requires like linear algebra and some topology

Two Hilbert space H_1 and H_1 are said to be **isomorphic** (maybe isometric) if there is a linear operator $U : H_1 \to H_2$ such that

$$\langle Ux, Uy \rangle_{H_2} = \langle x, y \rangle_{H_1}$$

for all $x, y \in H_1$. Such an operator U is called **unitary**.

Lemma 1. Let H be a Hilbert space, $M \leq H$ a closed subspace, and $x \in H$. Then there exists a unique element $z \in M$ with minimum distance to x.

Theorem 1 (projection theorem)**.** Let H be a Hilbert space and $M \leq H$ closed. Then $H = M \oplus M^{\perp}$.

Let H_1 and H_2 be Hilbert spaces. Denote the set

$$\mathcal{L}(H_1, H_2) := \{\text{bounded linear transformations } H_1 \to H_2\}.$$

This is a Banach space with the norm

$$\|T\| = \sup_{\|x\|_{H_1}=1} \|Tx\|_{H_2}$$

The **dual space** of a Hilbert space H is

$$H^* := \mathcal{L}(H, \mathbb{C}).$$

It's elements are called **continuous linear functionals**.

Theorem 2 (Riesz lemma)**.** For each $T \in H^*$ there is a unique $y_T \in H$ such that $Tx = \langle y_T, x \rangle$ for all $x \in H$ and $\|y_T\|_H = \|T\|_{H^*}$.

An **orthonormal basis** of a Hilbert space is a maximal orthonormal subset.

Theorem 3. Every Hilbert space H has an orthonormal basis.

A metric space is **separable** if it has a countable dense subset.

Theorem 4. A Hilbert space H is separable if and only if it has a countable orthonormal basis S. If there are $N < \infty$ elements in S, then H is isometric to $\mathbb{C}^N$. If there are countably infinitely many elements in S, then H is isometric to ℓ_2.

Let (M, μ) be a measure space and $p \geq 1$ a real number. Define the set

$$L^p(M, \mathrm{d}\mu) =$$

Theorem 5. Let $1 \leq p < \infty$.

(a) (Minkowski inequality) If $f, g \in L^p(M, \mathrm{d}\mu)$ then $\|f + g\|_p \leq \|f\|_p + \|g\|_p$.
(b) (Riesz-Fisher) $L^p(M, \mathrm{d}\mu)$ is complete.
(c) (Hölder inequality) Let $p, q, r \geq 1$ satisfy $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$. Suppose $f \in L^p(M, \mathrm{d}\mu)$, $g \in L^q(M, \mathrm{d}\mu)$. Then $fg \in L^r(M, \mathrm{d}\mu)$ and $\|fg\|_r \leq \|f\|_p \|g\|_q$.

We can define the space $\mathcal{L}(X, Y)$ of bounded linear operators for any normed spaces X and Y.

Dual space $X^* = \mathcal{L}(X, \mathbb{C})$.

Theorem 6. If Y is complete, $\mathcal{L}(X, Y)$ is a Banach space.

Theorem 7. A normed linear space is complete if and only if every absolutely summable sequence is summable.

Theorem 8. Let X be a Banach space. For each $x \in X$, let $\tilde{x} : X^* \to \mathbb{C}$ be the linear functional $\tilde{x}(\lambda) = \lambda(x)$ for all $\lambda \in X^*$. Then the map $J : X \to X^{**}$, $x \mapsto \tilde{x}$, is an isometric embedding of X as a subspace of X^{**}.

If J is surjective, say X is **reflexive**.

Theorem 9 (Hahn-Banach — real convex)**.** Let X be a real vector space and $p : X \to \mathbb{R}$ convex. Let $Y \leq X$ and $\lambda \in Y^*$ linear with $\lambda(y) \leq p(y)$ for all $y \in Y$. Then there exists $\Lambda \in X^*$ such that $\Lambda(x) \leq p(x)$ for all $x \in X$ and $\Lambda|_Y = \lambda$.

Theorem 10 (Hahn-Banach — complex convex)**.** Let X be a complex vector space and $p : X \to \mathbb{R}$ absolutely convex. Let $Y \leq X$ and $\lambda \in Y^*$ with $|\lambda(y)| \leq p(y)$ for all $y \in Y$. Then there exists $\Lambda \in X^*$ such that $|\Lambda(x)| \leq p(x)$ for all $x \in X$ and $\Lambda|_Y = \lambda$.

