Geometry cares about distance, so we study metric spaces.

Let X be a set.

A **metrix** on X is a function $d : X \times X \to \mathbb{R}$ such that

- $d(x, y) = 0$ if and only if $x = y$ (identity of indiscernibles / Leibniz's law);
- $d(x, y) = d(y, x)$ (symmetry);
- $d(x, z) \le d(x, y) + d(y, z)$ (triangle inequality).

Some authors require the following addition property (which follows from the firs three):

- $d(x, y) \ge 0$ (nonnegativity).

A **metric space** is a set X with a metric d_X.

Examples

- A normed vector space X (e.g., $\mathbb{R}^n$) has a metric $d(x, y) = \|x - y\|_X$.
- Any set has the **discrete metric** $d(x, x) = 0$ and $d(x, y) = 1$ for $x \ne y$.
- X a set and Y a metric space, the set of bounded functions $X \to Y$ is a metric space with $d(f, g) = \sup_{x\in X} d_Y(f(x), g(x))$. (This might be called the uniform metric or supremum (sup) metric.)
- X a metric space and $f : [0, \infty) \to [0, \infty)$ a nondecreasing concave function with $f^{-1}(0) = \{0\}$, then $f \circ d$ is a metric on X.
- X a set, Y a metric space, and $f : X \to Y$ an injection, then $d(x, y) = d_Y(f(x), f(y))$ is a metric on Y.

Important Example.

Let Γ be a connected graph, then set of vertices V has a metric given by the shortest path between two vertices.

$$d(u, v) = \min_{P_\Gamma(u,v)} |-| = \min\{|p| : p \in \Gamma[u, v]\}.$$

This is called the **path metric** of V with respect to Γ.

Given a group G and generating set S, the **word metric** d_S on G with respect to S is the path metric on G with respect to the Cayley graph $\mathrm{Cay}(G, S)$, i.e.,

$$d_S(g, h) = \min\{n \in \mathbb{N} \mid gs_1 \cdots s_n = h \text{ for some } s_i \in S\}.$$

The distance $d_S(1_G, g)$ is called the **word length** of g with respect to S.

Note

$$gs_1 \cdots s_n = h \iff s_1 \cdots s_n = g^{-1}h,$$

so $d_S(g,h) = d_S(1_G, g^{-1}h)$.

Let X be a metric space.

Given $r > 0$ and $x \in X$, we construct the **open ball** in X of radius r centered at x as follows:

$$B_r(x) = B_r(x; X) = B_r(x; d_X) = \{y \in X \mid d_X(x,y) < r\}.$$

The second and third notation may be used to specify the specific space/metric with respect to which the ball is constructed. Various notations are used for this set, including but not limited to:

$$B(x;r), \quad B_X(x;r), \quad B_r^X(x), \quad B(x;r;X), \quad B(x;r;d_X).$$

Hopefully the sort of pattern is clear—it is usually fairly easy to intuit any given authors convention. Usually any mix of a "B" and the necessary data denotes an open ball.

Sometimes this notation is also used for the following slightly modified construction. We construct the **closed ball** in X of radius r centered at x as follows:

$$\bar{B}_r(x) = \overline{B}_r(x) = \{y \in X \mid d_X(x,y) \leq r\}.$$

There are also all the corresponding notational variants featuring"$\bar{B}$" or "$\overline{B}$."

Every metric space X is 'naturally' a topological space, with a basis of open sets given by the collection of all open balls:

$$\{B_r(x) \subseteq X : r > 0, x \in X\}.$$

The topology generated by this basis is called the **metric topology** on X.

It it worth remarking that $\bar{B}_r(x) \neq \overline{B_r(x)}$ in general, where the former is the closed ball and the latter is the closure of the open ball in the metric topology. On the other hand, it is always true that $\overline{\bar{B}_r(x)} = \bar{B}_r(x)$, i.e, $\bar{B}_r(x)$ is closed in the metric topology.

