# Math 111C Midterm

Apr 29, 2021

- This exam is worth 35 points.
- No collaboration or discussion with others. No search of solutions online.
- Good luck!

1. (5 points) Show that $\mathbb{Q}(\sqrt{5}+\sqrt{2})$ is a splitting field for $(x^2-2)(x^2-10) \in \mathbb{Q}[x]$.
2. (5 points) Let $F$ be a field and $K \subseteq \overline{F}$ be the splitting field for a non-constant polynomial $f(x) \in F[x]$. Show that if $\gcd(f(x), f'(x)) = 1$, then $K/F$ is separable.
3. (5 points) Let $F$ be a field and $\alpha, \beta \in \overline{F}$. Show that $m_{\alpha,F}(x)$ is irreducible in $\big(F(\beta)\big)[x]$ iff $m_{\beta,F}(x)$ is irreducible in $\big(F(\alpha)\big)[x]$.
4. Let $f(x)$ be an irreducible polynomial in $\mathbb{F}_p[x]$.
   (a) (5 points) Show that if $\theta_1, \theta_2 \in \overline{\mathbb{F}}_p$ are both roots of $f(x)$, then $\mathbb{F}_p(\theta_1) = \mathbb{F}_p(\theta_2)$.
   (b) (5 points) Let $K \subseteq \overline{\mathbb{F}}_p$ be the splitting field for $f(x)$ over $\mathbb{F}_p$. Show that $[K : \mathbb{F}_p] = \deg f(x)$.
5. Let $K/F$ be an algebraic field extension.
   (a) (5 points) Let $E = \{\alpha \in K \mid \alpha \text{ is separable over } F\}$. Show that $E$ is subfield of $K$ containing $F$.
   (b) (3 points) Given $\alpha \in K$, show that there exists an integer $m \geq 0$ such that $\alpha^{p^n} \in E$ for all $n \geq m$.
   (c) (2 points) Suppose that $K/F$ is finite. Show that $[K : E]$ is a power of $p$.