1 Let λ be the Lebesgue measure and let $\{A_n\}_{n=1}^{\infty}$ be a sequence of Lebesgue-measurable subsets of $[0, 1]$. Assume the set B consists of those points $x \in [0, 1]$ that belong to infinitely many of the A_n.

(a) Prove that B is Lebesgue-measurable.

(b) Prove that if $\lambda(A_n) > \delta > 0$ for every $n \in \mathbb{N}$, then $\lambda(B) \geq \delta$.

(c) Prove that if $\sum_{n=1}^{\infty} \lambda(A_n) < \infty$, then $\lambda(B) = 0$.

(d) Give an example where $\sum_{n=1}^{\infty} \lambda(A_n) = \infty$, but $\lambda(B) = 0$.

2 Prove that if the set $A \subseteq \mathbb{R}$ is Lebesgue-measurable, with $\lambda(A) > 0$, then there is a subset of A that is not Lebesgue-measurable.

3 Let λ be the Lebesgue measure on $\mathbb{R}$.

(a) Let $A \subseteq \mathbb{R}$ be a set such that $\lambda(A) > 0$. Prove that for any $\varepsilon > 0$, there exists an interval $(a, b) \subseteq \mathbb{R}$ such that $\lambda(A) \cap (a, b)) > (1 - \varepsilon)(b - a)$.

(b) Construct a Borel set $B \subseteq \mathbb{R}$ such that $\lambda B > 0$ and $\lambda(B \cap I) < \lambda(I)$ for every non-degenerate interval $I \subseteq \mathbb{R}$.

4 Prove that if a Lebesgue-measurable set $A \subseteq \mathbb{R}$ has positive Lebesgue measure, then the set

$$A - A = \{a - b : a, b \in A\}$$

contains a neighborhood of the origin. Is the statement true if one only assumes $\lambda(A) > 0$ (i.e., A is not Lebesgue-measurable)?

5 Let $A \subseteq R$ be any set. Prove that the set

$$B = \bigcup_{x \in A} [x - 1, x + 1]$$

is Lebesgue-measurable.