Geometry & Topology in Low Dimensions
Fall 2022

this requires some topology, maybe algebraic topology, linear algebra, a little group theory

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we'll cover some hyperbolic geometry and topology, but will start with Euclidean

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Euclidean Geometry

Talking about $\mathbb{R}^n$ with norm $\|x\| = x_1^2 + \cdots + x_n^2$, then Euclidean metric $d(x, y) = \|x - y\|$.

A map $T : \mathbb{R}^n \to \mathbb{R}^n$ is an **isometry** if for all $x, y \in \mathbb{R}^n$ we have $d(Tx, Ty) = d(x, y)$.

Examples:

- translation: $Tx = x + t$ for some constant $t \in \mathbb{R}^n$.
- orthogonal: $Tx = Ax$ where $A \in M_n(\mathbb{R})$ and $n \times n$ matrix with $A^T A = I$. Check

  $$\|Ax\|^2 = \langle Ax, Ax\rangle = (Ax)^T(Ax) = x^T(A^T A)x = x^T x = \|x\|^2.$$

  Therefore,

  $$d(x, y)^2 = \|Tx - Ty\|^2 = \|Ax - Ay\|^2 = \|A(x - y)\|^2 = \|x - y\|^2 = d(x, y)^2$$

- composition. In fact, the set of isometries of $\mathbb{R}^n$ is a group under composition.

Theorem 1. If T is an isometry of $\mathbb{R}^n$ of then there exists $b \in \mathbb{R}^n$ and $A \in O(n)$ such that $Tx = Axb$.

Proof. Suppose T is an isometry, set $b = T(0)$.

Define $S(x) = x - b$, a translation.

Define $T' = S \circ T$, an isometry with $T'(0) = 0$.

Suffices to prove $T'x = Ax$.

Without loss of generality $T(0) = 0$.

In $\mathbb{R}^n$, the distances of a point to $0, e_1, \dots, e_n$ uniquely determines the point.

Define $v_i = T(e_i)$, then $\|v_i\| = 1$ since T isometry.

For $i \neq j$, we have $d(v_i, v_j) = d(e_i, e_j) = \sqrt{2}$, so $\langle v_i, v_j \rangle = 0$.

Hence, T sends e_i's to v_i's, which form an orthonormal basis.

Now define matrix

$$A = \begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{bmatrix}.$$

This matrix has $A^T A = I$ so $A \in O(n)$. Moreover,

$$(A^{-1} \circ T)e_i = A^{-1}(Te_i) = A^{-1}v_i = e_i.$$

Therefore, $A^{-1} \circ T$ fixes $0, e_1, \dots, e_n$, so $A^{-1} \circ T = I$, so $T = A$. $\square$

Classification of isometries

dim $n = 1$.

1. **reflection**: $Tx = 2c - x$ for some $c \in \mathbb{R}$
2. **translation**: $Tx = x + t$ for some $t \in \mathbb{R}$

notice that the composition of two reflections is a translation: For $Tx = x+c$ and $T'x = x+c'$, we have $(T \circ T')x = x + 2(c - c')$.

Group of isometries:

$$\operatorname{Isom}(\mathbb{R}) = \{x \mapsto ax + b \mid a = \pm 1, b \in \mathbb{R}\} \cong \begin{bmatrix} \pm 1 & \mathbb{R} \\ 0 & 1 \end{bmatrix} \cong Z_2 \ltimes \mathbb{R}.$$

Let S be a reflection, T a translation. Note $S = S^{-1}$, then

$$S \circ T \circ S^{-1} = S \circ T \circ S = T^{-1}.$$

dim $n = 2$

1. **reflection** across a line
2. **translation**
3. **rotation** around a point
4. **glide reflection**: reflect across a line and translate along the line

dim $n = 3$

1. **translation**
2. **rotation** around a line
3. **reflection** across a plane

4. **screw**: rotate around a line and translate along the line
5. **glide reflection**: reflect across a plane and translate along a line in the plane

dim $n = n$ (up to conjugacy)

- Case 1: there exists a fixed point. Then the isometry is conjugate to an orthogonal transformation.
- Case 2: no fixed point. Conjugate to $Tx = Ax + b$, with $A \in O(n)$ such that $Ab = b$.

  $$b^\perp = \{x \in \mathbb{R}^n \mid \langle x, b\rangle = 0\} \cong \mathbb{R}^{n-1}$$

  So A preserves the copy of $\mathbb{R}^{n-1}$ orthogonal to b. Say T "translates along an axis and 'rotates' (really orthogonal) in hyperplane orthogonal to axis"

HW 1: prove all this

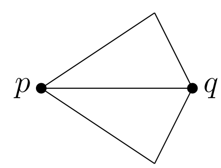
Corollary 1.

$$\mathrm{Isom}(\mathbb{R}^n) \cong O(n) \ltimes \mathbb{R}^n = \begin{bmatrix} A & b \\ 0 & 1 \end{bmatrix}$$

with $A \in O(n), b \in \mathbb{R}^n$.

