I worked with Joseph Sullivan and Gahl Shemy.

Exercise 1.4.8 Denote the map by $p : \mathbb{C} \to \mathbb{C}$. Then the derivative at a point $z \in \mathbb{C}$ is the linear map

$$\begin{aligned} \mathrm{d}p_z : \mathbb{C} &\longrightarrow \mathbb{C}, \\ v &\longmapsto p'(z)v. \end{aligned}$$

This map is surjective if and only if $p'(z)$ is nonzero. Note that

$$p'(z) = mz^{m-1} + (m-1)a_1 z^{m-2} + \cdots + a_{m-1}$$

is a complex polynomial, so the fundamental theorem of algebra tells us that $p'(z)$ has exactly $m-1$ roots in $\mathbb{C}$, counted with multiplicity. In particular, $p'(z)$ is nonzero except at finitely many points $z \in \mathbb{C}$.

Exercise 1.5.8 Let define the functions $h, s : \mathbb{R}^3 \to \mathbb{R}$ by

$$h(x,y,z) = x^2 + y^2 - z^2 - 1 \qquad \text{and} \qquad s(x,y,z) = x^2 + y^2 + z^2 - a.$$

At a point $(x,y,z) \in \mathbb{R}^3$, their Jacobians are

$$J_h(x,y,z) = \begin{bmatrix} 2x & 2y & -2z \end{bmatrix} \qquad \text{and} \qquad J_s(x,y,z) = \begin{bmatrix} 2x & 2y & 2z \end{bmatrix}.$$

We deduce that 0 is a regular value of both h and s, so they define the hyperbola $H = h^{-1}(0)$ and the sphere $S = s^{-1}(0)$ as submanifolds of $\mathbb{R}^3$. By Homework 5 Exercise 1.5.5, the tangent spaces are given by

$$T_{(x,y,z)}X = \ker \mathrm{d}h_{(x,y,z)} \qquad \text{and} \qquad T_{(x,y,z)}Y = \ker \mathrm{d}s_{(x,y,z)}.$$

Of course, these are the same as the kernels of the corresponding Jacobian matrices. We will use this fact to understand the tangent spaces.

The intersection $H \cap S$ is the set of points in $\mathbb{R}^3$ which are roots of both h and s. In particular, for or such a point we must have

$$1 + z^2 = x^2 + y^2 = a - z^2,$$

which implies $z = \pm\sqrt{(a-1)/2}$.

If $a < 1$, there is no real solution for z, so $H \cap S = \varnothing$. In this case, H and S are vacuously transverse.

If $a = 1$, then $z = 0$ and their intersection is the circle $\{x^2 + y^2 = 1\}$ in the xy-plane. However, H and S are not transverse at any of these points. For example, consider the point $e_1 = (1,0,0) \in H \cap S$. The tangent space of H at e_1 is

$$T_{e_1}H = \ker J_h(e_1) = \ker \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} = \langle e_2, e_3 \rangle$$

and the tangent space of S at e_1 is

$$T_{e_1}S = \ker J_s(e_1) = \ker \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} = \langle e_2, e_3 \rangle.$$

In other words, both tangent spaces equal the yz-plane. But these do not span $T_{e_1}\mathbb{R}^3 = \mathbb{R}^3$, so we conclude that H and S are not transverse for $a = 1$.

If $a > 1$, then there are two possible values of z—one positive and one negative. In this case, $H \cap S$ is the union of two circles $\{x^2 + y^2 = r\}$ with $r = a - z^2 = (a+1)/2$. Each circle is parallel to the xy-plane, but shifted up or down by the corresponding value of z. At all of theses points, H and S are transverse. A vertical reflection of $\mathbb{R}^3$ restricts to a diffeomorphism on each of H and S which preserves the overall geometry inside $\mathbb{R}^3$, so it suffices to check that H and S are transverse on just one of the two circles. Moreover, a rotation of $\mathbb{R}^3$ about the z axis restricts to the same sort of geometry-preserving diffeomorphism, so we need only check transversality at a single point of the intersection. We consider the point

$$p = (\sqrt{(a+1)/2}, 0, \sqrt{(a-1)/2}) \in H \cap S.$$

The tangent space of H at p is

$$T_pH = \ker \begin{bmatrix} \sqrt{2(a+1)} & 0 & -\sqrt{2(a-1)} \end{bmatrix} = \langle \sqrt{2(a-1)}e_1 + \sqrt{2(a+1)}e_3,\ e_2 \rangle$$

and the tangent space of S at p is

$$T_pS = \ker \begin{bmatrix} \sqrt{2(a+1)} & 0 & \sqrt{2(a-1)} \end{bmatrix} = \langle \sqrt{2(a-1)}e_1 - \sqrt{2(a+1)}e_3,\ e_2 \rangle.$$

These spaces do span $\mathbb{R}^3$, so we conclude that H and S are transverse for $a > 1$.

