Fix a base field $K$.

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matrix problems

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A **quiver** $Q$ consists of

- A set $Q_0$ of **vertices**;
- A set $Q_1$ of **arrows**;
- A function $s : Q_1 \to Q_0$ indicating the starting vertex of an arrow;
- A function $t : Q_1 \to Q_0$ indicating the ending vertex of an arrow.

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A **representation** of a quiver $Q$ (over $K$) consists of

- a finite dimensional $K$-vector space $V_i \in K\text{-}\mathsf{vect}$ for each vertex $i \in Q_0$;
- a linear transformation $f_\alpha : V_{s(\alpha)} \to V_{t(\alpha)}$ for each arrow $\alpha \in Q_1$

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A **path** of length $\ell$ is a $(\ell + 2)$-tuple written

$$w = (j|\alpha_\ell, \ldots \alpha_2, \alpha_1|i)$$

where $i, j \in Q_0$ and $\alpha_n \in Q_1$ are such that $s(\alpha_1) = i$, $t(\alpha_n) = s(\alpha_{n+1})$, and $t(\alpha_\ell) = j$.

Each vertex $i \in Q_0$ is identified with a trivial/identity path $e_i = (i||i)$.

Then $s$ and $t$ can be extended to all all paths by $s(w) = i$ and $t(w) = j$.

Define a concatenation

$$(k|\beta_m, \ldots, \beta_1|j) \circ (j|\alpha_n, \ldots, \alpha_1|i) = (k|\beta_m, \ldots, \beta_1, \alpha_n, \ldots, \alpha_1|i)$$

A **cycle** is a path $w$ with $s(w) = t(w)$.

A cycle of length 1 is a **loop**.

For $\ell \geq 0$, let $Q_\ell$ denote the set of paths of length $\ell$ in $Q$. This is consistent with $Q_0$ being the vertices and $Q_1$ being the arrows. Let

$$Q_\bullet = \bigcup_{\ell \geq 0} Q_\ell$$

be the set of all paths in $Q$.

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The **path category** or **free category** of a quiver $Q$ is the category whose objects are the vertices of $Q$ and whose morphisms are paths in $Q$.

Denote it by something like $\mathsf{cat}(Q)$.

This is a category enriched over $K$-vector spaces.

Then a representation of $Q$ is simply a functor $\mathsf{cat}(Q) \to K\text{-}\mathsf{vect}$.

Denote the functor category, $\mathsf{D}_K(Q) = \mathsf{rep}_K(Q) = [\mathsf{cat}(Q), K\text{-}\mathsf{vect}]$, and call it the category of representations of $Q$.

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For a quiver $Q$, the **path algebra** $KQ$ is defined as free $K$-module generated by the set $\mathrm{Mor}(\mathsf{cat}(Q))$ of all paths in $Q$ with multiplication

$$pq = \begin{cases} p \circ q & \text{if } s(p) = t(q), \\ 0 & \text{otherwise.} \end{cases}$$

$$KQ = \bigoplus_{i,j \in Q_0} K \cdot \mathsf{cat}(Q)(i,j).$$

$KQ = Q_\bullet^{(K)}$ all finite sums over $Q_\bullet$ with coefficients in $K$.

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Something to check that $KQ\text{-}\mathsf{mod} \cong \mathsf{rep}_K(Q)$

Given $M \in KQ\text{-}\mathsf{mod}$, define $M_i = e_i M$ for each $i \in Q_0$, then $M = \bigoplus_{i \in Q_0} M_i$ and for each arrow $\alpha : i \to j \in Q_1$, define

$$\begin{aligned} f_\alpha : M_i &\longrightarrow M_j \\ e_i m &\longmapsto \alpha e_i m = \alpha m. \end{aligned}$$

Then $((M_i)_{i \in Q_0}, (f_\alpha)_{\alpha \in Q_1})$ is a representation of $Q$.

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Conditions for ideal $I$ to give path algebra modulo relations.

An ideal $I \trianglelefteq KQ$ is called **admissible** if

- $I$ is generated over $KQ$ by paths of length at least 2;
- there exists $N \in \mathbb{N}$ such that all paths of length $N$ belong to $I$.

(In particular, $I$ is also generated over $K$ by some paths of length at least 2.)

If $I \trianglelefteq KQ$ is admissible, say $KQ/I$ is a path algebra modulo relations.

Have all good decomposition results for $KQ/I$.

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For $\Lambda \in K\text{-}\mathsf{alg}$ there is always a complete set of primitive orthogonal idempotents $e_1, \ldots e_n \in \Lambda$.

Then ${}_\Lambda\Lambda = \Lambda e_1 \oplus \cdots \oplus \Lambda e_n$ with each $\Lambda e_i$ indecomposable in $\Lambda\text{-}\mathsf{mod}$.

The **Jacobson radical** of $\Lambda$ is

$$J = J(\Lambda) = \bigcap \{\text{maximal left ideals of } \Lambda\}.$$

Up to isomorphism, $S_i = \Lambda e_i / J e_i$ are all simples in $\Lambda\text{-}\mathsf{mod}$.

For $M \in \Lambda\text{-}\mathsf{mod}$, have a decomposition

$$M = e_1 M \oplus \cdots \oplus e_n M.$$

Then $\dim_K e_i M$ is the multiplicity of $S_i$ as a composition factor of $M$.

