Homework 1

1 Let X be a nonempty set and let μ be a measure on X. We have a theorem on sequences of decreasing measurable sets that states the following: Assume $X \supseteq A_1 \supseteq A_2 \supseteq A_3 \supseteq \cdots$ are μ-measurable, such that $\mu(A_1) < \infty$. Then one has

$$\lim_{n\to\infty} \mu(A_n) = \mu(\textstyle\bigcap_{n=1}^{\infty} A_n).$$

Prove that in this theorem the condition $\mu(A_1) < \infty$ is necessary.

2 Does there exist an infinite σ-algebra that has countably many elements?

3 Is it true that if μ is a Borel measure on a nonempty set X, then for any sets $A, B \subset X$ with $\operatorname{dist}(A, B) > 0$, one has

$$\mu(A \cup B) = \mu(A) + \mu(B)?$$

4 Let X be an uncountable set and let $\mathcal{C}$ be the collection of all subsets A of X such that either A or A^c is at most countable. Prove that $\mathcal{C}$ is a σ-algebra.

Homework 2

1 Give an example of a topological space X and a measure μ on X so that μ is Borel but not Borel-regular.

2 Let X be a nonempty set and let $\{\mu_n\}_{n=1}^{\infty}$ be a sequence of measures on X. Assume for any subset $A \subseteq X$ the limit $\lim_{n\to\infty} \mu_n(A)$ exists and denote $\mu(A) = \lim_{n\to\infty} \mu_n(A)$.

(i) Is it true that μ is a measure on X if for any $A \subseteq X$ the sequence $\{\mu_n(A)\}$ is increasing?

(ii) Assume in addition that $\mu_1(X) < \infty$, and that each of the measures μ_n is Borel-regular. Is it true that μ is a measure on X if for any $A \subseteq X$ the sequence $\{\mu_n(A)\}$ is decreasing?

3 Let X be a nonempty set and F be a collection of functions $f : X \to \mathbb{R}$ with the following properties:

(i) The constant function $f(x) \equiv 1 \in F$, and if $f, g \in F$ and $c \in \mathbb{R}$, then $f+g, fg, cf \in F$.

(ii) If a sequence $\{f_n\} \subseteq F$ has as pointwise limit in X: $f(x) = \lim_{n\to\infty} f_n(x)$ for all $x \in X$, then $f \in F$.

Prove that the collection $\mathcal{A} = \{A \subseteq X : \chi_A \in F\}$ is a σ-algebra, where χ_A is the characteristic function of the set A.

4 Prove that any open subset of $\mathbb{R}^n$ can be expressed as a countable union of closed balls in $\mathbb{R}^n$

Remark. The statement is true for any separable metric space X.

Homework 3

1 Let λ be the Lebesgue measure and let $\{A_n\}_{n=1}^{\infty}$ be a sequence of Lebesgue-measurable subsets of $[0,1]$. Assume the set B consists of those points $x \in [0,1]$ that belong to infinitely many of the A_n.

(a) Prove that B is Lebesgue-measurable.

(b) Prove that if $\lambda(A_n) > \delta > 0$ for every $n \in \mathbb{N}$, then $\lambda(B) \geq \delta$.

(c) Prove that if $\sum_{n=1}^{\infty} \lambda(A_n) < \infty$, then $\lambda(B) = 0$.

(d) Give an example where $\sum_{n=1}^{\infty} \lambda(A_n) = \infty$, but $\lambda(B) = 0$.

2 Prove that if the set $A \subseteq \mathbb{R}$ is Lebesgue-measurable, with $\lambda(A) > 0$, then there is a subset of A that is not Lebesgue-measurable.

3 Let λ be the Lebesgue measure on $\mathbb{R}$.

(a) Let $A \subseteq \mathbb{R}$ be a set such that $\lambda(A) > 0$. Prove that for any $\varepsilon > 0$, there exists an interval $(a,b) \subseteq \mathbb{R}$ such that $\lambda(A \cap (a,b)) > (1-\varepsilon)(b-a)$.

(b) Construct a Borel set $B \subseteq \mathbb{R}$ such that $\lambda(B) > 0$ and $\lambda(B \cap I) < \lambda(I)$ for every non-degenerate interval $I \subseteq \mathbb{R}$.

4 Prove that if a Lebesgue-measurable set $A \subseteq \mathbb{R}$ has positive Lebesgue measure, then the set

$$A - A = \{a - b : a, b \in A\}$$

contains a neighborhood of the origin.

