Lemma 1. Let $M \in \Lambda\text{-}\mathsf{mod}$ with $\underline{\dim}\, M = \mathbf{d}$, and let $x \in \mathrm{Rep}_{\mathbf{d}}\Lambda$ such that M corresponds to the G-orbit $G.x$. If $U \subseteq M$ is a submodule, then the direct sum $U \oplus M/U$ corresponds to a G-orbit contained in $\overline{G.x}$.

Proof. Let $\mathbf{d}' = \underline{\dim}\, U$. Then $\mathbf{d}'' = \mathbf{d} - \mathbf{d}' = \underline{\dim}\, M/U$, since $e_i(M/U) \cong e_iM/e_iU$ canonically.

For each $i \leq n$, choose an ordered K-basis for e_iU and supplement it to an ordered basis for e_iM.

Suppose $x = (x_\alpha)_{\alpha \in Q_1}$. Since $\alpha U \subseteq U$ by hypothesis, the x_α have the following block format. If $\alpha : e_i \to e_j$ then

$$x_\alpha = \begin{bmatrix} A_\alpha & C_\alpha \\ 0 & B_\alpha \end{bmatrix}$$

where A_α is a $d_i' \times d_i'$ matrix and B_α is a $d_i'' \times d_i''$ matrix.

For each $c \in K$, define an element $g(c) \in \prod_{1 \leq i \leq n} M_{d_i}(K)$ as follows:

$$g(c) = (g(c)_1, \ldots, g(c)_n)$$

where

$$g(c)_i = \begin{bmatrix} cI_{d_i'} & 0 \\ 0 & I_{d_i''} \end{bmatrix}$$

where I_m is the $m \times m$ identity matrix.

Clearly, $g(c) \in G = \prod_{1 \leq i \leq n} \mathrm{GL}_{d_i}(K)$ whenever $c \in K^\times$.

Now consider the morphism of varieties

$$\begin{aligned} \psi : K &\longrightarrow \mathrm{Rep}_{\mathbf{d}}(\Lambda), \\ c &\longmapsto \left(\begin{bmatrix} A_\alpha & cC_\alpha \\ 0 & B_\alpha \end{bmatrix} \right)_{\alpha \in Q_1}. \end{aligned}$$

Observe: for $c \in K^\times$, we have

$$\psi(c) = \left(g(c)_{\mathrm{end}(\alpha)} x_\alpha g(c)_{\mathrm{start}(\alpha)} \right)_{\alpha \in Q_1} = g(c).x \in G.x$$

Since ψ is Zariski-continuous, $\psi^{-1}(\overline{G.x})$ is closed in K, and thus $\psi^{-1}(\overline{G.x}) = K$.

In other words,

$$\psi(0) = \left(\begin{bmatrix} A_\alpha & 0 \\ 0 & B_\alpha \end{bmatrix} \right)_{\alpha \in Q_1} \in \overline{G.x},$$

but clearly the orbit of $\psi(0)$ in $\overline{G.x}$ represents the direct sum $U \oplus M/U$.

□