(worked with Joseph Sullivan)

1 Exercise 1.14 Prove that the map $k[x,y] \to k[t]$ sending x to t^2 and y to t^3 induces an isomorphism

$$k[x,y]/(y^2 - x^3) \cong k[t^2, t^3] \subset k[t].$$

Proof. The map is surjective, so the quotient of $k[x,y]$ by the kernel is isomorphic to $k[t^2,t^3]$; it remains the prove that the ideal $\langle y^2 - x^3 \rangle$ is the map's kernel. This ideal is contained in the kernel, as $y^2 - x^3 \mapsto t^6 - t^6 = 0$. For $f \in k[x,y]$ in the kernel, $f(x,y) \mapsto f(t^2,t^3) = 0$. Since $y^2 - x^3$ is monic in y, then we perform long division in the ring $(k[x])[y]$ to obtain

$$f(x,y) = q(x,y)(y^2 - x^3) + r(x,y),$$

where the y-degree of r is less than 2, the y-degree of $y^2 - x^3$. We can rewrite r as

$$r(x,y) = r_0(x) + r_1(x)y,$$

then

$$0 = f(t^2,t^3) = q(t^2,t^3)(t^6 - t^6) + r(t^2,t^3) = r_0(t^2) + r_1(t^2)t^3.$$

We have the polynomials $r_0(t^2)$ and $r_1(t^2)t^3$ in $k[t]$, where the former has only even degrees of t and the latter has only odd. This implies that $r_0(t^2) = r_1(t^2) = 0$, so we must also have $r_0(x) = r_1(x) = 0$, since $k[x] \cong k[t^2]$. Hence, $r = 0$, and we conclude that $f \in \langle y^2 - x^3 \rangle$.

□

2 Exercise 1.15 A conic in the affine real plane $\mathbb{R}^2$ belongs to one of the following eight types:

a. The empty set

b. A single point

c. A line

d. The union of two coincident lines

e. The union of two parallel lines

f. A parabola

g. A hyperbola

h. An ellipse

(a) Show that in the complex affine plane $\mathbb{C}^2$ there are only five types of loci defined by equations of degree 2: Types a and b disappear, and types g and h coincide.

A quadratic in $\mathbb{C}[x, y]$ is of the form

$$ax^2 + bxy + cy^2 + dx + ey + f$$

If $a \neq 0$, then every value of $y \in \mathbb{C}$ gives a quadratic in x:

$$ax^2 + (by + d)x + (cy^2 + ey + f).$$

Since $\mathbb{C}$ is algebraically closed, this has two solutions in x, counting multiplicity. In other words, the quadratic has infinitely many solutions in $\mathbb{C}^2$. The same is true when $c \neq 0$. If $a = c = 0$, so $b \neq 0$, then the polynomial becomes

$$(by + d)x + ey + f.$$

Any value of $y \in \mathbb{C}$, other than $-d/b$, again has infinitely many solutions in x. Hence, cases a and b are not possible.

A hyperbola ($xy - 1 = 0$) can be transformed into the form of an ellipse by the invertible linear transformation of coordinates $x \mapsto x + iy$, $y \mapsto x - iy$ (giving $x^2 + y^2 - 1 = 0$).

(b) Show that in the complex projective plane $\mathbb{P}^2(\mathbb{C})$ there are only three types of loci represented by quadratic equations; they are represented by types c, d, and h on the above list. More generally, there are exactly n types of nonzero quadratic forms in n variables, classified by rank (where the rank of a quadratic form $\sum_{i<j} a_{ij}x_ix_j$ is defined to be the rank of the symmetric matrix (a_{ij})).

Proof. Each (homogeneous) quadratic $f \in \mathbb{C}[x_0, \dots, x_n]$ has a representation $f = x^T Ax$, where $A \in \mathbb{C}^{(n+1)\times(n+1)}$ and $x = [x_0 \cdots x_n]^T$. Without loss of generality, we may assume A is chosen to be symmetric. We define the rank of the quadratic form f to be the rank of the symmetric matrix A.

For quadratic forms $f = x^T Ax$ and $g = x^T Bx$, we define the relation $Z(f) \sim Z(g)$ if there is an $(n+1) \times (n+1)$ invertible matrix P with entries in $\mathbb{C}$, such that the map $Z(f) \to Z(g)$, where $x \mapsto Px$, is an isomorphism or, equivalently, if $A = P^T BP$ and write $A \sim B$. One can check that this is an equivalence relation on the loci of quadratic forms. We claim that $A \sim B$ (i.e., $Z(f) \sim Z(g)$) if and only if $\operatorname{rank} A = \operatorname{rank} B$.

If $A \sim B$, then $A = P^T BP$ for some invertible matrix P. Since rank is preserved under multiplication by an invertible matrix, $\operatorname{rank} A = \operatorname{rank} P^T BP = \operatorname{rank} B$.