Theorem 11 (Hahn-Banach — real quasi-seminorm)**.** Let X be a real vector space and $q : X \to \mathbb{R}$ a quasi-seminorm. Let $Y \leq X$ and $\lambda \in Y^*$ with $\lambda(y) \leq q(y)$ for all $y \in Y$. Then there exists $\Lambda \in X^*$ such that $\Lambda(x) \leq q(x)$ for all $x \in X$ and $\Lambda|_Y = \lambda$.

Theorem 12 (Hahn-Banach — complex quasi-seminorm)**.** Let X be a complex vector space and $q : X \to \mathbb{R}$ a quasi-seminorm. Let $Y \leq X$ and $\lambda \in Y^*$ with $\operatorname{Re} \lambda(y) \leq q(y)$ for all $y \in Y$. Then there exists $\Lambda \in X^*$ such that $\operatorname{Re} \Lambda(x) \leq q(x)$ for all $x \in X$ and $\Lambda|_Y = \lambda$.

Theorem 13 (Hahn-Banach — real seminorm)**.** Let X be a real vector space and $p : X \to \mathbb{R}$ a seminorm. Let $Y \leq X$ and $\lambda \in Y^*$ with $\lambda(y) \leq p(y)$ for all $y \in Y$. Then there exists $\Lambda \in X^*$ such that $\Lambda(x) \leq p(x)$ for all $x \in X$ and $\Lambda|_Y = \lambda$.

Theorem 14 (Hahn-Banach — complex seminorm)**.** Let X be a complex vector space and $p : X \to \mathbb{R}$ a seminorm. Let $Y \leq X$ and $\lambda \in Y^*$ with $|\lambda(y)| \leq p(y)$ for all $y \in Y$. Then there exists $\Lambda \in X^*$ such that $|\Lambda(x)| \leq p(x)$ for all $x \in X$ and $\Lambda|_Y = \lambda$.

Corollary 1. Let X be a normed linear space, $Y \leq X$, and $\lambda \in Y^*$. Then there exists $\Lambda \in X^*$ with $\Lambda|_Y = \lambda$ and $\|\Lambda\|_{X^*} = \|\lambda\|_{Y^*}$.

Corollary 2. Let X be a normed linear space and $y \in X$. Then there exists a nonzero $\Lambda \in X^*$ such that $\Lambda(y) = \|\Lambda\|_{X^*}\|y\|$.

Corollary 3. Let X be a normed linear space, $Z \leq X$, and $y \in X$ with $d = d(y, Z) > 0$. Then there exists $\Lambda \in X^*$ such that $\|\Lambda\| \leq 1$, $\Lambda(y) = d$, and $\Lambda|_Z = 0$.

Theorem 15. Let B be a Banach space. If B^* is separable, then B is separable.

Baire category

Theorem 16 (uniform boundedness principle)**.** Let B be a Banach space and V a normed space. Let $\mathcal{F} \subseteq \mathcal{L}(B, V)$ such that for all $x \in X$

$$\sup_{T \in \mathcal{F}} \|Tx\|_V < \infty$$

(i.e., the set $\{\|Tx\|_V : T \in \mathcal{F}\}$ is bounded). Then

$$\sup_{T \in \mathcal{F}} \|T\| < \infty$$

(i.e., the set $\{\|T\| : T \in \mathcal{F}\}$ is bounded).

Theorem 17 (open mapping)**.** Let B_1 and B_2 be Banach spaces and $T \in \mathcal{L}(B_1, B_2)$. If T is surjective then T is open.

Theorem 18 (inverse mapping)**.** Let B_1 and B_2 be Banach spaces and $T \in \mathcal{L}(B_1, B_2)$. If T is bijective then T^{-1} is bounded/continuous.

Theorem 19 (closed graph)**.** Let B_1 and B_2 be Banach spaces and $T : B_1 \to B_2$ linear. Then T is bounded if and only if $\Gamma(T)$ is closed in $B_1 \oplus B_2$

Corollary 4 (Hellinger-Toeplitz theorem)**.** Let H be a Hilbert space and $A \in \mathcal{L}(H)$ with $\langle x, Ay\rangle = \langle Ax, y\rangle$ for all $x, y \in H$. Then A is bounded.

