1 Exercise 11.1 If $f \geq 0$ and $\int_E f \, \mathrm{d}\mu = 0$, prove that $f(x) = 0$ almost everywhere on E.
Hint: Let E_n be the subset of E on which $f(x) > 1/n$. Write $A = \bigcup E_n$. Then $\mu(A) = 0$ if and only if $\mu(E_n) = 0$ for every n.

Proof. Since f is a measurable function, $\{x : f(x) \geq \frac{1}{n}\}$ is a measurable set, for all $n \in \mathbb{N}$. Then the intersection of measurable sets

$$E_n = \{x : f(x) \geq \tfrac{1}{n}\} \cap E = \{x \in E : f(x) \geq \tfrac{1}{n}\}$$

is measurable, and the countable union

$$A = \bigcup_{n=1}^{\infty} E_n = \{x \in E : f(x) > 0\}$$

is the measurable subset of E on which f is nonzero. For each $n \in \mathbb{N}$, Theorem 11.24 (i.e., additivity of integration over measurable sets) and Remark 11.23(b) give us

$$\int_E f \, \mathrm{d}\mu = \int_{E_n} f \, \mathrm{d}\mu + \int_{E \setminus E_n} f \, \mathrm{d}\mu \geq \frac{1}{n}\mu(E_n) + 0,$$

so

$$\mu(E_n) \leq n \int_E f \, \mathrm{d}\mu = n \cdot 0 = 0.$$

Therefore, $\mu(E_n) = 0$, and we obtain

$$\mu(A) \leq \sum_{n=1}^{\infty} \mu(E_n) = \sum_{n=1}^{\infty} 0 = 0.$$

Hence $\mu(A) = 0$, which is to say that f is nonzero on a measure zero subset of E, i.e., f is zero almost everywhere on E.

□

2 Exercise 11.2 If $\int_A f \,\mathrm{d}\mu = 0$ for every measurable subset A of a measurable set E, then $f(x) = 0$ almost everywhere on E.

Proof. Since f is measurable, $\{x : f(x) \geq 0\}$ is a measurable. Therefore, the intersection

$$A = \{x : f(x) \geq 0\} \cap E = \{x \in E : f(x) \geq 0\}$$

is a measurable subset of E, implying $\int_A f \,\mathrm{d}\mu = 0$. Moreover, A is precisely subset of E on which f is nonnegative, so we may apply Exercise 1 to deduce that f is zero almost everywhere on A.

Similarly,

$$B = \{x : f(x) \leq 0\} \cap E = \{x \in E : f(x) \leq 0\}$$

is the measurable subset of E on which f is nonpositve. Then $-f$ is a measurable function, and is nonnegative on B. Remark 11.23(d) gives us

$$\int_B -f \,\mathrm{d}\mu = -\int_B f \,\mathrm{d}\mu = -1 \cdot 0 = 0.$$

From Exercise 1, $-f$ is zero almost everywhere on B and, therefore, so is f.

By construction, $E = A \cup B$, so the subset of E on which f is nonzero is given by

$$E \setminus f^{-1}(0) = (A \setminus f^{-1}(0)) \cup (B \setminus f^{-1}(0)).$$

(We may consider a singleton as a closed interval, i.e., $\{0\} = [0, 0]$, so the preimage under f is a measurable set.) Taking the measure, we find

$$\mu\big(E \setminus f^{-1}(0)\big) = \mu\big(A \setminus f^{-1}(0)\big) + \mu\big(B \setminus f^{-1}(0)\big) = 0 + 0 = 0.$$

In other words, f is zero almost everywhere on E.

$\square$

3 Exercise 11.3 If $\{f_n\}$ is a sequence of measurable functions, prove that the set of points x at which $\{f_n(x)\}$ converges is measurable.