Is the statement true if one only assumes $\lambda(A) > 0$ (i.e., A is not Lebesgue-measurable)?

5 Let $A \subseteq \mathbb{R}$ be any set. Prove that the set

$$B = \bigcup_{x \in A} [x-1, x+1]$$

is Lebesgue-measurable.

Homework 4

1 Let X be a nonempty topological space and let μ be a measure on X. Prove that if the functions $f_n : X \to [-\infty, +\infty]$ are μ-measurable for $n = 1, 2, \ldots$, then the set

$$A = \{x \in X : \lim_{n\to\infty} f_n(x) \text{ exists}\}$$

is μ-measurable.

2 Prove that any Lebesgue-measurable function $f : \mathbb{R} \to \mathbb{R}$ that satisfies the relation

$$f(x+y) = f(x) + f(y) \quad \text{for all} \quad x, y \in \mathbb{R},$$

must be linear.

3 Let $f : (0,1) \to \mathbb{R}$ be such that for every $x \in (0,1)$ there exists $\delta > 0$ and a Borel-measurable function $g : \mathbb{R} \to \mathbb{R}$ (both dependent on x), such that $f(y) = g(y)$ for all $y \in (x-\delta, x+\delta) \cap (0,1)$. Prove that f is Borel-measurable. (You can assume that $f(x) = 0$ outside the interval $(0,1)$).

4 Give an example of a collection of Lebesgue-measurable nonnegative functions $\{f_\alpha\}_{\alpha\in A}$ ($f_\alpha : \mathbb{R} \to \mathbb{R}$) such that the function

$$g(x) = \sup_{\alpha\in A} f_\alpha(x), \quad x \in \mathbb{R}$$

is finite for all $x \in \mathbb{R}$ but g is not Lebesgue-measurable. Here A is a nonempty indexing set.

5 A function $f : \mathbb{R}^n \to \mathbb{R}$ is called lower semi-continuous at the point $x \in \mathbb{R}^n$ if, for any sequence $x_k \in \mathbb{R}^n$ with $x_k \to x$, one has

$$\liminf_{k\to\infty} f(x_k) \geq f(x).$$

Prove that any lower semi-continuous function on $\mathbb{R}^n$ is Borel-measurable.

Homework 5

1 (Integrability of the Product) Let X be a nonempty set and let μ be a measure on X. Prove that if μ-measurable functions $f, g : X \to [-\infty, \infty]$ are such that f is μ-summable on X and g is bounded on X ($|g(x)| \leq M$ for μ-a.e. $x \in X$), then the product fg is μ-summable and

$$\int_X |fg| \, \mathrm{d}\mu \leq M \int_X |f| \, \mathrm{d}\mu.$$

2 Let X be a nonempty set and let μ be a measure on X. Assume μ-summable functions $f, f_n : X \to [-\infty, \infty]$ are such that

$$f_n \longrightarrow f \qquad \mu\text{-a.e. in } X$$

and

$$\int_X |f_n| \, \mathrm{d}\mu \longrightarrow \int_X |f| \, \mathrm{d}\mu.$$

Prove that

$$\int_X |f_n - f| \, \mathrm{d}\mu \longrightarrow 0.$$

3 Let X be a topological space and let μ be a finite measure on X, i.e., $\mu(X) < \infty$. A family of μ-measurable functions $f_n : X \to \mathbb{R}$ is called **uniformly integrable** in X if for any $\varepsilon > 0$ there exists $M > 0$ such that

$$\int_{\{x\,:\,|f_n(x)|>M\}} |f_n(x)| \, \mathrm{d}\mu < \varepsilon \qquad \text{for all } n = 1, 2, \ldots.$$

Similarly $\{f_n\}$ is called **uniformly absolutely continuous** if for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any μ-measurable set $A \subseteq X$ with $\mu(A) < \delta$ one has

$$\left| \int_A f_n(x) \, \mathrm{d}\mu \right| < \varepsilon \qquad \text{for all } n = 1, 2, \ldots.$$

Prove that $\{f_n\}$ is uniformly integrable if and only if

$$\sup_n \int_X |f_n(x)| \, \mathrm{d}\mu < \infty$$

and $\{f_n\}$ is uniformly absolutely continuous.

4 Compute the limit

$$\lim_{n\to\infty} \int_0^n (1 - \frac{x}{n})^n \ln(2 + \cos(\frac{x}{n})) \, \mathrm{d}x$$