1.1.A. A category in which each morphism is an isomorphism is called a **groupoid**.

(a) A perverse definition of a **group** is: a groupoid with one object. Make sense of this.

Let $\mathcal{G}$ be a (locally-small) category with one object. Call the object $\bullet$ and assume all morphisms are isomorphisms, i.e.,

$$\mathrm{Mor}_{\mathcal{G}}(\bullet, \bullet) = \mathrm{Mor}_{\mathcal{G}}(\bullet) = \mathrm{Aut}_{\mathcal{G}}(\bullet) = \mathrm{Iso}_{\mathcal{G}}(\bullet).$$

We claim there is a group G whose underlying set is $\underline{G} = \mathrm{Mor}_G(\bullet)$ and whose operation is given by composition of morphisms. For any two elements $f, g \in G$, put $f \cdot g = f \circ g$, where composition is performed in $\mathcal{G}$. The distinguished identity element of G will be the identity morphism of $\bullet$; put $1_G = \mathrm{id}_\bullet$.

To see that this element satisfied the identity axioms of a group, we appeal to the axioms of a category. The associativity of the operation in G follows from the associativity of composition in $\mathcal{G}$:

$$(f \cdot g) \cdot h = (f \circ g) \circ h = f \circ (g \circ h) = f \cdot (g \cdot h).$$

The properties of 1_G in G "are the same as" the properties of $\mathrm{id}_\bullet$ in $\mathcal{G}$:

$$f \cdot 1_G = f \circ \mathrm{id}_\bullet = f \quad \text{and} \quad 1_G \cdot f = \mathrm{id}_\bullet \circ f = f.$$

The existence of inverses in G is guaranteed by the fact that all morphisms in $\mathcal{G}$ are isomorphisms: if $f \in \mathrm{Mor}_{\mathcal{G}}(\bullet)$ then there exists $f^{-1} \in \mathrm{Mor}_{\mathcal{G}}(\bullet)$ giving us

$$f \cdot f^{-1} = f \circ f^{-1} = \mathrm{id}_\bullet = 1_G \quad \text{and} \quad f^{-1} \cdot f = f^{-1} \circ f = \mathrm{id}_\bullet = 1_G.$$

Hence, G is a group.

This construction may be followed backwards from a group to obtain a category with one object, where all morphisms are isomorphisms.

(b) Describe a groupoid that is not a group.

We give the data of a category $\mathcal{N}$ as follows:

1. a single object $\bullet$,
2. a set of morphisms $\mathrm{Mor}_{\mathcal{N}}(\bullet) = \{\mathrm{id}_\bullet, s, s^2, \ldots, s^n, \ldots\}$, for which $s^n = s \circ \cdots \circ s$ is the n-times composition of s with itself, and all such compositions are distinct.

Putting $s^0 = \mathrm{id}_\bullet$ and $s^1 = s$ suggests an obvious bijection between the set of morphisms $\mathrm{Mor}_{\mathcal{N}}(\bullet)$ and the set of natural numbers $\mathbb{N} = \{0, 1, 2, \ldots, \}$.

Indeed, this construction of $\mathcal{N}$ mirrors the inductive construction of the natural numbers, in which the morphism s is the successor function.

Indeed, this is a monoid, where composition is the addition of natural numbers.

1.1.C, 1.1.D omitted

Did in MATH 237A Homework 7.

Didn't check naturality for D?

1.2.A. Show that any two initial objects are uniquely isomorphic. Show that any two final objects are uniquely isomorphic.

Let I and I' be two initial objects.

Let $f : I \to I'$ be the unique map from I to I', which exists since I is initial.

Let $g : I' \to I$ be the unique map from I' to I, which exists since I' is initial.

Then $g \circ f : I \to I$ is an endomorphism of I. But id_I is also an endomorphism of I, so by the universal property of the initial object I, we have $g \circ f = \mathrm{id}_I$.

By an analogous argument, $f \circ g = \mathrm{id}_{I'}$.

Hence, f and g are inverse morphisms, so I and I' are isomorphic.

The uniqueness of this isomorphism follows from the universal properties of the initial objects.

By duality, a final object in a category is an initial object in the opposite category. Initial objects in (opposite) categories are uniquely isomorphic, so final objects in the original category are uniquely isomorphic.

1.2.B omitted

Did in MATH 237A Homework 7.

1.2.C. Show that $\iota : A \to S^{-1}A$ ($a \mapsto a/1$) is injective iff S contains no zero divisors.

