Exercise 8.7 If $0 < x < \dfrac{\pi}{2}$, prove that

$$\frac{2}{\pi} < \frac{\sin x}{x} < 1.$$

Proof. We have that $\sin x > 0$ on $(0, \pi/2)$, so $\frac{\mathrm{d}^2}{\mathrm{d}x^2} \sin x = -\sin x < 0$ on the same interval. Hence, $\sin x$ is strictly concave on $(0, \pi/2)$, so

$$t = (1-t)\sin 0 + t \sin\frac{\pi}{2} < \sin\Big((1-t)0 + t\frac{\pi}{2}\Big) = \sin\Big(t\frac{\pi}{2}\Big),$$

for $t \in (0,1)$. Take $x = t\pi/2$, we rewrite this as

$$\frac{2}{\pi}x < \sin x \iff \frac{2}{\pi} < \frac{\sin x}{x},$$

for $x \in (0, \pi/2)$.

Since $\sin x$ is differentiable on $[0, \pi/2]$, then the mean value theorem tells us that for every $x \in (0, \pi/2)$, there is some $c_x \in (0, x)$ such that

$$\sin x = \sin x - \sin 0 = (x - 0)\cos c_x = x \cos c_x.$$

Since both sine and cosine are positive on $(0, \pi/2)$ and cosine is strictly decreasing, then $x \cos c_x < x$. So we obtain the remaining inequality

$$\sin x < x \iff \frac{\sin x}{x} < 1.$$

□

Exercise 8.12 Suppose $0 < \delta < \pi$, $f(x) = 1$ if $|x| \leq \delta$, $f(x) = 0$ if $\delta < |x| \leq \pi$, and $f(x + 2\pi) = f(x)$ for all x.

(a) Compute the Fourier coefficients of f.

First, we find

$$\begin{aligned} c_0 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-i0x}\, \mathrm{d}x \\ &= \frac{1}{2\pi} \int_{-\delta}^{\delta} \mathrm{d}x \\ &= \frac{1}{2\pi} \cdot 2\delta \\ &= \frac{\delta}{\pi}. \end{aligned}$$

Then for each nonzero $n \in \mathbb{Z}$, we find

$$\begin{aligned} c_n &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx}\, \mathrm{d}x \\ &= \frac{1}{2\pi} \int_{-\delta}^{\delta} e^{-inx}\, \mathrm{d}x \\ &= \frac{1}{2\pi} \left[\frac{-e^{-inx}}{in} \right|_{-\delta}^{\delta} \\ &= \frac{e^{in\delta} - e^{-in\delta}}{2\pi i n} \\ &= \frac{\sin(n\delta)}{\pi n}. \end{aligned}$$

Note that for nonzero $n \in \mathbb{Z}$, we have

$$c_{-n} = \frac{\sin(-\delta)}{-\pi n} = \frac{\sin(\delta)}{\pi n} = c_n.$$

(b) Conclude that

$$\sum_{n=1}^{\infty} \frac{\sin(n\delta)}{n} = \frac{\pi - \delta}{2}, \quad (0 < \delta < \pi).$$

The above coefficients tell us that

$$s_N(f; 0) = \sum_{n=-N}^{N} c_n e^{in0} = \frac{\delta}{\pi} + 2 \sum_{n=1}^{N} \frac{\sin(n\delta)}{\pi n}.$$

Taking constants $\delta > 0$ and $M = 0$, then for all $t \in (-\delta, \delta)$ we have

$$|f(0+t) - f(0)| = 0 \leq M|t|,$$

so $s_N(f;0) \to f(0) = 1$ as $N \to \infty$. Then

$$\begin{aligned}
1 &= \frac{\delta}{\pi} + 2\sum_{n=1}^{\infty} \frac{\sin(n\delta)}{\pi n} \\
\pi &= \delta + 2\sum_{n=1}^{N} \frac{\sin(n\delta)}{n} \\
\frac{\pi - \delta}{2} &= \sum_{n=1}^{\infty} \frac{\sin(n\delta)}{n}.
\end{aligned}$$

(c) Deduce from Parseval's theorem that

$$\sum_{n=1}^{\infty} \frac{\sin^2(n\delta)}{n^2\delta} = \frac{\pi - \delta}{2}.$$

