1 Let X be a noncompact, locally compact Hausdorff space, Y be any Hausdorff compactification, and X_∞ be the one-point compactification $X \cup \{\infty\}$. Show that the map $Y \to X_\infty$ which is the identity on X and collapses all points of $Y \setminus X$ to ∞ is a quotient map.

Lemma 1. If $U \subseteq X$ is open in X, then U is also open in Y.

Proof. Since $X \subseteq Y$ has the subspace topology, it suffices to prove that X is open in Y. Given $U \subseteq X$ open, $U = V \cap X$ for some open $V \subseteq Y$. If X is in fact open in Y, then U is open in Y, as the intersection of two open sets in Y.

Given $x \in X$, let K be a compact neighborhood of x and U be an open neighborhood of x contained in K. We claim that U is open in Y. Since $X \subseteq Y$ has the subspace topology, $U = V \cap X$ for some open $V \subseteq Y$. Suppose, for contradiction, that $U \neq V$, so there exists some $y \in V$. Since $U = V \cap X$, we must have $y \in Y \setminus X$; in particular, $y \in Y \setminus K$. Since K is compact, it is closed in Y, so $Y \setminus K$ is open. Therefore, $V \cap (Y \setminus K)$ is an open neighborhood of y, but $(V \cap (Y \setminus K)) \cap X = \varnothing$. This is a contradiction since $y \in Y = \overline{X}$, so in fact $U = V$ is open in Y. Hence, X is open in Y.

□

Proof. Let $f : Y \to X_\infty$ be the identity on X and collapse all points of $Y \setminus X$ to ∞. In order for this to be a quotient map, we require that the topology on X_∞ be the finest topology such that f is continuous. In other words, we require that $U \subseteq X_\infty$ is open if and only if its preimage $f^{-1}(U) \subseteq Y$ is open.

We will first suppose $U \subseteq X_\infty$ is open. By definition of the one-point compactification, there are two cases: (i) U is an open subset of X, or (ii) $U = (X \setminus K) \cup \{\infty\}$ with $K \subseteq X$ compact. In case (i), f is the identity on X and maps all points outside of X to ∞, so $f^{-1}(U) = U$. Since $f^{-1}(U) \subseteq X$ is open in X, Lemma 2 implies it is open in Y. In case (ii), we have

$$f^{-1}(U) = f^{-1}(X \setminus K) \cup f^{-1}(\{\infty\}) = (X \setminus K) \cup (Y \setminus X) = Y \setminus K.$$

Since K is compact and Y is Hausdorff, K is closed in Y, implying that $Y \setminus K$ is open in Y. Thus, in both cases, $f^{-1}(U)$ is open in Y.

Now, suppose $A \subseteq X_\infty$ is arbitrary, with $f^{-1}(A) \subseteq Y$ open. Again, we have two cases, depending on whether or not A contains ∞. If $\infty \notin A$, then $A \subseteq X$, which means $A = f^{-1}(A)$. Since A is open in Y and $X \subseteq Y$ has the subspace topology, $A = A \cap X$ is an open subset of X. Therefore, A is an open subset of X_∞. If $\infty \in A$, then $f^{-1}(A)$ contains $Y \setminus X$. Define the set

$$E = Y \setminus f^{-1}(A) = X \setminus A.$$

Since E is closed in the compact space Y, it is compact. Since $E \subseteq X$ is compact,

$$A = (X \setminus E) \cup \{\infty\}$$

is open in X_∞. Thus, in both cases, A is open in X_∞. □

2 Consider the quotient space $X = \mathbb{N} \times [0,1]/\mathbb{N} \times \{0\}$. We can give $\mathbb{N} \times [0,1]$ a metric by embedding it in $\mathbb{R}^2$.

(a) A space is *first-countable* if for every point x there is a countable set $\{U_i\}$ of open sets containing x such that every neighborhood of x contains some U_i. Show that X is not first-countable.

