1 A space X is *locally (path-)connected at x* if for every neighborhood N of x, there is a (path-)connected neighborhood $N' \subseteq N$ of x. It is *locally (path-)connected* if it is locally (path-)connected at every point.

(a) Show that a connected, locally path-connected space is path-connected.

Proof. For any $x \in X$, let $P_x \subseteq X$ be its path-connected component, i.e., P_x is the set of points in X with a path to x.

We claim that P_x is an open subset of X, and will prove this directly. Let $y \in P_x$, so there is a path between x and y. Since X is locally path-connected, there is some path-connected neighborhood N of y. For each $z \in N$, there is a path between y and z. We can then join the x-y path and the y-z path to obtain a path between x and z. This means that $z \in P_x$, hence $N \subseteq P_x$. This shows that $P_x \subseteq X$ is open for every $x \in X$.

Let $x \in X$; we claim that $P_x = X$. Assume not, then the set

$$U = \bigcup_{y \in X \setminus P_x} P_y$$

is nonempty. As the union of open sets, $U \subseteq X$ is open. Moreover, if $y \in X \setminus P_x$, then P_y must be disjoint from P_x; otherwise, there would be a path between x and y, constructed out of paths between each and a point in the intersection. This implies that U is disjoint from P_x. We now have $X = P_x \sqcup U$, where P_x and U are open. In other words, X is the disjoint union of nonempty open subsets, which is a contradiction. Hence, X is path-connected.

$\square$

(b) Construct a subspace of $\mathbb{R}^2$ which is path-connected, but not locally connected.

Define $L_0 = [0,1] \times \{0\} \subseteq \mathbb{R}^2$. For each $n \in \mathbb{N}$, let $L_n \subseteq \mathbb{R}^2$ be the line segment between the origin and the point $(1, 1/n) \in \mathbb{R}^2$. Define $X = \bigcup_{n=0}^{\infty} L_n$.

Each line segment L_n is trivially path-connected and each contains the origin. Hence, X is path-connected as any two points have paths to the origin and, therefore, to each other.

For any $n \in \mathbb{N}$, we can split X along the line $y = 2x/(2n+1)$. That is, define the open halfspaces

$$H_n = \{(x, y) \in \mathbb{R}^2 : y > 2x/(2n+1)\}$$

and

$$K_n = \{(x, y) \in \mathbb{R}^n : y < 2x/(2n+1)\}.$$

Then $X = \{0,0\} \sqcup (H_n \cap X) \sqcup (K_n \cap X)$, where $H_n \cap X$ and $K_n \cap X$ are open in the subspace topology on $X \subseteq \mathbb{R}^2$.

We will show that X is not locally connected. Consider the point $x = (1,0) \in X$, and any neighborhood $N \subseteq B_1(x) \cap X$ of x. Note that N does not contain the origin, so N can be written as the disjoint union of open subsets $N = (H_n \cap N) \sqcup (K_n \cap N)$. There is some $\varepsilon > 0$ such that $B_\varepsilon(x) \cap X \subseteq N$, and some $n \in \mathbb{N}$ such that $1/n < \varepsilon$. Then $L_n \cap B_\varepsilon(x) \subseteq H_n \cap N$ is nonempty, as it contains the point $(1, 1/n)$. Moreover, $K_n \cap N$ is nonempty, as it contains $x = (1,0)$. Hence, N is the disjoint union of nonempty open subsets, i.e., disconnected.

2
(a) Show that every countable metric space is totally disconnected.

Proof. Let (X, d) be a countable metric space.

Suppose $x, y \in X$ are distinct points, and let $R = d(x, y) > 0$.

We claim that there is some radius $r \in \mathbb{R}$ such that $0 < r < R$ and no $z \in X$ has $d(x, z) = r$. If there were some $z \in X$ with $d(x, z) = r$ for every $r \in (0, R)$, then we could construct an injection $(0, R) \to X$, using the axiom of choice, with $r \mapsto z$. This would imply that X is uncountable, as the real interval $(0, R)$ is uncountable.

