3 Let X be a compact metric space and $f : X \to X$ an isometric embedding. Show that f is surjective.

Hint: Suppose not, find a point x_0 not in the image, and consider the sequence $x_0, f(x_0), f(f(x_0)), \ldots$.

Proof. Let $x_0 \in X$ and inductively define a sequence $x_n = f(x_{n-1})$ for all $n \geq 1$. In other words, $x_n = f^n(x_0)$, where f^n denotes f composed with itself n times. Note that for $n, m \in \mathbb{N}$ with $n \leq m$ we have

$$d(x_n, x_m) = d(f^n(x_0), f^m(x_0)) = d(x_0, f^{m-n}(x_0)).$$

Since X is a compact metric space, it is sequentially compact; let $\{x_{n_k}\}_{k\in\mathbb{N}}$ be a convergent subsequence. In particular, this sequence is Cauchy. Given $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $k, \ell \geq N$ we have $d(x_{n_k}, x_{n_\ell}) < \varepsilon$. That is, if $k < \ell$ (so $n_\ell - n_k \geq 1$) then

$$d(x_0, f^{n_\ell - n_k}(x_0)) = d(x_k, x_\ell) < \varepsilon.$$

In particular, we have found the point $x_{n_\ell - n_k} \in B_\varepsilon(x_0) \cap f(X)$. Since we can find such a point for all $\varepsilon > 0$, this means x_0 is a limit point of $f(X)$. Since $f(X)$ is compact, this implies $x_0 \in f(X)$, hence f is surjective. $\square$

4 Define $\mathbb{R}^\infty = \bigcup_{n=1}^\infty \mathbb{R}^n$, where the points

$$(x_1, \ldots, x_n) \quad \text{and} \quad (x_1, \ldots, x_n, 0, \ldots, 0)$$

are identified, for any number of zeros. We topologize $\mathbb{R}^\infty$ as follow: a set in $\mathbb{R}^\infty$ is open iff its intersection with each $\mathbb{R}^n$ is open.

There is an obvious injection $f : \mathbb{R}^\infty \to \ell^\infty$, where $\ell^\infty = C_B(\mathbb{N})$ is the set of bounded sequences of real numbers with the sup norm.

For $a \in \mathbb{R}$ and $r > 0$ denote the interval $I_r(a) = (a - r, a + r) \subseteq \mathbb{R}$.

(a) Show f is continuous.

Proof. Note that ℓ^∞ has a basis of open balls $B_r(x) = \prod_{k=1}^\infty I_r(x_k)$ for $x \in \ell^\infty$ and $r > 0$. It suffices to check the continuity of f on these basis sets.

In order for a point $a \in \mathbb{R}^n$ to be in the preimage $f^{-1}(B_r(x))$, we must have $|a_k - x_k| < r$ for $k = 1, \ldots, n$ and $|x_k| < r$ for all $k > n$. So if it is the case that $|x_k| \geq r$ for some $k > n$, then we know $f^{-1}(B_r(x)) \cap \mathbb{R}^n = \varnothing$, which is open in $\mathbb{R}^n$.

If $|x_k| < r$ for all $i > n$ then

$$f^{-1}(B_r(x)) \cap \mathbb{R}^n = \prod_{k=1}^n I_r(x_k).$$

If we consider $\mathbb{R}^n \subseteq \mathbb{R}^\infty$ with the sup norm (which generates the usual topology), this is simply the open ball of radius r centered at $(x_1, \ldots, x_n) \in \mathbb{R}^n$.

We conclude that $f^{-1}(B_r(x)) \cap \mathbb{R}^n$ is open in $\mathbb{R}^n$ for all n. So in fact $f^{-1}(B_r(x))$ is open in the topology on $\mathbb{R}^\infty$, hence f is continuous. $\square$

(b) Is f an embedding? That is, is the subspace $f(\mathbb{R}^\infty) \subseteq \ell^\infty$ homeomorphic to $\mathbb{R}^\infty$? Justify your answer.

No.

Proof. Consider the set $U = \prod_{k=1}^\infty I_{1/k}(0) \subseteq \mathbb{R}^\infty$. Since $U \cap \mathbb{R}^n = \prod_{k=1}^n I_{1/k}(0)$ is open in $\mathbb{R}^n$ for all $n \in \mathbb{N}$, we know that U is open in $\mathbb{R}^\infty$. However, we will show that $f(U)$ is not open in $f(\mathbb{R}^\infty)$.

Assume for contradiction that $f(U)$ is open in the subspace $f(\mathbb{R}^\infty) \subseteq \ell^\infty$, i.e., there is some open set $V \subseteq \ell^\infty$ such that $f(U) = V \cap f(\mathbb{R}^\infty)$. Then V is an open neighborhood of $0 \in \ell^\infty$, so there is some $\varepsilon > 0$ such that $B_\varepsilon(0) \subseteq V$, implying $B_\varepsilon(0) \cap f(\mathbb{R}^\infty) \subseteq f(U)$.

