1 (Integrability of the Product) Let X be a nonempty set and let μ be a measure on X. Prove that if μ-measurable functions $f, g : X \to [-\infty, \infty]$ are such that f is μ-summable on X and g is bounded on X ($|g(x)| \leq M$ for μ-a.e. $x \in X$), then the product fg is μ-summable and

$$\int_X |fg| \,\mathrm{d}\mu \leq M \int_X |f| \,\mathrm{d}\mu.$$

Proof. First, assume f and g are nonnegative, then their product fg is nonnegative and μ-measurable—therefore μ-integrable. Denote $A = g^{-1}([-M, M])$, which is μ-measurable since g is μ-measurable. Note that $\mu(X \setminus A) = 0$ so $\int_X h \,\mathrm{d}\mu = \int_A h \,\mathrm{d}\mu$ for all μ-integrable functions $h : X \to [-\infty, \infty]$. Then

$$\int_X fg \,\mathrm{d}\mu = \int_A fg \,\mathrm{d}\mu \leq \int_A fM \,\mathrm{d}\mu = M \int_A f \,\mathrm{d}\mu = M \int_X f \,\mathrm{d}\mu.$$

Since f is μ-summable and $M < \infty$, we conclude that fg is μ-summable.

For f and g arbitrary write

$$fg = (f^+ - f^-)(g^+ - g^-) = f^+g^+ - f^+g^- - f^-g^+ + f^-g^-.$$

Then each $f^\pm$ is μ-summable and each $g^\pm$ is bounded μ-a.e. on X by M. An application of the first case tells us each $f^\pm g^\pm$ is μ-summable with $\int_X f^\pm g^\pm \,\mathrm{d}\mu \leq M \int_X f^\pm \,\mathrm{d}\mu$. It follows that fg is μ-summable as a finite $\mathbb{R}$-linear combination of μ-summable functions.

Similarly, $|f|$ is μ-summable and $|g|$ is bounded μ-a.e. on X by M, so the first case gives

$$\int_X |fg| \,\mathrm{d}\mu = \int_X |f||g| \,\mathrm{d}\mu \leq M \int_X |f| \,\mathrm{d}\mu.$$

□

2 Let X be a nonempty set and let μ be a measure on X. Assume μ-summable functions $f, f_n : X \to [-\infty, \infty]$ are such that

$$f_n \longrightarrow f \qquad \mu\text{-a.e. in } X$$

and

$$\int_X |f_n| \, \mathrm{d}\mu \longrightarrow \int_X |f| \, \mathrm{d}\mu.$$

Prove that

$$\int_X |f_n - f| \, \mathrm{d}\mu \longrightarrow 0.$$

Proof. For $n \in \mathbb{N}$ define the nonnegative μ-summable function $g_n = |f_n| + |f|$, then define

$$g = \liminf_{n\to\infty} g_n.$$

It follows that g and $2|f|$ agree μ-a.e. in X. If $A = (g - 2|f|)^{-1}(0)$ then $\mu(X \setminus A) = 0$, so

$$\int_X g \, \mathrm{d}\mu = \int_A g \, \mathrm{d}\mu = \int_A 2|f| \, \mathrm{d}\mu = \int_X 2|f| \, \mathrm{d}\mu.$$

For $n \in \mathbb{N}$ consider the nonnegative μ-summable function $g_n - |f_n - f|$. By Fatou's Lemma,

$$\int_X g \, \mathrm{d}\mu = \int_X \liminf_{n\to\infty} (g_n - |f_n - f|) \, \mathrm{d}\mu \le \liminf_{n\to\infty} \int_X (g_n - |f_n - f|) \, \mathrm{d}\mu.$$

Then

$$\int_X \liminf_{n\to\infty} g_n \, \mathrm{d}\mu = \int_X |f| \, \mathrm{d}\mu + \lim_{n\to\infty} \int_X |f_n| \, \mathrm{d}\mu = \int_X 2|f| \, \mathrm{d}\mu.$$