Something continuity something ε-δ.

Examples

- $\mathbb{R}^n$ with any usual metric induces Euclidean topology.
- discrete metric induces discrete topology.

Let X and Y be metric spaces.

An **isometric embedding** is a function $f : X \to Y$ such that $d_Y(f(x), f(y)) = d_X(x, y)$.

All isometric embeddings are injective.)

Every isometric embedding is a continuous with respect to the metric topologies.

An **isometry** is a surjective isometric embedding.

All isometries are bijective. Their inverses are isometric embeddings and isometries.

Every isometry is a homeomorphism with respect to the metric topologies.

Isometry is very strong—too strong...

Let $f : X \to Y$ be a function between metric spaces.

Say f is **Lipschitz** (or Lipschitz continuous) if there is a constant $C \in \mathbb{R}$ such that

$$d_Y(f(x), f(y)) \leq C d_X(x, y).$$

In this case, also say f is **C-Lipschitz**.

Since the metric is nonnegative, nontrivial C-Lipschitz functions only exist for $C > 0$, so we typically assume that K is positive (or at least nonnegative). Some authors require this in the definition, but it is a minor point in any case.

All lipschitz functions are continuous with respect to the metric topologies.

Say f is **bilipschitz** (or called a **bilipschitz embedding**) if there exists a (positive) constant $K \in \mathbb{R}$ such that

$$\frac{1}{C} d_X(x, y) \leq d_Y(f(x), f(y)) \leq C d_X(x, y).$$

In this case, also say f is **C-bilipschitz**.

A C-bilipschitz function is clearly C-Lipschitz.

All bilipschitz functions are injections.

A **bilipschitz equivalence** is a bilipschitz function with a bilipschitz inverse. (equiv, a surjective bilipschitz function).

Say two spaces are **bilipschitz equivalent** if there exists a bilipschitz equivalence between them.

Let $f : X \to Y$ be a function between metric spaces.

Say f is a **quasi-isometric embedding** if there exist constants $C, K \in \mathbb{R}$ such that

$$\frac{1}{C} d_X(x, y) - K \leq d_Y(f(x), f(y)) \leq C d_X(x, y) + K.$$

In this case, also say f is a **(C, K)-quasi-isometric embedding**. (slogan "almost injective")

Say f is a **quasi-isometry** if it is a (C, K)-quasi-isometric embedding and there exists a constant $D \in \mathbb{R}$ such that every point in Y is a distance at most D from a point in the image of f, i.e., for all $y \in Y$ there exists $x \in X$ such that $d_Y(y, f(x)) \leq D$. (slogan "almost surjective")

Say X and Y are **quasi-isometric**, written $X \sim_{\mathrm{QI}} Y$, if there exists a quasi-isometry between them.

It is equivalent to take a single constant $R = \max\{C, K\}$ in the definition of quasi-isometric embedding, and likewise $R = \max\{C, K, D\}$ in the definition of quasi-isometry.

"$\sim_{\mathrm{QI}}$" is an equivalence relation on metric spaces.

Some equivalent definition of quasi-isometry in terms of quasi-inverse and...

Say two functions $f, g : X \to Y$ between metric spaces are a **finite distance** from each other if there is a constant $D \in \mathbb{R}$ such that $d_Y(f(x), g(x)) \leq D$.

If $f : X \to Y$ and $g : Y \to X$ are maps such that $f \circ g$ is a finite distance to id_Y and $g \circ f$ is a finite distance to id_X, say f and g are **quasi-inverse** to each other.

Categories

Let $\mathsf{Met}_{\mathsf{isom}}$ and $\mathsf{Met}_{\mathsf{bilip}}$ be the categories whose objects are metric spaces and whose morphisms are isometric embeddings and bilipschitz embeddings, respectively. Then the isomorphisms in these categories are the respective equivalences between metric spaces.