Let X be a topological space and S any set. Let

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Let H be a Hilbert space.

An operator $B \in \mathcal{L}(H)$ is **positive**, written $B \geq 0$, if $\langle Bx, x \rangle \geq 0$ for all $x \in H$.

Lemma 2. If $B \in \mathcal{L}(H)$ is positive, then $B = B^*$.

Proof. Using polarization identity we deduce

$$\begin{aligned}\langle Bx, y\rangle + \langle By, x\rangle &= \frac{1}{2}\big(\langle B(x+y), x+y\rangle - \langle B(x-y), x-y\rangle\big) \\ &= \frac{1}{2}\big(\langle x+y, B(x+y)\rangle - \langle x-y, B(x-y)\rangle\big) \\ &= \langle x, By\rangle + \langle y, Bx\rangle\,.\end{aligned}$$

Rearranging, we find

$$\langle Bx, y\rangle - \overline{\langle Bx, y\rangle} = \langle x, By\rangle - \overline{\langle x, By\rangle}.$$

Replace y bu iy, get

$$\langle Bx, y\rangle + \overline{\langle Bx, y\rangle} = \langle x, By\rangle + \overline{\langle x, By\rangle}.$$

Adding these and dividing by 2, conclude $\langle Bx, y\rangle = \langle x, By\rangle$, so indeed $B = B^*$. □

Theorem 20 (Square Root Lemma)**.** If $A \in \mathcal{L}(H)$ is positive, then there exists a unique $B \in \mathcal{L}(H)$ such that $B^2 = A$ and $B \geq 0$. Moreover, for any $C \in \mathcal{L}(H)$, if $AC = CA$ then $BC = CB$.

Proof. Want to construct B. Without loss of generality, may assume $\|A\| \leq 1$; otherwise rescale. Recall that $A = A^*$ implies $\|A\| = \sup_{\|x\|=1} |\langle Ax, x\rangle|$. Now consider

$$\begin{aligned}\|I - A\| &= \sup_{\|x\|=1} |\langle (I-A)x, x\rangle| \\ &= \sup_{\|x\|=1} \big|\|x\|^2 - \langle Ax, x\rangle\big| \\ &\leq 1,\end{aligned}$$

since $\|A\| \leq 1$, know $0 \leq \langle Ax, x\rangle \leq 1$.

Lemma 3. $\sqrt{1-z}$ has power series absolutely convergent for $|z| \leq 1$. ($\sqrt{1-z} = \sum_{n=0}^{\infty} c_n z^n$ with $c_n < 0$ if $n \geq 1$.)

By Lemma, $B := \sum_{n=0}^{\infty} c_n (I-A)^n$ converges in the operator norm, so is well-defined. This construction gives $B^2 = I - (I - A) = A$.

Since B is defined in terms of A and I, then if A commutes with C, it is easy to check that B also commutes with C.

We now check B is positive; suffices to check for $\|x\| = 1$. Consider

$$\langle Bx, x\rangle = 1 + \sum_{n=1}^{\infty} c_n \left\langle (I-A)^n x, x\right\rangle .$$

Claim 1. $0 \leq \langle (I-A)^n x, x\rangle \leq 1$ for all $n \in \mathbb{N}$.

Proof. If $n = 2k$ is even, then $\langle (I-A)^n x, x\rangle = \|(I-A)^k x\|^2$.

If $n = 2k+1$ is odd,

$$\langle (I-A)^n x, x\rangle = \left\langle (I-A)((I-A)^k x), (I-A)^k x\right\rangle = \langle (I-A)y, y\rangle ,$$

where $y = (I-A)^k x$. Then by an above argument, this is between 0 and 1. $\square$

By the claim,

$$\langle Bx, x\rangle \geq 1 + \sum_{n=1}^{\infty} c_n = \sqrt{1-1} = 0,$$

hence $B \geq 0$.

