(worked with Joseph Sullivan)

1 Eisenbud Exercise 1.19 Let k be a field. Let $I \subseteq k[x,y,z,w]$ be the ideal generated by the 2×2 minors of the matrix

$$\begin{bmatrix} x & y & z \\ y & z & w \end{bmatrix},$$

that is, $I = \langle yw - z^2, xw - yz, xz - y^2 \rangle$.

Show that $R = k[x,y,z,w]/I$ is a finitely generated free module over $S = k[x,w]$. Exhibit a basis for R as an S-module.

Proof. We claim that each monomial in $R = S[y,z]/I$ has a representative sy^az^b, where $s \in S$ and $a, b \in \{0,1\}$. Let y^az^b be an arbitrary monomial in $S[y,z]/I$, so $a, b \in \mathbb{Z}_{\geq 0}$. We define a recursive procedure on the monomial to find a new representative.

If $a \geq 2$, proceed as follows. Write $a = a' + 2c$, where $c \geq 1$ and $a' \in \{0,1\}$. Then

$$y^a = y^{a'+2c} = y^{a'}(y^2)^c = y^{a'}(xz)^c = x^c y^{a'} z^c,$$

so $y^az^b = x^c y^{a'} z^{c+b}$, where $x^c \in S$ and $a' + c + b < a + b$.

If $b \geq 2$, proceed as follows. Write $b = b' + 2c$, where $c \geq 1$ and $b' \in \{0,1\}$. Then

$$z^b = z^{b'+2c} = z^{b'}(z^2)^c = y^{a'}(yw)^c = w^c y^c z^{b'},$$

so $y^az^b = w^c y^{a+c} z^{b'}$, where $w^c \in S$ and $a + c + b' < a + b$.

In either case, a representative in $S[y,z]/I$ is produced with a strictly smaller total degree. Since the total degree of the original monomial is finite, the procedure must terminate. When it does, we obtain a representative of the form sy^az^b, with $a, b \in \{0,1\}$.

Moreover, the monomial yz has the representative wx. It follows, then, that every monomial in $S[y,z]/I$ has a representative of either s, sy, or sz., for some $s \in S$. The leading terms (under a suitable choice of monomial order) of the generators of I are y^2, zy, and z^2, which are linearly independent in the S-module $S[y,z]$. This means that we cannot use I to further reduce the total degree, in the variables y and z. In other words, 1, y, and z are linearly independent in the S-module $S[y,z]/I$.

(Equivalently, we can notice that I is given with a Gröbner basis, so every polynomial in $S[y,z]$ has a unique representative in $S[y,z]/I$ such that no monomial terms are divisible by the leading terms of the generators. The leading terms of the generators are y^2, yz, z^2, which means that each monomial term has degree at most 1.)

Hence, $R = S[y,z]/I$ has a basis $\{1, y, z\}$ as an S-module, i.e., $R = S \oplus Sy \oplus Sz$.

$\square$

Show that there is a ring homomorphism $R \to k[s,t]$ such that $x \mapsto s^3$, $y \mapsto s^2t$, $z \mapsto st^2$, $w \mapsto t^3$.

Proof. Consider the k-algebra homomorphism $\varphi : k[x,y,z,w] \to k[s,t]$ defined by $x \mapsto s^3$, $y \mapsto s^2t$, $z \mapsto st^2$, $w \mapsto t^3$. We apply this map to the generators of I:

$$\begin{aligned} yw - z^2 &\longmapsto (s^2t)t^3 - (st^2)^2 &= 0, \\ xw - yz &\longmapsto s^3t^3 - (s^2t)(st^2) &= 0, \\ xz - y^2 &\longmapsto s^3(st^2) - (s^2t)^2 &= 0. \end{aligned}$$

This implies $I \subseteq \ker\varphi$, so φ factors through the natural projection

$$\pi : k[x,y,z,w] \to k[x,y,z,w]/I = R.$$

That is, there is a unique k-algebra homomorphism $\psi : R \to k[s,t]$ such that $\psi \circ \pi = \varphi$. Thus, ψ is the desired ring homomorphism.

$\square$

Use the basis you constructed to show that it is a monomorphism.

Proof. Suppose $f, g \in R$ such that $\psi(f) = \psi(g)$, i.e.,

$$f(s^3, s^2t, st^2, t^3) = g(s^3, s^2t, st^2, t^3) \in k[s,t].$$

Since $R = S \oplus Sy \oplus Sz$, we have

$$f = a_0 + a_1y + a_2z \quad \text{and} \quad g = b_0 + b_1y + b_2z,$$

for some $a_i, b_i \in S$.

For every $c \in S = k[x,w]$, we have $\psi(c) = c(s^3, t^3) \in k[s^3, t^3]$. In particular, both the s- and t-degree of every monomial term of $\psi(c)$ is a nonnegative multiple of 3, i.e., equivalent to 0 mod 3. Along similar lines, every monomial of $\psi(cy) = c(s^3,t^3)s^2t$ has s-degree equivalent to 2 mod 3 and t-degree equivalent to 1 mod 3. And every monomial of $\psi(cz) = c(s^3,t^3)st^2$ has s-degree equivalent to 1 mod 3 and t-degree equivalent to 2 mod 3.

