1 Let X and Y be topological spaces.

(a) Show that for any topological space T, a function $f : T \to X \times Y$ is continuous if and only if the compositions $p_X f : T \to X$, $p_Y f : T \to Y$ are continuous. Here p_X and p_Y are the obvious projection maps.

Proof. Suppose f is continuous. Let $U \subseteq X$ be open, then $p_X^{-1}(U) = U \times Y \subseteq X \times Y$ is open, as it is in the basis for the product topology. And since f is continuous,

$$(p_X f)^{-1}(U) = f^{-1}(U \times Y) \subseteq T$$

is open. This shows $p_X f$ is continuous, and it is the same to show that $p_Y f$ continuous.

Suppose both $p_X f$ and $p_Y f$ are continuous. It suffices to prove that the preimages under f of basis elements are open. Let $U \subseteq X$ and $V \subseteq Y$ be open, so $U \times V \subseteq X \times Y$ is an arbitrary basis element of the product topology. We can rewrite $U \times V$ as

$$U \times V = (U \times Y) \cap (X \times V) = p_X^{-1}(U) \cap p_Y^{-1}(V),$$

then

$$f^{-1}(U \times V) = (p_X f)^{-1}(U) \cap (p_Y f)^{-1}(V).$$

Since $p_X f$, $p_Y f$ are continuous and U, V are open, $f^{-1}(U \times V) \subseteq T$ is is also open, proving that f is continuous.

$\square$

(b) Let Z be a topological space with maps $g_X : Z \to X$ and $g_Y : Z \to Y$. Suppose that for every space T and pair of continuous functions $f_X : T \to X$ and $f_Y : T \to Y$, there is a unique continuous function $f : T \to Z$ such that $f_X = g_X f$ and $f_Y = g_Y f$.

Show that Z must be homeomorphic to $X \times Y$ with the product topology and g_X and g_Y are (taken by the homeomorphism to) p_X and p_Y.

Proof. Consider $X \times Y$ with the natural projections p_X and p_Y. By the stated universal property, there is a unique continuous function $p : X \times Y \to Z$ such that $p_X = g_X p$ and $p_Y = g_Y p$. We construct a function $g = (g_X, g_Y) : Z \to X \times Y$, where $z \mapsto (g_X(z), g_Y(z))$. We have $p_X g = g_X$ and $p_Y g = g_Y$, so 1(a) implies g is continuous. We claim that p and g are inverses and, in which case, describe a homeomorphism between $X \times Y$ and Z.

First, it can be seen that

$$gp = (g_X p,\, g_Y p) = (p_X,\, p_Y) = \mathrm{id}_{X\times Y}\,.$$

To prove the opposite composition is the identity on Z, we construct the continuous functions

$$f_X : Z \xrightarrow{g} X \times Y \xrightarrow{p} Z \xrightarrow{g_X} X \quad \text{and} \quad f_Y : Z \xrightarrow{g} X \times Y \xrightarrow{p} Z \xrightarrow{g_Y} Y.$$

By the universal property, there is a unique continuous function $f : Z \to Z$ such that $f_X = g_X f$ and $f_Y = g_Y f$. On one hand, the constructions of f_X and f_Y imply that $f = pg$. On the other hand, we have

$$f_X = g_X pg = p_X g = g_X \quad \text{and} \quad f_Y = g_Y pg = p_Y g = g_Y,$$

which would imply that $f = \mathrm{id}_Z$. Therefore, $pg = f = \mathrm{id}_Z$.

$\square$

2 Similarly to the previous problem, suppose that Z is a space equipped with maps $q_X : X \to Z$ and $q_Y : Y \to Z$, and that

> for every space T and pair of continuous functions $f_X : X \to T$ and $f_Y : Y \to T$, there is a unique continuous function $f : Z \to T$ such that $f_X = fq_X$ and $f_Y = fq_Y$.

Show that Z is homeomorphic to $X \sqcup Y$ and q_X and q_Y are (taken by the homeomorphism to) the obvious inclusions.

This shows that $X \sqcup Y$ is the coproduct of X and Y in the category of topological spaces.