It remains to prove uniqueness. Suppose $C^2 = A$ and $C \geq 0$. Have

$$CA = C^3 = AC \quad \Longrightarrow \quad CB = BC.$$

Also have

$$C^2 = A = B^2.$$

These facts imply

$$(C-B)(B+C)(C-B) = 0.$$

which implies

$$(C-B)B(C-B) = (C-B)C(C-B) = 0.$$

Can verify this by applying to an element and using commutativity. In particular, if sum of two positive operators is zero then both must be zero. It follows that $(C-B)^3 = 0$, so of course $(C-B)^n = 0$ for $n \geq 3$. Since B, C self-adjoint, deduce

$$\|(C-B)\|^4 = \|(C-B)^4\| = 0,$$

so indeed $B = C$. $\square$

From the Square Root Lemma, denote $\sqrt{A} = B$ for positive $A \in \mathcal{L}(H)$

Let $A \in \mathcal{L}(H)$ be any, define **absolute value** as $|A| = \sqrt{A^*A}$. (Note A^*A is positive so this makes sense: $\langle A^*Ax, x\rangle \geq 0$). Theorem gives $|A|^2 = A^*A$.

Remark. It is not true in general that $|AB| = |A||B|$; this is only notation.

Example. Let $T : \ell_2 \to \ell_2$ be the right shift operator $T(x_1, x_2, \dots) = (0, x_1, x_2, \dots)$. Adjoint T^* is left shift, then $T^*T = I$ implies $|T| = I \neq T$.

An operator $U \in \mathcal{L}(H)$ is called a **partial isometry** (or **partial unitary**) if U is an isometry on $(\ker U)^\perp$, i.e., $\|Ux\| = \|x\|$ for all $x \in (\ker U)^\perp$.

Fact. U is a partial isometry if $U^*U = P$ and $UU^* = Q$ where $P : H \to (\ker U)^\perp$ and $Q : H \to \operatorname{im} U$ are orthogonal projections

Theorem 21 (Polar Decomposition)**.** Let $A \in \mathcal{L}(H)$ arbitrary, then there exists a unique partial isometry U such that $\ker U = \ker A$ and $A = U|A|$. Moreover, $\operatorname{im} U = \overline{\operatorname{im} A}$.

Example. Let $T : \ell_2 \to \ell_2$ be right shift. Since $|T| = I$, then this theorem would imply $U = T$. Additionally, theorem says that $\operatorname{im} T$ should be closed.

Proof. Will construct U, prove properties, then prove uniqueness.

First, want to define $U : \operatorname{im}|A| \to \operatorname{im} A$ by $U(|A|x) = Ax$ for all $x \in H$, but must check well-defined. Notice

$$\||A|x\|^2 = \left\langle |A|^2 x, x \right\rangle = \langle A^*Ax, x\rangle = \|Ax\|^2,$$

which implies $\||A|x\| = \|Ax\|$ and $\ker|A| = \ker A$. So $|A|x_1 = |A|x_2$ implies $Ax_1 = Ax_2$, hence U is indeed well-defined on $\operatorname{im}|A|$.

Naturally extend to $U : \overline{\operatorname{im}|A|} \to \overline{\operatorname{im} A}$, and notice that this is an isometry.

Then we can simply define $U|_{\overline{\operatorname{im}|A|}^\perp} = 0$, i.e.,

$$\ker U = (\operatorname{im}|A|)^\perp = \ker|A| = \ker A$$

One can check second equality by self-adjointness of $|A|$.

We have so far constructed a partial isometry U with the desired properties. To show uniqueness, can suppose there is another and show equality... $\square$

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Let B_1, B_2 be Banach spaces.

A bounded operator $T \in \mathcal{L}(B_1, B_2)$ is called **compact** (or completely continuous) if for every bounded sequence $\{x_n\}$ in B_1 the image sequence $\{Tx_n\}$ in B_2 has a convergent subsequence.

Recall. If X is a metric space, a subset $K \subseteq X$ is compact (every open cover of K has a finite subcover) if and only if every sequence in K has a convergence subsequence. In other words, for metric spaces compactness is equivalent to sequential compactness.

A subset of a topological space is **precompact** if its closure in the larger space is compact.

Equivalently, bounded operator $T \in \mathcal{L}(B_1, B_2)$ is compact if and only if it maps any bounded set in B_1 to a precompact (relatively compact) set in B_2.

Remark. One can verify that if T is compact, it is bounded. To see this, consider the closed unit ball $D \subseteq B_1$, which is in particular a bounded set. Then by the equivalent definition, $\overline{T(D)} \subseteq B_2$ is compact, which is in particular bounded. So then

$$\|T\| = \sup_{\|x\|=1} \|Tx\| \leq \sup_{x \in D} \|Tx\| < \infty.$$

Example.

