

Groups of order $104 = 2^3 \cdot 13$.

By the Sylow theorems, $n_{13} \equiv 1 \pmod{13}$ and $n_{13} \mid 8$, which implies $n_{13} = 1$.

Let N be the unique sylow 13-subgroup of G . Then $|N| = 13$ and $N \trianglelefteq G$.

If H is a sylow 2-subgroup of G then $|H| = 8$.

We conclude that $|NH| = |N||H| = 13 \cdot 8 = |G|$, so $G \cong N \rtimes_{\varphi} H$ where $\varphi : H \rightarrow \text{Aut}(N)$.

Since $|N| = 13$, then $N \cong Z_{13}$ so $\text{Aut}(N) \cong Z_{12}$.

Note that $8 = |H| = |\ker \varphi| |\text{im } \varphi|$ and $\text{im } \varphi \leq \text{Aut}(N)$ so $|\text{im } \varphi|$ divides $\gcd(8, 12) = 4$.

In particular, if $\text{Aut}(N) = \langle \sigma \rangle \cong Z_{12}$ then $\text{im } \varphi \leq \langle \sigma^3 \rangle \cong Z_4$.

Case $H \cong Z_8$.

Say $H = \langle y \mid y^8 = 1 \rangle$.

If $\ker \varphi = H$ then (1) $G \cong Z_{13} \times Z_8$

If $\ker \varphi = \langle y^2 \rangle$ then $\text{im } \varphi = \langle \sigma^6 \rangle \cong Z_2$ and (2) $G \cong Z_{13} \rtimes Z_8$ with $xyx^{-1} = \sigma^6(x) = x^{12}$.

If $\ker \varphi = \langle y^4 \rangle$ then $\text{im } \varphi = \langle \sigma^3 \rangle \cong Z_4$ and (3) $G \cong Z_{13} \rtimes Z_8$ with $xyx^{-1} = \sigma^3(x) = x^8$.

Case $H \cong Z_4 \times Z_2$.

Say $H = \langle y, z \mid y^4 = z^2 = [y, z] = 1 \rangle$.

If $\ker \varphi = H$ then (4) $G \cong Z_{13} \times Z_4 \times Z_2$.

If $\ker \varphi = \langle y \rangle$ then $\text{im } \varphi = \langle \sigma^6 \rangle \cong Z_2$ and (5) $G \cong (Z_{13} \rtimes Z_2) \times Z_4$ with $zxz^{-1} = x^{12}$, i.e., $D_{2(13)} \times Z_4$.

If $\ker \varphi = \langle y^2, z \rangle$ then $\text{im } \varphi = \langle \sigma^6 \rangle \cong Z_2$ and (6) $G \cong (Z_{13} \rtimes Z_4) \times Z_2$ with $yzzy^{-1} = x^{12}$, i.e., $G \cong BD_{4(13)} \times Z_2$.

Where $BD_{4n} = \langle a, b \mid a^{2n} = 1, a^n = b^2, bab^{-1} = a^{-1} \rangle$ take $a = xy^2$, $b = y$.

If $\ker \varphi = \langle z \rangle$ then $\text{im } \varphi = \langle \sigma^3 \rangle \cong Z_4$ and (7) $G \cong (Z_{13} \rtimes Z_4) \times Z_2$ with $xyx^{-1} = x^8$.

Case $H \cong Z_2^3$.

Say $H \cong \langle y, z, w \mid y^2 = z^2 = w^2 = [y, z] = [y, w] = [z, w] = 1 \rangle$.

If $\ker \varphi = H$ then (8) $G \cong Z_{13} \times Z_2^3$.

If $\ker \varphi = \langle z, w \rangle$ then $\text{im } \varphi = \langle \sigma^6 \rangle \cong Z_2$ and (9) $G \cong (Z_{13} \rtimes Z_2) \times Z_2^2$ with $xyx^{-1} = x^{12}$, i.e., $D_{2(13)} \times Z_2^2$.

case $H \cong D_{2(4)}$.

Say $H = \langle y, z \mid y^4 = z^2 = 1, zyz = y^{-1} \rangle$,

If $\ker \varphi = H$ then (10) $G \cong Z_{13} \times D_{2(4)}$.

