1

Let $X = \mathbb{C}^2 \setminus \{0\}$.

Consider the projection $\pi : X \to \mathbb{C}\mathrm{P}^1$.

Since X is second-countable, it is clear that $\mathbb{C}\mathrm{P}^1$ is as well.

We check that π is an open map. Let $U \subseteq X$ be open, then

$$\pi^{-1}(\pi(U)) = \bigcup_{z \in U} [z] = \bigcup_{z \in U} \bigcup_{\lambda \neq 0} \{\lambda z\} = \bigcup_{\lambda \neq 0} \bigcup_{z \in U} \{\lambda z\} = \bigcup_{\lambda \neq 0} \lambda U.$$

For $\lambda \neq 0$, the map $z \mapsto \lambda z$ is a homeomorphism of X to itself. Therefore, λU as the image of an open set is open. Hence, $\pi^{-1}(\pi(U))$ is open, and we conclude that $\pi(U)$ is open.

We check that $\mathbb{C}\mathrm{P}^1$ is Hausdorff.

Define a relation on X by $z \sim w$ whenever $\pi(z) = \pi(w)$. Let $R = \{(z, w) \mid z \sim w\} \subseteq X \times X$ be the set of pairs of points that are identified under the projection π. Equivalently, $z \sim w$ if and only if $z^1 w^2 = z^2 w^1$. Consider the polynomial map

$$\begin{aligned} f : X \times X &\longrightarrow \mathbb{C}, \\ (z, w) &\longmapsto z^1 w^2 - z^2 w^1. \end{aligned}$$

This map is continuous, so its zero locus $R = f^{-1}(0)$ is closed in $X \times X$. It follows that the quotient space $\mathbb{C}\mathrm{P}^1 = X/{\sim}$ is Hausdorff.

Define the open cover $\{U_1, U_2\}$ of $\mathbb{C}\mathrm{P}^1$ as usual. Let $\varphi_1 : [1 : w] \mapsto w$ and $\varphi_2 : [z : 1] \to z$ be the usual charts. We check that these charts are smoothly compatible:

$$\varphi_2 \circ \varphi_1^{-1}(w) = \varphi_2([1 : w]) = \varphi_2([\tfrac{1}{w} : 1]) = \tfrac{1}{w}$$

is a smooth map $\varphi_1(U_1 \cap U_2) = \mathbb{C} \setminus 0 \to \varphi_2(U_1 \cap U_2) = \mathbb{C} \setminus \{0\}$ and

$$\varphi_1 \circ \varphi_2^{-1}(z) = \varphi_1([z : 1]) = \varphi_1([1 : \tfrac{1}{z}]) = \tfrac{1}{z}$$

is a smooth map $\varphi_2(U_1 \cap U_2) = \mathbb{C} \setminus 0 \to \varphi_1(U_1 \cap U_2) = \mathbb{C} \setminus \{0\}$. Hence, we have found a smooth structure on $\mathbb{C}\mathrm{P}^1$.

2

We claim that the differential $i_* : T_pS^n \to T_p\mathbb{R}^{n+1} \cong \mathbb{R}^{n+1}$ is injective. The radial projection $\pi : \mathbb{R}^{n+1} \setminus \{0\} \to S^n$ is a retraction, i.e., $\pi \circ \iota = \mathrm{id}$. Then functorality of the differential gives us $\pi_* \circ \iota_* = \mathrm{id}$, hence ι_* is injective.

3

a

Say $c \in I$ is a nonzero constant function and $f \in C^\infty(M)$ is arbitrary. Then $\frac{1}{c} \in C^\infty(M)$ is a nonzero constant function and therefore

$$f = \frac{f}{c} \cdot c \in I.$$

Hence, $I = C^\infty(M)$.

Now suppose $f \notin I_p$, i.e., $f(p) \neq 0$. Then $g = f(p) - f \in I_p$. However, $g + f = f(p)$ is a nonzero constant function so $I_p + \langle f \rangle$ must be all of $C^\infty(M)$. In other words, I_p is a maximal ideal.

b

Assume for contradiction that $I \trianglelefteq C^\infty(M)$ is a maximal ideal which is not of the form I_p. In particular, $I \nsubseteq I_p$ for all $p \in M$. For each $p \in M$, let $f_p \in I$ be such that $f_p(p) \neq 0$. We can choose some open neighborhood U_p of p on which f_p is nonzero, e.g., $f_p^{-1}(B_\varepsilon(f_p(p))$ for small $\varepsilon > 0$. Then $\{U_p\}$ is an open cover of M and we can select a finite subcover, indexed by $p_1, \ldots, p_k$. Now $f_{p_i}^2 \in I$ is nonnegative function which is positive on at least U_p, so the sum $f = \sum_{i=1}^k f_{p_i}^2 \in I$ is strictly positive on all of M. But then $\frac{1}{f} \in C^\infty(M)$, so the constant function $1 \equiv \frac{1}{f} \cdot f$ is an element of I. By part (a), we conclude that $I = C^\infty(M)$, which is a contradiction.