1 Lee 1.9 **Complex projective n-space**, denoted by $\mathbb{CP}^n$, is the set of all 1-dimensional complex-linear subspaces of $\mathbb{C}^{n+1}$, with the quotient topology inherited from the natural projection $\pi : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{CP}^n$. Show that $\mathbb{CP}^n$ is a compact $2n$-dimensional topological manifold, and show how to give it a smooth structure analogous to the one we constructed for $\mathbb{RP}^n$. (We use the correspondence

$$(x^1 + iy^1, \ldots, x^{n+1} + iy^{n+1}) \leftrightarrow (x^1, y^1, \ldots, x^{n+1}, y^{n+1})$$

to identify $\mathbb{C}^{n+1}$ with $\mathbb{R}^{2n+2}$.)

Denote $X = \mathbb{C}^{n+1} \setminus \{0\}$.

We first check that π is an open map.

Proof. Let $U \subseteq X$ be open. We compute

$$\pi^{-1}(\pi(U)) = \bigcup_{z\in U} [z] = \bigcup_{z\in U} \bigcup_{\lambda\neq 0} \{\lambda z\} = \bigcup_{\lambda\neq 0} \bigcup_{z\in U} \{\lambda z\} = \bigcup_{\lambda\neq 0} \lambda U.$$

For $\lambda \neq 0$, the map $z \mapsto \lambda z$ is a homeomorphism $X \to X$. Therefore, the image of U under this map, λU, is open. Hence, $\pi^{-1}(\pi(U))$ is open, which implies $\pi(U)$ is open since π is a quotient map. □

We now check that $\mathbb{CP}^n$ is Hausdorff.

Proof. Let $R \subseteq X \times X$ be the pairs of points that are identified under the projection π. That is, $R = \{(z, w) \mid z \sim w\}$ where $z \sim w$ if and only if $z = \lambda w$ for some $\lambda \neq 0$. Equivalently, $z \sim w$ if and only if $z^i w^j = z^j w^i$ for all i and j. For each i and j define the polynomial map

$$\begin{aligned} f_{ij} : X \times X &\longrightarrow \mathbb{C}, \\ (z, w) &\longmapsto z^i w^j - z^j w^i. \end{aligned}$$

In particular, these maps are continuous, so their zero loci are closed in $X \times X$. Moreover, we can write R as the intersection of all of these zero loci, i.e.,

$$R = \bigcap_{i,j} f_{ij}^{-1}(0).$$

Therefore, R is closed and by the lemma from class, we conclude that $\mathbb{CP}^n = X/{\sim}$ is Hausdorff. □

Lastly, we construct a smooth structure on $\mathbb{CP}^n$.

For $i = 1, \ldots, n + 1$ define the open set $\widetilde{U}_i = \{z \in X \mid z^i \neq 0\}$—these cover X. Since π is an open map, the images $U_i = \pi(\widetilde{U}_i) \subseteq \mathbb{CP}^n$ form an open cover of $\mathbb{CP}^n$. We now define continuous maps

$$\begin{aligned} \widetilde{\varphi}_i : \widetilde{U}_i &\longrightarrow \mathbb{C}^n, \\ z &\longmapsto \left(\tfrac{z^1}{z^i}, \ldots, \widehat{\tfrac{z^i}{z^i}}, \ldots, \tfrac{z^{n+1}}{z^i} \right). \end{aligned}$$

We claim that $\widetilde{\varphi}_i$ factors through the projection $\pi|_{\widetilde{U}_i}$. To see this, suppose $z, w \in \widetilde{U}_i$ are such that $[z] = [w]$. This is the case if and only if $z = \lambda w$ for some $\lambda \neq 0$, so

$$\begin{aligned}
\widetilde{\varphi}_i(z) &= \left(\tfrac{z^1}{z^i}, \dots, \widehat{\tfrac{z^i}{z^i}}, \dots, \tfrac{z^{n+1}}{z^i}\right) \\
&= \left(\tfrac{\lambda w^1}{\lambda w^i}, \dots, \widehat{\tfrac{\lambda w^i}{\lambda w^i}}, \dots, \tfrac{\lambda w^{n+1}}{\lambda w^i}\right) \\
&= \left(\tfrac{w^1}{w^i}, \dots, \widehat{\tfrac{w^i}{w^i}}, \dots, \tfrac{w^{n+1}}{w^i}\right) \\
&= \widetilde{\varphi}_i(w).
\end{aligned}$$