Proof. Let $\pi : \mathbb{N} \times [0,1] \to X$ be the natural projection and denote by $x_0 \in X$ the point to which $\mathbb{N} \times \{0\}$ is collapsed.

Suppose $U \subseteq X$ is an open neighborhood of x_0, which means $\pi^{-1}(U)$ is an open subset of the original space. Note that $\pi^{-1}(U)$ contains $\pi^{-1}(x_0) = \mathbb{N} \times \{0\}$. Since $\mathbb{N} \times [0,1]$ has the metric topology, there is an open ball around each $(n,0)$ contained in $\pi^{-1}(U)$, i.e., there is some $\varepsilon > 0$ such that $\{n\} \times [0,\varepsilon) \subseteq \pi^{-1}(U)$. For each $n \in \mathbb{N}$, define

$$a_n(U) = \max\{a \in (0,1] : \{n\} \times [0,a) \subseteq \pi^{-1}(U)\}.$$

This specifies a sequence of positive real numbers $\{a_n(U)\}_{n\in\mathbb{N}}$, such that $\pi^{-1}(U)$ contains the open set $\{n\} \times [0, a_n(U))$, for each $n \in \mathbb{N}$.

Let $\{U_k\}_{k\in\mathbb{N}}$ be a countable collection of open neighborhoods of x_0; we will show that this is not a neighborhood basis of x_0. To each U_k corresponds a sequence $\{a_n(U_k)\}_{n\in\mathbb{N}}$ of positive real numbers such that $\{n\} \times [0, a_n(U_k)) \subseteq \pi^{-1}(U_k)$. We employ a sort of diagonalization to construct an open neighborhood of x_0, which contains none of the U_k's. For each $t \in \mathbb{N}$, define $r_n = \frac{1}{2}a_n(U_n) > 0$. Then, define the set

$$U = \pi\left(\bigcup_{n\in\mathbb{N}} \{n\} \times [0, r_n)\right) \subseteq X.$$

Note that

$$\pi^{-1}(U) = \bigcup_{n\in\mathbb{N}} \{n\} \times [0, r_n)$$

is open in $\mathbb{N} \times [0,1]$, so U is open in X. In particular, U is an open neighborhood of x_0. But, by construction, U does not contain any U_k. Hence, the family of U_k's does not form a neighborhood basis of x_0, so X is not first-countable.

□

(b) Show that every metric space is first-countable. Conclude that X is not metrizable.

Proof. For each point x in a metric space, the collection of open balls $\{B_{1/n}(x)\}_{n\in\mathbb{N}}$ is a neighborhood basis This is because any open set containing x must contain a ball $B_\varepsilon(x)$, for some $\varepsilon > 0$. Then, choosing $n \in \mathbb{N}$ with $1/n < \varepsilon$, we have $B_{1/n}(x) \subseteq B_\varepsilon(x)$.

□

Since X is not first countable, its topology is not induced by any metric, i.e., not metrizable.

3 Let X be a metric space and A a closed subset. As in Homework 2, for a point $x \in X$, we define

$$d(x, A) = \inf_{a \in A} d(x, a).$$

(a) Define the *quotient metric* on X/A by

$$d_{X/A}(x, y) = \min\{d_X(x, y), d(x, A) + d(y, A)\}.$$

Show that this defines a metric.

Proof. Notice that both $d_X(x, y)$ and $d(x, A) + d(y, A)$ are nonnegative, so $d_{X/A}(x, y) \geq 0$.

If $x = y$, then $0 \leq d_{X/A}(x, y) \leq d_X(x, y) = 0$, implying $d_{X/A}(x, y) = 0$.

Suppose $d_{X/A}(x, y) = 0$, meaning either $d_X(x, y) = 0$ or $d(x, A) = d(y, A) = 0$. In the first case, we deduce $x = y$ from the fact that d_X is a metric. In the second case, we use Homework 2 Problem 3(b) to select $a, b \in A$ such that $d_X(x, a) = d_X(y, b) = 0$. Once more, this implies $x = a$ and $y = b$. In particular, x and y are both in A, which is collapsed to a single point in X, i.e., $x = y$ in X/A.