Let $r \in (0, R)$ such that $d(x, z) \neq r$ for all $z \in X$. Define the disjoint sets

$$U = \{z \in X : d(x, z) < r\} \quad \text{and} \quad V = \{z \in X : d(x, z) > r\}.$$

By choice of r, we know U and V cover X. Moreover, $x \in U$ and $y \in U$, so both are nonempty. Lastly, $U = B_r(x)$ is open, implying that $V = X \setminus \overline{U}$ is also open. Hence, X is the disjoint union of the nonempty open subsets U and V.

If $C \subseteq X$ is the connected component containing x, then we can write $C = (U \cap C) \sqcup (V \cap C)$, where $U \cap C$ and $V \cap C$ are open in the subspace topology on C. Since $U \cap C$ contains x, it is nonempty. Since C is connected, we must have $V \cap C$ be empty; in particular, $y \notin C$.

We have shown that $x, y \in X$ are in the same connected component of X if and only if $x = y$, which means that every connected component is a singleton. Hence, X is totally disconnected.

$\square$

(b) Show that if $A \subset \mathbb{R}^2$ is countable, then $\mathbb{R}^2 \setminus A$ is path-connected.

Proof. Denote $X = \mathbb{R}^2 \setminus A$.

Let $x \in X$ and define L to be the collection of lines in $\mathbb{R}^2$ passing through the point x. In other words, the sets in L correspond to the 1-dimensional subspaces of the real vector space $\mathbb{R}^2$, shifted by the vector $x \in \mathbb{R}^2$. Note that L is at least uncountable, since any choice of slope in $\mathbb{R}$ corresponds to a line through x with that slope. Moreover, any two distinct lines in L intersect only at x.

We claim that there is some line $\ell \in L$ such that $\ell \cap A = \varnothing$. If there were some $a \in \ell \cap A$ for every $\ell \in L$, the we could construct an injection $L \to A$ by $\ell \mapsto a$, using the axiom of choice. This would imply that A is uncountable, as L is uncountable. Let ℓ be a line in $\mathbb{R}^2$ passing through x and disjoint from A.

For any other $y \in X$, we use the same argument to construct a line ℓ' in $\mathbb{R}^2$ through y and disjoint from A. Without loss of generality, we can assume ℓ' is not parallel to ℓ; otherwise we repeat the argument excluding this line, as there are still uncountably many lines through y not parallel to ℓ.

We now have $x, y \in \ell \cup \ell' \subseteq X$. Since ℓ and ℓ' are not parallel, they intersect at some point $z \in \mathbb{R}^2$. Then X contains a line segment between x and z, and a line segment between z and y, which join to give a path in X between x and y. Hence, X is path-connected.

$\square$

3 Let X be a metric space and A a subset. For a point $x \in X$, we define

$$d(x, A) = \inf_{a \in A} d(x, a).$$

(a) Show that $d(x, A) = 0$ if and only if $x \in \bar{A}$.

Proof. Suppose $d(x, A) = 0$, so for every $\varepsilon > 0$, there is some $a \in A$ such that $d(x, a) < \varepsilon$. This implies $a \in B_\varepsilon(x)$, i.e., $B_\varepsilon(x) \cap A \neq \varnothing$. Since this holds for all $\varepsilon > 0$ and every neighborhood of x contains a ball of some positive radius, then every neighborhood of x has a nonempty intersection with A. Hence, $x \in \overline{A}$.

Suppose $x \in \overline{A}$. For every $\varepsilon > 0$, the open ball $B_\varepsilon(x)$ is a neighborhood of x, so $B_\varepsilon \cap (A) \neq \varnothing$. That is, there is some $a \in A$ such that $d(x, a) < \varepsilon$, implying $d(x, A) \leq \varepsilon$. Since this holds for all $\varepsilon > 0$, we in fact have $d(x, A) = 0$.

$\square$

(b) Show that if A is compact, $d(x, A) = d(x, a)$ for some $a \in A$.

Proof. For each $n \in \mathbb{N}$, there is some $a_n \in A$ such that $d(x, a_n) < d(x, A) + 1/n$, which means that $d(x, a_n) \to d(x, A)$. Then $\{a_n\}$ is a sequence in the compact set A, so there is a convergent subsequence $a_{n_k} \to a \in A$. Since $d(x, -)$ is a continuous function, we have convergence $d(x, a_{n_k}) \to d(x, a) = d(x, A)$.