In a stupid predicate logic way for fun.

$$\begin{aligned}
\iota \text{ is injective} &\iff \ker \iota = 0 && \text{classic algebra result} \\
&\iff \forall a \in A \text{ nonzero}, \iota(a) \neq 0 && \text{def of kernel} \\
&\iff \forall a \in A \text{ nonzero}, a/1 \neq 0 && \text{def of } \iota
\end{aligned}$$

Then

$$\begin{aligned}
a/1 = 0 &\iff \exists u \in S \text{ s.t. } ua = 0 && \text{def of `=' in } S^{-1}A \\
&\iff \neg \forall u \in S, ua \neq 0 && \text{quantifier duality}
\end{aligned}$$

Then

$$\begin{aligned}
\iota \text{ is injective} &\iff \forall a \in A \text{ nonzero}, \forall u \in S, ua \neq 0 && \text{substitution} \\
&\iff \forall u \in S, \forall a \in A \text{ nonzero}, ua \neq 0 && \forall\text{-commutativity} \\
&\iff \forall u \in S, \neg\exists a \in A \text{ nonzero s.t. } ua = 0 && \text{quantifier duality} \\
&\iff \forall u \in S, u \text{ is not a zero divisor} && \text{def of zero divisor} \\
&\iff \neg\exists u \in S \text{ s.t. } u \text{ is a zero divisor} && \text{quantifier duality} \\
&\iff S \text{ contains no zero divisors} && \text{rewriting}
\end{aligned}$$

1.2.D. Verify that $A \to S^{-1}A$ satisfies the following universal property: $S^{-1}A$ is initial among A-algebras B where every element of S is sent to an invertible element in B.

Define the map $f : S^{-1}A \to B$ by $a/s \mapsto s^{-1}a$ where $s^{-1} \in B$ is the inverse of $s \in S$.

Check well-definedness: show $a/s = b/t$ implies $s^{-1}a = t^{-1}b$. (falls out of equality in $S^{-1}A$).

Check that f is an A-Homomorphism. Show

- $f(a/s + b/t) = f(a/s) + f(b/t)$
- $f((a/s)(b/t)) = f(a/s)f(b/t)$
- $f(a(b/s)) = af(b/s)$

For another A-Hom $g : S^{-1}A \to B$

Check uniqueness: show that if $g : S^{-1}A \to B$ is an A-Hom which sends elements of S to invertible elements of B, then $g = f$. Use def of A-Hom and def of f.

1.2.I. Show that the tensor product $(T, t : M \times N \to T)$ defined by the universal property is unique up to unique isomorphism.

Consider the category $\mathcal{C}$ consisting of

- objects: pairs (T, t) where $t : M \times N \to T$ is A-bilinear,
- morphisms: a morphism $(T, t) \to (T, t')$ consists of an A-linear map $f : T \to T'$ such that the following diagram commutes:

$$\begin{tikzcd}
& M \times N \arrow[dl, "t"'] \arrow[dr, "t'"] & \\
T \arrow[rr, "f"] & & T'
\end{tikzcd}$$

Moreover, the identity morphism on (T, t) is the identity map on T, and composition of morphisms is composition of the linear maps.

If (T, t) and (T', t') are two tensor products of M and N, then there are morphisms $f : T \to T'$ and $g : T' \to T$, which both commute with t and t'. Then their compositions must be the endomorphisms of T and T' which make the relevant diagrams commute. But the universal property requires that these be the identity morphisms on T and T', respectively. Hence, (T, t) and (T', t') are isomorphic in $\mathcal{C}$. Additionally, the universal property of the tensor product guarantees that this isomorphism is unique.

1.2.J. no

Tensor Product Construction Let A be a commutative ring and let M and N be A-modules. For elements $m \in M$ and $n \in N$, define the **pure tensor** *of* m *and* n to be the symbol

$$m \otimes n.$$

We now consider the free A-module F generated by the set of all pure tensors:

$$F := \bigoplus_{\substack{m \in M \\ n \in N}} A\langle m \otimes n \rangle \cong A^{(M \times N)},$$

where each $A\langle m \otimes n \rangle$ is a copy of A with basis element $m \otimes n$. Now define the submodule K of F, generated by the following elements:

$$K := \left\langle \begin{array}{l} (m_1 + m_2) \otimes n - m_1 \otimes n - m_2 \otimes n, \\ m \otimes (n_1 + n_2) - m \otimes n_1 - m \otimes n_2, \\ a(m \otimes n) - am \otimes n, \\ a(m \otimes n) - m \otimes an \end{array} \right\rangle.$$

Now we define the **tensor product** of M and N over A to be the quotient module

$$M \otimes_A N := F/K.$$

In other words, $M \otimes_A N$ is the module whose elements are finite A-linear combinations of pure tensors, subject to the relations generating K.

