1 Let R, S be rings and ${}_SB_R$ an S-R-bimodule. Prove that the functors $G = B \otimes_R - :$ R-$\mathsf{Mod} \to S$-Mod and $F = \mathrm{Hom}_S(B, -) : S$-$\mathsf{Mod} \to R$-$\mathsf{Mod}$ form an adjoint pair (G, F).

We will construct a natural isomorphism

$$\begin{tikzcd} R\text{-}\mathsf{Mod} \times S\text{-}\mathsf{Mod} \arrow[r, bend left, "\mathrm{Hom}_S(B\otimes_R -,-)"] \arrow[r, bend right, "\mathrm{Hom}_R(-,\mathrm{Hom}_S(B,-))"'] \arrow[r, phantom, "\Downarrow\cong"] & \mathsf{Set} \end{tikzcd}$$

Fix a pair of left modules $M \in R$-Mod and $N \in S$-Mod.

We define the component

$$\alpha_{M,N} : \mathrm{Hom}_S(B \otimes_R M, N) \to \mathrm{Hom}_R(M, \mathrm{Hom}_S(B, N)).$$

Given a homomorphism of left S-modules $f : B \otimes_R M \to N$ and an element $m \in M$, we define a map $\alpha f(m) : B \to N$ by $b \mapsto f(b \otimes m)$. We check that $\alpha f(m)$ is a homomorphism of left S-modules:

$$\begin{aligned} \alpha f(m)(sb_1 + b_2) &= f((sb_1 + b_2) \otimes m) \\ &= f(s(b_1 \otimes m) + b_2 \otimes m) \\ &= s\alpha f(m)(b_1) + \alpha f(m)(b_2). \end{aligned}$$

Hence, $\alpha f : M \to \mathrm{Hom}_S(B, N)$ is a well-defined map. Next, we check that αf is a homomorphism of left R-modules:

$$\begin{aligned} \alpha f(rm_1 + m_2)(b) &= f((rm_1 + m_2) \otimes b) \\ &= f(r(m_1 \otimes b) + m_2 \otimes b) \\ &= r\alpha f(m_1)(b) + \alpha f(m_2)(b) \\ &= (r\alpha f(m_1) + \alpha f(m_2))(b). \end{aligned}$$

Hence, $\alpha_{M,N} : \mathrm{Hom}_S(B \otimes_R M, N) \to \mathrm{Hom}_R(M, \mathrm{Hom}_S(B, N))$ is a well-defined map.

We now define the component

$$\beta_{M,N} : \mathrm{Hom}_R(M, \mathrm{Hom}_S(B, N)) \to \mathrm{Hom}_S(B \otimes_R M, N).$$

Given a homomorphism of left R-modules $g : M \to \mathrm{Hom}_S(B, N)$, there is a map $\widetilde{g} : B \times M \to N$ given by $\widetilde{g}(b, m) = g(m)(b)$. We check that $\widetilde{g}$ is an R-balanced map:

$$\widetilde{g}(b, m_1 + m_2) = g(m_1 + m_2)(b) = g(m_1)(b) + g(m_2)(b) = \widetilde{g}(b, m_1) + \widetilde{g}(b, m_2),$$

$$\widetilde{g}(b_1 + b_2, m) = g(m)(b_1 + b_2) = g(m)(b_1) + g(m)(b_2) = \widetilde{g}(b_1, m) + \widetilde{g}(b_2, m),$$
$$\widetilde{g}(br, m) = g(m)(br) = rg(m)(b) = g(rm)(b) = \widetilde{g}(b, rm).$$

The second equality in the third condition follows from the fact that $\mathrm{Hom}_S(B, N)$ is a left R-module by $(rh)(b) = h(br)$ for $h \in \mathrm{Hom}_S(B, N)$. By the universal property of the tensor product, there is a unique homomorphism of left S-modules βg which makes the following diagram commute:

$$\begin{tikzcd}
B \times M \arrow[r, "\widetilde{g}"] \arrow[d, "\tau"'] & N \\
B \otimes_R M \arrow[ur, dashed, "\beta g"'] &
\end{tikzcd}$$

Here, τ is the canonical map from the product to the tensor product. By this construction, we have the characterization $\beta g(b \otimes m) = g(m)(b)$. Hence, $\beta_{M,N} : \mathrm{Hom}_R(M, \mathrm{Hom}_S(B, N)) \to \mathrm{Hom}_S(B \otimes_R M, N)$ is a well-defined map.

Lastly, we check that $\alpha_{M,N}$ and $\beta_{M,N}$ are inverses to each other:

$$(\beta\alpha f)(b \otimes m) = \alpha f(m)(b) = f(b \otimes m),$$

$$(\alpha\beta g)(m)(b) = \beta g(b \otimes m) = g(m)(b).$$

Hence, α and β describe the desired natural isomorphism.

