

1 Let X be a nonempty set and let μ be a measure on X . We have a theorem on sequences of decreasing measurable sets that states the following: Assume $X \supseteq A_1 \supseteq A_2 \supseteq A_3 \supseteq \cdots$ are μ -measurable, such that $\mu(A_1) < \infty$. Then one has

$$\lim_{n \rightarrow \infty} \mu(A_n) = \mu\left(\bigcap_{n=1}^{\infty} A_n\right).$$

Prove that in this theorem the condition $\mu(A_1) < \infty$ is necessary.

Proof. Consider $X = \mathbb{Z}$ with the cardinality measure: $\mu(E) = |E|$, for all $E \subseteq X$. One can check that this satisfies the necessary properties of a measure. Moreover, every subset of \mathbb{Z} is μ -measurable.

For $n \in \mathbb{N}$, define the subset of integers $A_n = \{k \in \mathbb{Z} : k \geq n\}$. Then $X \supseteq A_1 \supseteq A_2 \supseteq \cdots$ is a decreasing chain of μ -measurable subsets. Additionally, we have $\mu(A_n) = \infty$, for all $n \in \mathbb{N}$, and $\bigcap_{n=1}^{\infty} A_n = \emptyset$. Then

$$\lim_{n \rightarrow \infty} \mu(A_n) = \lim_{n \rightarrow \infty} \infty = \infty,$$

but

$$\mu\left(\bigcap_{n=1}^{\infty} A_n\right) = \mu(\emptyset) = 0.$$

□

2 Does there exist an infinite σ -algebra that has countably many elements?

No.

Proof. Assume, for contradiction, that \mathcal{A} is a countably infinite σ -algebra on a set X .

In order to derive a contradiction, we will construct an infinite collection $\mathcal{N} \subseteq \mathcal{A}$ of disjoint sets in \mathcal{A} . Then an injection $\mathbb{N} \rightarrow \mathcal{N}$ gives us a bijection $2^{\mathbb{N}} \rightarrow 2^{\mathcal{N}}$; in particular, we deduce that $2^{\mathcal{N}}$ is uncountable. For any $S \in 2^{\mathcal{N}}$, there is a set $\bigcup_{E \in S} E \in \mathcal{A}$, since $S \subseteq \mathcal{A}$ is at most countable. Because all the sets in \mathcal{N} are disjoint, different choices of $S \in 2^{\mathcal{N}}$ produce different unions, giving us an injection $2^{\mathcal{N}} \rightarrow \mathcal{A}$. However, this is a contradiction, as we assumed \mathcal{A} to be countable. It remains to construct the desired \mathcal{N} .

For each $x \in X$, we define the set $N_x = \bigcap_{x \in E \in \mathcal{A}} E$, i.e., the intersection of all sets in \mathcal{A} containing the point x . We claim that the collection $\mathcal{N} = \{N_x\}_{x \in X}$ is as desired.

First, since \mathcal{A} contains only countably many sets, each N_x is the intersection of at most countably many sets. Then, because \mathcal{A} is a σ -algebra, $N_x \in \mathcal{A}$, so in fact $\mathcal{N} \subseteq \mathcal{A}$.

Next, we check that the sets in \mathcal{N} are mutually disjoint. More specifically, we will prove that N_x and N_y are either equal or disjoint, for all $x, y \in X$, which would mean that \mathcal{N} describes a partition of X into the components N_x .

Suppose $x, y \in X$ such that $y \in N_x$. By construction, N_x is the intersection of all $E \in \mathcal{A}$ containing x . So $y \in N_x$ means that the collection of sets in \mathcal{A} containing x is a subset of the collection of those containing y , therefore $N_y \subseteq N_x$. Assume, for contradiction, $N_x \not\subseteq N_y$. By the contrapositive of the first inclusion, we must have $x \notin N_y$. This means that there is some set $E \in \mathcal{A}$ containing y but not x , so $E^c \in \mathcal{A}$ is a set containing x but not y . However, this means that $N_x \subseteq E^c$ but $y \notin E^c$, implying $y \notin N_x$. This is a contradiction, so $N_x \subseteq N_y$.

Lastly, we prove that \mathcal{N} is infinite. First, note that each set $E \in \mathcal{A}$ can be written as the union $E = \bigcup_{x \in E} N_x$. This suggests \mathcal{N} be interpreted as the “irreducible” sets in \mathcal{A} , and every other set in \mathcal{A} has an “irreducible decomposition” into sets in \mathcal{N} . Since \mathcal{A} is infinite, there must be infinitely many different combinations of sets in \mathcal{N} , which is only possible if \mathcal{N} itself is infinite.

□

3 Is it true that if μ is a Borel measure on a nonempty set X , then for any sets $A, B \subset X$ with $\text{dist}(A, B) > 0$, one has

$$\mu(A \cup B) = \mu(A) + \mu(B)?$$

Yes.

Proof. Let $r = \text{dist}(A, B) > 0$, and define the set $E = \bigcup_{x \in A} B_r(x) \subseteq X$. This construction gives us $A \subseteq E$ and $B \subseteq E^c$. As the union of open balls, E is an open subset of X and, therefore, μ -measurable. Hence,

$$\mu(A \cup B) = \mu((A \cup B) \cap E) + \mu((A \cup B) \setminus E) = \mu(A) + \mu(B).$$

□

4 Let X be an uncountable set and let \mathcal{C} be the collection of all subsets A of X such that either A or A^c is at most countable. Prove that \mathcal{C} is a σ -algebra.

Proof. Since $\emptyset = X^c$ is at most countable, $\emptyset, X \in \mathcal{C}$.

If $A \in \mathcal{C}$, then either A^c or $A = (A^c)^c$ is at most countable, so $A^c \in \mathcal{C}$.

Suppose $A_1, A_2, \dots \in \mathcal{C}$ and let $A = \bigcup_{n=1}^{\infty} A_n$. It is either the case that all A_n are at most countable or some A_N^c is at most countable. In the first case, A is countable, as a countable union of countable subsets. In the second case, $A^c = \bigcap_{n=1}^{\infty} A_n^c$ is contained in some countable set A_N^c , implying A^c is countable. In both cases, either A is at most countable or A^c is at most countable, so in fact $A \in \mathcal{C}$.

□