I worked with Joseph Sullivan and Gahl Shemy.

1 Exercise 1.3.6

(a)

Proof. If f and g are immersions, their derivatives are injective. Then $\mathrm{d}(f \times g)_(x, y) = \mathrm{d}f_x \times \mathrm{d}g_x$ is also injective, which implies $f \times g$ is an immersion. $\square$

(b)

Proof. If f and g are immersions, their derivatives are injective. Then $\mathrm{d}(g \circ f)_x = \mathrm{d}g_{f(x)} \circ \mathrm{d}f_0$ is also injective, which implies $g \circ f$ is an immersion. $\square$

(c)

Proof. Suppose M is the domain of f and $\iota : N \hookrightarrow M$ is an inclusion of manifolds. Then the derivative $\mathrm{d}\iota_x : T_x N \hookrightarrow T_x M$ is the inclusion of tangent spaces. In particular, $\mathrm{d}\iota_x$ is injective so ι is an immersion. Then by part (b), we know $f|_N = f \circ \iota$ is an immersion. $\square$

(d)

Proof. If $f : X \to Y$ is an immersion, then $\mathrm{d}f_x : T_x X \to T_{f(x)} Y$ is injective. Because

$$\dim T_x X = \dim X = \dim Y = \dim T_{f(x)} Y,$$

we know that $\mathrm{d}f_x$ is also surjective, and therefore an isomorphism of tangent spaces. By definition, f is a local diffeomorphism.

If $f : X \to Y$ is a local diffeomorphism, then $\mathrm{d}f_x$ is an isomorphism. In particular the derivative is injective, so f is an immersion. $\square$

2 Exercise 1.3.9

Lemma 1. Let $V \leq \mathbb{R}^n$ be a subspace of dimension k. Then the natural projection of V onto the subspace $\langle e_{i_1}, \ldots, e_{i_k} \rangle \leq \mathbb{R}^n$ is an isomorphism for some choice of e_{i_j}.

Proof. Let $\{v_1, \ldots, v_k\}$ be a basis of V and consider the $k \times n$ matrix with the v_j's as rows:

$$\begin{bmatrix} - & v_1 & - \\ & \vdots & \\ - & v_k & - \end{bmatrix}.$$

Performing any row operations on this matrix yields a matrix whose rows are still a basis of V. Put the matrix into reduced row echelon form:

$$\begin{bmatrix} * & 1 & * \cdots * & 0 & * \cdots * & 0 & \\ & & 0 \cdots 0 & 1 & * \cdots * & 0 & \\ & & & & 0 \cdots 0 & 1 & \\ & & & & & & \ddots \end{bmatrix}$$

Without loss of generality, we can choose the v_i's to be the rows of this matrix. Let i_j be the index of the jth pivot column in the matrix.

Then the natural projection $V \to \langle e_{i_1}, \ldots, e_{i_k} \rangle$ sends basis elements $v_j \mapsto e_{i_j}$. This is a linear surjection with the domain and codomain both of dimension k, so it must be an isomorphism of vector spaces. □

(a)

Proof. By Lemma 1, suppose the natural projection $T_x X \to \langle e_{i_1}, \ldots, e_{i_k} \rangle$ is an isomorphism of vector spaces. Then the coordinate function $F = (x_{i_1}, \ldots, x_{i_k}) : \mathbb{R}^N \to \mathbb{R}^k$ restricts to an isomorphism $T_x X \to \mathbb{R}^k$ and the restriction $f = F|_X : X \to \mathbb{R}^k$ has derivative

$$\mathrm{d}f_x = \mathrm{d}F_x|_{T_x X} = F|_{T_x X}.$$

So $\mathrm{d}f_x$ is an isomorphism, so f restricts to a diffeomorphism between an open neighborhood of x in X and an open subset of $\mathbb{R}^k$. In other words, f induces a smooth chart at x. □

(b)

