Problem 1.1 Let X be the set of all points $(x, y) \in \mathbb{R}^2$ such that $y = \pm 1$, and let M be the quotient of X by the equivalence relation generated by $(x, -1) \sim (x, 1)$ for all $x \neq 0$. Show that M is locally Euclidean and second-countable, but not Hausdorff.

Let $q : X \to M$ be the quotient map.

For nonzero $x \in \mathbb{R}$, denote the point (equivalence class) $\{(x, 1), (x, -1)\} \in M$ by $[x]$.

Denote the origins of M by 0_+ and 0_-.

We first show M is locally Euclidean and second-countable.

Proof. The space M is covered by two open sets: $U_\pm = M \setminus \{0_\mp\}$. We can then construct maps $\varphi_\pm : U_\pm \to \mathbb{R}$ which send $[x] \mapsto x$ for nonzero values of x and send $0_\pm \mapsto 0$. These maps are homeomorphisms, hence provide an atlas for M.

Note that $\mathbb{R}$ is second-countable; let $\mathcal{U}$ be a countable base for $\mathbb{R}$. We can construct a countable base for M by replacing each set in $\mathcal{U}$ which contains the origin by a pair of subsets, one containing 0_+ and one containing 0_-. $\square$

We show that M is not Hausdorff.

Proof. Consider the points $0_\pm \in M$. Let $U \subseteq M$ and $V \subseteq M$ be open neighborhoods of 0_+ and 0_-, respectively. Both U and V correspond to neighborhoods of the origin in $\mathbb{R}$. We can find some $\varepsilon > 0$ small enough that it is contained in both of these neighborhoods. In which case, we the points $[\varepsilon] \in M$ is an element of both U and V, hence the two are not disjoint. $\square$

Problem 1.2 Show that the disjoint union of uncountably many copies of $\mathbb{R}$ is locally Euclidean and Hausdorff, but not second-countable.

Proof. Let I be an uncountable indexing set. For each $\alpha \in I$, let $\mathbb{R}_\alpha$ be a copy of $\mathbb{R}$. Denote the disjoint union space $X = \bigsqcup_{\alpha \in I} \mathbb{R}_\alpha$. By construction, each $\mathbb{R}_\alpha$ is open subset of X homeomorphic to $\mathbb{R}$. Hence, X is locally Euclidean.

For two distinct points $x, y \in X$, either both points are in the same copy of $\mathbb{R}$ or they are in different copies. In the first case, then the fact that $\mathbb{R}$ is Hausdorff allows us to choose a pair of disjoint open neighborhoods. In the second case, the open neighborhoods are simply the respective copies of $\mathbb{R}$, say $\mathbb{R}_\alpha$ and $\mathbb{R}_\beta$. Hence, X is Hausdorff.

Let $\mathcal{U}$ be a base for X. For each $\alpha \in I$, there must be some $U_\alpha \in \mathcal{U}$ such that $U_\alpha \subseteq \mathbb{R}_\alpha$, since $\mathbb{R}_\alpha$ is an open set in X. But then $\{U_\alpha\}_{\alpha \in I}$ is an uncountable collection of distinct sets in $\mathcal{U}$, implying that $\mathcal{U}$ must be uncountable. Hence, no countable base for X exists $\square$

Problem 1.6 Let M be a nonempty topological manifold of dimension $n \geq 1$. If M has a smooth structure, show that it has uncountably many distinct ones. [Hint: first show that for any $s > 0$, $F_s(x) = |x|^{s-1}x$ defines a homeomorphism from B^n to itself, which is a diffeomorphism if and only if $s = 1$.]

Proof. Per the hint, we check the properties of F_s. Since the absolute value and power functions are smooth away from zero, so is F_s. Moreover, $F_{1/s}$ is the smooth inverse of F_s, away from zero. Hence F_s is always a diffeomorphism away from zero. Next, F_s is continuous at zero, since the limit as x approaches zero is indeed zero, so F_s is always a homeomorphism on B^n. In the case that $s = 1$, F_s is the identity and therefore a diffeomorphism on B^n. If $s \neq 1$ then either F_s or its inverse $F_{1/s}$ is not differentiable at the origin, hence F_s is not a diffeomorphism.

