1 (a) Prove that there is a nonzero bounded linear functional on $L^\infty(\mathbb{R})$ which vanishes on $C(\mathbb{R})$. Note that $C(\mathbb{R})$ consists of all bounded continuous functions on $\mathbb{R}$, thus a subspace of $L^\infty(\mathbb{R})$.

Proof. The Heaviside function $H \in L^\infty(\mathbb{R})$ is not continuous, i.e., $H \notin C(\mathbb{R})$. In fact, we claim that H is a positive distance from the subspace $C(\mathbb{R}) \leq L^\infty(\mathbb{R})$.

Let $f \in C(\mathbb{R})$ be given. For each $n \in \mathbb{N}$ we can find a null set $E_n \subseteq \mathbb{R}$ such that

$$\|H - f\|_\infty \leq \sup_{\mathbb{R}\setminus E_n} |H - f| \leq \|H - f\|_\infty + \frac{1}{n}.$$

Then the union $E = \bigcup_{n\in\mathbb{N}} E_n$ is a null set such that for all $n \in \mathbb{N}$,

$$\|H - f\|_\infty \leq \sup_{\mathbb{R}\setminus E} |H - f| \leq \sup_{\mathbb{R}\setminus E_n} |H - f| \leq \|H - f\|_\infty + \frac{1}{n}.$$

Letting $n \to \infty$, we obtain

$$\|H - f\|_\infty = \sup_{\mathbb{R}\setminus E} |H - f|.$$

In other words, $M = \mathbb{R} \setminus E$ is a full measure subset on which the supremum of $|H - f|$ is the same as the essential supremum on all of $\mathbb{R}$. In particular, note that M is dense in $\mathbb{R}$.

For $\varepsilon > 0$ choose $\delta > 0$ for the uniform continuity of f on the closed interval $[-1, 1]$. Since M is dense in $\mathbb{R}$, we can choose $x_0 \in (-\delta/2, 0) \cap M$ and $x_1 \in (0, \delta/2) \cap M$. This choice gives us $H(x_0) = 0$ and $H(x_1) = 1$ in addition to $|x_0 - x_1| < \delta$, so

$$\begin{aligned} 1 = |H(x_0) - H(x_1)| &\leq |H(x_0) - f(x_0)| + |f(x_0) - f(x_1)| + |f(x_1) - H(x_1)| \\ &\leq \sup_M |H - f| + \varepsilon + \sup_M |H - f| \\ &= 2\|H - f\|_\infty + \varepsilon. \end{aligned}$$

Letting $\varepsilon \to 0$, we obtain $\|H - f\|_\infty \geq 1/2$ for all $f \in C(\mathbb{R})$, so

$$d(H, C(\mathbb{R})) = \inf_{f\in C(\mathbb{R})} \|H - f\|_\infty \geq \frac{1}{2}.$$

(In fact $d(H, C(\mathbb{R})) = 1/2$ since the constant function $c(x) = 1/2$ has $\|H - c\|_\infty = 1/2$.)

Importantly, H is an element of $L^\infty(\mathbb{R})$ with $d = d(H, C(\mathbb{R})) > 0$. So by a corollary of the Hahn-Banach theorem, there exists $\Lambda \in L^\infty(\mathbb{R})^*$ such that $\|\Lambda\| \leq 1$, $\Lambda(H) = d$, and $\Lambda|_{C(\mathbb{R})} = 0$. In other words, Λ is of the desired form. $\square$

(b) Prove that there is a bounded linear functional Λ on $L^\infty(\mathbb{R})$, such that $\Lambda(f) = f(0)$ for any $f \in C(\mathbb{R})$.

Proof. Define the function $\lambda : C(\mathbb{R}) \to \mathbb{R}$ by $\lambda(f) = f(0)$. We check that λ is linear:

$$\lambda(\alpha f + g) = (\alpha f + g)(0) = \alpha f(0) + g(0) = \alpha\lambda(f) + \lambda(g).$$

To see that λ is bounded, let $f \in C(\mathbb{R})$ be nonzero. Similar to the proof of problem A, we can choose a full measure subset $M \subseteq \mathbb{R}$ for which

$$\|f\|_\infty = \sup_M |f|.$$

Given $\varepsilon > 0$ choose $\delta > 0$ for the continuity of f at 0. Since M is dense in $\mathbb{R}$, we can find a point $x \in (-\delta, \delta) \cap M$. Then

$$0 \le |f(0)| \le |f(x)| + |f(x) - f(0)| \le \sup_M |f| + \varepsilon = \|f\|_\infty + \varepsilon.$$

