1 Language

Definition 1.1. A **propositional logic language** L consists of the following data:

(a) a nonempty set of symbols called **propositional variables** (e.g., $P, Q, R, \dots$),

(b) a set of **logical connectives** (e.g., $\neg, \wedge, \vee, \rightarrow, \leftrightarrow$).

Alternate names for propositional logic include the following: propositional calculus, sentential logic (operating on sentences), statement logic, and zeroth-order logic.

Remark 1.2. In practice, the propositional variables are either be upper or lower case letters. If we run out of letters, we can use numerical subscripts (e.g., $P_1, P_2, \dots$).

Remark 1.3. For historical reasons, the following five logical connectives are considered canonical in some sense:

Symbol	Formal Name	Informal Name	Alternate Symbols
$\neg$	negation	"not"	$\sim, !, ', -$
$\wedge$	conjunction	"and"	$\&, \&\&, \cdot$
$\vee$	disjunction	"or"	$\vert, \vert\vert, +$
$\rightarrow$	implication	"implies" or "if-then"	$\implies, \supset$
$\leftrightarrow$	biconditional	"if and only if" or "iff"	$\iff, \equiv$

The alternate symbols may be found in older texts or for logic in other contexts. We will only use the symbols on the left of the above table to refer to the logical connectives proper. Any usage of a symbol on the right means something else. For example, " $\implies$ " and " $\iff$ " are used in their normal manner of mathematical English.

There are other logical connectives that are used in other contexts (e.g., XOR, NAND, NOR), but we will see later that they can be defined in terms of the five canonical connectives, so they are not strictly necessary. In fact, we don't even need all five of the canonical connectives, as we can define some in terms of others. Some equivalencies are sketched below and will be treated more formally later.

$$P \wedge Q \approx \neg(\neg P \vee \neg Q) \qquad\qquad P \vee Q \approx \neg(\neg P \wedge \neg Q)$$
$$P \rightarrow Q \approx \neg P \vee Q \qquad\qquad P \leftrightarrow Q \approx (P \rightarrow Q) \wedge (Q \rightarrow P)$$

For our purposes, we will take either $\{\neg, \vee\}$ or $\{\neg, \rightarrow\}$ as the minimal set of logical connectives.

Definition 1.4. Let L be a language of propositional logic. A **formula** of L, also called a **well-formed formula** or **wff**, is defined inductively as follows:

(a) Every propositional variable is a formula.

(b) If φ is a formula, then $(\neg\varphi)$ is a formula.

(c) If φ and ψ are formulas, then $(\varphi \star \psi)$ is a formula, where $\star$ is any binary logical connective, e.g., $\vee$ or $\rightarrow$.

Any sequence of symbols that is not obtained by one of the above rules is not a formula.

Remark 1.5. Implicitly, we are allowing the use of parentheses to disambiguate the order of operations. A proper treatment of this aspect of formal languages can be found elsewhere.

Remark 1.6. One way to characterize the set of formulas is as the smallest set of strings of symbols which contains all the propositional variables and is closed under finite applications of the logical connective operations. Let $\mathcal{V} = \mathcal{V}(L)$ be the set of propositional variables of L. Then the set $\mathcal{F} = \mathcal{F}(L)$ of formulas of L is the set generated by $\mathcal{V}$ under the following **formula building operations**:

$$\varepsilon_\neg(\varphi) := (\neg\varphi),$$
$$\varepsilon_\wedge(\varphi, \psi) := (\varphi \wedge \psi),$$
$$\varepsilon_\vee(\varphi, \psi) := (\varphi \vee \psi),$$
$$\varepsilon_\rightarrow(\varphi, \psi) := (\varphi \rightarrow \psi),$$
$$\varepsilon_\leftrightarrow(\varphi, \psi) := (\varphi \leftrightarrow \psi).$$

In other words, $\mathcal{F} = \overline{\mathcal{V}}$ is the closure of $\mathcal{V}$ under these operations.

2 Semantics

Definition 2.1. Let L be a language of propositional logic. A **truth valuation** on L is a map $v : \mathcal{V} \rightarrow \{T, F\}$ from the set of propositional variables to the set of truth values.