Proof. Part (a) gives us a local diffeomorphism $f : V \to U \subseteq \mathbb{R}^k$, which has $f(a_1, \ldots, a_N) = (a_1, \ldots, a_k)$ for all $(a_1, \ldots, a_N) \in V$. Then the smooth inverse $g = f^{-1} : U \to V$ has

$$(a_1, \ldots, a_N) = g(f(a_1, \ldots, a_N)) = g(a_1, \ldots, a_k) = (g_1(a), \ldots, g_N(a)),$$

so $g_i(a) = a_i$ for $i = 1, \ldots, k$, hence

$$g(a_1, \ldots, a_k) = (a_1, \ldots, a_k, g_{k+1}(a), \ldots, g_N(a)).$$

□

3 Exercise 1.3.10

4 Exercise 1.4.1

Proof. Let $x \in U$ be any point. Since f is a submersion, there are local parameterizations $\varphi : V \to X$ and $\psi : W \to Y$ at x and y, respectively, such that $F = \psi^{-1} \circ f \circ \varphi$ is the standard submersion. The intersection $U' = U \cap \varphi(V)$ is an open neighborhood of x in X. The parameterizations are homeomorphisms and the standard submersion is an open map, so

$$f(U') = f|_{\varphi(V)}(U') = (\psi \circ F \circ \varphi^{-1})(U')$$

is an open neighborhood of $f(x)$ in $\psi(W)$. Since $\psi(W)$ is open in Y, the image of U' is also open in Y. Hence, $f(U')$ is an open neighborhood of $f(x)$ contained in $f(U)$, so by definition $f(U)$ is open in Y. □

5 Exercise 1.4.2

(a)

Proof. By Problem 4 Exercise 1.4.1, $f(X)$ is an open subset of Y. Since X is compact, $f(X)$ is compact and therefore also closed. Since Y is connected and $f(X) \subseteq Y$ clopen, we either have $f(X) = \varnothing$ or $f(X) = Y$. Since X is nonempty, the image is nonempty. Hence $f(X) = Y$, i.e., f is surjective. □

(b)

Proof. Suppose $f : X \to \mathbb{R}^n$ is a smooth map from a compact manifold. Then $f(X) \subseteq \mathbb{R}^n$ is compact, but $\mathbb{R}^n$ is not. In particular, $f(X) \neq \mathbb{R}^n$, so f is not surjective. The contrapositive of part (a) tells us that f is not a submersion. □

6 Exercise 1.4.7

Proof. By the preimage theorem, $Z = f^{-1}(y)$ is a submanifold of X of dimension

$$\dim Z = \dim X - \dim Y = 0.$$

Each point $x \in Z$ has a neighborhood $U \subseteq Z$ diffeomorphic to $\mathbb{R}^0 = \{0\}$. So $Z \cap U = \{x\}$ is an open subset of Z for all $x \in Z$, i.e., Z has the discrete topology. Moreover, Z is a closed subset of the compact set X, which implies Z is compact. Therefore, Z must contain only finitely many points since the collection of all singletons forms an open cover.

Say $Z = f^{-1}(y) = \{x_1, \ldots, x_N\}$. Each x_i is a regular point of f, so there are neighborhoods $x \in U_i \subseteq X$ and $y \in V_i \subseteq Y$ (and suitable parameterizations) on which f is equivalent to the standard submersion. Since

$$\dim U_i = \dim X = \dim Y = \dim V_i,$$

then f is actually locally equivalent to the identity map on an open subset of Euclidean space. In particular, $f|_{U_i} : U_i \to V_i$ is a diffeomorphism.