Letting $\varepsilon \to 0$, we obtain $0 \le |f(0)| \le \|f\|_\infty$ for all nonzero $f \in C(\mathbb{R})$, so

$$\|\lambda\| = \sup_{\substack{f \in C(\mathbb{R}) \\ f \neq 0}} \frac{|\lambda(f)|}{\|f\|_\infty} \le 1.$$

In particular, λ is bounded, so indeed $\lambda \in C(\mathbb{R})^*$. By a corollary of the Hahn-Banach theorem, there exists $\Lambda \in L^\infty(\mathbb{R})^*$ such that $\Lambda|_{C(\mathbb{R})} = \lambda$ and $\|\Lambda\| = \|\lambda\|$. □

Then use this to show that $(L^\infty(\mathbb{R}))^* \neq L^1(\mathbb{R})$.

We claim that Λ does not correspond to an element of $L^1(\mathbb{R})$.

Proof. Suppose there did exist $\delta \in L^1(\mathbb{R})$ such that $\Lambda(f) = \int_{\mathbb{R}} f\delta \, \mathrm{d}x$ for all $f \in L^\infty(\mathbb{R})$. For $\varepsilon > 0$ let $f : \mathbb{R} \to [0, 1]$ be any choice of continuous bump function with $f(0) = 1$ and $f(x) = 0$ for all $|x| > \varepsilon$. Then $f \in C(\mathbb{R})$ so

$$1 = f(0) = \Lambda(f) = \int_{\mathbb{R}} f\delta \, \mathrm{d}x = \int_{-\varepsilon}^{\varepsilon} f\delta \, \mathrm{d}x \le \int_{-\varepsilon}^{\varepsilon} |f||\delta| \, \mathrm{d}x \le \int_{-\varepsilon}^{\varepsilon} |\delta| \, \mathrm{d}x.$$

However, $\delta \in L^1(\mathbb{R})$ means that δ is Lebesgue integrable in the usual sense, so we must have

$$\lim_{\varepsilon \to 0} \int_{-\varepsilon}^{\varepsilon} |\delta| \, \mathrm{d}x = 0.$$

This is a contradiction, so no such δ exists, i.e., $\Lambda \notin L^1(\mathbb{R})$ as a subspace of $L^\infty(\mathbb{R})^*$. □

2 Let X be a Banach space. Give an example of an everywhere-defined but discontinuous linear functional T.

Let X be any infinite-dimensional Banach space and $\beta = \{e_n\}_{n \in \mathbb{N}}$ be a countably infinite set of linearly independent unit vectors in X. (e.g., $X = \ell_1$ and $e_n \in \ell_1$ is the sequence with a 1 in the nth position and 0's elsewhere.)

We extend β to a Hamel basis $\mathcal{B}$ of X and define a linear functional $T : X \to \mathbb{R}$ on the basis by $Te_n = n$ and $Tv = 0$ for all $v \in \mathcal{B} \setminus \beta$. Then T is not bounded since

$$\frac{|Te_n|}{\|e_n\|_X} = \frac{n}{1} = n.$$

With X a Banach space, we conclude that T is not continuous.

Show directly that the graph of T is not closed.

For $n \in \mathbb{N}$, the point $x_n = (e_n/n, 1) \in X \times \mathbb{R}$ is in the graph of T. However, the limit of the sequence $\lim_{n\to\infty} x_n = (0, 1)$ is not in the graph of T since $T0 = 0 \neq 1$. Since a closed set in any topological space is also sequentially closed, the fact that the graph of T is not sequentially closed implies it is not closed.

3 Let X be a Banach space in either of the norms $\|\cdot\|_1$ or $\|\cdot\|_2$. Suppose that $\|\cdot\|_1 \leq C\|\cdot\|_2$ for some constant C. Prove that there is a constant D such that $\|\cdot\|_2 \leq D\|\cdot\|_1$.