1. Finite rank operators: $\dim \operatorname{im} T < \infty$. In this case, can write $\operatorname{im} T = \mathbb{C}y_1 \oplus \cdots \oplus \mathbb{C}y_n \leq B_2$. For a bounded sequence $\{x_k\}_k$ in B_1, we can write $Tx_k = \sum_{i=1}^n a_{i,k} y_i$. Then for each i, $\{a_{i,k}\}_k$ is a bounded sequence in $\mathbb{C}$, so there exists a convergent subsequence $\{a_{i,k_j}\}_j$. Then $\{Tx_{k_j}\}_j$ is a convergent subsequence in B_2. Hence, T is compact.
2. Classical integral operators. Given $k \in C([0,1] \times [0,1])$, consider $B_1 = B_2 = C[0,1]$ with $\|\cdot\|_\infty$. Then have operator $\hat{k} \in \mathcal{L}(C[0,1])$ defined by

$$(\hat{k}f)(x) = \int_0^1 k(x,y) f(y) \mathrm{d}y.$$

   It follows from this definition that $\|\hat{k}\| \leq \|k\|_\infty$. For $\varepsilon > 0$ choose $\delta > 0$ for uniform continuity of k. Then $|x - x'| < \delta$ gives

$$\left| \hat{k}f(x) - \hat{k}f(x') \right| \leq \sup_{y \in [0,1]} |k(x,y) - k(x',y)| \, \|f\|_\infty < \varepsilon \|f\|_\infty.$$

   Can verify that if $\{f_n\}$ is bounded sequence in $C[0,1]$ then $\{\hat{k}f_n\}$ is uniformly bounded. Moreover, $\{\hat{k}f_n\}$ is equicontinuous. By the Arzelà-Ascoli theorem $\hat{k}$ has a convergence subsequence.

Given sequence $\{x_n\}$ in B and $x \in B$:

- say $x_n \to x$ strongly if $\|x_n - x\| \xrightarrow{n \to \infty} 0$;
- say $x_n \to x$ weakly if $\ell(x_n) \to \ell(x)$ for all $\ell \in B^*$.

Theorem 22. Compact operators map weakly convergent sequences to strongly convergent.

Proof. Suppose $x_n \to x$ weakly in B_1, i.e., $\ell(x_n) \to \ell(x)$ for all $\ell \in B_1^*$. Identifying $B_1 \leq B_1^{**}$, we can view this as a sequence $\tilde{x}_n(\ell) \to \tilde{x}(\ell)$ for all $\ell \in B_1^*$. By the uniform boundedness principle, $\{\tilde{x}_n\}$ is bounded in B_1^{**}—equivalently $\{x_n\}$ is bounded in B_1 since $\|\tilde{y}\| = \|y\|$ for all $y \in B_1$.

Let $T \in \mathcal{L}(B_1, B_2)$ be a compact operator. Then $\{y_n = Tx_n\}$ in B_2 has a convergent subsequence $\{y_{n_k}\}$. Let $y = T_x$, then $y_n \to y$ weakly in B_2. For $\ell \in B_2^*$, we have

$$\ell(y_n) - \ell(y) = \ell(T(x_n)) - \ell(T(x)) = T'\ell(x_n - x)$$

with $T'\ell \in B_1^*$. This converges to zero by weak $x_n \to x$, hence $y_n \to y$ weakly in B_2.

Assuming $y_n \not\to y$ strongly, there exists a subsequence $\{y_{n_j}$ and $\varepsilon > 0$ such that $\|y_{n_j} - y\| \geq \varepsilon$. However, $y_{n_j} = Tx_{n_j}$ has a convergent subsequence, i.e., $y_{n_j} \to y' \neq y$. But then also have $y_{n_j} \to y'$ weakly, which contradicts weak convergence $y_{n_j} \to y$. Hence, $y_n \to y$ strongly. $\square$

Theorem 23. Let $T \in \mathcal{L}(B_1, B_2)$.

(a) If $\{T_k\}$ is a sequence of compact operators in $\mathcal{L}(B_1, B_2)$ such that $T_k \to T$ with respect to the operator norm, i.e., $\|T_k - T\| \to 0$, then T is also compact.
(b) T is compact if and only if T^* is compact.
(c) If $S \in \mathcal{L}(B_0, B_1)$ arbitrary and T is compact, then $TS \in \mathcal{L}(B_0, B_2)$ is compact.

