

**Yoneda Lemma** For any functor  $F : \mathcal{C} \rightarrow \mathbf{Set}$ , whose domain  $\mathcal{C}$  is locally small and any object  $c \in \mathcal{C}$ , there is a bijection

$$\mathrm{Nat}(\mathrm{Mor}_{\mathcal{C}}(c, -), F) \cong Fc$$

that associates a natural transformation  $\alpha : \mathrm{Mor}_{\mathcal{C}}(c, -) \rightarrow F$  with the element  $\alpha_c(1_c) \in Fc$ . Moreover, this correspondence is natural in both  $c$  and  $F$ .

*Proof.* The map  $\Phi : \mathrm{Nat}(\mathrm{Mor}_{\mathcal{C}}(c, -), F) \rightarrow Fc$  is easy to construct. Given a natural transformation  $\alpha : \mathrm{Mor}_{\mathcal{C}}(C, -) \Rightarrow F$ , we simply define

$$\Phi(\alpha) := \alpha_c(1_c),$$

where  $\alpha_c : \mathrm{Mor}_{\mathcal{C}}(c, c) \rightarrow Fc$  is the component of  $\alpha$  at  $c$ .

We now wish to construct  $\Phi$ 's inverse,  $\Psi : Fc \rightarrow \mathrm{Nat}(\mathrm{Mor}_{\mathcal{C}}(c, -), F)$ . That is, given an element  $x \in Fc$ , we must construct a natural transformation  $\Psi(x) : \mathrm{Mor}_{\mathcal{C}}(c, -) \Rightarrow F$ . To do this, we will construct its components  $\Psi(x)_a : \mathrm{Mor}_{\mathcal{C}}(c, a) \rightarrow Fd$  for each object  $a \in \mathcal{C}$ . Moreover, this construction must adhere to the naturality condition, i.e., for all morphisms  $f : a \rightarrow b$  in  $\mathcal{C}$ , the following diagram must commute:

$$\begin{array}{ccc} \mathrm{Mor}_{\mathcal{C}}(c, a) & \xrightarrow{\Psi(x)_a} & Fa \\ f_* \downarrow & & \downarrow Ff \\ \mathrm{Mor}_{\mathcal{C}}(c, b) & \xrightarrow{\Psi(x)_b} & Fb \end{array}$$

Here,  $f_* = \mathrm{Mor}_{\mathcal{C}}(c, f)$  is the function which takes a morphism  $g : c \rightarrow a$  and sends it to the composition  $f_*g = f \circ g : c \rightarrow b$ .

Let us look at what this diagram is saying in the particular case of  $a = c$  and  $f : c \rightarrow b$ . The diagram looks like this:

$$\begin{array}{ccc} \mathrm{Mor}_{\mathcal{C}}(c, c) & \xrightarrow{\Psi(x)_c} & Fc \\ f_* \downarrow & & \downarrow Ff \\ \mathrm{Mor}_{\mathcal{C}}(c, b) & \xrightarrow{\Psi(x)_b} & Fb \end{array}$$

Consider the identity  $1_c$  in the upper left corner. Following the left side of the square, we obtain

$$\Psi(x)_b(f_*1_c) = \Psi(x)_b(f \circ 1_c) = \Psi(x)_b(f).$$

Since we eventually want to this square to commute, this must be equal to the result of following the right side, i.e., we must have

$$\Psi(x)_b(f) = Ff(\Psi(x)_c(1_c)).$$

In other words, it would suffice to define  $\Psi(x)_c(1_c)$ .

Recall our definition of  $\Phi$ , for which we want  $\Psi$  to be an inverse. Plugging in  $\Psi(x)$  for  $\alpha$ , we have

$$\Psi(x)_c(1_c) = \Phi(\Psi(x)) = x.$$

Our hand is now forced to define

$$\Psi(x)_b(f) := Ff(x).$$

□