1 Exercise I.5.1 Locate the singular points and sketch the following curves in $\mathbb{A}^2$ (assume char $k \neq 2$). Which is in figure 4?

(a) $x^2 = x^4 + y^4$

Let $f = x^2 - x^4 - y^4$ and consider the rank of

$$\begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{bmatrix} = \begin{bmatrix} 2x - 4x^3 & -4y^3 \end{bmatrix}.$$

The singularities occur when the rank is less than 1, i.e., then the rank is zero. The rank is zero when

$$2x - 4x^3 = 0 \quad \text{and} \quad -4y^3 = 0.$$

This occurs when $x = 0, \pm 1/\sqrt{2}$ and $y = 0$. Note however that $(\pm 1/\sqrt{2}, 0) \notin Z(f)$, so the only singularity is at $(0, 0)$.

This is the Tacnode in Figure 4.

(b) $xy = x^6 + y^6$

Let $f = xy - x^6 - y^6$ and consider the rank of

$$\begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{bmatrix} = \begin{bmatrix} y - 6x^5 & x - 6y^5 \end{bmatrix}.$$

The singularities occur when

$$y = 6x^5 \quad \text{and} \quad x = 6y^5.$$

This occurs when $x = y = 0$, and we check that no other solution lies in $Z(f)$. If x is nonzero, y must also be nonzero and

$$x = 6y^5 = 6(6x^5)^5 = 6^6 x^{25},$$

which implies $x^{24} = 1/6^6$. However, in order to satisfy $f(x, y) = 0$, we must have

$$0 = xy - x^6 - y^6 = x(6x^5) - x^6 - (6x^5)^6 = 5x^6 - 6^6 x^{30},$$

which implies $x^{24} = 5/6^6$. Hence, the only singularity is at $(0, 0)$.

This is the Node in Figure 4.

(c) $x^3 = y^2 + x^4 + y^4$

Let $f = x^3 - x^4 - y^2 - y^4$ and consider the rank of

$$\begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{bmatrix} = \begin{bmatrix} 3x^2 - 4x^3 & -2y - 4y^3 \end{bmatrix}.$$

The singularities occur when

$$3x^2 - 4x^3 = 0 \quad \text{and} \quad -2y - 4y^3 = 0.$$

This occurs when $x = 0, 3/4$ and $y = 0, \pm i/\sqrt{2}$. One can check that the only combination of components satisfying $f(x,y) = 0$ is $(0,0)$.

This is the Cusp in Figure 4.

(d) $x^2y + xy^2 = x^4 + y^4$

Let $f = x^2y + xy^2 - x^4 - y^4$ and consider the rank of

$$\begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{bmatrix} = \begin{bmatrix} 2xy + y^2 - 4x^3 & x^2 + 2xy - 4y^3 \end{bmatrix}.$$

Let $g_1 = 2xy + y^2 - 4x^3$ and $g_2 = x^2 + 2xy - 4y^3$. The singularities occur when

$$g_1(x,y) = 0 \quad \text{and} \quad g_2(x,y) = 0.$$

In other words, the set of singularities can be written as

$$Z(f, g_1, g_2) = Z(\langle f, g_1, g_2 \rangle).$$

Using Buchberger's algorithm to compute a reduced Gröbner basis with respect to the lexicographic monomial order, we find

$$\langle f, g_1, g_2 \rangle = \langle y^3,\, x^2 + y^2,\, xy + \tfrac{1}{2}y^2 \rangle.$$

So we must have $y^3 = 0$, which implies $x = y = 0$. Hence, the only singularity is at $(0,0)$.

This is the Triple Point in Figure 4.

2 Exercise I.5.2 Locate the singular points and describe the singularities of the following surfaces in $\mathbb{A}^3$ (assume $\operatorname{char} k \neq 2$). Which is in figure 5?

