**1** A space is *Lindelöf* if every open cover has a countable subcover. Show that a Lindelöf metric space is second-countable.

*Proof.* Let $(X, d)$ be a Lindelöf metric space. For each $n \in \mathbb{N}$, the set of balls $\{B_{1/n}(x)\}_{x \in X}$ forms an open cover of $X$. Then there is a choice of points $x_{n,k} \in X$ for $k \in \mathbb{N}$ such that the collection $\{B_{1/n}(x_{n,k})\}_{k \in \mathbb{N}}$ is a countable subcover. We claim that the countable collection

$$\mathcal{B} = \{B_{1/n}(x_{n,k})\}_{n,k \in \mathbb{N}}$$

is a basis for $X$. Let $x \in X$ and $U \subseteq X$ be an open neighborhood of $x$. Then $B_r(x) \subseteq U$ for some radius $r > 0$. Choose $N \in \mathbb{N}$ such that $1/N < r/2$, then $x \in B_{1/N}(x_{N,k}) \in \mathcal{B}$ for some $k \in \mathbb{N}$. For all $y \in B_{1/N}(x_{N,k})$ we compute

$$d(x, y) \leq d(x, x_{N,k}) + d(x_{N,k}, y) < \frac{1}{N} + \frac{1}{N} < r,$$

so $y \in B_r(x)$, implying $B_{1/N}(x_{N,k}) \subseteq B_r(x)$. This proves $\mathcal{B}$ is a basis, hence $X$ is Lindelöf. $\square$

**2** Show that $\mathcal{C}([0,1])$ with the sup norm is separable (and therefore second-countable).

**Hint:** This is equivalent to showing that every continuous function $[0,1] \to \mathbb{R}$ can be approximated arbitrarily closely by functions from some countable set.

*Proof.* By the Weierstrass approximation theorem, every function in $\mathcal{C}([0,1])$ can be uniformly approximated within an arbitrary distance by a polynomial in $\mathbb{R}[x]$. We will show that every polynomial in $\mathbb{R}[x]$ can be uniformly approximated within an arbitrary distance by a polynomial in $\mathbb{Q}[x]$.

Let $f \in \mathbb{R}[x]$, i.e., $f = \sum_{k=0}^{d} a_k x^k$ where $d = \deg f$ and $a_k \in \mathbb{R}$. Given $\varepsilon > 0$, we can choose $q_k \in \mathbb{Q}$ such that $|a_k - q_k| < \varepsilon/(d+1)$ for $k = 0, \ldots, d$. Then $g = \sum_{k=0}^{d} q_k x^k$ is a polynomial in $\mathbb{Q}[x]$. For all $x \in [0,1]$, we find

$$\begin{aligned}
|f(x) - g(x)| &= \left| \sum_{k=0}^{d} (a_k - q_k) x^k \right| \\
&\leq \sum_{k=0}^{d} |a_k - q_k||x|^k \\
&< \sum_{k=0}^{d} \frac{\varepsilon}{d+1} \cdot 1 \\
&= \varepsilon.
\end{aligned}$$

Hence, polynomials in $\mathbb{R}[x]$ can be uniformly approximated within an arbitrary distance by polynomials in $\mathbb{Q}[x]$.

Given $f \in \mathcal{C}([0,1])$ and $\varepsilon > 0$, we can choose $g \in \mathbb{R}[x]$ such that $\|f - g\| < \varepsilon/2$ and $h \in \mathbb{Q}[x]$ such that $\|g - h\| < \varepsilon/2$. Then

$$\|f - h\| \leq \|f - g\| + \|g - h\| < \varepsilon.$$

Hence, functions in $\mathcal{C}([0,1])$ can be uniformly approximated within an arbitrary distance by polynomials in $\mathbb{Q}[x]$. In other words, $\mathbb{Q}[x]$ is a countable dense subset of $\mathcal{C}([0,1])$, so $\mathcal{C}([0,1])$ is separable. $\square$

**3** Let $\{X_\lambda\}_{\lambda\in\Lambda}$ be an indexed family of connected spaces. In this problem you will show that their product $X = \prod_{\lambda\in\Lambda} X_\lambda$ is connected.

