Notation

Let K denote an algebraically closed ground field.

Let $K[x_1, \ldots, x_n]$ to be the K-alegbra of polynomials, graded by degree. We ill mostly focus on $K[x, y]$.

For $n \in \mathbb{N}$, we call $\mathbb{A}^n = \mathbb{A}^n_K$ the **affine n-space** over K.

For $S \subseteq K[x_1, \ldots, x_n]$, call

$$V(S) = \{x \in \mathbb{A}^n : f(x) = 0 \text{ for all } f \in S\}$$

the (affine) **zero locus** of S. Subsets of $\mathbb{A}^n$ of this form are called **affine varieties**.

Definition 1. (Affine Curves)

(a) An **(affine plane algebraic) curve** is a nonconstant polynomial $F \in K[x, y]$ modulo units, i.e., modulo the equivalence relation $F \sim G$ if $F = \lambda G$ for some $\lambda \in K^\times$.

Call $V(F) = \{P \in \mathbb{A}^2 : F(P) = 0\}$ the **set of points** of F.

(b) The **degree** of a curve is its degree as a polynomial, denoted $\deg F$.

(c) A curve F is called **irreducible** if it is as a polynomial, and **reducible** otherwise. Similarly, if $F = F_1^{d_1} \cdots F_k^{d_k}$ is the irreducible decomposition of F as a polynomial, we will also call this the **irreducible decomposition** of the curve F. The curves $F_1, \ldots, F_k$ are called the **(irreducible) components** of F and $d_1, \ldots, d_k$ their multiplicities.

Lemma 1. Let F be an affine curve.

(a) If K is algebraically closed then $V(F)$ is infinite.

(b) If K is infinite then $\mathbb{A}^2_K \setminus V(F)$ is infinite.

Proposition 1. If two curves F and G have no common component then their intersection $V(F, G)$ is finite.

Corollary 1. Let F be a curve over an algebraically closed field. THe for any irreducible curve G we have $G \mid F$ if and only if $V(G) \subseteq V(F)$.

In particular, the irreducible components of F (but not their multiplicities) can be recovered from $V(F)$.

Notation 1. Due to the above correspondence between a curve F and its set of points $V(F)$, we will sometimes write

(a) $P \in F$ instead of $P \in V(F)$, i.e., $F(P) = 0$;

(b) $F \cap G$ instead of $V(F, G)$ for the points that lie on both F and G;

(c) $F \cup G$ for the curve FG;

(d) $G \subseteq F$ instead of $G \mid F$.

Definition 2. Let $a \in \mathbb{A}^2$ be a point.

(a) The **local ring** of $\mathbb{A}^2$ at P is defined as

$$\mathscr{O}_a = \mathscr{O}_{\mathbb{A}^2,a} = \left\{ \frac{g}{f} : f, g \in K[x,y] \text{ with } f(a) \neq 0 \right\} \subseteq K(x,y)$$

(b) It admits a well-defined ring homomorphism

$$\mathscr{O}_a \to K, \frac{g}{f} \mapsto \frac{g(a)}{f(a)}$$

which we call the **evaluation map**. Its kernel will be denoted by

$$I_a = I_{\mathbb{A}^2,a} = \{\varphi \in \mathscr{O}_a \mid \varphi(a) = 0\}$$

which is the unique maximal ideal in $\mathscr{O}_a$.

Definition 3. For a point $a \in \mathbb{A}^2$ and two curves F and G we define the **intersection multiplicity** of F and G at a to be

$$\mu_a(F,G) = \dim \mathscr{O}_a / \langle F, G \rangle \in \mathbb{N} \cup \{\infty\},$$

where dim denotes the dimension as a vector space over K.

Lemma 2. Let $a \in \mathbb{A}^2$ and let F and G be two curves. We have

(a) $\mu_a(F,G) \geq 1$ if and only if $a \in F \cap G$;

(b) $\mu_a(F,G) = 1$ if and only if $\langle F, G \rangle = I_a$ in $\mathscr{O}_a$.

Notation 2. For a polynomial $F \in K[x,y]$ of degree d and $i = 0, \ldots, d$, we define the **degree-i part** of F to be the sum of all terms of F of degree i. Hence all F_i are homogeneous, and we have $F = F_0 + \cdots + F_d$. We call F_0 the **constant** part, F_1 the **linear** part, and F_d the **leading** part of F.

Proposition 2. Let F and G be two curves through the origin. Then $\mu_0(F,G) = 1$ if and only if the linear parts of F and G are linearly independent.

1 projective

Definition 4. For $n \in \mathbb{N}$, we define the **projective n-space** over K as the set of all 1-dimensional linear subspaces of K^{n+1}. It is denoted by $\mathbb{P}^n_K$ or simply $\mathbb{P}^n$.