Since Z is a finite discrete subset of Euclidean, there is a positive radius for which the open balls $B_r(x_i)$ are all disjoint. Since X has the subspace topology, the intersection $B_r(x_i)X$ is an open neighborhood of x_i in X. Take $W_i = U_i \cap B_r(x_i)$ and $U = \bigcap_{i=1}^N f(W_i)$. Then $f^{-1}(U)$ is the disjoint union of $W_i' = W_i \cap f^{-1}(U)$, and each $W_i \cap f^{-1}(U)$ is mapped diffeomorphically to U. □

7 Exercise 1.4.10

Proof. Let $f : M(n) \to S(n)$ be the map $f(A) = AA^T$ so $O(n) = f^{-1}(I)$. Then

$$\mathrm{d}f_I(A) = AI^T + IA^T = A + A^T.$$

Then

$$T_I O(n) = \ker \mathrm{d}f_I = \{A \in M(n) : A^T = -A\}.$$

□

8 Exercise 1.4.12

Proof. Consider the map $f = \det|_{M(n)\setminus\{0\}}$; we must check that 0 is a regular value of f. Let $A \in f^{-1}(0)$ and write

$$A = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}.$$

Then the Jacobian of f at A is

$$J_f(A) = \begin{bmatrix} x_4 & -x_3 & -x_2 & x_1 \end{bmatrix}.$$

Since A is nonzero, some x_i is nonzero, so the Jacobian is full rank. Therefore, $\mathrm{d}f_A$ is surjective so A is a regular point of f. Hence 0 is a regular value of f, so $f^{-1}(0)$ is a manifold and is precisely the set of 2×2 matrices of ranks 1. □

9 Exercise 1.6.1

(a)

Proof. By definition, A and V are transverse if $\operatorname{im} \mathrm{d}A_x + T_{Ax}V = T_{Ax}\mathbb{R}^n$. But we have

$$\mathrm{d}A_x = A, \quad T_{Ax}V = V, \quad \text{and} \quad T_{Ax}\mathbb{R}^n = \mathbb{R}^n.$$

So indeed, A and V are transverse precisely when $A(\mathbb{R}^k) + V = \mathbb{R}^n$. □

(b)

Proof. By definition, V and W transverse if $T_xV + T_xW = T_x\mathbb{R}^n$. But we have

$$T_xV = V, \quad T_xW = W, \quad \text{and} \quad T_x\mathbb{R}^n = \mathbb{R}^n.$$

So indeed, V and W are transverse precisely when $V + W = \mathbb{R}^n$. □

10 Exercise 1.6.2

(a)

Transverse since the xy-plane is the span $\langle e_1, e_2 \rangle \leq \mathbb{R}^3$, the z-axis the the span $\langle e_3 \rangle \leq \mathbb{R}^3$ and

$$\langle e_1, e_2 \rangle + \langle e_3 \rangle = \langle e_1, e_2, e_3 \rangle = \mathbb{R}^3.$$

(b)

Transverse since xy-plane is the span $\langle e_1, e_2 \rangle \leq \mathbb{R}^3$ and the other plane contains $4e_2 - e_3$, so their sum also contains e_3 and is therefore all of $\mathbb{R}^3$.

(c)

Not transverse since both are contained in the xy-plane, which does not span $\mathbb{R}^3$.

(d)

Transverse if and only if $k + \ell \geq n$.

(e)

Transverse if and only if $k = n$ or $\ell = n$

(f)

Transverse since $(v, 0) \in V \times 0$ and $(v, v) \in \Delta(V)$ so $(0, v) \in V + \Delta(V)$. Then the vectors $(v, 0)$ and $(0, v)$ as for all $v \in V$ span $V \times V$.

(g)

Transverse since every matrix $A \in M(n)$ can be written as

$$A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T),$$

where $A + A^T$ is symmetric and $A - A^T$ is skew-symmetric, hence the two subspaces span $M(n)$.

11 Exercise 2.2.4

Proof. Per the hint, Exercise 1.1.4 gives us a diffeomorphism $\varphi : B_a \to \mathbb{R}^k$. Then there is a diffeomorphism $g : \mathbb{R}^k \to \mathbb{R}^k$ defined by $g(x) = x + e_1$. The composition $f = \varphi^{-1} \circ g \circ \varphi$ is thus a diffeomorphism $B_a \to B_a$.

