I worked with Joseph Sullivan and Gahl Shemy.

1 Exercise 1.1.1 If $k < \ell$ we can consider $\mathbb{R}^k$ to be the subset $\{(a_1, \dots, a_k, 0, \dots, 0)\}$ in $\mathbb{R}^\ell$. Show that the smooth functions on $\mathbb{R}^k$, considered as a subset of $\mathbb{R}^\ell$, are the same as usual.

Proof. Recall that a function $f = (f_1, \dots, f_m) : \mathbb{R}^n \to \mathbb{R}^m$ is smooth if and only if all its component functions $f_i : \mathbb{R}^n \to \mathbb{R}$ are smooth. Therefore, it suffices to consider only functions with codomain $\mathbb{R}$.

Let $\iota : \mathbb{R}^k \hookrightarrow \mathbb{R}^\ell$ be standard immersion and $\pi : \mathbb{R}^\ell \to \mathbb{R}^k$ be the standard submersion, which are both smooth maps.

Given $f : \mathbb{R}^k \to \mathbb{R}$ smooth in the usual sense, the composition $F = f \circ \pi : \mathbb{R}^\ell \to \mathbb{R}$ is again smooth. Moreover,

$$F|_{\mathbb{R}^k} = f \circ \pi|_{\mathbb{R}^k} = f \circ \mathrm{id}_{\mathbb{R}^k} = f,$$

so F is a smooth extension of f to $\mathbb{R}^\ell$ (an open neighborhood of $\mathbb{R}^k$). By definition, this means f is smooth on $\mathbb{R}^k$ as a subset of $\mathbb{R}^\ell$.

Given $f : \mathbb{R}^k \to \mathbb{R}$ smooth in the subset sense, let $F : U \to \mathbb{R}$ be a smooth local extension at a point $x \in \mathbb{R}^k$ to an open neighborhood $U \subseteq \mathbb{R}^\ell$. Then the composition

$$F \circ \iota = F|_{\mathbb{R}^k} = f|_{U \cap \mathbb{R}^k}$$

is smooth in the usual sense. In particular, f is smooth at x and therefore on all of $\mathbb{R}^k$. $\square$

2 Exercise 1.1.2 Suppose that X is a subset of $\mathbb{R}^N$ and Z is a subset of X. Show that the restriction to Z of any smooth map on X is a smooth map on Z.

Proof. As in Problem 1, it suffices to consider only functions with codomain $\mathbb{R}$.

Let $f : X \to \mathbb{R}$ be a smooth function and $z \in Z$ be any point. Since f is smooth at $z \in X$ (as a subset of $\mathbb{R}^N$), we can find a smooth local extension $F : U \to \mathbb{R}$ of f at z, where $U \subseteq \mathbb{R}^N$ is an open neighborhood of z. Then $F|_Z = f|_Z$ means that F is also a smooth local extension of $f|_Z$ at z. By definition, $f|_Z$ is smooth map on Z. $\square$

3 Exercise 1.1.3 Let $X \subseteq \mathbb{R}^N$, $Y \subseteq \mathbb{R}^M$, $Z \subseteq \mathbb{R}^L$ be arbitrary subsets, and let $f : X \to Y$, $g : Y \to Z$ be smooth maps. Then the composite $g \circ f : X \to Z$ is smooth.

Proof. Let $x \in X$ and $F : U \to \mathbb{R}^M$ be a smooth local extension of f at x. Next, let $G : V \to \mathbb{R}^L$ be a smooth local extension of g at $f(x)$. Set $W = U \cap F^{-1}(V)$, then $F|_W : W \to V$ is a smooth extension of f at x. Then the composition of smooth maps (in the usual sense) $G \circ F|_W : W \to \mathbb{R}^L$ is again smooth. Moreover, $G \circ F|_W$ is in fact a local extension of $g \circ f$ at x. Hence, $g \circ f$ is a smooth map on X. $\square$

If f and g are diffeomorphisms, so is $g \circ f$.

Proof. When f and g are diffeomorphisms, they have smooth inverses $f^{-1} : Y \to X$ and $g^{-1} : Z \to Y$, respectively. By the previous result, the function inverse $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$ is smooth. $\square$

4 Exercise 1.1.4(a) Let B_a be the open ball $\{x : |x|^2 < a^2\}$ in $\mathbb{R}^k$. ($|x|^2 = \sum x_i^2$) Show that the map

$$x \longmapsto \frac{ax}{\sqrt{a^2 - |x|^2}}$$

is a diffeomorphism of B_a onto $\mathbb{R}^k$. [Hint: Compute its inverse directly.]