Exercise 1.6.3 Per the hint, we check that path-connectivity defines an equivalence relation between points. The relation is reflexive since the constant map $I \to \{x\}$ is smooth. The relation is symmetric since if $f : I \to X$ is a smooth path from x to y then $f(1-t)$ gives a smooth path from y to x. Lastly, the relation is transitive since a smooth path $I \to X$ from x to y can be considered as a homotopy between the maps $\{0\} \to \{x\}$ and $\{0\} \to \{y\}$ from the 0-manifold $\{0\} = \mathbb{R}^0$. In other words, the transitivity of paths is a special case of the transitivity of homotopies, which we have from Homework 5 Exercise 1.6.2. We deduce that X is the disjoint union of its path-components.

We check that the path-components are open. Let $P \subseteq X$ be a path component and $x \in P$. Pick a chart $\varphi : U \to \mathbb{R}^k$ on X with $\varphi(x) = 0$. Then the image $\varphi(U) \subseteq \mathbb{R}^k$ is an open neighborhood of the origin, so it must contain some open ball of radius $\varepsilon > 0$. Since $B_\varepsilon(0)$ is path-connected and open in $\varphi(U)$, its image $\varphi^{-1}(B_\varepsilon(0))$ is similarly path-connected and open in U. Since this set is path-connected and contains x, it must be contained in P. And since U is open in X, the set is also open in X. Hence, we have found an open neighborhood of x contained in P, so P is open in X.

To summarize, we have found that every manifold is the disjoint union of its path-components, all of which are open. If X is connected, this is only possible if X has a single path-component, hence X must be path-connected.

Exercise 1.6.4 Suppose X is contractible and let $R : X \times I \to X$ be a smooth homotopy from the identity $R_0 = \mathrm{id}_X$ to a constant map $R_1 = \mathrm{c}_x : X \to \{x\}$. Let Y be an arbitrary manifold and $f : Y \to X$ a smooth map. Combining maps with smoothness-preserving operations, we construction a smooth map $H : Y \times I \to X$ as follows:

$$Y \times I \xrightarrow{f \times \mathrm{id}_Y} X \times I \xrightarrow{\quad R \quad} X$$

Then H is a homotopy from

$$H_0 = R_0 \circ f = \mathrm{id}_X \circ f = f$$

to

$$H_1 = R_1 \circ f = \mathrm{c}_x \circ f = \mathrm{c}_x.$$

This proves $f \simeq \mathrm{c}_x$ for all maps $f : Y \to X$. From Homework 5 Exercise 1.6.2, we know that homotopy is an equivalence relation. Then for all maps $f, g : Y \to X$ we have $f \simeq \mathrm{c}_x \simeq g$, which implies $f \simeq g$.

Conversely, assume that for any manifold Y all maps $Y \to X$ are homotopic. In particular, take $Y = X$ and consider the identity map id_X and a constant map c_x for some $x \in X$. By assumption, we have $\mathrm{id}_X \simeq \mathrm{c}_x$, so indeed X is contractible.

Exercise 1.6.5 We will apply Exercise 1.6.4. Define $M : \mathbb{R}^k \times I \to \mathbb{R}^k$ by scalar multiplication, i.e., $M(x,t) = tx$. Then M is smooth and thus defines a homotopy $M_0 \simeq M_1$. By construction, we have

$$M_1(x) = 1x = x = \mathrm{id}_{\mathbb{R}^k}(x)$$

and

$$M_0(x) = 0x = 0 = \mathrm{c}_0(x).$$

Thus, we have found a homotopy $\mathrm{id}_{\mathbb{R}^k} \simeq \mathrm{c}_0$, so indeed $\mathbb{R}^k$ is contractible.

Exercise 1.6.6 Assume X is contractible. For any pair of points $x, y \in X$, there is a homotopy between the maps $\{0\} \to \{x\}$ and $\{0\} \to \{y\}$, where $\{0\} = \mathbb{R}^0$ is a 0-manifold with a single point. These two maps are homotopic by Exercise 1.6.4, and a homotopy between them is precisely a path between x and y in X. Therefore, X is (path-)connected. (The proof of Exercise 1.6.3 shows that connectedness and path-connectedness are equivalent conditions for manifolds.) Another application of Exercise 1.6.4 tells us that every map $S^1 \to X$ is homotopic to a constant map, so indeed X is simply-connected.

