**1 Exercise 0.9** Show that a retract of a contractible space is contractible.

*Proof.* Suppose $X$ is a contractible space with $A \subseteq X$ a retract. Let $F : X \times I \to X$ be a contraction, i.e., a map with $F_0 = \mathrm{id}_X$ and $F_1(X) = \{x_0\}$ for some point $x_0 \in X$. And let $r : X \to A$ be a retract, i.e., a map with $r|_A = \mathrm{id}_A$.

We define a homotopy $G : A \times I \to A$ by the following composition:

$$A \times I \hookrightarrow X \times I \xrightarrow{F} X \xrightarrow{r} A.$$

By construction, for each $a \in A$, we have

$$G_0(a) = r(F_0(a)) = r(a) = a$$

and

$$G_1(a) = r(F_1(a)) = r(x_0).$$

In other words, $G_0 = \mathrm{id}_A$ and $G_1(A) = \{r(x_0)\}$, hence $G$ describes a contraction of $A$ to the point $r(x_0) \in A$. $\square$

**2 Exercise 0.10** Show that a space $X$ is contractible iff every map $f : X \to Y$, for arbitrary $Y$, is nullhomotopic.

*Proof.* Suppose $X$ has a contraction $R : X \times I \to X$, and consider a map $f : X \to Y$. Then the composition $fR = f \circ R : X \times I \to Y$ defines a homotopy between the maps

$$(fR)_0 = f \circ R_0 = f \circ \mathrm{id}_X = f$$

and

$$(fR)_1 = f \circ R_1,$$

the latter of which is a constant map since $R_1$ is constant. Hence, $f$ is nullhomotopic.

On the other hand, if every map $X \to Y$ is nullhomotopic, then in particular the identity $\mathrm{id}_X : X \to X$ is nullhomotopic, which means $X$ is contractible. □

Similarly, show $X$ is contractible iff every map $f : Y \to X$ is nullhomotopic.

*Proof.* Suppose $X$ has a contraction $R : X \times I \to X$, and consider a map $f : Y \to X$. Then the composition $Rf = R \circ (f \times \mathrm{id}_I) : Y \times I \to X$ defines a homotopy between the maps

$$(Rf)_0 = R_0 \circ f = \mathrm{id}_X \circ f = f$$

and

$$(fR)_1 = R_1 \circ f,$$

the latter of which is a constant map since $R_1$ is constant. Hence, $f$ is nullhomotopic.

On the other hand, if every map $Y \to X$ is nullhomotopic, then in particular the identity $\mathrm{id}_X : X \to X$ is nullhomotopic, which means $X$ is contractible. □

**3 Exercise 0.11** Show that $f : X \to Y$ is a homotopy equivalence if there exist maps $g, h : Y \to X$ such that $fg \simeq \mathrm{id}_Y$ and $hf \simeq \mathrm{id}_X$.

*Proof.* Suppose $G : Y \times I$ is a homotopy with $G_0 = f \circ g$ and $G_1 = \mathrm{id}_Y$. Similarly, suppose $H : X \times I \to X$ is a homotopy with $H_0 = \mathrm{id}_X$ and $H_1 = h \circ f$. We represent $G$ and $H$ with the following diagram (which commutes only in the sense that difference paths through the diagram are homotopic, rather than equal):

$$\begin{tikzcd}
Y \arrow[r, "g"] \arrow[rr, bend right=50, "\mathrm{id}_Y"'] \arrow[rr, phantom, bend right=25, "\Downarrow G"] & X \arrow[r, "f"] \arrow[rr, bend left=50, "\mathrm{id}_X"] \arrow[rr, phantom, bend left=25, "\Downarrow H"] & Y \arrow[r, "h"] & X
\end{tikzcd}$$

Intuitively, the diagram suggest a homotopy between $g$ (the top path in the diagram) and $h$ (the bottom path). To make this explicit, we construct the following "horizontal compositions" of homotopies (also used in Problem 2):

$$Hg = H \circ (g \times \mathrm{id}_I) : Y \times I \to X$$

and

$$hG = h \circ G : Y \times I \to X.$$

The first of these describes a homotopy between the maps

$$(Hg)_0 = H_0 \circ g = \mathrm{id}_X \circ g = g$$

and

$$(Hg)_1 = H_1 \circ g = (h \circ f) \circ g = h \circ f \circ g.$$

The second describes a homotopy between the maps

$$(hG)_0 = h \circ G_0 = h \circ (f \circ g) = h \circ f \circ g$$

and

$$(hG)_1 = h \circ G_1 = h \circ \mathrm{id}_Y = h.$$

The "vertical compositions" of these homotopies then gives us

$$g \simeq h \circ f \circ g \simeq h.$$

Lastly, if $K : Y \times I \to X$ is a homotopy with $K_0 = h$ and $K_1 = g$, then we could draw the following diagram of homotopies:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[rr, bend left=50, "\mathrm{id}_X"] \arrow[rr, phantom, bend left=25, "\Downarrow H"] & Y \arrow[r, "h"] \arrow[r, bend right=50, "g"'] \arrow[r, phantom, bend right=25, "\Downarrow K"] & X
\end{tikzcd}$$

