1

Proof. In particular, as f is continuous on $\mathbb{R}$, it is continuous on the compact interval $[-\pi, \pi]$ and, therefore, Lipschitz continuous on $[-\pi, \pi]$. Moreover, since f has a period of 2π and is continuous, then it is Lipschitz on all of $\mathbb{R}$. So there exists some $M > 0$ such that

$$|f(x) - f(y)| \leq M|x - y|$$

for all $x, y \in \mathbb{R}$. Then for any $x_0 \in \mathbb{R}$, we have

$$|f(x_0 + t) - f(x_0)| \leq M|t|$$

for all $t \in \mathbb{R}$. This condition implies that $s_n(f; x_0) \to f(x_0)$ as $n \to \infty$, i.e.,

$$f(x_0) = \sum_{n=-\infty}^{\infty} c_n e^{inx_0} = \sum_{n=-\infty}^{\infty} 0 e^{inx_0} = 0.$$

$\square$

2

2(a)

Proof. We first assume that both x and y are nonzero. The derivative of F at (x, y) is given by

$$[F'(x,y)] = \begin{bmatrix} (D_1F_1)(x,y) & (D_2F_1)(x,y) \\ (D_1F_2)(x,y) & (D_2F_2)(x,y) \end{bmatrix} = \begin{bmatrix} 2x & -2y \\ 2x & 2y \end{bmatrix},$$

so the Jacobian is

$$J_F(x,y) = \det F'(x,y) = (2x)(2y) - (2x)(-2y) = 4xy + 4xy = 8xy$$

Since, $x \neq 0$ and $Y \neq 0$, then $J_f(x,y) \neq 0$. In which case, $F'(x,y)$ is invertible, so the implicit function theorem implies the existence of a neighborhood of (x, y) on which F is injective.

□

2(b)

No.

Consider F along the x-axis, i.e., where $y = 0$. We have

$$F_1(x,0) = x^2 \quad \text{and} \quad F_2(x,0) = 0.$$

Then any neighborhood of $(0, 0)$ contains the points $(\pm\delta, 0)$ for some $\delta > 0$. Then

$$F(\delta, 0) = (\delta^2, 0) = F(-\delta, 0).$$

Therefore, F is not injective on any neighborhood of $(0, 0)$.

3

Since F is continuously differentiable the map $B^n \to \mathbb{R}$ which maps $x \mapsto \|F'(x)\|$ is a continuous function. Since it is a continuous function on the compact set B^n, then it attains a maximum at some point $a \in B^n$. Define $\delta = \|F'(a)\| < 1$. Then $\|F'(x)\| \leq \delta$ for all $x \in B^n$. Since B^n is a convex set on which F is continuously differentiable, then we deduce that

$$|F(x) - F(y)| \leq \delta|x - y|$$

for all $x, y \in B^n$. Since $\delta < 1$, this means that F is a contraction mapping on B^n. Therefore, there exists a fixed point $x_0 \in B^n$ of F, i.e., $F(x_0) = x_0$.

4

4(a)

Let $U \subseteq \mathbb{R}^n$ be an open subset and $F : U \to \mathbb{R}^n$ be a continuously differentiable function. If $x_0 \in U$ with $J_F(x_0) \neq 0$, then there exists an open neighborhood V of x_0 on which F is injective. Moreover, if we define $W = F(V)$ and $G = F^{-1}|_W$, then $G : W \to V$ is a continuously differentiable function.

4(b)

Let $U \subseteq \mathbb{R}^{n+m}$ be an open subset and $F : U \to \mathbb{R}^m$ be a continuously differentiable.

Assume $F = (F_1, \ldots, F_m)$, define

$$A_x = \begin{bmatrix} D_1F_1 & \cdots & D_nF_1 \\ \vdots & & \vdots \\ D_1F_m & \cdots & D_nF_m \end{bmatrix} \quad \text{and} \quad A_y = \begin{bmatrix} D_{n+1}F_1 & \cdots & D_mF_1 \\ \vdots & & \vdots \\ D_{n+1}F_m & \cdots & D_mF_m \end{bmatrix},$$

then $[F'] = \begin{bmatrix} A_x & A_y \end{bmatrix}$.

If $(x_0, y_0) \in U$ such that $F(x_0, y_0) = 0$ and $\det A_x \neq 0$, then there exists open neighborhoods V of $(x_0, y_0$ and W of y_0 such that for every $y \in W$, there exists a unique $x \in \mathbb{R}^n$ with $(x, y) \in V$ and $F(x, y) = 0$.

Moreover, we can define a map $G : W \to \mathbb{R}^n$ such that $(G(y), y) \in V$ and $F(G(y), y) = 0$ for all $y \in W$. Then G is continuously differentiable with $G'(y_0) = -(A_x)^{-1}A_y$.

4(c)

Proof. Given a continuously differentiable function $F : U \to \mathbb{R}^n$ with $U \subseteq \mathbb{R}^n$ open, we extend F to a function $(F, I) : \mathbb{R}^{2n} \to \mathbb{R}$ by $(x, y) \mapsto (F(x), y)$. Then for each point $x_0 \in U$ with $J_F(x_0) \neq 0$, this is condition $\det A_x \neq 0$ for the implicit function theorem. Then the local function G from the implicit function theorem is precisely a local inverse of F.

□

5

Proof. If $n \leq m$, then we can project F onto the first n coordinates, giving us a continuously differentiable map $U \to V \cap \mathbb{R}^n$. Then using the inverse function theorem, we can construct a local inverse for each point in U of this map. However, since we already have a global inverse F^{-1}, then we could not have $n < m$, since the projection of F is locally bijective $U \to V \cap \mathbb{R}^n$. If $m \leq n$, we make the same argument for F^{-1}, implying that we must have $n = m$.

□