1 Let R be any ring. For a left R-module M the following conditions are equivalent:

(A) M is projective;

(B) There exists a family $(m_i)_{i\in I}$ of element of M, together with a family of maps $(\Phi_i)_{i\in I}$ in $M^* = \operatorname{Hom}_R(M, R)$ such that

  (i) for each $m \in M$, there are only finitely many $i \in I$ with the property that $\Phi_i(m) \neq 0$;

  (ii) $\mathrm{id}_M = \sum_{i\in I} \Phi_i(-)m_i$, that is, $m = \sum_{i\in I} \Phi_i(m)m_i$ for every $m \in M$.

Prove only "(B) $\implies$ (A)."

Proof. Assume (B) holds. Consider the free left R-module $F = \bigoplus_{i\in I} Rx_i$. We define a morphism $\Psi : F \to M$ by $x_i \mapsto m_i$. It follows from property (ii) tells us that the m_i's generate M as a left R-module, so in fact Ψ is an epimorphism. This gives us the a short exact sequence in R-Mod:

$$0 \longrightarrow \ker\Psi \hookrightarrow F \xrightarrow{\Psi} M \longrightarrow 0$$

For each $m \in M$, we define $\Phi(m) = \sum_{i\in I} \Phi_i(m)x_i$, which is a finite sum by property (i) and therefore well-defined. This gives us a map $\Phi : M \to F$, which is in fact a homomorphism of left R-modules:

$$\begin{aligned}
\Phi(rm + m') &= \sum_{i\in I} \Phi_i(rm + m')x_i \\
&= \sum_{i\in I} (r\Phi_i(m) + \Phi_i(m'))x_i \\
&= r\sum_{i\in I} \Phi_i(m)x_i + \sum_{i\in I} \Phi_i(m')x_i \\
&= r\Phi(m) + \Phi(m').
\end{aligned}$$

Note that the third equality relies on the fact that each sum is finite by property (i). Now, for all $m \in M$, we have

$$\Psi\Phi(m) = \Psi\left(\sum_{i\in I} \Phi_i(m)x_i\right) = \sum_{i\in I} \Phi_i(m)\Psi(x_i) = \sum_{i\in I} \Phi_i(m)m_i = m.$$

In other words, the exact sequence above splits, so $F \cong \ker\Psi \oplus M$. In particular, M is a direct summand of a free module, so by definition M is projective. $\square$

2 This problem shows that projective modules (in contrast to free modules) need not be direct sums of finitely generated modules.

Let R be the ring of continuous real functions $f : [0,1] \to \mathbb{R}$ with the standard operations and M the ideal consisting of those functions $g \in R$ which vanish on some (variable) neighborhood of 0; that is

$$M = \{g \in R \mid \exists \text{ a neighborhood } N \text{ of } 0 \text{ such that } g|_N = 0\}.$$

(a) Prove that M is not finitely generated as an R-module.

Proof. Assume for contradiction that M is finitely generated by $g_1, \ldots, g_n \in R$, i.e., that we can write $M = \sum_{i=1}^n Rg_i$. For each $i = 1, \ldots, n$ choose $\varepsilon_i > 0$ such that $g_i|_{[0,\varepsilon_i)} = 0$.

Define $\varepsilon = \min\{\varepsilon_1, \ldots, \varepsilon_n\} > 0$, so that $[0,\varepsilon) \subseteq [0,\varepsilon_i)$ for all i. By assumption, for an arbitrary $g \in M$ we have $g = \sum_{i=1}^n a_i g_i$ for some coefficient functions $a_i \in R$. For any $x \in [0,\varepsilon)$ we have

$$g(x) = \sum_{i=1}^n a_i(x) g_i(x) = \sum_{i=1}^n a_i(x) \cdot 0 = 0.$$

In other words, $g|_{[0,\varepsilon)} = 0$ for all $g \in M$.

Define the function $f : [0,1] \to \mathbb{R}$ by

$$f(x) = \max\{0, x - \varepsilon/2\}.$$

As the composition of continuous functions, f is continuous, i.e., $f \in R$. More specifically, f is zero on the interval $[0, \varepsilon/2)$, which is an open neighborhood of 0 in $[0,1]$, so in fact $f \in M$. However, we also have

$$f(3\varepsilon/4) = \max\{0, 3\varepsilon/4 - \varepsilon/2\} = \max\{0, \varepsilon/4\} = \varepsilon/4 > 0,$$

which contradicts the fact that all functions in M should vanish on $[0,\varepsilon)$. $\square$

(b) Show that M is an indecomposable R-module, i.e., whenever $M = A \oplus B$ with submodules A, B (ideals of R contained in M in our case), then $A = 0$ or $B = 0$.

