1 Give an example of a topological space X and a measure μ on X so that μ is Borel but not Borel-regular.

Let $X = \{0, 1\}$ with the indiscrete topology and μ the cardinality measure. Clearly, μ is Borel, since $\varnothing$ and X, the only Borel sets, are always measurable. Consider $\{0\} \subseteq X$; the only Borel subset containing $\{0\}$ is X itself, but $\mu(\{0\}) = 1 \neq 2 = \mu(X)$. Hence, μ is not Borel-regular.

2 Let X be a nonempty set and let $\{\mu_n\}_{n=1}^{\infty}$ be a sequence of measures on X. Assume for any subset $A \subseteq X$ the limit $\lim_{n\to\infty} \mu_n(A)$ exists and denote $\mu(A) = \lim_{n\to\infty} \mu_n(A)$.

(i) Is it true that μ is a measure on X if for any $A \subseteq X$ the sequence $\{\mu_n(A)\}$ is increasing?

Yes.

Proof. First,

$$\mu(\varnothing) = \lim_{n\to\infty} \mu_n(\varnothing) = \lim_{n\to\infty} 0 = 0.$$

Suppose $A_1, A_2, \ldots \subseteq X$ and $A \subseteq \bigcup_{i=1}^{\infty} A_i$. By the increasing condition, $\mu_n(A_i) \leq \mu(A_i)$, for all $n, i \in \mathbb{N}$. Then, for all $n \in \mathbb{N}$,

$$\mu_n(A) \leq \sum_{i=1}^{\infty} \mu_n(A_i) \leq \sum_{i=1}^{\infty} \mu(A_i).$$

Letting $n \to \infty$, we obtain

$$\mu(A) = \lim_{n\to\infty} \mu_n(A) \leq \sum_{i=1}^{\infty} \mu(A_i).$$

This is the countable subadditivity and monotonicity, hence μ is a measure on X.

$\square$

(ii) Assume in addition that $\mu_1(X) < \infty$, and that each of the measures μ_n is Borel-regular. Is it true that μ is a measure on X if for any $A \subseteq X$ the sequence $\{\mu_n(A)\}$ is decreasing?

Yes.

Proof. First,

$$\mu(\varnothing) = \lim_{n\to\infty} \mu_n(\varnothing) = \lim_{n\to\infty} 0 = 0.$$

Next, μ is is monotone. Suppose $A \subseteq B \subseteq X$, then

$$\mu(A) = \lim_{n\to\infty} \mu_n(A) \leq \lim_{n\to\infty} \mu_n(B) = \mu(B).$$

Let $A_1, A_2, \cdots \subseteq X$ be mutually disjoint Borel sets. Since each μ_n is Borel-regular, then each A_i is μ_n-measurable. Then,

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \lim_{n\to\infty} \mu_n\left(\bigcup_{i=1}^{\infty} A_i\right) = \lim_{n\to\infty} \sum_{i=1}^{\infty} \mu_n(A_i).$$

Since $\mu_n(A_i) \leq \mu_1(A_i)$ and $\sum_{i=1}^{\infty} \mu_1(A_i) = \mu_1(\bigcup_{i=1}^{\infty} A_i) < \infty$, we may perform the interchange

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \lim_{n\to\infty} \sum_{i=1}^{\infty} \mu_n(A_i) = \sum_{i=1}^{\infty} \lim_{n\to\infty} \mu_n(A_i) = \sum_{i=1}^{\infty} \mu(A_i).$$

This shows μ is countably additive on disjoint Borel sets.

Now, suppose the A_i's are not necessarily disjoint Borel sets. Define the Borel sets $B_1 = A_1$ and $B_i = A_i \setminus B_{i-1}$ for $i \geq 2$. Then $\bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} B_i$ and $B_i \subseteq A_i$, so

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(B_i) \leq \sum_{i=1}^{\infty} \mu(A_i).$$

This shows μ is countably subadditive on Borel sets.