As a likely counterexample to the converse, consider S^2. In more general algebraic topology, we do not require the homotopies to be smooth—only continuous. Any smooth homotopy is also a continuous (not necessarily smooth) homotopy, but a priori the existence of a continuous homotopy does not in general imply the existence of a smooth one. I presume that the same techniques for showing S^2 is continuously simply-connected would work to show that is is smoothly simply connected, so long as we are careful to preserve smoothness. On the other hand, it is obvious that S^2 is not smoothly contractible since we know more generally that it is not continuously contractible.

Exercise 1.6.7 Per the hint, let $R_t : \mathbb{R}^2 \to \mathbb{R}^2$ be the the linear map defined by the rotation matrix

$$[R_t] = \begin{bmatrix} \cos \pi t & -\sin \pi t \\ \sin \pi t & \cos \pi t \end{bmatrix}.$$

Then R restricts to a homotopy of maps $S^2 \to S^2$ corresponding to the matrices

$$[R_0] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2 \qquad \text{and} \qquad [R_1] = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = -I_2.$$

The former is corresponds to the identity map and the latter to the antipodal map. Then for $k = 2n - 1$ with $n \in \mathbb{N}$, we have $S^k \subseteq \mathbb{R}^{2n}$ and define the linear map $F_t : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ by the block matrix built out of n copies of $[R_t]$ along the diagonal (and zeros elsewhere):

$$[F_t] = \begin{bmatrix} [R_t] & & 0 \\ & \ddots & \\ 0 & & [R_t] \end{bmatrix}.$$

As in the previous step, F restricts to a homotopy of maps $S^k \to S^k$ corresponding to

$$[F_0] = \begin{bmatrix} I_2 & & 0 \\ & \ddots & \\ 0 & & I_2 \end{bmatrix} = I_{2n} \qquad \text{and} \qquad [F_1] = \begin{bmatrix} -I_2 & & 0 \\ & \ddots & \\ 0 & & -I_2 \end{bmatrix} = -I_{2n}.$$

Also like the previous step, the former is corresponds to the identity map and the latter to the antipodal map.

Exercise 2.1.7 Let $x \in \partial X$ and take parameterizations $\varphi : U \to X$ and $\psi : V \to X$ with $\varphi(0) = x = \psi(0)$, where U and V are open subsets of H^k. By shrinking U and V we may assume without loss of generality that $\varphi(U) = \psi(V)$. Then $h = \psi^{-1} \circ \varphi : U \to V$ is a diffeomorphism, so its derivative $\mathrm{d}h_0$ is an isomorphism of tangent spaces

$$\mathbb{R}^k = T_0(U) \longrightarrow T_0(V) = \mathbb{R}^k.$$

We claim that $\mathrm{d}h_0(H^k) = H^k$. To show this, notice that we can write

$$H^k = \partial H^k + \mathbb{R}_{\geq 0} e_k,$$

where $\partial H^k = \mathbb{R}^{k-1}$ and $\mathbb{R}_{\geq 0} e_k$ is the nonnegative multiples of e_k. The linearity of $\mathrm{d}h_0$ gives

$$\mathrm{d}h_0(H^k) = \mathrm{d}h_0(\partial H^k + \mathbb{R}_{\geq 0} e_k) = \mathrm{d}h_0(\partial H^k) + \mathbb{R}_{\geq 0} \mathrm{d}h_0(e_k)$$

Now considering boundaries, the map $\partial h = \partial \psi^{-1} \circ \partial \varphi : \partial U \to \partial V$ is a diffeomorphism with ∂U and ∂V both open sets in $\partial H^k = \mathbb{R}^{k-1}$. In other words, ∂h is a parameterization of ∂V as a $(k-1)$-manifold, so

$$\mathrm{d}h_0(\partial H^k) = \operatorname{im} \mathrm{d}(\partial h)_0 = T_0 \partial V = T_0 \partial H^k = \partial H^k.$$

We now inspect the directional derivative

$$\mathrm{d}h_0(e_k) = \mathrm{D}_{e_k}h(0) = \lim_{t\to 0}\frac{h(0+te_k)-h(0)}{t} = \lim_{t\to 0}\frac{h(te_k)}{t}$$

For all $t > 0$ we have $te_k \in H^k$. Since U is an open neighborhood of 0 in H^k, then for small $t > 0$ we have $te_k \in U$. Then $h(te_k) \in V \subseteq H^k$, and scaling by $1/t > 0$ keeps the value inside of H^k. Since H^k is closed as a subset $\mathbb{R}^k$, we have

$$\mathrm{d}h_0(e_k) = \lim_{t\to 0}\frac{h(te_k)}{t} \in H^k.$$

Moreover, since e_k is not contained in the invariant subspace $\partial H^k = \mathrm{d}h_0(\partial H^k)$ and $\mathrm{d}h_0$ is a linear isomorphism, $\mathrm{d}h_0(e_k)$ must also not be in ∂H^k. In other words, $\mathrm{d}h_0(e_k)$ is is the (strictly) positive upper half-space $H^k \setminus \partial H^k$, so

$$\mathrm{d}h_0(H^k) = \partial H^k + \mathbb{R}_{\geq 0}\mathrm{d}h_0(e_k) = H^k.$$

