I worked with Joseph Sullivan and Gahl Shemy.

1 Let A be a Δ-complex and build a new Δ-complex X by adding a single new n-simplex D. Using simplicial homology, compute the difference between $H_*(A)$ and $H_*(X)$.

First note that $C_n(X) = C_n(A) \oplus \mathbb{Z}D$ and $C_i(X) = C_i(A)$ for $i \neq n$.

Because of this, the kernels and images of the boundary maps ∂_i for $i \neq n$ are entirely unchanged. Therefore, when $i > n$ or $i \leq n-2$ we have

$$H_i(X) = Z_i(X)/B_i(X) = Z_i(A)/B_i(A) = H_i(A).$$

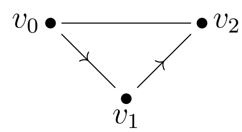
It remains to check H_n and H_{n-1}.

$H_{n-1}(X) = Z_{n-1}(X)/B_{n-1}(X) = Z_{n-1}(A)/(B_{n-1}(A) + \mathbb{Z}\partial_n(D))$

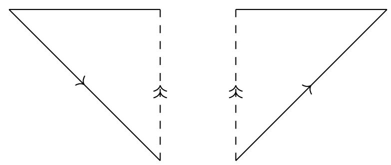
$H_n(X) = Z_n(X)/B_n(X) = Z_n(X)/B_n(A)$

2 Hatcher 2.1.1 What familiar space is the quotient Δ-complex of a 2-simplex $[v_0, v_1, v_2]$ obtained by identifying the edges $[v_0, v_1]$ and $[v_1, v_2]$, preserving the ordering of vertices?

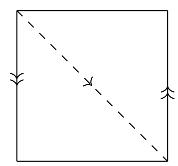
We want to form the following quotient:



Before identifying the edges, we make the following cut:



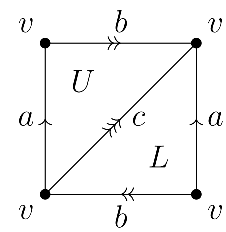
We now glue the original pair of edges to obtain the following:



Regluing the cut we made gives us the Möbius strip.

3 Hatcher 2.1.5 Compute the simplicial homology groups of the Klein bottle using the Δ-complex structure described at the beginning of this section.

We use the following Δ-complex construction:



We first compute the boundaries of the 1-simplices:

$$\partial_1(a) = \partial_1(b) = \partial_1(c) = v - v = 0.$$

This allows us to compute the 0th homology group

$$H_0 = \langle v \rangle / 0 \cong \mathbb{Z}.$$

We now compute the boundaries of the 2-simplices:

$$\begin{aligned} \partial_2(U) &= a + b - c, \\ \partial_2(L) &= -a + b + c. \end{aligned}$$

This gives us

$$H_1 = \langle a, b, c \rangle / \langle a + b - c, -a + b + c \rangle.$$

Consider the map

$$\begin{aligned} f : \mathbb{Z}^2 &\longrightarrow \mathbb{Z}^3 = \langle a, b, c \rangle, \\ e_1 &\longmapsto a + b - c, \\ e_2 &\longmapsto -a + b + c. \end{aligned}$$

Then we can write $H_2 = \mathbb{Z}^3 / \operatorname{im} f$. If $P : \mathbb{Z}^2 \to \mathbb{Z}^2$ and $Q : \mathbb{Z}^3 \to \mathbb{Z}^3$ are invertible linear maps, then we also have $H_2 = \mathbb{Z}^3 / \operatorname{im}(QfP)$. Write the linear map f as the matrix

$$\begin{bmatrix} 1 & -1 \\ 1 & 1 \\ -1 & 1 \end{bmatrix},$$

then P and Q correspond to row and column operations on this matrix. We find the following equivalent matrix:

$$\begin{bmatrix} 1 & -1 \\ 1 & 1 \\ -1 & 1 \end{bmatrix} \xrightarrow{r_1 = r_1 + r_3} \begin{bmatrix} 0 & 0 \\ 1 & 1 \\ -1 & 1 \end{bmatrix} \xrightarrow{r_3 = r_3 + r_2} \begin{bmatrix} 0 & 0 \\ 1 & 1 \\ 0 & 2 \end{bmatrix} \xrightarrow{c_2 = c_2 + c_1} \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 2 \end{bmatrix}.$$

Thus, we can now compute the 1st homology group

$$H_1 = \langle a, b, c \rangle / \langle b, 2c \rangle = \langle a \rangle \oplus \langle c \rangle / \langle 2c \rangle \cong \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}.$$

We compute the kernel of ∂_2:

