1(a)

Yes. Suppose $(x, y), (u, v) \in M \times M$. Then

$$\begin{aligned}
\varphi((x, y) + (u, v)) &= \varphi(x + u, y + v) \\
&= (x + u) - (y + v) \\
&= x + u - y - v \\
&= x - y + u - v \\
&= \varphi(x, y) + \varphi(u, v).
\end{aligned}$$

And for $r \in R$ and $(x, y) \in M \times M$, we find

$$\begin{aligned}
\varphi(r(x, y)) &= \varphi(rx, ry) \\
&= rx - ry \\
&= r(x - y) \\
&= r\varphi(x, y).
\end{aligned}$$

1(b)

No. Consider $1, x \in \mathbb{Q}[x]$. First, we have

$$\varphi(x1) = \varphi(x) = \frac{\mathrm{d}}{\mathrm{d}x}x = 1.$$

However,

$$x\varphi(1) = x\frac{\mathrm{d}}{\mathrm{d}x}1 = x0 = 0.$$

Thus, $\varphi(x1) \neq x\varphi(1)$, so φ is not a $\mathbb{Q}[x]$-module homomorphism.

2

Yes. It is a field if and only if the ideal is maximal in $\mathbb{Q}[x]$. Since $\mathbb{Q}[x]$ is a UFD, ideals are maximal if and only if prime, and elements are prime if and only if irreducible. The polynomial has integer coefficients with GCD 1, so it is irreducible in $\mathbb{Q}[x]$ if and only if it is irreducible in $\mathbb{Z}[x]$. Since it is monic and 3 divide all but the leading coefficient, but 3^2 does not divide the constant term, then by Eisenstein's criterion, the polynomial is irreducible in $\mathbb{Z}[x]$. Hence, the quotient ring is a field.

3

Here, all integers implicitly represent their equivalence class mod 11. First, we find the characteristic polynomial.

$$\begin{aligned}
c_A(x) &= \det(xI_3 - A) \\
&= \det \begin{bmatrix} x-1 & -2 & 0 \\ -3 & x-4 & -5 \\ -2 & 0 & x+1 \end{bmatrix} \\
&= (x-1)\det \begin{bmatrix} x-4 & -5 \\ 0 & x+1 \end{bmatrix} - (-2)\det \begin{bmatrix} -3 & -5 \\ -2 & x+1 \end{bmatrix} \\
&= (x-1)((x-4)(x+1) - (-5)0) + 2(-3(x+1) - (-5)(-2)) \\
&= (x-1)(x-4)(x+1) + 2(-3x-13) \\
&= (x^2-1)(x-4) - 6x - 26 \\
&= x^3 - 4x^2 - 7x + 11 \\
&= x(x^2 - 4x - 7) \\
&= x(x^2 - 4x + 4) \\
&= x(x-2)^2.
\end{aligned}$$

The possibilities for the minimal polynomial are therefore $x(x-2)^2$ or $x(x-2)$. We check if the latter evaluates to zero at A.

$$A(A - 2I_3) = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 4 & 5 \\ 2 & 0 & -1 \end{bmatrix} \begin{bmatrix} -1 & 2 & 0 \\ 3 & 2 & 5 \\ 2 & 0 & -3 \end{bmatrix} = \begin{bmatrix} 5 & * & * \\ * & * & * \\ * & * & * \end{bmatrix} \neq 0.$$

Therefore, the minimal polynomial is $m_A(x) = x(x-2)^2$.

3(a)

The invariant factor is $x(x-2)^2$.

3(b)

The elementary divisors are x and $(x-2)^2$.

3(c)

The Jordan canonical form is $\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

4(a)

Let $s \in R$ and $x, y \in M$. Then

$$\begin{aligned}
\varphi_r(sx + y) &= r(sx + y) \\
&= rsx + ry \\
&= srx + ry \\
&= s\varphi_r(x) + \varphi_r(y).
\end{aligned}$$

Since $1 \in R$, this proves φ_r is an R-module homomorphism.

4(b)

Let $r, s, t \in R$. We want to show that $f(rs + t) = rf(s) + f(t)$, i.e., that

$$\varphi_{rs+t} = r\varphi_s + \varphi_t.$$

Let $x \in M$, then

$$\begin{aligned}
\varphi_{rs+t}(x) &= (rs + t)(x) \\
&= rsx + tx \\
&= r\varphi_s(x) + \varphi_t(x) \\
&= (r\varphi_s + \varphi_t)(x).
\end{aligned}$$

Hence, f is an R-module homomorphism.

4(c)

Let $x \in M$ such that $M = Rx$ and $\varphi \in \mathrm{Hom}_R(M, M)$. Since M is cyclic, then for some $r \in R$ we have $rx = \varphi(x)$. We claim that $f(r) = \varphi_r = \varphi$. Let $sx \in M$ (arbitrary element since $M = Rx$), then

$$\begin{aligned}
\varphi_r(sx) &= rsx \\
&= srx \\
&= s\varphi(x) \\
&= \varphi(sx).
\end{aligned}$$

Hence, $f(r) = \varphi$, so f is surjective.

5

Let $I \subseteq R$ be an ideal. Since I is a free R-module, then there exists a basis $\{x_1, \ldots, x_n\}$ for I, with $x_1, \ldots, x_n \in R$ nonzero. Suppose for contradiction that $n > 1$, so $x_1, x_2 \in R$ nonzero. Then we have the R-linear combination of basis elements

$$(x_2)x_1 + (-x_1)x_2 + 0x_3 + \cdots + 0x_n = x_1x_2 - x_1x_2 = 0.$$

This implies that all the coefficients are zero, so $x_2 = -x_1 = 0$. This is a contradiction, since all basis elements are assumed nonzero. Therefore, $\{x_1\}$ is a basis for I, meaning that $I = Rx_1 = (x_1)$. Hence, all ideals of R are principal, so R is a PID.

6(a)

Since φ is an R-module homomorphism, its image $\varphi(M) \subseteq M$ is an R-submodule of M. Since M is irreducible, this implies that $\varphi(M) = 0$ or $\varphi(M) = M$. Since φ is nonzero, then we must have $\varphi(M) = M$, i.e., φ is surjective. Now, $\ker\varphi \subseteq M$ is also an R-submodule of M, so we must have $\ker\varphi = 0$ or $\ker\varphi = M$. Since φ is nonzero, then we cannot have $\ker\varphi = M$, as that would imply $\varphi(M) = 0$. Therefore, $\ker\varphi = 0$, so φ is injective. Thus, φ is a bijective R-module homomorphism, so it is an R-module isomorphism.

6(b)

Since V is irreducible it has only itself and 0 as $\mathbb{C}[x]$-submodules. The $\mathbb{C}[x]$-submodules correspond bijectively to the T_A-stable subspaces. That is, the only T_a-stable subspaces are 0 and V, so must have $A = 0$? If $A = \lambda I_n$ with λ nonzero, then the span of any basis vector would be T_A-stable, so maybe I'm missing something.