In other words, we have

$$\mathbb{P}^n = (K^{n+1} \setminus \{0\}) / \sim$$

with the equivalence relation $x \sim y$ if and only if $x = \lambda y$ for some $\lambda \in K^\times$. We denote the equivalence class of $(x_0, \ldots, x_n)$ by $[x_0 : \cdots : x_n] \in \mathbb{P}^n$. Call $x_0, \ldots, x_n$ the **homogeneous** or **projective coordinate** of the point $[x_0 : \cdots : x_n]$.

For a subset $S \subseteq K[x_0, \dots, x_n]$ of homogeneous polynomials we call

$$V(S) = \{P \in \mathbb{P}^n : f(P) = 0 \text{ for all } f \in S\} \subseteq \mathbb{P}^n$$

the projective **zero locus** of S. Subsets of $\mathbb{P}^n$ of this form are called **projective varieties**.

Definition 5. (Projective curves) A **(projective plane algebraic) curve** is a nonconstant homogeneous polynomial $F \in K[x, y, z]$ modulo units. We call $V(F) = \{P \in \mathbb{P}^2 : F(P) = 0\}$ is **set of points**.

The **degree** of a projective curve is its degree as a polynomial.

The notions of irreducible/reducible/reduced curves, as well as irreducible components and their multiplicities, are defined in the same way as for affine curves.

Construction 1. For $P \in \mathbb{P}^2$ we define the **local ring** of $\mathbb{P}^2$ at P as

$$\mathscr{O}_P = \mathscr{O}_{\mathbb{P}^2,P} = \{\frac{g}{f} : f, g \in K[x, y, z] \text{ homogeneous of the same degree with } f(P) \neq 0\}$$

and the unique maximal ideal

$$I_P = I_{\mathbb{P}^2,P} = \{\varphi \in \mathscr{O}_P : f(P) = 0\}.$$

There is an isomorphism $\mathscr{O}_{\mathbb{P}^2,[x_0:y_0:1]} \xrightarrow{\sim} \mathscr{O}_{\mathbb{A}^2,(x_0,y_0)}$ given by $\varphi \mapsto \varphi^{\mathrm{i}}$, which then restricts to $I_{\mathbb{P}^2,[x_0:y_0:1]} \xrightarrow{\sim} I_{\mathbb{A}^2,(x_0,y_0)}$.

Construction 2. Note that the local ring $\mathscr{O}_{\mathbb{P}^2,P}$ does not contain $K[x, y, z]$ as a subring. But for $F_1, \dots, F_k$ homogeneous there is still a generated ideal

$$\langle F_1, \dots, F_k\rangle = \left\{\frac{a_1}{b_1}F_1 + \cdots + \frac{a_k}{b_k}F_k : a_i = 0 \text{ or } a_i b_i \text{ homogeneous with } \deg(a_i F_i) = \deg b_i \text{ for all } i\right\}$$

in $\mathscr{O}_P$. As in the affine case we can therefore define **intersection multiplicity** of two curves F and G at a point $P \in \mathbb{P}^2$ as

$$\mu_P(F, G) = \dim \mathscr{O}_P/\langle F, G\rangle.$$

Definition 6. Let R be a ring. The set of all prime ideals of R is called the **spectrum** of R or the **affine scheme** associated to R. We denote it by $\operatorname{Spec} R$.

Definition 7. Let R be a ring, and let $P \in \operatorname{Spec} R$ be a point in the corresponding affine scheme, i.e., a prime ideal $P \trianglelefteq R$.

We denote by $K(P)$ the quotient field (fraction field) of the integral domain R/P. It is called the **residue field** of $\operatorname{Spec} R$ at P.

For any $f \in R$ we define the **value** of f at P, written as $f(P)$, to be the image of f under the composite ring homomorphism $R \to R/P \to K(P)$.

In particular, we have $f(P) = 0$ if and only if $f \in P$.

Definition 8. Let R be a ring.

For a subset $S \subseteq R$, we define the **zero locus** of S to be the set

$$V(S) := \{P \in \operatorname{Spec} R : f(P) = 0 \text{ for all } f \in S\} = \{P \in \operatorname{Spec} R : S \subseteq P\} \quad \subseteq \operatorname{Spec} R.$$

For a subset $X \subseteq \operatorname{Spec} R$, we define the **idea;** of X to be

$$I(X) := \{f \in R : f(P) = 0 \text{ for all } P \in X\} = \bigcap_{P \in X} P \quad \trianglelefteq R.$$

Definition 9. We define the **Zariski topology** on an affine scheme $\operatorname{Spec} R$ to be the topology whose closed sets are exactly the sets of the form $V(S) = \{P \in \operatorname{Spec} R : S \subseteq P\}$ for some $S \subseteq R$.

Lemma 3. (Scheme Nullstellensatz) Let R be a ring.