(a) $xy^2 = z^2$

Let $f = xy^2 - z^2$ and consider the rank of

$$\begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{bmatrix} = \begin{bmatrix} y^2 & -2xy & -2z \end{bmatrix}.$$

The singularities occur when

$$y^2 = 0, \quad -2xy = 0, \quad -2z = 0.$$

That is, the surface is singular when $y = z = 0$, i.e., alone the line $(x, 0, 0)$ for $x \in \mathbb{C}$.

This is the Pinch Point in Figure 5.

(b) $x^2 + y^2 = z^2$

Let $f = x^2 + y^2 - z^2$ and consider the rank of

$$\begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{bmatrix} = \begin{bmatrix} 2x & 2y & -2z \end{bmatrix}.$$

The singularities occur when

$$2x = 0, \quad 2y = 0, \quad -2z = 0.$$

That is, the only singularity is at $(0, 0, 0)$.

This is the Conical Double Point in Figure 5.

$\boxed{\textbf{(c)} \quad xy + x^3 + y^3 = 0}$

Let $f = xy + x^3 + y^3$ and consider the rank of

$$\begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{bmatrix} = \begin{bmatrix} y + 3x^2 & x + 3y^2 \end{bmatrix}.$$

The singularities occur when

$$y + 3x^2 = 0, \quad x + 3y^2 = 0.$$

In other words, the set of singularities can be written as

$$Z(f, y + 3x^2, x + 3y^2) = Z(\langle f, y + 3x^2, x + 3y^2 \rangle).$$

Using Buchberger's algorithm to compute a reduced Gröbner basis with respect to the lexicographic monomial order, we find

$$\langle f, y + 3x^2, x + 3y^2 \rangle = \langle x, y \rangle.$$

So we must have $x = y = 0$. Hence, the only singularity is at $(0, 0, 0)$.

This is the Double Line in Figure 5.

3 Exercise I.5.3 Let $Y \subseteq \mathbb{A}^2$ be a curve defined by the equation $f(x,y) = 0$. Let $P = (a,b)$ be a point of $\mathbb{A}^2$. Make a linear change of coordinates so that P becomes the point $(0,0)$. Then write f as a sum $f = f_0 + f_1 + \cdots f_d$, where f_i is the homogeneous polynomial of degree i in x and y. Then we define the *multiplicity* of P on Y, denoted $\mu_P(Y)$, to be the least r such that $f_r \neq 0$. (Note that $P \in Y \iff \mu_P(Y) > 0$.) The linear factors of f_r are called the *tangent directions* at P.

(a) Show that $\mu_P(Y) = 1$ if and only if P is a nonsingular point of Y.

Proof. After a linear change of coordinates, we

$$f = ax + by + \sum_{i=2}^{d} f_i,$$

where $a, b \in \mathbb{C}$ and $f_i \in \mathbb{C}[x,y]$ is homogeneous of degree i. Then to check if Y is singular at P, we consider the rank of

$$\begin{bmatrix} \frac{\partial f}{\partial x}\big|_{(0,0)} & \frac{\partial f}{\partial y}\big|_{(0,0)} \end{bmatrix} = \begin{bmatrix} a & b \end{bmatrix}.$$

We know that Y is nonsingular at P if and only if the rank of this matrix is 1. The rank of this matrix is 1 if and only if a, b are not both zero. And a, b are not both zero if and only if the linear part $ax + by$ of f at P is nonzero. Lastly, the linear part $ax + by$ of f at P is nonzero if and only if $\mu_Y(P) = 1$. □

(b) Find the multiplicity of each of the singular points in (Ex. 5.1) above.

(a) 2

(b) 2

(c) 2

(d) 3

4 Exercise II.1.15 Let $\mathcal{F}$, $\mathcal{G}$ be sheaves of abelian groups on X. For any open set $U \subseteq X$, show that the set $\operatorname{Hom}(\mathcal{F}|_U, \mathcal{G}|_U)$ of morphisms of the restricted sheaves has a natural structure of abelian group.

Proof. Note that $\mathcal{F}|_U(V) = \mathcal{F}(V)$ for all open sets $V \subseteq U$.