Therefore, there exists a unique continuous map $\varphi_i : U_i \to \mathbb{C}^n$ such that $\varphi_i \circ \pi|_{\widetilde{U}_i} = \widetilde{\varphi}_i$. This map is surjective since for any $z \in \mathbb{C}^n$ we have

$$\varphi_i([z^1 : \cdots : z^{i-1} : 1 : z^i : \cdots : z^n]) = z.$$

Additionally, this map is injective since $\varphi_i([z]) = \varphi_i([w])$ implies that $z^j/z^i = w^j/w^i$ for all j. In other words, $z = \lambda w$ with $\lambda = z^i/w^i$ so in fact $[z] = [w]$. Moreover, the inverse map $\varphi_i : \mathbb{C}^n \to U_i$ can be constructed as the composition of the continuous map

$$\begin{aligned}
\mathbb{C}^n &\longrightarrow \mathbb{C}^{n+1}, \\
z &\longmapsto (z^1, \dots, z^{i-1}, 1, z^i, \dots, z^n)
\end{aligned}$$

and the projection $\pi : X \to \mathbb{CP}^n$. Hence, the atlas $\{(U_i, \varphi_i)\}_{i=1}^{n+1}$ gives us a (complex) topological manifold structure on $\mathbb{CP}^n$. Since we have a homeomorphism $\mathbb{C}^1 \cong \mathbb{R}^2$ by splitting coordinates into real and imaginary parts, this complex n-manifold structure induces a real $2n$-manifold structure.

We now check smooth compatibility. For $i \neq j$ and $z \in \operatorname{im} \varphi_i$ we have

$$\begin{aligned}
\varphi_j \circ \varphi_i^{-1}(z) &= \varphi_j([z^1 : \cdots : z^{i-1} : 1 : z^i : \cdots : z^n]) \\
&= \varphi_j([w^1 : \cdots : w^{i-1} : 1 : w^{i+1} : \cdots : w^{n+1}]) \\
&= \left(\tfrac{w^1}{w^j}, \dots, \tfrac{w^{i-1}}{w^j}, \tfrac{1}{w^j}, \tfrac{w^{i+1}}{w^j}, \dots, \widehat{\tfrac{w^j}{w^j}}, \dots, \tfrac{w^{n+1}}{w^j}\right),
\end{aligned}$$

where we are denoting $w^k = z^k$ for $k < i$ and $w^{k+1} = z^k$ for $k \geq i$. We can rewrite this result in corresponding real coordinates by using

$$\frac{a + ib}{c + id} = \frac{(a + ib)(c - id)}{c^2 + d^2} = \frac{ac + bd}{c^2 + d^2} + i\frac{ad - bc}{c^2 + d^2}.$$

With $w^j \neq 0$, this is a smooth map $\mathbb{R}^{2n} \to \mathbb{R}^{2n}$, hence we have found a smooth manifold structure on $\mathbb{CP}^n$.

2 Lee 2.1 Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} 1, & x \geq 0, \\ 0, & x < 0. \end{cases}$$

Show that for every $x \in \mathbb{R}$, there are smooth coordinate charts (U, φ) containing x and (V, ψ) containing $f(x)$ such that $\psi \circ f \circ \varphi^{-1}$ is smooth as a map from $\varphi(U \cap f^{-1}(V))$ to $\psi(V)$, but f is not smooth in the sense we have defined in this chapter.

Take $U = \mathbb{R}$ and $\varphi = \mathrm{id}$.

For $x < 0$ choose $V = (-1, 1)$ and $\psi = \mathrm{id}$. Then

$$\varphi(U \cap f^{-1}(V)) = \mathrm{id}(\mathbb{R} \cap (-\infty, 0)) = (-\infty, 0),$$

and on this set $\psi \circ f \circ \varphi^{-1} = f$ restricts to the constant 0 map, which is smooth.