1.2.K. (a) If M is an A-module and $A \to B$ is a morphism of rings, give $B \otimes_A M$ the structure of a B-module (this is part of the exercise). Show that this describes a functor $\mathsf{Mod}_A \to \mathsf{Mod}_B$.

With $A \to B$ a morphism of rings, we consider B as an A-algebra and, in particular, as an A-module.

Fix a scalar $\beta \in B$. We now define the map

$$\begin{aligned} g_\beta : B \times M &\longrightarrow B \otimes_A M, \\ (b, m) &\longmapsto (\beta b) \otimes m. \end{aligned}$$

We claim that this map is A-bilinear, i.e., it is A-linear in each variable.

$$\begin{aligned} g_\beta(ab_1 + b_2, m) &= (\beta(ab_1 + b_2)) \otimes m \\ &= (\beta ab_1 + \beta b_2) \otimes m \\ &= (\beta ab_1) \otimes m + (\beta b_2) \otimes m \\ &= a(\beta b_1 \otimes m) + (\beta b_2 \otimes m) \\ &= ag_\beta(b_1, m) + g_\beta(b_2, m) \end{aligned}$$

$$\begin{aligned} g_\beta(b, am_1 + m_2) &= (\beta b) \otimes (am_1 + m_2) \\ &= (\beta b) \otimes am_1 + (\beta b) \otimes m_2 \\ &= a((\beta b) \otimes m_1) + (\beta b) \otimes m_2 \\ &= ag_\beta(b, m_1) + g_\beta(b, m_2) \end{aligned}$$

By the universal property of the tensor product, there is an A-linear map

$$\begin{aligned} \beta\cdot : B \otimes_A M &\longrightarrow B \otimes_A M, \\ b \otimes m &\longmapsto (\beta b) \otimes m. \end{aligned}$$

We call this the B-scalar multiplication on $B \otimes_A M$. We check that this indeed satisfies the requirements of a B-module structure:

(i) Because $\beta\cdot$ is A-linear on simple tensors, we get this for free:

$$\beta \cdot (b_1 \otimes m_1 + b_2 \otimes m_2) = \beta \cdot (b_1 \otimes m_1) + \beta \cdot (b_2 \otimes m_2).$$

(ii) Using the rules of tensor products, we have

$$\begin{aligned} (\beta_1 + \beta_2) \cdot (b \otimes m) &= ((\beta_1 + \beta_2)b) \otimes m \\ &= (\beta_1 b + \beta_2 b) \otimes m \\ &= (\beta_1 b) \otimes m + (\beta_2 b) \otimes m \\ &= \beta_1 \cdot (b \otimes m) + \beta_2 \cdot (b \otimes m). \end{aligned}$$

(iii) This property comes down to the associativity of multiplication in B:

$$\begin{aligned} (\beta_1\beta_2) \cdot (b \otimes m) &= ((\beta_1\beta_2)b) \otimes m \\ &= (\beta_1(\beta_2 b)) \otimes m \\ &= \beta_1 \cdot ((\beta_2 b) \otimes m) \\ &= \beta_1 \cdot (\beta_2 \cdot (b \otimes m)). \end{aligned}$$

(iv) Lastly,

$$1_B \cdot (b \otimes m) = (1_B b) \otimes m = b \otimes m.$$

Thus, we have indeed found a B-module structure on $B \otimes_A M$.

(In the case of noncommutative rings, this is a left B-module structure. We do not have a notion of scaling elements of M by elements of B, except those coming from $A \to B$. In other words, we have $\alpha b \otimes m = b \otimes \alpha m$, but $\beta b \otimes m \neq b \otimes \beta m$ (what would βm even mean?). We can think of the B in $B \otimes_A M$ as holding all the B-scalar data that cannot be directly applied to M. It is "the most general" way to extend the A-module structure of M to a B-module structure, while respecting the A-module structure of B.)

We now have a rule between the categories

$$\mathsf{Mod}_A \xrightarrow{B \otimes_A -} \mathsf{Mod}_B$$

$$M \longmapsto B \otimes_A M$$

We claim that this is a functor.

