Algebraic Topology

Fall 2022

requires homotopy stuff and some group theory. helpful to have some category theory.

9/22/2022

Why algebraic topology?

We want to learn about spaces, but spaces are squishy and complicated so we want to reduce them down to something more rigid.

Idea: look for algebraic 'invariants' (maps from spaces to algebraic things, where homeomorphic and homotopy equivalent spaces give the same thing).

Example: Euler Characteristic. Take a surface and tile it with polygons, then

$$\chi = (\#\text{vertices}) - (\#\text{edges}) + (\#\text{faces}).$$

Claim: this doesn't depend on the particular tiling, only the original surface. To prove this, show that certain moves/operations on the tilings doesn't change χ and that the set of moves is enough to get between any two tilings. Homology will tell us how this works out.

More generally: someone gives you a space defined implicitly, e.g., given (X, x_0) nice space define ΩX to be the set of loops in X based at x_0 with the C^0 topology (supremum distance between loops as functions $[0,1] \to X$).

What is homology?

Slogan: $H_n(X)$ (nth homology group of X) measures the "n-dimensional holes" of X.

What should This mean?

Example: n-sphere S^n should have an n-dimensional hole.

How to formalize this?

1. homotopy groups.

$$\pi_n(X, x_0) = \{\text{based maps } (S^n, s_0) \to (X, x_0)\}/\text{homotopy}$$

Problems: hard to compute; $\pi_2(S^2) = \mathbb{Z}$, $\pi_3(S^2) = \mathbb{Z}$, $\pi_4(S^2) = Z_2$, $\pi_5(S^2) = Z_2$, $\pi_6(S^2) = Z_{12}$, $\pi_7(S^2) = Z_2$, etc.

2. cycles and boundaries. An n-chain is some sort of n-dimensional subsurface. Say two n-chains are equivalent if together they form the boundary of an $(n+1)$-chain. Say an n-dimensional hole is a n-cycle which is not the boundary of an $(n+1)$-chain. Issues when trying to make this rigorous: multiplicity, orientation, intersections, need machinery to compute.

How is H_1 different from π_1? e.g., punctured torus deformation retracts to figure-eight with $\pi_1 \cong F_2$. But the embedded figure-eight is simply the boundary of the 2-chain formed by the rest of the surface; it turns out H_1 is abelian (in fact H_1 is the abelianization π_1/commutators).

9/27/22

Simplicial Homology

Idea: Impose strong restriction on what is a chain.

The **standard n-simplex** is constructed as

$$\Delta^n = \{(t_0, \dots, t_n) \in \mathbb{R}^{n+1} \mid t_i \geq 0, \textstyle\sum_i t_i = 1\}.$$

Equivalently, Δ^n is the convex hull of the vertices $e_1, \dots, e_{n+1}$.

Call the tuple $(t_0, \dots, t_n)$ the **Barycentric coordinates**.

A **face** of Δ^n is a copy of Δ^{n-1} obtained by setting $t_i = 0$ (called the i-th face for each i).

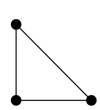
An **abstract simplicial complex** consists of

- a set V of vertices;
- a set D of finite subsets of V, closed under $\subseteq$ (taking faces/subsimplices), which tells you which simplices are included.

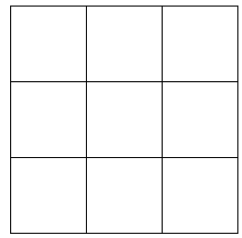
The **geometric realization** of an abstract simplicial complex is constructed by taking simplices corresponding to the elements of D and gluing faces according to the recipe. Give this space the quotient topology.

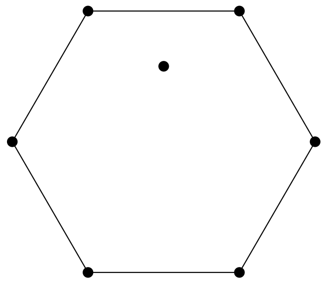
Examples

S^1 homeomorphic to



Torus T^2 homeomorphic to





An n-**chain** in a simplicial complex X is a formal linear combination of n-simplices.

There is a **boundary homomorphism**

$$C_n(X) \xrightarrow{\partial_n} C_{n-1}(X)$$

which sends each simplex to its boundary and extends by linearity.

$\partial[v_0, v_1] = [v_1] - [v_0]$.

$\partial[v_0, v_1, v_2] = [v_1, v_2] - [v_0, v_2] + [v_0, v_1]$.

$\partial[v_0, v_1, v_2, v_3] = [v_1, v_2, v_3] - [v_0, v_2, v_3] + [v_0, v_1, v_3] - [v_0, v_1, v_2]$.

In general,

$$\partial[v_0, \dots, v_n] = \sum_i (-1)^i [v_0, \dots, \widehat{v}_i, \dots, v_n].$$

To ways to fully formalize this:

1. Global ordering on vertices, giving fixed orientation on each simplex;
2. include ordering on the notation, adopt convention of putting arrows on edges and writing $[v_0, v_1] = -[v_1, v_0]$.
3. Then $[v_0, \dots, v_0] = \mathrm{sign}(\sigma)[v_{\sigma(0)}, \dots, v_{\sigma(n)}]$ for any permutation σ (equivalently, should be sign of determinant of transformation taking one to the other).

Let X be a simplicial complex.

Have n-chains $C_n(X)$.

Have n-cycles $Z_n(X) = \ker \partial_n \subseteq C_n(X)$.

Have n-boundaries $B_n(X) = \mathrm{im}\, \partial_{n+1} \subseteq Z_n(X)$.

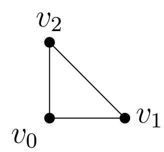
Have **chain complex**

$$\cdots \xrightarrow{\partial_{n+2}} C_{n+1}(X) \xrightarrow{\partial_{n+1}} C_n(X) \xrightarrow{\partial_n} C_{n-1}(X) \xrightarrow{\partial n-1} \cdots$$

The n**th simplicial homology group** of X is $H_n(X) = Z_n(X)/B_n(X)$. An element of $H_n(X)$ is called a **homology class**. Two cycles in the same class are said to be **homologous**, i.e., they differ by the boundary of something.

Example

S_1



$H_0(S^1) = \langle v_0, v_1, v_2 \rangle \,/\, \langle v_1 - v_0, v_2 - v_1, v_0 - v_2 \rangle \cong \mathbb{Z}$ with $[v_0] = [v_1] = [v_2]$.

$H_1(S^1) = Z_1(S^1)/B_1(S^1)$. Have $B_1(S^1) = 0$ because no 2-simplices, and

$$Z_1(S^1) = \langle [v_1, v_2] - [v_0, v_1] + [v_0, v_1] \rangle \cong \mathbb{Z}.$$

Δ-Complexes