For $x \geq 0$ choose $V = (0, 2)$ and $\psi = \mathrm{id}$. Then

$$\varphi(U \cap f^{-1}(V)) = \mathrm{id}(\mathbb{R} \cap [0, \infty)) = [0, \infty),$$

and on this set f restricts to the constant 1 map, which is smooth (by any suitable definition that supports non-open domains).

However, if we add in the restriction that the charts be chosen such that $f(U) \subseteq V$, we find that f is not smooth at 0. Say (U, φ) is a chart at 0 and (V, ψ) is a chart at $f(0) = 1$ such that $f(U) \subseteq V$. Without loss of generality, we may assume U is an open interval around zero; in particular, $\varphi(U)$ is a connected set in $\mathbb{R}$. On the other hand, the image of $f|_U$ is disconnected and therefore so will the image of $\psi \circ f \circ \varphi^{-1}$. In particular, this shows the map is not continuous and therefore not smooth.

3 Lee 2.6 Let $P : \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{R}^{k+1} \setminus \{0\}$ be a smooth function, and suppose that for some $d \in \mathbb{Z}$, $P(\lambda x) = \lambda^d P(x)$ for all $\lambda \in \mathbb{R} \setminus \{0\}$ and $x \in \mathbb{R}^{n+1} \setminus \{0\}$. (Such a function is said to be **homogeneous of degree d**.) Show that the map $\widetilde{P} : \mathbb{RP}^n \to \mathbb{RP}^k$ defined by $\widetilde{P}([x]) = [P(x)]$ is well defined and smooth.

We claim that the map $\pi \circ P : \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{RP}^k$ factors through the quotient map $\pi : \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{RP}^n$. Suppose $x, y \in \mathbb{R}^{n+1} \setminus \{0\}$ such that $[x] = [y]$, i.e., $x = \lambda y$ for some $\lambda \neq 0$. Then

$$P(x) = P(\lambda y) = \lambda^d P(y),$$

which implies $[P(x)] = [P(y)]$. Hence, there is a unique continuous map $\widetilde{P} : \mathbb{RP}^n \to \mathbb{RP}^k$ satisfying $\widetilde{P} \circ \pi = \pi \circ P$, i.e., $\widetilde{P}([x]) = [P(x)]$.

Given a point $[x] \in \mathbb{RP}^n$, choose a chart (U_i, φ_i) in the standard smooth atlas on $\mathbb{RP}^n$ containing $[x]$ and a chart (V_j, ψ_j) in the standard smooth atlas on $\mathbb{RP}^k$ containing $\widetilde{P}([x])$. Restrict the first chart to the open set $U = U_i \cap \widetilde{P}^{-1}(V_j)$ and $\varphi = \varphi_i|_U$. Then for $y \in \varphi(U) \subseteq \mathbb{R}^n$ we have

$$\begin{aligned}
\psi \circ \widetilde{P} \circ \varphi^{-1}(y) &= \psi \circ \widetilde{P}([y^1 : \cdots : y^{i-1} : 1 : y^i : \cdots : y^n]) \\
&= \psi([P_1(y^1, \ldots, 1, \ldots, y^n) : \cdots : P_{k+1}(y^1, \ldots, 1, \ldots, y^n)]) \\
&= \psi \circ \pi \circ P(y^1, \ldots, 1, \ldots, y^n).
\end{aligned}$$

The map $\mathbb{R}^n \to \mathbb{R}^{n+1}$ defined by $y \mapsto (y^1, \ldots, 1, \ldots, y^n)$ is smooth so it remains to check that $\psi \circ \pi : \widetilde{V}_j = \pi^{-1}(V)_j \to \mathbb{R}^k$ is smooth:

$$\psi \circ \pi(y) = \psi([y]) = \left(\tfrac{y^1}{y^j}, \ldots, \widehat{\tfrac{y^j}{y^j}}, \ldots, \tfrac{y^k}{y^j} \right).$$

Indeed, this is smooth provided $y_j \neq 0$, which is exactly the points in $\widetilde{V}_j$.

