1 Let X be a metric space and let r be a constant. You may assume the theorem that metric spaces are paracompact.

(a) Show that there is a simplicial complex P and a map $f : X \to P$ such that the preimage of each simplex has diameter at most r. (The *diameter* of a set in a metric space is the sup distance between points inside that set.)

Before beginning the proof, we make explicit the notion of a simplicial complex with arbitrarily many distinct vertices.

Definition 1. Let Λ be an arbitrary indexing set. To each finite subset $L = \{\lambda_0, \dots, \lambda_n\} \subseteq \Lambda$ we associate an n-simplex $\Delta^L \cong \Delta^n$. Recall the standard n-simplex

$$\Delta^n = \left\{x \in \mathbb{R}^{n+1} : x_i \geq 0 \text{ and } \textstyle\sum_{i=0}^{n} x_i = 1\right\}.$$

With this construction in mind, we can write Δ^L as the set of formal sums

$$\Delta^L = \left\{\textstyle\sum_{i=0}^{n} \alpha_i \lambda_i : \alpha \in \Delta^n\right\}.$$

For $K \subseteq L$ there is a natural (continuous) inclusion $\Delta^K \hookrightarrow \Delta^L$. Under this identification, the faces of Δ^L are the simplices $\Delta^{L \setminus \{\lambda_i\}}$.

Define the simplicial complex Δ^Λ as union of all simplices Δ^L with $L \subseteq \Lambda$ finite, quotiented by the inclusions $\Delta^K \hookrightarrow \Delta^L$ for all $K \subseteq L \subseteq \Lambda$. We can describe Δ^Λ as the set of formal sums $\sum_{\lambda \in \Lambda} \alpha_\lambda \lambda$ such that

(i) $\alpha_\lambda \geq 0$ for all $\lambda \in \Lambda$,

(ii) $\alpha_\lambda = 0$ for all but finitely many $\lambda \in \Lambda$ (i.e., $\alpha : \Lambda \to \mathbb{R}$ has finite support), and

(iii) $\sum_{\lambda \in \Lambda} \alpha_\lambda = 1$ with the sum taken over the nonzero α_λ's.

For each finite subset $L \subseteq \Lambda$ there is a natural inclusion $\Delta^L \hookrightarrow \Delta^\lambda$; the topology on Δ^Λ is the direct limit topology with respect to these inclusions.

Proof of (a). The collection of open balls $\{B_{r/4}(x)\}_{x \in X}$ is an open cover of X. Since X is paracompact, this cover has a locally finite refinement $\mathcal{U} = \{U_\lambda\}_{\lambda \in \Lambda}$. In other words, $\mathcal{U}$ is an open cover of X such that each U_λ has diameter at most $r/2$. Additionally, let $\{\tau_\lambda\}_{\lambda \in \Lambda}$ be a partition of unity subordinate to $\mathcal{U}$.

Define the map $f : X \to \Delta^\Lambda$ by

$$f(x) = \sum_{\lambda \in \Lambda} \tau_\lambda(x) \lambda.$$

We check that f is continuous. Given an open subset $U \subseteq \Delta^\Lambda$, we consider a point in its preimage $x \in f^{-1}(U)$. Since $\mathcal{U}$ is locally finite, there is an open neighborhood $V \subseteq X$ of x such that the set of indices $L = \{\lambda \in \Lambda : V \cap U_\lambda \neq \varnothing\}$ is a finite. If $\lambda \notin L$ then the support of τ_λ is contained in $U_\lambda \subseteq V^c$. So for $y \in V$ we have

$$f(y) = \sum_{\lambda \in L} \tau_\lambda(y) \lambda.$$

In other words, we can consider $f|_V$ as a map $V \to \Delta^L \subseteq \Delta^\Lambda$. Identifying Δ^L with a standard simplex (as in Definition 1), it is clear that $f|_V$ is continuous as the sum of continuous maps from the partition of unity. And since the inclusion $\Delta^L \hookrightarrow \Delta^\Lambda$ is continuous, $W = f|_V^{-1}(U)$ is an open subset of V containing x. Since V is an open subspace of X, we know W is also open in X. With $W \subseteq f^{-1}(U)$, this proves $f^{-1}(U)$ is open, hence f is continuous.

We construct a simplicial complex P as a subcomplex of Δ^Λ:

$$P = \bigcup \big\{ \Delta^L \subseteq \Delta^\Lambda : f(x) \in \operatorname{int} \Delta^L \text{ for some } x \in X \big\}.$$

(We consider the interior of a 0-simplex to be itself: $\operatorname{int} \Delta^{\{\lambda\}} = \Delta^{\{\lambda\}}$ for all $\lambda \in \Lambda$.) It is immediate that P is itself a simplicial complex since each finite simplex Δ^L includes its faces and intersections in P are the same as in Δ^Λ. Since $f(x)$ is always contained in some finite simplex and every point of a finite simplex is contained in the interior of a subsimplex, we know that $f(x)$ must be contained in the interior of some finite simplex which, by construction, is contained in P. Therefore, the image of f is contained in P so we may consider f as a continuous function $f : X \to P \subseteq \Delta^\Lambda$.