Proof. Denote the given map by $f : B_a \to \mathbb{R}^k$ and define $g : \mathbb{R}^k \to \mathbb{R}^k$ by

$$g(x) = \frac{ax}{\sqrt{a^2 + |x|^2}}.$$

We check that the image of g is contained in B_a:

$$\left| \frac{ax}{\sqrt{a^2 + |x|^2}} \right|^2 = \frac{a^2|x|^2}{a^2 + |x|^2} = \frac{a^2}{\frac{a^2}{|x|^2} + 1} \leq a^2.$$

This allows us to consider g as be a map $\mathbb{R}^k \to B_a$. As compositions of smooth functions, both f and g are smooth—we claim they are inverses. First, for any $x \in \mathbb{R}^k$,

$$f(g(x)) = \frac{a\frac{ax}{\sqrt{a^2+|x|^2}}}{\sqrt{a^2 - \left|\frac{ax}{\sqrt{a^2+|x|^2}}\right|^2}} = \frac{\frac{a^2x}{\sqrt{a^2+|x|^2}}}{\sqrt{a^2 - \frac{a^2|x|^2}{a^2+|x|^2}}} = \frac{a^2x}{\sqrt{a^4 + a^2|x|^2 - a^2|x|^2}} = x.$$

In particular, this tells us that f is surjective, since $g(x) \in B_a$ is a point in the domain of f mapping to x. Second, for any $x \in B_a$,

$$g(f(x)) = \frac{a\frac{ax}{\sqrt{a^2-|x|^2}}}{\sqrt{a^2 + \left|\frac{ax}{\sqrt{a^2-|x|^2}}\right|^2}} = \frac{a^2x}{\sqrt{(a^2 - |x|^2)\left(a^2 + \frac{a^2|x|^2}{a^2-|x|^2}\right)}} = \frac{a^2x}{\sqrt{a^4 - a^2|x|^2 + a^2|x|^2}} = x.$$

Hence, f is a diffeomorphism with smooth inverse g. $\square$

5 Exercise 1.1.6 A smooth bijective map of manifolds need not be a diffeomorphism. In fact, show that $f : \mathbb{R}^1 \to \mathbb{R}^1$, $f(x) = x^3$ is an example.

Proof. The function inverse of f is $g(x) = \sqrt[3]{x}$, but this is not differentiable at 0, since

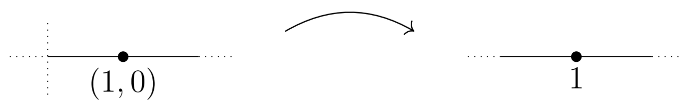
$$\lim_{h\to 0} \frac{g(0+h) - g(0)}{h} = \lim_{h\to 0} \frac{\sqrt[3]{h}}{h} = \lim_{h\to 0} \frac{1}{h^{2/3}} = \infty.$$

$\square$

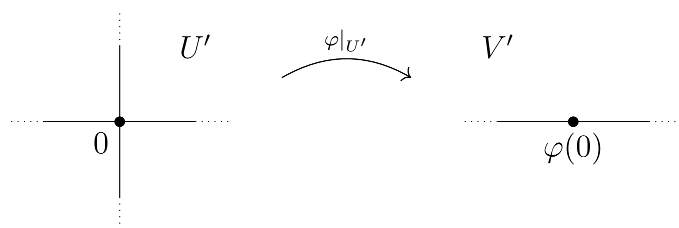
6 Exercise 1.1.7 Prove that the union of the two coordinate axes in $\mathbb{R}^2$ is not a manifold.

Proof. Assume in contradiction that $X = \{xy = 0\} \subseteq \mathbb{R}^2$ is a manifold.

We first show that X must be a 1-dimensional manifold. The point $(1, 0) \in X$ has an open neighborhood $(0, 2) \times 0 \subseteq X$, which is diffeomorphic to the open interval $(0, 2) \subseteq \mathbb{R}^1$ via projecting onto the first coordinate. In particular, X is locally 1-dimensional at the point $(1, 0)$, so by definition it must be globally 1-dimensional.