We deduce that $\psi(a_0)$, $\psi(a_1y)$, and $\psi(a_2z)$ share no monomial terms with each other, and the same is true for $\psi(b_0)$, $\psi(b_1y)$, and $\psi(b_2z)$. So $\psi(f) = \psi(g)$ implies that $\psi(a_0) = \psi(b_0)$, $\psi(a_1y) = \psi(b_1y)$, and $\psi(a_2z) = \psi(b_2z)$. Since

$$\psi(a_1)s^2t = \psi(a_1y) = \psi(b_1y) = \psi(b_1)s^2t$$

and

$$\psi(a_2)st^2 = \psi(a_2z) = \psi(b_2z) = \psi(b_1)st^2,$$

then in fact $\psi(a_i) = \psi(b_i)$ for $i = 0, 1, 2$.

Note that $\psi|_S : x \mapsto s^3, w \mapsto t^3$ describes a k-algebra isomorphism from $S = k[x,w]$ to $k[s^3,t^3]$. In particular, it is an injection $S \to k[s,t]$. We conclude that $a_i = b_i$ for $i = 0,1,2$, so indeed $f = g$. Hence, ψ is an injective homomorphism (monomorphism).

$\square$

Conclude that I is prime.

Proof. Since ψ is an injective homomorphism, $R \cong \operatorname{im}\psi \subseteq k[s,t]$. Recall that $\psi \circ \pi = \varphi$ and π is surjective, so $\operatorname{im}\psi = \operatorname{im}\varphi$. Since φ is a k-algebra homomorphism, we have

$$\begin{aligned}\operatorname{im}\varphi &= \varphi(k[x,y,z,w]) \\ &= k[\varphi(x), \varphi(y), \varphi(z), \varphi(w)] \\ &= k[s^3, s^2t, st^2, t^3].\end{aligned}$$

Hence, $R \cong k[s^3, s^2t, st^2, t^3]$. In particular, $R = k[x,y,z,w]/I$ is an integral domain, proving that I is a prime ideal.

□

From the rank of R as a free S-module, and the degrees of the generators, deduce the Hilbert function of R.

Proof. The monomials in $S = k[x,w]$ of degree $s \geq 0$ are $x^a w^{s-a}$ for $a = 0, \ldots, s$, so

$$H_S(s) = \begin{cases} s+1 & \text{if } s \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Recall that R as an S-module, is $S \oplus Sy \oplus Sz$. Note that Sy and Sz can be treated as copies of S with its degree shifted up by 1, i.e., $H_{Sy}(s) = H_{Sz}(s) = H_S(s-1)$. Hence,

$$H_R(s) = H_S(s) + 2H_S(s-1) = \begin{cases} 3s+1 & \text{if } s \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

□

Show that R is not finitely generated as a module over $k[x,y]$.

2 Hartshorne Exercise I.1.2 Let $Y \subseteq \mathbb{A}^3$ be the set $Y = \{(t, t^2, t^3) \mid t \in k\}$. Show that Y is an affine variety of dimension 1. Find generators for the ideal $I(Y)$. Show that $A(Y)$ is isomorphic to a polynomial ring in one variable over k. We say that Y is given by the parametric representation $x = t$, $y = t^2$, $z = t^3$.

Let $f = x^2 - y, g = x^3 - z \in k[x, y, z]$. We claim $Y = Z(f, g)$. For $P = (t, t^2, t^3) \in Y$, we have $f(P) = g(P) = 0$, so $P \in Z(f, g)$. On the other hand, for $P = (a, b, c) \in Z(f, g)$, we have $a^2 - b = a^3 - c = 0$, so $P = (a, a^2, a^3) \in Y$. Hence, $Y = Z(f, g)$.

Let $J = \langle f, g\rangle \trianglelefteq k[x, y, z]$. By the Nullstellensatz,

$$I(Y) = I(Z(J)) = \sqrt{J}.$$

We claim that J is a radical ideal, i.e., that $\sqrt{J} = J$.

Notice that $J = \langle y - x^2, z - x^3\rangle$ is simply the kernel of the evaluation $k[x]$-algebra homomorphism $(k[x])[y, z] \to k[x]$ defined by $y \mapsto x^2$, and $z \mapsto x^3$. This is a surjective map, so we obtain $k[x, y, z]/J \cong k[x]$. In particular, $k[x, y, z]/J$ is a reduced ring (has no nonzero nilpotent elements), so J is a radical ideal.

So, $A(Y) = k[x, y, z]/I(Y) \cong k[x]$, i.e., $A(Y) \cong A(\mathbb{A}^1)$ as k-algebras, so $Y \cong \mathbb{A}^1$ as varieties.

