1 Let $\mathcal{S}$ be the Schwartz space (functions of rapid decay), and δ is the Dirac delta function. Prove directly that the derivative of δ, denoted by δ', is in S' (the dual space of $\mathcal{S}$).

Proof. Note that $\delta \in \mathcal{S}'(\mathbb{R})$. By definition, $\delta' : \mathcal{S}(\mathbb{R}) \to \mathbb{C}$ is a linear map given by

$$\delta'(f) = \mathrm{D}^1\delta(f) = (-1)^1\delta(\mathrm{D}^1 f) = -\delta(f') = -f'(0).$$

Recall that $\mathcal{S}$ is a locally compact space with the seminorms $\|-\|_{\alpha,\beta}$. Trivially, $\mathbb{C}$ is also a locally compact space with its usual norm $|-|$. Therefore, boundedness follows from

$$|\delta'(f)| = |f'(0)| \leq \|f'\|_\infty = \sup_{x\in\mathbb{R}} |x^0\mathrm{D}^1 f(x)| = \|f\|_{0,1}.$$

□

Prove that δ' does not come from a measure.

Proof. Suppose there is a measure ν on $\mathbb{R}$ such that for all $f \in \mathcal{S}$

$$\int_{\mathbb{R}} f \,\mathrm{d}\nu = \delta'(f) = -f'(0).$$

Consider the function $f(x) = \int_0^x e^{-t^2}\mathrm{d}t$. This is an increasing smooth function whose derivatives near $\pm$ infinity go to zero faster than any polynomial grows, i.e., $x^\alpha\mathrm{D}^\beta f(x) \to 0$ as $x \to \pm\infty$ for all $\alpha, \beta \in \mathbb{N}$. Since each derivative is continuous, this implies $\|f\|_{\alpha,\beta} < \infty$, so in fact $f \in \mathcal{S}$. Additionally, $f'(0) = 1$.

Define the functions $f_n(x) = f(nx)/n$ for all $n \in \mathbb{N}$. Then $f_n \in \mathcal{S}$ and

$$\int_{\mathbb{R}} f_n \,\mathrm{d}\nu = \delta'(f_n) = -f_n'(0) = -1.$$

Note that $\|f\|_\infty \leq 1$ so $\|f_n\|_\infty \to 0$ as $n \to \infty$. Each f_n is integrable and $|f_n| \geq |f_{n+1}|$, so monotone convergence gives us

$$-1 = \lim_{n\to\infty} \int_{\mathbb{R}} f_n \,\mathrm{d}\nu = \int_{\mathbb{R}} \lim_{n\to\infty} f_n \,\mathrm{d}\nu = \int_{\mathbb{R}} 0 \,\mathrm{d}\nu = 0.$$

This is a contradiction, so no such measure ν exists. □

2 Let X and Y be Banach spaces. Prove that if $T_n \in \mathcal{L}(X,Y)$ and $\{T_n x\}$ is a Cauchy sequence for each $x \in X$, then there exists $T \in \mathcal{L}(X,Y)$ so that $T_n \to T$ in the strong operator.

Proof. Since Y is Banach, $T_n x$ converges to some point $Tx \in Y$. Then $T : X \to Y$ is a linear map:

$$T(ax+y) = \lim_{n\to\infty} T_n(ax+y) = a \lim_{n\to\infty} T_n x + \lim_{n\to\infty} T_n y = aTx + Ty.$$

By construction, $T_n x \to Tx$ for all $x \in X$. In particular,

$$\sup_{n\in\mathbb{N}} \|T_n x\| < \infty$$

for all $x \in X$. So the uniform boundedness principle gives us a bound on the operator norms

$$M = \sup_{n\in\mathbb{N}} \|T_n\| < \infty.$$

Then for all $x \in X$ we have

$$\|Tx\| = \lim_{n\to\infty} \|T_n x\| \leq \lim_{n\to\infty} \|T_n\|\|x\| \leq \lim_{n\to\infty} M\|x\| = M\|x\|,$$

Hence, T is bounded, i.e., $T \in \mathcal{L}(X,Y)$. $\square$

3 Let T_t be an operator on $L^2(\mathbb{R})$ with $T_t\varphi(x) := \varphi(x+t)$. What is the norm of T_t?

We have

$$\|T_t\varphi\|_2 = \int_{\mathbb{R}} |\varphi(x+t)|^2 \, \mathrm{d}x = \int_{\mathbb{R}} |\varphi(x)|^2 \, \mathrm{d}x = \|\varphi\|_2,$$

so $\|T_t\| = 1$.

To what operator does T_t converge as $t \to \infty$ and in what topology?

Claim that $T_t \to 0$ weakly.

Proof. Suppose $\varphi \in L^2(\mathbb{R})$ and $\ell \in L^2(\mathbb{R})^*$. Choose $\psi \in L^2(\mathbb{R})$ by the Riesz lemma for ℓ. Then

$$|\ell(T_t\varphi)| = \left| \int_{\mathbb{R}} \overline{\psi(x)}\varphi(x+t) \, \mathrm{d}x \right|.$$

Given $\varepsilon > 0$ choose $M > 0$ such that

$$\int_{|x|>M} |\psi(x)|^2 \mathrm{d}x < \varepsilon \quad \text{and} \quad \int_{|x|>M} |\varphi(x)|^2 \mathrm{d}x < \varepsilon.$$

