Exercise 9.16 Show that the continuity of $\mathbf{f}'$ at the point $\mathbf{a}$ is needed in the inverse function theorem, even in the case $n = 1$: If

$$f(t) = t + 2t^2 \sin\left(\frac{1}{t}\right)$$

for $t \neq 0$ and $f(0) = 0$, then $f'(0) = 1$, f' is bounded in $(-1, 1)$, but f is not one-to-one in any neighborhood of 0.

For $h \neq 0$, we find

$$\frac{f(h) - f(0)}{h - 0} = \frac{h + 2h^2 \sin(1/h)}{h} = 1 + 2h \sin(1/h).$$

And since $|2h \sin(1/h)| \leq 2|h|$, then taking the limit as $h \to 0$ gives us $f'(0) = 1$. Additionally, for nonzero $t \in (-1, 1)$ we have

$$\begin{aligned} |f'(t)| &= |1 - 2\cos(1/t) + 4t \sin(1/t)| \\ &\leq 1 + 2|\cos(1/t)| + 4|t||\sin(1/t)| \\ &\leq 1 + 2 + 4 \\ &= 7. \end{aligned}$$

However, f is not injective in any neighborhood of 0. Let $\varepsilon > 0$ be given and choose $N = \min\{n \in \mathbb{N} : 1/(2\pi n) < \varepsilon\}$. Then

$$f\left(\frac{1}{2\pi N}\right) = \frac{1}{2\pi N} + 2\left(\frac{1}{2\pi N}\right)^2 \sin(2\pi N) = \frac{1}{\pi N},$$

and

$$f'\left(\frac{1}{2\pi N}\right) = 1 - 2\cos(2\pi N) + 4(2\pi N)\sin(2\pi N) = -1.$$

Since the derivative of f at the point $1/(2\pi N)$ is continuous, then there is some open neighborhood of this point, on which f is strictly decreasing. In particular, this means we can find some point $c \in (0, 1/(2\pi N))$ with $f(c) > 1/(\pi N) > 0$. Then, by the continuity of f on $(0, c)$, there is some point $d \in (0, c)$ such that $f(d) = 1/(\pi N)$. So there are two distinct points, namely $1/(2\pi N)$ and d, inside $(-\varepsilon, \varepsilon)$ which have the same value under f. Therefore, f is not injective on any neighborhood of 0.

Exercise 9.17 Let $\mathbf{f} = (f_1, f_2)$ be the mapping of $\mathbb{R}^2$ into $\mathbb{R}^2$ given by

$$f_1(x, y) = e^x \cos y, \qquad f_2(x, y) = e^x \sin y.$$

(a) What is the range of $\mathbf{f}$?

Claim. $f(\mathbb{R}^2) = \mathbb{R}^2 \setminus \{0\}$.

Proof. For each $(x, y) \in \mathbb{R}^2$, we always have $e^x > 0$. On the other hand, we never have $\cos y = 0$ and $\sin y = 0$, simultaneously. That is, $\mathbf{f}(x, y) \neq (0, 0)$ so $\mathbf{f}(\mathbb{R}^2) \subseteq \mathbb{R}^2 \setminus \{0\}$.

Any point $(x, y) \in \mathbb{R}^2 \setminus \{0\}$, has a polar representation in terms of magnitude $r > 0$ and angle $\theta \in (-\pi, \pi]$ to the positive x-axis, so that $(x, y) = r(\cos\theta, \sin\theta)$. Then $(\log r, \theta) \in \mathbb{R}^2$ and we find

$$\mathbf{f}(\log r, \theta) = (e^{\log r} \cos\theta, e^{\log r} \sin\theta) = (r\cos\theta, r\sin\theta) = (x, y).$$

Hence, the image of $\mathbf{f}$ is precisely $\mathbb{R}^2 \setminus \{0\}$.

□

(b) Show that the Jacobian of $\mathbf{f}$ is not zero at any point of $\mathbb{R}^2$. Thus every point of $\mathbb{R}^2$ has a neighborhood in which $\mathbf{f}$ is one-to-one. Nevertheless, $\mathbf{f}$ is not one-to-one on $\mathbb{R}^2$.