Again, the diagram suggests a homotopy between $\mathrm{id}_X$ and $g \circ f$ which we will make explicit. Constructing the horizontal composition

$$Kf = K \circ (f \times \mathrm{id}_I) : X \times I \to X$$

gives us a homotopy between the maps

$$(Kf)_0 = K_0 \circ f = h \circ f$$

and

$$(Kf)_1 = K_1 \circ f = g \circ f.$$

Vertically composing this homotopy with $H$ allows us to conclude

$$\mathrm{id}_X \simeq h \circ f \simeq g \circ f.$$

By assumption, $f \circ g \simeq \mathrm{id}_Y$, so in fact $g$ is a homotopy inverse of $f$, hence $f$ is a homotopy equivalence. $\square$

More generally, show that $f$ is a homotopy equivalence if $fg$ and $hf$ are homotopy equivalences.

*Proof.* Suppose $k : Y \to Y$ and $\ell : X \to X$ are homotopy inverses to $f \circ g$ and $h \circ f$, respectively. Then let $G : Y \times I \to Y$ and $H : X \times I \to X$ be homotopies with

$$f \circ g \circ k = G_0 \simeq G_1 = \mathrm{id}_Y$$

and

$$\mathrm{id}_X = H_0 \simeq H_1 = \ell \circ h \circ f.$$

We describe the situation we the following diagram:

$$\begin{tikzcd} Y \arrow[r, "k"] \arrow[rrr, bend right, "\mathrm{id}_Y"'] & Y \arrow[r, "g"] \arrow[r, phantom, shift right=4ex, "\Downarrow G"] & X \arrow[r, "f"] \arrow[rrr, bend left, "\mathrm{id}_X"] & Y \arrow[r, "h"] \arrow[r, phantom, shift left=4ex, "\Downarrow H"] & X \arrow[r, "\ell"] & X \end{tikzcd}$$

Notice that this is essentially the same diagram as the one we used in the first part, but with $g$ and $h$ replaced with $g \circ k$ and $\ell \circ h$, respectively. Moreover, we can use the same techniques of horizontal and vertical composition of homotopies to obtain

$$g \circ k \simeq (g \circ k) \circ f \circ (\ell \circ h) \simeq \ell \circ h.$$

And again, if $K : Y \times I \to X$ is a homotopy with

$$\ell \circ h = K_0 \simeq K_1 = g \circ k,$$

then we can draw the following diagram of homotopies:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[rr, bend left=40, "\mathrm{id}_X"] \arrow[rr, phantom, bend left=20, "\Downarrow H"] & Y \arrow[r, "\ell\circ h"] \arrow[r, bend right=50, "g\circ k"'] \arrow[r, phantom, bend right=25, "\Downarrow K"] & X \end{tikzcd}$$

This, in turn, gives us a homotopy

$$\mathrm{id}_X \simeq (\ell \circ h) \circ f \simeq (g \circ k) \circ f.$$

By assumption, $f \circ (g \circ k) \simeq \mathrm{id}_Y$, so in fact $g \circ k$ is a homotopy inverse of $f$, hence $f$ is a homotopy equivalence. $\square$

**4 Exercise 0.12** Show that a homotopy equivalence $f : X \to Y$ induces a bijection between the set of path-components of $X$ and the set of path-components of $Y$, and that $f$ restricts to a homotopy equivalence from each path-component of $X$ to the corresponding path-component of $Y$.

*Proof.* We claim that a pair of points $a, b \in X$ are path-connected in $X$ if and only if their images $f(a)$ and $f(b)$ are path-connected in $Y$.

On one hand, if $\gamma : I \to X$ is a path from $a$ to $b$, then the composition $f \circ \gamma : I \to Y$ is a path from $f(a)$ to $f(b)$.