Parseval's theorem gives us

$$\begin{aligned}
\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2 \,\mathrm{d}x &= \sum_{n=-\infty}^{\infty} |c_n|^2 \\
\frac{1}{2\pi}\int_{-\delta}^{\delta} \mathrm{d}x &= \frac{\delta^2}{\pi^2} + 2\sum_{n=1}^{\infty} \frac{\sin^2(n\delta)}{\pi^2 n^2} \\
\frac{\pi}{2} \cdot 2\delta &= \delta^2 + 2\sum_{n=1}^{\infty} \frac{\sin^2(n\delta)}{n^2} \\
\pi &= \delta + 2\sum_{n=1}^{\infty} \frac{\sin^2(n\delta)}{n^2\delta} \\
\frac{\pi - \delta}{2} &= \sum_{n=1}^{\infty} \frac{\sin^2(n\delta)}{n^2\delta}.
\end{aligned}$$

(d) Let $\delta \to 0$ and prove that

$$\int_0^\infty \left(\frac{\sin x}{x}\right)^2 \mathrm{d}x = \frac{\pi}{2}.$$

Proof. Note that

$$\lim_{x\to 0} \frac{\sin x}{x} = 1,$$

so the function is integrable on $[0, b]$, for any $b > 0$. In particular, the integral over $[0, 1]$ converges, so we will show that the integral over $[1, +\infty)$ converges. If it exists, the improper integral is given by

$$\int_1^\infty \left(\frac{\sin x}{x}\right)^2 \mathrm{d}x = \lim_{b\to\infty} \int_1^b \left(\frac{\sin x}{x}\right)^2 \mathrm{d}x\,.$$

First, we bound the improper integral

$$\int_1^\infty \frac{\mathrm{d}x}{x^2} \le \sum_{n=1}^\infty \frac{1}{n^2}.$$

The series on the right converges since the power of n is greater than 1, implying that the integral converges. Then for any $b > 1$, we have

$$0 < \int_1^b \left(\frac{\sin x}{x}\right)^2 \mathrm{d}x \le \int_1^b \frac{\mathrm{d}x}{x^2} < \int_1^\infty \frac{\mathrm{d}x}{x^2}.$$

The right side is a finite constant, so letting $b \to \infty$ gives us convergence of the integral over $[1, +\infty)$. Hence, the improper integral

$$\int_0^\infty \left(\frac{\sin x}{x}\right)^2 \mathrm{d}x$$

converges. Then for any sequence of partitions with diameter tending to zero, the sequence of Riemann sums converges to the integral. In particular, for $k \in \mathbb{N}$, let $\delta_k = 1/k$. Define the partition of $[0, +\infty)$ by $x_n = n\delta_k$. Then

$$\begin{aligned}
\int_0^\infty \left(\frac{\sin x}{x}\right)^2 \mathrm{d}x &= \lim_{k\to\infty} \sum_{n=1}^\infty \left(\frac{\sin x_n}{x_n}\right)^2 \delta_k \\
&= \lim_{k\to\infty} \sum_{n=1}^\infty \frac{\sin^2(n\delta_k)}{n^2\delta_k} \\
&= \lim_{k\to\infty} \frac{\pi - \delta_k}{2} \\
&= \frac{\pi}{2}.
\end{aligned}$$

□

(e) Put $\delta = \pi/2$ in (c). What do you get?

$$\sum_{n=1}^{\infty} \frac{\sin^2(n\pi/2)}{n^2\pi/2} = \frac{\pi - \pi/2}{2}$$
$$\sum_{n=1}^{\infty} \frac{\sin^2(n\pi/2)}{n^2} = \frac{1}{2}.$$

Note that $\sin(\pi k) = 0$ for all $k \in \mathbb{N}$, so the even terms in the above series are all zero, giving us

$$\sum_{n=0}^{\infty} \frac{\sin^2((2n+1)\pi/2)}{(2n+1)^2} = \frac{1}{2}.$$

Then $\sin((2n+1)\pi/2) = \pm 1$, so its square is 1, and we obtain

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \frac{1}{2}.$$

Exercise 8.13 Put $f(x) = x$ if $0 \leq x < 2\pi$, and apply Parseval's theorem to conclude that
$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