Proof. The Jacobian of $\mathbf{f}$ at $(x, y) \in \mathbb{R}^2$ is given by

$$\begin{aligned}
[\mathbf{f}'(x, y)] &= \begin{bmatrix} (D_1 f_1)(x, y) & (D_2 f_1)(x, y) \\ (D_1 f_2)(x, y) & (D_2 f_2)(x, y) \end{bmatrix} \\
&= \begin{bmatrix} e^x \cos y & -e^x \sin y \\ e^x \sin y & e^x \cos y \end{bmatrix} \\
&= e^x \begin{bmatrix} \cos y & -\sin y \\ \sin y & \cos y \end{bmatrix}.
\end{aligned}$$

Then $\det[\mathbf{f}'(x, y)] = e^x(\cos^2 y + \sin^2 y) = e^x \neq 0$. So $\mathbf{f}'(x, y)$ is invertible, and the inverse function theorem implies the existence of an open neighborhood of (x, y) on which $\mathbf{f}$ is injective.

However, consider the points $(0, 0), (0, 2\pi) \in \mathbb{R}^2$. We have

$$\mathbf{f}(0, 0) = (e^0 \cos 0, e^0 \sin 0) = (1, 0)$$

and

$$\mathbf{f}(0, 2\pi) = (e^0 \cos 2\pi, e^0 \sin 2\pi) = (1, 0).$$

Hence, $\mathbf{f}$ is not injective.

□

(c) Put $\mathbf{a} = (0, \pi/3)$, $\mathbf{b} = \mathbf{f}(\mathbf{a})$, let $\mathbf{g}$ be the continuous inverse of $\mathbf{f}$, defined in a neighborhood of $\mathbf{b}$, such that $\mathbf{g}(\mathbf{b}) = \mathbf{a}$, Find an explicit formula for $\mathbf{g}$, compute $\mathbf{f}'(\mathbf{a})$ and $\mathbf{g}'(\mathbf{b})$, and verify the formula (52).

We will consider the restriction of $\mathbf{f}$ to the open set $U = \mathbb{R} \times (0, \pi)$, which contains $\mathbf{a}$. We show that $\mathbf{f}|_U$ is an injection. Suppose $(x_1, y_1), (x_2, y_2) \in U$ such that $\mathbf{f}(x_1, y_1) = \mathbf{f}(x_2, y_2)$. Then in particular $|\mathbf{f}(x_1, y_1)| = |\mathbf{f}(x_2, y_2)|$, so

$$\begin{aligned}
\sqrt{(e^{x_1}\cos y_1)^2 + (e^{x_1}\sin y_1)^2} &= \sqrt{(e^{x_2}\cos y_2)^2 + (e^{x_2}\sin y_2)^2} \\
\sqrt{e^{2x_1}(\cos^2 y_1 + \sin^2 y_1)} &= \sqrt{e^{2x_2}(\cos^2 y_2 + \sin^2 y_2)} \\
\sqrt{e^{2x_1}} &= \sqrt{e^{2x_2}} \\
e^{x_1} &= e^{x_2}.
\end{aligned}$$

So $x_1 = x_2$, which gives us $\cos y_1 = \cos y_2$. And since $y_1, y_2 \in (0, \pi)$, then in fact $y_1 = y_2$. Then the inverse of $\mathbf{f}|_U$ is given by

$$\mathbf{g}(x, y) = (g_1, g_2)(x, y) = \left(\log\sqrt{x^2 + y^2},\ \arctan(y/x)\right).$$

We now calculate the derivatives of $\mathbf{f}$ and $\mathbf{g}$ at $\mathbf{a}$. First, using the expression for the Jacobian of $\mathbf{f}$ from (b), we find

$$[\mathbf{f}'(\mathbf{a})] = e^0 \begin{bmatrix} \cos(\pi/3) & -\sin(\pi/3) \\ \sin(\pi/3) & \cos(\pi/3) \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix}.$$