4 Lee 2.10 For any topological space M, let $C(M)$ denote the algebra of continuous functions $f : M \to \mathbb{R}$. Given a continuous map $F : M \to N$, define $F^* : C(N) \to C(M)$ by $F^*(f) = f \circ F$.

(a) Show that F^* is a linear map.

Proof. Let $f, g \in C(N)$, $\alpha \in \mathbb{R}$, and $x \in N$. Then

$$F^*(\alpha f + g)(x) = (\alpha f + g)(F(x)) = \alpha f(F(x)) + g(F(x)) = \alpha F^* f(x) + F^* g(x),$$

hence $F^*(\alpha f + g) = \alpha F^* f + F^* g$. $\square$

(b) Suppose M and N are smooth manifolds. Show that $F : M \to N$ is smooth if and only if $F^*(C^\infty(N)) \subseteq C^\infty(M)$.

Proof. If F is smooth, then for $f \in C^\infty(N)$ the composition $f \circ F$ is smooth, so $F^* f \in C^\infty(M)$.

Assume $F^*(C^\infty(N)) \subseteq C^\infty(M)$. It suffices to prove that F is locally smooth. Given $x \in M$, choose a chart (V, ψ) of N such that $\psi(F(x)) = 0$ and $\psi(V) \supseteq B_2(0)$. Let $H : \mathbb{R}^k \to [0, 1]$ be a smooth cutoff function with $H \equiv 1$ on $B_1(0)$ and $H \equiv 0$ outside $B_2(0)$.

We now define $g : N \to \mathbb{R}^k$ by

$$g(y) = \begin{cases} H(y)\psi(y) & \text{if } y \in V, \\ 0 & \text{otherwise.} \end{cases}$$

Define the open set $V' = \psi^{-1}(B_1(0))$, on which g is the same as ψ. Defining $\psi' = \psi|_{V'}$ gives us a new restricted chart (V', ψ') at $F(x)$.

Now choose a chart (U, φ) of M at x such that $F(U) \subseteq V'$ (which can be done since F is continuous). Then on $\varphi(U)$ we have

$$\psi' \circ F \circ \varphi^{-1} = g \circ F \circ \varphi^{-1}.$$

Note that $g \circ F = (F^* g_1, \ldots, F^* g_k)$ where each $g_i \in C^\infty(N)$, so $g \circ F$ is smooth since each component is smooth. Since φ^{-1} is also smooth, so is their composition, hence F is a smooth map of manifolds. $\square$

(c) Suppose $F : M \to N$ is a homeomorphism between smooth manifolds. Show that it is a diffeomorphism if and only if F^* restricts to an isomorphism from $C^\infty(N)$ to $C^\infty(M)$.

Proof. Let $G : N \to M$ be the continuous inverse of F. By part (a), $G^*C(M) \to C(N)$ is also linear. For any $f \in C(M)$ we have

$$F^*G^*f = F^*(f \circ G) = f \circ G \circ F = f \circ \mathrm{id}_M = f$$

and for any $g \in C(N)$ we have

$$G^*F^*g = G^*(g \circ F) = g \circ F \circ G = g \circ \mathrm{id}_N = g.$$

This shows that G^* is the inverse of F^*, i.e., $(F^{-1})^* = (F^*)^{-1}$.

Suppose $F : M \to N$ is a diffeomorphism, so G is smooth. Applying part (b), the restrictions $F^* : C^\infty(N) \to C^\infty(M)$ and $G^* : C^\infty(M) \to C^\infty(N)$ are well-defined. Since they are also linear inverses, they describe an isomorphism of vector spaces.

Suppose the restriction $F^* : C^\infty(N) \to C^\infty(M)$ is an isomorphism. In particular, the restriction of its inverse $G^* : C^\infty(M) \to C^\infty(N)$ is well-defined. Applying part (b), both F and G must be smooth. Since they are also inverses, we conclude F is a diffeomorphism. □

Remark this result shows that in a certain sense, the entire smooth structure of M is encoded in the subset $C^\infty(M) \subseteq C(M)$. In fact, some authors *define* a smooth structure on a topological manifold M to be a subalgebra of $C(M)$ with certain properties.