I worked with Joseph Sullivan and Gahl Shemy.

1 Exercise 1.1.5 Let $V \leq \mathbb{R}^N$ be a linear subspace of dimension k. Choose a basis $\{v_1, \ldots, v_k\}$ of V and extend it to a basis $\{v_1, \ldots, v_k, u_{k+1}, \ldots, u_N\}$ of $\mathbb{R}^N$. Then there is a linear surjection $L : \mathbb{R}^N \to \mathbb{R}^k$ defined by $v_i \mapsto e_i$ and $u_i \mapsto 0$. This map is smooth, therefore the restriction $\varphi = L|_V : V \to \mathbb{R}^k$ is smooth. Moreover, this is an isomorphism of vector spaces, so there is a linear inverse $\varphi^{-1}\mathbb{R}^k \to V$. Since φ^{-1} is a linear map $\mathbb{R}^k \to \mathbb{R}^N$, it is smooth, hence φ is a diffeomorphism. That is, φ is a global parameterization giving V the structure of a manifold, diffeomorphic to $\mathbb{R}^k$.

2 Exercise 1.2.1 Take parameterizations $\varphi : U \to X$ and $\psi : V \to Y$ where $U \subseteq \mathbb{R}^k$ and $V \subseteq \mathbb{R}^\ell$ are open sets such that $\varphi(0) = x = \psi(0)$. Without loss of generality, we may assume U is chosen small enough that $\varphi(U) \subseteq \psi(V)$, giving us the following commutative diagram:

$$\begin{tikzcd} X \arrow[r, hook, "\iota"] & Y \\ U \arrow[u, "\varphi"] \arrow[r, dashed, "h=\psi^{-1}\circ\varphi"'] & V \arrow[u, "\psi"'] \end{tikzcd}$$

Taking derivatives, we obtain

$$\begin{tikzcd} T_xX \arrow[r, "\mathrm{d}\iota_0"] & T_xY \\ \mathbb{R}^k \arrow[u, "\mathrm{d}\varphi_0"] \arrow[r, "\mathrm{d}h_0"'] & \mathbb{R}^\ell \arrow[u, "\mathrm{d}\psi_0"'] \end{tikzcd}$$

Note that we can write $\varphi = \psi \circ h$, so

$$\mathrm{d}\varphi_0^{-1} = \mathrm{d}(\psi \circ h)_0^{-1} = (\mathrm{d}\psi_{h(0)} \circ \mathrm{d}h_0)^{-1} = \mathrm{d}h_0^{-1} \circ \mathrm{d}\psi_0^{-1}.$$

Then we compute

$$\mathrm{d}\iota_x = \mathrm{d}\psi_0 \circ \mathrm{d}h_0 \circ \mathrm{d}(\varphi)_0^{-1} = \mathrm{d}\psi_0 \circ \mathrm{d}h_0 \circ \mathrm{d}h_0^{-1} \circ \mathrm{d}\psi_0^{-1} = \mathrm{id}\,.$$

That is, $\mathrm{d}\iota_x$ acts as the identity on T_xX, so it must be the inclusion $T_xX \hookrightarrow T_xY$.

3 Exercise 1.2.4 Let $\varphi : U \to X$ be a local parameterization with $U \subseteq \mathbb{R}^k$ an open set such that $\varphi(0) = x$. In particular, φ is a diffeomorphism $U \to \varphi(U)$. Since f is a diffeomorphism, the composition $f \circ \varphi$ is a diffeomorphism $U \to f(\varphi(U)) \subseteq Y$. In other words, $\psi = f \circ \varphi : U \to Y$ is a local parameterization with $\psi(0) = f(x)$. Hence, we have the following commutative diagram:

$$\begin{tikzcd} X \arrow[r, "f"] & Y \\ U \arrow[u, "\varphi"] \arrow[r, dashed, "\mathrm{id}_U"'] & U \arrow[u, "\psi"'] \end{tikzcd}$$

Taking derivatives, we obtain

$$\begin{array}{ccc} T_xX & \xrightarrow{\mathrm{d}f_x} & T_{f(x)}Y \\ {\scriptstyle \mathrm{d}\varphi_0}\uparrow & & \uparrow{\scriptstyle \mathrm{d}\psi_0} \\ \mathbb{R}^k & \xrightarrow[\mathrm{d}(\mathrm{id}_U)_0=\mathrm{id}_{\mathbb{R}^k}]{} & \mathbb{R}^k \end{array}$$

Since each of $\mathrm{d}\varphi_0$, $\mathrm{d}\psi_0$, and $\mathrm{id}_{\mathbb{R}^k}$ are isomorphisms, then $\mathrm{d}f_x$ is necessarily an isomorphism.

