1 Exercise 5.6.1 Find all closed points of the real affine plane $\mathbb{A}^2_{\mathbb{R}}$. What are their residue fields?

As an affine scheme, we consider $\mathbb{A}^2_{\mathbb{R}} = \operatorname{Spec} \mathbb{R}[x, y]$, whose closed are precisely the maximal ideals of $\mathbb{R}[x, y]$, i.e., the elements of $\operatorname{mSpec} \mathbb{R}[x, y]$.

Denote $R = \mathbb{R}[x, y]$ and $C = \mathbb{C}[x, y]$.

The inclusion $R \hookrightarrow C$ induces a morphism $\operatorname{Spec} C \to \operatorname{Spec} R$ sending $\mathfrak{p} \mapsto \mathfrak{p} \cap R$. Since C is an integral ring extension of R by i, where $i^2 - 1 = 0$, the "Lying Over" property (Gathmann, Commutative Algebra, Proposition 9.18) tells us that for every $\mathfrak{p} \in \operatorname{Spec} R$ there is a point $\mathfrak{q} \in \operatorname{Spec} C$ such that $\mathfrak{p} = \mathfrak{q} \cap R$. In other words, the intersection map $\operatorname{Spec} C \to \operatorname{Spec} R$ is surjective. As maximality is preserved (Gathmann, Commutative Algebra, Corollary 9.21(b)), this restricts to a surjective map $\operatorname{mSpec} C \to \operatorname{mSpec} R$.

In other words, the maximal ideals of $\mathbb{R}[x, y]$ are simply the maximal ideals of $\mathbb{C}[x, y]$, restricted to their real elements.

Since $\mathbb{C}$ is algebraically closed, the maximal ideals of $\mathbb{C}[x, y]$ are the ideals of the form $\langle x - a, y - b \rangle$ with $a, b \in \mathbb{C}$.

If a and b are real, then the intersection with $\mathbb{R}[x, y]$ is simply the ideal of $\mathbb{R}[x, y]$ with the same generators: $\langle x - a, y - b \rangle$. In which case the residue field is $\mathbb{R}$.

If a is complex and b is real, then the intersection with $\mathbb{R}[x, y]$ is the ideal $\langle (x-a)(x-\overline{a}), y-b \rangle$, where $\overline{a}$ is the complex conjugate of a. In which case the residue field is $\mathbb{C}$.

If both a and b are complex, a change of variables gives the first case, again.

2 Exercise 5.6.2 Let $f(x,y) = y^2 - x^2 - x^3$. Describe the affine scheme $X = \operatorname{Spec} R/\langle f\rangle$ set-theoretically for the following rings R:

By the correspondence theorem for rings, there is a natural bijection between the ideals of $R/\langle f\rangle$ and the ideals of R containing $\langle f\rangle$. Moreover, an ideal $I \trianglelefteq R$ containing $\langle f\rangle$ is prime if and only if its quotient $I/\langle f\rangle \trianglelefteq R/\langle f\rangle$ is prime. Hence, there is a natural bijection between the elements of the set $\operatorname{Spec} R/\langle f\rangle$ and the prime ideals of R containing $\langle f\rangle$.

(i) $R = \mathbb{C}[x,y]$ (the standard polynomial ring),

As an affine variety, $X = Z(f) \subseteq \mathbb{A}^2_{\mathbb{C}}$ is irreducible and 1-dimensional. Therefore, the only proper irreducible closed subvarieties of X are 0-dimensional. By the Nullstellensatz, this equivalently means that the only prime ideals of $\mathbb{C}[x,y]$ strictly containing $\langle f\rangle$ are maximal. The maximal ideals of $\mathbb{C}[x,y]$ are of the form $\langle x-a, y-b\rangle$ with $a,b \in \mathbb{C}$, and in order for such an ideal to contain $\langle f\rangle$, we must have $f(a,b) = 0$. In other words, the maximal ideals of $\mathbb{C}[x,y]$ containing $\langle f\rangle$ correspond bijectively with the points $(a,b) \in \mathbb{A}^2_{\mathbb{C}}$ which lie on the curve defined by f.

