1 Prove that for a function $f : \mathbb{R} \to \mathbb{R}$ the definition of differentiable at x given in class is equivalent to the usual definition that $\lim_{h\to 0}(f(x+h) - f(x))/h$ exists.

Proof. Suppose f is differentiable at x in the sense of the definition from class, i.e., there is a linear map $T : \mathbb{R} \to \mathbb{R}$ such that

$$0 = \lim_{\substack{\|\Delta x\|\to 0 \\ \Delta x \in \mathbb{R}\setminus\{0\}}} \frac{\|f(x+\Delta x) - f(x) - T(\Delta x)\|}{\|\Delta x\|} = \lim_{h\to 0} \frac{|f(x+h) - f(x) - T(h)|}{|h|}.$$

Since $T : \mathbb{R} \to \mathbb{R}$ is linear, there exists $a \in \mathbb{R}$ such that $T(y) = ay$ for all $y \in \mathbb{R}$. Then

$$\lim_{h\to 0} \left| \frac{f(x+h) - f(x)}{h} - a \right| = \lim_{h\to 0} \frac{|f(x+h) - f(x) - T(h)|}{|h|} = 0.$$

That is, $f'(x) = a$ in the usual sense.

Suppose now that f is differentiable at x in the usual sense—say $f'(x) = a$. Define the linear map $T : \mathbb{R} \to \mathbb{R}$ by $T(y) = ay$ for all $y \in \mathbb{R}$. Then

$$\lim_{h\to 0} \frac{f(x+h) - f(x) - T(h)}{h} = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} - a = f'(x) - a = 0.$$

Taking the absolute value, this limit is still zero, so we obtain the desired limit. □

2 Suppose that $f : \mathbb{R}^n \to \mathbb{R}^m$ is the affine map $\vec{x} = A\vec{x} + \vec{b}$ where A an $m \times n$ matrix and $\vec{b} \in \mathbb{R}^m$. Prove that f is differentiable everywhere and that $\mathrm{d}f = A$.

Proof. For any point $x \in \mathbb{R}^n$ and nonzero $\Delta x \in \mathbb{R}^n$ we have

$$\frac{\|f(x+\Delta x) - f(x) - A\Delta x\|}{\|\Delta x\|} = \frac{\|A(x - \Delta x) + b - (Ax + b) - A\Delta x\|}{\|\Delta x\|} = \frac{\|0\|}{\|\Delta x\|} = 0.$$

The limit as $\|\Delta x\| \to 0$ is hence zero, so indeed $\mathrm{d}f = A$. □

3 Given an integer n the function $f_n : \mathbb{R}^2 \to \mathbb{R}$ is given in polar coordinates by $f_n(r, \theta) = r\sin(n\theta)$ for $r \neq 0$ and $f_n(0, 0) = 0$. Define a surface

$$S_n = \{(r\cos\theta, r\sin\theta, f_n(r, \theta)) : r \geq 0, \theta \in \mathbb{R}\} \subseteq \mathbb{R}^3.$$

(i) Sketch S_2.

(ii) Prove that every directional derivative of f_n exists at the origin and find it.

Proof. Let $v \in \mathbb{R}^2$ nonzero be given by the polar coordinates $(r; \theta)$. Then the derivative of f_n at the origin in the direction of v is given by

$$\mathrm{D}_v f(0) = \lim_{h\to 0} \frac{f_n(0 + hv) - f_n(0)}{h} = \lim_{h\to 0} \frac{hr\sin(n\theta) - 0}{h} = \lim_{h\to 0} r\sin(n\theta) = r\sin(n\theta).$$

□

(iii) Use the definition in class to prove that f_2 is not differentiable at the origin.