Now, suppose the A_i's are arbitrary sets. For each n, i, there is a Borel set $B_{n,i} \subseteq X$ such that $A_i \subseteq B_{n,i}$ and $\mu_n(A_i) = \mu_n(B_{n,i})$. Define the Borel set $B_i = \bigcap_{n=1}^{\infty} B_{n,i}$, then $A_i \subseteq B_i \subseteq B_{n,i}$. So

$$\mu_n(A_i) \leq \mu_n(B_i) \leq \mu_n(B_{n,i}) = \mu_n(A_i),$$

implying $\mu_n(A_i) = \mu_n(B_i)$. Letting $n \to \infty$, we find $\mu(A_i) = \mu(B_i)$. Then

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \mu\left(\bigcup_{i=1}^{\infty} B_i\right) \leq \sum_{i=1}^{\infty} \mu(B_i) = \sum_{i=1}^{\infty} \mu(A_i).$$

This shows μ is countably subadditive on all sets.

$\square$

3 Let X be a nonempty set and F be a collection of functions $f : X \to \mathbb{R}$ with the following properties:

(i) The constant function $f(x) \equiv 1 \in F$, and if $f, g \in F$ and $c \in \mathbb{R}$, then $f+g, fg, cf \in F$.

(ii) If a sequence $\{f_n\} \subseteq F$ has as pointwise limit in X: $f(x) = \lim_{n\to\infty} f_n(x)$ for all $x \in X$, then $f \in F$.

Prove that the collection $\mathcal{A} = \{A \subseteq X : \chi_A \in F\}$ is a σ-algebra, where χ_A is the characteristic function of the set A.

Proof. Since $\chi_X = 1 \in F$, we know $X \in \mathcal{A}$.

If $A \in F$, then $\chi_{A^c} = 1 - \chi_A \in F$, implying $A^c \in \mathcal{A}$.

Suppose $A_1, A_2, \ldots \in \mathcal{A}$ and let $A = \bigcap_{i=1}^{\infty} A_i$. For $n \in \mathbb{N}$, define $B_n = \bigcap_{i=1}^{n} A_i$, then

$$\chi_{B_n} = \prod_{i=1}^{n} \chi_{A_i} \in F.$$

Then $\chi_{B_n} \to \chi_A$ pointwise in X, so $\chi_A \in F$, implying $A \in \mathcal{A}$.

□

4 Prove that any open subset of $\mathbb{R}^n$ can be expressed as a countable union of closed balls in $\mathbb{R}^n$

Remark. The statement is true for any separable metric space X.

Proof. Let X be a separable metric space and let $Y \subseteq X$ be a countable dense subset. Let $U \subseteq X$ be an open subset. For each $x \in U$, define the radius $r_x = d(x, U^c) > 0$ and the closed ball $E_x = \overline{B_{r_x/2}(x)}$. We claim that $U = \bigcup_{y \in U\cap Y} E_y$ (a countable union of closed balls).

By construction, $E_y \subset B_{r_y}(y) \subseteq U$ for all $y \in U \cap Y$, so $\bigcup_{y\in U\cap Y} E_y \subseteq U$.

For each $x \in U$, consider the open neighborhood $B_{r_x/4}(x) \subseteq U$ of x. Since Y is a dense subset of X, we can find some $y \in B_{r_x/4}(x)$. Then

$$r_x \leq d(x, y) + d(y, U^c) = d(x, y) + r_y \leq \tfrac{1}{4} r_x + r_y,$$

so $\frac{3}{4} r_x \leq r_y$. Then

$$d(x, y) \leq \tfrac{1}{4} r_x \leq \tfrac{1}{2} \cdot \tfrac{3}{4} r_x \leq \tfrac{1}{2} r_y,$$

so $x \in E_y$. Hence, $U \subseteq \bigcup_{y\in U\cap Y} E_y$, proving the equality.

□