1 Let λ be the Lebesgue measure and let $\{A_n\}_{n=1}^{\infty}$ be a sequence of Lebesgue-measurable subsets of $[0,1]$. Assume the set B consists of those points $x \in [0,1]$ that belong to infinitely many of the A_n.

(a) Prove that B is Lebesgue-measurable.

Proof. Note that the set of Lebesgue-measurable subsets of $[0,1]$ form a σ-algebra.

For $n \in \mathbb{N}$, define the Lebesgue-measurable set $B_n = \bigcup_{i=n}^{\infty} A_i$. In other words, B_n is the set of points which appear in some A_i with $i \geq n$. Then the intersection of all the B_n's is the set of points that belong to infinitely many A_i's. That is, $B = \bigcap_{n=1}^{\infty} B_n$, which is also Lebesgue-measurable. □

(b) Prove that if $\lambda(A_n) > \delta > 0$ for every $n \in \mathbb{N}$, then $\lambda(B) \geq \delta$.

Proof. Note that we have a decreasing sequence $B_n \supseteq B_{n+1}$. Then a theorem that tells us

$$\lambda(B) = \lambda\left(\bigcap_{n=1}^{\infty} B_n\right) = \lim_{n\to\infty} \lambda(B_n).$$

Additionally, $B_n \supseteq A_n$, so monotonicity implies

$$\lambda(B) \geq \lim_{n\to\infty} \lambda(A_n) \geq \lim_{n\to\infty} \delta = \delta.$$

□

(c) Prove that if $\sum_{n=1}^{\infty} \lambda(A_n) < \infty$, then $\lambda(B) = 0$.

Proof. From part (a), we have $B \subseteq B_n$, so $\lambda(B) \leq \lambda(B_n)$. Assuming $\sum_{n=1}^{\infty} \lambda(A_n) < \infty$, then for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $\sum_{i=N}^{\infty} \lambda(A_i) < \varepsilon$. Then

$$\lambda(B) \leq \lambda(B_N) \leq \sum_{i=N}^{\infty} \lambda(A_i) < \varepsilon.$$

Letting $\varepsilon \to 0$, we obtain $\lambda(B) = 0$. □

(d) Give an example where $\sum_{n=1}^{\infty} \lambda(A_n) = \infty$, but $\lambda(B) = 0$.

Proof. Define $A_n = [0, 1/n]$ for all $n \in \mathbb{N}$. Then $B_n = A_n$, so $B = \{0\}$. Then

$$\sum_{n=1}^{\infty} \lambda(A_n) = \sum_{n=1}^{\infty} \frac{1}{n} = \infty,$$

but $\lambda(B) = \lambda(\{0\}) = 0$. □

2 Prove that if the set $A \subseteq \mathbb{R}$ is Lebesgue-measurable, with $\lambda(A) > 0$, then there is a subset of A that is not Lebesgue-measurable.

Proof. We perform a construction similar to a Vitali set.

Consider the additive quotient $A/\mathbb{Q}$, where $\overline{x} = \overline{y}$ if $x - y \in \mathbb{Q}$. Using the axiom of choice, construct a set $V \subseteq A$ with one representative from each equivalence class $\overline{x} \in A/\mathbb{Q}$.

First, assume A is bounded by the interval $[-M, M]$. Then

$$A \subseteq \bigcup_{x \in [-2M,2M] \cap \mathbb{Q}} x + V \quad \subseteq [-3M, 3M].$$

Assume, for contradiction, that V is measurable, then $\lambda(V) = \lambda(x + V)$ for all $x \in \mathbb{R}$. And since the elements in V are in different equivalence classes in $A/\mathbb{Q}$, then shifted sets $x + V$ are mutually disjoint over $x \in \mathbb{Q}$. So we have

$$\lambda\left(\bigcup_{x \in [-2M,2M] \cap \mathbb{Q}} x + V\right) = \sum_{x \in [-2M,2M] \cap \mathbb{Q}} \lambda(x + V) = \sum_{x \in [-2M,2M] \cap \mathbb{Q}} \lambda(V) \quad .$$

Hence,

$$0 < \lambda(A) \leq \sum_{x \in [-2M,2M] \cap \mathbb{Q}} \lambda(V) \quad \leq \lambda([-3M, 3M]) = 6M.$$

But, what is $\lambda(V)$? We must have $\lambda(V) > 0$, since the above sum is positive. But if this is the case, the sum of countably many copies of this positive number would be infinite. This is a contradiction, so V is not Lebesgue-measurable.

This proves that every bounded Lebesgue-measurable set of positive measure contains a non-measurable subset. For an unbounded measurable set $A \subseteq \mathbb{R}$, we can find some interval $[-M, M]$ for which $A \cap [-M, M]$ has positive measure and repeat the process to construct a non-measurable subset of A.

$\square$

3 Let λ be the Lebesgue measure on $\mathbb{R}$.

(a) Let $A \subseteq \mathbb{R}$ be a set such that $\lambda(A) > 0$. Prove that for any $\varepsilon > 0$, there exists an interval $(a, b) \subseteq \mathbb{R}$ such that $\lambda(A \cap (a, b)) > (1 - \varepsilon)(b - a)$.

