Homework 2

MATH 118B

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1

Suppose that $f : (a, b) \to \mathbb{R}$ is differentiable, and $f'(x) > 0$ in (a, b) (note that similar results hold if $f' < 0$).

1(a)

Prove that the limits

$$m = \lim_{x \to a^+} f(x) \tag{1}$$

and

$$M = \lim_{x \to b^-} f(x) \tag{2}$$

exist (allowing the possibilities $m = -\infty$, $M = +\infty$).

Lemma 1. f is strictly increasing on (a, b).

Proof. Let $x, y \in (a, b)$ with $x < y$. Then f is differentiable on $[x, y]$ with positive derivative. By Homework 1 Problem 1(c), f is strictly increasing on $[x, y]$, so $f(x) < f(y)$.

□

Proposition 1. $M = \lim_{x \to b^-} f(x)$ for some $M \in (-\infty, +\infty]$.

Proof. We claim $M = \sup f$, allowing the possibility that $M = +\infty$. Let $M_0 < M$ be given. By definition of supremum, there is some $x_0 \in (a, b)$ with $f(x_0) \in (M_0, M]$. Define $\delta = b - x_0$. Then for all $x \in (b - \delta, b)$ we have $x_0 < x$, implying $f(x_0) < f(x)$, so $f(x) \in (M_0, M]$.

□

Proposition 2. $m = \lim_{x \to a^+} f(x)$ for some $m \in [-\infty, +\infty)$.

Proof. We claim $m = \inf f$, allowing the possibility that $m = -\infty$. Let $m_0 > m$ be given. By definition of infimum, there is some $x_0 \in (a, b)$ with $f(x_0) \in [m, m_0)$. Define $\delta = x_0 - a$. Then for all $x \in (a, a + \delta)$ we have $x < x_0$, implying $f(x) < f(x_0)$, so $f(x) \in [m, m_0)$.

□

1(b)

Prove that $f((a,b)) = (m, M)$.

Proof. Let $y \in (a,b)$ and choose $x \in (a,y)$ and $z \in (y,b)$. Since f is strictly increasing, we must have $f(x) < f(y) < f(z)$. Since $m = \inf f$ and $M = \sup f$, then $m \leq f(x)$ and $f(z) \leq M$. Therefore, $f(y) \in (m, M)$, giving us $f((a,b)) \subseteq (m, M)$.

Now let $y \in (m, M)$. Since $m = \inf f$ and $M = \sup f$, there exist $x, z \in (a,b)$ such that $m \leq f(x) < y$ and $y < f(z) \leq M$. Then the intermediate value theorem tells us that there is some $c \in (x, z)$ with $f(c) = y$. Therefore, $y \in f((a,b))$, giving us $(m, M) \subseteq f((a,b))$.

$\square$

1(c)

Prove that f has an inverse, $g : (m, M) \to (a, b)$.

Proof. Problem 1(b) tells us that $f : (a,b) \to (m, M)$ is surjective. Suppose $x, y \in (a,b)$ such that $x \neq y$. Without loss of generality, assume $x < y$. Because f is strictly increasing, $f(x) < f(y)$, so $f(x) \neq f(y)$. Thus, f is injective and, therefore, has an inverse.

$\square$

1(d)

Prove that g is continuous.

Proof. Since f and g are inverses, then g is continuous if and only if f is an open map, i.e., maps open sets to open sets. Since a subset of $\mathbb{R}$ is open if and only if it is an arbitrary union of open intervals (i.e., the open intervals are a base for the topology), it suffices to prove that f maps open intervals to open intervals. Let $(a', b') \subseteq (a, b)$. Then f is differentiable on (a', b') and $f'(x) > 0$ for all $x \in (a', b')$. As an instance of 1(b), the image of this interval is $f((a', b')) = (m', M')$ (where m' and M' would be the infimum and supremum, respectively, of the restriction of f to (a', b')).