For this problem, we will integrate over $[0, 2\pi]$ instead of $[-\pi, \pi]$, so that $f(x) = x$. First,
$$\begin{aligned} c_0 &= \frac{1}{2\pi} \int_0^{2\pi} f(x) e^{-i0x} \, \mathrm{d}x \\ &= \frac{1}{2\pi} \int_0^{2\pi} x \, \mathrm{d}x \\ &= \frac{1}{2\pi} \cdot \frac{4\pi^2}{2} \\ &= \pi. \end{aligned}$$

Then for nonzero $n \in \mathbb{Z}$, we compute
$$c_n = \frac{1}{2\pi} \int_0^{2\pi} x e^{-inx} \, \mathrm{d}x \, .$$

For integration by parts, take
$$u(x) = x \quad \text{and} \quad v(x) = \frac{-1}{in} e^{-inx},$$
so $u'(x) = 1$ and $v'(x) = e^{-inx}$. Then
$$\begin{aligned} \int_0^{2\pi} x e^{-inx} \, \mathrm{d}x &= \left[\frac{-x e^{-inx}}{in} \right|_0^{2\pi} - \int_0^{2\pi} e^{-inx} \, \mathrm{d}x \\ &= \frac{-2\pi e^{-2\pi i n}}{in} + \frac{0 e^{-in0}}{in} - 0 \\ &= \frac{2\pi i}{n}. \end{aligned}$$

So $c_n = i/n$. Then by Parseval's theorem,
$$\begin{aligned} \sum_{n=-\infty}^{\infty} |c_n|^2 &= \frac{1}{2\pi} \int_0^{2\pi} |f(x)|^2 \, \mathrm{d}x \\ \pi^2 + 2 \sum_{n=1}^{\infty} \frac{1}{n^2} &= \frac{1}{2\pi} \int_0^{2\pi} x^2 \, \mathrm{d}x \\ \sum_{n=1}^{\infty} \frac{1}{n^2} &= \frac{1}{4\pi} \cdot \frac{8\pi^3}{3} - \frac{\pi^2}{2} \\ \sum_{n=1}^{\infty} \frac{1}{n^2} &= \frac{\pi^2}{6}. \end{aligned}$$

Exercise 8.14 If $f(x) = (\pi - |x|)^2$ on $[-\pi, \pi]$, prove that

$$f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2} \cos nx$$

and deduce that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}, \quad \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}.$$

First,

$$\begin{aligned} c_0 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-i0x} \,\mathrm{d}x \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} (\pi - |x|)^2 \,\mathrm{d}x \\ &= \frac{1}{\pi} \int_{0}^{\pi} (\pi - x)^2 \,\mathrm{d}x \,. \end{aligned}$$

For change of variables, take $t = \pi - x$, so

$$c_0 = \frac{1}{\pi} \int_0^{\pi} t^2 \,\mathrm{d}t = \frac{1}{\pi} \cdot \frac{\pi^3}{3} = \frac{\pi^2}{3}.$$

Now for nonzero $n \in \mathbb{Z}$, we find

$$\begin{aligned} c_n &= \frac{1}{2\pi} \int_{-\pi}^{\pi} (\pi - |x|)^2 e^{-inx} \,\mathrm{d}x \\ &= \frac{1}{2\pi} \left(\int_{-\pi}^{0} (\pi + x)^2 e^{-inx} \,\mathrm{d}x + \int_{0}^{\pi} (\pi - x)^2 e^{-inx} \,\mathrm{d}x \right) \\ &= \frac{1}{2\pi} \left(\int_{0}^{\pi} (\pi - x)^2 e^{inx} \,\mathrm{d}x + \int_{0}^{\pi} (\pi - x)^2 e^{-inx} \,\mathrm{d}x \right) \\ &= \frac{1}{2\pi} \int_{0}^{\pi} (\pi - x)^2 \big(e^{inx} + e^{-inx} \big) \,\mathrm{d}x \\ &= \frac{1}{\pi} \int_{0}^{\pi} (\pi - x)^2 \cos(nx) \,\mathrm{d}x \\ &= \pi \int_{0}^{\pi} \cos(nx) \,\mathrm{d}x - 2 \int_{0}^{\pi} x \cos(nx) \,\mathrm{d}x + \frac{1}{\pi} \int_{0}^{\pi} x^2 \cos(nx) \,\mathrm{d}x \\ &= 0 - 2 \int_{0}^{\pi} x \cos(nx) \,\mathrm{d}x + \frac{1}{\pi} \int_{0}^{\pi} x^2 \cos(nx) \,\mathrm{d}x \,. \end{aligned}$$