Proof. Let $\varepsilon > 0$ be given. Assume, for contradiction, that $\lambda(A \cap (a, b)) < (1 - \varepsilon)(b - a)$ for all intervals $(a, b) \subseteq \mathbb{R}$. By definition of the Lebesgue measure, there is a cover of A by disjoint open intervals $(a_i, b_i) \subseteq \mathbb{R}$ such that

$$\lambda(A) \leq \sum_{i=1}^{\infty} (b_i - a_i) < \lambda(A) + \varepsilon\lambda(A).$$

Denote $U = \bigcup_{i=1}^{\infty}(a_i, b_i)$, so

$$\begin{aligned}
\lambda(A) &= \lambda(A \cap U) + \lambda(A \setminus U) \\
&= \lambda\left(\bigcup_{i=1}^{\infty} A \cap (a_i, b_i)\right) + \lambda(\varnothing) \\
&\leq \sum_{i=1}^{\infty} \lambda(A \cap (a_i, b_i)) \\
&\leq \sum_{i=1}^{\infty} (1-\varepsilon)(b_i - a_i) \\
&= (1-\varepsilon)\lambda(U) \\
&< (1-\varepsilon)(\lambda(A) + \varepsilon\lambda(A)) \\
&= (1-\varepsilon^2)\lambda(A).
\end{aligned}$$

This is a contradiction.

$\square$

(b) Construct a Borel set $B \subseteq \mathbb{R}$ such that $\lambda(B) > 0$ and $\lambda(B \cap I) < \lambda(I)$ for every non-degenerate interval $I \subseteq \mathbb{R}$.

Let $\{a_k\}_{k\in\mathbb{N}}$ be a sequence indexing all the rationals $\mathbb{Q}$. For each $k \in \mathbb{N}$, define the interval

$$A_k = (a_k - 2^{-k},\ a_k + 2^{-k}) \subseteq \mathbb{R},$$

and $A = \bigcup_{k=1}^{\infty} A_k$. As the union of countable many intervals, A is a Borel set. Then its complement $B = \mathbb{R} \setminus A$ is also Borel. Since the measure of A is bounded above by

$$\lambda(A) \leq \sum_{k=1}^{\infty} \frac{2}{2^k} = 2,$$

its complement must have infinite—in particular positive—measure, i.e., $\lambda(B) > 0$.

On the other hand, we now consider the intersection of B with a non-degenerate interval $I \subseteq \mathbb{R}$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, I must contain some rational $a_k \in I$. Then $J = I \cap A_k$ is some non-degenerate interval. Then

$$B \cap I \subseteq I \setminus A_k = I \setminus J,$$

so $\lambda(B \cap I) = \lambda(I) - \lambda(J) < \lambda(I)$.

4 Prove that if a Lebesgue-measurable set $A \subseteq \mathbb{R}$ has positive Lebesgue measure, then the set
$$A - A = \{a - b : a, b \in A\}$$
contains a neighborhood of the origin.

Proof. Since λ is radon, there is a compact set $K \subseteq A$ such that $0 < \lambda(K) \le \lambda(A)$. Then $K - K \subseteq A - A$, so it suffices to show $K - K$ contains a neighborhood of the origin.

Again, since λ is radon, there is an open set $U \supseteq K$ such that $\lambda(K) \le \lambda(U) < 2\lambda(K)$. Since U is an open neighborhood of the compact set K, there is some $\varepsilon > 0$ such that the ε-neighborhood of K
$$B_\varepsilon(K) = \bigcup_{x \in K} B_\varepsilon(x) = \bigcup_{r \in (-\varepsilon, \varepsilon)} (r + K)$$
is still contained in U. For any $r \in (-\varepsilon, \varepsilon)$, we know that $\lambda(r + K) = \lambda(K)$, so
$$\lambda(K \cup (r + K)) \le \lambda(U) < 2\lambda(K) = \lambda(K) + \lambda(r + K).$$
This implies $K \cap (r+K)$ must have positive measure. In particular, $K \cap (r+K)$ is nonempty, so there is some $x, y \in K$ such that $x = r + y$, i.e., $r = x - y \in K - K$. In other words, we have shown that $(-\varepsilon, \varepsilon) \subseteq K - K$. $\square$

Is the statement true if one only assumes $\lambda(A) > 0$ (i.e., A is not Lebesgue-measurable)?

No.

If V is a Vitali set, then $\lambda(V) > 0$, but by construction $(V - V) \cap \mathbb{Q} = \varnothing$. Since every neighborhood of the origin contains rational numbers, $V - V$ cannot contain a neighborhood of the origin.

5 Let $A \subseteq \mathbb{R}$ be any set. Prove that the set

$$B = \bigcup_{x \in A} [x - 1, x + 1]$$

is Lebesgue-measurable.

Proof. For each $x \in B$, there is an interval containing x and contained in B; let I_x be the maximal such interval. We say that I_x is a connected component of B, in the topological sense. We claim that B has countably many connected components. Each connected component of B contains an interval $[x - 1, x + 1]$ for some $x \in A$. That means we can choose a representative from the interior of each connected component. Let C be a set of representatives from each connected component of B such that for each $x \in C$ we have $x \in \operatorname{int} I_x$. Then $|C|$ is the number of connected components of B. Additionally, note that C is a discrete subset of $\mathbb{R}$, so it must be countable. Hence, B has countably many connected components. And since

$$B = \bigcup_{x \in C} I_x,$$

where each I_x is an interval and C is countable, then in fact B is a Borel set. Since the Lebesgue measure is Borel, we conclude that B is Lebesgue-measurable.

$\square$