1 Show that any open subset in $\mathbb{R}^n$ can be written as a disjoint union of rectangles, and consequently, any open subset is Lebesgue measurable.

In order to show Lebesgue measurability, we specifically want a *countable* union of rectangles; otherwise, each point $u \in \mathbb{R}^n$ could be identified with a trivial closed rectangle $R_u = \{x \in \mathbb{R} : u_i \leq x_i \leq u_i\} = \{u\}$. In which case, any subset $U \subseteq \mathbb{R}^n$ could be written as the disjoint union of rectangles $U = \bigcup_{u \in U} R_u$.

To prove the countable case, we use the same technique as Stein Theorem 1.4.

Proof. For each $a \in \mathbb{Z}^n$ and $k \in \mathbb{Z}_{\geq 0}$, define the rectangle

$$R_{a,k} = \{x \in \mathbb{R}^n : a_i \leq 2^k x_i < a_i + 1\},$$

whose lower vertex is at $a/2^k$ and has side length $1/2^k$. Then, the collection $\mathcal{R}_k = \{R_{a,k}\}_{a \in \mathbb{Z}^n}$ forms a partition of $\mathbb{R}^n$ into countably many rectangles. Moreover, the rectangle $R_{a,k}$ is partitioned by the set of 2^n rectangles

$$\{R_{2a+s,k+1} : s \in \{0,1\}^n\}.$$

In particular, each rectangle of $\mathcal{R}_{k+1}$ is contained in exactly one rectangle of $\mathcal{R}_k$, meaning that $\mathcal{R}_{k+1}$ is a refinement of $\mathcal{R}_k$ (as a partition of $\mathbb{R}^n$) into rectangles of exactly half the side length. We will use this fact to make increasingly precise approximations of an open set.

Given an open subset $U \subseteq \mathbb{R}^n$, define $U_0 = U$ and denote the the collection of unit rectangles contained in U_0 by

$$A_0 = \{R \in \mathcal{R}_0 : R \subseteq U_0\}.$$

We consider $\bigcup A_0$ (used as a set-theoretic shorthand for $\bigcup_{R \in A_0} R$) to be our first approximation of U, with a "resolution" of 1. Since $\mathcal{R}_0$ is a countable set of disjoint rectangles, so is $A_0 \subseteq \mathcal{R}_0$. Intentionally, this approximation is a subset of U, as will be the case for further approximations, as we approach U from below.

For the second approximation of U, define $U_1 = U_0 \setminus \bigcup A_0$, which is the subset of U_0 that failed to be captured by the first approximation. Then

$$A_1 = \{R \in \mathcal{R}_1 : R \subseteq U_1\}$$

is a set of rectangles of side length $1/2$, each fully contained in U_1. Then $\bigcup(A_0 \cup A_1)$ is our second approximation of U, with a resolution of $1/2$. By construction, U_0 and U_1 are disjoint, so the rectangles of A_0 and A_1 are also disjoint.

The inductive construction follows. For all $k \geq 1$, define $U_k = U_{k-1} \setminus \bigcup A_{k-1}$ and

$$A_k = \{R \in \mathcal{R}_k : R \subseteq U_k\}.$$

Interpreted, U_k is the subset of U which is not captured by any approximation down to a resolution of $1/2^{k-1}$, and $\bigcup A_k$ is the approximation of U_k with a resolution of $1/2^k$. Then an approximation of all of U with a resolution of $1/2^k$ is given by $\bigcup(A_0 \cup \cdots \cup A_k)$, which is a disjoint union of countably many rectangles, contained in U.

As the limiting case of these successive approximations, define

$$A = \bigcup_{k=0}^{\infty} A_k = A_0 \cup A_1 \cup A_2 \cup \cdots ,$$

which is a countable union of sets, each containing countably many rectangles; therefore, A is a countable. Moreover, we know that A is a disjoint set of rectangles; since any pair of overlapping rectangles would need to occur at some finite stage, but by construction $A_0 \cup \cdots \cup A_k$ is a disjoint set of rectangles for all $k \in \mathbb{Z}_{\geq 0}$. Also by construction, we know $\bigcup A \subseteq U$, since each finite step is a subset of U.

