**Exercise 6.28** Let $L_1, L_2 \subset \mathbb{P}^3$ be two disjoint lines (i.e. 1-dimensional linear subspaces in the sense of Example 6.12(b)), and let $a \in \mathbb{P}^3 \setminus (L_1 \cup L_2)$. Show that there is a unique line $L \subset \mathbb{P}^3$ through $a$ that intersects both $L_1$ and $L_2$.

**Construction.** Let $L \subset \mathbb{P}^3$ be a line not containing $a$. Taking for granted that the results of dimension carry over from affine varieties, we can write $L = V_{\mathrm{p}}(f_1, f_2)$ for some homogenous linear polynomials $f_1, f_2 \in K[x_0, \dots, x_3]$. Moreover, $f_1$ and $f_2$ are linearly independent; otherwise their ideals would be equal and $L$ would be the zero locus of a single polynomial, which is not the case. We consider the cone over $L$, given by

$$C(L) = C(V_{\mathrm{p}}(f_1, f_2)) = V_{\mathrm{a}}(f_1, f_2) \subset \mathbb{A}^4.$$

As a subset of the vector space $K^4$, we see that $C(L)$ is a 2-dimensional linear subspace. Let $A \in M_{2\times 4}(K)$ be the $2 \times 4$ matrix with entries in $K$, such that the entry $a_{ij}$ is the coefficient in $f_i$ of $x_{j-1}$. Then for all $v \in K^4$, we have $f_1(v) = f_2(v) = 0$ if and only if $Av = 0$. That is, $C(L)$ is precisely $\ker A$, which is a 2-dimensional linear subspace of $K^4$ since the linear independence of $f_1$ and $f_2$ means that the rank of $A$ is 2.

By construction of the projective space $\mathbb{P}^3$, it is natural to interpret $C(a)$ as a 1-dimensional linear subspace of $K^4$. And since $a \notin L = V_{\mathrm{p}}(f_1, f_2)$, then $Av \neq 0$ for all nonzero $v \in C(a)$, which means $C(a) \cap C(L) = \{0\}$. This implies that the linear subspace $H = C(a) + C(L)$ of $K^4$ has

$$\dim_K H = \dim_K C(a) + \dim_K C(L) = 3.$$

In particular, $H$ is a hyperplane in $K^4$.

---

*Proof.* Let $H_1, H_2 \subset K^4$ be the hyperplanes obtained from applying the above construction to $L_1$, $L_2$, respectively. We interpret $C(L_1)$ and $C(L_2)$ as 2-dimensional linear subspaces of $K^4$, then their intersection is also a linear subspace of $K^4$. Since $L_1$ and $L_2$ are disjoint, then we must have $C(L_1) \cap C(L_2) = \{0\}$; otherwise their intersection would contain a 1-dimensional linear subspace of $K^4$, corresponding (under projectivization) to a point in the intersection of $L_1$ and $L_2$.

Therefore,

$$H_1 + H_2 \supset C(L_1) + C(L_2) = K^4,$$

which implies $\dim_K(H_1 + H_2) = 4$. Let $W = H_1 \cap H_2$, which is a linear subspace of $K^4$ with

$$\dim_K W = \dim_K H_1 + \dim_K H_2 - \dim_K(H_1 + H_2) = 2.$$

By the constructions of $H_1$ and $H_2$, we know $C(a) \subset W$, implying that $a \in \mathbb{P}(W)$. Since $\dim_K C(L_1) = 2$ and $\dim_K H_2 = 3$, then $\dim_K(C(L_1) \cap H_2) \geq 1$. And since

$$C(L_1) \cap H_2 \subset C(L_1) \cap H_2 = W,$$

then the intersection of $C(L_1)$ and $W$ is at least dimension 1, and the same can be said of $C(L_2)$ and $W$. That is, the projectivization $\mathbb{P}(W)$ intersects both $L_1$ and $L_2$.

As a 2-dimensional linear subspace of $K^4$, $W$ can be written as $W = \ker A$ for some $A \in M_{2\times 4}(K)$. Then we define the homogeneous linear polynomials $f_1, f_2 \in K[x_0, \ldots, x_3]$, where the coefficient in $f_i$ of $x_{j-1}$ is the entry $a_{ij}$ of $A$, so

$$W = \ker A = V_{\mathrm{a}}(f_1, f_2).$$

Let $L = \mathbb{P}(W)$, so

$$L = \mathbb{P}(V_{\mathrm{a}}(f_1, f_2)) = V_{\mathrm{p}}(f_1, f_2).$$

Since $f_1, f_2$ are homogeneous degree 1, then $L$ is a linear subspace of $\mathbb{P}^3$. And since $W$ is a cone in $K^4$ of dimension 2, then $L$ has dimension 1 in $\mathbb{P}^3$. That is, $L$ is a line in $\mathbb{P}^3$ through $a$ and intersecting both $L_1$ and $L_2$.

To show that $L$ is the unique such line, suppose that $L' \subset \mathbb{P}^3$ is a line through $a$ and intersecting both $L_1$ and $L_2$. We consider the cone $C(L')$ as a subset of $K^4$, in fact a 2-dimensional linear subspace. Since $a \in L'$, then we know $C(a) \subset C(L')$. Similarly, since $L_1 \cap L' \neq \varnothing$, then $C(L_1)$ and $C(L')$ have nontrivial intersection. Any nonzero vector in $C(a)$ is linearly independent with any vector in $C(L_1)$. Choosing a vector from $C(a)$ and one from $C(L_1) \cap C(L')$, we obtain two linearly independent vectors in $C(L')$. Since $\dim_K C(L') = 2$, then these form a basis for $C(L')$, implying that

$$C(L') \subset C(a) + C(L_1) = H_1.$$

By the same argument, $C(L') \subset H_2$. Therefore, $C(L') \subset H_1 \cap H_2 = W$. Since both $C(L')$ and $W$ are 2-dimensional linear subspaces of $K^4$, then they must be equal. Hence, $C(L') = W = C(L)$, so in fact $L' = L$.