**Lemma 1.** If $Y$ and $Z$ are connected subspaces of a topological space such that $Y \cap Z \neq \varnothing$, then $Y \cup Z$ is connected.

*Proof.* Assume—for contradiction—that $Y \cup Z$ is disconnected, i.e., $Y \cup Z = U \cup V$ where $U, V \subseteq Y \cup Z$ are nonempty disjoint open sets. We can write

$$Y = (Y \cup Z) \cap Y = (U \cap Y) \cup (V \cap Y),$$

where $U \cap Y$ and $V \cap Y$ are disjoint open subsets of $Y$. Similarly,

$$Z = (Y \cup Z) \cap Z) = (U \cap Z) \cup (V \cap Z),$$

where $U \cap Z$ and $V \cap Z$ are disjoint open subsets of $Z$.

Since $Y$ is connected, we can assume without loss of generality that

$$U \cap Y = Y \quad \text{and} \quad V \cap Y = \varnothing.$$

This implies $V \subseteq Z$, i.e., $V \cap Z = V$. Since $V$ is nonempty and $Z$ is connected,

$$V \cap Z = Z \quad \text{and} \quad V \cap Z = \varnothing.$$

This now tells us $Y = U$ and $Z = V$. However, this is a contradiction since we assumed $Y$ and $Z$ to overlap but $U$ and $V$ to be disjoint. $\square$

**(a)** Fix a point $\mathbf{a} = (a_\lambda)_{\lambda\in\Lambda} \in X$. Given a finite subset $K \subset \Lambda$, let $X_K$ denote the subspace of points in $X$ whose coordinates are all $a_\lambda$ except perhaps for $\lambda \in K$. Show that $X_K$ is connected.

*Proof.* Given $\lambda_0 \in \Lambda$ the space $X_{\lambda_0}$ is homeomorphic to the subspace $X_{\{\lambda_0\}} \subseteq X$. Explicitly, for each point $x \in X_{\lambda_0}$, there is a point $(x_\lambda)_{\lambda\in\Lambda} \in X$ with $x_{\lambda_0} = x$ and $x_\lambda = a_\lambda$ for all $\lambda \neq \lambda_0$. Then the map $x \mapsto (x_\lambda)_{\lambda\in\Lambda}$ is the desired homeomorphism. In particular, $X_{\{\lambda_0\}}$ is connected since $X_{\lambda_0}$ is connected.

Note that $X_K$ is the finite union of connected subspaces $X_{\{\lambda\}}$ for $\lambda \in K$, which all contain the point $\mathbf{a}$. As an immediate corollary to Lemma 1, the finite union of overlapping connected subspaces is connected, hence $X_K$ is connected. $\square$

**(b)** Show that the union $Y$ of all the $X_K$ is connected.

*Proof.* Assume—for contradiction—that $Y$ is disconnected. Then $Y = U \cup V$ where $U$ and $V$ are nonempty disjoint open subsets of $Y$. Let $x \in U$ and $y \in V$. By construction of $Y$, we have $x \in X_{K_1}$ and $y \in X_{K_2}$ for some finite subsets $K_1, K_2 \subseteq \Lambda$. Then $K = K_1 \cup K_2$ is also a finite subsets of $\Lambda$ with $x, y \in X_K$. However, we also have

$$X_K = Y \cap X_K = (U \cap X_K) \cup (V \cap X_K),$$

where $U \cap X_K$ and $V \cap X_K$ are disjoint open subsets of $X_K$. And since $x \in U \cap X_K$ and $y \in V \cap X_K$, then we conclude that $X_K$ is disconnected. However, this contradicts part (a) which tells us that $X_K$ must be connected. $\square$

**(c)** Show that the closure of a connected subset of any space is connected.

*Proof.* Let $C$ be a connected subset of a space $Z$. If $C = \varnothing$, then the result is trivial, so we assume $C$ is nonempty.

Assume—for contradiction—that $\overline{C}$ is disconnected. Then $\overline{C} = U \cup V$ for some nonempty disjoint open subsets $U, V \subseteq \overline{C}$. Without loss of generality, assume $U \cap C \neq \varnothing$. If $V \cap C$ were also nonempty, then we could write $C = (U \cap C) \cup (V \cap C)$, with $U$ and $V$ being nonempty disjoint open subsets of $C$. Since $C$ is connected, this is not possible, so $V \cap C$ must be empty.