> **2** Let K be a field. Show that the contravariant functor $D = \operatorname{Hom}_K(-, K) : K\text{-}\mathsf{Mod} \to K\text{-}\mathsf{Mod}$ is not a natural equivalence. In particular, show that the vector spaces V and $D(V)$ fail to be isomorphic if $V \cong K^{(\mathbb{N})}$.

We consider the vector space $V = K^{(\mathbb{N})} = \bigoplus_{i \in \mathbb{N}} K$. It has a countable basis given by $\{e_i\}_{i \in \mathbb{N}}$. In particular, the dimension of V as a K-vector space is countably infinite.

Consider its dual space $D(V) = \operatorname{Hom}_K(V, K)$. A linear functional on V is determined by where in K it sends the basis vectors of V. Moreover, any choice of image in K for each basis vector of V determines a linear functional on V. In other words, a choice of linear functional on V is equivalent to a choice of element in K for each natural number, so $D(V) \cong K^{\mathbb{N}}$ as K-vector spaces.

We claim that the dimension of $K^{\mathbb{N}}$ as a K-vector space is uncountably infinite. In particular, this would mean $\dim_K V \neq \dim_K D(V)$, implying that V and $D(V)$ could not possibly be isomorphic.

Let $F = K_0$ be the prime subfield of K. Then $F = \mathbb{Q}$ or $F = \mathbb{Z}/p\mathbb{Z}$ for some prime p—in particular, F is countable.

Lemma 1. $\dim_F F^{\mathbb{N}}$ is uncountably infinite.

Proof. The dimension of $F^{(\mathbb{N})}$ as an F-vector space is countably infinite since it has a countably infinite basis given by $\{e_i\}_{i \in \mathbb{N}}$.

The dimension of $F^{\mathbb{N}}$ is at least countably infinite, since $\{e_i\}_{i \in \mathbb{N}}$ is an F-linearly independent set in $F^{\mathbb{N}}$ (though not a basis). It other words, $\dim_F F^{\mathbb{N}} \geq \dim_F F^{(\mathbb{N})}$; it remains to prove that this inequality is strict.

We can write $F^{(\mathbb{N})}$ as the direct limit $F^{(\mathbb{N})} = \varinjlim F^n$ with respect to the natural inclusions $F^n \hookrightarrow F^{n+1}$ for each $n \in \mathbb{N}$. This gives $F^{(N)} = F^1 \cup F^2 \cup F^3 \cup \cdots$, which is a countable union of countable sets, therefore the cardinality of $F^{(\mathbb{N})}$ is countable. On the other hand, the cardinality of $F^{\mathbb{N}}$ is given by $|F^{\mathbb{N}}| = |F|^{|\mathbb{N}|}$, which is uncountable.

Since the cardinalities of $F^{(\mathbb{N})}$ and $F^{\mathbb{N}}$ as sets are different, there can be no bijection between them. In particular, there can be no isomorphism between the respective F-vector spaces. It follows that their dimensions must be different, so indeed $\dim_F F^{\mathbb{N}} > \dim_F F^{(\mathbb{N})}$. □

By Lemma 1, let $\mathcal{S} \subseteq F^{\mathbb{N}}$ be an uncountable F-linearly independent set. Under the natural set inclusion $F^{\mathbb{N}} \hookrightarrow K^{\mathbb{N}}$, we can interpret $\mathcal{S}$ as a set of vectors in $K^{\mathbb{N}}$. We claim that $\mathcal{S}$ is also K-linearly independent.

To prove that $\mathcal{S}$ is K-linearly independent, we will prove the equivalent condition that every finite subset of $\mathcal{S}$ is K-linearly independent. Let $S = \{x_1, \ldots, x_n\} \subseteq \mathcal{S}$ be an arbitrary finite subset. Of course, S is F-linearly independent.

Lemma 2. There exists $m \in \mathbb{N}$ such that $S_m = \{x_1^{(m)}, \ldots, x_n^{(m)}\}$ is F-linearly independent, where $x_i^{(m)} \in F^m$ is the truncation of x_i to the first m components.

Proof. For $m \in \mathbb{N}$ define the following subspace of F^n:

$$D_m = \{a \in F^n \mid \textstyle\sum_{i=1}^n a_i x_i^{(m)} = 0\}.$$

Notice that $\dim_F D_m < \infty$ and $D_m \subseteq D_{m+1}$ for all $m \in \mathbb{N}$. Moreover, if it happens for some $m \in \mathbb{N}$ that $\dim_F D_m = 0$ then S_m would be F-linearly independent. We will show that such an m can be found.

