1 Let (M, μ) be a measure space, show that $L^p(M, \mathrm{d}\mu)$ is a Banach space for any $1 < p < \infty$.

Proof. Normed vector space easy; show complete.

By Problem 4, to show $L^p(M, \mathrm{d}\mu)$ is complete, it suffices to check that every absolutely summable sequence is summable. Suppose $\{f_n\}$ is an absolutely summable sequence, which means $\sum_{n=1}^{\infty} \|f_n\|_p < \infty$.

For each $m \in \mathbb{N}$, define $g_n = |f_n|^p$ which has L^1 norm

$$\|g_n\|_1 = \int_M |g_n| \, \mathrm{d}\mu = \int_M |f_n|^p \, \mathrm{d}\mu = \|f_n\|_p^p < \infty,$$

so $g_n \in L^1(M, \mathrm{d}\mu)$. Moreover, $\{g_n\}$ is absolutely summable since

$$\sum_{n=1}^{\infty} \|g_n\|_1 = \sum_{n=1}^{\infty} \|f_n\|_p^p = \sum_{n=1}^{N-1} \|f_n\|_p^p + \sum_{n=N}^{\infty} \|f_n\|_p^p \le \sum_{n=1}^{N-1} \|f_n\|_p^p + \sum_{n=N}^{\infty} \|f_n\|_p < \infty,$$

where $N \in \mathbb{N}$ is chosen such that $\|f_n\|_p \le 1$ for all $n \ge N$. Since $L^1(M, \mathrm{d}\mu)$ is complete, Problem 4 implies $\{g_n\}$ is summable, so we can define

$$G = \sum_{n=1}^{\infty} g_n = \sum_{n=1}^{\infty} |f_n|^p \in L^1(M, \mathrm{d}\mu).$$

For all $x \in M$ and $m \in \mathbb{N}$, we have

$$\left| \sum_{n=1}^{m} |f_n(x)| \right|^p \le \sum_{n=1}^{m} |f_n(x)|^p \le G(x)$$

so

$$\sum_{n=1}^{\infty} |f_n(x)| = \lim_{m \to \infty} \sum_{n=1}^{m} |f_n(x)| \le G(x)^{1/p}.$$

Then $\int_M G \, \mathrm{d}\mu < \infty$ implies that $G(x)$ is finite for μ-a.e. $x \in M$; in which case, the sequence $\{f_n(x)\}$ in $\mathbb{C}$ is absolutely summable. Since $\mathbb{C}$ is complete, Problem 4 tells us absolutely summable sequences are summable. So for μ-a.e. $x \in M$, we can define

$$F(x) = \sum_{n=1}^{\infty} f_n(x).$$

Moreover, $F \in L^p(M, \mathrm{d}\mu)$ since

$$\|F\|_p^p = \int_M |F|^p \, \mathrm{d}\mu = \int_M \left| \sum_{n=1}^{\infty} f_n \right|^p \mathrm{d}\mu \le \int_M \sum_{n=1}^{\infty} |f_n|^p \, \mathrm{d}\mu = \int_M G \, \mathrm{d}\mu < \infty.$$

Since $\int_M |G| \mathrm{d}\mu < \infty$, then for μ-a.e. $x \in M$ we have

$$\sum_{n=1}^{m} |f_n(x)| \le \lim_{m \to \infty} \sum_{n=1}^{m} |f_n(x)|^p = |G(x)| < \infty.$$

Thus, $\{f_n\}$ is summable with $F = \sum_{n=1}^{\infty} f_n$, so $L^p(M, \mathrm{d}\mu)$ is complete. $\square$

2 Let M be a subspace of a Hilbert space H, and $f : M \to \mathbb{C}$ a bounded linear functional on M with bound C. Prove that there is a **unique** extension of f to a bounded linear functional on H with the same bound C.