Since $V$ is nonempty, we can choose some $x \in V$. Since $\overline{C} \subseteq Z$ has the subspace topology, we have $V = \overline{C} \cap W$ for some open set $W \subseteq Z$. But then $W$ is an open neighborhood of $x$ outside of $C$. This is a contradiction since we assumed $x \in V \subseteq \overline{C}$. $\square$

**(d)** Show that the closure of $Y$ is $X$. Conclude that $X$ is connected.

*Proof.* Let $x \in X$ and consider an open neighborhood $U$ of $x$. Without loss of generality, we may assume $U$ is in the basis for the product topology on $X$. This means there is a finite subset $K \subseteq \Lambda$ and open neighborhoods $U_\lambda$ of $x_\lambda$ for each $\lambda \in K$ such that

$$U = \prod_{\lambda \in K} U_\lambda \times \prod_{\lambda \in \Lambda \setminus K} X_\lambda.$$

Then $U$ and $Y$ overlap at some point $(y_\lambda)_{\lambda \in \Lambda} \in X_K$, chosen such that $y_\lambda = a_\lambda$ for $\lambda \in \Lambda \setminus K$ and $y_\lambda \in U_\lambda$ for $\lambda \in K$. This implies $x \in \overline{Y}$, so in fact $\overline{Y} = X$. Applying part (c), we conclude that $X$ is connected. $\square$

**4**

**(a)** Show that if $X$ is a first-countable space, then for every $A \subset X$, every point in the closure of $A$ is the limit of a sequence in $A$.

*Proof.* Let $x \in \overline{A}$ and let $\{U_n\}_{n\in\mathbb{N}}$ be a countable neighborhood basis for $x$ with $U_n \supseteq U_{n+1}$. Since $x \in \overline{A}$, we can choose a point $x_n \in U_n \cap A$. We check that $x_n \to x$. For any open neighborhood $U$ of $x$, there is some $N \in \mathbb{N}$ such that $U_N \subseteq U$. Moreover, for any $n \geq N$ we have

$$x_n \in U_n \subseteq U_N \subseteq U,$$

hence $x_n \to x$. $\square$

**(b)** Using the previous problem, show that this is not true for spaces that aren't first-countable.

Let $\Lambda = \mathbb{R}$ and $X_\lambda = \mathbb{R}$ for $\lambda \in \Lambda$. So $X = \mathbb{R}^\mathbb{R}$ is the space of functions $\mathbb{R} \to \mathbb{R}$ with the product topology. Choose the base point $\mathbf{a} = \mathbf{0} \in X$ where $a_\lambda = \mathbf{0}(x) = 0 \in \mathbb{R}$ for all $x \in \mathbb{R}$. Then $Y$ can be described as the set of functions with finite support.

Consider the function $\mathbf{1} \in X$ with $\mathbf{1}(x) = 1 \in \mathbb{R}$ for all $x \in \mathbb{R}$. We claim that no sequence in $Y$ converges to $\mathbf{1}$.

Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of points in $Y$, i.e., each $f_n$ is a function $\mathbb{R} \to \mathbb{R}$ with finite support. For each $n \in \mathbb{N}$, denote the support of $f_n$ by $\operatorname{supp} f_n$. Since each support is finite, the countable union

$$S = \bigcup_{n\in\mathbb{N}} \operatorname{supp} f_n = \{x \in \mathbb{R} : f_n(x) \neq 0 \text{ for some } n \in \mathbb{N}\}$$

is a countable subset of $\mathbb{R}$. Then there is some $x_0 \in \mathbb{R} \setminus S$, because $\mathbb{R}$ is uncountable. There is an open neighborhood of $\mathbf{1}$ given by

$$U = \{f \in X : f(x_0) \in B_{1/2}(1)\}.$$

However, for all $n \in \mathbb{N}$, we have $f_n(x_0) = 0$, implying $f_n \notin U$. Hence, the sequence $(f_n)_{n\in\mathbb{N}}$ does not converge to $\mathbf{1}$.