Suppose for a given $m \in \mathbb{N}$ that $\dim_F D_m > 0$. Choose any nonzero element $a \in D_m$, then Ka is a 1-dimensional subspace of D_m. Since S is F-linearly independent and a is nonzero, we must have $x = \sum_{i=1}^n a_i x_i \neq 0$. By assumption, the first m components of x are zero, but in order for x to be nonzero there must be some $m' \geq m$ such that the m'th component is nonzero. Then $a \notin D_{m'}$ and we compute

$$\begin{aligned}
\dim_F D_{m'} &= \dim_F(D_{m'} + Ka) - \dim_F Ka + \dim_F(D_{m'} \cap Ka) \\
&= \dim_F(D_{m'} + Ka) - 1 + 0 \\
&\leq \dim_F D_m - 1 \\
&< \dim_F D_m.
\end{aligned}$$

Since D_m is always finite dimensional, we can repeat this process which strictly reduces the dimension a finite number of times to find $m \in \mathbb{N}$ large enough that $\dim_F D_m = 0$. $\square$

Choose $m \in \mathbb{N}$ as in Lemma 2. We will show that S_m is K-linearly independent as a set of vectors in K^m. Extend S_m to an F-basis $\beta = S_m \cup \{y_1, \ldots, y_{m-n}\}$ of F^m. The fact that β is a basis is equivalent to the invertibility of the following matrix:

$$A = \begin{bmatrix} | & & | & | & & | \\ x_1^{(m)} & \ldots & x_n^{(m)} & y_1 & \ldots & y_{m-n} \\ | & & | & | & & | \end{bmatrix} \in M_m(F).$$

So there exists an inverse matrix $A^{-1} \in M_m(F)$. Under the inclusion $M_m(F) \hookrightarrow M_m(K)$, we may consider both A and A^{-1} to be matrices in $M_m(K)$. Moreover, it is still true in $M_m(K)$ that $AA^{-1} = A^{-1}A = I_m$, i.e., that A is invertible. Equivalently, the columns of A—the elements of β under the inclusion $F^m \hookrightarrow K^m$—form a K-basis of K^m. In particular, S_m is K-linearly independent.

From this, we deduce that S is K-linearly independent, since for any nonzero $a \in K^n$ we must have $\sum_{i=1}^n a_i x_i^{(m)} \neq 0$, which implies $\sum_{i=1}^n a_i x_i \neq 0$. Therefore, every finite subset of $\mathcal{S}$ is K-linearly independent, which means $\mathcal{S}$ is K-linearly independent. Thus, we have found an uncountable set of K-linearly independent vectors in $K^{\mathbb{N}}$, so indeed $\dim_K K^{\mathbb{N}}$ is uncountably infinite. In particular, $\dim_K K^{\mathbb{N}} > \dim_K K^{(\mathbb{N})}$ so $K^{\mathbb{N}} \ncong K^{(\mathbb{N})}$.

Lemma 3. Let $\{X_i\}_{i \in I}$ be a collection of objects in a category. Suppose there exists a coproduct $X = \coprod_{i \in I} X_i$ with inclusion morphisms $\iota_i : X_i \to X$. If there exists an isomorphism $\alpha : X \to \widetilde{X}$, then $\widetilde{X}$ with inclusion morphisms $\alpha \circ \iota_i : X_i \to \widetilde{X}$ is also a categorical coproduct of the X_i's.

Proof. Suppose $f_i : X_i \to Y$ is a collection of morphisms. Then by the universal property of the coproduct X, there is a unique morphism $f : X \to Y$ such that $f \circ \iota_i = f_i$ for all $i \in I$. Then the composition $\widetilde{f} = f \circ \alpha^{-1} : \widetilde{X} \to Y$ is a morphism satisfying $\widetilde{f} \circ (\alpha \circ \iota_i) = f_i$. In other words, the following diagram commutes for all $i \in I$:

$$\begin{tikzcd}
\widetilde{X} \arrow[ddr, dashed, "\widetilde{f}"] \\
X \arrow[u, "\alpha", "\cong"'] \arrow[dr, dashed, "f"] \\
X_i \arrow[u, "\iota_i"] \arrow[r, "f_i"'] & Y
\end{tikzcd}$$

To check the uniqueness of $\widetilde{f}$, suppose $g : \widetilde{X} \to Y$ is another morphism satisfying the same diagram. But then, considering $g \circ \alpha$, we have the following commutative diagram:

$$\begin{tikzcd}
\widetilde{X} \arrow[ddr, dashed, "g"] \\
X \arrow[u, "\alpha", "\cong"'] \arrow[dr, dashed, "g\circ\alpha"] \\
X_i \arrow[u, "\iota_i"] \arrow[r, "f_i"'] & Y
\end{tikzcd}$$

By the universal property of the coproduct X, we must have $g \circ \alpha = f$, so $g = f \circ \alpha^{-1} = \widetilde{f}$. $\square$

Lemma 4. Suppose $\mathcal{C}$ and $\mathcal{D}$ are equivalent categories via functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$, and natural isomorphisms $\alpha : \mathrm{id}_{\mathcal{C}} \Rightarrow GF$ and $\beta : \mathrm{id}_{\mathcal{D}} \Rightarrow FG$. If $\iota_i : X_i \to X$ describes a categorical coproduct in $\mathcal{C}$, then $F\iota_i : FX_i \to FX$ describes a categorical coproduct in $\mathcal{D}$.