We can parameterize this curve by $\mathbb{A}^1_{\mathbb{C}}$ where $t \mapsto (t^2-1, t^3-t^2) \in \mathbb{A}^2_{\mathbb{C}}$. (This is injective everywhere except for $\pm 1 \mapsto (0,0)$.) Then the underlying set of X can be written as

$$\{\langle 0\rangle\} \cup \{\langle x - t^2 + 1, y - t^3 + t^2\rangle \mid t \in \mathbb{A}^1_{\mathbb{C}}\},$$

where the ideals are generated in $\mathbb{C}[x,y]/\langle f\rangle$.

(ii) $R = \mathbb{C}[x,y]_{\langle x,y\rangle}$ (the localization of the polynomial ring at the origin),

Here, the elements of R correspond to the prime ideals of $\mathbb{C}[x,y]$ contained in $\langle x,y\rangle$.

Then $\operatorname{Spec} R/\langle f\rangle$ is the curve from part (i) without the origin. The elements of the set correspond to the prime ideals of $\mathbb{C}[x,y]$ contained in $\langle x,y\rangle$ and containing $\langle f\rangle$.

(iii) $R = \mathbb{C}[[x,y]]$ (the ring of formal power series).

3 Exercise 5.6.3 For each of these cases below give an example of an affine scheme X with that property, or prove that such an X does not exist:

(i) X has infinitely many points, and $\dim X = 0$.

Let k be a field and Λ be an infinite indexing set. Consider $R = k^\Lambda = \{(a_\lambda)_{\lambda\in\Lambda} \mid a_\lambda \in k\}$, i.e., the product ring of copies of k indexed by Λ.

For each $\lambda \in \Lambda$ there is a maximal ideal

$$\mathfrak{m}_\lambda = \{a \in R \mid a_\lambda = 0\}.$$

(I was originally assuming all the ideals of R were products of ideals in each component, but this is not true for infinite product rings. Every prime ideal which happens to be a product of ideals is in fact maximal, which points toward the Krull dimension of R being zero. I cannot find any prime ideals not of this form which are also not maximal. So this may still work, but I would not know how to prove it.)

(ii) X has exactly one point, and $\dim X = 1$.

Does not exist.

If $X = \operatorname{Spec} R$ has exactly one point, then R has a unique prime ideal. In which case, the maximum length of an ascending chain of prime ideals is 1, i.e., $\dim X = \dim R = 0$.

(iii) X has exactly two points, and $\dim X = 1$.

Take the localization $R = \mathbb{C}[x]_{\langle x\rangle}$. Then R has only the prime ideals $\langle 0\rangle \subset \langle x\rangle$. The corresponding scheme $X = \operatorname{Spec} R$ therefore has exactly two points and $\dim X = \dim R = 1$.

(iv) $X = \operatorname{Spec} R$ with $R \subseteq \mathbb{C}[x]$, and $\dim X = 2$.

Take $R = \mathbb{Q}[x, c]$, where $c \in \mathbb{C}$ is not algebraic over $\mathbb{Q}$, e.g., e or π. Then R is isomorphic to the ring of polynomials over $\mathbb{Q}$ in two variables, and therefore has krull dimension 2. We obtain the scheme $X = \operatorname{Spec} R \cong \mathbb{A}^2_{\mathbb{Q}}$ with dimension 2.

4 Exercise 5.6.4 Let X be a scheme, and let Y be an irreducible closed subset of X. If η_Y is the generic point of Y, we write $\mathcal{O}_{X,Y}$ for the stalk $\mathcal{O}_{X,\eta_Y}$. Show that $\mathcal{O}_{X,Y}$ is "the ring of rational functions on X that are regular at a general point of Y," i.e., it is isomorphic to the ring of equivalence classes of pairs (U, φ), where $U \subseteq X$ is open with $U \cap Y \neq \varnothing$ and $\varphi \in \mathcal{O}_X(U)$, and where two such pairs (U, φ) and (U', φ') are called equivalent if there is an open subset $V \subseteq U \cap U'$ with $V \cap Y \neq \varnothing$ such that $\varphi|_V = \varphi'|_{V'}$.