Given $\varphi, \psi \in \operatorname{Hom}(\mathcal{F}|_U, \mathcal{G}|_U)$ and $V \subseteq U$ open, define

$$\begin{aligned}(\varphi + \psi)(V) : \mathcal{F}|_U(V) &\longrightarrow \mathcal{G}|_U(V) \\ f &\longmapsto \varphi(V)(f) + \psi(V)(f).\end{aligned}$$

One can check that this describes an abelian group structure on $\operatorname{Hom}(\mathcal{F}|_U, \mathcal{G}|_U)$. □

Show that the presheaf $U \mapsto \operatorname{Hom}(\mathcal{F}|_U, \mathcal{G}|_U)$ is a sheaf. It is called the *sheaf of local morphisms* of $\mathcal{F}$ into $\mathcal{G}$, "sheaf hom" for short, and is denoted $\mathcal{H}om(\mathcal{F}, \mathcal{G})$.

Proof. Let $\{V_i\}$ be an open cover of an open set $U \subseteq X$ and $\varphi_i \in \operatorname{Hom}(\mathcal{F}|_{V_i}, \mathcal{G}|_{V_i})$ such that $\varphi_i|_{V_i \cap V_j} = \varphi_j|_{V_i \cap V_j}$. We want to construct $\varphi \in \operatorname{Hom}(\mathcal{F}|_U, \mathcal{G}|_U)$ such that $\varphi|_{V_i} = \varphi_i$. For an open subset $V \subseteq U$ we have an open cover $\{W_i = V \cap V_i\}$ of V. Then given $f \in \mathcal{F}|_U(V)$, consider the images $\varphi_i(W_i)(f|_{W_i}) \in \mathcal{G}|_U(W_i) = \mathcal{G}(W_i)$. We have

$$\begin{aligned}\varphi_i(W_i)(f|_{W_i})|_{W_i \cap W_j} &= \varphi_i(V \cap V_i)(f|_{V \cap V_i})|_{V \cap V_i \cap V_j} \\ &= \varphi_i(V \cap V_i \cap V_j)(f|_{V \cap V_i \cap V_j}) \\ &= \varphi_i|_{V_i \cap V_j}(V \cap V_i \cap V_j)(f|_{V \cap V_i \cap V_j}) \\ &= \varphi_j|_{V_i \cap V_j}(V \cap V_i \cap V_j)(f|_{V \cap V_i \cap V_j}) \\ &= \varphi_j(V \cap V_i \cap V_j)(f|_{V \cap V_i \cap V_j}) \\ &= \varphi_j(V \cap V_j)(f|_{V \cap V_j})|_{V \cap V_i \cap V_j} \\ &= \varphi_j(W_j)(f|_{W_j})|_{W_i \cap W_j}.\end{aligned}$$

Since $\mathcal{G}|_U$ is a sheaf, there is a unique section in $\mathcal{G}|_U(V)$ which restricts to $\varphi_i(W_i)(f|_{W_i})$ on each W_i; denote this section by $\varphi(f)$. One can check that that $\varphi : \mathcal{F}|_U \to \mathcal{G}|_U$ defines a morphism of sheaves.

Let $\{V_i\}$ be an open cover of an open set $U \subseteq X$ and $\varphi \in \operatorname{Hom}(\mathcal{F}|_U, \mathcal{G}|_U)$ such that $\varphi|_{V_i} = 0$. Given an open set $V \subseteq U$, consider $\varphi(V) : \mathcal{F}(V) \to \mathcal{G}(V)$. For $f \in \mathcal{F}(V)$, we have $\varphi(V)(f) \in \mathcal{G}(V)$. Then

$$\varphi(V)(f)|_{V \cap V_i} = \varphi(V \cap V_i)(f|_{V \cap V_i}) = \varphi|_{V_i}(V \cap V_i)(f|_{V \cap V_i}) = 0.$$

Since $\mathcal{G}|_U$ is a sheaf, this implies $\varphi(V)(f) = 0$. So in fact $\varphi(V) = 0$ for all $V \subseteq U$ open, and we conclude that $\varphi = 0$.

□