $a > 0$ such that $\varepsilon/2 \leq a\|x\| < \varepsilon$. Then

$$|f(x)| = \frac{|f(ax)|}{a} \leq \frac{2\|x\|}{\varepsilon} \cdot 2M = \frac{4M}{\varepsilon}\|x\|.$$

$\square$

3

(a) Let Z and X be locally compact Hausdorff spaces. Suppose $K \subseteq Z \times X$ is a compact set contained in an open set U. Show that K is covered by a finite number of compact boxes $A_i \times B_i \subseteq U$.

Proof. Consider a point $(z, x) \in K \subseteq U$. Then there are open neighborhoods $V \subseteq Z$ and $W \subseteq X$ of z and x, respectively, such that $V \times W \subseteq U$. Since Z and X are locally compact and Hausdorff, we can choose a compact neighborhood $A_z \subseteq V$ of x and $B_x \subseteq W$ of y. The product of interiors $\operatorname{int} A_z \times \operatorname{int} B_x \subseteq A_z \times B_x$ is an open neighborhood of (z, x), hence $A_z \times B_x$ is a compact box neighborhood of (z, x). Importantly, we have found a compact box neighborhood of (z, x) contained in U.

Then the collection of interior boxes $\{\operatorname{int} A_z \times \operatorname{int} B_x\}_{(z,x)\in K}$ is an open cover of K. If we choose $\{\operatorname{int} A_i \times \operatorname{int} B_i\}_{i=1}^n$ to be a finite subcover of K, then $\{A_i \times B_i\}_{i=1}^n$ is the desired cover by compact boxes contained in U. $\square$

(b) In class, we showed that if X is a locally compact Hausdorff space, then for any spaces Z and Y,

$$C(Z, C(X, Y)) \text{ and } C(Z \times X, Y)$$

are in bijection as sets, where $C(X, Y)$ is given the compact-open topology. Show that if Z is also locally compact Hausdorff, then this bijection is a homeomorphism if both sets are considered as spaces with the compact-open topology.

Hint: As always, a good strategy to show two spaces are homeomorphic is to show that a subbasis for the topology of one is open in the other and vice versa.

Lemma 1. If $\mathcal{S}$ is a subbasis of open sets for Y then

$$\{V(K, U) : K \subseteq X \text{ compact}, U \in \mathcal{S}\}$$

is a subbasis of open sets for $C(X, Y)$ with the compact open topology.

Proof. Let $\mathcal{B}$ be the collection of finite intersections of sets in $\mathcal{S}$. Then $\mathcal{B}$ is a basis of open sets for Y. Consider an arbitrary standard subbasis set $V(K, U) \subseteq C(X, Y)$, i.e., $K \subseteq X$ is compact and $U \subseteq Y$ is open. Then $U = \bigcup_{\alpha\in I} U_\alpha$ for some $U_\alpha \in \mathcal{B}$ so

$$V(K, U) = \bigcup_{\alpha\in I} V(K, U_\alpha).$$

In other words,

$$\{V(K, U) : K \subseteq X \text{ compact}, U \in \mathcal{B}\}$$

generates the standard subbasis for $C(X, Y)$ under arbitrary unions and is therefore itself a subbasis. Given a basis set $U \in \mathcal{B}$, we can write $U = \bigcap_{i=1}^n U_i$ for some $U_i \in \mathcal{S}$ so

$$V(K, U) = \textstyle\bigcap_{i=1}^n V(K, U_i).$$

That is, the collection subbasis sets in the statement of the Lemma generates a subbasis for $C(X, Y)$ under finite intersections and arbitrary unions, hence the collection is itself a subbasis. $\square$

Lemma 2. The collection

$$\{V(A \times B, U) : A \subseteq Z \text{ and } B \subseteq X \text{ compact}, U \subseteq Y \text{ open}\}$$

is a subbasis of open sets for $C(Z \times X, Y)$ with the compact open topology.

Proof. Let $V(K, U)$ be a standard subbasis set and $f \in V(K, U)$. Then $K \subseteq Z \times X$ is a compact set contained in the open set $f^{-1}(U)$. As in part (a), let $\{A_i \times B_i\}_{i=1}^n$ be a cover of K by compact boxes contained in $f^{-1}(U)$. Then $f \in K(A_i \times B_i, U)$ for $i = 1, \ldots, n$ and

$$\bigcap_{i=1}^n V(A_i \times B_i, U) = V(\bigcup_{i=1}^n (A_i \times B_i), U) \subseteq V(K, U).$$

This is an open neighborhood of f contained in $V(K, U)$. We can then write $V(K, U)$ as the union of all such intersections for $f \in V(K, U)$. That is, the collection in question generates the standard subbasis under finite intersections and arbitrary unions and is therefore itself a subbasis. □

Proof (a). Lemma 1 tells us that

$$\{V(A, V(B, U)) : A \subseteq Z \text{ and } B \subseteq X \text{ compact}, U \subseteq Y \text{ open}\}$$

is a subbasis of open sets for $C(Z, C(X, Y))$ with the compact open topology. Applying Lemma 2, the bijection induces a correspondence of subbasis sets:

$$\begin{aligned} C(Z, C(X, Y)) &\longleftrightarrow C(Z \times X, Y) \\ V(A, V(B, U)) &\longleftrightarrow V(A \times B, U) \end{aligned}$$

□