Proof. Suppose $\Delta x \in \mathbb{R}^2$ nonzero is given in polar coordinates by $(r;\theta)$ and $T : \mathbb{R}^2 \to \mathbb{R}$ is a linear map. Then

$$\frac{f_2(0+\Delta x) - f_2(0) - T(\Delta x)}{\|\Delta x\|} = \frac{r\sin(2\theta) - 0 - rT(1;\theta)}{r} = \sin(2\theta) - T(1;\theta).$$

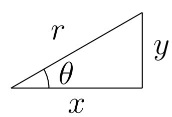
If we assume that f is differentiable at the origin with $T = \mathrm{d}f$, then we would have

$$0 = \lim_{\substack{\|\Delta x\| \to 0 \\ \Delta x \in \mathbb{R}\setminus\{0\}}} \frac{\|f(x+\Delta x) - f(x) - T(\Delta x)\|}{\|\Delta x\|} = \|\sin(2\theta) - T(1;\theta)\|.$$

That is, $T(1;\theta) = \sin(2\theta)$ for all angles θ. In particular, the unit vectors in $\mathbb{R}^2$ are mapped to $T(e_1) = T(1;0) = 0$ and $T(e_2) = T(1;\pi) = 0$. It follows that T is the zero map. However, along $\theta = \pi/2$ we have $\sin\pi = 1 \neq 0 = T(1;\pi/2)$. This is a contradiction, so f must not be differentiable at the origin. $\square$

(iv) Express f_2 in terms of (x, y) coordinates.

Given a nonzero point $(x, y) \in \mathbb{R}^2$, consider the triangle



Applying trigonometric identities, we compute

$$f_2(x,y) = r\sin(2\theta) = r \cdot 2\sin\theta\cos\theta = 2r \cdot \frac{y}{r} \cdot \frac{x}{r} = \frac{2xy}{\sqrt{x^2+y^2}}.$$

(v) Do you think $\partial f_2/\partial x$ is continuous at the origin?

No.

We compute

$$\frac{\partial f_2}{\partial x} = \frac{\partial}{\partial x}\frac{2xy}{(x^2+y^2)^{1/2}} = \frac{2y^3}{(x^2+y^2)^{3/2}}.$$

Along the y-axis, this gives

$$\left.\frac{\partial f_2}{\partial x}\right|_{x=0} = \frac{2y^3}{(y^2)^{3/2}} = \frac{2y^3}{|y|^3} = \frac{2y}{|y|}.$$

For $y > 0$ this is 2 and for $y < 0$ this is -2, so

$$\lim_{y\to 0^+}\frac{\partial f_2}{\partial x} = 2 \neq -2 = \lim_{y\to 0^-}\frac{\partial f_2}{\partial x}.$$

Hence, $\partial f_2/\partial x$ is not continuous at the origin.

4 Suppose that $f : \mathbb{R}^2 \to \mathbb{R}$ and $\partial f/\partial y = 0$ everywhere. Prove that there is $g : \mathbb{R} \to \mathbb{R}$ such that $f(x, y) = g(x)$.

Proof. For $x \in \mathbb{R}$ define the function $f_x : \mathbb{R} \to \mathbb{R}$ by $f_x(y) = f(x, y)$. Then f_x is differentiable everywhere with $f_x'(y) = \partial f/\partial y = 0$. Let $a, b \in \mathbb{R}$ with $a < b$. The mean value theorem tells us there is some point $c \in (a, b)$ such that $f_x(a) - f_x(b) = f_x'(c)(a - b)$. But $f_x'(c) = 0$ implies $f_x(a) = f_x(b)$, hence f_x is constant on $\mathbb{R}$. We now define $g(x) = f_x(0)$, and it follows that $f(x, y) = g(x)$ for all $(x, y) \in \mathbb{R}^2$. □

Deduce that if $\mathrm{d}f = 0$ everywhere on $\mathbb{R}^2$ then f is constant.

Proof. Assuming $\mathrm{d}f = 0$, then the Jacobian matrix

$$J_f = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{bmatrix}$$

is zero. In particular, both partial derivatives of f are zero everywhere. By the previous part, there are functions $g_1, g_2 : \mathbb{R} \to \mathbb{R}$ such that $f(x, y) = g_1(x) = g_2(y)$. Then for all $(x, y) \in \mathbb{R}^2$ we have

$$f(x, y) = g_1(x) = f(x, 0) = g_2(0) = f(0, 0),$$

hence f is constant. □

What happens if the *domain* of f is only $\{(x, y) : x \neq 0\}$?