We now conclude that

$$\mathrm{d}\varphi_0(H^k) = \mathrm{d}\psi_0(\mathrm{d}h_0(H^k)) = \mathrm{d}\psi_0(H^k).$$

Exercise 2.1.8 Since $T_x\partial X$ is a subspace of codimension in T_xX, then the orthogonal complement $(T_x\partial X)^\perp$ is a 1-dimensional subspace of T_xX. A 1-dimensional real vector space has exactly two unit vectors and they are opposites of each other. That is, if $v \in (T_x\partial X)^\perp$ is a unit vector, the other unit vector is $-v$. In the proof of Exercise 2.1.7, we found that H_xX is a half-space of T_xX whose boundary is precisely $T_x\partial X$. By construction, exactly one of $\pm v$ is contained in this half-space.

For $X = H^k$, we have $\vec{n}_{H^k}(x) = -e^k$ for all $x \in \partial H^k$. In other words, $\vec{n}_{H^k}$ is a constant map, which is in particular smooth. For $x \in \partial X$, let $\varphi : U \to X$ be a local parameterization with $\varphi(0) = x$. For each $u \in U$, the derivative $\mathrm{d}\varphi_u : \mathbb{R}^k \to T_{\varphi(u)}X$ is an isomorphism of vector spaces. There is a map

$$\begin{aligned} U &\longrightarrow X \times (\mathbb{R}^N)^k \\ u &\longmapsto \big(\varphi(u), (\mathrm{d}\varphi_u(e_1), \ldots, \mathrm{d}\varphi_u(e_k))\big), \end{aligned}$$

where each point $\varphi(u)$ is paired with a basis of the tangent space $T_{\varphi(u)}X$. This map is smooth, as it can be constructed out of restrictions of the global derivative map $\mathrm{d}\varphi : T(U) \to R(X)$ between tangent bundles.

For any vector space V and linearly independent set $v_1, \ldots, v_k$, we can perform the Gram-Schmidt process to obtain an orthonormal set $w_1, \ldots, w_k$ such that the span of $v_1, \ldots, v_i$ is the same as the span of $w_1, \ldots, w_i$ for each $i = 1, \ldots, k$. The subset $Y \subseteq (\mathbb{R}^N)^k$ consing only of k-tuples of linearly independent vectors forms a manifold and the map $Y \to Y$ which performs the Gram-Schmidt process is a smooth map.

Applying this process to $\mathrm{d}\varphi_u(e_1), \ldots, \mathrm{d}\varphi_u(e_k)$ gives us $w_1, \ldots, w_k$ with $w_1, \ldots, w_{k-1}$ spanning $T_{\varphi(u)}\partial X$ and w_k a unit vector in $T_{\varphi(u)}X$ orthogonal to $T_{\varphi(u)}\partial X$. Moreover, from the proof of Exercise 2.1.7, we know that $\mathrm{d}\varphi_u(e_k)$ and therefore w_k are both contained in $H_{\varphi(u)}X$, so in fact $\vec{n}(\varphi(u)) = -w_k$.

Composing all the necessary maps gives $\vec{n}$ as a smooth map.

Exercise 2.1.9

(a) We prove that $\operatorname{int} X = X \setminus \partial X$ is open. Let $x \in \operatorname{int} X$ and choose a local parameterization $\varphi : U \to X$ where U is an open set in H^k. Since x is not on the boundary of X, the point $u \in U$ with $\varphi(u) = x$ must not be on the boundary of H^k. Since $\partial H^k = \mathbb{R}^{k-1}$ is a closed subspace of $\mathbb{R}^k$, there is a small enough neighborhood $V \subseteq \mathbb{R}^k$ of u such that $V \cap \mathbb{R}^{k-1} = \varnothing$. Then $W = U \cap V$ is an open set in H^k with $W \cap \partial H^k = \varnothing$.

Then $\varphi(W)$ is an open neighborhood of x in X. Moreover, φ^{-1} gives a chart of $\varphi(W)$ such that $\varphi^{-1}(y) \notin \partial H^k$ for all $y \in \varphi(W)$. By Exercise 2.1.1, we conclude that $\varphi(W)$ and ∂X are disjoint so $\varphi(W) \in \operatorname{int} X$, hence $\operatorname{int} X$ is open.

(b) Trivially, any non-compact manifold without boundary, since the emptyset is compact.

The 1-manifold $H^1 \subseteq \mathbb{R}^1$ is not compact since it is unbounded, but $\partial H^1 = \{0\}$ is compact.