For integration by parts, take

$$u(x) = x \quad \text{and} \quad v(x) = \frac{\sin(nx)}{n},$$

so $u'(x) = 1$ and $v'(x) = \cos(nx)$. Then

$$\begin{aligned}
\int_0^{\pi} x\cos(nx)\,\mathrm{d}x &= \left[\frac{x\sin(nx)}{n}\right|_0^{\pi} - \frac{1}{n}\int_0^{\pi}\sin(nx)\,\mathrm{d}x \\
&= \frac{\pi\sin(n\pi)}{n} - \frac{1}{n}\left[\frac{-\cos(nx)}{n}\right|_0^{\pi} \\
&= 0 + \frac{\cos(n\pi) - \cos(n0)}{n^2} \\
&= \frac{\cos(n\pi) - 1}{n^2}.
\end{aligned}$$

For integration by parts, take

$$u(x) = x^2 \quad \text{and} \quad v(x) = \frac{\sin(nx)}{n},$$

so $u'(x) = 2x$ and $v'(x) = \cos(nx)$. Then

$$\begin{aligned}
\int_0^{\pi} x^2\cos(nx)\,\mathrm{d}x &= \left[\frac{x^2\sin(nx)}{n}\right|_0^{\pi} - \frac{2}{n}\int_0^{\pi} x\sin(nx)\,\mathrm{d}x \\
&= \frac{\pi^2\sin(n\pi)}{n} - \frac{2}{n}\int_0^{\pi} x\sin(nx)\,\mathrm{d}x \\
&= 0 - \frac{2}{n}\int_0^{\pi} x\sin(nx)\,\mathrm{d}x\,.
\end{aligned}$$

For integration by parts, take

$$u(x) = x \quad \text{and} \quad v(x) = \frac{-\cos(nx)}{n},$$

so $u'(x) = 1$ and $v'(x) = \sin(nx)$. Then

$$\begin{aligned}
\int_0^{\pi} x^2\cos(nx)\,\mathrm{d}x &= \frac{-2}{n}\left(\left[\frac{-x\cos(nx)}{n}\right|_0^{\pi} + \frac{1}{n}\int_0^{\pi}\cos(nx)\,\mathrm{d}x\right) \\
&= \frac{-2}{n}\left(\frac{-\pi\cos(n\pi)}{n} + 0\right) \\
&= \frac{2\pi\cos(nx)}{n^2}.
\end{aligned}$$

Now, we have

$$c_n = -2\cdot\frac{\cos(n\pi) - 1}{n^2} + \frac{1}{\pi}\cdot\frac{2\pi\cos(nx)}{n^2} = \frac{2}{n^2}.$$

In particular, we can see that $c_n = c_{-n}$. So we have the Fourier expansion

$$\sum_{n=-\infty}^{\infty} c_n e^{inx} = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{2}{n^2}\left(e^{inx} + e^{-inx}\right) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4}{n^2}\cos(nx).$$

We now show that $f(x)$ is equal to its Fourier expansion on $[-\pi, \pi]$. First, f is differentiable for $0 < |x| < \pi$, with $|f'(x)| \leq 2\pi$. Therefore, f is Lipschitz on both $(-\pi, 0)$ and $(0, \pi)$, implying that $f(x)$ equals its Fourier expansion at these points. It remains to show that the conditions of Theorem 8.14 are satisfied at the points $0, \pm\pi$. Note that f is at least continuous at these points, but not necessarily differentiable. For 0 and any nonzero $t \in (-\pi, \pi)$, the interval from 0 to t is contained in either $[-\pi, 0]$ or $[0, \pi]$. In either case, we use the continuity of f on these intervals and the differentiability on the open subintervals to select a point c between 0 and t such that

$$f(t) - f(0) = f'(c)(t - 0).$$

Taking the absolute value, we obtain

$$|f(0 + t) - f(0)| = |f'(c)||t| \leq 2\pi|t|.$$

Hence, $f(0)$ is equal to its Fourier expansion. By similar argument, f is continuous at $\pm\pi$, and differentiable in a radius of π around each point, with derivative bounded by 2π. Thus,

$$f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2} \cos(nx), \qquad x \in [-\pi, \pi].$$

