Math 111C Final

- This exam is worth 50 points.
- No collaboration or discussion with others. No search of solutions online.
- Good luck!

1. Let ζ_n be a primitive n-th root of unity and K be a subfield of $\mathbb{Q}(\zeta_n)$.
   (a) (3 points) Show that $K/\mathbb{Q}$ is Galois.
   (b) (3 points) Show that $\mathrm{Gal}(K/\mathbb{Q})$ is abelian.

2. Let $\alpha = \sqrt{5+\sqrt{5}} \in \overline{\mathbb{Q}}$.
   (a) (4 points) Find the minimal polynomial $m_{\alpha,\mathbb{Q}}(x)$.
   (b) (4 points) Show that $\mathbb{Q}(\alpha)/\mathbb{Q}$ is Galois.
   (c) (4 points) Show that there exists $\sigma \in \mathrm{Gal}(\mathbb{Q}(\alpha)/\mathbb{Q})$ such that $\sigma\left(\sqrt{5+\sqrt{5}}\right) = \sqrt{5-\sqrt{5}}$.
   (d) (4 points) Show that the automorphism σ in (c) generates $\mathrm{Gal}(\mathbb{Q}(\alpha)/\mathbb{Q})$.

3. Let K be the splitting field of $x^4 - 5x^2 + 6 \in \mathbb{Q}[x]$.
   (a) (3 points) Show that $K/\mathbb{Q}$ is Galois.
   (b) (5 points) Show that $\mathrm{Gal}(K/\mathbb{Q}) = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.
   (c) (5 points) Find all the subfileds of K.

4. Let $f(x)$ be a polynomial in $\mathbb{Q}[x]$ of degree n and $K \subseteq \overline{\mathbb{Q}}$ be the splitting field of $f(x)$.
   (a) (5 points) Show that if $f(x)$ is irreducible in $\mathbb{Q}[x]$ and $\mathrm{Gal}(K/\mathbb{Q})$ is abelian, then $[K:\mathbb{Q}] = n$.
   (b) (2 points) If $f(x)$ is not necessarily irreducible in $\mathbb{Q}[x]$ and $\mathrm{Gal}(K/\mathbb{Q})$ is abelian, can we deduce that $[K:\mathbb{Q}] \leq n$? Justify your answer.

5. (5 points) Let $K \subseteq \overline{\mathbb{F}}_7$ be the splitting field of $x^3 - \bar{2}$ over $\mathbb{F}_{49}$. Show that $\mathrm{Gal}(K/\mathbb{F}_7) \cong \mathbb{Z}/6\mathbb{Z}$.

6. (3 points) Let $K/\mathbb{Q}$ be a finite field extension with $[K:\mathbb{Q}] = n \geq 3$ and $L \subseteq \overline{\mathbb{Q}}$ be the Galois closure of K over $\mathbb{Q}$. Show that if $\mathrm{Gal}(L/\mathbb{Q}) \cong S_n$, then $\mathrm{Aut}(K/\mathbb{Q}) = \{\mathrm{id}_K\}$.