Next

$$\begin{aligned}
[\mathbf{g}'(x, y)] &= \begin{bmatrix} (D_1 g_1)(x, y) & (D_2 g_1)(x, y) \\ (D_1 g_2)(x, y) & (D_2 g_2)(x, y) \end{bmatrix} \\
&= \begin{bmatrix} \frac{x}{x^2+y^2} & \frac{y}{x^2+y^2} \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} \end{bmatrix} \\
&= \frac{1}{x^2 + y^2}\begin{bmatrix} x & y \\ -y & x \end{bmatrix},
\end{aligned}$$

so

$$\begin{aligned}
[\mathbf{g}'(\mathbf{b})] &= [\mathbf{g}'(\mathbf{f}(\mathbf{a}))] \\
&= [\mathbf{g}'(1/2, \sqrt{3}/2)] \\
&= \frac{1}{(1/2)^2 + (\sqrt{3}/2)^2}\begin{bmatrix} 1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & 1/2 \end{bmatrix} \\
&= \frac{1}{2}\begin{bmatrix} 1 & \sqrt{3} \\ -\sqrt{3} & 1 \end{bmatrix},
\end{aligned}$$

which is precisely the inverse of $[\mathbf{f}'(\mathbf{a})] = [\mathbf{f}'(\mathbf{g}'(\mathbf{b}))]$.

(d) What are the images under $\mathbf{f}$ of lines parallel to the coordinate axes?

We write $\mathbf{f}(x, y) = e^x(\cos y, \sin y)$. A choice of $y \in \mathbb{R}$ gives a point $(\cos y, \sin y)$ on the unit circle in $\mathbb{R}^2$ centered at the origin. And a choice of $x \in \mathbb{R}$ gives a radius $e^x \in (0, +\infty)$. Which means that the point (x, y) is the point with magnitude e^x and angle y to the positive x-axis. So for a fixed x, the line parallel to the y-axis has the image of a circle centered at the origin with radius e^x. And for a fixed $y \in \mathbb{R}$, the image of the line parallel to the x-axis has the image of a ray starting at (but not including) the origin with angle y.

Exercise 9.19 Show that the system of equations

$$3x + y - z + u^2 = 0$$
$$x - y + 2z + u = 0$$
$$2x + 2y - 3z + 2u = 0$$

can be solved for x, y, u in terms of z; for x, z, u in terms of y; for y, z, u in terms of x; but not for x, y, z in terms of u.

Define the functions

$$\begin{aligned} f_1(x, y, z, u) &= 3x + y - z + u^2, \\ f_2(x, y, z, u) &= x - y + 2z + u, \\ f_3(x, y, z, u) &= 2x + 2y - 3z + 2u, \end{aligned}$$

and $F = (f_1, f_2, f_3) : \mathbb{R}^{3+1} \to \mathbb{R}^3$. We see F is continuously differentiable, as its partial derivatives exist and are continuous, and that $F(0) = 0$. In particular,

$$[F'(x, y, z, u)] = \begin{bmatrix} 3 & 1 & -1 & 2u \\ 1 & -1 & 2 & 1 \\ 2 & 2 & -3 & 2 \end{bmatrix}.$$

Define the matrix B_z to be the Jacobian of F at 0 excluding the 'z' row, i.e.,

$$B_z = \begin{bmatrix} 3 & 1 & 0 \\ 1 & -1 & 1 \\ 2 & 2 & 2 \end{bmatrix}.$$

Then

$$\det B_z = 3 \begin{vmatrix} -1 & 1 \\ 2 & 2 \end{vmatrix} - \begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix} = 3(-2 - 2) - (2 - 2) = -12,$$

so B_z is invertible. Then by the implicit function theorem with $n = 3$ and $m = 1$, there exist open neighborhoods $0 \in V \subseteq \mathbb{R}^{3+1}$ and $0 \in W \subseteq \mathbb{R}^1$, such that to each $z \in W$ there corresponds a unique $(x, y, u) \in V$ such that $F(x, y, z, u) = 0$. The map $z \mapsto (x, y, u)$ is precisely the solving of the system of equations for x, y, u in terms of z. Repeating this process for y and x, we find

$$\det B_y = \begin{vmatrix} 3 & -1 & 0 \\ 1 & 2 & 1 \\ 2 & -3 & 2 \end{vmatrix} = 3 \begin{vmatrix} 2 & 1 \\ -3 & 2 \end{vmatrix} - (-1) \begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix} = 3(4 + 3) + (2 - 2) = 21$$

and

$$\det B_x = \begin{vmatrix} 1 & -1 & 0 \\ -1 & 2 & 1 \\ 2 & -3 & 2 \end{vmatrix} = 1 \begin{vmatrix} 2 & 1 \\ -3 & 2 \end{vmatrix} - (-1) \begin{vmatrix} -1 & 1 \\ 2 & 2 \end{vmatrix} = 1(4 + 3) + (-2 - 2) = 3.$$

Hence, we can use any one of x, y, z to express the remaining values.