Remark 2.2. A truth valuation may also be called a truth assignment or structure or interpretation of L. All of these terms express the fact that v provides **semantics** for L. In the context of logic, semantics refers to something like the meaning of the words. Semantics for a language tell us what the words and sentences refer to. etc

Remark 2.3. A truth valuation v can be extended to a map $\overline{v} : \mathcal{F} \rightarrow \{T, F\}$ by defining it recursively on formulas of L:

$$\overline{v}(\neg\varphi) := \begin{cases} T & \text{if } \overline{v}(\varphi) = F, \\ F & \text{otherwise,} \end{cases}$$

$$\overline{v}(\varphi \wedge \psi) := \begin{cases} T & \text{if } \overline{v}(\varphi) = T \text{ and } \overline{v}(\psi) = T, \\ F & \text{otherwise,} \end{cases}$$

$$\overline{v}(\varphi \vee \psi) := \begin{cases} T & \text{if } \overline{v}(\varphi) = T \text{ or } \overline{v}(\psi) = T, \\ F & \text{otherwise,} \end{cases}$$

$$\overline{v}(\varphi \rightarrow \psi) := \begin{cases} T & \text{if } \overline{v}(\varphi) = F \text{ or } \overline{v}(\psi) = T, \\ F & \text{otherwise,} \end{cases}$$

$$\overline{v}(\varphi \leftrightarrow \psi) := \begin{cases} T & \text{if } \overline{v}(\varphi) = \overline{v}(\psi), \\ F & \text{otherwise.} \end{cases}$$

Displayed below in a truth table format, where the rows correspond to the possible truth values of the relevant propositional variables:

φ	ψ	$\neg\varphi$	$\varphi \wedge \psi$	$\varphi \vee \psi$	$\varphi \rightarrow \psi$	$\varphi \leftrightarrow \psi$
T	T	F	T	T	T	T
T	F	F	F	T	F	F
F	T	T	F	T	T	F
F	F	T	F	F	T	T

These definitions are chosen in such a way that they mirror the meanings of these concepts as they appear in our metalanguage of mathematical English.

It should be noted that the uniqueness of the extension $\bar{v}$ is something worth proving. Taking uniqueness for granted, we will only refer to v, identifying it with its extension $\bar{v}$.

In this manner, we consider a truth valuation to be a map $v : \mathcal{F} \rightarrow \{T, F\}$ which is characterized by the restriction $v|_\mathcal{V} : V \rightarrow \{T, F\}$ to propositional variables.

Definition 2.4. Let L be a language of propositional logic, v a truth valuation on L.

For a formula $\varphi \in \mathcal{F}$, we say that v **satisfies** φ if $v(\varphi) = T$. Equivalently, we might say φ is **true** in v when $v(\varphi) = T$, and **false** if $v(\varphi) = F$.

For a set of formulas $\Sigma \subseteq \mathcal{F}$, say v **satisfies** Σ if v satisfies all the formulas in Σ.

Definition 2.5. Σ **tautologically implies** τ (written $\Sigma \vDash \tau$) if every truth valuation satisfying Σ also satisfies τ.

In case $\varnothing \vDash \tau$, say τ is a **tautology** and write $\vDash \tau$.

In case $\{\sigma\} \vDash \tau$, write $\sigma \vDash \tau$.

If $\varphi \vDash \psi$ and $\psi \vDash \varphi$, say φ and ψ are **tautologically equivalent** and write $\varphi \equiv \psi$.

Remark 2.6. When $\Sigma \vDash \tau$, it may also be said that Σ **semantically implies** τ, or that τ is a **tautological/semantic consequence** of Σ. Similarly, $\varphi \equiv \psi$ could be read as semantic equivalence.

Remark 2.7. Recall that truth valuations on L are characterized by their valuation on the set of variables $\mathcal{V} = \mathcal{V}(L)$. So there is a bijection between the set of truth valuations on L and the set of functions

$$\{T, F\}^\mathcal{V} = \{\text{functions } \mathcal{V} \rightarrow \{T, F\}\}.$$

This set has cardinality $2^{|\mathcal{V}|}$, hence there are as many truth valuations on L.

In particular, if $\mathcal{V}$ is a finite set then there are only finitely many truth valuations on L. In which case, it is "straightforward" to simply check all possible truth valuations to determine if $\Sigma \vDash \tau$ or indeed if $\vDash \tau$. However, the number of cases to check grows exponentially with the number of variables, so it quickly becomes practically intractable.