Let $\Delta^L \subseteq P$ be a finite subsimplex and $z \in X$ such that $f(z) \in \operatorname{int} \Delta^L$. Recall that

$$f(z) = \sum_{\lambda \in \Lambda} \tau_\lambda(z) \lambda,$$

so we must have $\tau_\lambda(z) \neq 0$ if and only if $\lambda \in L$, which implies $z \in \bigcap_{\lambda \in L} U_\lambda$. By construction we have $f^{-1}(\Delta^L) \subseteq \bigcup_{\lambda \in L} U_\lambda$. Then for $x, y \in f^{-1}(\Delta^L)$ we find

$$d(x, y) \leq d(x, z) + d(z, y) < \frac{r}{2} + \frac{r}{2} = r.$$

Hence, the diameter of $f^{-1}(\Delta^L)$ is at most r. $\square$

(b) Let P be as in the previous part and equip $C(X)$ with the sup norm. Construct a map $g : P \to C(X)$ such that $g \circ f(x)$ is at most distance $2r$ from the Kuratowski embedding.

Fix a point $x_0 \in X$ and let $\Phi : X \to C_B(X)$ be the Kuratowski embedding defined by

$$\Phi(x)(y) = d(x, y) - d(x_0, y).$$

For $\lambda \in \Lambda$ choose a representative point $x_\lambda \in U_\lambda$.

Suppose $u \in \Delta^L \subseteq P$ with $u = \sum_{\lambda \in L} \alpha_\lambda \lambda$. Define $g(u) : X \to \mathbb{R}$ by

$$g(u)(y) = \sum_{\lambda \in \Lambda} \alpha_\lambda d(x_\lambda, y) - d(x_0, y),$$

where $\alpha_\lambda = 0$ for $\lambda \notin L$. As the composition of continuous functions $g(u) \in C(X)$.

For $x \in X$ recall that $f(x) = \sum_{\lambda \in \Lambda} \tau_\lambda(x)\lambda$ so

$$(g \circ f)(x)(y) = \sum_{\lambda \in \Lambda} \tau_\lambda(x) d(x_\lambda, y) - d(x_0, y).$$

Then

$$\begin{aligned}
|\Phi(x)(y) - (g \circ f)(x)(y)| &= \left| d(x, y) - d(x_0, y) - \sum_{\lambda \in \Lambda} \tau_\lambda(x) d(x_\lambda, y) - d(x_0, y) \right| \\
&= \sum_{\lambda \in \Lambda} \tau_\lambda(x) \big| d(x, y) - d(x_\lambda, y) \big| \\
&\leq \sum_{\lambda \in \Lambda} \tau_\lambda(x) d(x, x_\lambda).
\end{aligned}$$

If $f(x) \in \Delta^L \subseteq P$ then $\tau_\lambda(x) = 0$ for all $\lambda \notin L$ so

$$\sum_{\lambda \in \Lambda} \tau_\lambda(x) d(x, x_\lambda) = \sum_{\lambda \in L} \tau_\lambda(x) d(x, x_\lambda).$$

By construction of P, we can choose a point $z \in X$ such that $f(z) \in \operatorname{int} \Delta^L$, i.e., $\tau_\lambda(z) \neq 0$ if and only if $\lambda \in L$. Then

$$\sum_{\lambda \in L} \tau_\lambda(x) d(x, x_\lambda) \leq \sum_{\lambda \in L} \tau_\lambda(x) \big(d(x, z) + d(z, x_\lambda) \big) = d(x, z) + \sum_{\lambda \in L} \tau_\lambda(x) d(z, x_\lambda).$$

Note that $x, z \in f^{-1}(\Delta^L)$ so part (a) implies $d(x, z) \leq r$. Additionally, $z \in U_\lambda$ for all $\lambda \in L$ and each U_λ has a diameter of at most $r/2$, so

$$\sum_{\lambda \in L} \tau_\lambda(x) d(z, x_\lambda) \leq \sum_{\lambda \in L} \tau_\lambda(x) \frac{r}{2} = \frac{r}{2} \leq r.$$

Hence, $\|\Phi(x) - (g \circ f)(x)\|_\infty \leq 2r$.