$\square$

1(e)

Prove that g is differentiable, and that

$$g'(f(x)) = \frac{1}{f'(x)}, \quad \forall\, x \in (a,b). \tag{3}$$

Proof. Let $y_0 \in (m, M)$, we want to show that

$$\lim_{y \to y_0} \frac{g(y) - g(y_0)}{y - y_0} = \frac{1}{f'(g(y_0))}.$$

To prove the limit, it suffices to prove for arbitrary sequences. Suppose $y_n \to y_0$ with $y_n \neq y_0$ for all $n \in \mathbb{N}$, and consider the sequence given by

$$\frac{g(y_n) - g(y_0)}{y_n - y_0} = \frac{g(y_n) - g(y_0)}{f(g(y_n)) - f(g(y_0))} = \frac{1}{\dfrac{f(g(y_n)) - f(g(y_0))}{g(y_n) - g(y_0)}}.$$

Since g is continuous, then $g(y_n) \to g(y_0)$. Since f is differentiable at $g(y_0)$, then the limit definition of the derivative of f at $g(y_0)$ holds for arbitrary sequences converging to $g(y_0)$. Then with $f'(g(y_0)) > 0$, we have

$$\frac{1}{f'(g(y_0))} = \frac{1}{\displaystyle\lim_{x \to g(y_0)} \frac{f(x) - f(g(y_0))}{x - g(y_0)}} = \lim_{n \to \infty} \frac{1}{\dfrac{f(g(y_n)) - f(g(y_0))}{g(y_n) - g(y_0)}}.$$

Thus,

$$g'(y) = \frac{1}{f'(g(y))}, \quad y \in (m, M),$$

which is equivalent to equation (3) with the fact that f and g are inverses.

□

1(f)

Show that the logarithm function $\log : (0, +\infty) \to \mathbb{R}$ is differentiable and that

$$\frac{d}{dx} \log x = \frac{1}{x}, \quad \forall\, x \in (0, +\infty). \tag{4}$$

Proof. From Homework 1 Problem 2, the exponential is differentiable on $\mathbb{R}$, its derivative is always positive, and its inverse is the logarithm. As an instance of Problem 1(e), we have

$$\frac{\mathrm{d}}{\mathrm{d}x} \log x = \frac{1}{e^{\log x}} = \frac{1}{x}, \quad x \in (0, +\infty).$$

□

2

Evaluate the following limits:

2(a)

$$\lim_{x\to 0} \frac{\tan x - x}{x^3}. \tag{5}$$

L'Hôpital's rule gives us

$$\lim_{x\to 0} \frac{\tan x - x}{x^3} = \lim_{x\to 0} \frac{\frac{\mathrm{d}}{\mathrm{d}x}(\tan x - x)}{\frac{\mathrm{d}}{\mathrm{d}x}x^3}.$$

We find the derivative of the numerator:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\sin x}{\cos x} - x\right) &= \frac{\cos x \frac{\mathrm{d}}{\mathrm{d}x}\sin x - \sin x \frac{\mathrm{d}}{\mathrm{d}x}\cos x}{\cos^2 x} - 1 \\
&= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} - 1 \\
&= \frac{1 - \cos^2 x}{\cos^2 x} \\
&= \frac{\sin^2 x}{\cos^2 x}.
\end{aligned}$$

And $\frac{\mathrm{d}}{\mathrm{d}x}x^3 = 3x^2$. Thus,

$$\begin{aligned}
\lim_{x\to 0} \frac{\tan x - x}{x^3} &= \lim_{x\to 0} \frac{\sin^2 x}{3x^2 \cos^2 x} \\
&= \lim_{x\to 0} \left(\frac{1}{3}\left(\frac{\sin x}{x}\right)^2 \left(\frac{1}{\cos x}\right)^2\right) \\
&= \frac{1}{3} \cdot 1^2 \cdot \left(\frac{1}{1}\right)^2 \\
&= \frac{1}{3}.
\end{aligned}$$

2(b)

$$\lim_{x \to +\infty} \frac{x^3}{e^x}. \tag{6}$$

Applying L'Hôpital's rule thrice, we find

$$\lim_{x\to+\infty} \frac{x^3}{e^x} = \lim_{x\to+\infty} \frac{\frac{\mathrm{d}^3}{\mathrm{d}x^3} x^3}{\frac{\mathrm{d}^3}{\mathrm{d}x^3} e^x} = \lim_{x\to+\infty} \frac{\frac{\mathrm{d}^2}{\mathrm{d}x^2} 3x^2}{\frac{\mathrm{d}^2}{\mathrm{d}x^2} e^x} = \lim_{x\to+\infty} \frac{\frac{\mathrm{d}}{\mathrm{d}x} 6x}{\frac{\mathrm{d}}{\mathrm{d}x} e^x} = \lim_{x\to+\infty} \frac{6}{e^x} = 0.$$

