1 Exercise I.4.3
(a) Let f be the rational function on $\mathbb{P}^2$ given by $f = x_1/x_0$. Find the set of points where f is defined and describe the corresponding regular function.

The rational function f is defined on the open set $U_0 = \{x_0 \neq 0\} = \mathbb{P}^2 \setminus Z(x_0)$, and is represented everywhere in U_0 by the quotient x_1/x_0 of degree 1 homogeneous polynomials $x_0, x_1 \in k[x_0, x_1, x_2]$, where x_0 is never zero on U_0.

Identifying $\mathbb{A}^2$ in the coordinates x_1, x_2 with U_0, we can think of f as the projection on the first coordinate, i.e., $[1 : x_1 : x_2] = (x_1, x_2) \longmapsto x_1$.

(b) Now think of this function as a rational map from $\mathbb{P}^2$ to $\mathbb{A}^1$. Embed $\mathbb{A}^1$ in $\mathbb{P}^1$, and let $\varphi : \mathbb{P}^2 \to \mathbb{P}^1$ be the resulting rational map. Find the set of points where φ is defined, and describe the corresponding morphism.

The rational map φ is defined on the set $U_0 = \mathbb{P}^2 \setminus Z(x_0)$, and the morphism $\varphi|_{U_0}$ is

$$\begin{array}{ccccc} U_0 & \xrightarrow{f} & \mathbb{A}^1 & \hookrightarrow & \mathbb{P}^1 \\ [1 : x_1 : x_2] & \longmapsto & x_1 & \longmapsto & [1 : x_1]. \end{array}$$

2 Exercise I.4.4 A variety Y is *rational* if it is birationally equivalent to $\mathbb{P}^n$ for some n (or, equivalently by (4.5), if $K(Y)$ is a pure transcendental extension of k).

(a) Any conic in $\mathbb{P}^2$ is a rational curve.

Proof. A conic in the projective plane is a projective closure $X = \overline{Z(f)} \subseteq \mathbb{P}^2$, where $f \in k[x, y]$ is some quadratic. Let $(x_0, y_0) \in Z(f) \subseteq \mathbb{A}^2$ be any point on the affine conic. Consider the lines $L_t = Z((y - y_0) - t(x - x_0)) \subseteq \mathbb{A}^2$, parameterized by $t \in \mathbb{A}^1$. The intersection of $Z(f)$ and L_t is the solutions of the polynomial

$$f(x, y_0 + t(x - x_0)) = 0.$$

For all but finitely many $t \in \mathbb{A}^1 = k$, this is a quadratic in $k[x]$, with one root being x_0. To solve for the other root write

$$f(x, y_0 + t(x - x_0)) = A_t x^2 + B_t x + C_t,$$

where A_t, B_t, C_t are polynomials in t. Denoting the other root by x_t, we have

$$x^2 + \frac{B_t}{A_t} x + \frac{C_t}{A_t} = (x - x_0)(x - x_t),$$

so $x_t = -x_0 - B_t/A_t$. Moreover, $y_t = y_0 + t(x_t - x_0)$ is therefore also a rational function in t. This means we have a rational map

$$\begin{aligned} \mathbb{A}^1 &\overset{\varphi}{\dashrightarrow} Z(f) \\ t &\longmapsto (x_t, y_t), \end{aligned}$$

since B_t/A_t is a rational function in t. This map is injective since distinct choices of t give distinct lines L_t, which intersect the conic $Z(f)$ at distinct points away from (x_0, y_0). It remains to construct a rational inverse.

For $(x, y) \in Z(f)$, there is some $t_{x,y} \in \mathbb{A}^1$ such that $L_{t_{x,y}}$ is the line between (x, y) and (x_0, y_0). Since $(x, y) \in L_{t_{x,y}}$, we can solve for $t_{x,y}$ to obtain $t_{x,y} = (y - y_0)/(x - x_0)$, which is a rational function in x, y. Hence, we have a rational map

$$\begin{aligned} Z(f) &\overset{\psi}{\dashrightarrow} \mathbb{A}^1 \\ (x, y) &\longmapsto t_{x,y}, \end{aligned}$$

which is a rational inverse to φ.

The open embeddings $\mathbb{A}^1 \hookrightarrow \mathbb{P}^1$ and $Z(f) \hookrightarrow X$ are birational maps, which allows us to extend φ and ψ to a birational equivalence between $\mathbb{P}^1$ and X.

$\square$

(b) The cuspidal cubic $y^2 = x^3$ is a rational curve.

Let $X = Z(y^2 - x^3)$. There is an isomorphism

$$\begin{aligned} \mathbb{A}^1 \setminus \{0\} &\longleftrightarrow X \setminus \{0\} \\ t &\longmapsto (t^2, t^3) \\ y/x &\longleftarrow\!\shortmid (x, y), \end{aligned}$$

Embedding $\mathbb{A}^1$ in $\mathbb{P}^1$, this defines a birational equivalence between $\mathbb{P}^1$ and X.