Proof. Suppose we have a family of morphisms $g_i : FX_i \to Y$ in $\mathcal{D}$. Then G gives us a family of morphisms $Gg_i : GFX_i \to GY$ in $\mathcal{C}$. Lemma 3 and the naturality of α gives us a unique morphism $\widetilde{g}$ which makes the following diagram in $\mathcal{D}$ commute for all $i \in I$:

$$\begin{tikzcd}
X \arrow[r, "\alpha_X", "\cong"'] & GFX \arrow[dr, dashed, "\widetilde{g}"] \\
X_i \arrow[u, "\iota_i"] \arrow[r, "\cong", "\alpha_{X_i}"'] & GFX_i \arrow[u, "GF\iota_i"'] \arrow[r, "Gg_i"'] & GY
\end{tikzcd}$$

Mapping the triangle of this diagram under F, and using the naturality of β, we obtain the following commutative diagram in $\mathcal{C}$:

$$\begin{tikzcd}
FX \arrow[r, "\beta_{FX}", "\cong"'] & FGFX \arrow[dr, "F\widetilde{g}"] & & \\
FX_i \arrow[u, "F\iota_i"] \arrow[r, "\cong", "\beta_{FX_i}"'] & FGFX_i \arrow[u, "FGF\iota_i"'] \arrow[r, "FGg_i"'] & FGY \arrow[r, "\beta_Y^{-1}"] & Y
\end{tikzcd}$$

Now, $g = \beta_Y^{-1} \circ F\widetilde{g} \circ \beta_{FX} : FX \to Y$ is a morphism satisfying $g \circ F\iota_i = \beta_Y^{-1} \circ FGg_i \circ \beta_{FX_i}$, which equals g_i by the naturality of β, for all $i \in I$.

It remains to prove that g is the unique such morphism. Suppose $h : FX \to Y$ is a morphism satisfying $h \circ F\iota_i = g_i$ for all $i \in I$. Mapping under G, we find that Gh makes the following diagram in $\mathcal{C}$ commute for all $i \in I$:

$$\begin{tikzcd}
X \arrow[r, "\alpha_X", "\cong"'] & GFX \arrow[dr, "Gh"] & \\
X_i \arrow[u, "\iota_i"] \arrow[r, "\cong", "\alpha_{X_i}"'] & GFX_i \arrow[u, "GF\iota_i"'] \arrow[r, "Gg_i"'] & GY
\end{tikzcd}$$

But then the universal property of GFX as a coproduct of X_i's tells us that $Gh = \widetilde{g}$, so

$$FGh = F\widetilde{g} = \beta_Y \circ g \circ \beta_{FX}^{-1}.$$

Combining this with the naturality of β, we find that the following diagram in $\mathcal{D}$ commutes:

$$\begin{tikzcd}
FX \arrow[r, "\beta_{FX}", "\cong"'] \arrow[d, "h"'] & FGFX \arrow[d, "FGh"'] & FX \arrow[l, "\beta_{FX}"', "\cong"] \arrow[d, "g"] \\
Y \arrow[r, "\cong", "\beta_Y"'] & FGY & Y \arrow[l, "\cong"', "\beta_Y"]
\end{tikzcd}$$

The perimeter of this diagram gives us $h = g$, and we conclude that $F\iota_i : FX_i \to FX$ indeed describes a categorical coproduct in $\mathcal{D}$. $\square$

Finally, suppose for contradiction that D is part of a natural equivalence of categories. Consider the K-vector space $K^{(\mathbb{N})} = \bigoplus_{i\in\mathbb{N}} K$ as above. Notice that $K^{(\mathbb{N})}$ is the categorical coproduct of countably infinitely many copies of K. Applying Lemma 4, we compute

$$K^{\mathbb{N}} \cong D(K^{(\mathbb{N})}) \cong \bigoplus_{i\in\mathbb{N}} D(K) \cong \bigoplus_{i\in\mathbb{N}} K = K^{(\mathbb{N})}$$

But this is a contradiction, as we have already shown these two vector spaces to be non-isomorphic.