2(c)

$$\lim_{x\to+\infty} \left(1 + \frac{1}{x}\right)^x. \tag{7}$$

With L'Hôpital's rule and the derivative of the logarithm from Problem 1(f), we find the following limit:

$$\begin{aligned}
\lim_{x\to+\infty} x \log\left(1 + \frac{1}{x}\right) &= \lim_{x\to+\infty} \frac{\log\left(1+\frac{1}{x}\right)}{\frac{1}{x}} \\
&= \lim_{x\to+\infty} \frac{\frac{\mathrm{d}}{\mathrm{d}x}\log\left(1+\frac{1}{x}\right)}{\frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{x}} \\
&= \lim_{x\to+\infty} \frac{\frac{1}{1+\frac{1}{x}} \cdot \frac{\mathrm{d}}{\mathrm{d}x}\left(1+\frac{1}{x}\right)}{\frac{-1}{x^2}} \\
&= \lim_{x\to+\infty} \frac{\frac{1}{1+\frac{1}{x}} \cdot \frac{-1}{x^2}}{\frac{-1}{x^2}} \\
&= \lim_{x\to+\infty} \frac{1}{1+\frac{1}{x}} \\
&= 1.
\end{aligned}$$

Since the exponential function is continuous, we obtain

$$\lim_{x\to+\infty} \left(1+\frac{1}{x}\right)^x = \lim_{x\to+\infty} e^{x\log\left(1+\frac{1}{x}\right)} = e^1 = e.$$

3

Consider the function $f(x) = \sin(x)$. Show that if $|x| \le M$, then

$$|f(x) - T_n(x;0)| \le \frac{M^{n+1}}{(n+1)!}, \tag{8}$$

and use this to prove that

$$\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}, \quad \forall\, x \in \mathbb{R}. \tag{9}$$

Lemma 2.

$$f^{(n)}(x) = \begin{cases} \sin x & \text{if } n \equiv 0 \pmod 4 \\ \cos x & \text{if } n \equiv 1 \pmod 4 \\ -\sin x & \text{if } n \equiv 2 \pmod 4 \\ -\cos x & \text{if } n \equiv 3 \pmod 4 \end{cases}$$

Proof. We find the first four derivatives:

$$f^{(1)}(x) = \cos x, \qquad f^{(2)}(x) = -\sin x, \qquad f^{(3)}(x) = -\cos x, \qquad f^{(4)}(x) = \sin x.$$

Then assuming $f^{(4k)}(x) = \sin x$ for some $k \in \mathbb{N}$ we find

$$f^{(4(k+1))} = \frac{\mathrm{d}^4}{\mathrm{d}x^4} f^{(4k)}(x) = \frac{\mathrm{d}^4}{\mathrm{d}x^4} \sin x = \sin x.$$

By induction, $f^{(4k)}(x) = \sin x$ for all $k \in \mathbb{N}$. Suppose $n, m \in \mathbb{N}$ such that $n \equiv m \pmod 4$. Without loss of generality, assume $n > m$ (if $n = m$, there is nothing to prove), so $n = 4k+m$ for some $k \in \mathbb{N}$. Therefore, we have

$$f^{(n)}(x) = \frac{\mathrm{d}^m}{\mathrm{d}x^m} f^{(4k)}(x) = \frac{\mathrm{d}^m}{\mathrm{d}x^m} \sin x = f^{(m)}(x).$$

This, with the values of the first four derivatives, is the desired result. □

As an immediate corollary, we have $|f^{(n)}(x)| \le 1$, since $\max|\sin x| = \max|\cos x| = 1$.

Proposition 3. If $|x| \le M$, then

$$|f(x) - T_n(x;0)| \le \frac{M^{n+1}}{(n+1)!},$$

and

$$\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}, \quad \forall\, x \in \mathbb{R}.$$

Proof. Suppose $x \in \mathbb{R}$ with $|x| \le M$. Note that Lemma 3 implies $f(x)$ has derivatives of all orders. By Taylor's theorem, there exists some point c between 0 and x such that

$$f(x) = T_n(x;0) + \frac{f^{(n+1)}(c)}{(n+1)!}x^{n+1}.$$

Then we obtain the first result

$$\begin{aligned} |f(x) - T_n(x;0)| &= \left|\frac{f^{(n+1)}(c)}{(n+1)!}x^{n+1}\right| \\ &= \left|f^{(n+1)}(c)\right| \frac{|x|^{n+1}}{(n+1)!} \\ &\le \frac{M^{n+1}}{(n+1)!}. \end{aligned}$$

Now let $x \in \mathbb{R}$ be arbitrary and let

$$a_n = \frac{|x|^{n+1}}{(n+1)!}, \quad n \in \mathbb{N}$$

define a sequence, then

$$|f(x) - T_n(x;0)| \le a_n.$$

Let $N \in \mathbb{N}$ with $N \ge |x|$, then for all $n \ge N$, we have the recursive relation between terms

$$a_n = \frac{|x|^{n+1}}{(n+1)!} = a_{n-1}\left(\frac{|x|}{n+1}\right) \le a_{n-1}\left(\frac{N}{N+1}\right).$$