Let $\varphi : U \to V$ be a smooth chart of X around the origin, i.e., $U \subseteq X$ is an open neighborhood of the origin, $V \subseteq \mathbb{R}^1$ is open, and φ is a diffeomorphism. Since $X \subseteq \mathbb{R}^2$ has the subspace topology, we may restrict our attention to a small open ball $B_r(0) \subseteq \mathbb{R}^2$ whose intersection $U' = B_r(0) \cap X$ is contained in U. Set $V' = \varphi(U') \subseteq V$, then $U' \subseteq U$ being open implies that the restriction $\varphi|_{U'} : U' \to V'$ is still a diffeomorphism, i.e., a smooth chart.



On one hand, U' is star-shaped (with all line segments to the origin) so it is a connected space. The homeomorphism $\varphi|_{U'}$ preserves connectedness, therefore $V' \subseteq \mathbb{R}^1$ must be an open interval. On the other hand, $U' \setminus \{0\}$ is an open subset with four connected components—namely $\{\pm x > 0\} \cap U$ and $\{\pm y > 0\} \cap U$. Restricting φ gives a diffeomorphism to

$$\varphi(U' \setminus \{0\}) = \varphi(U') \setminus \{\varphi(0)\} = V' \setminus \{\varphi(0)\}.$$

However, removing a single point from an open interval leaves us with only two disjoint intervals, which make up its two connected components. This is a contradiction since homeomorphisms preserve the number of connected components. □

7 Exercise 1.1.8 Prove that the paraboloid in $\mathbb{R}^3$, defined by $x^2 + y^2 - z^2 = a$, is a manifold if $a > 0$.

Proof. Define $f(x, y, z) = x^2 + y^2 - z^2$, so $f : \mathbb{R}^3 \to \mathbb{R}$ is a smooth function. We claim that $a \neq 0$ is a regular value of f. Suppose $p = (x, y, z) \in f^{-1}(a)$, then we have the Jacobian

$$J_f(p) = \begin{bmatrix} 2x & 2y & -2z \end{bmatrix}.$$

Since $x^2 + y^2 - z^2 = a \neq 0$, then some component of p must be nonzero. In particular, the Jacobian has rank 1, so $\mathrm{d}f_p$ is surjective. This means a is a regular value of f, therefore the paraboloid $f^{-1}(a) \subseteq \mathbb{R}^3$ is a smooth manifold of dimension 2. □

$\boxed{\text{Why doesn't } x^2 + y^2 - z^2 = 0 \text{ define a manifold?}}$

Note that when $a = 0$, the origin $0 \in f^{-1}(a)$ has Jacobian $J_f(0) = 0$. In particular, a is not a regular value of f, so the above argument does not work to show $f^{-1}(0)$ is a manifold. We will prove more explicitly that it is not a manifold.

Proof. Denote $X = f^{-1}(0) \subseteq \mathbb{R}^3$. Consider the open subsets $V = \{z > 0\} \cap X$ of X and $U = \mathbb{R}^2 \setminus \{0\}$ of $\mathbb{R}^2$. There is a smooth surjection $U \to V$ defined by

$$(x, y) \longmapsto \left(x, y, \sqrt{x^2 + y^2}\right).$$

In fact, this is a diffeomorphism whose inverse is projection onto the first two components. Assuming X is a manifold, this provides a local parameterization for any point in V. In particular, X must be a manifold of dimension 2.

Suppose $\varphi : U \to V \subseteq \mathbb{R}^2$ is a local chart at $0 \in X$; without loss of generality, assume U is the intersection of X and a ball in $\mathbb{R}^3$ around the origin. Since f is a homogeneous polynomial (of degree 2), we have $f(ap) = a^2 f(p)$ for all $a \in \mathbb{R}$ and $p \in \mathbb{R}^3$, which implies that X is closed under scalar multiplication. It follows that U is star-shaped (having all line segments to the origin) and therefore connected. Since φ is a homeomorphism, V is also connected.

Removing the origin from U disconnects the space, yielding the components $\{\pm z > 0\} \cap U$. But removing $\varphi(0)$ from the open set $V \subseteq \mathbb{R}^2$ results in a connected space (see Lemma 1 below). This is a contradiction since φ restricts to a homeomorphism between these spaces, preserving the number of connected components. $\square$

Lemma 1. If $U \subseteq \mathbb{R}^2$ is open connected and $x \in U$, then $U \setminus \{x\}$ is still connected.

