Fix a base field K.

An **algebraic group** G is a group object in the category of algebraic varieties.

In other words, G is an algebraic variety endowed with the the following structural data:

- an **identity** element $1 \in G$;
- a **multiplication** morphism $\mu : G \times G \to G$ of varieties, denoted $\mu(x, y) = xy$;
- an **inversion** morphism $i : G \to G$ of varieties, denoted $i(x) = x^{-1}$;

such that $(G, 1, \mu)$ specifies a group which is coherent with the inversion i.

A **morphism of algebraic groups** is a morphism of the underlying varieties which is also a group homomorphism on the groups.

Basic Properties:

Let G be an algebraic group.

The inversion is an automorphism of G as an algebraic group with $i^2 = \mathrm{id}_G$.

The left and right multiplication maps (also called left/right translation maps) are isomorphisms of algebraic groups:

$$\begin{aligned} \lambda_x = (x \cdot -) : G &\to G \\ y &\mapsto xy \end{aligned} \qquad\qquad \begin{aligned} \rho_y = (- \cdot y) : G &\to G \\ x &\mapsto xy \end{aligned}$$

In particular, if G has any 'local' geometric properties at a point $x \in G$ then such properties hold at any other point $y \in G$, since the translation $\lambda_{yx^{-1}} : G \to G$ is an isomorphism of varieties which sends x to y. In other words, 'local properties of algebraic groups are global.'

Lemma 1. Let G be an algebraic group.

(i) G has precisely one irreducible component G^*, containing 1.

(ii) G^* is a closed normal subgroup of finite index in G.

(iii) The irreducible components of G are precisely the cosets of G^*.

It follows that the irreducible and connected components of G coincide.

Denote by $G^0 = G^\circ = G^*$ the **identity component** of G, which is the unique connected component of G containing the identity 1.

Theorem 1. Let G be an algebraic group.

(1) G^0 is a closed normal subgroup of G with finite index.

(2) The irreducible components of G are precisely the cosets of G^0.

(3) If $H \leq G$ is a closed subgroup of finite index, then $G^0 \subseteq H$.

(4) G^0 is smooth.

Let T be a topological space.

A subset $D \subseteq T$ is **locally closed** if $D = U \cap E$ for some U open and E closed in T. (Equivalently if D is open in $\overline{D}$.)

A subset of T is **constructible** if it is the union of finitely many locally closed subsets.

The set of constructible subsets of T is the boolean algebra generated by all the open and closed sets in T.

Theorem 2 (Chevalley)**.** If $\varphi : X \to Y$ is a morphism of (quasi-projective?) varieties, then $\operatorname{im}\varphi = \varphi(X)$ is a constructible subset of Y.

Moreover, if X and Y are irreducible and φ is **dominant** ($\overline{\varphi(X)} = Y$), then there exists a dense open subset $U \subseteq Y$ such that for all $u \in U \cap \varphi(X)$, we have

$$\dim \varphi^{-1}(u) = \dim X - \dim Y.$$

Lemma 2. Let T be any Noetherian topological space and $C \subseteq T$ constructible. Then there exists $U \subseteq C$ open with $\overline{U} = \overline{C}$.

Lemma 3. Let G be an algebraic group, $U, V \subseteq G$ open dense, $H \leq G$ not necessarily closed.

(i) $U \cdot V = G$.

(ii) $\overline{H} \leq G$.

(iii) If H is constructible, then H is closed.

Theorem 3. Let $\varphi : G \to G'$ be a morphism of algebraic groups.

(1) $\ker\varphi \leq G$ and $\operatorname{im}\varphi \leq G'$ are both closed.

(2) $\varphi(G^0) = (\operatorname{im}\varphi)^0$.

(3) $\dim G = \dim \operatorname{im}\varphi + \dim\ker\varphi$.

Morphic actions

For $V, W \subseteq X$, define the **transporter set**

$$\operatorname{Tran}_G(V, W) = \{g \in G \mid g \cdot V \subseteq W\}.$$

For $g \in G$, define the fixed something

$$\mathrm{Fix}_X(g) = \{x \in X \mid g \cdot x = x\}$$

Lemma 4. Let G be an algebraic group acting morphically on a (quasi-projective?) variety X.

(1) If $W \subseteq X$ is closed in X, then $\mathrm{Tran}_G(V, W)$ is closed in G.

(2) $\mathrm{Fix}_X(g)$ is closed in X for all $g \in G$.

(3) For any closed $H \leq G$, the normalizer $N_G(H)$ and the centralizer $C_G(H)$ are both closed in X.

Theorem 4. Let G be an algebraic group acting morphically on a(quasi-projective?) variety X. For $x \in X$,

(1) $G \cdot x$ is locally closed in X, so it is a quasi-projective variety.

(2) $G \cdot x$ is smooth.

(3) $\overline{G \cdot x} \setminus G \cdot x$ is a union of orbits, all of which have dimension less than $G \cdot x$.

(4) $\dim G \cdot x = \dim G - \dim \mathrm{Stab}_G(x)$.