In this case, f is constant on both the open left and right half-planes, though their respective values may differ.

5 Using the definition of derivative given in class prove that $f = (f_1, f_2) : \mathbb{R} \to \mathbb{R}^2$ is differentiable at $a \in \mathbb{R}$ iff f_1 and f_2 are differentiable at a.

Proof. Suppose f is differentiable at a and let $\mathrm{d}f_a = T = (T_1, T_2) : \mathbb{R} \to \mathbb{R}^2$ be the derivative. We consider the max norm $\|(x, y)\| = \max\{|x|, |y|\}$ on $\mathbb{R}^2$. Then

$$\begin{aligned} 0 &= \lim_{h \to 0} \frac{\|f(x+h) - f(x) - T(h)\|}{|h|} \\ &= \lim_{h \to 0} \frac{\max\{f_1(x+h) - f(x) - T_1(h), f_2(x+h) - f_2(x) - T_2(h)\}}{|h|} \\ &= \lim_{h \to 0} \max\left\{\frac{|f_1(x+h) - f_1(x) - T_1(h)|}{|h|}, \frac{|f_2(x+h) - f_2(x) - T_2(h)|}{|h|}\right\}. \end{aligned}$$

Since this is an upper bound for both limits, it follows that

$$\lim_{h \to 0} \frac{|f_1(x+h) - f_1(x) - T_1(h)|}{|h|} = \lim_{h \to 0} \frac{|f_2(x+h) - f_2(x) - T_2(h)|}{|h|} = 0.$$

Hence, both f_1 and f_2 are differentiable at a with $(\mathrm{d}f_i)_a = T_i$.

Suppose now that f_1 and f_2 are differentiable at a with derivatives $(\mathrm{d}f_i)_a = T_i : \mathbb{R} \to \mathbb{R}$. Define the linear map $T = (T_1, T_2) : \mathbb{R} \to \mathbb{R}^2$. Then

$$\begin{aligned}
&\lim_{h\to 0} \frac{\|f(x+h) - f(x) - T(h)\|}{|h|} \\
&\quad = \lim_{h\to 0} \max\left\{\frac{|f_1(x+h) - f_1(x) - T_1(h)|}{|h|}, \frac{|f_2(x+h) - f_2(x) - T_2(h)|}{|h|}\right\} \\
&\quad \leq \lim_{h\to 0} \frac{|f_1(x+h) - f_1(x) - T_1(h)|}{|h|} + \lim_{h\to 0} \frac{|f_2(x+h) - f_2(x) - T_2(h)|}{|h|} \\
&\quad = 0.
\end{aligned}$$

Hence, f is differentiable at a with $\mathrm{d}f_a = ((\mathrm{d}f_1)_a, (\mathrm{d}f_2)_a)$. $\square$

6 The function $f : \mathbb{R} \to \mathbb{R}$ is defined by

$$f(x) = \begin{cases} e^{-1/x^2} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Prove that f is smooth at the origin.

Proof. Consider the ring of Laurent polynomials $\mathbb{R}[x, x^{-1}]$. We claim that for all $n \geq 0$ there exists a Laurent polynomial $p_n(x) \in \mathbb{R}[x, x^{-1}]$ such that

$$f^{(n)}(x) = p_n(x) e^{-1/x^2},$$

where the derivative is take away from the origin. We will prove this by induction on n. The base case is immediate since $f^{(0)}(x) = f(x) = 1e^{-1/x^2}$. Assuming the result holds for derivatives less than n, we find

$$f^{(n)}(x) = \frac{\partial}{\partial x} f^{(n-1)}(x) = \frac{\partial}{\partial x} p_{n-1}(x) e^{-1/x^2} = (p'_{n-1}(x) + p_{n-1}(x) \cdot 2x^{-3}) e^{-1/x^2}.$$

Since the ring $\mathbb{R}[x, x^{-1}]$ is closed under taking derivatives, we have

$$p_n(x) = p'_{n-1}(x) + p_{n-1}(x) \cdot 2x^{-3} \in \mathbb{R}[x, x^{-1}].$$

This completes the induction.