Let QMet' be the category whose objects are metric spaces and whose morphisms are quasi-isometric embeddings. The isomorphisms in this category are bijective quasi-isometric embeddings whose inverse is also a quasi-isometric embedding, not the quasi-isometries. The relation of "finite distance from" describes an equivalence relation on $\mathsf{QMet}'(X, Y)$. We define the quotient category

$$\mathsf{QMet} = \mathsf{QMet}'/\text{finite distance},$$

whose objects are metric spaces and whose morphisms are equivalence classes of quasi-isometric embeddings modulo finite distance. The isomorphisms in QMet are precisely the quasi-isometries, as desired.

Natural inclusions (essentially subcategories with less morphisms):

$$\mathsf{Met}_{\mathsf{isom}} \hookrightarrow \mathsf{Met}_{\mathsf{bilip}} \hookrightarrow \mathsf{QMet}.$$

For $X \in \mathsf{QMet}$ (any metric space), the **quasi-isometry group** of X is

$$\mathrm{QI}(X) := \mathrm{Aut}_{\mathsf{QMet}}(X).$$

In other words, the group of quasi-isometries $X \to X$ modulo finite distance.

Let X be a metric space.

A **geodesic segment** in X is an isometric embedding $[a, b] \to X$ of an interval $[a, b] \subseteq \mathbb{R}$.

A **geodesic** (line) in X is an isometric embedding $\mathbb{R} \to X$.

We often identify geodesics (and geodesic segments) with their images in X.

We say X is a **geodesic metric space** if for any $x, y \in X$, there is a geodesic segment $\gamma : [a, b] \to X$ such that $\gamma(a) = x$ and $\gamma(b) = y$. In other words, any two points in X are connected by a geodesic segment that realizes their distance.

Geodesic metric spaces.

- $\mathbb{R}^n$ with Euclidean metric; geodesics are straight lines.
- Any convex set in $\mathbb{R}^n$.
- $\mathbb{R}^n \setminus \{0\}$ is not geodesic as v and $-v$ have no stright line between them for $v \neq 0$.
- For G finitely generated group, $\mathrm{Cay}(G, S)$ with the path metric is geodesic. When G is a free group over S, $\mathrm{Cay}(G, S)$ is a tree so there is a unique path between any pair of points.

A metric space X is **proper** if for all $x \in X$ and $r > 0$ the closed ball $\bar{B}_r(x)$ is a compact subset of X.

Let X be a metric space.

A **quasi-geodesic segment** in X is a quasi-isometric embedding $[a, b] \to X$.

A **quasi-geodesic** (line) in X is a quasi-isometric embedding $\mathbb{R} \to X$.

Let G be a group and X a metric space.

An **action by isometries** of G is a group homomorphism $G \to \mathrm{Isom}(X)$.

Let G be a group and $X \subseteq G$.

Say X **generates** G, written $G = \langle X \rangle$ if every element $g \in G$ can be expressed as a product of elements in $X \cup X^{-1}$, where $X^{-1} = \{x^{-1} | x \in X\}$. That is, there exists $x_i \in X$ and $\varepsilon_i \in \{\pm 1\}$ such that

$$g = x_1^{\varepsilon_1} x_2^{\varepsilon_2} \cdots x_n^{\varepsilon_n}.$$

Say X is **symmetric** if $X = X^{-1}$, i.e., $x^{-1} \in X$ for all $x \in X$. If this is the case, we can simply express $g \in G$ as $g = x_1 x_2 \cdots x_n$ for some $x_i \in X$.