For any closed subset $X \subseteq \operatorname{Spec} R$, we have $V(I(X)) = X$.

For any ideal $J \trianglelefteq R$, we have $I(V(J)) = \sqrt{J}$.

In particular, $V(-)$ and $I(-)$ induce and inclusion-reversing bijection

$$\{\text{closed subsets of } \operatorname{Spec} R\} \longleftrightarrow \{\text{radical ideals in } R\}.$$

Might be true that a closed $Y \subseteq \operatorname{Spec} R$ is irreducible iff $I(Y) \trianglelefteq J$ is prime. So in particular,

$$\{\text{irreducible closed subsets of } \operatorname{Spec} R\} \longleftrightarrow \{\text{prime ideals in } R\} = \operatorname{Spec} R,$$

and, in which case, the topological dimension $\dim \operatorname{Spec} R$ equals the Krull dimension $\dim R$.

Definition 10. For a ring R and an element $f \in R$, we call

$$D(f) := \operatorname{Spec} R \setminus V(f) = \{P \in \operatorname{Spec} R : f \notin P\}.$$

the **distinguished open subset** of f in $\operatorname{Spec} R$. $A \trianglelefteq B$ $A \trianglelefteq B$

Definition 11. Let R be a ring, and let U and an open subset of the affine scheme $\operatorname{Spec} R$. A **regular function** on U is a family $\varphi = (\varphi_P)_{P \in U}$ with $\varphi_P \in R_P$ for all $P \in U$, such that, for every $P \in U$, there is an open neighborhood $P \in U_P \subseteq U$ and $f_P, g_P \in R$, such that for all $Q \in U_P$, we have

$$\varphi_Q = \frac{g_P}{f_P} \quad \in R_Q.$$

In particular, we require $f_P(Q) \neq 0 \iff f_P \notin Q$ for all $Q \in U_P$, i.e., $U_P \subseteq D(f_P)$.

Stated obtusely:

$$\forall P \in U,\ \exists U_P \in \mathcal{N}_{\operatorname{Spec} R}(P),\ \exists f, g \in R: \quad \forall Q \in U_P,\ \varphi_Q = \frac{g}{f} \ \in R_Q.$$

The set of all such regular function on U is clearly a ring, and we denote it by $\mathscr{O}_{\operatorname{Spec} R}(U)$. Moreover, the condition imposed on φ is local and it is obvious that $\mathscr{O}_{\operatorname{Spec} R}$ is a sheaf; it is called the **structure sheaf** of $\operatorname{Spec} R$.

Lemma 4. Let R be a ring. Then for any point $P \in \operatorname{Spec} R$ the stalk $\mathscr{O}_{\operatorname{Spec} R, P}$ of the structure sheaf $\mathscr{O}_{\operatorname{Spec} R}$ at P is isomorphic to the localization R_P.

Lemma 5. Let R be a ring and $f \in R$, then $\mathscr{O}_{\operatorname{Spec} R}(D(f)) \cong R_f$ as rings.

In particular, $\mathscr{O}_{\operatorname{Spec} R}(\operatorname{Spec} R) \cong R$.

Definition 12. A **locally ringed space** is a ringed space $(X, \mathscr{O}_X)$ such that each stalk $\mathscr{O}_{X,P}$ for $P \in X$ is a local ring.

A **morphism** of locally ringed spaces from $(X, \mathscr{O}_X)$ to $(Y, \mathscr{O}_Y)$ is given by the following data:

- a continuous map $f : X \to Y$,
- for every open subset $U \subseteq Y$ a ring homomorphism $f_U^* : \mathscr{O}_Y(U) \to \mathscr{O}_X(f^{-1}(U))$ called the pullback on U,

such that the following two conditions hold:

- The pullback maps are compatible with restrictions, i.e., $f_U^*(\varphi_U) = (f_V^*\varphi)|_{f^{-1}(U)}$ for all $U \subseteq V \subseteq Y$ and $\varphi \in \mathscr{O}_Y(V)$. In particular, this implies that there are induced ring homomorphisms $f_P^* : \mathscr{O}_{Y,f(P)} \to \mathscr{O}_{X,P}$ on the stalks for all $P \in X$.
- For all $P \in X$, we have $(f_P^*)^{-1}(I_P) = I_{f(P)}$, where I_P and $I_{f(P)}$ denote the maximal ideals in the local rings $\mathscr{O}_{X,P}$ and $\mathscr{O}_{Y,f(P)}$, respectively.

Lemma 6. For any two rings R and S there is a bijection

$$\{\text{morphism } \operatorname{Spec} R \to \operatorname{Spec} S\} \longleftrightarrow \{\text{ring homomorphism } S \to R\}$$

In particular, this means that there is a natural bijection

$$\{\text{affine schemes}\}/\text{isomorphisms} \longleftrightarrow \{\text{rings}\}/\text{isomorphisms}$$