Next, we claim that for any Laurent polynomial $p(x) \in \mathbb{R}[x, x^{-1}]$,

$$\lim_{h\to 0} p(h) e^{-1/h^2} = 0.$$

One can check that $e^{1/h^2} < e^{1/|h|}$ for all nonzero $|h| < 1$. Additionally, $|p(h)| \leq p(|h|)$ by applying the triangle inequality to each term in the polynomial. So for $p(x) \in \mathbb{R}[x, x^{-1}]$ and nonzero $|h| < 1$ we have

$$|p(h) e^{-1/h^2}| = |p(h)| e^{-1/h^2} \leq p(|h|) e^{-1/|h|}.$$

Taking the limit as $h \to 0$, we write

$$\lim_{h\to 0} |p(h)e^{-1/h^2}| \leq \lim_{h\to 0} \frac{p(|h|)}{e^{1/|h|}} = \lim_{y\to\infty} \frac{p(1/y)}{e^y}.$$

Note that $p(1/y) = p(y^{-1}) \in \mathbb{R}[y, y^{-1}]$ is still a (Laurent) polynomial. Since the exponential function is faster than any polynomial function, we conclude that this limit is zero.

In particular, this proves

$$\lim_{x\to 0} f^{(n)}(x) = p_n(x)e^{-1/x^2} = 0.$$

We now prove $f^{(n)}(0) = 0$ by induction on n. Since $f^{(0)}(0) = f(0) = 0$, the base case holds. Assume the result holds for derivatives less than n. Then the n-derivative of f at the origin is given by

$$\lim_{h\to 0} \frac{f^{(n-1)}(0+h) - f^{(n-1)}(0)}{h} = \lim_{h\to 0} \frac{f^{(n-1)}(h)}{h} = \lim_{h\to 0} \frac{p_{n-1}(h)}{h} e^{-1/h^2}.$$

Since $x^{-1}p_{n-1}(x) \in \mathbb{R}[x, x^{-1}]$, we have seen that this limit is zero, so indeed $f^{(n)}(0) = 0$. □

7 The function $f : \mathbb{R}^2 \to \mathbb{R}^2$ is given by $f(x, y) = (e^x + e^y - 2, x^5 - y^5 + x)$. Show that there is $\varepsilon > 0$ and $\delta > 0$ such that for all a, b with $|a| < \varepsilon$ and $|b| < \varepsilon$ there are unique x, y with $x^2 + y^2 < \delta$ and $f(x, y) = (a, b)$.

Proof. Note that $f = (f_1, f_2)$ is smooth with $f(0) = 0$. The Jacobian of f at zero is

$$J_f(0) = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{bmatrix}\Bigg|_0 = \begin{bmatrix} e^x & e^y \\ 5x^4 + 1 & -5y^4 \end{bmatrix}\Bigg|_0 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}.$$

This matrix is full rank, so $(\mathrm{d}f)_0$ invertible. By the inverse function theorem, there are open neighborhoods $U, V \subseteq \mathbb{R}^2$ of the origin such that $f|_U : U \to V$ is a homeomorphism.

Considering $\mathbb{R}^2$ with the Euclidean norm $\|\cdot\|_2$, we choose an open ball $B_\delta(0; \|\cdot\|_2)$ around the origin contained in U. Since f_U is an open map, the image $f(B_\delta(0; \|\cdot\|_2)) \subseteq V$ is an open neighborhood of the origin. Considering $\mathbb{R}^2$ with the max/sup norm $\|\cdot\|_\infty$, we choose an open ball $B_\varepsilon(0; \|\cdot\|_\infty) \subseteq f(B_\delta(0; \|\cdot\|_2))$.

Hence, f induces a bijection $f^{-1}(B_\varepsilon(0; \|\cdot\|_\infty)) \to B_\varepsilon(0; \|\cdot\|_\infty)$, the domain of which is contained in $B_\delta(0; \|\cdot\|_2)$. Taking $\delta' = \delta^2$ gives us the desired $\varepsilon > 0$ and $\delta' > 0$. □