(worked w/ Joseph Sullivan)

1 Exercise I.3 Let G be a group. A **commutator** in G is an element of the form $aba^{-1}b^{-1}$ with $a, b \in G$. Let G^c be the subgroup generated by the commutators. Then G^c is called the **commutator subgroup**. Show that G^c is normal.

Proof. Let $x \in G$ and $y \in G^c$, then there is a commutator $xyx^{-1}y^{-1} \in G^c$. So, as the product of two elements in G^c,

$$xyx^{-1} = (xyx^{-1}y^{-1})y \in G^c.$$

Hence, $xG^cx^{-1} \subseteq G^c$ for all $x \in G$, so in fact $G^c \trianglelefteq G$.

□

Show that any homomorphism of G into an abelian group factors through G/G^c.

Proof. Let $\varphi : G \to A$ be a group homomorphism to an abelian group A and $\pi : G \to G/G^c$ be the natural projection. We want to find a group homomorphism $\psi : G/G^c \to A$ such that $\psi \circ \pi = \varphi$. In order for such a ψ to exist, φ must be constant on the equivalence classes in G/G^c. In other words, we must show that $\overline{x} = \overline{y} \in G/G^c$ implies $\varphi(x) = \varphi(y)$. For any $x, y \in G$,

$$\varphi(xy) = \varphi(x)\varphi(y) = \varphi(y)\varphi(x) = \varphi(yx),$$

where the commutation of $\varphi(x)$ and $\varphi(y)$ is permitted in A. Then

$$1 = \varphi(xy)\varphi(yx)^{-1} = \varphi(xyx^{-1}y^{-1}).$$

In other words, all the commutators in G are in the kernel of φ. Since G^c is generated by the commutators, we conclude that $G^c \subseteq \ker \varphi$. This means that if $\overline{x} = \overline{y} \in G/G^c$, i.e., $xy^{-1} \in G^c$, then $1 = \varphi(xy^{-1}) = \varphi(x)f(y)^{-1}$, giving us $\varphi(x) = \varphi(y)$.

Since φ is constant on equivalence classes, then we obtain a group homomorphism ψ from G/G^c to A, with $\overline{x} \mapsto \varphi(x)$, as desired.

□

2 Exercise I.6 Prove that the group of inner automorphisms of a group G is normal in $\mathrm{Aut}(G)$.

Proof. Denote the subgroup of $\mathrm{Aut}(G)$ consisting of the inner automorphisms by

$$\mathrm{Inn}(G) = \{c_x \mid x \in G\}.$$

(Note c_x is the conjugation $y \mapsto xyx^{-1}$.) Let $\sigma \in \mathrm{Aut}(G)$ and $c_x \in \mathrm{Inn}(G)$, then for $y \in G$,

$$\begin{aligned}(\sigma c_x \sigma^{-1})(y) &= \sigma(x\sigma^{-1}(y)x^{-1}) \\ &= \sigma(x)y\sigma(x)^{-1} \\ &= c_{\sigma(x)}(y).\end{aligned}$$

As $c_{\sigma(x)} \in \mathrm{Inn}(G)$, this proves $\sigma\,\mathrm{Inn}(G)\sigma^{-1} \subseteq \mathrm{Inn}(G)$, hence $\mathrm{Inn}(G) \trianglelefteq \mathrm{Aut}(G)$.

□

3 Exercise I.7 Let G be a group such that $\mathrm{Aut}(G)$ is cyclic. Prove that G is abelian.

Proof. Note that G is abelian if and only if $\mathrm{Inn}(G)$ is trivial. Assume, for contradiction, that $\mathrm{Inn}(G)$ is not trivial. Since $\mathrm{Inn}(G)$ is a subgroup of the cyclic group $\mathrm{Aut}(G)$, it is cyclic and generated by some $c_x \neq \mathrm{id}_G$. Then there is some $y \in G$ such that $c_x(y) \neq y$, i.e., $xy \neq yx$. However, as $c_y \in \mathrm{Inn}(G) = \langle c_x \rangle$, then we have $c_y = c_x^k$ for some $k \in \mathbb{Z}$. Therefore,

$$\begin{aligned} c_y(x) &= c_x^k(x), \\ yxy^{-1} &= x^k x x^{-k}, \\ yx &= xy, \end{aligned}$$

which is a contradiction.

$\square$

4 Exercise I.12 Let G be a group and let H, N be subgroups with N normal. Let γ_x be conjugation by an element $x \in G$.

(a) Show that $x \mapsto \gamma_x$ induces a homomorphism $f : H \to \mathrm{Aut}(N)$.