Proof. For each $S \subseteq R$, define the support of S in $[0,1]$ by

$$\operatorname{supp} S = \{x \in [0,1] \mid \exists f \in S \text{ such that } f(x) \neq 0\}.$$

Assume for contradiction that $M = A \oplus B$. On one hand, $A \subseteq M$ and $B \subseteq M$, from which it follows that $\operatorname{supp} A \cup \operatorname{supp} B \subseteq \operatorname{supp} M$. On the other hand, if $x \in \operatorname{supp} M$ then there is some $f \in M$ such that $f(x) \neq 0$. Since $M = A \oplus B$, we can write $f = f_A + f_B$ with $f_A \in A$ and $f_B \in B$. Then $f(x) = f_A(x) + f_B(x) \neq 0$, so either $f_A(x) \neq 0$ or $f_B(x) \neq 0$. Therefore, either $x \in \operatorname{supp} A$ or $x \in \operatorname{supp} B$, so in fact $\operatorname{supp} M = \operatorname{supp} A \cup \operatorname{supp} B$.

We claim $\operatorname{supp} A$ and $\operatorname{supp} B$ are disjoint. Suppose $x \in \operatorname{supp} A \cap \operatorname{supp} B$, which means there exists $f \in A$ adn $g \in B$ such that $f(x)$ and $g(x)$ are both nonzero. But then their product

fg is an element of $A \cap B$, which must be trivial. That is, $fg = 0$, but we must have $fg(x) = f(x)g(0) \neq 0$. This is a contradiction, so $\operatorname{supp} A$ and $\operatorname{supp} B$ are disjoint.

Since $\operatorname{supp} M$ connected, and $\operatorname{supp} A$ and $\operatorname{supp} B$ are disjoint open sets, we must have either $\operatorname{supp} A = \varnothing$ or $\operatorname{supp} B = \varnothing$. But this implies either $A = 0$ or $B = 0$. ☐

(c) Prove that M is a projective R-module.

Proof. We will apply Problem 1, i.e., we will construct a dual basis for M. Define $m_n \in M$ as per the hint. Additionally, define $\Phi_n \in \operatorname{Hom}_R(M, R)$ by $\Phi_1(f) = f$ and $\Phi_n(f) = (1 - m_{n-1})f$ for $n \geq 2$.

For every $f \in M$, there is some $\varepsilon > 0$ such that $f|_{[0,\varepsilon)} = 0$. By construction, $1 - m_n$ is zero on the interval $[\frac{1}{n}, 1]$. Then for $n > \frac{1}{\varepsilon}$, we have $1 - m_n$ zero on the interval $[\varepsilon, 0]$, which implies $\Phi_{n+1}(f) = (1 - m_n)f$ is zero on all of $[0, 1]$. In particular, $\Phi_n(f)$ is nonzero for finitely many $n \in \mathbb{N}$, i.e., condition (i) is satisfied.

For $x \in (\frac{1}{i+1}, \frac{1}{i})$, we have $m_n(x) = 0$ for $n \leq i - 1$ and $m_n(x) = 1$ for $n \geq i + 1$. Then

$$\begin{aligned}
\sum_{n \in \mathbb{N}} (1 - m_{n-1}(x))m_n(x) &= (1 - m_{i-1}(x))m_i(x) + (1 - m_i(x))m_{i+1}(x) \\
&= (1 - 0)m_i(x) + (1 - m_i(x)) \cdot 1 \\
&= m_i(x) + (1 - m_i(x)) \\
&= 1.
\end{aligned}$$

Therefore, for $f \in M$ and $x \in [0, 1]$, we have

$$\left(\sum_{n \in \mathbb{N}} \Phi_n(f) m_n \right)(x) = \sum_{n \in \mathbb{N}} (1 - m_{n-1}(x)) f(x) m_n(x) = f(x).$$

In other words, condition (ii) is satisfied. This means we have successfully constructed a dual basis for M, hence M is projective. ☐

3 Prove that, up to isomorphism, the divisible abelian groups are precisely the direct sums of copies of $\mathbb{Q}$ and Prüfer groups $\mathbb{Z}(p^\infty)$, for primes p.

Lemma 1. If A is a torsionfree divisible abelian group, $A \cong \mathbb{Q}^{(I)}$.