Taking u as as constant, we solve the linear system of equations.

$$\begin{bmatrix} 3 & 1 & -1 & u^2 \\ 1 & -1 & 2 & u \\ 2 & 2 & -3 & 2u \end{bmatrix} \sim \begin{bmatrix} 0 & 4 & -7 & u^2 - 3u \\ 1 & -1 & 2 & u \\ 0 & 4 & -7 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & 4 & -7 & u^2 - 3u \\ 1 & -1 & 2 & u \\ 0 & 0 & 0 & -u^2 + 3u \end{bmatrix}$$

Therefore, $-u(u-3) = 0$, so u must be 0 or 3. hence, the possible solution space over is a 2-dimensional subspace of $\mathbb{R}^3$, meaning we cannot solve for x, y, z as a function of u.

Exercise 9.20 Take $n = m = 1$ in the implicit function theorem, and interpret the theorem (as well as its proof) graphically.

Suppose $U \subseteq \mathbb{R}^2$ is open and $f : U \to \mathbb{R}$ is a C^1 function. The graph of f is the set of points $\{(x, y, f(x, y)) : (x, y) \in U\}$, which defines a 2-dimensional surface in $\mathbb{R}^3$. Suppose there is a point $(a, b) \in U$ such that $f(a, b) = 0$ and $\frac{\partial f}{\partial x}(a, b) = (D_1 f)(a, b) \neq 0$. These condition on (a, b) ensure f is not constantly zero around (a, b), so that the set of zeros of f, which is the intersection of the graph of f and the xy-plane, is locally a curve around the point (a, b). Additionally, we can deduce that the tangent line to this curve at (a, b) is not parallel to the x-axis, and we will be able to select a small neighborhood around (a, b) where this curve can be injectively projected onto the y-axis.

The implicit function theorem states that, in some neighborhood $V \subseteq U$ of (a, b), this curve can be represented by a a C^1 function $g : W \to \mathbb{R}$, where the open set $W \subseteq \mathbb{R}$ is the projection V onto the y-axis. The graph of g is the set of points $\{(g(y), y) : y \in W\}$, which defines a curve in $\mathbb{R}^2$, and is precisely the set of zeros of f inside V. Geometrically, we can think of g as the local inverse of the projection from the set of zeros of f to the y-axis.

Exercise 9.23 Define f in $\mathbb{R}^3$ by

$$f(x, y_1, y_2) = x^2 y_1 + e^x + y_2.$$

Show that $f(0, 1, -1) - 0$, $(D_1 f)(0, 1, -1) \neq 0$, and that there exists therefore a differentiable function g in some neighborhood of $(1, -1)$ in $\mathbb{R}^2$, such that $g(1, -1) = 0$ and

$$f(g(y_1, y_2), y_1, y_2) = 0.$$

Find $(D_1 g)(1, -1)$ and $(D_2 g)(1, -1)$.

Evaluating f at $(0, 1, -1)$, we find

$$f(0, 1, -1) = (0)^2(1) + e^0 + (-1) = 0 + 1 - 1 = 0.$$

The first partial derivative of f is given by $D_1 f = 2xy_1 + e^x$, so

$$(D_1 f)(0, 1, -1) = 2(0)(1) + e^0 = 0 + 1 = 1.$$

Then the implicit function theorem tells us that such a function g exists. The Jacobian of f at $(0, 1, -1)$ is given by

$$[f'(0, 1, -1)] = \begin{bmatrix} A_{\mathbf{x}} & A_{\mathbf{y}} \end{bmatrix},$$

where

$$A_{\mathbf{x}} = [D_1 f(0, 1, -1)] = [1]$$

and

$$A_{\mathbf{y}} = \begin{bmatrix} D_2 f(0, 1, -1) & D_3 f(0, 1, -1) \end{bmatrix}.$$

We have $D_2 f = x^2$ and $D_3 f = 1$, so $A_{\mathbf{y}} = \begin{bmatrix} 0 & 1 \end{bmatrix}$. Then the implicit function theorem tells us that

$$[g'(1, -1)] = -(A_{\mathbf{x}})^{-1} A_{\mathbf{y}} = -[1] \begin{bmatrix} 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \end{bmatrix}.$$

So $(D_1 g)(1, -1) = 0$ and $(D_2 g)(1, -1) = -1$.