Evaluating at $x = 0$, we find

$$\begin{aligned} f(0) &= \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2} \cos(n0) \\ \pi^2 &= \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{1}{n^2} \\ \frac{\pi^2}{6} &= \sum_{n=1}^{\infty} \frac{1}{n^2}. \end{aligned}$$

Applying Parseval's theorem, we find

$$\begin{aligned} \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 \, \mathrm{d}x &= \sum_{n=-\infty}^{\infty} |c_n|^2 \\ \frac{1}{\pi} \int_{0}^{\pi} (\pi - x)^4 \, \mathrm{d}x &= \frac{\pi^4}{9} + 2\sum_{n=1}^{\infty} \frac{2^2}{n^4} \\ \frac{1}{\pi} \left[\frac{-(\pi - x)^5}{5} \right|_0^{\pi} - \frac{\pi^4}{9} &= 8\sum_{n=1}^{\infty} \frac{1}{n^4} \\ \frac{1}{\pi} \cdot \frac{\pi^5}{5} - \frac{\pi^4}{9} &= 8\sum_{n=1}^{\infty} \frac{1}{n^4} \\ \frac{\pi^4}{90} &= \sum_{n=1}^{\infty} \frac{1}{n^4}. \end{aligned}$$

Exercise 8.15 With D_n as defined in (77), put

$$K_N(x) = \frac{1}{N+1}\sum_{n=0}^{N} D_n(x).$$

Prove that

$$K_N(x) = \frac{1}{N+1}\cdot\frac{1-\cos(N+1)x}{1-\cos x}$$

and that

Let $z = e^{ix}$, then the Dirichlet kernel is given by

$$D_n(x) = \frac{\sin(n+1/2)x}{\sin(x/2)} = \frac{z^{n+1/2} - z^{-n-1/2}}{z^{1/2} - z^{-1/2}} = \frac{z^{-n} - z^{n+1}}{1-z}.$$

Then

$$\begin{aligned}
(N+1)K_N(x) &= \sum_{n=0}^{N} \frac{z^{-n} - z^{n+1}}{1-z} \\
&= \frac{1}{1-z}\left(\sum_{n=0}^{N} z^{-n} - z\sum_{n=0}^{N} z^{n}\right) \\
&= \frac{1}{1-z}\left(\frac{1-z^{-(N+1)}}{1-z^{-1}} - z\cdot\frac{1-z^{N+1}}{1-z}\right) \\
&= \frac{-z}{1-z}\left(\frac{1-z^{-(N+1)}}{1-z} + \frac{1-z^{N+1}}{1-z}\right) \\
&= \frac{1}{1-z^{-1}}\cdot\frac{2-\left(z^{N+1}+z^{-(N+1)}\right)}{1-z} \\
&= \frac{2-2\cos(N+1)x}{2-(z+z^{-1})} \\
&= \frac{1-\cos(N+1)x}{1-\cos x}.
\end{aligned}$$

Hence,

$$K_N(x) = \frac{1}{N+1}\cdot\frac{1-\cos(N+1)x}{1-\cos x}.$$

(a) $K_N \geq 0$

Since $\cos x \leq 1$ for all $x \in \mathbb{R}$, then we must also have $1 - \cos x \geq 0$. So in fact

$$K_N(x) = \frac{1}{N+1}\cdot\frac{1-\cos(N+1)x}{1-\cos x} \geq 0.$$

(b)

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} K_N(x)\,\mathrm{d}x = 1,$$

First, we see that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} D_n(x)\,\mathrm{d}x = \sum_{k=-n}^{n}\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ikx}\,\mathrm{d}x = 1.$$

So

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} K_N(x)\,\mathrm{d}x = \frac{1}{N+1}\sum_{n=0}^{N}\frac{1}{2\pi}\int_{-\pi}^{\pi} D_n(x)\,\mathrm{d}x = \frac{1}{N+1}\sum_{n=0}^{N} 1 = 1.$$

(c)

$$K_N(x) \le \frac{1}{N+1}\cdot\frac{2}{1-\cos\delta} \quad \text{if } 0 < \delta \le |x| \le \pi.$$

