By an **element** y of an object Y (notation $y \in Y$) of an abelian category $\mathcal{A}$ we mean an equivalence class of pairs (X, h), $X \in \mathcal{A}$, $h : X \to Y$, by the following equivalence relation: $(X, h) \sim (X', h')$ if and only if there exists $Z \in \mathcal{A}$ and epimorphisms $u : Z \to X$ and $u' : Z \to X'$ such that $hu = h'u'$.

We will use the following properties from the previous exercise.

(a) $f : Y_1 \to Y_2$ is a monomorphism if and only if $f(y) = 0$ for $y \in Y_1$ implies $y = 0$.

(b) $f : Y_1 \to Y_2$ is a monomorphism if and only if $f(y) = f(y')$ for $y, y' \in Y_1$ implies $y = y'$.

(c) $f : Y_1 \to Y_2$ is an epimorphism if and only if for any $y \in Y_2$ there exists $y' \in Y_1$ such that $f(y') = y$.

(d) $f : Y_1 \to Y_2$ is the zero morphism if and only if $f(y) = 0$ for all $y \in Y_1$.

(e) A sequence $Y_1 \xrightarrow{f} Y \xrightarrow{g} Y_2$ is exact at Y if and only if $gf = 0$ and for any $y \in Y$ with $g(y) = 0$ there exists $y' \in Y_1$ with $f(y') = y$.

(f) Assume we are given a morphism $g : Y_1 \to Y_2$ and elements $y, y' \in Y_1$ such that $g(y) = g(y')$. Then there exists $z \in Y_1$ such that $g(z) = 0$ and, moreover, for any $f : Y_1 \to Y$ with $f(y) = 0$ we have $f(z) = -f(y')$ and for any $f' : Y_1 \to Y$ with $f'(y') = 0$ we have $f'(z) = f'(y)$. (The element z is an analogue of the difference $y - y'$.)

1 Exercise II.5.6 (Five Lemma) Assume we are given a commutative diagram

$$\begin{tikzcd}
X_1 \arrow[r, "a_1"] \arrow[d, "f_1"] & X_2 \arrow[r, "a_2"] \arrow[d, "f_2"] & X_3 \arrow[r, "a_3"] \arrow[d, "f_3"] & X_4 \arrow[r, "a_4"] \arrow[d, "f_4"] & X_5 \arrow[d, "f_5"] \\
Y_1 \arrow[r, "b_1"'] & Y_2 \arrow[r, "b_2"'] & Y_3 \arrow[r, "b_3"'] & Y_4 \arrow[r, "b_4"'] & Y_5
\end{tikzcd}$$

with exact rows. Assume also that f_1 is an epimorphism, f_5 is a monomorphism, and f_2 and f_4 are isomorphisms. Then f_3 is also an isomorphism.

Proof. We prove that f_3 is an isomorphism by showing that it is both a monomorphism and and epimorphism.

Suppose $x \in \ker f_3$, i.e., $x \in X_3$ such that $f_3(x) = 0$. Then commutativity of the third square gives us

$$0 = b_3 f_3(x) = f_4 a_3(x).$$

Since f_4 is an monomorphism and $f_4(0) = 0$, we deduce that $a_3(x) = 0$. By exactness at X_3, there exists $x' \in X_2$ such that $a_2(x') = x$. Then commutativity of the second square gives us

$$0 = f_3(x) = f_3 a_2(x') = b_2 f_2(x').$$

So, by exactness at Y_2, there exists $y \in Y_1$ such that $b_1(y) = f_2(x')$. Since f_1 is an epimorphism, there exists $x'' \in X_1$ such that $f_1(x'') = y$. Then commutativity of the first square gives us

$$f_2 a_1(x'') = b_1 f_1(x'') = b_1(y) = f_2(x').$$

And since f_2 is a monomorphism, we deduce that $a_1(x'') = x'$. But exactness at X_2 gives us $a_2 a_1 = 0$, so in fact

$$x = a_2(x') = a_2 a_1(x'') = 0.$$

This proves that f_2 is a monomorphism.

Suppose $y \in Y_3$. Exactness at Y_4 gives us $b_4 b_3(y) = 0$. Since f_4 is an epimorphism, there exists $x \in X_4$ such that $f_4(x) = b_3(y)$. By commutativity of the fourth square, we have

$$0 = b_4 b_3(y) = b_4 f_4(x) = f_5 a_4(x).$$

This f_5 is a monomorphism and $f_5(0) = 0$, we deduce that $a_4(x) = 0$. By exactness at X_4, there exists $x' \in X_3$ such that $a_3(x') = x$. Then commutativity of the third square gives us

$$b_3 f_3(x') = f_4 a_3(x') = f_4(x) = b_3(y).$$

In other words, we do not know if $f_3(x')$ and y are equal, but they are sent to the same element by b_3. Let $z \in Y_3$ be as in (f), i.e., z is an analogue of the difference $f_3(x') - y$. In particular, $b_3(z) = 0$, so exactness at Y_3 tells us there exists $y' \in Y_2$ such that $b_2(y') = z$. Since f_2 is an epimorphism, there exists $x'' \in X_2$ such that $f_2(x'') = y'$. Commutativity of the second square gives us

$$z = b_2(y') = b_2 f_2(x'') = f_3 a_2(x'').$$

We now apply (f) to obtain $\widetilde{x} \in X_3$ as an analogue of the difference $x' - a_2(x'')$.

Then we should get something like the following:

$$f_3(\widetilde{x}) = f_3(x') - f_3 a_2(x'') = f_3(x') - z \approx f_3(x') - (f_3(x') - y) = y.$$

However, I am not sure how to make this rigorous given the current setup with (f) handling the differences. Assuming this works out, we conclude that f_3 is an epimorphism. $\square$