4 Exercise 1.2.12 Fix $x \in X$ and choose a local parameterization $\varphi : U \to X$ with $U \subseteq \mathbb{R}^k$ open and $\varphi(0) = x$. Then φ is a diffeomorphism from U to the open set $\varphi(U) \subseteq X$, so its derivative is an isomorphism of tangent spaces

$$\mathbb{R}^k = T_0U \xrightarrow[\cong]{\mathrm{d}\varphi_0} T_x\varphi(U) = T_xX.$$

Given $v \in T_xX$, set $u = \mathrm{d}\varphi_0^{-1}(v) \in \mathbb{R}^k$ and define a linear map $\gamma : \mathbb{R}^1 \to \mathbb{R}^k$ by $\gamma(t) = tu$, so

$$\mathrm{d}\gamma_0(t) = tu.$$

Choose $\varepsilon > 0$ such that $(-\varepsilon, \varepsilon) \subseteq \gamma^{-1}(U)$ and define the curve $c = \varphi \circ \gamma|_{(-\varepsilon,\varepsilon)} : (-\varepsilon, \varepsilon) \to X$. Then the velocity vector of c at 0 is

$$\mathrm{d}c_0(1) = \mathrm{d}(\varphi \circ \gamma)_0(1) = \mathrm{d}\varphi_0(\mathrm{d}\gamma_0(1)) = \mathrm{d}\varphi_0(u) = v.$$

Hence, every vector of X is the velocity vector of some curve in X.

Conversely, by definition, every velocity vector lives in some tangent space of X.

5 Exercise 1.4.11

(a) The determinant $\det : M(n) \to \mathbb{R}$ is a polynomial with respect to the entries of the matrices, and is therefore smooth. By definition, $SL(n) = \det^{-1}(1) \subseteq M(n)$.

Let $A \in M(n)$ with $\det A \neq 0$. Consider the line $\gamma : \mathbb{R} \to M(n)$ defined by $\gamma(t) = tA$. Define the map $f = \det \circ \gamma : \mathbb{R} \to \mathbb{R}$, which we can write as

$$f(t) = \det(tA) = t^n \det A.$$

Then the derivative of f at $1 \in \mathbb{R}$ is

$$f'(t) = nt^{n-1} \det A.$$

For $t = 1$ this is nonzero, which means the derivative is a surjective map on tangent spaces. And since $\mathrm{d}f_1 = \mathrm{d}(\det)_A \circ \mathrm{d}\gamma_1$, then in particular $\mathrm{d}(\det)_A$ is surjective. In other words, A is a regular point of the determinant, so all nonzero values are regular values. Hence, $SL(n)$ is a manifold by the preimage theorem.

(b) Since $SL(n) = \det^{-1}(1) \subseteq M(n)$, the tangent space is given by

$$T_{I_n} SL(n) = \ker \mathrm{d}(\det)_{I_n}.$$

Since det is smooth, we can use the directional derivative

$$\mathrm{d}(\det)_{I_n}(A) = \mathrm{D}_A(I_n) = \lim_{t \to 0} \frac{\det(I_n + tA) - \det I_n}{t}.$$

Note that A is similar to some (possibly complex) upper triangular matrix U, i.e., there is some (possibly complex) $P \in GL(n)$ with

$$PAP^{-1} = U = \begin{bmatrix} u_1 & & * \\ & \ddots & \\ 0 & & u_n \end{bmatrix}.$$

Since determinant is invariant under similarity, we have

$$\det(I_n + tA) = \det(P(I_n + tA)P^{-1}) = \det(I_n + tU).$$

Then $t_n + tU$ is upper triangular, so its determinant is the product of the diagonal entries:

$$\det(I_n + tU) = (1 + tu_1) \cdots (1 + tu_n) = 1 + t(u_1 + \cdots + u_n) + O(t^2).$$

We can compute the derivative

$$\mathrm{d}(\det)_{I_n}(A) = \lim_{t \to 0} \frac{t(u_1 + \cdots + u_n) + O(t^2)}{t} = u_1 + \cdots + u_n,$$

but this is simply the trace, which is invariant under similarity, so

$$\mathrm{d}(\det)_{I_n}(A) = \mathrm{tr}(U) = \mathrm{tr}(P^{-1}UP) = \mathrm{tr}(A).$$

Hence, the tangent space is

$$T_{I_n} SL(n) = \ker \mathrm{d}(\det)_{I_n} = \ker \mathrm{tr} = \{A \in M(n) \mid \mathrm{tr}(A) = 0\}.$$

