Exercise 3.12 Show that Example 3.11 can be generalized as follows: Let Y be a nonempty irreducible subvariety of an affine variety X, and set $U = X \setminus Y$.

(a) Assume that $A(X)$ is a unique factorization domain. Show that $\mathscr{O}_X(U) = A(X)$ if and only if $\operatorname{codim} Y \geq 2$.

Proof. Suppose $\operatorname{codim} Y \geq 2$. We have $Y = V(f_1, \ldots, f_k)$ for some $f_1, \ldots, f_k \in A(X)$ and $k \in \mathbb{N}$. Note that we must have $k \geq 2$, otherwise $Y = V(f_1)$ is irreducible and, therefore, has codimension 1 in X. Then

$$Y = V(f_1, \ldots, f_k) = V(f_1) \cap \cdots \cap V(f_k),$$

and assume that $k \geq 2$ is the smallest integer such that this is true. Additionally, we may assume that each pair f_i, f_j is coprime for $i \neq j$. To see why, suppose that $f_i = hg_i$ and $f_j = hg_j$ for some $h \in A(X)$ and $i \neq j$. Then we have

$$V(f_i) \cap V(f_j) = (V(h) \cup V(g_i)) \cap (V(h) \cup V(g_j)) = V(h) \cup V(g_i, g_j).$$

This gives us

$$\begin{aligned} Y &= (V(h) \cup V(g_i, g_j)) \cap V(\{f_\ell : \ell \neq i, j\}) \\ &= V(\{h\} \cup \{f_\ell : \ell \neq i, j\}) \cup V(\{g_i, g_j\} \cup \{f_\ell : \ell \neq i, j\}), \end{aligned}$$

which is an expression of Y as the union of two closed sets. Since Y is irreducible, we cannot have both of these be proper subsets, so one must equal Y. Since k was chosen to be minimal, and the left term is the zero locus of only $k-1$ polynomials, it cannot equal Y. Therefore, we can express Y as the right term, which is the zero locus of the same polynomials, but f_i, f_j replaced with g_i, g_j, the latter pair being coprime. Then we now have

$$U = D(f_1) \cup \cdots \cup D(f_k).$$

Let $\varphi \in \mathscr{O}_X(U)$. For some $i \neq j$ and $n, m \in \mathbb{N}$, we have

$$\varphi|_{D(f_i)} = \frac{g_i}{f_i^n} \quad \text{and} \quad \varphi|_{D(f_j)} = \frac{g_j}{f_j^m}.$$

for some $g_i, g_j \in A(X)$. We can assume that $f_i \nmid g_i$ and $f_j \nmid g_j$. Then in $D(f_i) \cap D(f_j)$, this means $g_i f_j^m - g_j f_i^n = 0$. Therefore, $D(f_i) \cap D(f_j)$ is a subset of the closed set $V(g_i f_j^m - g_j f_i^n)$. And since X is irreducible, then the closure of $D(f_i) \cap D(f_j)$ is precisely X, implying that $V(g_i f_j^m - g_j f_i^n) = X$. Thus, we have $g_i f_j^m = g_j f_i^n$ as polynomials in $A(X)$.

We now claim that $n = m = 0$. Suppose, to the contrary, that $n \geq 1$, then f_i divides f_i^n, which divides $g_i f_j^m$. However, since f_i and f_j are coprime, and $A(X)$ is a UFD, then $f_i \mid g_i$, which is a contradiction. Therefore, $n = 0$ and, by the same argument, $m = 0$.

This now gives us $g_i = g_j = \varphi$ on the open set $D(f_i) \cap D(f_j)$. Since X is irreducible, Remark 3.5 tells us, then, that this holds on all of X, so $\varphi \in A(X)$. This proves the inclusion $\mathscr{O}_X(U) \subseteq A(X)$, and the opposite inclusion is obvious, giving us equality.

Now suppose $\operatorname{codim} Y < 2$, we will show that $\mathscr{O}_X(U) \neq A(X)$. If $Y = X$, then $\mathscr{O}_X(U) = \mathscr{O}_X(\varnothing) = \{0\}$, which cannot be $A(X)$ since X is nonempty, and we have the constant polynomial $1 \in A(X)$.

If $Y \neq X$, then we have the chain $Y = Y_0 \subsetneq Y_1 = X$, so th codimension of Y in X is at least 1, so in fact $\operatorname{codim} Y = 1$. Then by Proposition 2.37, $A(X)$ being a UFD means that $Y = V(f)$ for some irreducible (particularly non-unit) $f \in A(X)$. Then we have $\mathscr{O}_X(U) = \mathscr{O}_X(D(f))$, which contains $\frac{1}{f}$. However, since $\frac{1}{f}$ is the multiplicative inverse of f in the localization $A(X)_f$, but f is not a unit in $A(X)$, then $\frac{1}{f} \notin A(X)$. Hence $\mathscr{O}_X(U) \neq A(X)$.
$\square$

(b) Show by example that the equivalence of (a) is in general false if $A(X)$ is not assumed to be a unique factorization domain.