Hence, $d_{X/A}(x, y) = 0$ if and only if $x = y$ in X/A.

By the symmetry of d_X and commutativity of addition in $\mathbb{R}$,

$$\begin{aligned} d_{X/A}(x, y) &= \min\{d_X(x, y), d(x, A) + d(y, A)\} \\ &= \min\{d_X(y, x), d(y, A) + d(x, A)\} \\ &= d_{X/A}(y, x). \end{aligned}$$

Note that $\min\{a + b, c\} \leq \min\{a, c\} + \min\{b, c\}$ for all $a, b, c \geq 0$. Then

$$\begin{aligned} d_{X/A}(x, z) &= \min\{d_X(x, z), d(x, A) + d(z, A)\} \\ &\leq \min\{d_X(x, y) + d_X(y, z),\ d(x, A) + 2d(y, A) + d(z, A)\} \\ &= d_{X/A}(y, x). \end{aligned}$$

For $x, y, z \in X/A$, it is clear that

$$d_{X/A}(x, z) \leq d_X(x, z) \leq d_X(x, y) + d_X(y, z)$$

and

$$d_{X/A}(x, z) \leq d(x, A) + d(z, A) \leq d(x, A) + 2d(y, A) + d(z, A).$$

It remains to check the case where $d_{X/A}(x, y) = d(x, y)$ and $d_{X/A}(y, z) = d(y, A) + d(z, A)$. In this case,

$$d_{X/A} \leq d(x, A) + d(z, A) \leq d(x, y) + (y, A) + d(z, A),$$

so in fact, $d_{X/A}(x, z) \leq d_{X/A}(x, y) + d_{X/A}(y, z)$.

$\square$

(b) Show that when A is compact, the topology induced by the quotient metric coincides with the quotient topology. (Use Homework 2.)

Proof. Let $\pi : X \to X/A$ denote the natural projection. Let $x \in X/A$; we will show that every ball centered at x contains a neighborhood in the quotient topology, and vice versa.

For any $r > 0$, consider the open ball $B_{X/A}(x; r) = \{y \in X/A : d_{X/A}(x, y) < r\}$. If $A \notin B_{X/A}(x; r)$, then $d(x, A) \geq r$. So for all $y \in B_{X/A}(x; r)$,

$$d_X(x, y) < r \leq d(x, A) + d(y, A),$$

so $d_{X/A}(x, y) = d_X(x, y)$. Thus, y is in the open ball $B_X(x; r) = \{y \in X : d_X(x, y) < r\}$. In other words, $\pi^{-1}(B_{X/A}(x; r)) = B_X(x; r)$ is open in X, so $B_{X/A}(x; r)$ is open in the quotient topology.

If U is an open neighborhood of x in the quotient topology X/A, $\pi^{-1}(U)$ is open in X. Then there is some $r > 0$ such that $B_X(x; r) \subseteq \pi^{-1}(U)$. If $x \notin A$, then we can assume $r < d(x, A)$, so $B_{X/A}(x; r) = \pi(B_X(x; r)) \subseteq U$, i.e., U is open in the metric topology.

If $x \in A$, then $U(A, r) \subseteq \pi^{-1}(U)$ (in the notation of Homework 2), so $B_{X/A}(x; r) = \pi(U(A, r)) \subseteq U$. Hence, U is open in the metric topology.

$\square$

(c) For the space in the previous problem, compare the quotient topology and the topology induced by the quotient metric. (Is one finer than the other?)

Topology induced by metric is coarser: open ball around x_0 requires a minimum $\varepsilon > 0$ such that preimage contains $\mathbb{N} \times [0, \varepsilon)$. On the other hand, quotient topology contains the open set $\bigcup_{n \in \mathbb{N}}(\{n\} \times [0, \frac{1}{n}))$ which contains no $\mathbb{N} \times [0, \varepsilon)$ for any $\varepsilon > 0$.