By definition, we have the stalk

$$\mathcal{O}_{X,\eta_Y} = \{\overline{(U, \varphi)} \mid \eta_Y \in U \subseteq X \text{ open}, \varphi \in \mathcal{O}_X(U)\},$$

where $(U, \varphi) \sim (V, \psi)$ if $\varphi|_W = \psi|_W$ for some $\eta_Y \in W \subseteq U \cap V$ open. Also by definition, we have $Y = \overline{\{\eta_Y\}} \subseteq X$, so an open subset $U \subseteq X$ contains η_Y if and only if $U \cap Y \neq \varnothing$. We can simply rewrite the stalk as

$$\mathcal{O}_{X,Y} = \{\overline{(U, \varphi)} \mid U \subseteq X \text{ open}, U \cap Y \neq \varnothing, \varphi \in \mathcal{O}_X(U)\},$$

where $(U, \varphi) \sim (V, \psi)$ if $\varphi|_W = \psi|_W$ for some $W \subseteq U \cap V$ open with $W \cap Y \neq \varnothing$.

5 Exercise 5.6.5 Let X be a scheme of finite type over an algebraically closed field k. Show that the closed points of X are dense in every closed subset of X.

First, consider the affine case $X = \operatorname{Spec} R$, where R is a finitely generated k-algebra. Consider a nonempty distinguished open set $X_f = \operatorname{Spec} R_f$, i.e., $f \in R$ such that R_f is not the zero ring or, equivalently, f is not nilpotent. By Gathmann Corollary 10.13, the fact that R is a finitely generated k-algebra implies that

$$\sqrt{\langle 0 \rangle} = \bigcap_{\mathfrak{m} \in \operatorname{mSpec} R} \mathfrak{m}.$$

($f \notin \sqrt{\langle 0 \rangle}$), so there is some $\mathfrak{m} \in \operatorname{mSpec} R$ such that $f \notin \mathfrak{m}$. That is, $f(\mathfrak{m}) \neq 0$ so $\mathfrak{m} \in X_f$. Since the distinguished open subsets in X form a basis for the Zariski topology and every distinguished open subset contains a closed point, we conclude that every open subset of X contains a closed point. Hence, the closed points of X are dense in X.

Explicitly, if we denote the set of closed points of X by

$$X_0 = \{\mathfrak{p} \in X \mid \overline{\mathfrak{p}} = \{\mathfrak{p}\}\},$$

then we have shown $\overline{X_0} = X$.

We may consider a closed subset $Z(I) \subseteq X$ as an affine subscheme $\operatorname{Spec} R/I \to X$. Moreover, the corresponding ring homomorphism $R \to R/I$ gives R/I as a finitely generated k-algebra, so $Y = \operatorname{Spec} R/I$ is an affine scheme of finite type over k. It follows from the first case that the closed points of Y are dense in Y. Additionally, the subscheme inclusion $Y \to X$ describes topological embedding whose image is $Z(I)$, so the closed points of Y correspond to the closed points of X contained in $Z(I)$, which are similarly dense.

For a general scheme X of finite type over k, we choose a finite affine open cover $\{U_i\}_{i=1}^n$ with $U_i = \operatorname{Spec} R_i$ such that each R_i is a finitely generated k-algebra. As each $U_i \subseteq X$ has the subspace topology, the closed points of U_i are simply the closed points of X contained in U_i, i.e., $(U_i)_0 = X_0 \cap U_i$. As as instance of the second case, the closed points of U_i are dense in $Y \cap U_i$, so we conclude that

$$Y = \bigcup_{i=1}^n (Y \cap U_i) = \bigcup_{i=1}^n \overline{Y \cap (U_i)_0} = \overline{\bigcup_{i=1}^n (Y \cap (U_i))_0} = \overline{Y \cap X_0}.$$

Conversely, give an example of a scheme X such that the closed points of X are not dense in X.

From Problem 3(iii), consider $R = \mathbb{C}[x]_{\langle x \rangle}$ with the prime ideals $\langle 0 \rangle, \langle x \rangle$. Then $\langle x \rangle$ is a closed point of $X = \operatorname{Spec} R$, but $\{\langle 0 \rangle\}$ is an open subset of X containing no closed points. Hence, the closed points of X are not dense in X.