Proof. First, note that f is Lipschitz on M, since for all $x, y \in M$ we have

$$|f(x) - f(y)| = |f(x - y)| \leq C\|x - y\|.$$

In particular, f is uniformly continuous on M, so it uniquely extends to a continuous function on the closure $V = \overline{M} \leq H$ (a subspace of H. Denote the extension by $\tilde{f} : V \to \mathbb{C}$; we check that $\tilde{f}$ is linear. Let $x, y \in V$ and $a \in \mathbb{C}$. Since $V = \overline{M}$, there are sequences $\{x_n\}$ and $\{y_n\}$ in M converging to x and y, respectively. Since f is linear and $\tilde{f}$ is a continuous extension, we have

$$\tilde{f}(ax + y) = \lim_{n\to\infty} f(ax_n + y_n) = a \lim_{n\to\infty} f(x_n) + \lim_{n\to\infty} f(y_n) = a\tilde{f}(x) + \tilde{f}(y).$$

Hence, $\tilde{f}$ is a bounded linear extension of f to V. We now check that $\|\tilde{f}\|_{V^*} = C$. Since $M \subseteq V$, it is clear that $\|\tilde{f}\|_{V^*} \geq C$. Given $x \in V$ nonzero, let $\{x_n\}$ be a nonzero sequence in M converging to x, then

$$\frac{|\tilde{f}(x)|}{\|x\|} = \lim_{n\to\infty} \frac{|f(x_n)|}{\|x_n\|} \leq \lim_{n\to\infty} C = C.$$

This implies $\|\tilde{f}\|_{V^*} \leq C$, so we in fact have equality. Thus, we have shown that f extends uniquely to a bounded linear functional $\tilde{f}$ on $V = \overline{M}$, with the same bound C.

Since V is a closed subspace of a Hilbert space, it too is a Hilbert space. By the Riesz representation theorem, there is a unique $y \in V$ such that $\tilde{f} = \langle -, y\rangle$ and $\|y\| = C$. Naturally, we define $F = \langle -, y\rangle : H \to \mathbb{C}$, which has

$$\|F\|_{H^*} = \|y\| = C.$$

Hence, F is a bounded linear extension of f with the same bound C. To show that F is unique, suppose $G : H \to \mathbb{C}$ is another bounded linear extension of f with the same bound C. Since H is a Hilbert space, the Riesz representation theorem tells us there is some $z \in H$ such that $G = \langle -, z\rangle$ for all $x \in H$ and $\|z\| = \|G\|_{H^*} = C$. Since V is a closed subspace of H, we have $H = V \oplus V^\perp$; write $z = v + w$ for $v \in V$ and $w \in V^\perp$. For all $x \in V$, we have

$$\langle x, v\rangle = \langle x, v + w\rangle = G(x) = F(x) = \langle x, y\rangle.$$

This means $\langle -, v\rangle = \langle -, y\rangle$ as linear functionals on V, so the Riesz representation theorem implies $v = y$. Lastly, since $v \perp w$, we have

$$\|y\|^2 = \|z\|^2 = \|v + w\|^2 = \|v\|^2 + \|w\|^2 = \|y\|^2 + \|w\|^2.$$

This implies $w = 0$ so $y = z$, hence $F = G$. $\square$

3 Show that the unit ball in an infinite dimensional Hilbert space contains infinitely many **disjoint** ball of radius $\sqrt{2}/4$.

Proof. Let $\{x_\alpha\}_{\alpha \in A}$ be an orthonormal basis of H. Since H is infinite dimensional, this basis is infinite. Define $y_\alpha = (1 - \sqrt{2}/4)x_\alpha$ and balls

$$B_\alpha = B_{\sqrt{2}/4}(y_\alpha) \subseteq B_1(0).$$

For $\alpha \neq \beta$, we have $y_\alpha \perp y_\beta$ so

$$\|y_\alpha - y_\beta\|^2 = \|y_\alpha\|^2 + \|y_\beta\|^2 = 2(1 - \sqrt{2}/4)^2.$$

Then

$$\|y_\alpha - y_\beta\| = \sqrt{2}(1 - \sqrt{2}/4) > 2 \cdot \sqrt{2}/4,$$

which implies $B_\alpha \cap B_\beta = \varnothing$. □

4 Prove that a normed linear space is complete if and only if every absolutely summable sequence is summable.

Proof. Suppose X is a Banach space and $\{x_n\}$ is an absolutely summable sequence in X, i.e., have $\sum_{n=1}^{\infty} \|x_n\| < \infty$. The sequence in question is summable if and only if the sequence of partial sums $y_m = \sum_{n=1}^{m} x_n$ converges in X. And for $m \geq k$ we have

$$\|y_m - y_k\| = \left\| \sum_{n=k+1}^{m} x_n \right\| \leq \sum_{n=k}^{\infty} \|x_n\|.$$

The tail tends to zero as $k \to \infty$, which implies $\{y_m\}$ is Cauchy. Since X is complete, the sequence converges, hence $\{x_n\}$ is summable.