Note, however, that for any given formula $\varphi \in \mathcal{F}$, there are only finitely many propositional variables–say $\mathcal{V}(\varphi) = \{A_1, \ldots, A_n\}$–which occur in φ. Informally, we should expect that the

truth valuation of φ to only depend on the truth valuations of the A_i's. One way to formalize this claim is as follows: if two truth valuations agree on $\mathcal{V}(\varphi)$ then they must also agree on φ, i.e., if $v|_{\mathcal{V}(\varphi)} = v'|_{\mathcal{V}(\varphi)}$ then $v(\varphi) = v'(\varphi)$.

In practice, this means that we can determine whether or not $\Sigma \vDash \tau$, only needing to consider variables occurring in τ and formulas of Σ. To account for this, we could consider equivalence classes of truth valuations. Or, equivalently, we could modify our definition of truth valuations so that they are relative to a subset $S \subseteq \mathcal{V}$ of variables of L, instead of all variables.

Example 2.8. A selected list of tautologies (like most of these notes, lifted from Enderton)

1. Associativity and Commutativity.

$$(A \wedge (B \wedge C)) \leftrightarrow ((A \wedge B) \wedge C) \qquad (A \wedge B) \leftrightarrow (B \wedge A)$$
$$(A \vee (B \vee C)) \leftrightarrow ((A \vee B) \vee C) \qquad (A \vee B) \leftrightarrow (B \vee A)$$
$$(A \leftrightarrow (B \leftrightarrow C)) \leftrightarrow ((A \leftrightarrow B) \leftrightarrow C) \qquad (A \leftrightarrow B) \leftrightarrow (B \leftrightarrow A)$$

2. Distributivity.

$$(A \wedge (B \vee C)) \leftrightarrow ((A \wedge B) \vee (A \wedge C))$$
$$(A \vee (B \wedge C)) \leftrightarrow ((A \vee B) \wedge (A \vee C))$$

3. Negation.

$$(\neg(\neg A)) \leftrightarrow A$$
$$(\neg(A \rightarrow B)) \leftrightarrow (A \wedge (\neg B))$$
$$(\neg(A \leftrightarrow B)) \leftrightarrow ((A \wedge (\neg B)) \vee ((\neg A) \wedge B))$$

4. De Morgan's Laws.

$$(\neg(A \wedge B)) \leftrightarrow ((\neg A) \vee (\neg B))$$
$$(\neg(A \vee B)) \leftrightarrow ((\neg A) \wedge (\neg B))$$

5. Other

Excluded Middle	$A \vee (\neg A)$
Non-Contradiction	$\neg(A \wedge (\neg A))$
Contraposition	$(A \rightarrow B) \leftrightarrow ((\neg B) \rightarrow (\neg A))$
Exportation	$((A \wedge B) \rightarrow C) \leftrightarrow (A \rightarrow (B \rightarrow C))$

Claim 2.9. For $\Sigma \subseteq \mathcal{F}$ and $\alpha, \beta \in \mathcal{F}$,

(i) $\Sigma \cup \{\alpha\} \vDash \beta$ iff $\Sigma \vDash (\alpha \to \beta)$,

(ii) $\alpha \equiv \beta$ iff $\vDash (\alpha \leftrightarrow \beta)$.

3   Compactness

Theorem 3.1 (Compactness for Propositional Logic). A set of formulas is satisfiable iff every finite subset is satisfiable.

Proof. The ($\implies$) direction of the proof is clear: if v satisfies Σ then it also satisfies every finite subset of Σ.

For the ($\impliedby$) direction, let $\Sigma \subseteq \mathcal{F}$ be a (possibly infinite) set of formulas. Moreover, assume Σ is **finitely satisfiable**, i.e., every finite subset of Σ is satisfiable.