We claim that f has no fixed points. Suppose $x \in B_a$ is a fixed point, so

$$x = f(x) = \varphi^{-1}(g(\varphi(x))),$$

which gives us

$$\varphi(x) = g(\varphi(x)) = \varphi(x) + e_1.$$

But this implies $e_1 = 0$, which is a contradiction. $\square$

12 Exercise 2.2.6

Proof. Let $f : B \to B$ be a continuous map from the closed unit n-ball to itself. Let $\varepsilon > 0$ be given and use the Weierstrass approximation theorem to choose a polynomial $p : \mathbb{R}^n \to \mathbb{R}^n$ such that $\|f - p\|_B < \varepsilon$. The result is trivial if $f = 0$, so we assume $f \neq 0$ and ε is small enough so that $p \neq 0$. In particular, $\|f\|_B$ and $\|p\|_B$ are nonzero with

$$|\|f\|_B - \|p\|_B| \leq \|f - p\|_B < \varepsilon.$$

Define a new polynomial

$$q = \frac{\|f\|_B}{\|p\|_B} p,$$

which has $\|q\|_B = \|f\|_B \leq 1$. A priori, we do not know whether p maps the ball back into itself B, but we do know $q(B) \subseteq B$. Moreover,

$$\|f - q\|_B \leq \|f - p\|_B + \|p - q\|_B < \varepsilon + \|p - q\|_B.$$

We now estimate

$$\|p - q\|_B = \left| 1 - \frac{\|f\|_B}{\|p\|_B} \right| \|p\|_B = |\|p\|_B - \|f\|_B| < \varepsilon,$$

hence $\|f - q\|_B < 2\varepsilon$. In other words, f is approximable by a polynomial $q : B \to B$. Since q is smooth, it has a fixed point: $x \in B$ with $q(x) = x$. Then

$$|f(x) - x| = |f(x) - q(x)| \leq \|f - q\|_B < 2\varepsilon,$$

which means

$$0 \leq \inf\{|f(y) - y| : y \in B\} \leq |f(x) - x| < 2\varepsilon.$$

Since this bound holds for all $\varepsilon > 0$, we conclude that

$$\inf\{|f(y) - y| : y \in B\} = 0.$$

Since this B is compact and $g(y) = |f(y) - y|$ is continuous, the infimum is attained somewhere on B. Hence, there is some $x \in B$ such that $|f(x) - x| = 0$, so $f(x) = x$ is a fixed point of f. $\square$

13 Exercise 2.2.7

Proof. Per the hint let $f : \mathbb{R}^n \to \mathbb{R}^n$ be the map $v \mapsto Av/|Av|$. Consider the set

$$Q = \{x = (x_1, \ldots, x_n) \mid x_i \geq 0 \text{ and } \|x\|_2 = 1\}.$$

Then for $x \in Q$, we have

$$Ax = \sum_{i=1}^{n} x_i A e_i = \sum_{i=1}^{n} x_i \left(\sum_{j=1}^{n} a_{ji} e_j \right) = \sum_{j=1}^{n} \left(\sum_{i=1}^{n} x_i a_{ji} \right) e_j.$$

Since $x_i \geq 0$ and $a_{ji} \geq 0$, we know that $\sum_{i=1}^{n} x_i a_{ji} \geq 0$. Since $f(x) \in S^{n-1}$, i.e., $\|f(x)\|_2 = 1$, so indeed $f(x) \in Q$. Let $\varphi : B^{n-1} \to Q$ be a diffeomorphism, then $g = \varphi^{-1} \circ f \circ \varphi$ is a smooth map from B^{n-1} to itself. By the fixed point theorem, there is some $x \in B^{n-1}$ such that $g(x) = x$, then setting $y = \varphi(x) \in Q$ we get $f(y) = y$. Hence, $Ay = |Ay|y$, so $|Ay|$ is a positive eigenvalue of A. $\square$