It remains to show $U \subseteq \bigcup A$. For any $x \in U$, there is some $\delta > 0$ such that $B_\delta(x) \subseteq U$. For $k \in \mathbb{Z}_{\geq 0}$, the diagonal of any rectangle $R \in \mathcal{R}_k$ is given by $\sqrt{n}/2^k$. If we choose k such that $\sqrt{n}/2^k < \delta$, then (using the fact that $\mathcal{R}_k$ is a partition of R^n) the rectangle $R \in \mathcal{R}_k$ containing x is entirely contained in $B_\delta(x)$, in turn contained in U. In particular, this shows the existence of a nonnegative integer k, for which the unique rectangle of $\mathcal{R}_k$ containing x is also contained in U. Therefore, it makes sense to define the minimum such integer:

$$N = \min\{k \in \mathbb{Z}_{\geq 0} : x \in R \subseteq U \text{ for some } R \in \mathcal{R}_k\}.$$

By construction, we know that $x \in U_N$, since the rectangles of $\mathcal{R}_0, \ldots, \mathcal{R}_{N-1}$ containing x are not contained in U (i.e., x is not captured by any approximation of U down to a resolution of $1/2^{k-1}$). By definition, there is some $R \in \mathcal{R}_N$ such that $x \in R \subseteq U$. For any other point $y \in R$, the unique rectangles of $\mathcal{R}_0, \ldots, \mathcal{R}_{N-1}$ containing y are precisely the same as those containing x, implying that $y \in U_N$ for the same reasons. Hence, $R \subseteq U_N$, meaning that R is one of the rectangles used in the approximation of U_N, i.e., $R \in A_N$. And since $A_N \subseteq A$,

$$x \in R \subseteq \bigcup A_N \subseteq \bigcup A.$$

Thus, $U \subseteq \bigcup A$, implying equality: $U = \bigcup A$. As previously noted, A is a countable set of disjoint rectangles, so this gives U as the union of countably many disjoint rectangles.

Since rectangles are elementary open sets, this gives U as the union of countably many finitely Lebesgue measurable sets; hence, U is Lebesgue measurable.

$\square$

2 Show that R^n is Lebesgue measurable, and therefore any closed subset is Lebesgue measurable.

Proof. For any elementary set $A \in \mathcal{E}$, trivially, there is a sequence $\{A_n = A\}$ of elementary sets such that

$$d(A_n, A) = d(A, A) = 0 \xrightarrow{n\to\infty} 0.$$

That is, A (and therefore any elementary set) is finitely Lebesgue measurable. For each $k \in \mathbb{N}$, define the rectangle

$$R_k = \{x \in \mathbb{R}^n : -n < x_i < n\},$$

which is an elementary set and, therefore, a finitely Lebesgue measurable set. Then

$$\mathbb{R}^n = \bigcup_{k\in\mathbb{N}} R_k$$

is a countable union of finitely Lebesgue measurable sets. By definition, this shows $\mathbb{R}^n$ to be a Lebesgue measurable set, i.e., $\mathbb{R}^n \in \mathfrak{M}(m)$.

If $E \subseteq \mathbb{R}^n$ is closed, then $E^C \subseteq \mathbb{R}^n$ is open. Since $\mathfrak{M}(m)$ is a σ-ring, containing E^C and $\mathbb{R}^n$,

$$E = \mathbb{R}^n \setminus E^C \in \mathfrak{M}(m).$$

Equivalently, E is Lebesgue measurable.

□

3 Prove that for any Lebesgue measurable E, any $\varepsilon > 0$, there exists an open set G, a closed set F, such that $F \subseteq E \subseteq G$ and $m(G \setminus F) < \varepsilon$.

