1 Let X be a nonempty set and let μ be a measure on X. We have a theorem on sequences of decreasing measurable sets that states the following: Assume $X \supseteq A_1 \supseteq A_2 \supseteq A_3 \supseteq \cdots$ are μ-measurable, such that $\mu(A_1) < \infty$. Then one has

$$\lim_{n\to\infty} \mu(A_n) = \mu(\textstyle\bigcap_{n=1}^{\infty} A_n).$$

Prove that in this theorem the condition $\mu(A_1) < \infty$ is necessary.

Proof. Consider $X = \mathbb{Z}$ with the cardinality measure: $\mu(E) = |E|$, for all $E \subseteq X$. One can check that this satisfies the necessary properties of a measure. Moreover, every subset of $\mathbb{Z}$ is μ-measurable.

For $n \in \mathbb{N}$, define the subset of integers $A_n = \{k \in \mathbb{Z} : k \geq n\}$. Then $X \supseteq A_1 \supseteq A_2 \supseteq A_3 \supseteq \cdots$ is a decreasing chain of μ-measurable subsets. Additionally, we have $\mu(A_n) = \infty$, for all $n \in \mathbb{N}$, and $\bigcap_{n=1}^{\infty} A_n = \varnothing$. Then

$$\lim_{n\to\infty} \mu(A_n) = \lim_{n\to\infty} \infty = \infty,$$

but

$$\mu\big(\textstyle\bigcap_{n=1}^{\infty} A_n\big) = \mu(\varnothing) = 0.$$

□

2 Does there exist an infinite σ-algebra that has countably many elements?

No.

Proof. Assume, for contradiction, that $\mathcal{A}$ is a countably infinite σ-algebra on a set X.

In order to derive a contradiction, we will construct an infinite collection $\mathcal{N} \subseteq \mathcal{A}$ of disjoint sets in $\mathcal{A}$. Then an injection $\mathbb{N} \to \mathcal{N}$ gives us a bijection $2^{\mathbb{N}} \to 2^{\mathcal{N}}$; in particular, we deduce that $2^{\mathcal{N}}$ is uncountable. For any $S \in 2^{\mathcal{N}}$, there is a set $\bigcup_{E \in S} E \in \mathcal{A}$, since $S \subseteq \mathcal{A}$ is at most countable. Because all the sets in $\mathcal{N}$ are disjoint, different choices of $S \in 2^{\mathcal{N}}$ produce different unions, giving us an injection $2^{\mathcal{N}} \to \mathcal{A}$. However, this is a contradiction, as we assumed $\mathcal{A}$ to be countable. It remains to construct the desired $\mathcal{N}$.

For each $x \in X$, we define the set $N_x = \bigcap_{x \in E \in \mathcal{A}} E$, i.e., the intersection of all sets in $\mathcal{A}$ containing the point x. We claim that the collection $\mathcal{N} = \{N_x\}_{x \in X}$ is as desired.

First, since $\mathcal{A}$ contains only countably many sets, each N_x is the intersection of at most countably many sets. Then, because $\mathcal{A}$ is a σ-algebra, $N_x \in \mathcal{A}$, so in fact $\mathcal{N} \subseteq \mathcal{A}$.

Next, we check that the sets in $\mathcal{N}$ are mutually disjoint. More specifically, we will prove that N_x and N_y are either equal or disjoint, for all $x, y \in X$, which would mean that $\mathcal{N}$ describes a partition of X into the components N_x.

Suppose $x, y \in X$ such that $y \in N_x$. By construction, N_x is the intersection of all $E \in \mathcal{A}$ containing x. So $y \in N_x$ means that the collection of sets in $\mathcal{A}$ containing x is a subset of the collection of those containing y, therefore $N_y \subseteq N_x$. Assume, for contradiction, $N_x \nsubseteq N_y$. By the contrapositive of the first inclusion, we must have $x \notin N_y$. This means that there is some set $E \in \mathcal{A}$ containing y but not x, so $E^c \in \mathcal{A}$ is a set containing x but not y. However, this means that $N_x \subseteq E^c$ but $y \notin E^c$, implying $y \notin N_x$, This is a contradiction, so $N_x \subseteq N_y$.

Lastly, we prove that $\mathcal{N}$ is infinite. First, note that each set $E \in \mathcal{A}$ can be written as the union $E = \bigcup_{x \in E} N_x$. This suggests $\mathcal{N}$ be interpreted as the “irreducible” sets in $\mathcal{A}$, and every other set in $\mathcal{A}$ has an “irreducible decomposition” into sets in $\mathcal{N}$. Since $\mathcal{A}$ is infinite, there must be infinitely many different combinations of sets in $\mathcal{N}$, which is only possible if $\mathcal{N}$ itself is infinite.

$\square$

3 Is it true that if μ is a Borel measure on a nonempty set X, then for any sets $A, B \subset X$ with $\operatorname{dist}(A, B) > 0$, one has

$$\mu(A \cup B) = \mu(A) + \mu(B)?$$

Yes.

Proof. Let $r = \operatorname{dist}(A, B) > 0$, and define the set $E = \bigcup_{x \in A} B_r(x) \subseteq X$. This construction gives us $A \subseteq E$ and $B \subseteq E^c$. As the union of open balls, E is an open subset of X and, therefore, μ-measurable. Hence,

$$\mu(A \cup B) = \mu((A \cup B) \cap E) + \mu((A \cup B) \setminus E) = \mu(A) + \mu(B).$$

$\square$

4 Let X be an uncountable set and let $\mathcal{C}$ be the collection of all subsets A of X such that either A or A^c is at most countable. Prove that $\mathcal{C}$ is a σ-algebra.

Proof. Since $\varnothing = X^c$ is at most countable, $\varnothing, X \in \mathcal{C}$.

If $A \in \mathcal{C}$, then either A^c or $A = (A^c)^c$ is at most countable, so $A^c \in \mathcal{C}$.

Suppose $A_1, A_2, \ldots \in \mathcal{C}$ and let $A = \bigcup_{n=1}^{\infty} A_n$. It is either the case that all A_n are at most countable or some A_N^c is at most countable. In the first case, A is countable, as a countable union of countable subsets. In the second case, $A^c = \bigcap_{n=1}^{\infty} A_n^c$ is contained in some countable set A_N^c, implying A^c is countable. In both cases, either A is at most countable or A^c is at most countable, so in fact $A \in \mathcal{C}$.

$\square$