Let $\mathcal{U}$ be a smooth (maximal) atlas on M. Fix a point $p \in M$ and a chart $(U, \varphi) \in \mathcal{U}$ at p. After restricting the codomain and scaling, we may assume that the image of φ is the unit ball $B^n \subseteq \mathbb{R}^n$. We now modify $\mathcal{U}$ by removing the point p from all the other charts; denote this new collection of charts by $\mathcal{U}'$. Since U still covers p and the other charts cover the rest of M, $\mathcal{U}'$ is another atlas on M.

For a given $s > 0$, we create a collection of charts $\mathcal{U}_s$ which contains all the same charts at $\mathcal{U}'$. but replacing (U, φ) with $(U, F_s \circ \varphi)$. Now for any other chart $(V, \psi) \in \mathcal{U}_s$, we know that $p \notin U \cap V$. Moreover, F_s is a diffeomorphism away from the origin, so

$$\psi \circ (F_s \circ \varphi)^{-1} = \psi \circ \varphi^{-1} \circ F_s^{-1}$$

is a diffeomorphism by composition from $\varphi(U \cap V)$ to $\psi(U \cap V)$. In other words, $(U, F_s \circ \varphi)$ is smoothly compatible with the rest of the charts in $\mathcal{U}_s$. And since the rest of the charts are unchanged, we know they remain smoothly compatible with each other.

To see that the $\mathcal{U}_s$'s are distinct, we look near the point p. We check to see if the charts $(U, F_s \circ \varphi)$ and $(U, F_t \circ \varphi)$ are smoothly compatible for $s \neq t$. Consider

$$(F_s \circ \varphi) \circ (F_t \circ \varphi)^{-1} = F_s \circ \varphi \circ \varphi^{-1} \circ F_t^{-1} = F_s \circ F_{1/t} = F_{s/t}.$$

Since $s \neq t$, then $s/t \neq 1$, so $F_{s/t}$ is not a diffeomorphism. Hence, $\mathcal{U}$ adn $\mathcal{U}'$ are distinct atlases. □

Problem 1.7 Let N denote the north pole $(0, \ldots, 0, 1) \in S^n \subseteq \mathbb{R}^{n+1}$, and let S denote the south pole $(0, \ldots, 0, -1)$. Define the stereographic projection $\sigma : S^n \setminus \{N\} \to \mathbb{R}^n$ by

$$\sigma(x^1, \ldots, x^{n+1}) = \frac{(x^1, \ldots, x^n)}{1 - x^{n+1}}.$$

Let $\tilde{\sigma}(x) = -\sigma(-x)$ for $x \in S^n \setminus \{S\}$.

(a)

The line through N and x is parallel to the vector

$$x - N = (x^1, \ldots, x^n, x^{n+1} - 1).$$

We parameterize the line as $(x - N)t + N$ for $t \in \mathbb{R}$. We solve for where this line crosses the $x^{n+1} = 0$ plane:

$$(x^{n+1} - 1)t + 1 = 0 \implies t = \frac{1}{1 - x^{n+1}}.$$

The point of intersection is then

$$\frac{(x^1, \ldots, x^n, 0)}{1 - x^{n+1}}.$$

This point is precisely $\sigma(x)$.

The argument is the same for $\tilde{\sigma}$, but with some signs flipped.

(b)

We check that the given function is indeed the inverse of σ.

$$\begin{aligned}
\sigma\left(\frac{(2u^1, \ldots, 2u^n, |u|^2 - 1)}{|u|^2 + 1}\right) &= \frac{(2u^1, \ldots, 2u^n}{|u|^2 + 1} \cdot \frac{1}{1 - \frac{|u|^2 - 1}{|u|^2 + 1}} \\
&= \frac{(2u^1, \ldots, 2u^n)}{|u|^2 + 1 - (|u|^2 - 1)} \\
&= \frac{(2u^1, \ldots, 2u^n)}{2} \\
&= (u^1, \ldots, u^n).
\end{aligned}$$

(c)

(d)