Proof. (a) Use diagonal argument. Let $\{x_n\}$ be a bounded sequence in B_1. Each T_k is bounded, so $\{T_k x_n\}_n$ has a convergent subsequence $\{T_k x_{n_{k,j}}\}_j$. Then get subsequences

$$\begin{array}{c|cccc} & x_1 & x_2 & x_3 & \cdots \\ \hline T_1 & T_1(x_{n_{1,1}}) & T_1(x_{n_{1,2}}) & T_1(x_{n_{1,3}}) & \\ T_2 & T_2(x_{n_{2,1}}) & T_2(x_{n_{2,2}}) & T_2(x_{n_{2,3}}) & \\ T_3 & T_3(x_{n_{3,1}}) & T_3(x_{n_{3,2}}) & T_3(x_{n_{3,3}}) & \\ \vdots & & & & \ddots \end{array}$$

Then diagonal $\{x_{n_{k,k}}\}_k$ is a subsequence such that $\{T_j x_{n_{k,k}}\}_k$ is convergent for all j. Then

$$\begin{aligned} \|Tx_{n_{k,k}} - Tx_{n_{\ell,\ell}}\| &\leq \|Tx_{n_{k,k}} - T_j x_{n_{k,k}}\| + \|T_j x_{n_{k,k}} = T_j x_{n_{\ell,\ell}}\| + \|T_j x_{n_{\ell,\ell}} = Tx_{n_{\ell,\ell}}\| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

First two for large j and middle for large k, ℓ. Therefore, $Tx_{n,k}$ is convergent, so T is compact.

$\square$

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Proof. (b) For $B_1 = B_2 = H$, $T : H \to H$, $T^* : H \to H$.
Lemma 4. For $T \in \mathcal{L}(H)$ with T^*T compact, then T is compact.
Proof. Given $\{x_n\} \subseteq H$ bounded, get

$$\|T(x_n - x_m)\|^2 = \langle T^*T(x_n - x_m), x_n - x_m\rangle \leq C\|T^*T(x_n - x_m)\|.$$

Then some subsequence $\{T^*Tx_{n_k}\}$ is Cauchy. Plugging in to the above inequality implies $\{Tx_{n_k}\}$ is Cauchy, therefore convergent in H. Hence, T is compact. $\square$

Then if T is compact, then (c) implies $TT^* = (T^*)^*T^*$ is compact, then Lemma implies T^* is compact.
Remains to show for general Banach spaces and converse.

□

Lemma 5. If $T \in \mathcal{L}(H)$ is compact, then $\operatorname{im} T$ is separable.

Proof. Idea: $T(B_n(0))$ is precompact so $\overline{T(B_n(0))}$ is compact. Compact sets in metric space are separable (Look at center points of each finite cover by balls of radius $1/n$). Then $\overline{T(B_n(0))}$ is separable, and so is $T(B_n(0))$. Then can write $H = \bigcup_{n=1}^{\infty} B_n(0)$, and $T(H) \subseteq \bigcup_{n=1}^{\infty}$ so $\operatorname{im} T$ is separable.

To complete this proof, must give stronger argument for each step. □

Lemma 6. If $T \in \mathcal{L}(H)$ is compact, then $(\ker T)^{\perp}$ is separable.

Proof. Verify that $(\ker T)^{\perp} = \overline{\operatorname{im} T^*}$. Then T compact implies T^*, so previous lemma implies $\operatorname{im} T^*$ is separable, and therefore so is its closure. □

Theorem 24. Let $T \in \mathcal{L}(H)$ compact. Then there exists a sequence $\{T_n\}$ of finite rank operators such that $T_n \to T$ in the operator norm.

Proof. By Lemma, $(\ker T)^{\perp}$ has a countable orthonormal basis $\{x_n\}$. Since T acting on the kernel is zero, only care about this orthogonal complement. Define space

$$H_n = \ker T \oplus \mathbb{C}x_1 \oplus \cdots \oplus \mathbb{C}x_n.$$

Then $\{H_n\}$ is increasing sequence of subspaces, so $\{H_n^{\perp}\}$ is a shrinking sequence of subspaces. Let

$$\lambda_n = \sup_{\substack{x \in H_n^{\perp} \\ \|x\|=1}} \|Tx\|$$

This $\{\lambda_n\}$ is a decreasing nonnegative sequence, therefore convergent $\lambda = \lim_{n\to\infty} \lambda_n \geq 0$.