Proof. We want to define a $\mathbb{Q}$-module structure on A. For $\frac{a}{b} \in \mathbb{Q}$ and $x \in A$, choose $y \in A$ such that $x = by$, then define $\frac{a}{b} \cdot x = ay$.

For this to be well-defined, we check that the choice of y is unique. Suppose $x = by = by'$, then $b(y - y') = 0$. Since $b \in \mathbb{Z}_{>0}$ and A is torsionfree, we must have $y - y' = 0$, i.e., $y = y'$.

We now check that the scalar multiplication above defines a $\mathbb{Q}$-module structure on A:

$$\tfrac{1}{1} \cdot x = 1x = x,$$

$$\tfrac{a}{b} \cdot (x + x') = \tfrac{a}{b} \cdot (by + by') = \tfrac{a}{b} \cdot b(y + y') = a(y + y') = ay + ay' = \tfrac{a}{b} \cdot x + \tfrac{a}{b} \cdot x',$$

$$(\tfrac{a}{b} + \tfrac{a'}{b'}) \cdot x = \tfrac{ab' + a'b}{bb'} \cdot bb'y = (ab' + a'b)y = ab'y + a'by = \tfrac{a}{b} \cdot bb'y + \tfrac{a'}{b'} \cdot bb'y = \tfrac{a}{b} \cdot x + \tfrac{a'}{b'} \cdot x,$$

$$\tfrac{a}{b} \cdot (\tfrac{a'}{b'} \cdot x) = \tfrac{a}{b} \cdot (\tfrac{a'}{b'} \cdot bb'y) = \tfrac{a}{b} \cdot a'by = aa'y = \tfrac{aa'}{b} \cdot bb'y = (\tfrac{a}{b} \cdot \tfrac{a'}{b'}) \cdot x.$$

Hence, A is a $\mathbb{Q}$-vector space and therefore we have an isomorphism $A \cong \mathbb{Q}^{(I)}$. □

Lemma 2. If A is a nonzero divisible torsion abelian group, then $A = \bigoplus_{p \text{ prime}} T_p(A)$, where

$$T_p(A) = \{x \in A \mid \exists n \in \mathbb{N} \text{ such that } p^n x = 0\}.$$

Proof. It is quick to check that each $T_p(A)$ is a subgroup. Given $x, y \in T_p(A)$, say $p^n x = 0$ and $p^m x = 0$. Then $p^{n+m}(x + y) = p^m(p^n x) + p^n(p^m y) = 0 + 0 = 0$, hence $x + y \in T_p(A)$.

We first show that $A = \sum_{p \text{ prime}} T_p(A)$. Given $x \in A$, say $p^n x = 0$. Take a prime decomposition $n = p_1^{k_1} p_2^{k_2} \cdots p_m^{k_m}$, with all the p_i's distinct. We perform induction on m to show that $x \in \sum_{i=1}^m T_{p_i}(A)$. Trivially, if $m = 1$ then $p_1^{k_1} x = 0$ so indeed $x \in T_{p_1}(A)$.

For the inductive hypothesis, assume that the conclusion is true whenever the prime decomposition has at most $m \geq 1$ distinct primes. Suppose $n = p^k n_0$ where $n_0 = p_1^{k_1} \cdots p_m^{k_m}$, so that n has $m+1$ distinct primes in its prime factor decomposition. Note that p^k and n_0 are coprime, so Bézout's Lemma tells us there exist $u, v \in \mathbb{Z}$ such that $un_0 + vp^k = 1$. Then we can write $x = un_0 x + vp^k x$ and notice that $p^k(n_0 x) = nx = 0$, so $n_0 x \in T_p(A)$. Additionally, $n_0(p^k x) = nx = 0$, so $p^k x$ is an element of A which is annihilated by an integer n_0 which has m distinct primes in its prime factorization. Therefore, the inductive hypothesis gives us

$$x = un_0 x + vp^k x \in T_p(A) + \sum_{i=1}^m T_{p_i}(A).$$

This completes the induction, thus $A = \sum_{p \text{ prime}} T_p(A)$.

To prove that the summation is direct, we must show that every element $x \in A$ has a unique representation $x = x_1 + \cdots + x_n$ with $x_i \in T_{p_1}(A)$ for some finite collection of primes $p_1, \ldots, p_n$. It suffices to prove that $0 \in A$ has a unique representation, i.e., that whenever

$x_1 + \cdots + x_n = 0$ with $x_i \in T_{p_i}(A)$, we must have $x_i = 0$ for all i. If this is the case and we have $x = x_1 + \cdots + x_n = y_1 + \cdots + y_n$ for an arbitrary $x \in A$, with $x_i, y_i \in T_{p_i}(A)$, then $0 = (x_1 - y_1) + \cdots + (x_n - y_n)$ with each $x_i - y_i \in T_{p_i}(A)$. If indeed zero has a unique representation, then it must be the case that $x_i = y_i$ for all i, so indeed x would have a unique representation.