We extend Σ to a maximal set of formulas Δ using Zorn's Lemma. Consider the set

$$\mathcal{S} := \{U \mid \Sigma \subseteq U \subseteq \mathcal{F} \text{ and } U \text{ is finitely satisfiable}\},$$

ordered with the usual subset relation $\subseteq$. Let $C \subseteq \mathcal{S}$ be a chain (a totally ordered subset), and construct the union

$$\mathcal{U} := \bigcup_{U \in C} U.$$

Clearly, $\mathcal{U}$ is an upper bound of C and $\Sigma \subseteq \mathcal{U} \subseteq \mathcal{F}$. We claim that $\mathcal{U}$ is finitely satisfiable, whence $\mathcal{U} \in \mathcal{S}$. Let $\{\alpha_1, \ldots, \alpha_n\} \subseteq \mathcal{U}$ be an arbitrary finite subset of formulas. By construction of $\mathcal{U}$, each α_i is an element of some $U_i \in C$. Since C is totally ordered with respect to the subset relation, there is a maximum $U = \max_\subseteq \{U_1, \ldots, U_n\}$. Then $\{\alpha_1, \ldots, \alpha_n\} \subseteq U$ and U is finitely satisfiable, so $\{\alpha_1, \ldots, \alpha_n\}$ is satisfiable. Thus, $\mathcal{U}$ is finitely satisfiable and is therefore an upper bound of C in $\mathcal{S}$.

By Zorn's Lemma, a maximal $\Delta \in \mathcal{S}$ exists. Automatically, we know $\Sigma \subseteq \Delta$ and Δ is finitely satisfiable.

Claim 3.2. For all $\alpha \in \mathcal{F}$, either $\alpha \in \Delta$ or $(\neg\alpha) \in \Delta$.

Proof of Claim. waaahhhhhh $\qquad\qquad\square$

We will postpone the proof of Claim 3.2 for the moment.

We now wish to construct a truth valuation v satisfying Δ. For propositional variables $A \in \mathcal{V}$, put

$$v(A) := T \quad \text{iff} \quad A \in \Delta.$$

Of course, this characterizes v on all of $\mathcal{F}$, but it remains to show that v satisfies Δ. In fact, not only does v satisfy Δ, but we can say something slightly stronger.

Claim 3.3. For all $\alpha \in \mathcal{F}$, v satisfies α iff $\alpha \in \Delta$.

We first prove the ($\implies$) direction of the claim. Suppose v satisfies α. By Claim 3.2, either α or $\neg\alpha$ is in Δ. Assume for contradiction that $(\neg\alpha) \in \Delta$. Additionally, let $\mathcal{V}(\alpha)$ be the set of propositional variables occurring in α–in particular, $\mathcal{V}(\alpha)$ is finite. Define the set

$$V := \{A \mid A \in \mathcal{V}(\alpha) \text{ and } A \in \Delta\} \cup \{(\neg A) \mid A \in \mathcal{V}(\alpha) \text{ and } (\neg A) \in \Delta\}$$

Then $\mathcal{V}(\alpha) \cup \{\neg\alpha\}$ is a finite subset of Δ, so the finite satisfiability of Δ tells us that there exists a truth valuation v' which satisfies $\mathcal{V}(\alpha) \cup \{\neg\alpha\}$. Then v and v' agree on the propositional variables occurring in α, so they must agree on α. Since v satisfies α, we must have $v'(\alpha) = v(\alpha) = T$, but this implies $v'(\neg\alpha) = F$. This is a contradiction since we assumed v' satisfied α, hence $\alpha \in \Delta$.

fix this using V instead of $\mathcal{V}(\alpha)$

We now prove the ($\impliedby$) direction. Assume $\alpha \in \Delta$. Then $V \cup \{\alpha\}$ is a finite subset of Δ and is therefore satisfiable; let v' be a truth valuation satisfying $V \cup \{\alpha\}$. Then the value of v' on $A \in \mathcal{V}(\alpha)$ can be determined by $v'(A) = T$ iff $A \in \Delta$ and $v'(A) = F$ iff $(\neg A) \in \Delta$. But this is also true of v, so the two must agree on α. Since v' satisfies α, we conclude that $v(\alpha) = T$.

This completes the proof of Claim 3.3, hence compactness. $\qquad\square$

Corollary 3.4. If $\Sigma \vDash \tau$ then there exists a finite $\Sigma_0 \subseteq \Sigma$ such that $\Sigma_0 \vDash \tau$.

Proof. Note that $\Sigma \vDash \tau$ iff $\Sigma \cup \{\neg\tau\}$ is not unsatisfiable.

Assume for contradiction that there does not exist a finite $\Sigma_0 \subseteq \Sigma$ such that $\Sigma_0 \vDash \tau$. Equivalently, for every every finite $\Sigma_0 \subseteq \Sigma$, the set $\Sigma_0 \cup \{\neg\tau\}$. $\qquad\square$