On the other hand, suppose $\gamma : I \to Y$ is a path from $f(a)$ to $f(b)$. Additionally, let $g : Y \to X$ be a homotopy inverse of $f$, and let $H : X \times I \to X$ be a homotopy

$$\mathrm{id}_X = H_0 \simeq H_1 = g \circ f.$$

We construct maps $\alpha, \beta : I \to X$ by

$$\alpha(t) = H_t(a) \quad \text{and} \quad \beta(t) = H_t(b).$$

That is $\alpha$ is a path from

$$\alpha(0) = H_0(a) = \mathrm{id}_X(a) = a \quad \text{to} \quad \alpha(1) = H_1(a) = g(f(a))$$

and $\beta$ is a path from

$$\beta(0) = H_0(b) = \mathrm{id}_X(b) = b \quad \text{to} \quad \beta(1) = H_1(b) = g(f(b)).$$

Lastly, $g \circ \gamma : I \to X$ is a path from $g(f(a))$ to $g(f(b))$. Hence, there is a path from $a$ to $b$ in $X$, obtained as the product of paths

$$\alpha \cdot (g \circ \gamma) \cdot \overline{\beta},$$

where $\overline{\beta}$ is the inverse path of $\beta$, i.e., $\overline{\beta}(t) = \beta(1 - t)$.

Therefore, it is indeed true that $a, b \in X$ are path-connected in $X$ if and only if $f(a)$ and $f(b)$ are path-connected in $Y$. It follows that there is a well-defined injective function

$$\begin{aligned} \tilde{f} : \{PC(x) \mid x \in X\} &\longrightarrow \{PC(y) \mid y \in Y\}, \\ PC(x) &\longmapsto PC(f(x)), \end{aligned}$$

where $PC(-)$ denotes the path-component of a point in its respective space. (Note $\tilde{f}$ is well-defined and injective since $PC(a) = PC(b)$ if and only if $PC(f(a)) = PC(f(b))$.) By the same argument, there is a well-defined injective function

$$\begin{aligned} \tilde{g} : \{PC(y) \mid y \in Y\} &\longrightarrow \{PC(x) \mid x \in X\}, \\ PC(y) &\longmapsto PC(g(y)). \end{aligned}$$

In fact, $\tilde{f}$ and $\tilde{g}$ are inverses of each other. A portion of the above argument shows that any given point $x \in X$ is path-connected to $g(f(x))$, by a restriction of the homotopy $\mathrm{id}_X \simeq g \circ f$. In particular, this means

$$PC(x) = PC(g(f(x))) = \tilde{g}(\tilde{f}(PC(x)))$$

for all $x \in x$, and similarly that

$$PC(y) = PC(f(g(y))) = \tilde{f}(\tilde{g}(PC(y))),$$

for all $y \in Y$. Hence, $\tilde{f}$ is a bijection with $\tilde{g}$ its inverse (where $g$ is any homotopy inverse). $\square$

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| Prove also the corresponding statements with components instead of path-components. |

*Proof.* If $C \subseteq X$ is a connected component, the continuous image under $f$ is connected in $Y$. Then $f(C)$ is contained in some connected component of $Y$, say $\hat{f}(C)$. This gives us a well-defined function

$$\begin{aligned} \hat{f} : \{\text{connected components of } X\} &\longrightarrow \{\text{connected components of } Y\}, \\ C &\longmapsto \hat{f}(C). \end{aligned}$$

Similarly, we have another function

$$\begin{aligned} \hat{g} : \{\text{connected components of } Y\} &\longrightarrow \{\text{connected components of } X\}, \\ C &\longmapsto \hat{g}(C). \end{aligned}$$

For a connected component $C \subseteq X$ and a point $x \in C$, we have

$$f(x) \subseteq f(C) \subseteq \hat{f}(C).$$

Note that since $\hat{f}(C)$ is a connected component and $PC(f(x))$ is a connected subspace of $Y$ intersecting $\hat{f}(C)$, then we must have

$$\tilde{f}(PC(x)) = PC(f(x)) \subseteq \hat{f}(C).$$

By the same argument and the fact that $\tilde{f}$ and $\tilde{g}$ are inverses, we have

$$PC(x) = \tilde{g}(\tilde{f}(PC(x))) \subseteq \hat{g}(\hat{f}(C)).$$

Since $C$ and $\hat{g}(\hat{f}(C))$ are connected components of $X$ which intersect (at least at $x$), then in fact $C = \hat{g}(\hat{f}(C))$. Since $\hat{f}$ and $\hat{g}$ are symmetric, we conclude that they are inverses, and therefore describe a bijection. $\square$

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| Deduce from this that if the components and path-components of a space coincide, then the same is true for any homotopy equivalent space. |

Since a homotopy equivalence induces a bijection between both the path-components and the connected components of each space, the two types of components coincide for one space if and only if they coincide for the other.