3 Given $M \in R\text{-}\mathsf{Mod}$, consider the co- and contravariant hom-functors $F_1, F_2 : R\text{-}\mathsf{Mod} \to \mathbb{Z}\text{-}\mathsf{Mod}$, given by $F_1 = \mathrm{Hom}_R(M, -)$ and $F_2 = \mathrm{Hom}_R(-, M)$. For either choice of $k \in \{1, 2\}$, prove those of the following statements which are true in general, and provide a counterexample for those that are not. (Here $(N_i)_{i\in I}$ is an arbitrary family of left R-modules.)

(a) 1 $\mathrm{Hom}_R(M, \bigoplus_{i\in I} N_i) \cong \bigoplus_{i\in I} \mathrm{Hom}_R(M, N_i)$

Probably false.

(a) 2 $\mathrm{Hom}_R(\bigoplus_{i\in I} N_i, M) \cong \bigoplus_{i\in I} \mathrm{Hom}_R(N_i, M)$

False.

Lemma 5. $\mathrm{Hom}_R(\bigoplus_{i\in I} N_i, M) \cong \prod_{i\in I} \mathrm{Hom}_R(N_i, M)$.

Proof. For $i \in I$, let $\iota_i : N_i \hookrightarrow \bigoplus_{i\in I} N_i$ be the canonical projection. Define the map

$$\begin{aligned} \varphi : \mathrm{Hom}_R(\textstyle\bigoplus_{i\in I} N_i, M) &\to \textstyle\prod_{i\in I} \mathrm{Hom}_R(N_i, M) \\ f &\mapsto (f \circ \iota_i)_{i\in I}. \end{aligned}$$

We check that φ is a $\mathbb{Z}$-module homomorphism:

$$\begin{aligned} \varphi(af + g)(n) &= ((af + g)\iota_i(n))_{i\in I} \\ &= (af\iota_i(n) + g\iota_i(n))_{i\in I} \\ &= a(f\iota_i(n))_{i\in I} + (g\iota_i(n))_{i\in I} \\ &= a\varphi(f)(n) + \varphi(g)(n) \\ &= (a\varphi(f) + \varphi(g))(n). \end{aligned}$$

The bijectivity of of φ follows from the universal property of the direct sum (categorical coproduct) of $\mathbb{Z}$-modules, i.e., given a family of $\mathbb{Z}$-module homomorphisms $f_i : N_i \to M$ with $i \in I$, there is a unique $\mathbb{Z}$-module homomorphism $f : \bigoplus_{i\in I} N_i \to M$ such that $f_i = f \circ \iota_i$ for all $i \in I$. In other words, $\varphi(f) = (f_i)_{i\in I}$. The existence in this condition gives us surjectivity and the uniqueness gives us injectivity. Hence, φ is an isomorphism of $\mathbb{Z}$-modules. □

Take $R = \mathbb{Z}$, $M = \mathbb{Z}$, $N_i = \mathbb{Z}$, and $I = \mathbb{N}$ By 5, we have

$$\mathrm{Hom}_{\mathbb{Z}}(\textstyle\bigoplus_{i\in\mathbb{N}} \mathbb{Z}, \mathbb{Z}) \cong \prod_{i\in I} \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}) \cong \prod_{i\in I} \mathbb{Z} = \mathbb{Z}^{\mathbb{N}}.$$

However, we also have

$$\bigoplus_{i\in\mathbb{N}} \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}) \cong \bigoplus_{i\in\mathbb{N}} \mathbb{Z} = \mathbb{Z}^{(\mathbb{N})}.$$

The set $\mathbb{Z}^{\mathbb{N}}$ is uncountable whilst $\mathbb{Z}^{(\mathbb{N})}$ is countable. In particular, the cardinalities of the two sets are different, so there can be no bijection between them. Therefore, there can be no isomorphism between the respective $\mathbb{Z}$-modules.

(b) 1 $\mathrm{Hom}_R(M, \prod_{i\in I} N_i) \cong \prod_{i\in I} \mathrm{Hom}_R(M, N_i)$

True.

For $i \in I$, let $\pi_i : \prod_{i\in I} N_i \to N_i$ be the canonical projection. Define the map

$$\begin{aligned}\varphi : \operatorname{Hom}_R(M, \textstyle\prod_{i\in I} N_i) &\to \textstyle\prod_{i\in I} \operatorname{Hom}_R(M, N_i)\\ f &\mapsto (\pi_i \circ f)_{i\in I}.\end{aligned}$$

