Exercise 1.21 Determine the radical of the ideal $\langle x_1^3 - x_2^6,\ x_1x_2 - x_2^3\rangle \trianglelefteq \mathbb{C}[x_1, x_2]$.

Let $J = \langle x_1^3 - x_2^6,\ x_1x_2 - x_2^3\rangle$ and let $z = (z_1, z_2) \in V(J)$. Then we have

$$z_1^3 = z_2^6 \quad \text{and} \quad z_1z_2 = z_2^3.$$

If either z_1 or z_2 is zero, then the first condition implies that they are both zero. Then if $z_2 \neq 0$, the second condition is equivalent to $z_1 = z_2^2$. So $z = (w^2, w)$ where $w = z_2$.

Now consider the point $z = (w^2, w)$ for some $w \in \mathbb{C}$. We have

$$(w^2)^3 - w^6 = 0 \quad \text{and} \quad w^2w - w^3 = 0,$$

so in fact $z \in V(J)$. Thus, we have that

$$V(J) = \{(w^2, w) \mid w \in \mathbb{C}\}.$$

This is precisely the zero locus of the polynomial $x_1 - x_2^2 \in \mathbb{C}[x_1, x_2]$, since $x_1 - x_2^2 = 0$ if and only if $x_1 = x_2^2$. Therefore, we have

$$\sqrt{J} = I(V(J)) = I(V(\langle x_1 - x_2^2\rangle)) = \sqrt{\langle x_1 - x_2^2\rangle}.$$

If we can show that $x_1 - x_2^2$ is irreducible in the unique factorization domain $\mathbb{C}[x_1, x_2]$, then its ideal is prime and, therefore, radical. Suppose for contradiction that $x_1 - x_2^2$ is reducible into two nonzero, non-unit polynomials $p, q \in \mathbb{C}[x_1, x_2]$. Since $\mathbb{C}$ is a field, then the degrees of both p and q must be at least 1. And since

$$2 = \deg(x_1 + x_2^2) = \deg pq = \deg p + \deg q,$$

then we must have $\deg p = \deg q = 1$. Then p and q are of the form

$$p = a_0 + a_1x_1 + a_2x_2 \quad \text{and} \quad q = b_0 + b_1x_1 + b_2x_2.$$

Since $pq = x_1 - x_2^2$ has no constant term, then either $a_0 = 0$ or $b_0 = 0$. Without loss of generality, assume $b_0 = 0$, then

$$\begin{aligned} x_1 - x_2^2 &= pq \\ &= (a_0 + a_1x_1 + a_2x_2)(b_1x_1 + b_2x_2) \\ &= a_0b_1x_1 + a_0b_2x_2 + a_1b_1x_1^2 + (a_1b_2 + a_2b_1)x_1x_2 + b_1b_2x_2^2 \end{aligned}$$

Comparing the coefficient of x_1 on both sides, we see that $1 = a_0b_1$, implying that $a_0 \neq 0$. Similarly, the coefficient of x_2^2 is $-1 = b_1b_2$, so $b_2 \neq 0$. Lastly, the coefficient of x_2 is given by $0 = a_0b_2$. This is a contradiction since a_0 and b_2 are nonzero. Hence, the ideal of $x_1 - x_2^2$ is radical, so

$$\sqrt{J} = \sqrt{\langle x_1 - x_2^2\rangle} = \langle x_1 - x_2^2\rangle.$$

Exercise 1.22 Let $X \subset \mathbb{A}^3$ be the union of the three coordinate axes. Compute the generators for the ideal $I(X)$, and show that $I(X)$ cannot be generated by fewer than three elements.

Proposition 1. Let $S = \{x_1x_2,\ x_1x_3,\ x_2x_3\}$. Then $I(X) = \langle S\rangle$ and cannot be generated by a smaller set.

Proof. In general, the jth coordinate axis of $\mathbb{A}^n$ is given by

$$X_j = \{a \in \mathbb{A}^n \mid a_i = 0 \text{ for all } i \neq j\{.$$

In particular, we are considering $X = X_1 \cup X_2 \cup X_3 \subset \mathbb{A}^3$, which has the ideal

$$I(X) = I(X_1) \cap I(X_2) \cap I(X_3).$$

Given any $a \in X_1$, we know that $a_2 = a_3 = 0$, so

$$a_1a_2 = a_1a_3 = a_2a_3 = 0.$$

This means that $f(a) = 0$ for all $f \in S$, implying $\langle S\rangle \subseteq I(X_1)$. The same can be said for X_2 and X_3, so $\langle S\rangle \subseteq I(X)$.

