1 Let M be a smooth manifold.

(a) Use partition of unity to show that, if F_1, F_2 are two closed subsets of M such that $F_1 \cap F_2 = \varnothing$, then there is a smooth function f on M such that $f = 1$ on F_1 and $f = 0$ on F_2.

Note that $\{M \setminus F_1, M \setminus F_2\}$ is an open cover of M. Let $\{\psi_1, \psi_2\}$ be a smooth partition of unity subordinate to this cover. By construction, the support of ψ_1 is contained in $M \setminus F_1$, which means that $\psi_1 \equiv 0$ on F_1. Similarly, $\psi_2 \equiv 0$ on F_2. Since $\psi_1 + \psi_2 \equiv 1$, we deduce that $\psi_2 \equiv 1$ on F_1. Hence, $f = \psi_2$ is the desired function.

(b) Now let U be an open set of M (which then inherits a smooth manifold structure) and F be a closed subset such that $F \subseteq U$. Show that for any smooth function f on U, there is a smooth function $\overline{f}$ on M such that $\overline{f}|_F = f|_F$. (We have made use of this extension result in HW2.)

Applying part (a), choose $h \in C^\infty(M)$ such that $h \equiv 1$ on F and $h \equiv 0$ on $M \setminus U$. We now define the smooth function $\overline{f} = f \cdot h$ on U, which we can extend to all of M by defining $\overline{f} \equiv 0$ outside of U.

By construction, $\overline{f}$ is smooth inside of U. For a point $x \in M \setminus U$, we have $x \notin \operatorname{supp} f$. Since the support of f is a closed set, there is a neighborhood V of x which is entirely outside of $\operatorname{supp} f$. In which case, $f \equiv 0$ on V, which is clearly smooth.

2 Let S^n be the standard sphere in $\mathbb{R}^{n+1}$. Show that, for any $p \in S^n$, T_pS^n can be naturally identified with the subspace of vectors in $\mathbb{R}^{n+1}$ perpendicular to p.

Consider the inclusion $\iota : S^n \hookrightarrow \mathbb{R}^{n+1}$.

We claim that the differential $\iota_* : T_pS^n \to T_p\mathbb{R}^{n+1} \cong \mathbb{R}^{n+1}$ is injective. The radial projection $\pi : \mathbb{R}^{n+1} \setminus \{0\} \to S^n$ is a retraction, i.e., $\pi \circ \iota = \text{id}$. Then functorality of the differential gives us $\pi_* \circ \iota_* = \text{id}$, hence ι_* is injective.

3 Recall that the graph Γ of the function $f(x) = |x|$, $x \in \mathbb{R}$, is a smooth manifold.

(a) Show that it is diffeomorphic to $\mathbb{R}$.

Note that $\mathbb{R}$ is a smooth manifold with a global chart $\mathrm{id}_{\mathbb{R}} : \mathbb{R} \to \mathbb{R}$.

There is a global chart $\varphi : \Gamma \to \mathbb{R}$ given by $(x, |x|) \mapsto x$. By definition, this is a smooth map of manifolds.

Moreover, the inverse $\varphi^{-1} : \mathbb{R} \to \Gamma$ is also smooth:

$$\widehat{\varphi^{-1}} = \varphi \circ \varphi^{-1} \circ \mathrm{id}_{\mathbb{R}}^{-1} = \mathrm{id}_{\mathbb{R}}$$

Hence, φ is a diffeomorphism.

(b) Show that, however, there is no diffeomorphism $F : \mathbb{R}^2 \to \mathbb{R}^2$ taking $\mathbb{R}$ to Γ (or vice versa). Here you can take $\mathbb{R}$ to be the real axis of $\mathbb{R}^2$. (**Hint**: By contradiction, assume that there is such an F. Without loss of generality, we can assume that $F(0,0) = (0,0)$ (why?). Write $F(x,y) = (g(x,y), h(x,y))$. What does F taking $\mathbb{R}$ to Γ tell you about the form of $F(x,0)$? This will depend on which side you are approaching the origin. Now what does this information translate to $F_*(\frac{\partial}{\partial x})$?)

Per the hint, assume F is such a diffeomorphism. We may assume $F(0,0) = (0,0)$ since translation on $\mathbb{R}^2$ is a diffeomorphism.

The fact that F takes $\mathbb{R}$ to Γ tells us that $F(x,0) = (g(x,0), |g(x,0)|)$.

At the point $(a,0)$ we compute

$$F_*(\tfrac{\partial}{\partial x})(x) = \tfrac{\partial}{\partial x}\Big|_{(a,0)} g \quad \text{and} \quad F_*(\tfrac{\partial}{\partial x})(y) = \tfrac{\partial}{\partial x}\Big|_{(a,0)} |g|.$$

Therefore, we have

$$F_*(\tfrac{\partial}{\partial x}) = \left(\tfrac{\partial}{\partial x}\Big|_{(a,0)} g\right) \tfrac{\partial}{\partial x} + \left(\tfrac{\partial}{\partial x}\Big|_{(a,0)} |g|\right) \tfrac{\partial}{\partial y}.$$

For $a < 0$ we get

$$F_*(\tfrac{\partial}{\partial x}) = \left(\tfrac{\partial}{\partial x}\Big|_{(a,0)} g\right) \tfrac{\partial}{\partial x} - \left(\tfrac{\partial}{\partial x}\Big|_{(a,0)} g\right) \tfrac{\partial}{\partial y}$$

and for $a > 0$ we get

$$F_*(\tfrac{\partial}{\partial x}) = \left(\tfrac{\partial}{\partial x}\Big|_{(a,0)} g\right) \tfrac{\partial}{\partial x} + \left(\tfrac{\partial}{\partial x}\Big|_{(a,0)} g\right) \tfrac{\partial}{\partial y}.$$

Since F is a diffeomorphism, F_* is an isomorphism, so these are nonzero for all values of a. But then taking the limit as $a \to 0$ gives us different answers from the left and the right, which is a contradiction.

4 Lee 3-5 (second part only) Let $S^1 \subseteq \mathbb{R}^2$ be the unit circle, and let $K \subseteq \mathbb{R}^2$ be the boundary of the square of side 2 centered at the origin: $K = \{(x, y) : \max(|x|, |y|) = 1\}$. Show that there is a homeomorphism $F : \mathbb{R}^2 \to \mathbb{R}^2$ such that $F(S^1) = K$, but there is no *diffeomorphism* with the same property. [Hint: let γ be a smooth curve whose image lies in S^1, and consider the action of $\mathrm{d}F(\gamma'(t))$ on the coordinate functions x and y.]