Thus, for all $n \ge N$, we have

$$|f(x) - T_n(x;0)| \le a_n \le a_N\left(\frac{N}{N+1}\right)^{n-N}.$$

Since $N/(N+1) < 1$, then the limit of the geometric sequence is zero. Thus,

$$f(x) = \lim_{n\to\infty} T_n(x;0) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n.$$

From Lemma 3, $f^{(n)}(0) = \pm \sin 0 = 0$ for all even n, so

$$\sin x = \sum_{n=0}^{\infty} f^{(2n+1)}(0)\frac{x^{2n+1}}{(2n+1)!}.$$

Moreover, $f^{(2n+1)}(0) = \cos 0 = 1$ when n is even and $f^{(2n+1)}(0) = -\cos 0 = -1$ when n is odd. Hence,

$$\sin x = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}.$$

□

4

Suppose f is a real function on $(-\infty, \infty)$. We say that $x \in \mathbb{R}$ is a *fixed point* for f if $f(x) = x$.

4(a)

If f is differentiable and $f'(t) \neq 1$ for every real t, prove that f has at most one fixed point.

Proof. Suppose, for contradiction, that f has real, distinct fixed points x and y. The mean value theorem tells us that there is some point c between x and y such that

$$f(x) - f(y) = f'(c)(x - y).$$

However, $f(x) - f(y) = x - y \neq 0$ implies that $f'(c) = 1$, which is a contradiction.

□

4(b)

Show that the function f defined by

$$f(t) = t + \frac{1}{1 + e^t}$$

has no fixed point, although $0 < f'(t) < 1$ for all real t.

Proof. Suppose, for contradiction, that f has a real fixed point x. Then

$$0 = f(x) - x = \frac{1}{1 + e^x}.$$

However, zero is not the reciprocal of any real number, so this is a contradiction. The derivative of f is

$$\begin{aligned}
f'(t) &= \frac{\mathrm{d}}{\mathrm{d}t}\left(t + \frac{1}{1 + e^t}\right) \\
&= 1 + \frac{(1 + e^t)\frac{\mathrm{d}}{\mathrm{d}t}1 - 1\frac{\mathrm{d}}{\mathrm{d}t}(1 + e^t)}{(1 + e^t)^2} \\
&= 1 + \frac{-e^t}{(1 + e^t)^2} \\
&= \frac{1 + e^t + e^{2t}}{1 + 2e^t + e^{2t}}.
\end{aligned}$$

Since the exponential function is always positive, we have

$$0 < f'(t) < \frac{1 + 2e^t + e^{2t}}{1 + 2e^t + e^{2t}} = 1.$$

□

4(c)

Prove that if there is a constant $0 < A < 1$ such that $|f'(t)| \leq A$ for all $t \in \mathbb{R}$, then f has a fixed point x. To do this, given $x_1 \in \mathbb{R}$ arbitrary, construct the sequence

$$x_{n+1} = f(x_n) \quad n \geq 1,$$

and prove that the sequence converges to some point x. Then prove that x is the fixed point.

Proof. Suppose $0 < A < 1$ with $|f'(t)| \leq A$ for all $t \in \mathbb{R}$. Without loss of generality, assume $f(0) > 0$. If $f(0) = 0$, then 0 is a fixed point. If $f(0) < 0$, then $g(t) = -f(-t)$ is differentiable and $|g'(t)| = |f'(-t)| \leq A$. Moreover, $g(0) = -f(0) > 0$ and if g has a fixed point x, then $f(-x) = -g(x) = -x$, i.e., f has the fixed point $-x$. Thus, it suffices to prove the case that $f(0) > 0$.

By the mean value theorem, if $t > 0$ then there exists some $c \in (0, t)$ such that

$$f(t) - f(0) = f'(c)(t - 0) \leq At,$$

implying $f(t) \leq f(0) + At$. We define

$$b = \frac{f(0)}{1 - A},$$

which is positive since $f(0) > 0$ and $A < 1$, so $f(b) \leq f(0) + Ab = b$. If $f(b) = b$, then b is a fixed point. Otherwise, $h(t) = t - f(t)$ is a continuous function with $h(0) < 0 < h(b)$. The intermediate value theorem gives us $x \in (0, b)$ with $h(x) = 0$, i.e., $f(x) = x$, so x is a fixed point for f.

$\square$