If $\ker \varphi = \langle y \rangle \cong Z_4$ then $\operatorname{im} \varphi = \langle \sigma^6 \rangle \cong Z_2$ and (11) $G \cong Z_{13} \rtimes D_{2(4)}$ with $xyx^{-1} = x$ and $zxz^{-1} = x^{12}$, i.e., $D_{2(52)}$ where $\tau = xy$ and $\sigma = z$

If $\ker \varphi = \langle y^2, z \rangle \cong \langle y^2, yz \rangle \cong Z_2 \times Z_2$ then $\operatorname{im} \varphi = \langle \sigma^6 \rangle \cong Z_2$ and (12) $G \cong Z_{13} \rtimes D_{2(4)}$ with $xyx^{-1} = x^{12}$ and $zxz^{-1} = x$.

Case $H \cong Q_8$.

Say $H = \langle y, z \mid y^4 = 1, y^2 = z^2, zyz^{-1} = y^{-1} \rangle$.

If $\ker \varphi = H$ then (13) $G \cong Z_{13} \times Q_8$.

If $\ker \varphi = \langle y \rangle \cong Z_4$ then $\operatorname{im} \varphi = \langle \sigma^6 \rangle \cong Z_2$ and (14) $G \cong Z_{13} \rtimes Q_8$ with $xyx^{-1} = x^{12}$ and $zxz^{-1} = x$, i.e., $BD_{4(26)}$ where $a = xz$ and $b = y$.

1. $Z_{13} \times Z_8$
2. $Z_{13} \rtimes Z_8$ by $Z_8/Z_4 = Z_2$
3. $Z_{13} \rtimes Z_8$ by $Z_8/Z_2 = Z_4$
4. $Z_{13} \times Z_4 \times Z_2$
5. $D_{2(13)} \times Z_4$
6. $BD_{4(13)} \times Z_2$
7. $(Z_{13} \rtimes Z_4) \times Z_2$ by $Z_4/1 = Z_4$
8. $Z_{13} \times Z_2^3$
9. $D_{2(13)} \times Z_2^2$
10. $Z_{13} \times D_{2(4)}$
11. $D_{2(52)}$
12. $Z_{13} \rtimes D_{2(4)}$ by $D_{2(4)}/Z_2^2 = Z_2$
13. $Z_{13} \times Q_8$
14. $BD_{4(26)}$

1. $Z_{13} \rtimes Z_8$
 - (a) $Z_{13} \times Z_8$ or Z_{104}
 - (b) $Z_8/Z_2 \rightarrow Z_4$
 - (c) $Z_8/Z_4 \rightarrow Z_2$
2. $Z_{13} \rtimes (Z_4 \times Z_2)$
 - (a) $Z_{13} \times Z_4 \times Z_2$ or $Z_{52} \times Z_2$
 - (b) $(Z_4 \times Z_2)/(Z_4 \times 1) \rightarrow Z_2$ or $D_{2(13)} \times Z_4$
 - (c) $(Z_4 \times Z_2)/(Z_2 \times Z_2) \rightarrow Z_2$ or $BD_{4(13)} \times Z_2$
 - (d) $(Z_4 \times Z_2)/(1 \times Z_2) \rightarrow Z_4$ or $(Z_{13} \rtimes Z_4) \times Z_2$
3. $Z_{13} \rtimes (Z_2^3)$
 - (a) $Z_{13} \times Z_2^3$
 - (b) $Z_2^3/(Z_2^2 \times 1) \rightarrow Z_2$ or $D_{2(13)} \times Z_2^2$
4. $Z_{13} \rtimes D_{2(4)}$
 - (a) $Z_{13} \times D_{2(4)}$
 - (b) $D_{2(4)}/\langle \tau \rangle = D_{2(4)}/Z_4 \rightarrow Z_2$ or $D_{2(52)}$
 - (c) $D_{2(4)}/\langle \sigma, \tau^2 \rangle = D_{2(4)}/Z_2^2 \rightarrow Z_2$
5. $Z_{13} \rtimes Q_8$
 - (a) $Z_{13} \times Q_8$
 - (b) $Q_8/\langle i \rangle = Q_8/Z_4 \rightarrow Z_2$ or $BD_{4(26)}$