Exercise 8.6.1 Let X be a smooth projective curve. For any point $P \in X$ consider the exact skyscraper sequence of sheaves on X

$$0 \longrightarrow \omega_X \longrightarrow \omega_X \otimes \mathcal{O}_X(P) \longrightarrow k_P \longrightarrow 0.$$

Show that the induces sequence of global sections is not exact, i.e., the last map $\Gamma(\omega_X \otimes \mathcal{O}_X(P)) \to \Gamma(k_P)$ is not surjective.

Note that as $\dim X = 1$, we have $\omega_X = \omega_X$.

We obtain a long exact sequence of cohomologies

$$\begin{aligned} 0 \longrightarrow H^0(\omega_X) &\longrightarrow H^0(\omega_X \otimes \mathcal{O}_X(P)) \longrightarrow H^0(k_P) \\ \longrightarrow H^1(\omega_X) &\longrightarrow H^1(\omega_X \otimes \mathcal{O}_X(P)) \longrightarrow H^1(k_P) \\ \longrightarrow \cdots & \end{aligned}$$

The 0th cohomology is precisely the global sections, so we have the exact sequence

$$\Gamma(\omega_X \otimes \mathcal{O}_X(P)) \longrightarrow \Gamma(k_P) \longrightarrow H^1(\omega_X) \longrightarrow H^1(\omega_X \otimes \mathcal{O}_X(P)) \longrightarrow H^1(k_P).$$

Fix an affine open cover $\{U_i\}_{i\in I}$ of X (with I ordered). Without loss of generality, assume $P \in U_{i_0}$ and and $P \notin U_{i_1}$ for all $i_1 \neq i_0$. (If U_{i_1} does contain P, then $U_{i_1} \setminus \overline{\{P\}}$ is an open subset of X, i.e., it is an open subscheme. We can then replace U_{i_1} in the cover of X by an affine open cover of U_{i_1}.) By definition of the skyscraper sheaf, we find that

$$C^1(k_P) = \prod_{i_0<i_1} k_P(U_{i_0} \cap U_{1_i}) = 0,$$

and it follows that $H^1(k_P) = 0$. This gives us the exact sequence

$$\Gamma(\omega_X \otimes \mathcal{O}_X(P)) \longrightarrow \Gamma(k_P) \longrightarrow H^1(\omega_X) \longrightarrow H^1(\omega_X \otimes \mathcal{O}_X(P)) \longrightarrow 0.$$

Not sure if there is anywhere to go from here in general, so I will simplify to an easy case.

Consider $X = \mathbb{P}^1$ with the fixed affine open cover $U_i = \{x_i \neq 0\}$ for $i = 0, 1$. Additionally, we assume $P = [0 : 1]$ so that $P \in U_0$ but $P \notin U_1$. Then

$$\Gamma(k_P) = H^0(k_P) = k_P(U_0) \times k_P(U_1) = k \times 0 = k,$$

which means in particular that $h^0(k_P) = 1$. By Example 8.4.3,

$$h^1(\omega_X) = h^1(\mathbb{P}^1, \omega_{\mathbb{P}^1}) = 1.$$

This means that the map $\Gamma(k_P) \to H^1(\omega_X)$ is either the zero map or surjective; we want to show that it is not the zero map, so we need $H^1(\omega_X \otimes \mathcal{O}_X(P)) = 0$. By Lemma 7.4.15,

$$\omega_X = \omega_{\mathbb{P}^1} = \mathcal{O}_{\mathbb{P}^1}(-1-1) = \mathcal{O}_X(-2).$$

Additionally, since the line bundles on $\mathbb{P}^1$ are precisely the twisting sheaves $\mathcal{O}_{\mathbb{P}^1}(n)$, then

$$\mathcal{O}_X(P) = \mathcal{O}_{\mathbb{P}^1}(\deg P) = \mathcal{O}_{\mathbb{P}^1}(1) = \mathcal{O}_X(1).$$

Hence,

$$\omega_X \otimes \mathcal{O}_X(P) = \mathcal{O}_X(-2) \otimes \mathcal{O}_X(1) = \mathcal{O}_X(-1).$$

Note that $\mathbb{P}^1$ is a genus $g = 0$ curve and $\deg \mathcal{O}_X(-1) = -1 \geq 2g - 1$. By the Kodaira vanishing theorem, we conclude that

$$H^1(\omega_X \otimes \mathcal{O}_X(P)) = H^1(\mathcal{O}_X(-1)) = 0.$$

Hence, the map $\Gamma(k_P) \to H^1(\omega_X)$ is surjective, with each of dimension 1, so the kernel is proper. In other words, the image of the map $\Gamma(\omega_X \otimes \mathcal{O}_X(P)) \to \Gamma(k_P)$ is proper, hence the map is not surjective.

Exercise 8.6.3 Compute the cohomology groups $H^i(\mathbb{P}^1 \times \mathbb{P}^1, p^*\mathcal{O}_{\mathbb{P}^1}(a)) \otimes q^*\mathcal{O}_{\mathbb{P}^1}(b))$ for all $a, b \in \mathbb{Z}$, where p and q denote the two projection maps from $\mathbb{P}^1 \times \mathbb{P}^1$ to $\mathbb{P}^1$.

A result from a paper online gives

$$H^i(\mathbb{P}^1 \times \mathbb{P}^1, p^*\mathcal{O}_{\mathbb{P}^1}(a) \otimes q^*\mathcal{O}_{\mathbb{P}^1}(b)) \cong H^i(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(a)) \otimes H^i(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(b)).$$

We have computed

$$H^i(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d)) = \begin{cases} k[x_0, x_1]_d & \text{if } i = 0, \\ k[x_0, x_1]_{-2-d} & \text{if } i = 1, \\ 0 & \text{otherwise}, \end{cases}$$

so

$$H^i = \begin{cases} k[x_0, x_1]_a \otimes k[y_0, y_1]_b & \text{if } i = 0, \\ k[x_0, x_1]_{-2-a} \otimes k[y_0, y_1]_{-2-b} & \text{if } i = 1, \\ 0 & \text{otherwise}. \end{cases}$$