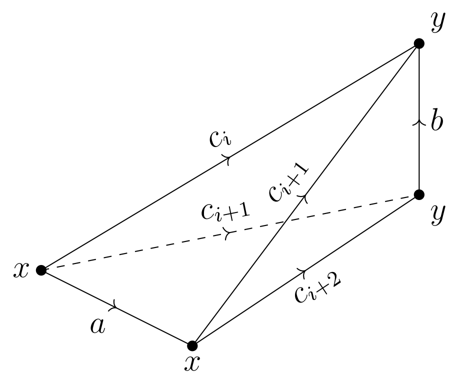
$$\begin{aligned} 0 &= \partial_2(\alpha U + \beta L) \\ &= \alpha(a + b - c) + \beta(-a + b + c) \\ &= (\alpha - \beta)a + (\alpha + \beta)b + (-\alpha + \beta)c. \end{aligned}$$

This implies $\alpha = \beta = -\beta$, so we must have $\alpha = \beta = 0$. Hence, we compute the 2nd homology group to be

$$H_2 = 0.$$

4 Hatcher 2.1.8 Construct a 3-dimensional Δ-complex X from n tetrahedra $T_1, \dots, T_n$ by the following two steps. First, arrange the tetrahedra in a cyclic pattern as in the figure, so that each T_i shares a common vertical face with its two neighbors T_{i-1} and T_{i+1}, subscripts being taken mod n. Then identify the bottom face of T_i with the top face of T_{i+1} for each i. Show the simplicial homology groups of X in dimensions 0, 1, 2, 3 are $\mathbb{Z}$, $\mathbb{Z}/n\mathbb{Z}$, 0, $\mathbb{Z}$, respectively.

After making the prescribed identifications, we may label the simplices of T_i as follows:



Additionally, we label the top face U_i, the bottom face U_{i+1}, the back/left face L_i, and the front/right face L_{i+1}.

We compute the boundaries of the 1-simplices:

$$\partial_1(a) = x - x = 0,$$
$$\partial_1(b) = y - y = 0,$$
$$\partial_1(c_i) = y - x.$$

We compute the 0th homology group

$$H_0 = \langle x, y \rangle / \langle y - x \rangle = \langle x \rangle \cong \mathbb{Z}.$$

We compute the kernel of ∂_1:

$$0 = \partial_1(\alpha a + \beta b + \textstyle\sum_i \gamma_i c_i) = \sum_i \gamma_i (y - x).$$

This implies $\sum_i \gamma_i = 0$ so $\gamma_1 = -\sum_{i=2}^n \gamma_i$. So in $C_2 = \langle a, b, c_1, \dots, c_n \rangle$, the kernel of of the boundary map consists of elements represented as vectors of the form

$$\begin{bmatrix} \alpha \\ \beta \\ -\sum_{i=2}^{n} \gamma_i \\ \gamma_2 \\ \vdots \\ \gamma_n \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \gamma_2 \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \gamma_2 \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \gamma_2 \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{aligned} &= \alpha a + \beta b + \gamma_2(c_2 - c_1) + \gamma_3(c_3 - c_1) + \cdots + \gamma_n(c_n - c_1) \\ &= \alpha a + \beta b + \sum_{i=2}^{n} \gamma_i(c_i - c_1). \end{aligned}$$

So $\ker \partial_1 = \langle a, b, c_2 - c_1, c_3 - c_1, \ldots, c_n - c_1 \rangle$.

We can manipulate the generators to get $\ker \partial_1 = \langle a, b, c_2 - c_1, c_3 - c_2, \ldots, c_n - c_{n-1} \rangle$.

Define the 1-chains $d_i = c_{i+1} - c_i$ for $i = 1, \ldots, n-1$. The d_i's are linearly independent, so we can write $\ker \partial_1 = \langle a, b, d_1, \ldots, d_{n-1} \rangle \cong \mathbb{Z}^{n+1}$.

We compute the boundaries of the 2-simplices:

$$\begin{aligned} \partial_2(U_i) &= a + c_{i+1} - c_i = a + d_i, \\ \partial_2(L_i) &= b + c_{i+1} - c_i = b + d_i. \end{aligned}$$

Then the 1st homology group is

$$H_1 = \langle a, b, d_1, \ldots, d_{n-1} \rangle / \langle a + d_1, \ldots, a + d_n, b + d_1, \ldots, b + d_n \rangle,$$

where $d_n = -d_1 - \cdots - d_{n-1}$. We consider the kernel as the image of a linear map $\mathbb{Z}^{2n} \to \mathbb{Z}^{n+1}$ which we represent as the matrix

$$\begin{array}{c} a \\ b \\ d_1 \\ d_2 \\ \vdots \\ d_{n-1} \end{array} \begin{bmatrix} 1 & 1 & \cdots & 1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 0 & \cdots & 0 & -1 & 1 & 0 & \cdots & 0 & -1 \\ 0 & 1 & \cdots & 0 & -1 & 0 & 1 & \cdots & 0 & -1 \\ \vdots & & \ddots & \vdots & \vdots & & & \ddots & & \vdots \\ 0 & 0 & \cdots & 1 & -1 & 0 & 0 & \cdots & 1 & -1 \end{bmatrix} = \begin{array}{c} a \\ b \\ d_i \end{array} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ I_{n-1} & -1 & I_{n-1} & -1 \end{bmatrix}.$$