If G is finitely generated, we define the **rank** of G, denoted $\mathrm{rk}(G)$, to be the minimal cardinality of a finite generating subset, i.e.,

$$\mathrm{rk}(G) = \min\{|X| : G = \langle X \rangle\}.$$

If $H \leq G$ is finite index, then G is finitely generated if and only if H is finitely generated. In particular,

$$\mathrm{rk}(G) \leq [G : H] + \mathrm{rk}(G) - 1$$

and

$$\mathrm{rk}(H) \leq [G : H] \cdot \mathrm{rk}(G).$$

Theorem 1 (Ping-Pong Lemma)**.** Let G be a group generated by $X \subseteq G$ with $|X| \geq 2$. Suppose G acts on a set E and there is a family $\{A_x\}_{x \in X}$ of nonempty subsets of E such that for all distinct $x, y \in X$ we have $A_y \nsubseteq A_x$ but $x^k A_y \subseteq A_x$ for all nonzero $k \in \mathbb{Z}$. Then G is a free group with basis X.

Let G be a group.

A **series/chain** in G is a collection of subgroups $G_i \leq G$ for $i \in \mathbb{Z}$ such that $G_i \leq G_{i+1}$.

We usually care about **ascending** chains, i.e.,

$$1 = G_0 \leq G_1 \leq G_2 \leq \cdots \leq G,$$

and **descending** chains, i.e.,

$$G = G_0 \geq G_{-1} \geq G_{-2} \geq \cdots \geq 1,$$

A (normal) series $\{G_i \trianglelefteq G\}_{i \in \mathbb{Z}}$ is **central** if and G_{i+1}/G_i is contained in the center of G/G_i.

Let G be a group.

The **commutator** of two elements $a, b \in G$ is the element

$$[a, b] := a^{-1} b^{-1} a b \in G.$$

Also define $a^b := b^{-1}ab$.

For $a_i \in G$ recursively define

$$[a_1, \ldots, a_n] := [[a_1, \ldots, a_{n-1}], a_n],$$

called a **simple commutator** of **weight** n.

Some identities:

(i) $[a, b]^{-1} = [b, a]$

(ii) $[ab, c] = [a, c]^b[b, c] = [a, c][a, c, b][b, c]$

(iii) $[a, bc] = [a, c][a, b]^c = [a, c][b, a, c][a, b]$

(iv) $[a, b^{-1}] = ([a, b]^{b^{-1}})^{-1}$

(v) $[a^{-1}, b] = [b, a]^{a^{-1}}$

(vi) $[a, b^{-1}, c]^b[b, c^{-1}, a]^c[c, a^{-1}, b]^a = 1$ (Hall-Witt)

In this notation, a subgroup $H \leq G$ is normal if and only if $h^g \in H$ for all $h \in H$ and $g \in G$.

Say H is **characteristic** in G if $\varphi(H) \subseteq H$ for all $\varphi \in \mathrm{Aut}(G)$.

Given subgroups $H, K \leq G$, define **commutator subgroup**

$$[H, K] := \langle [h, k] : h \in H, k \in K \rangle \leq G.$$

Notice $[H, K] = [K, H]$.

$[H, K] \trianglelefteq H$ whenever $H \trianglelefteq G$, and $[H, K] \trianglelefteq G$ whenever $H, K \trianglelefteq G$.

Lemma 1 (three subgroup)**.** Let $N \trianglelefteq G$ and $H, K, L \leq G$. If $[[H, K], L]$ and $[[K, L], H]$ are contained in N, then so is $[[L, H], K]$.

Let G be a group.

The **lower central series** of G is the descending central series $\{\gamma_k(G)\}_{k \geq 1}$ defined recursively by

$$\gamma_1(G) := G \quad \text{and} \quad \gamma_{k+1}(G) := [\gamma_k(G), G].$$

(Check that this is indeed a central series).

Say G is **nilpotent** if its lower central series is finite. In which case, its length is referred to as the **nilpotency class** of G.

Lemma 2. $[\gamma_i(G), \gamma_j(G)] \subseteq \gamma_{i+j}(G)$ for all $i, j \geq 1$.

Lemma 3. If $G \geq G_1 \geq G_2 \geq \cdots$ is a descending central series, then $G_i \supseteq \gamma_i(G)$ for all i.

Proposition 1. A group is nilpotent if and only if it admits a finite central series.