We check that φ is a $\mathbb{Z}$-module homomorphism:

$$\begin{aligned}\varphi(af+g)(m) &= (\pi_i(af+g)(m))_{i\in I}\\ &= (\pi_i(af(m)+g(m)))_{i\in I}\\ &= (a\pi_i f(m) + \pi_i g(m))_{i\in I}\\ &= a(\pi_i f(m))_{i\in I} + (\pi_i g(m))_{i\in I}\\ &= a\varphi(f)(m) + \varphi(g)(m)\\ &= (a\varphi(f)+\varphi(g))(m).\end{aligned}$$

The bijectivity of of φ follows from the universal property of the direct product (categorical product) of $\mathbb{Z}$-modules, i.e., given a family of $\mathbb{Z}$-module homomorphisms $f_i : M \to N_i$ with $i \in I$, there is a unique $\mathbb{Z}$-module homomorphism $f : M \to \prod_{i\in I} N_i$ such that $f_i = \pi_i \circ f$ for all $i \in I$. In other words, $\varphi(f) = (f_i)_{i\in I}$. The existence in this condition gives us surjectivity and the uniqueness gives us injectivity. Hence, φ is an isomorphism of $\mathbb{Z}$-modules.

(b) 2 $\operatorname{Hom}_R(\prod_{i\in I} N_i, M) \cong \prod_{i\in I} \operatorname{Hom}_R(N_i, M)$

Probably false.

4 Let $\mathcal{C}$ be an additive category; in particular, finite direct sums and products of objects in $\mathcal{C}$ exist. Prove that for any choice of objects $C_1, \ldots, C_n$ of $\mathcal{C}$, the direct sum $\coprod_{1 \le i \le n} C_i$ is isomorphic to the direct product $\prod_{1 \le i \le n} C_i$.

Proof. Let $X = \prod_{i=1}^n C_i$ be the product with projection maps $\pi_i : X \to C_i$. For all i and j, define the morphism $\lambda_{i,j} \in \mathrm{Hom}(C_i, C_j)$ by

$$\lambda_{i,j} = \begin{cases} \mathrm{id}_{C_i} & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases}$$

For a fixed i, the universal property of the product gives us a unique morphism λ_i which makes the following diagram commute for all j:

$$\begin{tikzcd} & X \arrow[d, "\pi_j"] \\ C_i \arrow[ur, dashed, "\lambda_i"] \arrow[r, "\lambda_{i,j}"'] & C_j \end{tikzcd}$$

Consider the morphism $\sigma = \sum_{j=1}^n (\lambda_j \circ \pi_j) \in \mathrm{Hom}(X, X)$. Composing with π_i gives

$$\pi_i \circ \sigma = \sum_{j=1}^n (\pi_i \circ \lambda_j) \circ \pi_j = \sum_{j=1}^n \lambda_{j,i} \circ \pi_j = \mathrm{id}_{C_i} \circ \pi_i. = \pi_i.$$

In other words, σ is a morphism which makes the following diagram commute for all i:

$$\begin{tikzcd} & X \arrow[d, "\pi_i"] \\ X \arrow[ur, dashed, "\sigma"] \arrow[r, "\pi_i"'] & C_i \end{tikzcd}$$

However, id_X makes the same diagram commute, so the universal property of the product X gives us $\sigma = \mathrm{id}_X$.

We claim that the morphisms $\lambda_i : C_i \to X$ describe a coproduct. Let $f_i : C_i \to Y$ be an arbitrary collection of morphisms. Define $f = \sum_{j=1}^n (f_j \circ \pi_j) \in \mathrm{Hom}(X, Y)$, for which we compute

$$f \circ \lambda_i = \sum_{j=1}^n f_j \circ (\pi_j \circ \lambda_i) = \sum_{j=1}^n f_j \circ \lambda_{i,j} = f_i \circ \mathrm{id}_{C_i} = f_i.$$

That is, f is a morphism which makes the following diagram commute:

$$\begin{tikzcd} X \arrow[dr, dashed, "f"] & \\ C_i \arrow[u, "\lambda_i"] \arrow[r, "f_i"'] & Y \end{tikzcd}$$

Suppose $h : X \to Y$ is a morphism which makes the same diagram commute, i.e., $h \circ \lambda_i = f_i$ for all i. Then

$$f = \sum_{i=1}^{n} (f_i \circ \pi_i) = \sum_{i=1}^{n} (h \circ \lambda_i) \circ \pi_i = h \circ \sum_{i=1}^{n} (\lambda_i \circ \pi_i) = h \circ \sigma = h \circ \mathrm{id}_X = h.$$

We conclude that $\lambda_i : C_i \to X$ is indeed a coproduct.

By the uniqueness of coproducts, there is a unique isomorphism $\alpha : \coprod_{i=1}^{n} C_i \to X$.