Proof. We first prove the result for finitely Lebesgue measurable sets. Let A be finitely Lebesgue measurable and $\varepsilon > 0$. There exists a sequence $\{A_n\}$ of elementary sets such that $A_n \to A$; choose $N \in \mathbb{N}$ such that $d(A_N, A) < \varepsilon$. Then $S(A_N, A)$ and $A_N \cap A$ are disjoint, with the union containing A. So

$$\begin{aligned} m(A) &\leq m(S(A_N, A) \cup (A_N \cap A)) \\ &= m(S(A_N, A)) + m(A_N \cap A) \\ &= d(A_N, A) + m(A_N \cap A) \\ &\leq \varepsilon + m(A_N). \end{aligned}$$

By definition of the Lebesgue outer measure, there is a countable open cover $\{A_j\}$ of A by open elementary sets such that

$$m(A) \leq \sum_{j=1}^{\infty} m(A_j) < m(A) + \varepsilon.$$

Then $G = \bigcup_{j=1}^{\infty} A_j$ is an open set containing A, with

$$m(G \setminus A) = m(G) - m(A) \leq \sum_{j=1}^{\infty} m(A_j) - m(A) < \varepsilon.$$

Now, for any lebesgue measurable E and $\varepsilon > 0$, we can write $E = \bigcup_{j=1}^{\infty} A_j$, where each A_j is a finitely Lebesgue measurable. For each $j \in \mathbb{N}$, by the above result, there is some open set $G_j \supseteq A_j$ such that $m(G_j \setminus A_j) < \varepsilon/2^j$. Then $G = \bigcup_{j=1}^{\infty} G_j$ is an open set containing E with

$$m(G \setminus E) \leq \sum_{j=1}^{\infty} m(G_j \setminus E) \leq \sum_{j=1}^{\infty} m(G_j \setminus A_j) < \sum_{j=1}^{\infty} \frac{\varepsilon}{2^j} = \varepsilon.$$

Since $\mathbb{R}^n$ is Lebesgue measurable, then so is $E^C = \mathbb{R}^n \setminus E$. Then there is some open set H containing E^C such that $m(E^C \setminus H) < \varepsilon$. Note that

$$E^C \setminus H = E^C \cap H^C = H^C \setminus E.$$

Take $F = H^C$, which is a closed set contained in E. Then

$$G \setminus F = ((G \setminus E) \cup E) \setminus F = (G \setminus E) \setminus F \cup (E \setminus F) = (G \setminus E) \cup (E \setminus F),$$

so

$$m(G \setminus F) = m(G \setminus E) + m(E \setminus F) < \varepsilon + \varepsilon = 2\varepsilon.$$

$\square$

4 Prove that if $E \subset \mathbb{R}^n$ is a Lebesgue measurable subset, $x \in \mathbb{R}^n$, then $x + E$ is Lebesgue measurable and $m(E) = m(x + E)$.

Proof. Let $c \in \mathbb{R}^n$.

Suppose $R \subseteq \mathbb{R}^n$ is the rectangle defined by the opposite vertices $a, b \in \mathbb{R}^n$, i.e.,

$$R = \{x \in \mathbb{R}^n : a_i \leq x_i \leq b_i\}.$$

(Note that R need not be a closed rectangle as written above, any combination of '$<$' and '$\leq$' is proved in the same way.) Then we have

$$\begin{aligned} c + R &= \{c + x : x \in R\} \\ &= \{x \in \mathbb{R}^n : x - c \in R\} \\ &= \{x \in \mathbb{R}^n : a_i \leq x_i - c_i \leq b_i\} \\ &= \{x \in \mathbb{R}^n : (a + c)_i \leq x_i \leq (b + c)_i\}. \end{aligned}$$

In other words, $c + R$ is the rectangle defined by the opposite vertices $a + c$ and $b + c$. Moreover,

$$m(c + R) = \prod_{j=1}^{n} [(b + c)_j - (a + c)_j] = \prod_{j=1}^{n} [b_j - a_j] = m(R).$$