Proof. By Theorem 11.17, the functions

$$\underline{f}(x) = \liminf_{n\to\infty} f_n(x) \quad \text{and} \quad \overline{f}(x) = \limsup_{n\to\infty} f_n(x)$$

are measurable, so the intersection

$$E = \{x : -\infty < \underline{f}(x)\} \cap \{x : \overline{f}(x) < +\infty\}$$

is a measurable set. Explicitly, E is the set of points x for which both the pointwise inferior and superior limits of $\{f_n(x)\}$ are finite. It can then be seen that $\{f_n(x)\}$ converges if and only if $x \in E$ and $\underline{f}(x) = \overline{f}(x)$; in which case, $f_n(x) \to \underline{f}(x) = \overline{f}(x)$.

Define the function $g = \underline{f} - \overline{f}$ on E, which is measurable on E, by Theorem 11.18. Then the preimage

$$g^{-1}([0,0]) = \{x \in E : g(x) = 0\} = \{x \in E : \underline{f} = \overline{f}\}$$

is measurable and is precisely the set of points x at which $\{f_n(x)\}$ converges.

$\square$

4 Exercise 11.4 If $f \in L(\mu)$ on E and g is bounded and measurable on E, then $fg \in L(\mu)$ on E.

Proof. By Theorem 11.18, fg is measurable on E, so we must check that its integral over E is finite. Define $M = \sup_E |g|$, so $-M \leq g \leq M$ on E. Then $\pm Mf$ are integrable functions such that, on E, we have

$$-Mf \leq fg \leq Mf.$$

Applying Remarks 11.23(b) and (d), we obtain

$$-M \int_M f \, \mathrm{d}\mu = \int_E -Mf \, \mathrm{d}\mu \leq \int_E fg \, \mathrm{d}\mu \leq \int_E Mf \, \mathrm{d}\mu = M \int_E f \, \mathrm{d}\mu.$$

Since M and $\int_E f \, \mathrm{d}\mu$ is finite, then so is $\int_E fg \, \mathrm{d}\mu$, hence $fg \in L(\mu)$ on E.

□

5 Exercise 11.5 Put

$$\begin{aligned} g(x) &= \begin{cases} 0 & \\ 1 & \end{cases} && \begin{aligned} &(0 \le x \le \tfrac{1}{2}), \\ &(\tfrac{1}{2} < x \le 1), \end{aligned} \\ f_{2k}(x) &= g(x) && (0 \le x \le 1), \\ f_{2k+1}(x) &= g(1-x) && (0 \le x \le 1). \end{aligned}$$

Show that

$$\liminf_{n\to\infty} f_n(x) = 0 \qquad (0 \le x \le 1),$$

but

$$\int_0^1 f_n(x)\,\mathrm{d}x = \frac{1}{2}.$$

[Compare with (77).]

Proof. For all $n \in \mathbb{N}$, $f_n \ge 0$, i.e., $\liminf f_n(x) \ge 0$ for all $x \in [0,1]$. So for a given x, we have equality if there exist infinitely many $n \in \mathbb{N}$ such that $f_n(x) = 0$. If $x \in [0, \frac{1}{2}]$, there are infinitely many even $n \in \mathbb{N}$, so that $f_n(x) = 0$. If $x \in (\frac{1}{2}, 1]$, there are infinitely many odd $n \in \mathbb{N}$, so that $f_n(x) = 0$. Hence, $\liminf f_n(x) = 0$ for all $x \in [0,1]$.

If $n \in \mathbb{N}$ is even, then $f_n = \chi_{[0,\frac{1}{2}]}$ (indicator function), so

$$\int_0^1 f_n(x)\,\mathrm{d}x = \int_{[0,1]} \chi_{[0,\frac{1}{2}]}\,\mathrm{d}m = m\big([0,\tfrac{1}{2}]\big) = \frac{1}{2} - 0 = \frac{1}{2}.$$

If $n \in \mathbb{N}$ is odd, then $f_n = \chi_{(\frac{1}{2},1]}$, so

$$\int_0^1 f_n(x)\,\mathrm{d}x = \int_{[0,1]} \chi_{(\frac{1}{2},1]}\,\mathrm{d}m = m\big((\tfrac{1}{2},1]\big) = 1 - \frac{1}{2} = \frac{1}{2}.$$

$\square$