(c) Deduce that if x and y are two points of X, then

$$|d(x,y) - d(g \circ f(x), g \circ f(y))| \leq 4r.$$

Proof. Note that The Kuratowski embedding is an isometry, i.e., $\|\Phi(x) - \Phi(y)\|_\infty = d(x,y)$ for all $x, y \in X$.

Denote $\Psi = g \circ f$. We compute

$$\begin{aligned} d(x,y) &= \|\Phi(x) - \Phi(y)\|_\infty \\ &\leq \|\Phi(x) - \Psi(x)\|_\infty + \|\Psi(x) - \Psi(y)\|_\infty + \|\Psi(y) - \Phi(y)\|_\infty \\ &\leq \|\Psi(x) - \Psi(y)\|_\infty + 4r. \end{aligned}$$

This implies

$$d(x,y) - d(\Psi(x), \Psi(y)) \leq 4r.$$

Similarly,

$$\begin{aligned} \|\Psi(x) - \Psi(y)\|_\infty &\leq \|\Psi(x) - \Phi(x)\|_\infty + \|\Phi(x) - \Phi(y)\|_\infty + \|\Phi(y) - \Psi(y)\|_\infty \\ &\leq \|\Phi(x) - \Phi(y)\|_\infty + 4r \\ &= d(x,y) + 4r. \end{aligned}$$

This implies

$$d(\Psi(x), \Psi(y)) - d(x,y) \leq 4r.$$

We conclude that

$$\left| d(x,y) - d(\Psi(x), \Psi(y)) \right| \leq 4r.$$

□

2 Prove the Remark on p. 124 of Jänich: a locally compact Hausdorff space which is a countable union of compact subspaces is paracompact.

Hint. Use (without proof, this time) the lemma from Homework 6: a compact subspace of a locally compact space is contained in the interior of a bigger compact subspace.

Proof. Let X be a locally compact Hausdorff space such that $X = \bigcup_{n\in\mathbb{N}} K_n$ with $K_n \subseteq X$ compact. Applying the lemma from Homework 6, there is a compact set $L_1 \subseteq X$ such that $K_1 \subseteq \operatorname{int} L_1$. Then can then replace K_2 with $K_2 \cup L$, i.e., we can assume $K_1 \subseteq \operatorname{int} K_2$. Continuing inductively, we can assume $K_n \subseteq \operatorname{int} K_{n+1}$ for all $n \in \mathbb{N}$.

Define the compact sets

$$E_n = K_n \setminus \operatorname{int} K_{n-1},$$

where $K_0 = \varnothing$. Then $K_n \subseteq \bigcup_{k=1}^n E_k$ so we have $X = \bigcup_{n\in\mathbb{N}} E_n$. Define the open sets

$$V_n = \operatorname{int} K_{n+1} \setminus K_{n-2},$$

where $K_{-1} = K_0 = \varnothing$; then $E_n \subseteq V_n$. For $m \geq n+3$ we have

$$\begin{aligned} V_n \cap V_m &= (\operatorname{int} K_{n+1} \setminus K_{n-2}) \cap (\operatorname{int} K_{m+1} \setminus K_{m-2}) \\ &\subseteq K_{n+1} \cap (X \setminus K_{m-2}) \\ &\subseteq K_{n+1} \cap (X \setminus K_{n+1}) \\ &= \varnothing. \end{aligned}$$

In other words, with $n \in \mathbb{N}$ fixed, $V_n \cap V_m = \varnothing$ for all but finitely many $m \in \mathbb{N}$.

Let $\mathcal{U}$ be an open cover of X. For $n \in \mathbb{N}$ define

$$\mathcal{U}_n = \{U \cap V_n : U \in \mathcal{U}\},$$

which is an open cover of the compact set E_n contained in V_n. Let $\mathcal{B}_n \subseteq \mathcal{U}_n$ be a finite subcover of E_n. We claim that $\mathcal{B} = \bigcup_{n\in\mathbb{N}} \mathcal{B}_n$ is a locally finite refinement of $\mathcal{U}$.

As $\mathcal{B}_n$ covers E_n and $X = \bigcup_{n\in\mathbb{N}} E_n$, we know that $\mathcal{B}$ is a cover of X.

Since each open set in $\mathcal{B}_n$ is of the form $U \cap V_n$ for some $U \in \mathcal{U}$, it is also clear that $\mathcal{B}$ is a refinement of $\mathcal{U}$.

Any point $x \in X$ is contained in some E_n. Then V_n is a neighborhood of x which meets only finitely many other V_m's. And V_n intersects $U \in \mathcal{B}$ only if V_n meets V_m and $U \in \mathcal{B}_m$. Since each $\mathcal{B}_m$ is finite, we conclude that V_n intersects finitely many open sets in $\mathcal{B}$.

□