Proof. Suppose not, then $U \setminus \{x\} = V \cup W$ for nonempty disjoint open subsets V and W of $U \setminus \{x\}$. Since $U \setminus \{x\}$ is open in U, so are V and W. Since $U \subseteq \mathbb{R}^2$ is open, $B_r(x) \subseteq U$ for some radius $r > 0$.

If $B_r(x) \subseteq V \cup \{x\}$, then taking $V' = V \cup B_r(x)$ we can write $X = V' \cup W$, where V' and W are nonempty disjoint open subsets of V. This is not possible since X is connected. By the same argument, $B_r(x)$ is also not contained in $W \cup \{x\}$.

It follows that $D = B_r(x) \setminus \{x\}$ contains points from both V and W; say v and w, respectively. Since D (a punctured disc) is path-connected, there is a path $\gamma : I \to D$ from v to w. Then $I = \gamma^{-1}(V) \cup \gamma^{-1}(W)$ is a decomposition of I into two nonempty disjoint open subsets. This is a contradiction since I is connected. $\square$

8 Exercise 1.1.14 If $f : X \to X'$ and $g : Y \to Y'$ are smooth maps, define a *product map* $f \times g : X \times Y \to X' \times Y'$ by

$$(f \times g)(x, y) = (f(x), g(y)).$$

Show that $f \times g$ is smooth.

Proof. Say $X \subseteq \mathbb{R}^n$, $X' \subseteq \mathbb{R}^k$, $Y \subseteq \mathbb{R}^m$, and $Y' \subseteq \mathbb{R}^\ell$. Given a point $(x, y) \in X \times Y$, choose smooth local extensions $F : U \to \mathbb{R}^k$ of f at x and $G : V \to \mathbb{R}^\ell$ of g at y. Then the product map $F \times G : U \times V \to \mathbb{R}^k \times \mathbb{R}^\ell$ is a smooth local extension of $f \times g$ at (x, y), hence $f \times g$ is smooth. $\square$

9 Exercise 1.2.3 Let V be a vector subspace of $\mathbb{R}^N$. Show that $T_x(V) = V$ if $x \in V$.

Proof. Let $v_1, \ldots, v_n \in V$ form a basis and define a parameterization $\varphi : \mathbb{R}^M \to V$ in terms of basis vectors $e_i \mapsto v_i$. This is indeed a diffeomorphism since it is linear and therefore smooth, with a smooth inverse $v_i \mapsto e_i$. Given a point $x \in V$, let $y = \varphi^{-1}(x) \in \mathbb{R}^M$. Then the fact that φ is linear implies $\mathrm{d}\varphi_y = \varphi : \mathbb{R}^M \to \mathbb{R}^N$. Therefore, we compute the tangent space to be

$$T_x(V) = \operatorname{im} \mathrm{d}\varphi_y = \operatorname{im} \varphi = V.$$

$\square$

10 Exercise 1.2.6 The tangent space to S^1 at a point (a, b) is a one-dimensional subspace of $\mathbb{R}^2$. Explicitly calculate the subspace in terms of a and b.

The function $(\cos, \sin) : \mathbb{R}^1 \to \mathbb{R}^2$ defined by $x \mapsto (\cos x, \sin x)$ is a smooth surjection on S^1; choose $p \in \mathbb{R}^1$ such that $p \mapsto (a, b) \in S^1$. On the interval $U = (p - \pi, p + \pi) \subseteq \mathbb{R}^1$, this map restricts to an injection, denoted by $\varphi : U \to V = S^1 \setminus \{(-a, -b)\}$. The inverse of φ is a smooth map $V \to U$ given by the appropriate branches of the inverse sine and cosine functions, hence φ is a diffeomorphism.

We now compute the derivative of φ at p:

$$\mathrm{d}\varphi_p(x) = \mathrm{d}(\cos, \sin)_p(x) = (-\sin, \cos)_p(x) = (-\sin p, \cos p) \cdot x = (-bx, ax).$$

Hence, we have the tangent space

$$T_{(a,b)}(S^1) = \operatorname{im} \mathrm{d}\varphi_p = \{(-bx, ax) : x \in \mathbb{R}^1\} = \operatorname{span}_{\mathbb{R}}\{-be_1 + ae_2\}$$

11 Exercise 1.2.8 What is the tangent space to the paraboloid defined by $x^2 + y^2 - z^2 = a$ at $(\sqrt{a}, 0, 0)$, where $(a > 0)$?