**5 Exercise 0.13** Show that any two deformation retractions $r_t^0$ and $r_t^1$ of a space $X$ onto a subspace $A$ can be joined by a continuous family of deformation retractions $r_t^s$, $0 \leq s \leq 1$, of $X$ onto $A$, where continuity means that the map $X \times I \times I \to X$ sending $(x, s, t)$ to $r_t^s(x)$ is continuous.

*Proof.* Denote $Y = X \times I$, so that $r^0, r^1 : Y \to X$. We define a map $F : Y \times I \to X$ by

$$F^s(x,t) = r_t^0(r_{st}^1(x)).$$

(One may construct $F$ explicitly as the result of combining continuous maps with continuity-preserving operations such as composition, product, multiplication, etc.) Then $F$ describes a sort of "second-order" homotopy between

$$F^0 = r^0 \circ r_0^1 = r^0 \circ \mathrm{id}_X = r^0$$

and

$$F^1 = r^0 \circ r^1.$$

Note that each $F^s : X \times I \to X$ is a homotopy with $F_t^s(x) = F^s(x,t)$. We find that

$$F_0^s = r_0^0 \circ r_0^1 = \mathrm{id}_X \circ \mathrm{id}_X = \mathrm{id}_X .$$

For all $a \in A$ we have

$$F_t^s(a) = r_t^0(r_{st}^1(a)) = r_t^0(a) = a,$$

since $r_t^0|_A = r_t^1|_A = \mathrm{id}_A$ for all $t$, i.e., $F_t^s|_A = \mathrm{id}_A$ for all $t$. And for all $x \in X$ we have

$$F_1^s(x) = R_s(x,1) = r_1^0(r_s^1(x)) \in A,$$

since $r_s^1(x) \in X$ and $r_1^0(X) \subseteq A$. In other words, each $F^s : X \times I \to X$ is a deformation retraction of $X$ onto $A$. We conclude that $R$ describes a second-order homotopy $r^0 \simeq r^0 \circ r^1$, where each intermediate homotopy $F^s$ is a deformation retraction of $X$ onto $A$.

We define another map $G : Y \times I \to X$ by

$$G^s(x,t) = r_{st}^0(r_t^1(x)),$$

which describes a second-order homotopy

$$r^1 = G^0 \simeq G^1 = r^0 \circ r^1.$$

One can check that, once again, each intermediate homotopy $G^s : X \times I \to X$ is a deformation retraction of $X$ onto $A$. Composing these second-order homotopies, we obtain

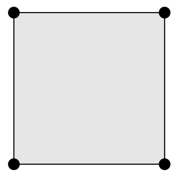
$$r^0 \simeq r^0 \circ r^1 \simeq r^1,$$

via deformation retractions of $X$ onto $A$. $\square$

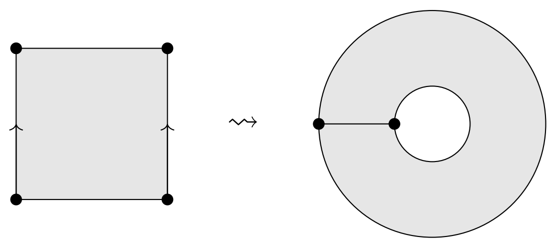
**6 Exercise 0.17** Construct a 2-dimensional cell complex that contains both an annulus $S^1 \times I$ and a Möbius band as deformation retracts.

Note that a quotient of a cell complex where we identify a pair of closed $n$-cells is still a cell complex, as we could reconstruct the new cell complex by adding in only a single $n$-cell for the chosen pair, identifying smaller cells as necessary and modifying all higher attaching maps to reflect this.

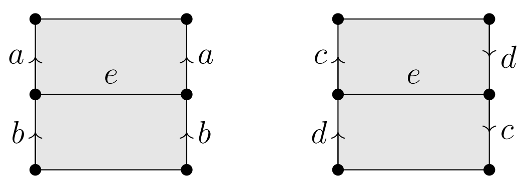
For example, we could construct a filled-in square as a 2-dimensional cell complex as follows:



We could then identify the pair of horizontally opposing 1-cells to obtain an annulus:



With this in mind, we construct a cell complex as the quotient of the following cell complex, where 1-cells are identified by name:




The left component becomes the annulus and the right component becomes the Möbius band. Each component can deformation retract onto the shared centerline $e$. Performing one of these deformation retractions while fixing the other component induces a deformation retraction of the whole complex onto the fixed component.