To prove that zero has a unique representation, suppose $x_1 + \cdots + x_n = 0$ with $x_i \in T_{p_i}(A)$. We perform induction on n. The base case is trivial. Suppose the result holds for $n \geq 1$ and that $0 = x + x_1 + \cdots + x_n$ with $x_i \in T_{p_i}(A)$ and $x \in T_p(A)$; say $p^k x = 0$. Then

$$0 = -p^k x = p^k x_1 + \cdots + p^k x_n,$$

where $p^k x_i \in T_{p_i}(A)$. By the inductive hypothesis, we must have $p^k x_i = 0$ for all i. But this implies $x_i \in T_p(A) \cap T_{p_i}(A)$. say $p_i^{k_i} x_i = 0$, then p^k and $p_i^{k_i}$ are coprime and Bézout's Lemma gives us $u, v \in \mathbb{Z}$ such that $up^k + vp_i^{k_i} = 1$. Then

$$x_i = up^k x_i + vp_i^{k_i} = 0 + 0 = 0,$$

from which we deduce

$$x = x + x_1 + \cdots + x_n = 0.$$

This completes the induction, thus $A = \bigoplus_{p \text{ prime}} T_p(A)$. □

Lemma 3. If A is a nonzero divisible p-torsion abelian group, then there exists a subgroup of A isomorphic to the Prüfer group, i.e., there is an embedding $\mathbb{Z}(p^\infty) \hookrightarrow A$. In particular, $\mathbb{Z}(p^\infty)$ is a direct summand of A, so $A \cong \mathbb{Z}(p^\infty) \oplus B$ for some subgroup B of A.

Proof. Given $y \in A$ nonzero, say $p^n y = 0$. Define $x_1 = p^{n-1} y \in A$ so the order of x_1 is p. Then the cyclic subgroup $\langle x_1 \rangle \leq A$ is isomorphic to $\mathbb{Z}/p\mathbb{Z}$.

Given $x_i \in A$ with $\langle x_i \rangle \cong \mathbb{Z}/p^i\mathbb{Z}$, choose $x_{i+1} \in A$ such that $x_i = px_{i+1}$. Then $\langle x_{i+1} \rangle \cong \mathbb{Z}/p^{i+1}\mathbb{Z}$ with an embedding $\langle x_i \rangle \hookrightarrow \langle x_{i+1} \rangle$ given by $x_i \mapsto px_i = x_{i+1}$.

This is an inductive construction of a system of inclusions $\langle x_i \rangle \hookrightarrow \langle x_{i+1} \rangle$ for which $\langle x_i \rangle \cong \mathbb{Z}/p^i\mathbb{Z}$. Taking the direct limit of this system gives us a subgroup $X \leq A$. Moreover, this construction is the same as our construction of $\mathbb{Z}(p^\infty)$ so in fact $X \cong \mathbb{Z}(p^\infty)$. □

Lemma 4. If A is a divisible p-torsion abelian group, $A \cong \mathbb{Z}(p^\infty)^{(I)}$.

Proof. Define the set

$$\mathcal{U} = \{(U_i)_{i \in I} \mid \mathbb{Z}(p^\infty) \cong U_i \leq A \text{ and } \textstyle\sum_{i \in I} U_i = \bigoplus_{i \in I} U_i\}.$$

We define a partial order on $\mathcal{U}$ by $(U_i)_{i \in I} \leq (V_j)_{j \in J}$ whenever there is an inclusion $I \hookrightarrow J$ of index sets and $U_i = V_i$ for all $i \in I$.

Let $C \subseteq U$ be a chain. Take the index set $I_0 = \bigcup\{I \mid (U_i)_{i \in I} \in C\}$, where we identify indices using the inclusions implied by the partial order. Then the upper bound of C is simply $(U_i)_{i \in I_0}$. The fact that $(U_i)_{i \in I_0}$ is an element of $\mathcal{U}$ follows from the fact that to check if a sum is direct, it suffices to check that each finite "sub-sum" is direct. And the fact that a given finite sub-sum is direct reduces to the condition for some $(U_i)_{i\in I} \in C$.