Consider a polynomial $f \in K[x_1, x_2, x_3]$. The restriction $f|_{X_1}$ is a polynomial function on the first coordinate axis, which can be found explicitly by 'evaluating' f at the point $(x_1, 0, 0)$. That is, $f|_{X_1}$ can be thought of as a polynomial in $K[x_1]$ given the terms of f which contain neither x_2 nor x_3.

If $f \in I(X_1)$, then we know that $f|_{X_1} = 0$, meaning that f has no terms which contain neither x_2 nor x_3. In other words, any term of f which has a factor of x_1, must also have a factor of either x_2 or x_3. Applying the same argument to the other axes, we conclude that any polynomial f has no terms containing only a single indeterminate and neither of the other two. So each term of f is a multiple of either x_1x_2, x_1x_3, or x_2x_3, i.e., $f \in \langle S\rangle$. Thus, we have $I(X) = \langle S\rangle$.

We now show that no fewer than three elements can generate $I = \langle S\rangle$. Let $M = I(0) = \langle x_1, x_2, x_3\rangle$ be the maximal ideal whose zero locus is the origin. Then we have the product of ideals

$$MI = \langle x_1, x_2, x_3\rangle\langle S\rangle = \langle x_1x_2x_3,\ x_1^2x_2,\ x_1x_2^2,\ x_1^2x_3,\ x_1x_3^2,\ x_2^2x_3,\ x_2x_3^2\rangle,$$

which is the set of polynomials $f \in K[x_1, x_2, x_3]$ such that each term of f is of degree at least 3 and has a factor in S. Since $I = \langle S\rangle$ is precisely the polynomials in $K[x_1, x_2, x_3]$ with each term having a factor in S, then MI is the polynomials in I with each term of degree at least 3. Therefore, the quotient I/MI is the (equivalence classes of) polynomials in I with degree 2.

Let $R = K[x_1, x_2, x_3]$, then I/MI is an R-module with the action $r\overline{f} = \overline{rf}$ for $r \in R$ and $\overline{f} \in I/MI$. Given $m \in M$ and $f \in I$, we know that $mf \in MI$, implying that $m\overline{f} = \overline{mf} = \overline{0}$ in I/MI. In other words, the M-action on I/MI, as a subset of the R-action, is annihilated. Therefore, the (R/M)-action given by by $(r + M)\overline{f} = r\overline{f}$ is well-defined on I/MI, making it an (R/M)-module.

We have

$$R/M = K[x_1, x_2, x_3]/\langle x_1, x_2, x_3\rangle \cong K,$$

and we regard I/MI as the set of polynomials

$$V = \{k_1x_1x_2 + k_2x_1x_3 + k_3x_2x_3 \mid k_1, k_2, k_3 \in K\}.$$

Then V is a K-module with the action of left-multiplication, and is 'the same as' the (R/M)-module I/MI. In fact, this makes V a K-vector space with addition from the polynomial ring and scalar multiplication from the (R/M)-action. Note that S is a generating set for the vector space V, and if

$$k_1x_1x_2 + k_2x_1x_3 + k_3x_2x_3 = 0,$$

then we must have $k_1 = k_2 = k_3 = 0$, so S is a basis for V. Hence, $\dim_K V = 3$.

Now suppose for contradiction that $I = \langle p, q\rangle$. We can map p and q into V by restricting each to only its terms of degree 2. Since they generate I, then in particular, they generate V as a subset of I. Let $f \in V$ and suppose $f = ap + bq$ for some $a, b \in R$. Since $p, q \in I$, then their terms must be of degree at least 2. So the value of $ap + bq$ depends only on the degree 2 terms of p and q and the constant terms of a and b. Therefore, f must be some K-linear combination of p and q restricted to their terms of degree 2. This means that the restrictions of p and q generate V as a K-vector space, implying that $\dim_K V \leq 2$, which is a contradiction.

$\square$