1 For a function $f : [a, b] \to \mathbb{R}$ define for every $x \in [a, b)$

$$D^+ f(x) = \limsup_{h \to 0+} \frac{f(x+h) - f(x)}{h}.$$

Prove that if $f : [a, b] \to \mathbb{R}$ is continuous and $D^+ f(x) \geq 0$ for all $x \in [a, b)$, then $f(b) \geq f(a)$.

2 Suppose $f_n : [0, 1] \to [0, \infty)$ is a sequence of increasing and right-continuous function. Let

$$f(x) = \sum_{n=1}^{\infty} f_n(x), \quad x \in [0, 1],$$

and assume that $f(1)$ is finite. Prove that

$$f'(x) = \sum_{n=1}^{\infty} f_n'(x)$$

for almost every $x \in [0, 1]$ (in the sense of the Lebesgue measure).

3 Find an increasing function $f : [0, 1] \to \mathbb{R}$ such that $f'(x) = 0$ almost everywhere in $[0, 1]$, but f is not constant on any open subinterval of $[0, 1]$.

4 If $f : [0, 1] \to \mathbb{R}$ is differentiable at each point of $[0, 1]$, if f necessarily absolutely continuous on $[0, 1]$?

5 Suppose $f : [0, 1] \to \mathbb{R}$ is continuous and absolutely continuous in $[a, 1]$ for all $a \in (0, 1)$. Is f necessarily absolutely continuous on $[0, 1]$? If f is in addition of bounded variation on $[0, 1]$ is it necessarily absolutely continuous on $[0, 1]$?