Composition of Rotation

In a plane, take points p and q



Euclidean Manifolds

A metric manifold M^n is **Euclidean** if it is locally isometric to Euclidean space.

Theorem 2. Every complete simply-connected Euclidean manifold is isometric to $\mathbb{R}^n$.

Proof Sketch. Let N be simply-connected complete Euclidean manifold.

Goal: construct isometry $\mathrm{dev} : N \to \mathbb{R}^n$, called the "developing map."

Know every point has a little neighborhood $U \subseteq N$ with isometry $h : U \to h(U) \subseteq \mathbb{R}^n$.

For another such isometry $k : V \to k(V) \subseteq \mathbb{R}^n$.

Then look at $h(U \cap V)$ and $k(U \cap V)$; would be great if these were the same in $\mathbb{R}^n$.

There exists an isometry $T : \mathbb{R}^n \to \mathbb{R}^n$ such that $T|_{k(U\cap V)} = h \circ k^{-1}$.

We then map V using $T \circ k$ instead of k.

Can continue this process for countably many open sets and charts.

Take an atlas of isometries and adjust the charts so that the charts of intersection open sets agree.

Simply-connected will ensure that this is well-defined.

Completeness implies the map is surjective.

Geometry gives injectivity. If two points in N were sent to the same point in $\mathbb{R}^n$ then straight lines bad. $\square$

Corollary 2. If M^n is a closed (complete?) Euclidean manifold, then universal cover $\widetilde{M}$ is a simply-connected complete Euclidean manifold. Therefore, $\widetilde{M} = \mathbb{R}^n$.

Covering transformations of $\widetilde{M}$ are isometries of $\mathbb{R}^n$. Then holonomy map $\pi_1 M \to \mathrm{Isom}(\mathbb{R}^n)$. Then

$$M = \mathbb{R}^n/\mathrm{hol}(\pi_1 M).$$

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Examples

1. $\mathbb{R}^n$, $G = 1$
2. $S^1 = \mathbb{R}/\mathbb{Z}$
3. $T^2 = \mathbb{R}^2/\mathbb{Z}^2$
4. Klein Bottle: $K = \mathbb{R}^2/G$ where $G = \langle \alpha, \beta : \text{ relations} \rangle$ with

   $$\beta(x, y) = (x, y + 2) \quad \text{and} \quad \alpha(x, y) = (x + 2, -y),$$

   a translation and glide reflection, respectively. There is a twofold (and cyclic) cover $T^2 \to K = \mathbb{R}^2/G$. $T^2 = \mathbb{R}^2/H$ with $H = \mathrm{ab}\, \langle \alpha^2, \beta \rangle \leq G$.

Exercise: only compact Euclidean 2-manifolds are torus and Klein bottle (up to homeomorphism).

Theorem 3 (Baberbach)**.**
1. Every closed Euclidean n-manifold is finitely covered by an n-torus ($T^n = S^1 \times \cdots \times S^1$).
2. Up to homeomorphism, there are finitely many.

dim	# closed Euclidean n-manifolds
1	1
2	2
3	10
4	74
5	1060
6	38746

Example: M^3 place diagram $G = \langle \alpha, \beta, \gamma \rangle \leq \mathrm{Isom}(\mathbb{R}^3)$ with

$$\begin{aligned}
\alpha(x, y, z) &= (x + 1, y, z) \\
\beta(x, y, z) &= (x, y + 1, z) \\
\gamma(x, y, z) &= (-y, x, z + 1).
\end{aligned}$$

Here, γ is a screw motion.

There is a 4-fold cyclic cover $T^3 \to M^3$.

Example: Hexagonal torus. place picture

Torus Bundle

$$\begin{tikzcd}
T^{n-1} \arrow[r] & M^n \arrow[d, "p"] \\
& S^1
\end{tikzcd}$$

p is a submersion; for all $x \in S^1$, $p^{-1}(x) \cong T^{n-1}$.