Exercise 3.23 Let $\mathscr{F}$ be a sheaf on a topological space X, and let Y be a nonempty irreducible closed subset of X. We define the *stalk of* $\mathscr{F}$ *at* Y to be

$$\mathscr{F}_Y := \{(U, \varphi) : U \text{ is an open subset of } X \text{ with } U \cap Y \neq \varnothing, \text{ and } \varphi \in \mathscr{F}(U)\}/\sim$$

where $(U, \varphi) \sim (U', \varphi')$ if and only if there is a small open set $V \subseteq U \cap U'$ with $V \cap Y \neq \varnothing$ and $\varphi|_V = \varphi'|_V$. It therefore describes functions in an arbitrarily small neighborhood of an arbitrary dense open subset of Y.
If Y is a nonempty irreducible subvariety of an affine variety X, prove that the stalk $\mathscr{O}_{X,Y}$ of $\mathscr{O}_X$ at Y is a K-algebra isomorphic to the localization $A(X)_{I(Y)}$ (hence giving a geometric meaning to this algebraic localization).

Proof. We consider the map

$$\begin{aligned} A(X)_{I(Y)} &\to \mathscr{O}_{X,Y} \\ \tfrac{g}{f} &\mapsto \overline{\left(D(f), \tfrac{g}{f}\right)}. \end{aligned}$$

We show that this map is well-defined. For each $\frac{g}{f} \in A(X)_{I(Y)}$, we have $f \in A(X) \setminus I(Y)$ so f is not identically zero on Y, so $D(f) \cap Y \neq \varnothing$. And since $\frac{g}{f} \in \mathscr{O}_X(D(f))$, this map does in fact have $\mathscr{O}_{X,Y}$ as its codomain. Now suppose $\frac{g_1}{f_1} = \frac{g_2}{f_2}$ in $A(X)_{I(Y)}$, so there exists some $h \in A(X)$ such that $h(g_1 f_2 - g_2 f_1) = 0$. We consider the open set $U = D(f_1 f_2 h) \subseteq D(f_1) \cap D(f_2)$. We have h nonzero everywhere in U, so $g_1 f_2 - g_2 f_1 = 0$ in U. This means that $\frac{g_1}{f_1} = \frac{g_2}{f_2}$ in U, implying that their images in the above map are the same.

This K-algebra homomorphism is surjective. An equivalence class in $\mathscr{O}_{X,Y}$ is represented by some $U \subseteq X$ with $U \cap Y \neq \varnothing$ and $\varphi \in \mathscr{O}_X(U)$. Then for some $a \in U \cap Y$, we have $\varphi = \frac{g}{f}$ on some open neighborhood U_a of a contained in U and some $g, f \in A(X)$ with f nowhere zero on U_a. In particular, f is not identically zero on Y, since $a \in U_a \cap Y \neq \varnothing$, so $f \in A(X) \setminus I(Y)$. Then letting $V = U_a \cap D(f)$, we see that $V \cap Y \neq \varnothing$ and $\varphi = \frac{g}{f}$ on V, so

$$\tfrac{g}{f} \mapsto \overline{\left(D(f), \tfrac{g}{f}\right)} = \overline{(U, \varphi)}.$$

Next we see that this homomorphism is injective, by showing its kernel is zero. Suppose we have

$$\tfrac{g}{f} \mapsto \overline{\left(D(f), \tfrac{g}{f}\right)} = \overline{\left(X, \tfrac{0}{1}\right)},$$

where the right hand side is precisely zero of $\mathscr{O}_{X,Y}$. Then for some open subset $V \subseteq D(f) \cap X$, we have $V \cap Y \neq \varnothing$ and $\frac{g}{f} = \frac{0}{1}$ on V. In particular, g is identically zero on V. Since V is open, it is the union of finitely many distinguished open sets; suppose $D(h) \subseteq V$ is one such distinguished open subset. Then h is zero in the complement $X \setminus D(h)$, which contains $X \setminus V$. Therefore, $hg = 0$ on all of X, implying that $\frac{g}{f} = \frac{0}{1} = 0$ in the localization ring $A(X)_{I(Y)}$. Hence, the kernel of this map zero, and we conclude that it is an isomorphism of K-algebras.

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