6 Exercise 1.5.4 By Problem 2 Exercise 1.2.1, the derivatives of the inclusions $X \cap Z \hookrightarrow X$ and $X \cap Z \hookrightarrow Z$ are inclusions $T_y(X \cap Z) \hookrightarrow T_y X$ and $T_y(X \cap Z) \hookrightarrow T_y Z$, hence

$$T_y(X \cap Z) \subseteq T_y X \cap T_y Z.$$

We now count dimensions to prove equality. Note that $X \cap Z = \iota^{-1}(Z)$ where $\iota : X \to Y$ is the inclusion, so

$$\mathrm{codim}_X(X \cap Z) = \mathrm{codim}_X \, \iota^{-1}(Z) = \mathrm{codim}_Y Z.$$

This means the dimension $\dim T_y(X \cap Z) = \dim(X \cap Z)$ is given by

$$\dim X - \mathrm{codim}_Y Z = \dim X - (\dim Y - \dim Z) = \dim X + \dim Z - \dim Y.$$

The fact that X and Z are transverse tells us $T_yX + T_yZ = T_yY$, so

$$\dim T_yX + \dim T_yZ = \dim T_yY - \dim(T_yX \cap T_yZ),$$

which implies

$$\dim(T_yX \cap T_yZ) = \dim T_yX + \dim T_yZ - \dim T_yY = \dim X + \dim Z - \dim Y.$$

We conclude that

$$\dim T_y(X \cap Z) = \dim X + \dim Z - \dim Y = \dim(T_yX \cap T_yZ),$$

so we must have equality

$$T_y(X \cap Z) = T_yX \cap T_yZ.$$

7 Exercise 1.5.5 Denote $y = f(x)$. By construction, f restricts to a map $W \to Z$, which means $\mathrm{d}f_x$ restricts to a map $T_xW \to T_yZ$, so we have

$$T_xW \subseteq \mathrm{d}f_x^{-1}(T_yZ).$$

We now count dimensions to prove equality. First, $\operatorname{codim}_X W = \operatorname{codim}_Y Z$ gives us

$$\dim T_xW = \dim W = \dim X - \operatorname{codim}_X W = \dim X - \operatorname{codim}_Y Z.$$

The fact that f and Z are transverse means $\operatorname{im} \mathrm{d}f_x$ and T_yZ are transverse subspaces of T_yY.

Note that for any linear map $L : V \to V'$ of vector spaces and $U \leq V'$ a subspace, it follows from the first isomorphism theorem that

$$\operatorname{codim}_V L^{-1}(U) = \operatorname{codim}_{V/\ker L}(L^{-1}(U)/\ker L) = \operatorname{codim}_{\operatorname{im} L}(U \cap \operatorname{im} L).$$

Applying this to $\mathrm{d}f_x : T_xX \to T_yY$ and $T_yZ \leq T_yY$, we get

$$\begin{aligned}
\dim \mathrm{d}f_x^{-1}(T_yZ) &= \dim T_xX - \operatorname{codim}_{T_xX} \mathrm{d}f_x^{-1}(T_yZ) \\
&= \dim X - \operatorname{codim}_{\operatorname{im} \mathrm{d}f_x}(T_yZ \cap \operatorname{im} \mathrm{d}f_x) \\
&= \dim X - \operatorname{codim}_{T_yZ + \operatorname{im} \mathrm{d}f_x} T_yZ \\
&= \dim X - \operatorname{codim}_{T_yY} T_yZ \\
&= \dim X - \operatorname{codim}_Y Z \\
&= \dim T_xW.
\end{aligned}$$

Hence, we have equality

$$T_xW = \mathrm{d}f_x^{-1}(T_yZ).$$

(Exercise 1.5.4 is the case of f being the inclusion $X \hookrightarrow Y$.)

8 Exercise 1.5.6 Consider $Y = \mathbb{R}^4$ with subspaces $X = \langle e_1, e_2 \rangle$ and $Z = \langle e_1, e_3 \rangle$. Then $X \cap Z = \langle e_1 \rangle$ is still a manifold but

$$T_0X + T_0Z = X + Z = \langle e_1, e_2, e_3 \rangle \neq Y,$$

which means X and Z are not transverse. Moreover,

$$\operatorname{codim}_Y(X \cap Z) = 3 \neq 4 = \operatorname{codim}_Y X + \operatorname{codim}_Y Z.$$

Lastly, if U and W are linear subspaces of V such that their intersection has codimension $\operatorname{codim}_V U + \operatorname{codim}_V W$, then U and V must be transverse. To see this, consider

$$\begin{aligned} \operatorname{codim}_V(U \cap W) &= \operatorname{codim}_V U + \operatorname{codim}_V W \\ \dim V - \dim(U \cap W) &= (\dim V - \dim U) + (\dim V - \dim W) \\ \dim V + \dim(U + W) - \dim U - \dim W &= 2 \dim V - \dim U - \dim W \\ \dim(U + W) &= \dim V. \end{aligned}$$

Since $U + W$ is a subspace of V, this implies $U + W = V$, i.e., U and V are transverse. Applying this fact to the tangent spaces, we deduce that if X and Z are submanifolds of Y such that their intersection is a submanifold of codimension $\operatorname{codim}_Y X + \operatorname{codim}_Y Z$, then X and Z must be transverse.