Then $M = T^{n-1} \times [0,1]/\sim$

IF M^n is a closed Euclidean manifold, then $M = \mathbb{R}^n/G$ and $G \leq \mathrm{Isom}(\mathbb{R}^n)$. There exists a short exact sequence of groups

$$0 \longrightarrow \mathbb{R}^n \longrightarrow \mathrm{Isom}(\mathbb{R}^n) \longrightarrow O(n) \longrightarrow 1$$

$$\begin{bmatrix} I & b \\ 0 & 1 \end{bmatrix} \longmapsto \begin{bmatrix} A & b \\ 0 & 1 \end{bmatrix} \longmapsto A$$

The A is called the linear or rotational part of isometry.

Theorem 4 (Baberbach)**.** There is a maximal subgroup $\mathbb{Z}^n \leq G$ and there is a short exact sequence

$$0 \longrightarrow \mathbb{Z}^n \longrightarrow G \longrightarrow F \longrightarrow 1$$

where F is a finite subgroup of $O(n)$.

Remark: these isometries of R^n never have a fixed point. for, say, a rotation, the point of rotation will have a nasty neighborhood.

A manifold Q is a **Euclidean Orbifold** if $Q = \mathbb{R}^n/G$ and G is a discrete subgroup of $\text{Isom}(\mathbb{R}^n)$.

Example

$n = 1$, $G = \langle \sigma, \tau : \sigma^2 = \tau^2 = 1 \rangle \cong D_\infty$.

$\sigma(x) = -x$, $\tau(x = 2 - x)$, $\sigma\tau(x) = x - 2$ (translation)

$\mathbb{R}^1/G = [0, 1]$.

$p^{-1}(0) = 2\mathbb{Z}$

Given $x \in \mathbb{R}$, $\text{Stab}(x) = \{g \in G : g(x) = x\}$. Zero is a singular point, i.e., does not have a little manifold neighborhood around it.

Example

$n = 2$. If Q is compact, G is called a **wallpaper group** (there are 17 up to conjugacy). There are 17 compact Euclidean orbifolds.

Could have $(\mathbb{R}/G) \times (\mathbb{R}/G) = \mathbb{R} \times \mathbb{R}/G \times G$ where G is from previous example.

This is $[0, 1]^2$... put drawing

Every point in a EUclidean orbifold has some stabilizer group associated to it. most (generic) points have the trivial group and therefore have a little euclidean neighborhood. Other points do not, and these are called **singular locus** points.

In 2-dimensions, singular points can be mirror, corner, or cone.

Example

H is orientation-preserving subgroup of $G \times G$.

This contains π rotations about $\mathbb{Z} \oplus \mathbb{Z}$.

Then $\mathbb{R}^2/H = S^2$ with singular points of Z_2; cone points.

Cone points are D^2 quotient by a rotation of finite order.

Orbifold fundamental group

Let $Q = \mathbb{R}^n/G$ be a Euclidean orbifold.

If Q happens to be a manifold, then $\mathbb{R}^n$ is universal cover and G is group of covering transformations, then G is the fundamental group of Q.

The orbifold fundamental group of Q is defined as

$$\pi_1^{\mathrm{orb}}(Q) = G.$$

Remark: if G acts freely, then G is the group of covering transformations of Q as a manifold, so $G = \pi_1(Q)$.

Let $\pi : \mathbb{R}^n \to Q = \mathbb{R}^n/G$ be orbifold projection

The **singular locus** of Q is the set

$$\Sigma(Q) = \pi\{x \in \mathbb{R}^n : \text{there exists } 1 \neq g \in G \text{ such that } g(x) = x\}.$$

For $n = 2$ there are three kinds of singular points:

1. cone point: D^2 mod rotation by $2\pi/n$
2. mirror neighborhood: D^2 mod reflection
3. corner point: D^2/D_{2n}

To see that these are the only possibilities, check the finite subgroups of $O(2)$.

Calculating the orbifold fundamental group in a special case

Q a surface of genus 2 with cone points $x_1, x_2, \ldots, x_k$ of orders $n_1, n_2, \ldots, n_k$.

$$\pi_1^{\mathrm{orb}}(Q) = \langle \pi_1(Q \setminus \Sigma(Q)) \mid \alpha_i^{n_1} = 1 \rangle$$

Example

Pillowcase

Theorem 5 (Holden, Lozano, Montesinos)**.** Every closed orientable 3-manifold is a branched cover of S^3 branched over Boromean rings