Proposition 2. If nilpotent G is has nilpotency class c, then all subgroups and quotients of G have nilpotency class at most c.

The **center** of a group G is the set

$$Z(G) := \{z \in G \mid zg = gz \text{ for all } g \in G\}.$$

The **upper central series** of G is the ascending central series defined recursively by

$$Z_0(G) := 1 \quad \text{and} \quad Z_{i+1}(G) := \pi_i^{-1}(Z(G/Z_i(G))),$$

where $\pi_i : G \to G/Z_i(G)$ is the quotient map. Then we have

$$Z(G/Z_i(G)) = Z_{i+1}(G)/Z_i(G).$$

In particular, $Z_1(G) = Z(G)$.

Lemma 4. If $G = G_1 \geq G_2 \geq \cdots \geq G_r = 1$ is a finite central series, then $Z_i(G) \supseteq G_{r-i}$ for all i. In particular, $Z_{r-1}(G) = G$.

Proposition 3. A group is nilpotent if and only if its upper central series is finite.

A group is **torsion-free** if it has no nontrivial elements of finite order.

Lemma 5. Let G be a torsion-free nilpotent group with upper central series $\{Z_k(G)\}_{k\geq 0}$. Then all factors $Z_{i+1}(G)/Z_i(G)$ are torsion-free abelian groups.

Theorem 2 (Malcev)**.** Every finitely generated torsion-free nilpotent group embeds into a group of upper triangular integer matrices, i.e., $G \hookrightarrow \mathrm{UT}_n(\mathbb{Z})$ for some n.

Lemma 6. It suffices to embed $G \in \mathrm{UT}_n(\mathbb{Q})$.

Let G be a group with a finite symmetric generating set S.

The **growth function** of G relative to S is the function $\beta_{G,S} : \mathbb{N} \to \mathbb{N}$ defined by

$$\beta_{G,S}(n) := |B_n^{G,S}(e)| = \#\{g \in G \mid d_S(g, e) \leq n\}.$$

Fixing G and S, we briefly omit them from notation.

Note that $B_0(e) = \{e\}$, so $\beta(0) = 1$ and $\beta(n) \leq \beta(n+1)$ for all $n \in \mathbb{N}$.

We also have $B_1(e) = S \cup \{e\}$, so $\beta(1) = |S \cup \{e\}|$. (Either $|S|$ or $|S| + 1$ depending on whether $e \in S$ or $e \notin S$, respectively.)

The product map gives a surjection

$$\begin{aligned} (S \cup \{1\})^{\times n} &\longrightarrow B_n(e) \\ (s_1, \ldots, s_n) &\longmapsto s_1 \cdots s_n \end{aligned}$$

so $\beta(n) \le |S \cup \{1\}|^n$.

Some properties:

(1) (Sub-multiplicativity) $\beta(m+n) \le \beta(m) \cdot \beta(n)$.

(2) If G is infinite, then $\beta(n) < \beta(n+1)$ for all $n \in \mathbb{N}$. In particular, $\beta(n) \ge n$.

(3) $\beta_{G,S}(n) \le \beta_{F,S}(n)$, where F is the free group on S.

A **generalized growth function** is an increasing function $[0,\infty) \to [0,\infty)$ of nonnegative real numbers.

Let $f, g : [0,\infty) \to [0,\infty)$ be generalized growth functions.

Say g **quasi-dominates** f, written $f \preceq g$, if there exist constants C, K such that

$$f(r) \le Cg(Cr + K) + K$$

for all $r \in [0,\infty)$.

Say f and g are **quasi-equivalent**, written $f \sim g$, if $f \preceq g$ and $g \preceq f$.

For G generated by S, have induced generalized growth function $r \mapsto \beta_{G,S}(\lceil r \rceil)$.

Examples

Use x as shorthand for identity in function expressions.

For $a \ge 0$, the power function $x^a : r \mapsto r^a$ is a generalized growth function.