Proof. Consider $X \sqcup Y$ with the inclusions $i_X : X \hookrightarrow X \sqcup Y$ and $i_Y : Y \hookrightarrow X \sqcup Y$. By the universal property, there is a unique continuous function $i : Z \to X \sqcup Y$ such that $i_X = iq_X$ and $i_Y = iq_Y$. We construct a function $q = q_X \sqcup q_Y : X \sqcup Y \to Z$, where $x \mapsto q_X(x)$ for all $x \in X$ and $y \mapsto q_Y(y)$ for all $y \in Y$. We claim that i and q are inverses and, in which case, describe a homeomorphism between $X \sqcup Y$ and Z.

First, it can be seen that

$$iq = iq_X \sqcup iq_Y = i_X \sqcup i_Y = \mathrm{id}_{X \sqcup Y}\,.$$

To prove the opposite composition gives the identity on Z, we construct the continuous functions

$$f_X : X \xrightarrow{q_X} Z \xrightarrow{i} X \sqcup Y \xrightarrow{q} Z \quad \text{and} \quad f_Y : Y \xrightarrow{q_Y} Z \xrightarrow{i} X \sqcup Y \xrightarrow{q} Z.$$

By the universal property, there is a unique continuous function $f : Z \to Z$ such that $f_X = fq_X$ and $f_Y = fq_Y$. On one hand, the constructions of f_X and f_Y imply that $f = qi$. On the other hand, we have

$$f_X = qiq_X = qi_X = q_X \quad \text{and} \quad f_Y = qiq_Y = qi_Y = q_Y,$$

which would imply that $f = \mathrm{id}_Z$. Therefore, $qi = f = \mathrm{id}_Z$.

$\square$

Lemma 1. The map $f : \mathbb{R}^2 \to \mathbb{R}$, $(x, y) \mapsto x + y$ is continuous.

Proof. It suffices to prove that the preimages under f of basis elements (open intervals) in $\mathbb{R}$ are open. Suppose $(a, b) \subseteq \mathbb{R}$, where $a \in [-\infty, \infty)$ and $b \in (-\infty, \infty]$. For $(x, y) \in f^{-1}((a, b))$, there is some $\varepsilon > 0$ such that $B_\varepsilon(x + y) \subseteq (a, b)$. Then, for any $(x', y') \in B_{\varepsilon/2}(x) \times B_{\varepsilon/2}(y)$,

$$a < (x - \varepsilon/2) + (y - \varepsilon/2) < x' + y' < (x + \varepsilon/2) + (y + \varepsilon/2) < b,$$

so $(x', y') \in f^{-1}((a, b))$. That is, we have found an open neighborhood of (x, y) contained in $f^{-1}((a, b))$, namely the basis element $B_{\varepsilon/2}(x) \times B_{\varepsilon/2}(y)$ in the product topology. Hence, $f^{-1}((a, b)) \subseteq \mathbb{R}^2$ is open, so f is continuous.

□

Lemma 2. The map $f : \mathbb{R} \to \mathbb{R}$, $x \mapsto -x$ is continuous.

Proof. Suppose $(a, b) \subseteq \mathbb{R}$ is a possibly unbounded open interval. Then $f^{-1}((a, b))$ is the set of points $x \in \mathbb{R}$ such that $a < -x < b$ or, equivalently, such that $-b < x < -a$. That is, $f^{-1}((a, b)) = (-b, -a)$, which is an open interval in $\mathbb{R}$, so f is continuous.

□

Lemma 3. The map $f : \mathbb{R} \to \mathbb{R}$, $x \mapsto |x|$ is continuous.

Proof. Suppose $(a, b) \subseteq \mathbb{R}$ is a possibly unbounded open interval, then $f^{-1}((a, b)) = (a, b) \cup (-b, -a)$. As the union of open intervals, $f^{-1}((a, b))$ is open, so f is continuous.

□

3 Let X be a topological space and let $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}$ be continuous. Show that $\min(f, g)$ is a continuous function.

Proof. By lemmas 1, 2, and 3, and the fact that the composition of continuous functions is continuous, we have that $\min(f, g) = \frac{1}{2}(f + g - |f - g|)$ is continuous.

□

4 Let X and Y be metric spaces. Show that the metrics

$$d_\infty((x,y),(x',y')) = \max(d(x,x'),d(y,y'))$$
$$d_1((x,y),(x',y')) = d(x,x') + d(y,y')$$

both induce the product topology on $X \times Y$.