Since $\cos x$ is decreasing on $[0,\pi]$, then $0 < \delta \le |x| \le \pi$ implies $\cos x \le \cos\delta$. Additionally, since $\cos x \ge -1$ for all $x \in \mathbb{R}$, then $1 - \cos x \le 2$. So

$$K_N(x) = \frac{1}{N+1}\cdot\frac{1-\cos(N+1)x}{1-\cos x} \le \frac{1}{N+1}\cdot\frac{2}{1-\cos\delta}.$$

If $s_N = s_N(f;x)$ is the Nth partial sum of the Fourier series of f, consider the arithmetic means

$$\sigma_N = \frac{s_0 + s_1 + \cdots + s_N}{N+1}.$$

Prove that

$$\sigma_N(f;x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)K_N(t)\,\mathrm{d}t\,,$$

We have that

$$s_n(f;x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)D_n(t)\,\mathrm{d}t\,,$$

so

$$\begin{aligned}
\sigma_N(f;x) &= \frac{1}{N+1}\sum_{n=0}^{N} s_n(f;x) \\
&= \frac{1}{N+1}\sum_{n=0}^{N}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)D_n(t)\,\mathrm{d}t\right) \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)\left(\frac{1}{N+1}\sum_{n=0}^{N} D_n(t)\right)\mathrm{d}t \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)K_N(t)\,\mathrm{d}t\,.
\end{aligned}$$

and hence prove Fejér's theorem: If f is continuous, with period 2π, then $\sigma_N(f;x) \to f(x)$ uniformly on $[-\pi,\pi]$. Hint: use properties (a), (b), (c), to proceed as in Theorem 7.26.

Proof. Since f is continuous on the compact interval $[-\pi,\pi]$, then it is uniformly continuous. Additionally f is bounded; let $M > 0$ such that $|f(x)| \le M$ for all $x \in [-\pi,\pi]$.

Let $\varepsilon > 0$ be given. To show uniform convergence, we want to find some $N \in \mathbb{N}$ such that

$$n \ge N, \implies |\sigma_n(f;x) - f(x)| < \varepsilon,$$

for all $x \in [-\pi,\pi]$. Using (a) and (b), we find

$$\begin{aligned}
|\sigma_n(f;x) - f(x)| &= \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-t) K_n(t)\, \mathrm{d}t - f(x) \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} K_n(t)\, \mathrm{d}t \right) \right| \\
&= \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} \big[f(x-t) - f(x) \big] K_n(t)\, \mathrm{d}t \right| \\
&\le \frac{1}{2\pi} \int_{-\pi}^{\pi} \big| f(x-t) - f(x) \big| K_n(t)\, \mathrm{d}t\,.
\end{aligned}$$

By the uniform continuity of f, let $\delta > 0$ such that

$$|x - y| < \delta \implies |f(x) - f(y)| < \frac{\varepsilon}{2}.$$

Additionally, assume $\delta < \pi$. We will estimate the above integral in two parts: first for $|t| \le \delta$ and second for $\delta \le |t| \le \pi$. Using (a) and the choice of δ, we find

$$\begin{aligned}
\frac{1}{2\pi} \int_{-\delta}^{\delta} \big| f(x-t) - f(x) \big| K_n(t)\, \mathrm{d}t &\le \frac{1}{2\pi} \int_{-\delta}^{\delta} \frac{\varepsilon}{2} K_n(t)\, \mathrm{d}t \\
&\le \frac{\varepsilon}{2} \cdot \frac{1}{2\pi} \int_{-\pi}^{\pi} K_n(t)\, \mathrm{d}t \\
&= \frac{\varepsilon}{2}.
\end{aligned}$$

Now using (c) and the choice of M, we find

$$\begin{aligned}
\frac{1}{2\pi} \int_{\delta \le |t| \le \pi} \big| f(x-t) - f(x) \big| K_n(t)\, \mathrm{d}t &\le \frac{1}{2\pi} \int_{\delta \le |t| \le \pi} 2M \cdot \frac{1}{n+1} \cdot \frac{2}{1 - \cos\delta}\, \mathrm{d}t \\
&\le \frac{1}{n+1} \cdot \frac{4M}{1 - \cos\delta} \cdot \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathrm{d}t \\
&= \frac{1}{n+1} \cdot \frac{4M}{1 - \cos\delta}.
\end{aligned}$$

Then we can choose $N \in \mathbb{N}$ such that

$$\frac{1}{N} < \frac{\varepsilon}{2} \cdot \frac{1 - \cos\delta}{4M}.$$

In which case, for any $n \geq N$ and $x \in [-\pi, \pi]$ we have

$$|\sigma_n(f;x) - f(x)| \leq \frac{\varepsilon}{2} + \frac{1}{n+1} \cdot \frac{4M}{1-\cos\delta} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus, $\sigma_n(f;x) \to f(x)$ uniformly on $[-\pi, \pi]$.