9 Exercise 1.5.7 Fix a point $x \in X$ and denote $y = f(x)$, $z = g(y)$.

Assume f is transverse to $g^{-1}(W)$, which means

$$\operatorname{im} \mathrm{d}f_x + T_y g^{-1}(W) = T_y Y.$$

Notice that $T_y g^{-1}(W) = \mathrm{d}g_y^{-1}(T_z W)$, so taking the image under $\mathrm{d}g_y$, we obtain

$$\operatorname{im} \mathrm{d}(g \circ f)_x + T_z W = \operatorname{im} \mathrm{d}g_y.$$

We add $T_z W$ to both sides, and the fact that g is transverse to W gives us

$$\operatorname{im} \mathrm{d}(g \circ f)_x + T_z W = \operatorname{im} \mathrm{d}g_y + T_z W = T_z Z.$$

Hence, $g \circ f$ is transverse to W.

Assume $g \circ f$ is transverse to W, which means

$$\operatorname{im} \mathrm{d}(g \circ f)_x + T_z W = T_z Z.$$

Note that $\operatorname{im} \mathrm{d}(g \circ f)_x = \mathrm{d}g_y(\operatorname{im} \mathrm{d}f_x)$, then taking the preimage under $\mathrm{d}g_y$ gives us

$$\mathrm{d}g_y^{-1}(\mathrm{d}g_y(\operatorname{im} \mathrm{d}f_x) + T_z W) = \mathrm{d}g_y^{-1}(T_z Z) = T_y Y.$$

If $v \in T_y Y$, then we can find $u \in T_x X$ and $w \in T_z W$ such that

$$\mathrm{d}g_y(v) = \mathrm{d}g_y(\mathrm{d}f_x(u)) + w.$$

Then

$$\mathrm{d}g_y(v - \mathrm{d}f_x(u)) = \mathrm{d}g_y(v) - \mathrm{d}g_y(\mathrm{d}f_x(u)) = w,$$

which means

$$v - \mathrm{d}f_x(u) \in \mathrm{d}g_y^{-1}(T_z W) = T_y g^{-1}(W).$$

We deduce that

$$v \in \operatorname{im} \mathrm{d}f_x + T_y g^{-1}(W),$$

so

$$T_y Y \subseteq \operatorname{im} \mathrm{d}f_x + T_y g^{-1}(W).$$

The reverse inclusions is clear since $T_y Y$ is the codomain of $\mathrm{d}f_x$ and $g^{-1}(W)$ is a submanifold of Y. Hence, f is transverse to $g^{-1}(W)$.

10 Exercise 1.6.1 From Homework 3 Exercise 1.1.18, let $h : \mathbb{R} \to \mathbb{R}$ be a smooth function such that $h(x) = 0$ for $x \leq 1/4$, $h(x) = 1$ for $x \geq 3/4$, and $0 < h(x) < 1$ for $1/4 < x < 3/4$. We may consider h as a smooth map $I \to I$.

Let $F : X \times I \to Y$ be a smooth homotopy from f_0 to f_1. We define $\tilde{F}$ as the following composition:

$$X \times I \xrightarrow{\mathrm{id}_X \times h} X \times I \xrightarrow{F} Y$$

Then $\tilde{F}$ is a smooth map, given by $\tilde{F}(x,t) = F(x, h(t))$. So for $t \leq 1/4$

$$\tilde{F}(x,t) = F(x, h(t)) = F(x, 0) = f_0(x)$$

and for $t \geq 3/4$

$$\tilde{F}(x,t) = F(x, h(t)) = F(x, 1) = f_1(x).$$

11 Exercise 1.6.2 Let $F, G : X \times I \to Y$ be smooth homotopies from f to g and from g to h respectively which satisfy the conditions of Problem 10 Exercise 1.6.1. Define the function $H : X \times I \to Y$ by

$$H(x,t) = \begin{cases} F(x, 2t) & \text{if } t \leq 1/2, \\ G(x, 2t-1) & \text{if } t \geq 1/2. \end{cases}$$

We obtain immediately that H is smooth for $t \neq 1/2$. Moreover,

$$F(x, 2(1/2)) = F(x, 1) = g(x) = G(x, 0) = G(x, 2(1/2) - 1),$$

so $H(x, 1/2) = g(x)$ is well-defined. In fact, for all $t \in (3/8, 5/8)$ we have $H(x,t) = g(x)$. In other words,

$$H|_{X \times (3/8, 5/8)} = g \circ \pi_X,$$

where $\pi_X : X \times I \to X$ is the projection onto X. Since g and π_X are smooth, so is their composition, hence H is smooth at $t = 1/2$.