(A good exercise is to visualize the balls in $\mathbb{R} \times \mathbb{R}$ in both of these topologies.)

Proof. To prove a pair of topologies are the same, we show that every neighborhood, in one topology, around a point contains a neighborhood, in the other topology, of the same point. (If this is the case, then every open set U, in one topology, is the union $\bigcup_{x\in U} U_x$, where U_x is an open set, in the other topology, chosen such that $x \in U_x \subseteq U$.)

We first prove that d_∞ induces the product topology on $X \times Y$. Let $W \subseteq X \times Y$ be open in the product topology and let $(x,y) \in W$. There is an element in the basis for the product topology $U \times V \subseteq X \times Y$, with $U \subseteq X$ open, $V \subseteq Y$ open, and $(x,y) \in U \times V$. Then there is some radius $r_X > 0$ such that $B_{r_X}(x) \subseteq U$, and some radius $r_Y > 0$ such that $B_{r_Y}(y) \subseteq V$. Let $r = \min\{r_X, r_Y\}$, then for any $(x',y') \in B_r((x,y);d_\infty)$, we have

$$d(x,x') \le d_\infty((x,y),(x',y')) < r \le r_X$$

and

$$d(y,y') \le d_\infty((x,y),(x',y')) < r \le r_Y,$$

so

$$(x',y') \in B_{r_X}(x) \times B_{r_Y}(y) \subseteq U \times V \subseteq W.$$

Thus, $(x,y) \in B_r((x,y);d_\infty) \subseteq W$, which proves that the topology induced by d_∞ is at least as fine as the product topology.

Let $W \subseteq X \times Y$ be open in the topology induced by d_∞ and let $(x,y) \in W$. There is some radius $r > 0$ such that $(x,y) \in B_r((x,y);d_\infty) \subseteq W$. Then for any $(x',y') \in B_r(x) \times B_r(y)$, we have $d(x,x') < r$ and $d(y,y') < r$, so $d_\infty((x,y),(x',y')) < r$, implying that $(x',y') \in B_r((x,y);d_\infty) \subseteq W$. Hence, we have found a neighborhood of (x,y) in the product topology, namely $B_r(x) \times B_r(Y)$, contained in W. This proves that the product topology is at least as fine as the topology induced by d_∞, so we conclude that they are the same topology.

Next, we will prove that d_∞ and d_1 induce the same topology. To do so, we will show that each open ball under one metric contains an open ball in the other metric. Let $B_r((x,y);d_\infty)$ be an open d_∞-ball. Then, for all $(x',y') \in B_{r/2}((x,y);d_1)$, we have

$$\max(d(x,x'),d(y,y')) \le d(x,x') + d(y,y') < r/2 + r/2 = r.$$

That is, $B_{r/2}((x,y);d_1) \subseteq B_r((x,y);d_\infty)$, so d_1 generates a topology at least as fine as d_∞. Now, consider an open d_1-ball $B_r((x,y);d_1)$, with $r > 0$. For all $(x',y') \in B_{r/2}((x,y);d_\infty)$,

$$d(x,x') + d(y,y') \le 2\max(d(x,x'),d(y,y')) < 2 \cdot r/2 = r.$$

That is, $B_{r/2}((x,y);d_\infty) \subseteq B_r((x,y);d_1)$, so d_∞ generates a topology at least as fine as d_1.

$\square$

5 Give an example of a subset $A \subset \mathbb{R}$ such that the following sets are all different:

$$A, \quad \overline{A}, \quad \operatorname{int} A, \quad \operatorname{int}\left(\overline{A}\right), \quad \overline{\operatorname{int} A}, \quad \overline{\operatorname{int}(\overline{A})}, \quad \operatorname{int}\left(\overline{\operatorname{int} A}\right).$$

Consider the set

$$A = [0,1) \cup (1,2] \cup \big([2,3] \cap \mathbb{Q}\big) \cup \{4\}.$$

We have

$$\begin{aligned} \overline{A} &= [0,3] \cup \{4\}, & \operatorname{int} A &= (0,1) \cup (1,2), \\ \operatorname{int}(\overline{A}) &= (0,3), & \overline{\operatorname{int} A} &= [0,2], \\ \overline{\operatorname{int}(\overline{A})} &= [0,3], & \operatorname{int}(\overline{\operatorname{int} A}) &= (0,2). \end{aligned}$$