$\square$

Is the corresponding statement for lines and points in $\mathbb{A}^3$ true as well?

No, in $\mathbb{A}^3$, we could choose $L_1$ and $L_2$ to be a pair of parallel lines and $a$ to be a point not coplanar with those lines. Then no single line in $\mathbb{A}^3$ could pass through all three items.

**Exercise 7.13** Let $X \subset \mathbb{P}^2$ be a cubic curve. Moreover, let $U \subset X \times X$ be the set of all $(a, b) \in X \times X$ such that $a \neq b$ and the unique line through the two points $a$ and $b$ meets $X$ in exactly three distinct points; we will denote the third one by $f(a, b) \in X$.
Show that $U \subset X \times X$ is open, and that $f : U \to X$ is a morphism.

*Proof.* Given a pair $(a, b) \in X \times X$, we parameterize the line passing through $a$ and $b$ by

$$L = \{[p_0(t), p_1(t), p_2(t)] : t \in K\} \subset \mathbb{P}^3,$$

where $p_j(t) = (1 - t)a_j + tb_j$ for $j = 0, 1, 2$. (This misses a certain point at infinity, depending on the particular representative coordinates of $a$ and $b$. I think this can be fixed by considering two parameterizations, simultaneously, and adjusting the following argument, accordingly.) Suppose that $X = V_{\mathrm{p}}(g)$, where $g \in K[x_1, x_2, x_3]$ is a homogeneous cubic polynomial. Then the composition

$$h(t) = g(p_0(t), p_1(t), p_2(t)) \in K[t]$$

is an inhomogeneous cubic polynomial, whose coefficients are determined by $a$ and $b$. We know that $h$ has roots at $t = 0, 1$, which parameterize the points $a$ and $b$, respectively, in $L$. And $h$ has a third root, distinct from 0 and 1, if and only if $L$ intersects $X$ at some third point distinct from $a$ and $b$. Let $c_0, \ldots, c_3 \in K$ be the coefficients of $h$, i.e.,

$$h(t) = c_0 + c_1 t + c_2 t^2 + c_3 t^3.$$

The discriminant of $h$ is given by

$$\mathrm{Disc}_t(h) = c_1^2 c_2^2 - 4c_1^3 c_3 - 4c_0 c_2^3 - 27c_0^2 c_3^2 + 18c_0 c_1 c_2 c_3,$$

and has the property of being equal to zero if and only if $h$ has a multiple root. Notice that we can consider the discriminant as a function of the coefficients of $h$; in particular, define the polynomial $d \in K[x_0, x_1, x_2, x_3]$ such that $\mathrm{Disc}_t(h) = d(c_0, c_1, c_2, c_3)$. Moreover, we see that $d$ is in fact a homogenous polynomial of degree 4.

Recall that $p_j(t) = (1-t)a_j + tb_j$, which can be considered as a homogeneous linear polynomial in terms of $a_j$ and $b_j$. Explicitly, for a fixed $t$, we define

$$q_j = (1 - t)x_j + ty_j \in K[x_0, x_1, x_2, y_0, y_1, y_2],$$

for $j = 0, 1, 2$. Then the composition

$$g(q_0, q_1, q_2) \in K[x_0, x_1, x_2, y_0, y_1, y_2]$$

is a homogeneous polynomial of degree 4. Separating the terms by $t$, we find that the coefficients of $h$ are each homogeneous polynomials of degree 4 in the same ring. Lastly, composing with $d$, we obtain a homogeneous polynomial of degree 8. Therefore, we conclude that the discriminant of $h$ is a homogenous polynomial on $X \times X$, denote it by $u \in K[x_0, x_1, x_2, y_0, y_1, y_2]$.

Then for all $(a, b) \in X \times X$ with $a \neq b$, we know that the unique line through $a$ and $b$ intersects $X$ in exactly three distinct points if and only if $u(a, b) \neq 0$. Additionally, our construction ensures that if $a = b$, then we still have $u(a, b) = 0$. Thus,

$$U = (X \times X) \setminus V_{\mathrm{p}}(u),$$

which is an open subset of $X \times X$.

□

(Not sure if the following would be the correct route, but it feels close to working.)

To show that $f : U \to X$ is a morphism, we show that its components are locally fractions of homogeneous polynomials of the same degree. For $(a, b) \in U$, we know that the discriminant of the related $h$ is $u(a, b) \neq 0$. That is, $h$ has exactly the roots $0, 1, t_c$, where $t_c \in K$ depends on $a$ and $b$ such that

$$f(a, b) = [p_0(t_c), p_1(t_c), p_2(t_c)].$$

We can (maybe) find $t_c$ in terms of the determinant by

$$u(a, b) = c_3^4(0 - 1^2)(1 - t_c)^2(t_c - 0)^2 = c_3^4(1 - t_c)^2 t_c^2.$$

Recall that $u$ is a homogenous polynomial of degree 8 and $c_3^4$ is a homogeneous polynomial of degree of 4. The last thing necessary would be a way to solve for $t_c$ while keeping homogeneity, then composing with the homogenous linear polynomials $p_j$ to obtain $f$ as being a coordinate-wise regular function on $U$.