Moreover, it follows from Lemma 3 that the inclusions $\lambda_i : C_i \to X$ we constructed are precisely the inclusions $\alpha \circ \iota_i$ induced on X by the isomorphism α, coming from the coproduct's inclusions $\iota_i : C_i \to \coprod_{i=1}^{n} C_i$. $\square$

5 Let G be a group, K a field. Consider the category $\mathsf{Rep}_K G$ whose objects are the group homomorphisms $\rho : G \to \mathrm{GL}(V)$, $g \mapsto \rho_g$, where V is any vector space over K; a morphism from $\rho : G \to \mathrm{GL}(V)$ to $\sigma : G \to \mathrm{GL}(V)$ is a map $f \in \mathrm{Hom}_K(V, W)$ such that $f \circ \rho_g = \sigma_g \circ f$ for all $g \in G$. Clearly, $\mathsf{Rep}_K G$ is a pre-additive category.

Moreover, consider the group algebra KG, defined as follows: As a K-vector space, KG is the vector space on basis G, whence its elements can be represented as finite sums $\sum k_g g$ with $k_g \in K$ and $g \in G$. The algebra multiplication on KG mimics the multiplication of G, namely

$$\left(\sum_{g\in G,\text{finite}} k_g g\right)\left(\sum_{h\in G,\text{finite}} l_h h\right) := \sum_{u\in G,\text{finite}} \left(\sum_{gh=u} k_g l_h\right) u.$$

Show that the categories $\mathsf{Rep}_K G$ and KG-Mod are naturally equivalent by way of additive functors (they are even isomorphic as categories, a rare phenomenon).

We will construct a functor $\Phi : \mathsf{Rep}_K G \to KG\text{-}\mathsf{Mod}$.

On objects ($\rho : G \to \mathrm{GL}(V) \in \mathsf{Rep}_K G$), we define $\Phi(\rho)$ as follows. The group homomorphism $\rho : G \to \mathrm{GL}(V)$ induces a K-algebra homomorphism $\widetilde{\rho} : KG \to \mathrm{End}_K(V)$. Here, $\widetilde{\rho}$ is a K-linear map characterized on the basis G by $\widetilde{\rho}(g) = \rho_g$. Moreover, we get a left KG-multiplication on V:

$$(\sum k_g g) \cdot v = \sum k_g \rho_g(v).$$

This multiplication makes $\Phi(\rho)$ a left KG-module with V as its underlying set.

On morphisms $f \in \mathrm{Hom}(\rho : G \to \mathrm{GL}(V), \sigma : G \to \mathrm{GL}(W))$, we define $\Phi(f)$ as follows. The data of f is a K-linear map $V \to W$. We use this as the underlying map of $\Phi(f) : \Phi(\rho) \to \Phi(\sigma)$. We already know that f is K-linear, so to ensure that it is a homomorphism of left KG-modules, we only need to check that it commutes with multiplication by elements of the basis G:

$$f(g \cdot v) = f(\rho_g(v)) = (f \circ \rho_g)(v) = (\sigma_g \circ f)(v) = \sigma_g(f(v)) = g \cdot f(v).$$

Having defined Φ on objects and morphisms, we now check that it is indeed a functor.

The identity morphism on $\rho : G \to \mathrm{GL}(V)$ consists of the identity map on V. The data of $\Phi(\rho)$ is also the identity map on V, which happens to also be the identity of the left KG-module $\Phi(\rho)$. Thus, Φ preserves identity morphisms.

The fact that Φ preserves composition of morphisms follows similarly from the fact that the underlying data of morphisms in both $\mathsf{Rep}_K G$ and KG-Mod are functions between the underlying sets, which Φ preserves. Moreover, the composition of morphisms in each category is precisely the composition of the underlying functions.

We now define a functor in the reverse direction: $\Psi : KG\text{-}\mathsf{Mod} \to \mathsf{Rep}_K G$.

There is a forgetful functor $U : KG\text{-}\mathsf{Mod} \to K\text{-}\mathsf{Mod}$ which remembers only the addition and multiplication by K. For an object $M \in KG$-Mod, the multiplication on M is given by a K-algebra homomorphism $m : KG \to \mathrm{End}_K(UM)$. We define $\Psi(M)$ to be the representation

$m|_G : G \to \mathrm{GL}(UM)$; since the elements of G are invertible in KG and m is a homomorphism, the images $m(g)$ are invertible in $\mathrm{End}_K(UM)$, hence the map is well-defined.

On morphisms $f \in \mathrm{Hom}(M, N)$, the data of $\Psi(f)$ is provided by the K-linear map $Uf : UM \to UN$. We check that Uf commutes with images of the representations:

$$(Uf \circ m_g)(v) = f(g \cdot v) = g \cdot f(v) = n_g(f(v)) = (n_g \circ Uf)(v).$$

The fact that Ψ is a functor follows similarly to the previous case in that the data of morphisms are functions between the underlying sets, which Ψ preserves.

