

1 Give an example of a topological space X and a measure μ on X so that μ is Borel but not Borel-regular.

Let $X = \{0, 1\}$ with the indiscrete topology and μ the cardinality measure. Clearly, μ is Borel, since \emptyset and X , the only Borel sets, are always measurable. Consider $\{0\} \subseteq X$; the only Borel subset containing $\{0\}$ is X itself, but $\mu(\{0\}) = 1 \neq 2 = \mu(X)$. Hence, μ is not Borel-regular.

2 Let X be a nonempty set and let $\{\mu_n\}_{n=1}^\infty$ be a sequence of measures on X . Assume for any subset $A \subseteq X$ the limit $\lim_{n \rightarrow \infty} \mu_n(A)$ exists and denote $\mu(A) = \lim_{n \rightarrow \infty} \mu_n(A)$.

(i) Is it true that μ is a measure on X if for any $A \subseteq X$ the sequence $\{\mu_n(A)\}$ is increasing?

Yes.

Proof. First,

$$\mu(\emptyset) = \lim_{n \rightarrow \infty} \mu_n(\emptyset) = \lim_{n \rightarrow \infty} 0 = 0.$$

Suppose $A_1, A_2, \dots \subseteq X$ and $A \subseteq \bigcup_{i=1}^\infty A_i$. By the increasing condition, $\mu_n(A_i) \leq \mu(A_i)$, for all $n, i \in \mathbb{N}$. Then, for all $n \in \mathbb{N}$,

$$\mu_n(A) \leq \sum_{i=1}^\infty \mu_n(A_i) \leq \sum_{i=1}^\infty \mu(A_i).$$

Letting $n \rightarrow \infty$, we obtain

$$\mu(A) = \lim_{n \rightarrow \infty} \mu_n(A) \leq \sum_{i=1}^\infty \mu(A_i).$$

This is the countable subadditivity and monotonicity, hence μ is a measure on X .

□

(ii) Assume in addition that $\mu_1(X) < \infty$, and that each of the measures μ_n is regular. Is it true that μ is a measure on X if for any $A \subseteq X$ the sequence $\{\mu_n(A)\}$ is decreasing?

Yes?

Proof. First,

$$\mu(\emptyset) = \lim_{n \rightarrow \infty} \mu_n(\emptyset) = \lim_{n \rightarrow \infty} 0 = 0.$$

Suppose $A_1, A_2, \dots \subseteq X$ and $A \subseteq \bigcup_{i=1}^\infty A_i$. By the decreasing condition, $\mu(A) \leq \mu_n(A)$, for all $n \in \mathbb{N}$, so

$$\mu(A) \leq \mu_n(A) \leq \sum_{i=1}^\infty \mu_n(A_i).$$

This is the countable subadditivity and monotonicity, hence μ is a measure on X .

□

3 Let X be a nonempty set and F be a collection of functions $f : X \rightarrow \mathbb{R}$ with the following properties:

- (i) The constant function $f(x) \equiv 1 \in F$, and if $f, g \in F$ and $c \in \mathbb{R}$, then $f+g, fg, cf \in F$.
- (ii) If a sequence $\{f_n\} \subseteq F$ has as pointwise limit in X : $f(x) = \lim_{n \rightarrow \infty} f_n(x)$ for all $x \in X$, then $f \in F$.

Prove that the collection $\mathcal{A} = \{A \subseteq X : \chi_A \in F\}$ is a σ -algebra, where χ_A is the characteristic function of the set A .

Proof. Since $\chi_X = 1 \in F$, we know $X \in \mathcal{A}$.

If $A \in \mathcal{A}$, then $\chi_{A^c} = 1 - \chi_A \in F$, implying $A^c \in \mathcal{A}$.

Suppose $A_1, A_2, \dots \in \mathcal{A}$ and let $A = \bigcap_{i=1}^{\infty} A_i$. For $n \in \mathbb{N}$, define $B_n = \bigcap_{i=1}^n A_i$, then

$$\chi_{B_n} = \prod_{i=1}^n \chi_{A_i} \in F.$$

Then $\chi_{B_n} \rightarrow \chi_A$ pointwise in X , so $\chi_A \in F$, implying $A \in \mathcal{A}$.

□

4 Prove that any open subset of \mathbb{R}^n can be expressed as a countable union of closed balls in \mathbb{R}^n

Remark. The statement is true for any separable metric space X .

Proof. Let X be a separable metric space and let $Y \subseteq X$ be a countable dense subset. Let $U \subseteq X$ be an open subset. For each $x \in U$, define the radius $r_x = d(x, U^c) > 0$ and the closed ball $E_x = \overline{B_{r_x/2}(x)}$. We claim that $U = \bigcup_{y \in U \cap Y} E_y$, which is a countable union of closed balls.

By construction, $E_y \subset B_{r_y}(y) \subseteq U$ for all $y \in U \cap Y$, so $\bigcup_{y \in U \cap Y} E_y \subseteq U$.

For each $x \in U$, consider the open neighborhood $B_{r_x/4}(x) \subseteq U$ of x . Since Y is a dense subset of X , we can find some $y \in B_{r_x/4}(x)$. Then

$$r_x \leq d(x, y) + d(y, U^c) = d(x, y) + r_y \leq \frac{1}{4}r_x + r_y,$$

so $3r_x/4 \leq r_y$. Then

$$d(x, y) \leq \frac{1}{4}r_x \leq \frac{3}{8}r_x \leq \frac{1}{2}r_y,$$

so $x \in E_y$. Hence, $U \subseteq \bigcup_{y \in U \cap Y} E_y$

□