Proof. The identity map $I = \mathrm{id}_X : X \to X$ is an isomorphism of vector spaces. Consider the Banach spaces $X_1 = (X, \|\cdot\|_1)$ and $X_2 = (X, \|\cdot\|_2)$, which have the same underlying vector space but different norms. We consider I to be a linear operator $X_2 \to X_1$ and compute its operator norm:

$$\|I\| = \sup_{x \neq 0} \frac{\|Ix\|_1}{\|x\|_2} = \sup_{x \neq 0} \frac{\|x\|_1}{\|x\|_2} \leq \sup_{x \neq 0} \frac{C\|x\|_2}{\|x\|_2} = C.$$

Hence, $I : X_2 \to X_1$ is a bounded linear operator. Since I is also surjective, it must be open, which means that its inverse $I^{-1} : X_1 \to X_2$ is continuous. Since I^{-1} is also linear (it acts as the identity on X), we conclude that I^{-1} is bounded and take $D = \|I^{-1}\|$. $\square$

4

We will first prove two lemmas.

For $T : X \to Y$ a bounded operator of Banach spaces and $\ell \in Y^*$, the adjoint can be written as the composition $T'\ell = \ell \circ T$.

Lemma 1. If $T : X \to Y$ is a linear isometry of Banach spaces, so is $T' : Y^* \to X^*$ (adjoint).

Proof. We have already seen that that T' is a linear map, so it remains to prove T' is an isometry. Let $S = T^{-1} : Y \to X$ be the inverse of T. For any $\ell \in Y^*$, we have

$$(S' \circ T')\ell = S'(T'\ell) = (\ell \circ T) \circ S = \ell \circ (T \circ S) = \ell \circ \mathrm{id}_Y = \ell.$$

Hence, $S' \circ T' = \mathrm{id}_{Y^*}$, and a symmetric argument shows $T' \circ S' = \mathrm{id}_{X^*}$. This proves T' is a bijection. Lastly, we check that T' preserves the norm. For $\ell \in Y^*$ we have

$$\|T'\ell\|_{X^*} = \sup_{x \neq 0} \frac{|T'\ell(x)|}{\|x\|_X} = \sup_{x \neq 0} \frac{|\ell(Tx)|}{\|STx\|_X} = \sup_{y \neq 0} \frac{|\ell(y)|}{\|Sy\|_X} = \sup_{y \neq 0} \frac{|\ell(y)|}{\|y\|_Y} = \|\ell\|_{Y^*}.$$

□

Lemma 2. Let $J : X \hookrightarrow X^{**}$ be the usual embedding of a Banach space into its double dual. Then its image $J(X)$ is a closed subspace of X^{**}.

Proof. We have already seen that $J(X)$ is a subspace of X^{**}, so it remains to prove that $J(X)$ is closed. Since X^{**} is a metric space, a subset being closed in equivalent to it being sequentially closed. Let $\{y_n\}$ be a sequence in $J(X)$ converging to a point $y \in X^{**}$. Since J is an isometric embedding, it gives an isometry $X \to J(X)$, which has an isometry inverse $J^{-1} : J(X) \to X$. Since $\{y_n\}$ is a Cauchy sequence in $J(X)$, its image $\{J^{-1}y_n\}$ is a Cauchy sequence in X. Since X is complete, the image sequence converges to some point $x \in X$. Mapping this sequence back under the isometric embedding J, we deduce that $\{y_n\}$ converges to the point $Jx \in J(X)$. Hence, $J(X)$ is sequentially closed and therefore closed. □

Prove that a Banach space X is reflexive if and only if its dual space X^* is reflexive. (Hint: if $X \neq X^{**}$, find a bounded linear functional on X^{**} which vanishes on X.)

Proof. Assume X is reflexive, i.e., the usual embedding $J : X \to X^{**}$ is a linear isometry. By Lemma 1, the adjoint $J' : X^{***} \to X^*$ is a linear isometry. We claim that it is the inverse to the usual isometric embedding $\tilde{J} : X^* \hookrightarrow X^{***}$.

We next check that $J' \circ \tilde{J} = \mathrm{id}_{X^{***}}$. Indeed, for all $\ell \in X^{***}$ and $x \in X^{**}$, we have

$$(\tilde{J} \circ J')\ell(x) = \tilde{J}(J'\ell)(x) = x(J'\ell) = J(J^{-1}x)(J'\ell) = J'\ell(J^{-1}x) = \ell(J(J^{-1}x)) = \ell(x).$$

From this, we also deduce

$$\tilde{J} \circ J' = \left((J')^{-1} \circ J'\right) \circ \tilde{J} \circ J' = (J')^{-1} \circ \mathrm{id}_{X^{***}} \circ J' = \mathrm{id}_{X^*}\,.$$

Hence, $\tilde{J}$ is a linear isometry with $\tilde{J}^{-1} = J'$. In particular, $\tilde{J}$ is surjective, so X^* is reflexive.

