This one is bad. Don't look please.

1 Define $r := \sqrt{x_1^2 + x_2^2 + x_3^2}$. Let $\omega := (\frac{1}{r})^3(x_3 dx_1 \wedge dx_2 - x_2 dx_1 \wedge dx_3 + x_1 dx_2 \wedge dx_3)$ be a 2-form on $\mathbb{R}^3 \setminus (0,0,0)$.

(a) Show that $d\omega = 0$.

Note that

$$(\mathrm{d}x_i \wedge \mathrm{d}x_j) \wedge \mathrm{d}x_i = -(\mathrm{d}x_i \wedge \mathrm{d}x_i) \wedge \mathrm{d}x_j = 0,$$

so if $\{i, j, k\} = \{1, 2, 3\}$, then

$$\begin{aligned} \mathrm{d}\Big(\frac{x_i}{r^3}\Big) \wedge \mathrm{d}x_j \wedge \mathrm{d}x_k &= \mathrm{D}_i \frac{x_i}{r^3} \mathrm{d}x_i \wedge \mathrm{d}x_j \wedge \mathrm{d}x_k + \mathrm{D}_j \frac{x_i}{r^3} \mathrm{d}x_j \wedge \mathrm{d}x_j \wedge \mathrm{d}x_k + \mathrm{D}_k \frac{x_i}{r^3} \mathrm{d}x_k \wedge \mathrm{d}x_j \wedge \mathrm{d}x_k \\ &= \mathrm{D}_i \frac{x_i}{r^3} \mathrm{d}x_i \wedge \mathrm{d}x_j \wedge \mathrm{d}x_k + 0 + 0. \end{aligned}$$

Then

$$\begin{aligned} \mathrm{d}\omega &= \mathrm{d}\Big(\frac{x_3}{r^3}\Big) \wedge \mathrm{d}x_1 \wedge \mathrm{d}x_2 + \mathrm{d}\Big(\frac{x_2}{r^3}\Big) \wedge \mathrm{d}x_1 \wedge \mathrm{d}x_3 + \mathrm{d}\Big(\frac{x_1}{r^3}\Big) \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3 \\ &= \mathrm{D}_3 \frac{x_3}{r^3} \mathrm{d}x_3 \wedge \mathrm{d}x_1 \wedge \mathrm{d}x_2 - \mathrm{D}_2 \frac{x_2}{r^3} \mathrm{d}x_2 \wedge \mathrm{d}x_1 \wedge \mathrm{d}x_3 + \mathrm{D}_1 \frac{x_1}{r^3} \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3 \\ &= \Big(\mathrm{D}_1 \frac{x_1}{r^3} + \mathrm{D}_2 \frac{x_2}{r^3} + \mathrm{D}_3 \frac{x_3}{r^3}\Big) \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3 \\ &= \left(\frac{-2x_1^2 + x_2^2 + x_3^2}{r^5} + \frac{x_1^2 - 2x_2^2 + x_3^2}{r^5} + \frac{x_1^2 + x_2^2 - 2x_3^2}{r^5}\right) \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3 \\ &= 0 \, \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3 \\ &= 0. \end{aligned}$$

(b) Let $B := \{(x_1, x_2, x_3) : (x_1 - 2)^2 + x_2^2 + x_3^2 = 3\}$ be a sphere in $\mathbb{R}^3$, find the integral $\int_B \omega$.

Note that $y = Ix = x$ and $\det(I)$, so

$$\mathrm{d}y = \mathrm{d}x = \det(I)\,\mathrm{d}x.$$

The wedge product satisfies the multilinearity and alternating of the determinant, which uniquely characterizes it

3 Let D be the closed unit disk in $\mathbb{R}^2$ and f be a continuous function on D. Show that for any $\epsilon > 0$, there exists a number n and functions $f_1, f_2, \ldots, f_n$ such that $f = f_1 + \ldots + f_n$ on D and the support of f_i has Lebesgue measure less than ϵ, for any $i = 1, \ldots, n$. State any theorem you use.

Given $\varepsilon > 0$, choose $n \in \mathbb{N}$ with $n > 4\pi/\varepsilon$. For $k = 0, \ldots, n-1$, define the open annulus

$$A_k = \{x \in \mathbb{R}^2 : \tfrac{k-1}{n} < |x| < \tfrac{k+1}{n}\},$$

Area in $\mathbb{R}^2$ coincides with the Lebesgue measure. By construction, for all k,

$$m(A_k) \leq m(A_{n-1})) = \pi\left((\tfrac{n}{n})^2 - (\tfrac{n-2}{n})^2\right) = \pi\left(\tfrac{4}{n} - \tfrac{4}{n^2}\right) = \tfrac{4\pi}{n}(1 - \tfrac{1}{n}) \leq \tfrac{4\pi}{n} < \varepsilon.$$

The collection $\{A_k\}_{k=0}^n$ forms an open cover of the unit disc in $\mathbb{R}^2$. Then there exists a partition of unity $\{\psi_j\}_{j=1}^m$ with each ψ_j having its support contained in some A_k. Define $f_j = \psi_j f$, then $f = f_1 + \cdots + f_m$ and each f_j has its support in some A_k, in particular, the support of f_j has Lebesgue measure at most $m(A_k) < \varepsilon$.

5 Prove that a subset E of $\mathbb{R}^n$ is Lebesgue measurable if and only if for any $\epsilon > 0$, there exists an open set $U \subset \mathbb{R}^n$ such that $E \subset U$ and $m(U \setminus E) < \epsilon$.

We know that m is regular on Lebesgue measurable sets, i.e., there exists an open set $U \subseteq \mathbb{R}^n$ containing E such that $m(U \setminus E) < \varepsilon$.

If such a U exists, then U and $U \setminus E$ are measurable, so $U \setminus (U \setminus E) = E$ is measurable.

5 Let $\{f_n\}$ be a sequence of measurable functions and define $f := \liminf_n f_n$. Is f measurable? If yes, justify your answer. If no, give a counterexample.

Yes.

Define $g_n = -f_n$ measurable for all $n \in \mathbb{N}$, then we have that $g = \limsup_n g_n$ is measurable function. Therefore, so is $f = \liminf_n f_n = -\limsup_n g_n = -g$.

6 Let $\{f_n\}$ be a uniformly convergent and uniformly bounded sequence of Lebesgue integrable functions on $\mathbb{R}^1$ and let $f := \lim_n f_n$ be the limit. Is it true that

$$\lim_n \int_{\mathbb{R}^1} f_n dm = \int_{\mathbb{R}^1} f dm?$$

If yes, justify your answer. If no, give a counterexample. All integrals are Lebesgue integrals.

No

$$f_n(x) = \begin{cases} 1/n & 0 \le x \le n, \\ 0 & \text{otherwise.} \end{cases}$$

Then $f_n \to 0$ uniformly on $\mathbb{R}^1$ and uniformly bounded by 1. But $\int_{\mathbb{R}^1} f_n = 1$ and $\int_{\mathbb{R}^1} 0 = 0$.