4 Say a quotient map $X \to X/\sim$ satisfies the *local lifting property* if for every space Z and every map $\varphi : Z \to X/\sim$ and every point $y \in Z$, there is a neighborhood U of y in Z such that $\varphi|_U$ lifts to a map $\Phi_U : Z \to X$.

(a) Give an example of a collapse map $X \to X/A$ that doesn't have the local lifting property (and an example map to show this).

Let $X = \{\alpha, \beta\} \times [0,1]$ be the disjoint union of two line segments, and let $A = \{(\alpha, 1), (\beta, 0)\}$. Then $X/A \cong [0,1]$, where the collapse map sends the segment $\{\alpha\} \times [0,1]$ to $[0, \frac{1}{2}]$ and the segment $\{\beta\} \times [0,1]$ to $[\frac{1}{2}, 1]$, which means A is sent to $\frac{1}{2}$.

Let $Z = [0,1]$ and $\varphi : Z \to X/A$ be the induced homeomorphism. In other words, φ specifies a path in X/A between $(\alpha, 0)$ and $(\beta, 1)$.

However, for the point $y = \frac{1}{2} \in Z$, every neighborhood U of y contains a point $a \in [0, \frac{1}{2})$ and a point $b \in (\frac{1}{2}, 1]$. Then $\varphi(a) \in \pi(\{\alpha\} \times [0,1])$ and $\varphi(b) \in \pi(\{\beta\} \times [0,1])$. Then, a lift of $\varphi|_U$ would induce a continuous map $\Phi : [a,b] \to X$ (a path), with $\Phi(a) \in \{\alpha\} \times [0,1]$ and $\Phi(b) \in \{\beta\} \times [0,1]$. But the α and β components of X are not path-connected, so this is not possible.

(b) Give an example of a quotient by a finite group action that doesn't have the local lifting property (and an example map to show this).

Let $X = \mathbb{R}^2$ and have $G = \mathbb{Z}/2\mathbb{Z}$ act on X by rotating halfway about the origin. The the quotient X/G is homeomorphic to the real plane $\mathbb{R}^2$, but the identity $\mathbb{R}^2 \to X/G$ must locally lift to a choice of either the top or bottom half of the place in X. However, in any neighborhood of the origin, a local lift must choose some half-disc in X, which would not be a continuous map.

5 Construct a simplicial complex and a CW complex homeomorphic to $$\mathbb{R}P^2 = S^2/x \sim -x.$$

We construct $\mathbb{R}P^2$ as a CW complex.

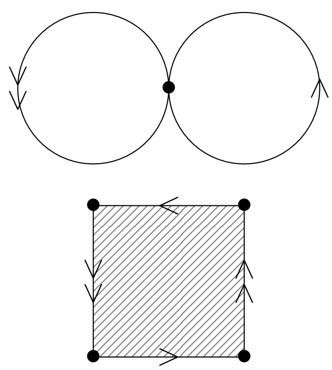
Let $X^{(0)} = \{\bullet\}$ be a single point.

For $X^{(1)}$, we add two segments, $A = [0_A, 1_A]$ and $B = [0_B, 1_B]$, with endpoints glued to $\bullet$.

For $X^{(2)}$, we add a single square $[0,1] \times [0,1]$, with the edges glued as follows:

- $[0,1] \times \{0\} \sim A$ with $(0,0) \sim 0_A$ and $(1,0) \sim 1_A$,
- $\{1\} \times [0,1] \sim B$ with $(1,0) \sim 0_B$ and $(1,1) \sim 1_B$,
- $[0,1] \times \{1\} \sim A$ with $(1,1) \sim 1_A$ and $(0,1) \sim 0_A$,
- $\{0\} \times [0,1] \sim B$ with $(0,1) \sim 1_B$ and $(0,0) \sim 0_B$,

Graphically:



As a simplicial complex, we can take the square above and subdivide it into a 3×3 grid, with each cell having two triangles. Then identifying edges in the same way, we obtain a simplicial complex construction of $\mathbb{R}P^2$, similar to that of the torus.