Assume X is not reflexive, i.e., there exists some $y \in X^{**} \setminus J(X)$. Lemma 2 tells us that $J(X)$ is a closed subspace, so $d = d(y, J(X))$ must be positive. Therefore, a corollary of the Hahn-Banach theorem gives $\Lambda \in X^{***}$ such that $\|\Lambda\| = 1$, $\Lambda(y) = d$, and $\Lambda|_{J(X)} = 0$. If we assume in addition that $\tilde{J} : X^* \hookrightarrow X^{***}$ is a linear isometry, then there is some $\ell \in X^*$ such that $\tilde{J}\ell = \Lambda$. Then for all $x \in X$ we have

$$0 = \Lambda(Jx) = \tilde{J}\ell(Jx) = Jx(\ell) = \ell(x).$$

But this implies $\ell = 0 \in X^*$ and the linearity of $\tilde{J}$ implies $\Lambda = \tilde{J}\ell = 0 \in X^{***}$. This is a contradiction since $\Lambda(y) = d \neq 0$, so X^* must not be reflexive. $\square$

5 (a) Prove that a locally convex space has a topology given by a single norm if the topology is generated by finitely many seminorms.

Proof. Let X be a locally convex space and let $\rho_1, \ldots, \rho_n$ be the seminorms which generate the natural topology, i.e., $0 \in X$ has a neighborhood subbasis of sets of the form

$$U_{i,\varepsilon} = \rho_i^{-1}([0, \varepsilon)) = \{x \in X : \rho_i(x) < \varepsilon\},$$

for $i = 1, \ldots, n$ and $\varepsilon > 0$. It follows that $0 \in X$ has a neighborhood basis of sets

$$U_\varepsilon = \bigcap_{i=1}^{n} U_{i,\varepsilon}$$

for $\varepsilon > 0$, since every finite intersection of subbasis sets contains some U_ε. We now define a function $\eta : X \to [0, \infty)$ as the maximum over the seminorms

$$\eta(x) = \max_i \rho_i(x).$$

Note that the family of seminorms $\{\rho_i\}$ must be separating since the natural topology on X is Hausdorff (every nonzero point is in the complement of some U_ε). So

$$\eta(x) = 0 \quad \iff \quad \rho_1(x) = \cdots = \rho_n(x) = 0 \quad \iff \quad x = 0.$$

For any scalar α, we have

$$\eta(\alpha x) = \max_i \rho_i(\alpha x) = \max_i |\alpha| \rho_i(x) = |\alpha| \max_i \rho_i(x) = |\alpha| \eta(x).$$

Lastly,

$$\eta(x + y) = \max_i \rho_i(x + y) \leq \max_i(\rho_i(x) + \rho_i(y)) \leq \max_i \rho_i(x) + \max_i \rho_i(y) = \eta(x) + \eta(y).$$

Hence, η is a norm on X. The metric topology which η generates on X is described by the open balls of the origin, which are of the form

$$B_\varepsilon^\eta(0) = \{x \in X : \eta(x) < \varepsilon\} = U_\varepsilon.$$

In other words, the U_ε's form a neighborhood basis of $0 \in X$ in both the natural topology and the metric topology, so the two topologies are the same. $\square$

(b) Prove that a locally convex space has a topology generated by a single norm if and only if 0 has a bounded neighborhood.

Proof. Let X be a locally convex space.

Assume the natural topology on X is generated by a norm $\|\cdot\|$. We claim that $B_1(0)$ is a bounded neighborhood of the origin. If N is a neighborhood of 0, it must contain some ball of radius $\varepsilon > 0$ around the origin. But then

$$\varepsilon B_1(0) = B_\varepsilon(0) \subseteq N,$$

so $B_1(0) \subseteq (1/\varepsilon)N$. Hence $B_1(0)$ is bounded.

Assume 0 has a bounded neighborhood, say N, and define

$$\eta(x) = \inf\{\alpha > 0 : x \in \alpha N\}.$$

Then the metric topology generated by η is described by the balls

$$B^\eta_\varepsilon(0) = \{x \in X : \eta(x) < \varepsilon\} = \{x \in X : x \in \varepsilon N\} = \varepsilon N.$$

Since the εN's form a neighborhood basis in the natural topology, the two topologies are the same. $\square$