Since f is μ-summable we can subtract the integral of $2|f|$ from both sides to obtain

$$0 \le -\liminf_{n\to\infty} \int_X |f_n - f| \, \mathrm{d}\mu,$$

or equivalently

$$\limsup_{n\to\infty} \int_X |f_n - f| \, \mathrm{d}\mu \le 0.$$

Hence, $\int_X |f_n - f| \, \mathrm{d}\mu \to 0$. $\square$

3 Let X be a topological space and let μ be a finite measure on X, i.e., $\mu(X) < \infty$. A family of μ-measurable functions $f_n : X \to \mathbb{R}$ is called **uniformly integrable** in X if for any $\varepsilon > 0$ there exists $M > 0$ such that

$$\int_{\{x\,:\,|f_n(x)|>M\}} |f_n(x)|\,\mathrm{d}\mu < \varepsilon \qquad \text{for all } n = 1, 2, \ldots.$$

Similarly $\{f_n\}$ is called **uniformly absolutely continuous** if for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any μ-measurable set $A \subseteq X$ with $\mu(A) < \delta$ one has

$$\left|\int_A f_n(x)\,\mathrm{d}\mu\right| < \varepsilon \qquad \text{for all } n = 1, 2, \ldots.$$

Lemma 1. If $\{f_n\}$ is uniformly absolutely continuous then so are $\{f_n^+\}$ and $\{f_n^-\}$ (with the same choice of δ).

Proof. Given $\varepsilon > 0$ let $\delta > 0$ be as in the definition. For $n \in \mathbb{N}$ define the set

$$U_n = f_n^{-1}([0,\infty)) = \{x \in X : f_n(x) \geq 0\},$$

which is μ-measurable since f_n is μ-measurable, so that $f_n^+ = f_n\chi_{U_n}$. Given a μ-measurable set $A \subseteq X$ with $\mu(A) < \delta$ we have

$$\int_A f_n^+\,\mathrm{d}\mu = \int_A f_n\chi_{U_n}\,\mathrm{d}\mu = \int_{A\cap U_n} f_n\,\mathrm{d}\mu.$$

Then $A \cap U_n$ is μ-measurable with $\mu(A \cap U_n) \leq \mu(A) < \delta$, hence

$$\left|\int_A f_n^+\,\mathrm{d}\mu\right| = \left|\int_{A\cap U_n} f_n\,\mathrm{d}\mu\right| < \varepsilon.$$

The proof for $\{f^-\}$ is similar with $L_n = f_n^{-1}((-\infty,0])$ and $f_n^- = -f_n\chi_{L_n}$. $\square$

Prove that $\{f_n\}$ is uniformly integrable if and only if

$$\sup_n \int_X |f_n(x)|\,\mathrm{d}\mu < \infty$$

and $\{f_n\}$ is uniformly absolutely continuous.

Proof. Suppose $\{f_n\}$ is uniformly integrable. Choose $\varepsilon = 1$ and let $M > 0$ be as in the definition. For $n \in \mathbb{N}$ define the set

$$A_n = f_n^{-1}([-M,M]) = \{x \in X : |f_n(x)| \leq M\},$$

which is μ-measurable since f_n is μ-measurable. We split the integral over X into two parts:

$$\int_X |f_n|\,\mathrm{d}\mu = \int_{A_n} |f_n|\,\mathrm{d}\mu + \int_{X\setminus A_n} |f_n|\,\mathrm{d}\mu.$$

By assumption, we have

$$\int_{X\setminus A_n} |f_n| \,\mathrm{d}\mu < 1.$$

Monotonicity of the μ-integral gives us

$$\int_{A_n} |f_n| \,\mathrm{d}\mu \leq \int_{A_n} M \,\mathrm{d}\mu = M \int_X \chi_{A_n} \,\mathrm{d}\mu = M\mu(A_n) \leq M\mu(X).$$

Letting $R = M\mu(X) + 1 < \infty$, we deduce that $\int_X |f_n| \,\mathrm{d}\mu < R$ for all $n \in \mathbb{N}$. Passing to the supremum, we obtain

$$\sup_n \int_X |f_n| \,\mathrm{d}\mu \leq R < \infty.$$

For an unknown $\delta > 0$ suppose that $A \subseteq X$ is a μ-measurable set with $\mu(A) < \delta$. Then

$$\left|\int_A f_n \,\mathrm{d}\mu\right| \leq \int_A |f_n| \,\mathrm{d}\mu \leq \int_A M \,\mathrm{d}\mu = M\mu(A) < M\delta.$$

So given $\varepsilon > 0$ we can choose $\delta \leq \varepsilon/M$, hence $\{f_n\}$ is uniformly absolutely continuous.