(c) Let Y be the nodal cubic curve $y^2 z = x^2(x+z)$ in $\mathbb{P}^2$. Show that the projection φ from the point $P = (0,0,1)$ to the line $z = 0$ (Ex. 3.14) induces a birational map from Y to $\mathbb{P}^1$. Thus Y is a rational curve.

Given $a = [x : y : z] \in Y$, the line through P and a is the set of points $[tx : ty : (1-t) + tz]$, parameterized by $t \in \mathbb{A}^1 = k$. Intersection with the line $\{z = 0\}$ occurs when $t = 1/(1-z)$, assuming a representative of a is chosen such that $z \neq 1$. In which case, $t \neq 0$, so

$$\varphi(a) = [tx : ty : 0] = [x : y : 0].$$

This suggests a rational map

$$\begin{aligned} Y &\dashrightarrow \mathbb{P}^1 \\ [x : y : z] &\longmapsto [x : y], \end{aligned}$$

defined on $Y \setminus Z(x, y)$, which is nonempty and open in Y. Solving for z in the defining polynomial of Y, we find a rational inverse

$$\begin{aligned} \mathbb{P}^1 &\dashrightarrow Y \\ [x : y] &\longmapsto \left[x : y : \tfrac{x^3}{y^2 - x^2}\right], \end{aligned}$$

defined on $\mathbb{P}^1 \setminus Z(y - x, y + x)$.

3 Exercise I.4.6 A birational map of $\mathbb{P}^2$ into itself is called a *plane Cremona transformation*. We give an example, called a *quadratic transformation*. It is the rational map $\varphi : \mathbb{P}^2 \to \mathbb{P}^2$ given by $(a_0, a_1, a_2) \to (a_1a_2, a_0a_2, a_0a_1)$ when no two of a_0, a_1, a_2 are 0.

(a) Show that φ is birational and its own inverse.

We compute

$$(\varphi \circ \varphi)([x : y : z]) = [x^2yz : xy^2z : xyz^2] = [x : y : z],$$

which is well-defined on the open set $\mathbb{P}^2 \setminus Z(x, y, z)$. That is, $\varphi \circ \varphi = \mathrm{id}_{\mathbb{P}^2}$ as rational maps, so in fact φ is birational and its own inverse.

(b) Find open sets $U, V \subseteq \mathbb{P}^2$ such that $\varphi : U \to V$ is an isomorphism.

Define the open set $U = \mathbb{P}^2 \setminus Z(x, y, z)$. Then for $[x : y : z] \in U$,

$$\varphi([x : y : z]) = [yz : xz : xy] = \left[\tfrac{1}{x} : \tfrac{1}{y} : \tfrac{1}{z}\right] \in U.$$

It can be seen that φ is a bijective morphism on U, and part (a) tells us φ is its own inverse morphism. Hence, $\varphi|_U : U \to U$ is an isomorphism.

(c) Find the open sets where φ and φ^{-1} are defined, and describe the corresponding morphisms. See also (V, 4.2.3).

On U from part (b), $\varphi = \varphi^{-1}$ is defined. The morphism inverts each coordinate, i.e.,

$$[x : y : z] \longmapsto \left[\tfrac{1}{x} : \tfrac{1}{y} : \tfrac{1}{z}\right].$$

4 Exercise I.4.10 Let Y be the cuspidal cubic curve $y^2 = x^3$ in $\mathbb{A}^2$. Blow up the point $O = (0,0)$, let E be the exceptional curve, and let $\tilde{Y}$ be the strict transform of Y. Show that E meets $\tilde{Y}$ in one point, and that $\tilde{Y} \cong \mathbb{A}^1$. In this case the morphism $\varphi : \tilde{Y} \to Y$ is bijective and bicontinuous, but is not an isomorphism.

Proof. We consider the total inverse image

$$\varphi^{-1}(Y) = \{((x,y),[z,w]) \in \mathbb{A}^2 \times \mathbb{P}^1 \mid y^2 = x^3, xw = yz\},$$

which is covered by the open sets $U_z = \{z \neq 0\}$ and $U_w = \{w \neq 0\}$.

In U_z, we set $z = 1$ and obtain the relations $y^2 = x^3$ and $xw = y$. Substituting, we have $x^2(w^1 - x) = 0$. Therefore, $\varphi^{-1}(Y)$ has two components here: one defined by $x = y = 0$ and w arbitrary, which is E, and the other defined by $w^2 - x = 0$, which is $\tilde{Y}$. Note that $\tilde{Y}$ meets E when $w = 0$.

In U_w, we set $w = 1$ and obtain the relations $y^2 = x^3$ and $x = yz$. Substituting, we have $y^2(1 - yz^3) = 0$. Therefore, $\varphi^{-1}(Y)$ has two components here: one defined by $x = y = 0$ and z arbitrary, which is E, and the other defined by $1 - yz^3 = 0$, which is $\tilde{Y}$. Note that $\tilde{Y}$ does not meet E here.

Hence, $\tilde{Y}$ meets E at only the point $((0,0),[1:0])$.

□