We claim that $\lambda = 0$.

Suppose $y_n \in H_n^{\perp}$ satisfying $\|y_n\| = 1$ and $\|Ty_n\| \geq \lambda/2$.

Then claim $y_n \to 0$ weakly, i.e., $\ell(y_n) \to 0$ for all $\ell \in H^*$.

By Riesz lemma, can write $\ell(y_n) = \langle y_n, x\rangle$ for some $x \in H$. Write $y_n = \sum_{i=n+1}^{\infty} a_i x_i$ then

$$\langle y_n, x\rangle = \sum_{i=n+1}^{\infty} a_i \langle x_i, x\rangle . \quad (a_{n,i}?)$$

The tail goes to zero as $n \to \infty$, so indeed $\ell(y_n) \to 0$.

Then T compact implies $Ty_n \to 0$ strongly, so must have $\lambda = 0$.

Now construct finite rank operators. Let $T_n x = \sum_{i=1}^n \langle x, x_i \rangle T x_i$ for $x \in (\ker T)^\perp$ and set $T_n|_{\ker T} = 0$. Then T_n is finite rank.

Should be clear from construction that $T_n \to T$. To verify, look at

$$\|(T - T_n)x\| \leq \lambda_n \|x\|$$

for $x \in (\ker T)^\perp$. Can write $(T - T_n)x = T z_n$ for some $z_n \in H_n^\perp$. Then $\lambda_n \to 0$ implies $\|T - T_n\| \to 0$. $\square$

Theorem 25 (Analytic Fredholm)**.** Let $D \subseteq \mathbb{C}$ open and $f : D \to \mathcal{L}(H)$ be analytic (i.e., $\lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$ exists for all $x \in D$). If $f(z)$ is compact for all $z \in D$, then either

(a) $(I - f(z))^{-1}$ does not exits for all $z \in D$ or
(b) there exists $S \subseteq D$ discrete such that $(I - f(z))^{-1}$ exists for all $z \in D \setminus S$. Moreover, $(I - f(z))^{-1}$ is meromorphic on D (analytic on $D \setminus S$), and $f(z)x = x$ has a nontrivial solution in H for all $z \in S$.

Proof. Pick any $z_0 \in D$ and choose radius $r > 0$ such that $B_r(z_0) \subseteq D$ and $\|f(z) - f(z_0)\| < 1/2$ for all $z \in B_r(z_0)$, by analyticity of f. By previous theorem applied to $f(z_0)$, choose a finite rank operator F such that $\|f(z_0) - F\| < 1/2$. Then $\|f(z) - F\| < 1$ for all $z \in B_r(z_0)$. Then $(I - f(z) + F)^{-1}$ exists for $z \in B_r(0)$ since can write as a power series. Moreover, this inverse is analytic for reasons...

Denote $g(z) = F(I - f(z) + F)^{-1}$, then

$$I - f(z) = (I - g(z))(I - f(z) + F).$$

Then $I - f(z)$ is invertible if and only if $I - g(z)$ is invertible. Since F is finite rank, so is $g(z)$.

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F is finite rank, so there exist $y_1, \ldots y_N \in H$ such that

$$F(x) = \sum_{i=1}^N \ell_i(x) y_i = \sum_{i=1}^N \langle x, x_i \rangle y_i.$$

(Second equality comes from Riesz Lemma.) Let $\varphi_n(z) = ((I - f(z) + F)^{-1})^*(x_n)$, then

$$g(z) = F(I - f(z) + F)^{-1}(z) = \sum_{n=1}^N \langle -, \varphi_n(z) \rangle y_n.$$

To see if $I - g(z)$ is invertible, we are asking if it has any nontrivial zeros, i.e., whether there are any nontrivial solutions for $g(z)x = x$. If x is such a solution, then in particular $x = g(z)x \in \operatorname{im} F$, so we can write $x = \sum_{k=1}^N c_k y_k$ giving

$$g(z)x = \sum_{n=1}^N \sum_{k=1}^N \langle c_k y_k, \varphi_n(z) \rangle y_n.$$

Can check that $g(z)x = x$ has a nontrivial solution if and only if

$$\det Q = 0 \quad \text{where} \quad Q = [\delta_{kn} - \langle y_k, \varphi_n(z) \rangle].$$

