1 Let $F : S^2 \to \mathbb{C}\mathrm{P}^1$ be the smooth map constructed in class. Show that F is actually a diffeomorphism by constructing explicitly F^{-1} and checking that it is also smooth. (**Hint:** try solve for F^{-1} on each of the two charts. The quadratic formula comes handy here.)

On the open set $U_1 = \{[1 : w]\} \subseteq \mathbb{C}\mathrm{P}^1$, we have

$$F^{-1}([1 : w]) = \left(\frac{1 - |w|^2}{1 + |w|^2}, \frac{2 \operatorname{Re} w}{1 + |w|^2}, \frac{2 \operatorname{Im} w}{1 + |w|^2} \right).$$

This map is clearly smooth since U_1 is diffeomorphic to $\mathbb{C}$ via the map $[1 : w] \mapsto w$, and this map is built out of smooth maps in the variable w.

On the open set $U_2 = \{[z : 1]\} \subseteq \mathbb{C}\mathrm{P}^1$, we have

$$F^{-1}([z : 1]) = \left(\frac{|z|^2 - 1}{|z|^2 + 1}, \frac{2 \operatorname{Re} z}{|z|^2 + 1}, \frac{2 \operatorname{Im} z}{|z|^2 + 1} \right).$$

This map is clearly smooth since U_2 is diffeomorphic to $\mathbb{C}$ via the map $[z : 1] \mapsto z$, and this map is built out of smooth maps in the variable z.

One can check that these maps agree on the overlap $U_1 \cap U_2$.

2 Let M^n be a smooth manifold of dimension n. Denote by $C^\infty(M)$ the space of C^∞ functions on M. Recall that this is a vector space with the usual addition, and scalar multiplication.

(a) Show that $C^\infty(\mathbb{R}^n)$ $(n > 0)$ is a vector space of infinite dimension.

Proof. Any polynomial function on $\mathbb{R}^n$, i.e., an element of $\mathbb{R}[x_1, \ldots, x_n]$, is in particular a smooth function. In other words, we have a subspace $\mathbb{R}[x_1, \ldots, x_n] \leq C^\infty(\mathbb{R}^n)$. The space of polynomial functions is an infinite-dimensional real vector space with basis given by all monomials of the form $x_1^{a_1} \cdots x_n^{a_n}$ for $a_i \in \mathbb{Z}_{\geq 0}$. Therefore, the entire space $C^\infty(\mathbb{R}^n)$ must be infinite-dimensional. $\square$

(b) Show that $C^\infty(M)$ $(n > 0)$ is a vector space of infinite dimension.

Proof. Choose a chart (U, φ) on M such that $\varphi(U) = \mathbb{R}^n$. For $k \in \mathbb{N}$ set $p_k = \varphi^{-1}(ke_1)$ and $U_k = \varphi^{-1}(B_{1/4}(ke_1))$. By this construction, the U_k's are completely disjoint sets, and in fact even their closures are disjoint. Applying a variant of Problem 3, let $f_k \in C^\infty(M)$ be constructed such that $f(p_k) = 1$ and $f_k \equiv 0$ on $M \setminus U_k$. In particular, we have $\operatorname{supp} f_k \subseteq U_k$, so the supports of all the f_k's are completely disjoint from one another.

It follows that $\{f_k\}$ is a linearly independent subset of $C^\infty(M)$. Suppose $\sum_k a_k f_k = 0$ for some $a_k \in \mathbb{R}$. Since the supports of the f_k's are pairwise disjoint, the only way for this to be possible is if $a_k = 0$ for all k. Hence, the f_k's are linearly independent. We have found infinitely many linearly independent elements in $C^\infty(M)$, so the space must be infinite-dimensional. $\square$

3 Let M be a smooth manifold. If U is an open set of M, and $p \in U$, show that there is a smooth function $f \in C^\infty(M)$ such that $f = 1$ on $M \setminus U$ and $f(p) = 0$.

Choose a chart $\varphi : U \to \mathbb{R}^n$ such that $\varphi(p) = 0$ and $\varphi(U) \supseteq B_2(0)$. Let $H : \mathbb{R}^n \to [0,1]$ be a smooth bump function satisfying $H \equiv 1$ on $B_1(0)$ and $H \equiv 0$ outside $B_2(0)$. Then we define the smooth function $f = 1 - H \circ \varphi : U \to \mathbb{R}$ which satisfies $f \equiv 0$ on $\varphi^{-1}(B_1(0))$ (in particular, $f(p) = 0$) and $f \equiv 1$ on $U \setminus \varphi^{-1}(B_2(0))$. We can now extend f to the rest of M by defining $f \equiv 1$ on $M \setminus U$.

4 Let M be a compact smooth manifold and $h : M \to \mathbb{R}$ a continuous function. Show that, for any $\varepsilon > 0$, there is $f \in C^\infty(M)$ such that

$$|h(p) - f(p)| < \varepsilon$$

for any $p \in M$. (**Hint:** recall that the (n-dimensional) Weierstrass approximation theorem says that a continuous function on a bounded domain in $\mathbb{R}^n$ can be approximated by polynomials.)

Choose a smooth atlas $\mathcal{U} = \{(U_\alpha, \varphi_\alpha)\}$ for M such that each $\varphi_\alpha(U_\alpha)$ is bounded, e.g., is contained in $B_1(0)$. Let $\{\psi_\alpha\}$ be a partition of unity subordinate to $\mathcal{U}$. Then $h \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha) \to \mathbb{R}$ is a continuous function on a bounded domain in $\mathbb{R}^n$. By the Weierstrass approximation theorem, let $f_\alpha : \varphi_\alpha(U_\alpha) \to \mathbb{R}$ be a polynomial approximation which is within a supremum distance of ε, i.e.,

$$\|f_\alpha - h \circ \varphi_\alpha^{-1}\| = \sup\{|f_\alpha(y) - (h \circ \varphi_\alpha^{-1})(y)| : y \in \varphi_\alpha(U_\alpha)\} < \varepsilon.$$

We now define

$$f = \sum_\alpha \psi_\alpha \cdot (f_\alpha \circ \varphi_\alpha) \in C^\infty(M).$$

We check that this approximates h. For a given $x \in M$, say $x \in \operatorname{supp} \psi_{\alpha_i}$ for $i = 1, \dots n$ (since the partition of unity has locally finite support). Then

$$\begin{aligned} |f(x) - h(x)| &= \left| \sum_{i=1}^n \psi_{\alpha_i}(x) f_{\alpha_i}(\varphi_{\alpha_i}(x)) - \sum_{i=1}^n \psi_{\alpha_i}(x) h(x) \right| \\ &= \sum_{i=1}^n \psi_{\alpha_i}(x) \big| f_{\alpha_i}(\varphi_{\alpha_i}(x)) - h(x) \big|. \end{aligned}$$

Denote $y_i = \varphi_{\alpha_i}(x) \in \varphi_{\alpha_i}(U_{\alpha_i})$, then

$$\begin{aligned} |f(x) - h(x)| &= \sum_{i=1}^n \psi_{\alpha_i}(x) \big| f_{\alpha_i}(y_i) - h(\varphi_{\alpha_i}^{-1}(y_i)) \big| \\ &\le \sum_{i=1}^n \psi_{\alpha_i}(x) \big\| f_{\alpha_i} - h \circ \varphi_{\alpha_i}^{-1} \big\| \\ &< \sum_{i=1}^n \psi_{\alpha_i}(x) \varepsilon \\ &= \varepsilon. \end{aligned}$$