

idk what is wanted, but imma fix a base field  $\mathbb{k}$ , which is likely  $\mathbb{R}$  or  $\mathbb{C}$ . I think any subfield of  $\mathbb{C}$  is fine.

A **vector space** (over  $\mathbb{k}$ ) is a  $\mathbb{k}$ -module.

A **norm** on a vector space  $X$  is a function  $\|-\| : X \rightarrow \mathbb{R}$  such that

- (positive definite)  $\|x\| = 0$  implies  $x = 0$ , for all  $x \in X$ ;
- (absolute homogeneity)  $\|cx\| = |c|\|x\|$  for all  $c \in \mathbb{k}$  and  $x \in X$ ;
- (triangle inequality)  $\|x + y\| \leq \|x\| + \|y\|$  for all  $x, y \in X$ .

One can check that these conditions also imply

- $\|x\| = 0$  if and only if  $x = 0$ , for all  $x \in X$ ;
- $\|x\| \geq 0$  for all  $x \in X$ .

These slightly stronger conditions are sometimes added into the definition of a norm.

A **normed vector space** is a vector space  $X$  with a norm  $\|-\|_X$ .

Let  $X$  and  $Y$  be normed vector spaces.

A linear map  $T : X \rightarrow Y$  is called **bounded** if there exists  $C \in \mathbb{R}$  such that

$$\|Tx\|_Y \leq C\|x\|_X, \quad \text{for all } x \in X.$$

(Necessarily, such a  $C$  would be positive.) We also call  $T$  a **bounded linear operator**.

If  $T : X \rightarrow Y$  is a bounded linear operator, define

$$\|T\| := \inf\{C \in \mathbb{R} : \|Tx\|_Y \leq C\|x\|_X \text{ for all } x \in X\}.$$

One can check that

$$\|T\| = \sup_{\substack{x \in X \\ x \neq 0}} \frac{\|Tx\|_Y}{\|x\|_X} = \sup_{\substack{x \in X \\ 0 < \|x\|_X \leq 1}} \frac{\|Tx\|_Y}{\|x\|_X} = \sup_{\substack{x \in X \\ 0 < \|x\|_X < 1}} \frac{\|Tx\|_Y}{\|x\|_X} = \sup_{\substack{x \in X \\ \|x\|_X = 1}} \|Tx\|_Y$$

Let  $X$  and  $Y$  be normed vector spaces and  $U \subseteq X$  be an open subset.

A function  $f : U \rightarrow Y$  is called **differentiable at**  $x \in U$  if there exists a bounded linear operator  $T : X \rightarrow Y$  satisfying

$$\lim_{\substack{\|h\|_X \rightarrow 0 \\ h \in X \setminus \{0\}}} \frac{\|f(x+h) - f(x) - T(h)\|_Y}{\|h\|_X} = 0.$$

$$Q(f, x, h, T) = \frac{\|f(x+h) - f(x) - T(h)\|_Y}{\|h\|_X}$$

In which case, we say  $T$  is the\* **derivative of  $f$  at  $x$** , written  $df_x = d(f)_x = T$ .

For any  $V \subseteq U$ ,  $f$  is called **differentiable on  $V$**  if it is differentiable at every point of  $V$ .

$f$  is called **differentiable** if it is differentiable on  $U$ .

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Let  $U \subseteq X$  be open and  $f : U \rightarrow Y$  be differentiable.

$\mathcal{L}(X, Y)$ .

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$f$  is called **continuously differentiable at  $x$**