This determinant function is analytic with respect to $z \in B_r(z_0)$. By complex analysis, we know that either $\det Q \equiv 0$ (= const_0 constantly zero) or $\det Q = 0$ only on a discrete set. (In other words, $(\det Q)^{-1}(0)$ is either $B_r(z_0)$ or discrete.) The result now follows. □

Corollary 5 (Fredholm alternative)**.** For compact $A \in \mathcal{L}(H)$, either $(I - A)^{-1}$ exists or $A\psi = \psi$ has a nonzero solution.

Proof. To apply Fredholm theorem, we must construct an analytic function $f : D \to \mathcal{L}(H)$.

Choosing constant $f \equiv A$, then it might simply happen that we are in case (a), where $I - A$ is simply not invertible.

We choose $f(z) = z \cdot A$ and $D \subseteq \mathbb{C}$ a neighborhood of zero, because $I - f(0) = I$ is of course invertible. Hence, we are in case (b), so $(I - f(z))^{-1}$ is invertible for all $z \in D \setminus S$ for a discrete set $S \subseteq D$.

If $1 \notin S$, then $I - f(1) = I - A$ is invertible.

If $1 \in S$, then $x = f(1)x = Ax$ has nontrivial solutions. □

Theorem 26 (Fredholm 1st)**.** Let $A \in \mathcal{L}(H)$ compact and $\lambda \neq 0$. Then λ is an eigenvalue for A if and only if $\overline{\lambda} \neq 0$ is an eigenvalue for A^*.

Proof. In previous proofs, reduce the problem to the invertibility of a finite rank square matrix. Something about relationship between invertibility and existence of nontrivial zeros.

Let $f(z) = z \cdot A$. Consider $z = 1/\lambda$ for Analytic Fredholm theorem; have

$$I - f(z) = I - zA = I - \tfrac{1}{\lambda}A = \tfrac{1}{\lambda}(\lambda I - A).$$

Then λ is an eigenvalue for A if and only if $I - g(z)$ is not invertible, if and only if $\det Q = 0$, if and only if $\det \overline{Q} = 0$, if and only if $I - g(z)^*$ is not invertible, if and only if $\overline{\lambda}$ is an eigenvalue of A^*. □

Theorem 27 (Fredholm 2nd)**.** Let $A \in \mathcal{L}(H)$ compact.

(a) For $\lambda \neq 0$, $A - \lambda I$ is invertible if and only if $\ker(A - \lambda I) = 0$;
(b) For $y \in H$, $(A - \lambda I)x = y$ is solvable if and only $y \perp \ker(A^* - \overline{\lambda} I)$.

Proof. (b)

Write

$$\ker(A^* - \overline{\lambda} I)^{\perp} = \overline{\operatorname{im}(A^* - \overline{\lambda} I)^*} = \overline{\operatorname{im}(A - \lambda I)}.$$

Claim that this equals $\operatorname{im}(A - \lambda I)$, i.e., image is closed. Using previous techniques, there is finite rank G (like $g(z)$) and invertible P (like $(I - f(z) + F)^{-1}$) such that

$$A - \lambda I = (I - G)P.$$

Then $\operatorname{im}(A - \lambda I)$ is closed if and only if $\operatorname{im}(I - G)$ is closed. Can write

$$H = \ker G \oplus H_1$$

for finite dimensional H_1 (verify by linear algebra since G finite rank). Then

$$\operatorname{im}(I - G) = \ker G \oplus F$$

for some finite dimensional F. Finite dimensional implies closed for subspaces, so F closed. Kernels probably closed (something to check), so $\ker G$ is closed. Then $\ker G$ and F both closed implies $\operatorname{im}(I - G)$ is closed. $\square$

Remark. $\dim \ker(A - \lambda I) < \infty$.

Theorem 28 (Fredholm 3rd)**.** Let $A \in \mathcal{L}(H)$ compact and $\lambda \neq 0$. Then $\dim \ker(A - \lambda I) = \dim \ker(A^* - \overline{\lambda} I)$.

Idea. Put $f(z) = zA$ and apply previous stuff to $z = 1/\lambda$. Reduce things to finite dimensional linear algebra. $\square$