$\square$

Exercise 8.17 Assume f is bounded and monotonic on $[-\pi, \pi)$, with Fourier coefficients c_n, as given by (62).

(a) Use Exercise 17 of Chap. 6 to prove that $\{nc_n\}$ is a bounded sequence.

By definition, we have

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} \, \mathrm{d}x \,.$$

We take $g(x) = e^{-inx}$ and $G(x) = ie^{-inx}/n$ for Exercise 6.17, giving us

$$\begin{aligned}\left| \int_{-\pi}^{\pi} f(x) e^{-inx} \, \mathrm{d}x \right| &= \left| f(\pi)G(\pi) - f(-\pi)G(-\pi) - \int_{-\pi}^{\pi} G \, \mathrm{d}f \right| \\ &\leq |f(\pi)G(-\pi)| + |f(-\pi)G(\pi)| + \int_{-\pi}^{\pi} |G||\mathrm{d}f|.\end{aligned}$$

Note that $|G(x)| = 1/n$ for all real x, so

$$\begin{aligned}\left| \int_{-\pi}^{\pi} f(x) e^{-inx} \, \mathrm{d}x \right| &\leq \frac{1}{n}|f(\pi)| + \frac{1}{n}|f(-\pi)| + \frac{1}{n}|f(\pi) - f(-\pi)| \\ &\leq \frac{2|f(\pi)| + 2|f(-\pi)|}{n}.\end{aligned}$$

Hence,

$$|nc_n| \leq n \cdot \frac{1}{2\pi} \cdot \frac{2|f(\pi)| + 2|f(-\pi)|}{n} = \frac{2|f(\pi)| + 2|f(-\pi)|}{\pi},$$

which is a uniform bound of the terms by a constant.

(b) Combine (a) with Exercise 16 and with Exercise 14(e) of Chap. 3, to conclude that

$$\lim_{N \to \infty} s_N(f; x) = \tfrac{1}{2}[f(x+) + f(x-)]$$

for every x.

Given x, we take $a_n = c_n e^{inx}$. Then $|na_n| = |c_n|$ is bounded as shown in (a). Let $s_N = s_N(f; x)$ and $\sigma_N = \sigma_N(f; x)$. Then from Exercises 3.14(e) and 8.16, we know that

$$\lim_{N \to \infty} s_N(f; x) = \lim_{N \to \infty} \sigma_N(f; x) = \tfrac{1}{2}[f(x+) + f(x-)].$$

(c) Assume only that $f \in \mathcal{R}$ on $[-\pi, \pi]$ and that f is monotonic in some segment $(\alpha, \beta) \subset [-\pi, \pi]$. Prove that the conclusion of (b) holds for every $x \in (\alpha, \beta)$. (This is an application of the localization theorem.)

Since $f \in \mathcal{R}[-\pi, \pi]$, then f is bounded on $[-\pi, \pi]$. In particular, f is bounded on (α, β); define $a = \inf_{(\alpha,\beta)} f$ and $b = \sup_{(\alpha,\beta)} f$. Then define the function

$$g(x) = \begin{cases} a & x \in [-\pi, \alpha], \\ f(x) & x \in (\alpha, \beta), \\ b & x \in [\alpha, \pi]. \end{cases}$$

Then $g(x) = f(x)$ for all $x \in (\alpha, \beta)$, so the localization theorem tells us that

$$\lim_{N\to\infty} s_N(f; x) = \lim_{N\to\infty} s_N(g; x), \quad x \in (\alpha, \beta).$$

Additionally, g is monotonic on $[-\pi, \pi]$, so for any $x \in (\alpha.\beta)$ we find

$$\begin{aligned} \lim_{N\to\infty} s_N(f; x) &= \lim_{N\to\infty} s_N(g; x) \\ &= \tfrac{1}{2}[g(x+) + g(x-)] \\ &= \tfrac{1}{2}[f(x+) + f(x-)]. \end{aligned}$$