Define **dimension vector**

$$\underline{\dim}\, M = (\dim_K e_1 M, \ldots, \dim_K e_n M).$$

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For $\Lambda = KQ/I$ and dimension vector $\underline{d} = (d_1, \ldots, d_n)$, define

$$\mathrm{Rep}_{\underline{d}}(\Lambda) = \left\{ x \in \prod_{\alpha \in Q_1} M_{d_{t(\alpha)} \times d_{s(\alpha)}}(K) \;\middle|\; \text{“}x \text{ satisfies relations in } I\text{”} \right\}.$$

Each element $\gamma \in I$ is a finite sum

$$\gamma = \sum_{\substack{p \in \mathsf{cat}(Q) \\ \mathrm{len}(p) \geq 2}} c_p p,$$

with $c_p \in K$. If $\ell = \mathrm{len}(p)$, then $p = \alpha_\ell \cdots \alpha_1$ for some $\alpha_i \in Q_1$.

Given $x = (x_\alpha)_{\alpha \in Q_1}$, write

$$x_p = x_{\alpha_\ell} \cdots x_{\alpha_1}$$

and

$$\hat{\gamma} = \sum c_p \hat{p}.$$

Then say “$x$ satisfies relations in $I$” if $\hat{\gamma} = 0$ for all $\gamma \in I$.

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For a quiver $Q$ and dimension vector $\underline{d} = (d_1, \ldots, d_n)$ define the set

$$\mathbb{M}_{\underline{d}}(Q) = \prod_{\alpha \in Q_1} M_{d_{t(\alpha)} \times d_{s(\alpha)}}(K).$$

This is essentially the affine space $\mathbb{A}_K^N$ with $N = \sum_{\alpha \in Q_1} d_{t(\alpha)} \cdot d_{s(\alpha)}$.

For $x = (x_\alpha)_{\alpha \in Q_1} \in \mathbb{M}_{\underline{d}}(Q)$ and $p = (j|\alpha_\ell, \ldots, \alpha_1|i) \in Q_\bullet$, define

$$x_p = x_{\alpha_\ell} \cdots x_{\alpha_1} \in M_{d_j \times d_i}(K).$$

By convention, $x_{(i||i)} = I_{d_i}$ for all trivial paths $(i||i) \in Q_0$.

For $\gamma = \sum_{p \in Q_\bullet} c_p p \in KQ$, define some good notion of

$$\gamma(x) = \sum_{p \in Q_\bullet} c_p x_p.$$

I guess for each $p \in Q_\bullet$, we have $c_p x_p \in M_{d_{t(p)} \times d_{s(p)}}(K)$, so this will be an element of something like

$$\bigoplus_{d, d' \in \geq 0} M_{d \times d'}(K).$$

Define

$$\operatorname{Rep}_{\underline{d}}(\Lambda) = \{x \in \mathbb{M}_{\underline{d}}(Q) \mid \gamma(x) = 0 \text{ for all } \gamma \in I\} = Z(I) \subseteq \mathbb{M}_{\underline{d}}(Q).$$

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For each $x \in \operatorname{Rep}_{\underline{d}}(\Lambda)$, we construct $M_x \in \Lambda\text{-}\mathsf{mod}$ as the set

$$M_x = \bigoplus_{i=1}^n K^{d_i}$$

with $\Lambda$ scalar multiplication in defined for each arrow $\alpha \in Q_1$ by

$$\alpha \cdot (m_1 + \cdots + m_n) = x_\alpha m_{s(\alpha)} \in K^{t(\alpha)} \subseteq M_x.$$

The map

$$\begin{aligned} \Phi : \operatorname{Rep}_{\underline{d}}(\Lambda) &\longrightarrow \{M \in \Lambda\text{-}\mathsf{mod} \mid \underline{\dim}\, M = \underline{d}\}/\text{iso} \\ x &\longmapsto [M_x] \end{aligned}$$

is a surjection whose fibers are the orbits of the group $G = \prod_{i=1}^n \mathrm{GL}_{d_i}(K)$ under the action

$$\begin{aligned} G \times \operatorname{Rep}_{\underline{d}}(\Lambda) &\longrightarrow \operatorname{Rep}_{\underline{d}}(\Lambda) \\ (g, x) &\longmapsto \left(g_{t(\alpha)} x_\alpha g_{s(\alpha)}^{-1}\right)_{\alpha \in Q_1}. \end{aligned}$$

This is understood as $g \cdot x$ referring to a change of basis for each linear map $K^{s(\alpha)} \to K^{t(\alpha)}$ corresponding to the matrix $x_\alpha$.

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Let $\Lambda = KQ/I$ be path algebra modulo relations.

Let $M \in \Lambda\text{-}\mathsf{mod}$ with $\underline{d} = \underline{\dim}\, M$ and let $x \in \operatorname{Rep}_{\underline{d}}(\Lambda)$ such that $M$ corresponds to the orbit $G \cdot x \subseteq \operatorname{Rep}_{\underline{d}}(\Lambda)$. If $U \leq M$ is a submodule, then $U \oplus M/U$ corresponds to an orbit contained in $\overline{G \cdot x}$.