**7 Exercise A.3** Show that a CW complex is path-connected iff its 1-skeleton is path-connected.

*Proof.* Assume $X = \bigcup_k X^k$ is a path-connected CW complex.

Let $\gamma : I \to X$ be a path in $X$. Since $I$ is compact and $\gamma$ is continuous, the image of $\gamma$ is compact subspace of $X$. By Hatcher Proposition A.1, $\gamma(I)$ is contained in some finite subcomplex of $X$—in particular, in some $k$-skeleton. We conclude that every path in $X$ is contained in some $k$-skeleton.

Let $x, y \in X^1$ be arbitrary distinct points in the 1-skeleton. Choose the minimum $k \geq 1$ such that $X^k$ contains a path between $x$ and $y$; let $\gamma : I \to X^k$ be such a path. (Note that $k$ is well-defined since some path in $X$ exists, and the previous result tells us the path is contained in some skeleton.) Assume, for contradiction, that $k > 1$. Again, $\gamma(I) \subseteq X$ is compact, and therefore contained in finitely many cells. In particular, $\gamma$ passes through finitely many $k$-cells $e_1^k, \ldots, e_n^k$.

Since $k \geq 2$, the boundary $\partial D_i^k$ of each of these $k$-cells is path-connected. Moreover, $x$ and $y$ are not in any of these $k$-cells, which means $\gamma$ does not start or end inside of a $k$-cell. Therefore, we can construct a new path between $x$ and $y$ by following $\gamma$ inside of $X^{k-1}$, but anywhere $\gamma$ would pass through a $k$-cell $e_i^k$, we replace with a path through the boundary $\partial D_i^k$. This is possible since $\gamma$ can only enter/exit the $k$-cell through its boundary, and any time $\gamma$ enters a $k$-cell, it must eventually exit.

Since the boundaries $\partial D_i^k$ are contained in $X^{k-1}$, this new path is completely contained in $X^{k-1}$. But this contradicts the choice of $k$, implying that $k = 1$. Hence, $x$ and $y$ are connected via a path in $X^1$, and we conclude that $X^1$ is path-connected.

Assume now that $X = \bigcup_k X^k$ is a CW complex with $X^1$ path-connected. We show by induction on $k$ that each $k$-skeleton is path-connected. Assume $X^{k-1}$ is path-connected, and $X^k$ is formed by attaching $k$-cells $e_\alpha^k$ to $X^{k-1}$. For a point $x \in X^k$, either $x \in X^{k-1}$ or $x \in e_\alpha^k$ for some $\alpha$. In the latter case, $e_\alpha^k$ is path-connected to its boundary, which is contained in $X^{k-1}$. Hence, every point of $X^k$ is path-connected to $X^{k-1}$. Then a path between any two points of $X^k$ is obtained by first connecting each point to a point in $X^{k-1}$ (if necessary), then finding a path connecting these points $X^{k-1}$. Thus, $X^k$ is path connected, which completes the induction. Then any two points in $X$ are contained in some $k$-skeleton, which we have just shown to be path-connected. The path in $X^k$ is also a path in $X$, hence $X$ is path-connected. $\square$

**8 Exercise A.4** Show that a CW complex is locally compact iff each point has a neighborhood that meets only finitely many cells.

*Proof.* Let $X$ be a CW complex. In particular, $X$ is Hausdorff, so we have the following equivalent criteria for $X$ to be locally compact:

(i) Each point of $X$ has a compact neighborhood contained in any given neighborhood.

(ii) Each point of $X$ has a compact neighborhood.

Definition (i) is given by Hatcher, but we will use (ii) for this proof since, as noted, they are equivalent for Hausdorff spaces.

Suppose $X$ is locally compact. Let $x \in X$ and let $K \subseteq X$ be a compact neighborhood of $x$. By Proposition A.1, we know that $K$ is contained in a finite subcomplex of $X$. In particular, $K$ meets only finitely many cells of $X$.

Suppose $x \in X$ has a neighborhood $N \subseteq X$ that meets only finitely many cells $e_1, \dots, e_n$ of $X$. If $e_i$ is a $k$-cell, then its closure $\overline{e_i} \subseteq X$ is homeomorphic to a closed ball in $\mathbb{R}^k$, which is compact. Therefore the finite union of compact sets $K = \overline{e_i} \cup \cdots \cup \overline{e_n}$ is compact. And since $N$ is a neighborhood of $x$ contained in $K$, then $K$ is a compact neighborhood of $x$, as desired. $\square$