We now perform row and column operations on this matrix to produce a more useful presentation of the quotient. Note that when a column is repeated, this corresponds to a repeated relation, so we can safely remove that any repeated columns. Additionally, if a column contains all 0's except for a 1 in the jth row, this corresponds to the jth generator reducing to zero in the quotient. In this case, we can safely remove that column and the jth row and discard that generator.

$$
\begin{aligned}
\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ I_{n-1} & -1 & I_{n-1} & -1 \end{bmatrix} &\to \begin{bmatrix} 1 & 1 & -1 & -1 \\ 0 & 0 & 1 & 1 \\ I_{n-1} & -1 & 0 & 0 \end{bmatrix} && \text{column operations} \\
&\to \begin{bmatrix} 1 & 1 & -1 \\ 0 & 0 & 1 \\ I_{n-1} & -1 & 0 \end{bmatrix} && \text{remove repeat columns} \\
&\to \begin{bmatrix} 0 & n & -1 \\ 0 & 0 & 1 \\ I_{n-1} & -1 & 0 \end{bmatrix} && \text{row operations} \\
&\to \begin{bmatrix} n & -1 \\ 0 & 1 \end{bmatrix} && \text{remove null generators} \\
&\to \begin{bmatrix} n & 0 \\ 0 & 1 \end{bmatrix} && \text{row operations} \\
&\to \begin{bmatrix} n \end{bmatrix} && \text{remove null generators}
\end{aligned}
$$

Thus, the 1st homology group is

$$H_1 = \langle a \rangle / \langle na \rangle \cong \mathbb{Z}/n\mathbb{Z}.$$

We compute the kernel of ∂_2:

$$
\begin{aligned}
\partial_2\left(\textstyle\sum_{i=1}^n \alpha_i U_i + \sum_{i=1}^n \beta_i L_i\right) &= \sum_{i=1}^n \alpha_i(a + c_{i+1} - c_i) + \beta_i(b + c_{i+1} - c_i) \\
&= \sum_{i=1}^n \alpha_i a + \sum_{i=1}^n \beta_i b + \sum_{i=1}^n (\alpha_{i-1} - \alpha_i + \beta_{i-1} - \beta_i) c_i.
\end{aligned}
$$

Elements of the kernel are of the form

$$
\begin{aligned}
\begin{bmatrix} \sum_i \alpha_i \\ \sum_i \beta_i \\ \alpha_n - \alpha_1 + \beta_n - \beta_1 \\ \alpha_1 - \alpha_2 + \beta_1 - \beta_2 \\ \vdots \\ \alpha_{n-1} - \alpha_n + \beta_{n-1} - \beta_n \end{bmatrix} &= \alpha_1 \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \cdots + \alpha_n \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ -1 \end{bmatrix} + \beta_1 \begin{bmatrix} 0 \\ 1 \\ -1 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \cdots + \beta_n \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ \vdots \\ 0 \\ -1 \end{bmatrix} \\
&= \sum_{i=1}^n \alpha_i(a + c_{i+1} - c_i) + \sum_{i=1}^n \beta_i(b + c_{i+1} - c_i).
\end{aligned}
$$

Then

$$\ker \partial_2 = \langle a + c_{i+1} - c_i, b + c_{i+1} - c_i \rangle.$$

We compute the image of ∂_3:

$$\begin{aligned}
\partial_3(U_i) &= a + c_{i+1} - c_i, \\
\partial_3(L_i) &= b + c_{i+1} - c_i.
\end{aligned}$$

Hence, the 2nd homology group is

$$H_2 = \langle a + c_{i+1} - c_i, b + c_{i+1} - c_i \rangle / \langle a + c_{i+1} - c_i, b + c_{i+1} - c_i \rangle = 0.$$

We compute the kernel of ∂_3:

$$\begin{aligned}
0 &= \partial_3(\textstyle\sum_i \alpha_i T_i) \\
&= \sum_{i=1}^{n} \alpha_i (L_{i+1} - L_i + U_i - U_{i+1}) \\
&= \sum_{i=1}^{n} (\alpha_{i-1} - \alpha_i) L_i + \sum_{i=}^{n} (\alpha_i - \alpha_{i-1}) U_i
\end{aligned}$$

This implies $\alpha_1 = \cdots = \alpha_n$, so $\ker \partial_3 = \langle T_1 + \cdots + T_n \rangle$. Hence, the 3rd homology group is

$$H_3 = \langle T_1 + \cdots + T_n \rangle / 0 \cong \mathbb{Z}.$$