$\square$

(c) Define the *ε-neighborhood* of A in X to be the set

$$U(A, \varepsilon) = \{x : d(x, A) < \varepsilon\}.$$

Show that $U(A, \varepsilon)$ is the union of the open balls $B_d(a, \varepsilon)$ for $a \in A$.

Proof. Given $x \in U(A, \varepsilon)$, we have $d(x, A) < \varepsilon$. That is, there is some $\delta > 0$ such that $d(x, A) = \varepsilon - \delta$. Then, for all $\eta > 0$, there is some $a \in A$ such that $d(x, a) < (\varepsilon - \delta) + \eta$. In particular, take $\eta = \delta$, so there is some $a \in A$ such that $d(x, a) < (\varepsilon - \delta) + \delta = \varepsilon$. In other words, $x \in B_\varepsilon(a)$.

If $x \in B_\varepsilon(a)$ for some $a \in A$, then $d(x, A) \leq d(x, a) < \varepsilon$, implying that $x \in U(A, \varepsilon)$.e

$\square$

(d) If A is compact and U is an open set containing A, show that some ε-neighborhood of A is contained in U.

Proof. Assume, for contradiction, that $U(A,\varepsilon) \cap U^c \neq \varnothing$ for all $\varepsilon > 0$. Then, for every $n \in \mathbb{N}$, there is some $x_n \in U^c$ such that $d(x_n, A) < \frac{1}{n}$. Moreover, since A is compact, there is some $a_n \in A$ such that $d(x_n, a_n) = d(x_n, A)$. Then, $\{a_n\}$ is a sequence in the compact set A, so there is a convergence subsequence $a_{n_k} \to a \in A$. The triangle inequality gives us

$$d(x_{n_k}, a) \leq d(x_{n_k}, a_{n_k}) + d(a_{n_k}, a),$$

which converges to zero as $k \to \infty$. This means that for every $\varepsilon > 0$, there is some $k \in \mathbb{N}$ such that $x_{n_k} \in B_\varepsilon(a)$, i.e., $B_\varepsilon(a) \cap U^c \neq \varnothing$. However, this contradicts the fact that $a \in A \subseteq U$ and U open implies that there is some $\varepsilon > 0$ such that $B_\varepsilon(x) \subseteq U$.

□

(e) Show that the result of (d) need not hold if A is closed but not compact.

Consider $X = \mathbb{R}^2$ and let A be the x-axis, a closed set. Define

$$U = \{(x, y) : x > 0 \text{ and } y < 1/x\}$$

Then for any $\varepsilon > 0$, $U(A,\varepsilon)$ contains the points (x, ε) for all $x > 0$. However, $1/\varepsilon \geq 1/\varepsilon$, which implies that the point $(1/\varepsilon, 1/\varepsilon)$ is in $U(A,\varepsilon)$ but not in U.

4 Given a nice enough path-connected subset $X \subseteq \mathbb{R}^n$, we can define the *intrinsic distance* between two points of X by

$$d(x,y) = \inf\left\{\int_0^1 |\dot\gamma(t)|dt : \gamma \text{ is an almost everywhere differentiable path from } x \text{ to } y\right\}.$$

Give an example of an $X \subset \mathbb{R}^2$ which is compact and path-connected, yet the intrinsic distance takes every positive real value (and hence is infinite for some pairs of points).

The moral is that a path being "infinitely long" is not a topological property and does not prevent you from being path-connected!

Optional: Is there an ambient self-homeomorphism of $\mathbb{R}^2$ that takes your infinite path to a finite one? Show that such a homeomorphism can't be differentiable.

Take the topologist's sine curve for $x \in (0,1]$, including the segment from -1 to 1 on the y-axis. This is a closed bounded subset of $\mathbb{R}^2$ and, therefore, compact. Add a path from the rightmost point of the curve, looping around the entire curve, and connecting to the origin on the left. This is, again, a compact set.

Moreover, it is path-connected. However, no path can pass from $x > 0$ to $x = 0$ through the sine curve; it must move through the added loop. Then for very small values of $x > 0$, the minimum length of a path from the point on the curve with that value of x to the origin is arbitrarily large, so the intrinsic distance attains every positive real.

(Note: I'm not sure how this would imply infinite intrinsic distance for some pairs of points, because it seems like every pair of points has some finite path, utilizing the added loop.)