Choose $n \in \mathbb{N}$ such that $1/n < \varepsilon/2$, and let $x = (0, \ldots, 0, \varepsilon/2) \in \mathbb{R}^n$. Then by construction $x \notin U$ since $x_n \notin I_{1/n}(0)$, so $f(x) \notin f(U)$. On the other hand, we have $f(x) \in B_\varepsilon(0) \cap f(\mathbb{R}^\infty)$, which is a contradiction.

In particular, we conclude that f is not an open map and therefore not a homeomorphism to its image. $\square$

5 (a) Show that a set of open subsets of a topological space X is a basis if and only if it contains a neighborhood basis for every point $x \in X$.

Proof. Let $\mathcal{U}$ be a collection of open subsets of X.

Suppose $\mathcal{U}$ is a basis and $x \in X$ is any point. We claim that $\mathcal{U}_x = \{U \in \mathcal{U} : x \in U\}$ is a neighborhood basis for x. If $N \subseteq X$ is a neighborhood of x it contains an open neighborhood $V \subseteq N$ of x. As $\mathcal{U}$ is a basis, we can write $V = \bigcup_{\alpha \in I} U_\alpha$ for some $U_\alpha \in \mathcal{U}$. Since $x \in V$, we must have $x \in U_\alpha$ for some $\alpha \in I$. Then $U_\alpha \in \mathcal{U}_x$ with $x \in U_\alpha \subseteq V \subseteq N$, hence $\mathcal{U}_x$ is a neighborhood basis for x.

Suppose $\mathcal{U}$ contains a neighborhood basis for every point $x \in X$. Let $V \subseteq X$ be an open subset. For each point $x \in V$ we can choose some $U_x \in \mathcal{U}$ such that $x \in U_x \subseteq V$. Then we can write $V = \bigcup_{x \in V} U_x$, hence $\mathcal{U}$ is a basis. $\square$

(b) Show that every compact totally separated space has a basis of clopen sets.

Proof. Let X be a compact totally separated space. For each pair of distinct points $a, b \in X$ choose some clopen set $A_{a,b} \subseteq X$ such that $a \in A_{a,b}$ and $b \in A_{a,b}^c$. Let $\mathcal{A}$ be the collection of all finite intersections of such clopen sets; we claim that $\mathcal{A}$ is a basis.

As per part (a), it suffices to show $\mathcal{A}$ contains a neighborhood basis for each point. Let $x \in X$ and $U \subseteq X$ be an open neighborhood of x. The collection $\{A_{x,y}^c\}_{y \in U^c}$ forms a clopen cover of U^c. As a closed subset of a compact space, U^c is compact, so we can choose a finite subcover $\{A_i^c\}_{i=1}^n$. Then $U^c \subseteq \bigcup_{i=1}^n A_i^c$, which implies

$$A = \bigcap_{i=1}^n A_i \subseteq \left(\bigcup_{i=1}^n A_i^c\right)^c \subseteq U.$$

And since $x \in A_i$ for $i = 1, \dots, n$, we know that $x \in A$. With $A \in \mathcal{A}$, we conclude that $\mathcal{A}$ contains a neighborhood basis for x and is therefore a basis (of clopen sets). $\square$

6 Prove or disprove:

(a) The arbitrary product of path-connected spaces is path-connected.

Yes.

Proof. Let $\{X_\lambda\}_{\lambda\in\Lambda}$ be a collection of path-connected spaces and $X = \prod_{\lambda\in\Lambda} X_\lambda$ have the product topology with natural projections $\pi_\lambda : X \to X_\lambda$. (Write $x_\lambda = \pi_\lambda(x)$ for each $x \in X$.) Given $x, y \in X$, let γ_λ be a path from x_λ to y_λ in X_λ, i.e., a continuous map $[0,1] \to X_\lambda$ with $\gamma_\lambda(0) = x_\lambda$ and $\gamma_\lambda(1) = y_\lambda$. By the universal property of topological products, there is a continuous map $\gamma : [0,1] \to X$ such that $\pi_\lambda \circ \gamma = \gamma_\lambda$. The fact that $\pi_\lambda(\gamma(0)) = x_\lambda$ and $\pi_\lambda(\gamma(1)) = y_\lambda$ for all $\lambda \in \Lambda$ implies $\gamma(0) = x$ and $\gamma(1) = y$. That is, γ is a path from x to y in X, hence X is path-connected. $\square$

(b) The arbitrary product of locally path-connected spaces is locally path-connected.

No.

With the discrete topology, $\{0,1\}$ is disconnected—and therefore path-disconnected—since each singleton is clopen. However, $\{0,1\}$ is locally path-connected since each singleton is path-connected.

Consider $X = \{0,1\}^{\mathbb{N}}$ with the product topology.

We claim X is totally path-disconnected, i.e., no two distinct points in X have a path between them. If $x, y \in X$ are distinct, we must have $x_n \neq y_n$ for some $n \in \mathbb{N}$; assume $x_n = 0$ and $y_n = 1$. If $\gamma : [0,1] \to X$ were a path from x to y, then $\pi_n \circ \gamma$ would be a path from 0 to 1 in $\{0,1\}$, which is not possible. Therefore, no path from x to y exists in X.

Since no singleton of X is open, every open set must contain at least two points. Thus, no open subset of X is path-connected, so X is not locally path-connected.