Let $M = \{x^2 + y^2 - z^2 = a\} \subseteq \mathbb{R}^3$ be the manifold in question. Since $x^2 + y^2 = z^2 + a > 0$, there is a well-defined function $f : M \to \mathbb{R}^3$ where

$$f(x, y, z) = \left(\tfrac{x}{\sqrt{x^2+y^2}}, \tfrac{y}{\sqrt{x^2+y^2}}, z\right) = \left(\tfrac{x}{\sqrt{z^2+a}}, \tfrac{y}{\sqrt{z^2+a}}, z\right).$$

The first two components describe a unit vector in $\mathbb{R}^2$, which means we may consider this to be a smooth map $f : M \to N = S^1 \times \mathbb{R}^1$. In fact, this is a diffeomorphism with smooth inverse $g : S^1 \times \mathbb{R}^1 \to M$ defined by

$$g((u,v),z) = \left(u\sqrt{z^2+a}, v\sqrt{z^2+a}, z\right).$$

Then at each point $p \in N$, the derivative $\mathrm{d}g_p : T_p(N) \to T_{g(p)}(M)$ is an isomorphism of tangent spaces (Exercise 1.1.4). We compute the Jacobian matrix at $p = ((u,v),z)$ to be

$$J_g(p) = \begin{bmatrix} \sqrt{z^2+a} & 0 & \frac{uz}{\sqrt{z^2+a}} \\ 0 & \sqrt{z^2+a} & \frac{vz}{\sqrt{z^2+a}} \\ 0 & 0 & 1 \end{bmatrix}.$$

Moreover, (Textbook Exercise 1.2.9 and) Problem 10 give us

$$T_p(N) = T_{((u,v),z)}(S^1 \times \mathbb{R}^1) = T_{(u,v)}(S^1) \times T_z(\mathbb{R}^1) = \operatorname{span}_{\mathbb{R}}\{-ve_1 + ue_2, e_3\}.$$

Taking the image under $\mathrm{d}g_p$ yields

$$T_{g(p)}(M) = \operatorname{im} \mathrm{d}g_p = J_g(p) \cdot T_p(N) = \operatorname{span}_{\mathbb{R}} \left\{ \begin{bmatrix} -v\sqrt{z^2+a} \\ u\sqrt{z^2+a} \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{uz}{\sqrt{z^2+a}} \\ \frac{vz}{\sqrt{z^2+a}} \\ 1 \end{bmatrix} \right\}.$$

At the point $(\sqrt{a}, 0, 0) = g((1,0),0) \in M$, we have

$$T_{(\sqrt{a},0,0)}(M) = \operatorname{span}_{\mathbb{R}} \left\{ \begin{bmatrix} -0\sqrt{0^2+a} \\ 1\sqrt{0^2+a} \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{1\cdot 0}{\sqrt{0^2+a}} \\ \frac{0\cdot 0}{\sqrt{0^2+a}} \\ 1 \end{bmatrix} \right\} = \operatorname{span}_{\mathbb{R}}\{e_2, e_3\} = 0 \times \mathbb{R}^2.$$

12 Exercise 1.2.10
(a) Let $f : X \to X \times X$ be the mapping $f(x) = (x,x)$. Check that $\mathrm{d}f_x(v) = (v,v)$.

Proof. Suppose $X \subseteq \mathbb{R}^n$ and $L : \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^n = \mathbb{R}^{n+n}$ is the linear map defined on bases by $e_i \mapsto (e_i, e_i) = e_i + e_i'$, where $\{e_i' = e_{i+n}\}$ is the standard basis for the second copy of $\mathbb{R}^n$. Then f is simply the restriction of L to X, so for all $x \in X$ its derivative is $\mathrm{d}f_x = L$. Hence, for all $v \in \mathbb{R}^n$, we indeed have $\mathrm{d}f_x(v) = L(v) = (v,v)$. □

(b) If Δ is the diagonal of $X \times X$, show that its tangent space $T_{(x,x)}(\Delta)$ is the diagonal of $T_x(X) \times T_x(X)$.

Proof. Note that $f : X \to \Delta$ from (a) is a diffeomorphism, whose inverse is projection onto either component, hence the derivative $\mathrm{d}f_x : T_x(X) \to T_{(x,x)}(\Delta)$ is an isomorphism of tangent spaces. Applying the result of (a), we find

$$T_{(x,x)}(\Delta) = \operatorname{im} \mathrm{d}f_x = \{(v,v) : v \in T_x(X)\},$$

which is precisely the diagonal of $T_x(X) \times T_x(X)$. □