By Zorn's Lemma, let $(U_i)_{i\in I}$ be a maximal element of $\mathcal{U}$. Define $U = \bigoplus_{i\in I} U_i$, which is a direct sum of divisible abelian groups and therefore an injective abelian group. So $A = U \oplus B$ for some $B \leq A$; we claim that B is trivial.

Note that B is a divisible p-torsion abelian group, so by Lemma 3, if B is nonzero then it must have a subgroup $X \leq B$ isomorphic to $\mathbb{Z}(p^\infty)$. In particular, $B = X \oplus C$ for some $C \leq B$. But then $A = U \oplus X \oplus C$, and we could add X into the collection $(U_i)_{i\in I}$ and get a strictly larger element of $\mathcal{U}$. This would contradict the maximality of $(U_i)_{i\in I}$, so B must be trivial. Hence, $A = U = \bigoplus_{i\in I} U_i \cong \mathbb{Z}(p^\infty)^{(I)}$. □

Proposition 1. If A is a divisible abelian group, then $A \cong \mathbb{Q}^{(I)} \oplus \bigoplus_{p \text{ prime}} \mathbb{Z}(p^\infty)^{(I_p)}$.

Proof. Let $T(A)$ be the torsion subgroup of A.

We check that $T(A)$ is divisible. Let $x \in T(A)$ and $n \in \mathbb{Z}_{>0}$. Since A is divisible, there is some $y \in A$ such that $x = ay$. Since x is torsion, there is some $m \in \mathbb{Z}_{>0}$ such that $mx = 0$. But then $0 = mx = (mn)y$, so y is torsion. That is, $y \in T(A)$, so $T(A)$ is a divisible group.

Hence, $T(A)$ is an injective $\mathbb{Z}$-module, so we can write $A = A_0 \oplus T(A)$, where $A_0 \cong A/T(A)$. We apply Lemma 1 to A_0, Lemma 2 to $T(A)$, and Lemma 4 to each $T_p(A) = T_p(T(A))$:

$$A \cong A_0 \oplus T(A) \cong \mathbb{Q}^{(I)} \oplus \bigoplus_{p \text{ prime}} \mathbb{Z}(p^\infty)^{(I_p)}$$

□

$\boxed{\textbf{4} \quad \text{Show that an abelian group } A \text{ is a flat } \mathbb{Z}\text{-module if and only if } A \text{ is torsionfree.}}$

Proof. For each $n \in \mathbb{Z}_{>0}$, there is an isomorphism of abelian groups $\mathbb{Z} \cong n\mathbb{Z}$ given by $1 \mapsto n$, i.e., the multiplication by n map.

Suppose A is flat. For any ideal inclusion $\iota : n\mathbb{Z} \hookrightarrow \mathbb{Z}$, tensoring with A gives a monomorphism $A \otimes_Z n\mathbb{Z} \to A \otimes_Z \mathbb{Z}$. Therefore, the following sequence of maps is a monomorphism:

$$\begin{array}{ccccccccc} A & \xrightarrow{\cong} & A \otimes_{\mathbb{Z}} \mathbb{Z} & \xrightarrow{\cong} & A \otimes_{\mathbb{Z}} n\mathbb{Z} & \xrightarrow{A\otimes\iota} & A \otimes_{\mathbb{Z}} \mathbb{Z} & \xrightarrow{\cong} & A \\ x & \longmapsto & x \otimes 1 & \longmapsto & x \otimes n & \longmapsto & x \otimes n & \longmapsto & nx \end{array}$$

In other words, multiplying by n is an injective operation on A for all $n \in \mathbb{Z}_{>0}$, hence A is torsionfree.

Suppose A is torsionfree. Let $\iota : n\mathbb{Z} \hookrightarrow \mathbb{Z}$ be the inclusion of any ideal. We know that multiplication by n is an injective operation on A, so the following sequence of maps is a monomorphism:

$$\begin{array}{ccccccccc} A \otimes_{\mathbb{Z}} n\mathbb{Z} & \xrightarrow{\cong} & A \otimes_{\mathbb{Z}} \mathbb{Z} & \xrightarrow{\cong} & A & \xrightarrow{\cdot n} & A & \xrightarrow{\cong} & A \otimes_{\mathbb{Z}} \mathbb{Z} \\ x \otimes n & \longmapsto & x \otimes 1 & \longmapsto & x & \longmapsto & nx & \longmapsto & nx \otimes 1 = x \otimes n \end{array}$$

But this is precisely the map $A \otimes \iota$, so in fact A is flat. $\square$