For $a, b \ge 0$, $x^a \preceq x^b$ if and only if $a \le b$, so $x^a \sim x^b$ if and only if $a = b$.

If $f : [0,\infty) \to [0,\infty)$ is a polynomial function of degree $d \ge 0$, then $f \sim x^d$.

For $a > 1$, the exponential function $a^x : r \mapsto a^r$ is a generalized growth function.

For $a, b > 1$, $a^x \sim b^x$. In particular, $a^x \sim \exp$.

For $a \ge 0$, $x^a \prec \exp$ (strict, i.e., $x^a \preceq \exp$ but $x^a \not\sim \exp$).

Proposition 4. Let G and H be groups with finite generating sets S and T, respectively.

(1) If there exists a quasi-isometric embedding $(G, d_S) \to (H, d_T)$, then $\beta_{G,S} \preceq \beta_{H,T}$.

(2) In particular, if G and H are quasi-isometric then $\beta_{G,S} \sim \beta_{H,T}$.

Corollary 1. Let G be a group with finite generating sets S and T. Then $\beta_{G,S} \sim \beta_{G,T}$. Moreover, $\beta_{G,S} \preceq \exp$.

The **growth type** of finitely generated group G is the equivalence of growth functions associated with its the finite symmetric generating sets; denote it by β_G. This is either an equivalence class or representative as necessary.

Say G is of **polynomial growth** if $\beta_G \sim x^a$ for some $a \geq 0$.

Say G is of **exponential growth** if $\beta_G \sim \exp$.

Say G is of **intermediate growth** if it is neither polynomial nor exponential.

Proposition 5. ...

(1) Let $f : [0, \infty) \to [0, \infty)$ be a generalized growth function with $f(0) > 0$. Then $f \sim 1$ (constant) if and only if f is bounded.

(2) Let G be a finitely generated group. Then $\beta_G \sim 1$ if and only if G is finite. In particular, all finite groups have the same growth type—namely constant 1.

(3) Let G be a finitely generated group. Then G is infinite if and only if $x \succeq \beta_G$.

(4) Every finitely generated group of sub-polynomial growth (in particular of polynomial growth) has sub-exponential growth.

Let P be a predicate of groups.

Say a group G is **virtually** P if there is a finite index subgroup $H \leq G$ which is P.

e.g., a group is virtually nilpotent if it contains a nilpotent subgroup of finite index.

e.g., a group is virtually solvable if it contains a solvable subgroup of finite index.

Theorem 3. Let G be a finitely generated nilpotent group with nilpotency class $n \in \mathbb{N}$ (length of lower central series). Then G has polynomial growth of degree

$$\sum_{k=1}^{n-1} (k+1) \cdot \mathrm{rk}_{\mathbb{Z}}(\gamma_k(G)/\gamma_{k+1}(G)).$$

where $\mathrm{rk}_{\mathbb{Z}}\, A$ is torsion-free rank of abelian group A.

Theorem 4 (Gromov?)**.** A finitely generated group has polynomial growth if and only if it is virtually nilpotent.

Corollary 2. Virtual nilpotency is QI invariant (geometric property?) for finitely generated groups.

Let M be a smooth manifold of dimension n. Denote its tangent bundle by TM. Given a local parameterization $\varphi : U \to M$, $U \subseteq \mathbb{R}^n$ open, induces vector fields $\partial_{x_1}, \ldots, \partial_{x_n}$ and the map $U \times \mathbb{R}^n \to TM$, $(x, v) \mapsto v_1\partial_{x_1} + \cdots + v_n\partial_{x_n}$, is a local parameterization of TM. (local frame? whatever that is.)

A **field of distributions** on M is a subset Δ

A **Lie algebra** $\mathfrak{g}$ is a vector space equipped with a bilinear operation $[-,-] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ satisfying

- (anti-commutativity) $[X, Y] = -[Y, X]$ for all $X, Y \in \mathfrak{g}$
- (Jacobi identity) $[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$