Suppose $\operatorname{rank} A = \operatorname{rank} B$. By the spectral theorem, A and B are unitarily diagonalizable into $A = P^T D_A P$ and $B = Q^T D_B Q$, where D_A, D_B are diagonal and P, Q are unitary. In particular, $A \sim D_A$ and $B \sim D_B$, and it remains to prove $D_A \sim D_B$. We have

$$\operatorname{rank} D_A = \operatorname{rank} A = \operatorname{rank} B = \operatorname{rank} D_B,$$

then write

$$D_A = \begin{bmatrix} \alpha_1 & & & \\ & \ddots & & \\ & & \alpha_r & \\ & & & 0 \end{bmatrix} \quad \text{and} \quad D_B = \begin{bmatrix} \beta_1 & & & \\ & \ddots & & \\ & & \beta_r & \\ & & & 0 \end{bmatrix}.$$

Since $\mathbb{C}$ is algebraically closed, we can construct

$$C_A = \begin{bmatrix} 1/\sqrt{\alpha_1} & & & \\ & \ddots & & \\ & & 1/\sqrt{\alpha_r} & \\ & & & I_{n-r} \end{bmatrix} \quad \text{and} \quad C_B = \begin{bmatrix} 1/\sqrt{\beta_1} & & & \\ & \ddots & & \\ & & 1/\sqrt{\beta_r} & \\ & & & I_{n-r} \end{bmatrix},$$

where principle square roots are taken. Then

$$C_A^T D_A C_A = \begin{bmatrix} I_r & \\ & 0 \end{bmatrix} = C_B^T D_B C_B,$$

so $D_A \sim D_B$, implying $A \sim B$.

$\square$

(c) Show that the different types in part (a) correspond to the relative placement of the conic and the line at infinity, in the sense that a parabola is a rank-3 conic tangent to the line at infinity, while an ellipse/hyperbola is a rank-3 conic meeting the line at infinity at two distinct points.

Consider $\mathbb{P}^2_{\mathbb{C}}$ in the coordinates $[x, y, z]$. The line at infinity (for $\mathbb{C}^2$ in (x, y)) is where $z = 0$.

The line $x^2 = 0$ intersects at the point $[0, 1, 0]$.

The union of two coincident lines $xy = 0$ intersects the points $[0, 1, 0]$ and $[1, 0, 0]$.

The union of two parallel lines $x(x-1) = 0$ homogenizes to $x(x-z) = 0$, which means $x = 0$ or $x = z$, so intersects at the point $[0, 1, 0]$.

A parabola $x^2 - y = 0$ homogenizes to $x^2 - yz = 0$. When $z = 0$, we must have $x^2 = 0$, so intersects at the point $[0, 1, 0]$.

A hyperbola $xy - 1 = 0$ homogenizes to $xy - z^2 = 0$. When $z = 0$, we must have $xy = 0$, so intersects at the points $[0, 1, 0]$ and $[1, 0, 0]$.

3 Exercise 1.17 Let $I \subset k[x_1, x_2, x_3]$ be the ideal $(x_1^2 + x_2, x_1^2 + x_3)$, and let $X \subset \mathbb{A}^3$ be the affine algebraic set $Z(I)$. Let $\overline{X} \subset \mathbb{P}^3$ be the projective closure of X. Show that the homogeneous ideal $I(\overline{X})$ is not generated by the homogenizations of $x_1^2 + x_2$ and $x_1^2 + x_3$.

Proof. The homogeneous ideal $I(\overline{X})$ is the ideal of $k[x_0, \ldots, x_3]$ generated by the homogenizations of every polynomial in I. In particular, we know that $x_1^2 + x_2, x_1^2 + x_3 \in I$, so the ideal generated by their homogenizations is contained in $I(\overline{X})$, i.e.,

$$J = \langle x_1^2 + x_0 x_2, x_1^2 + x_0 x_3 \rangle \subseteq I(\overline{X}).$$

Since J is generated by homogeneous polynomials of degree 2, all homogeneous polynomials in J have degree at least 2. However, I contains the element

$$(x_1^2 + x_2) - (x_1^2 + x_3) = x_2 - x_3,$$

which is homogeneous of degree 1, so $x_2 - x_3 \in I(\overline{X}) \setminus J$.

$\square$

4 Exercise 1.18 Let k be a field. Compute the Hilbert function and polynomial for the ring

$$k[x,y,z,w]/(x,y)\cap(z,w)$$

corresponding to the disjoint union of two lines in projective 3-space. Compare these to the Hilbert function and polynomial of the ring corresponding to one projective line, $k[x,y]$.

We can rewrite the ideal as

$$\langle x,y\rangle \cap \langle z,w\rangle = \langle xz, xw, yz, yw\rangle,$$

so

$$M = k[x,y,z,w]/\langle x,y\rangle \cap \langle z,w\rangle = k[x,y,z,w]/\langle xz, xw, yz, yw\rangle.$$

Then the monomials of degree s in M are $x^a y^{s-a}$ and $z^a w^{s-a}$, for $a = 0, 1, \ldots, s$. For $s \geq 1$, are exactly $2s+2$ of these monomials, so the Hilbert function is given by

$$H_M(s) = \begin{cases} 2s+2 & \text{if } s > 0, \\ 1 & \text{if } s = 0, \\ 0 & \text{if } s < 0. \end{cases}$$

This means that the Hilbert polynomial is $P_M(s) = 2s+2$, agreeing with $H_M(s)$ when $s \geq 1$.

The monomials of degree s in $N = k[x,y]$ are $x^a y^{s-a}$ for $a = 0, 1, \ldots, s$. For $s \geq 1$, there are exactly $s+1$ of these monomials, so the Hilbert function is given by

$$H_N(s) = \begin{cases} s+1 & \text{if } s > 0, \\ 1 & \text{if } s = 0, \\ 0 & \text{if } s < 0. \end{cases}$$

The Hilbert polynomial is $H_N(S) = s+1$, agreeing with $H_N(s)$ when $s \geq 1$.

So $H_M(s) = 2H_N(s)$ for $s \geq 1$, and $H_M(s) = H_N(s)$ otherwise. Similarly, $P_M(s) = 2P_N(s)$.