Suppose X is a normed linear space and every absolutely summable sequence is summable. Let $\{x_n\}$ be a Cauchy sequence in X. As usual, we may assume without loss of generality that $\|x_n - x_{n-1}\| < 1/2^n$ (otherwise choose a subsequence which does satisfy this). Define the consecutive difference $y_n = x_n - x_{n+1}$, then

$$\sum_{n=1}^{m} \|y_n\| = \sum_{n=1}^{m} \|x_n - x_{n-1}\| \leq \sum_{n=1}^{m} \frac{1}{2^n} \leq 1,$$

so the sequence $\{y_n\}$ is absolutely summable. By assumption, this means the sequence is summable so we have

$$y = \sum_{n=1}^{\infty} y_n = \lim_{n\to\infty} \sum_{k=1}^{n} y_k.$$

Note that

$$x_n = x_1 - \sum_{k=1}^{n} (x_k - x_{k+1}) = x_1 - \sum_{k=1}^{n} y_k.$$

Define $x = x_1 - y$, then

$$\lim_{n\to\infty} \|x - x_n\| = \lim_{n\to\infty} \left\| y - \sum_{k=1}^{n} y_k \right\| = 0.$$

Hence, $x_n \to x$ in X and we conclude that X is complete. $\square$

5 Let

$$c_0 := \{\{a_n\}_{n=1}^{\infty} : \lim_{n\to\infty} a_n = 0\}.$$

Prove that if $\{c_n\}_{n=1}^{\infty} \in \ell_1$, then the linear function on c_0 defined by

$$T(\{a_n\}_{n=1}^{\infty}) = \sum_{n=1}^{\infty} c_n a_n$$

has the norm $\sum_{n=1}^{\infty} |c_n|$.

Proof. We consider c_0 as a subspace of ℓ_∞. Given $a \in c_0$, we have $|a_n| \le \|a\|_\infty < \infty$, so

$$|Ta| \le \sum_{n=1}^{\infty} |c_n||a_n| \le \sum_{n=1}^{\infty} |c_n|\|a\|_\infty = C\|a\|_\infty,$$

where $c \in \ell_1$ gives us

$$C = \sum_{n=1}^{\infty} \|c_n\| < \infty.$$

Therefore, T is a bounded linear functional on c_0 with bound $\|T\|_{c_0^*} \le C$. To show equality, we define sequences $a_k = \{a_{k,n}\} \in c_0$ for $k \in \mathbb{N}$ by

$$a_{k,n} = \begin{cases} \frac{|c_n|}{c_n} & n \le k \text{ and } c_n \ne 0, \\ 0 & \text{otherwise.} \end{cases}$$

If $C = 0$, the overall result is trivial, so we assume $C > 0$. In particular, this implies that some c_N is nonzero, so $\|a_k\|_\infty = 1$ for all $k \ge N$. Moreover, given $\varepsilon > 0$, $c \in \ell_1$ tells us there is some $M \in \mathbb{N}$ such that

$$\sum_{n=1}^{M} |c_n| \ge C - \varepsilon.$$

Then for $k \ge \max\{N, M\}$, we have

$$\|T\|_{c_0^*} \ge \frac{|Ta_k|}{\|a_k\|} = |Ta_k| = \left|\sum_{n=1}^{\infty} c_n a_{k,n}\right| = \left|\sum_{n=1}^{k} |c_n|\right| \ge C - \varepsilon.$$

Letting $\varepsilon \to 0$, we obtain $\|T\|_{c_0^*} \ge C$ and, therefore, equality. □