We now show the reverse direction—suppose $R = \sup_n \int_X |f_n| \,\mathrm{d}\mu < \infty$ and $\{f_n\}$ is uniformly absolutely continuous. For $n \in \mathbb{N}$ and an unknown $M > 0$ define the set

$$B_n = X \setminus f_n^{-1}([-M, M]) = \{x \in X : |f_n(x)| > M\},$$

which is μ-measurable since f_n is μ-measurable. Then

$$\mu(B_n) = \frac{1}{M} \int_{B_n} M \,\mathrm{d}\mu \leq \frac{1}{M} \int_{B_n} |f_n| \,\mathrm{d}\mu \leq \frac{1}{M} \int_X |f_n| \,\mathrm{d}\mu \leq \frac{R}{M}.$$

Let $\varepsilon > 0$ be given. By Lemma 1 we can choose $\delta > 0$ such that for all μ-measurable sets $A \subseteq X$ with $\mu(A) < \delta$ we have

$$\int_A f_n^+ \,\mathrm{d}\mu < \frac{\varepsilon}{2} \quad \text{and} \quad \int_A f_n^- \,\mathrm{d}\mu < \frac{\varepsilon}{2}.$$

Choosing $M > R/\delta$ gives us $\mu(B_n) < \delta$, so

$$\int_{B_n} |f_n| \,\mathrm{d}\mu = \int_{B_n} (f_n^+ + f_n^-) \,\mathrm{d}\mu = \int_{B_n} f_n^+ \,\mathrm{d}\mu + \int_{B_n} f_n^- \,\mathrm{d}\mu < \varepsilon.$$

Hence, $\{f_n\}$ is uniformly integrable. $\square$

4 Compute the limit

$$\lim_{n\to\infty}\int_0^n \left(1-\frac{x}{n}\right)^n \ln\left(2+\cos\left(\frac{x}{n}\right)\right)\,\mathrm{d}x$$

For $n \in \mathbb{N}$ and $x \in [0,\infty)$ define the function

$$f_n(x) = \begin{cases} \left(1-\frac{x}{n}\right)^n \ln\left(2+\cos\left(\frac{x}{n}\right)\right) & \text{if } x \le n, \\ 0 & \text{otherwise.} \end{cases}$$

Then we can write

$$\int_0^n \left(1-\frac{x}{n}\right)^n \ln\left(2+\cos\left(\frac{x}{n}\right)\right)\,\mathrm{d}x = \int_0^\infty f_n(x)\,\mathrm{d}x.$$

Note that $\cos x$ is positive and decreasing on $[0,\pi/2]$, so for $n \in \mathbb{N}$ and $x \in [0,n]$ we have

$$\cos\left(\frac{x}{n}\right) \le \frac{x}{n+1}.$$

Since the logarithm is increasing on $(0,\infty)$, we obtain

$$\ln\left(2+\cos\left(\frac{x}{n}\right)\right) \le \ln\left(2+\cos\left(\frac{x}{n+1}\right)\right).$$

Check that $0 \le (1-\frac{x}{n})^n \le (1-\frac{x}{n+1})^{n+1}$ for all $n \in \mathbb{N}$ and $x \in [0,n]$.

It follows that $0 \le f_n(x) \le f_{n+1}(x)$ for all $n \in \mathbb{N}$ and $x \in [0,\infty)$. Monotone convergence of the Lebesgue integral gives us

$$\lim_{n\to\infty}\int_0^\infty f_n(x)\,\mathrm{d}x = \int_0^\infty e^{-x}\ln(3+\cos 0)\,\mathrm{d}x = \ln 3\int_{-\infty}^0 e^x\,\mathrm{d}x = \ln 3\cdot e^0 = \ln 3.$$