We now show that Φ and Ψ are inverse to each other.

Let $\rho : G \to \mathrm{GL}(V) \in \mathsf{Rep}_K G$. Then $\Phi(\rho)$ is a left KG-module M with underlying set V. Moreover, the multiplication on M is characterized by the representation ρ. Then, $\Psi\Phi(\rho)$ is a representation of G in $UM = V$, characterized by the multiplication on M, which is precisely the representation ρ. Hence, $\Psi\Phi(\rho) = \rho$.

Conversely, let $M \in KG\text{-}\mathsf{Mod}$. Then $\Psi(M)$ is a representation of G in UM, characterized by the multiplication on M. Then, $\Phi\Psi(M)$ is a left KG-module with underlying set UM, characterized by the representation $\Psi(M)$, which is precisely the multiplication on M. Hence, $\Phi\Psi(M) = M$.

Thus, we have strongly inverse functors $\Psi\Phi = \mathrm{id}_{\mathsf{Rep}_K G}$ and $\Phi\Psi = \mathrm{id}_{KG\text{-}\mathsf{Mod}}$.

Moreover, both Φ and Ψ are additive functors since the addition of the underlying functions is commutative and both maps simply preserve the underlying sets and functions.

Deduce that $\mathsf{Rep}_K G$ is an abelian category.

Since $\Psi : KG\text{-}\mathsf{Mod} \to \mathsf{Rep}_K G$ is additive, it carries over the zero object and commutes with finite biproducts. Moreover, since Ψ is part of an equivalence of categories, it preserves monomorphisms and epimorphisms. Additionally, the fact that Ψ is part of an equivalence also means it is an exact functor and therefore preserves kernels and cokernels. Thus, Ψ carries over all the abelian structure of $KG\text{-}\mathsf{Mod}$ to $\mathsf{Rep}_K G$.

6 (a) Let $M \in R\text{-}\mathsf{Mod}$. Verify that the contravariant functor $\mathrm{Hom}_R(-, M) : R\text{-}\mathsf{Mod} \to \mathbb{Z}\text{-}\mathsf{Mod}$ is left exact.

Let

$$A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

be an exact sequence in R-Mod and consider the image sequence

$$0 \longrightarrow \mathrm{Hom}_R(C, M) \xrightarrow{g^*} \mathrm{Hom}_R(B, M) \xrightarrow{f^*} \mathrm{Hom}_R(A, M)$$

in $\mathbb{Z}$-Mod.

Suppose $\varphi \in \ker g^*$, so $0 = g^*\varphi = \varphi \circ g$. This means that $\ker \varphi \supseteq \operatorname{im} g = C$, so in fact φ is zero on all of C. Hence, $\ker g^* = 0$.

We have $f^* \circ g^* = (g \circ f)^* = 0$, so $\operatorname{im} g^* \subseteq \ker f^*$. It remains to check the opposite inclusion.

Suppose $\psi \in \ker f^*$, so $0 = f^*\psi = \psi \circ f$. This means that $\ker \psi \supseteq \operatorname{im} f = \ker g$. We define a map $\varphi : C \to M$ as follows: for $c \in C$ pick any $b \in g^{-1}(c)$ then put $\varphi(c) = \psi(b)$. In order for this to be well-defined, we must check that $g(b) = g(b')$ implies $\psi(b) = \psi(b')$ for all $b \in B$. Indeed, if $g(b) = g(b')$, then we have $g(b - b') = 0$ so $b - b' \in \ker g \subseteq \ker \psi$. Then $\psi(b - b') = 0$ which implies $\psi(b) = \psi(b')$. By construction, this gives $\psi = \varphi \circ g = g^*\varphi$. Therefore, $\psi \in \operatorname{im} g^*$ and we conclude that $\operatorname{im} g^* = \ker f^*$.

(b) Now let $R = M = \mathbb{Z}$. Show that the functor $\mathrm{Hom}_{\mathbb{Z}}(-, \mathbb{Z}) : \mathbb{Z}\text{-}\mathsf{Mod} \to \mathbb{Z}\text{-}\mathsf{Mod}$ fails to be right exact.

Consider the short exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\cdot n} \mathbb{Z} \xrightarrow{q} \mathbb{Z}/n\mathbb{Z} \to 0$$

Taking the image under the contravariant hom-functor, we obtain the sequence

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}) \arrow[r, "q^*"] \arrow[d, equal] & \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}) \arrow[r, "(\cdot n)^*"] \arrow[d, "\cong"] & \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}) \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & 0 \arrow[r] & \mathbb{Z} \arrow[r, "\cdot n"'] & \mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

But this sequence is not exact in general since multiplying my n for $n \neq \pm 1$ is not an isomorphism.