3 Hartshorne Exercise I.1.4 If we identify $\mathbb{A}^2$ with $\mathbb{A}^1 \times \mathbb{A}^1$ in the natural way, show that the Zariski topology on $\mathbb{A}^2$ is not the product topology of the Zariski topologies of the two copies of $\mathbb{A}^1$.

Proof. The closed subsets of $\mathbb{A}^1$ under the Zariski topology are, in addition to $\mathbb{A}^1$ itself, precisely the finite subsets. (In other words, the Zariski topology on $\mathbb{A}^1$ is the cofinite topology.) This means that the closed subsets of $\mathbb{A}^1 \times \mathbb{A}^1$ (under the product topology) are finite unions of subsets of the form $X_1 \times X_1$, where $X_1, X_2 \subseteq \mathbb{A}^1$ are closed in the Zariski topology, i.e., either finite or all of $\mathbb{A}^1$.

By this characterization, we see that the set $X = \{(x, x) \mid x \in k\} \subseteq \mathbb{A}^1 \times \mathbb{A}^1$ is not closed in the product topology. This is because X contains no copies of $\mathbb{A}^1 \times \{x\}$ or $\{x\} \times \mathbb{A}^1$. This means that the only way X can be written as the union of closed sets in the product topology is as the infinite union $X = \bigcup_{x \in k} \{(x, x)\}$.

However, $X = Z(x - y) \subseteq \mathbb{A}^2$ is closed in the Zariski topology.

$\square$

4 Hartshorne Exercise I.2.9 If $Y \subseteq \mathbb{A}^n$ is an affine variety, we identify $\mathbb{A}^n$ with an open set $U_0 \subseteq \mathbb{P}^n$ by the homeomorphism φ_0. Then we can speak of $\overline{Y}$, the closure of Y in $\mathbb{P}^n$, which is called the *projective closure* of Y.

(a) Show that $I(\overline{Y})$ is the ideal generated by $\beta(I(Y))$, using the notation of the proof of (2.2).

Proof. By definition, $I(\overline{Y})$ is the ideal generated by the homogeneous polynomials which are zero on $\overline{Y}$. Let $f \in I(\overline{Y})$ be homogeneous, so $f(P) = 0$ for all $P \in \overline{Y}$. In particular, $f(P) = 0$ for all points $P \in \varphi_0(Y)$, i.e.,

$$\alpha(f)(a_1, \ldots, a_n) = f(1, a_1, \ldots, a_n) = 0$$

for all $(a_1, \ldots, a_n) \in Y$. Therefore, $\alpha(f) \in I(Y)$, so in fact $f = \beta(\alpha(f)) \in \beta(I(Y))$. Since this holds for all the generators, we conclude that $I(\overline{Y}) \subseteq \langle \beta(I(Y)) \rangle$.

If $f \in \beta(I(Y))$, then $\alpha(f) \in I(Y)$. So for all $(a_1, \ldots, a_n) \in Y$, we have

$$f(1, a_1, \ldots, a_n) = \alpha(f)(a_1, \ldots, a_n) = 0.$$

This means that $f(P) = 0$ for all $P \in \varphi_0(Y)$. In other words, $\varphi_0(Y) \subseteq Z(f) \subseteq \mathbb{P}^n$. Since $Z(f)$ is a closed subset, this implies that $\overline{Y} = \overline{\varphi_0(Y)} \subseteq Z(f)$. By the Nullstellensatz,

$$f \in \sqrt{\langle f \rangle} = I(Z(f)) \subseteq I(\overline{Y}).$$

Since all generators are contained in $I(\overline{Y})$, we conclude that $\langle \beta(I(Y)) \rangle \subseteq I(\overline{Y})$.

□

(b) Let $Y \subseteq \mathbb{A}^3$ be the twisted cubic of (Ex. 1.2). Its projective closure $\overline{Y} \subseteq \mathbb{P}^3$ is called the *twisted cubic curve* in $\mathbb{P}^3$. Find generators for $I(Y)$ and $I(\overline{Y})$, and use this example to show that if $f_1, \ldots, f_r$ generate $I(Y)$, then $\beta(f_1), \ldots, \beta(f_r)$ do not necessarily generate $I(\overline{Y})$.

Proof. In the proof of Problem 2, we showed that $I(Y) = \langle y - x^2, z - x^3 \rangle \trianglelefteq k[x, y, z]$.

Let t be the fourth projective coordinate in $\mathbb{P}^3$, then

$$\beta(y - x^2) = yt - x^2 \quad \text{and} \quad \beta(z - x^3) = zt^2 - x^3.$$

Define the homogeneous ideal $I = \langle yt - x^2, zt^2 - x^3 \rangle \trianglelefteq k[x, y, z, t]$, then $I \subseteq I(\overline{Y})$.

We have

$$xy - z = x(y - x^2) - (z - x^3) \in I(Y),$$

so $xy - zt \in I(\overline{Y})$.

However, $xy - zt \notin I$, since the only homogeneous degree 2 generator of I is $yt - x^2$, which is also the homogeneous generator of least degree.

□