Proof. Since N is normal, then for any $x \in H$, we have $\gamma_x(N) = xNx^{-1}$. So γ_x is a bijection, therefore automorphism, on N. For $x, y \in H$ and $n \in N$, we find

$$\gamma_{xy}(n) = (xy)n(xy)^{-1} = x(yny^{-1})x^{-1} = (\gamma_x\gamma_y)(n),$$

so f is in fact a group homomorphism.

□

(b) If $H \cap N = \{e\}$, show that the map $H \times N \to HN$ given by $(x, y) \mapsto xy$ is a bijection, and that this map is an isomorphism if and only if f is trivial, i.e., $f(x) = \mathrm{id}_N$ for all $x \in H$.

Proof. By definition, HN is the set of points xy such that $x \in H$ and $y \in N$, so the map is surjective. Suppose $x_1, x_2 \in H$ and $y_1, y_2 \in N$ such that $x_1y_1 = x_2y_2$. Then

$$x_1^{-1}x_2 = y_1^{-1}y_2 \in H \cap N = \{e\},$$

so $x_1^{-1} = x_2^{-1}$ and $y_1^{-1} = y_2^{-1}$. That is, $(x_1, y_1) = (x_2, y_2)$, so the map is injective.

Denote by m the map $(x, y) \mapsto xy$. If m is an isomorphism, then for $x \in H$ and $y \in N$,

$$\gamma_x(y) = xyx^{-1} = m(x, y)m(x^{-1}, 1) = m(xx^{-1}, y) = m(1, y) = y.$$

That is, $f(x) = \gamma_x = \mathrm{id}_N$ for all $x \in H$.

□

(c) We define G to be the **semidirect product** of H and N if $G = NH$ and $H \cap N = \{e\}$.

Conversely, let N, H be groups, and let $\psi : H \to \mathrm{Aut}(N)$ be a given homomorphism. Construct a semidirect product as follows. Let G be the set of pairs (x, h) with $x \in N$ and $h \in H$. Define the composition law

$$(x_1, h_1)(x_2, h_2) = (x_1\psi(h_1)x_2, h_1h_2).$$

Show that this is a group law, and yields a semidirect product of N and H, identifying N with the set of elements $(x, 1)$ and H with the set of elements $(1, h)$.

Proof. We write $\psi_h = \psi(h)$, so the fact that ψ is a group homomorphism tells us

$$\psi_{h_1h_2} = \psi_{h_1} \circ \psi_{h_2} \in \mathrm{Aut}(N).$$

Let $x, y, z \in N$ and $g, h, k \in H$, then

$$\begin{aligned}\big((x,g)(y,h)\big)(z,k) &= \big(x\psi_g(y),\, gh\big)(z,k)\\ &= \big(x\psi_g(y)\psi_{gh}(z),\, ghk\big)\\ &= \big(x\psi_g(y)\psi_g\big(\psi_h(z)\big), ghk\big)\\ &= \big(x\psi_g\big(y\psi_h(z)\big),\, ghk\big)\\ &= (x,g)\big(y\psi_h(z),\, hk\big)\\ &= (x,g)\big((y,h)(z,k)\big).\end{aligned}$$

That is, the composition law is associative.

We claim that $(1,1)$ is the identity element. Let $x \in N$ and $h \in H$. Since ψ is a group homomorphism, $\psi_1 = \mathrm{id}_N$, so

$$(1,1)(x,h) = (1\psi_1(x), 1h) = (x,h).$$

Since $\psi_h \in \mathrm{Aut}(N)$ is a group homomorphism from N to itself, $\psi_h(1) = 1$, so

$$(x,h)(1,1) = (x\psi_h(1), h1) = (x,h).$$

Lastly, for $x \in N$ and $h \in H$, we see that $(x,h)^{-1} = (\varphi_h^{-1}(x^{-1}), h^{-1})$, since

$$(x,h)(\psi_h^{-1}(x^{-1}), h^{-1}) = (x\psi_h(\psi_h^{-1}(x^{-1})), hh^{-1}) = (xx^{-1}, 1) = (1,1).$$

Hence, this is a group law.

We now show that this group is a semidirect product of N and H. For any (x,h) in the underlying set, we have $(x,1) \in N$ and $(1,h) \in H$ under the given identification. Then

$$(x,1)(1,h) = (x\psi_1(1), 1h) = (x,h),$$

so NH is the entire set. Moreover, the identification gives us $N \cap H = \{(1,1)\}$. Lastly, we check that N is normal. For any (x,h) in the group and $y \in N$, we find

$$\begin{aligned}(x,h)(y,1)(x,h)^{-1} &= (x\psi_h(y), h)(\psi_h^{-1}(x^{-1}), h^{-1})\\ &= (x\psi_h(y)x^{-1}, 1).\end{aligned}$$

Since $x\psi_h(y)x^{-1} \in N$, this shows N is normal.

$\square$