Then Cauchy-Schwarz gives us

$$\begin{aligned}
\left| \int_{\mathbb{R}} \overline{\psi(x)}\varphi(x-t) \, \mathrm{d}x \right| &\leq \left| \int_{|x|>M} \overline{\psi(x)}\varphi(x+t) \, \mathrm{d}x \right| + \left| \int_{|x|\leq M} \overline{\psi(x)}\varphi(x+t) \, \mathrm{d}x \right| \\
&\leq \left(\int_{|x|>M} |\psi(x)|^2 \mathrm{d}x \right)^{1/2} \left(\int_{|x|>M} |\varphi(x+t)|^2 \mathrm{d}x \right)^{1/2} \\
&\qquad + \left(\int_{|x|\leq M} |\psi(x)|^2 \mathrm{d}x \right)^{1/2} \left(\int_{|x|\leq M} |\varphi(x+t)|^2 \mathrm{d}x \right)^{1/2} \\
&\leq \varepsilon\|\varphi\|_\infty + \|\psi\|_\infty \left(\int_{|x|\leq M} |\varphi(x+t)|^2 \mathrm{d}x \right)^{1/2} \\
&\leq \varepsilon\|\varphi\|_\infty + \|\psi\|_\infty \left(\int_{|x-t|\leq M} |\varphi(x)|^2 \mathrm{d}x \right)^{1/2}.
\end{aligned}$$

If we choose t large enough, $|x-t| \leq M$ will imply $|x| > M$, giving us

$$|\ell(T_t\varphi)| = \left| \int_{\mathbb{R}} \overline{\psi(x)}\varphi(x-t) \, \mathrm{d}x \right| \leq \varepsilon\|\varphi\|_\infty + \|\psi\|_\infty\varepsilon.$$

Hence, we have convergence $T_t \to 0$ in the weak topology. $\square$

4 (a) Let A be a self-adjoint operator on a Hilbert space $\mathcal{H}$. Prove that

$$\|A\| = \sup_{\|x\|=1} |\langle Ax, x\rangle|.$$

Following the hint, assume $\|x\| = \|y\| = 1$, then we find

$$\begin{aligned}
4|\operatorname{Re}\langle x, Ay\rangle| &\leq |\langle x+y, A(x+y)\rangle| + |\langle x-y, A(x-y)\rangle| \\
&\leq \|x+y\|^2 \sup_{\|u\|=1} \langle u, Au\rangle + \|x-y\|^2 \sup_{\|u\|=1} |\langle u, Au\rangle| \\
&= (2\|x\|^2 + 2\|y\|^2) \sup_{\|u\|=1} |\langle u, Au\rangle| \\
&= 4 \sup_{\|u\|=1} |\langle u, Au\rangle|.
\end{aligned}$$

Consider the unit vector $w = \overline{\langle x, Ay\rangle}/|\langle x, Ay\rangle|$, then $w\langle x, Ay\rangle = \langle wx, Ay\rangle$ is real, so applying the above inequality we get

$$|\langle x, Ay\rangle| = |w\langle x, Ay\rangle| = |\operatorname{Re}\langle wx, Ay\rangle| \leq \sup_{\|u\|=1} |\langle u, Au\rangle|.$$

We now write

$$\|A^2\| = \|A\|^2 = \sup_{\|x\|=1} \|Ax\|^2 = \sup_{\|x\|=1} |\langle Ax, Ax\rangle| = \sup_{\|x\|=1} |\langle x, A^2x\rangle|.$$

I did not complete the proof.

b Find an example which shows that the conclusion of (a) need not be true if A is not self-adjoint.

Let $\mathcal{H} = \mathbb{R}^2$ and consider the rotation $A(x_1, x_2) = (-x_2, x_1)$.

Then $\|A\| = 1$ but $\langle Ax, x\rangle = 0$ for all $x \in \mathbb{R}^2$.

5 Show that the spectral radius of the Volterra integral operator

$$(Tf)(x) = \int_0^x f(y)\mathrm{d}y$$

as a map on $C[0,1]$, with the supremum norm, is equal to zero.

Proof. Note that $C[0,1]$ with the supremum norm is a Banach space, so the spectral radius is given by $\lim_{n\to\infty} \|T^n\|^{1/n}$. We claim that for $f \in C[0,1]$ with $\|f\|_\infty = 1$ and $x \in [0,1]$

$$|T^n f(x)| \leq \frac{x^n}{n!}.$$

We perform induction on $n \geq 0$. The base case of is trivial with $|f(x)| \leq \|f\|_\infty = 1$. The inductive step is shown by

$$|T^n f(x)| = \left| \int_0^x T^{n-1} f(y)\,\mathrm{d}y \right| \leq \int_0^x |T^{n-1} f(y)|\,\mathrm{d}y \leq \int_0^x \frac{y^{n-1}}{(n-1)!}\,\mathrm{d}y = \frac{x^n}{n!}.$$

We now compute

$$\|T^n\| = \sup_{\|f\|_\infty = 1} \|T^n f\|_\infty \leq \sup_{\|f\|_\infty = 1} \sup_{x\in[0,1]} \frac{x^n}{n!} = \frac{1}{n!},$$

so then

$$\lim_{n\to\infty} \|T^n\|^{1/n} \leq \lim_{n\to\infty} \frac{1}{\sqrt[n]{n!}} = 0.$$

Hence, the spectral radius must be zero. □

What is the norm of T?

In the first part, we found the upper bound $\|T\| \leq 1$. To see that we in fact have equality, consider the constant function $f(x) = 1$:

$$\|Tf\|_\infty = \sup_{x\in[0,1]} \int_0^x f(y)\mathrm{d}y = \sup_{x\in[0,1]} \int_0^x 1\mathrm{d}y = \sup_{x\in[0,1]} x = 1.$$

Hence, $\|T\| = 1$.