For any elementary set $A \in \mathcal{E}$, we have $A = \bigcup_{j=1}^{k} R_j$ where each R_j is a rectangle. Then

$$c + A = c + \bigcup_{j=1}^{k} R_j = \bigcup_{j=1}^{k} (c + R_j),$$

where each $c + R_j$ is a rectangle, implying that $c + A \in \mathcal{E}$. Without loss of generality, we may assume that the rectangles $R_1, \ldots, R_k$ are disjoint. In which case, the rectangles $c + R_1, \ldots, c + R_j$ are also disjoint, and we obtain

$$m(c + A) = \sum_{j=1}^{k} m(c + R_j) = \sum_{j=1}^{k} m(R_j) = m(A).$$

For any finitely Lebesgue measurable subset $A \subseteq \mathbb{R}^n$, there is some sequence $\{A_k\}$ of elementary sets such that $A_k \to A$. Then $\{c + A_k\}$ is a sequence of elementary sets, and we claim $c + A_k \to c + A$. For each $k \in \mathbb{N}$,

$$d(c + A_k, c + A) = m^*(S(c + A_k, c + A)) = m^*(c + S(A_k, A)).$$

The first equality follows by definition and the second follows simply by expanding the symmetric difference in a set-theoretic manner. For any countable cover $\{B_j\}$ of $S(A_k, A)$ by open elementary sets, $\{c + B_j\}$ is countable cover of $c + S(A_k, A)$ by open elementary sets, with

$$m^*(c + S(A_k, A)) \leq \sum_{j=1}^{\infty} m(c + B_j) = \sum_{j=1}^{\infty} m(B_j).$$

Taking the infimum over all covers of $S(A_k, A)$ by open elementary sets, we obtain

$$m^*(c + S(A_k, A)) \leq m^*(S(A_k, A))$$

By the same reasoning,

$$m^*(S(A_k, A)) = m^*(-c + c + S(A_k, A)) \leq m^*(c + S(A_k, A)),$$

giving us

$$d(c + A_k, c + A) = m^*(c + S(A_k, A)) = m^*(S(A_k, A)) = d(A_k, A).$$

Taking the limit as $k \to \infty$, we conclude that $c + A_k \to c + A$. In other words, $c + A$ is finitely Lebesgue measurable, with

$$m(c + A) = \lim_{k\to\infty} m(c + A_k) = \lim_{k\to\infty} m(A_k) = m(A).$$

Lastly, if $A \subseteq \mathbb{R}^n$ is a Lebesgue measurable set, then $A = \bigcup_{k=1}^{\infty} A_k$ where each A_k is finitely Lebesgue measurable. Then

$$c + A = c + \bigcup_{k=1}^{\infty} A_k = \bigcup_{k=1}^{\infty} (c + A_k),$$

where each $c + A_k$ is finitely Lebesgue measurable, implying $c + A$ is Lebesgue measurable. Without loss of generality, we may assume that sets $c + A_k$ are all disjoint (if not, simply define $A_1' = A_1$ and $A_k' = A_k \setminus A_{k-1}'$ for $k \geq 1$; then all A_k' are disjoint and finitely Lebesgue measurable). In which case, we obtain

$$m(c + A) = \sum_{k=1}^{\infty} m(c + A_k) = \sum_{k=1}^{\infty} m(A_k) = m(A).$$

$\square$

5 Prove that the set of all Lebesgue measure 0 subsets is a σ-ring.

Proof. Suppose $A, B \in \mathfrak{M}(m)$ with $m(A) = m(B) = 0$. First,

$$0 \leq m(A \cup B) \leq m(A \cup B) + m(A \cap B) = m(A) + m(B) = 0,$$

implying that $m(A \cup B) = 0$. Second, $A \setminus B \subseteq A$ implies

$$0 \leq m(A \setminus B) \leq m(A) = 0,$$

so $m(A \setminus B) = 0$. This proves that the set of all Lebesgue measure 0 subsets is a ring of sets.

If A_n is a Lebesgue measure 0 set for all $n \in \mathbb{N}$, define $A = \bigcup_{n=1}^{\infty} A_n$ (which is a Lebesgue measurable set), and we find

$$0 \leq m(A) \leq \sum_{n=1}^{\infty} m(A) = \sum_{n=1}^{\infty} 0 = 0.$$

Hence, $m(A) = 0$.

$\square$