To define $B \otimes_A -$ on morphisms, consider $f : M \to N$ an A-homomorphism and the A-bilinear maps characterizing the tensor products in following diagram:

$$\begin{tikzcd}
B \times M \arrow[r, "\mathrm{id}_B \times f"] \arrow[d] & B \times N \arrow[d] \\
B \otimes_A M & B \otimes_A N
\end{tikzcd}$$

Composing the top and right arrows gives us an A-bilinear map as follows:

$$\begin{tikzcd}
B \times M \arrow[r, "\mathrm{id}_B \times f"] \arrow[d] \arrow[dr, dashed] & B \times N \arrow[d] \\
B \otimes_A M & B \otimes_A N
\end{tikzcd}$$

Now, the universal property of the tensor product gives us a unique A-linear map which makes the following diagram commute:

$$\begin{tikzcd}
B \times M \arrow[r, "\mathrm{id}_B \times f"] \arrow[d] & B \times N \arrow[d] \\
B \otimes_A M \arrow[r, dashed, "\exists!"'] & B \otimes_A N
\end{tikzcd}$$

This map is what we will take as $B \otimes_A f$. On pure tensors, this is the map $b \otimes m \mapsto b \otimes f(m)$. However, as we want this to be a morphism in Mod_B, we need to check that it is B-linear. We get additivity for free as we already know the map to be A-linear, so we need only check that it commutes with arbitrary scalars in B. Let $\beta \in B$, then

$$\begin{aligned}(B \otimes_A f)(\beta(b \otimes m)) &= (B \otimes_A f)(\beta b \otimes m) \\ &= \beta b \otimes f(m) \\ &= \beta(b \otimes f(m)) \\ &= \beta(B \otimes_A f)(b \otimes m).\end{aligned}$$

We now have the data of a functor $B \otimes_A -$ from Mod_A to Mod_B:

- For each A-module M, a B-module $B \otimes_A M$.
- For each A-homomorphism $f : M \to N$, a B-homomorphism $B \otimes_A f$ from $B \otimes_A M$ to $B \otimes_A N$.

We now check the functorial properties. Temporarily denote the functor by $T = B \otimes_A -$.

(i) *Preserves composition*. Let $f : M \to N$ and $g : N \to P$ be A-homomorphisms. We check how $T(g \circ f)$ acts on pure tensors:

$$\begin{aligned}(T(g \circ f))(b \otimes m) &= b \otimes (g(f(m))) \\ &= (Tg)(b \otimes f(m)) \\ &= (Tg)((Tf)(b \otimes m)) \\ &= (Tg \circ Tf)(b \otimes m).\end{aligned}$$

So in fact, $T(g \circ f) = Tg \circ Tf$.

(ii) *Preserves identities*. Let M be an A-module, then

$$(T\mathrm{id}_M)(b \otimes m) = b \otimes \mathrm{id}_M(m) = b \otimes m = \mathrm{id}_{TM}(b \otimes m).$$

Hence, $T = B \otimes_A -$ is a functor from Mod_A to Mod_B.

1.2.K. (b) If further $A \to C$ is another morphism of rings, show that $B \otimes_A C$ has a natural structure of a ring. Hint: multiplication will be given by $(b_1 \otimes c_1)(b_2 \otimes c_2) = (b_1 b_2) \otimes (c_1 c_2)$.

Sketch:

map $(b_1, c_1, b_2, c_2) \mapsto b_1 b_2 \otimes c_1 c_2$ is multilinear so should factor good. See this by looking at diagram

$$\begin{array}{c} B \times C \times B \times C \\ \big\downarrow \text{transposition} \\ B \times B \times C \times C \\ \big\downarrow m_B \times m_C \\ B \times C \\ \big\downarrow \tau \\ B \otimes_A C \end{array}$$

Then by universal property has to factor thru multi tensor product

$$B \times C \times B \times C \longrightarrow B \otimes_A C \otimes_A B \otimes_A C$$

Which should factor thru

$$B \times C \times B \times C \longrightarrow (B \otimes_A C) \times (B \otimes_A C)$$

Whole thing can prolly just be shown elementwise easy.

1.2.R. Given morphisms $X_1 \to Y$, $X_2 \to Y$, and $Y \to Z$, show that there is a natural morphism $X_1 \times_Y X_2 \to X_1 \times_Z X_2$, assuming that both the fibered products exist.

We consider the fiber product $X_1 \times_Z X_2$ to relative to the morphisms obtained by composition, i.e.,

$$X_1 \to Y \to Z \quad \text{and} \quad X_2 \to Y \to Z.$$

The universal property of this fiber product lets us fill in the following commutative diagram:

$$\begin{tikzcd}
X_1 \times_Y X_2 \arrow[dr, dashed, "\exists!"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& X_1 \times_Z X_2 \arrow[r] \arrow[d] & X_2 \arrow[d] & \\
& X_1 \arrow[r] & Y \arrow[dr] & \\
& & & Z
\end{tikzcd}$$

1.2.S. The Diagonal Base Change Diagram. Suppose we are given morphism $X_1, X_2 \to Y$ and $Y \to Z$. Show that the following diagram is a Cartesian square.

$$\begin{tikzcd}
X_1 \times_Y X_2 \arrow[r] \arrow[d] & X_1 \times_Z X_2 \arrow[d] \\
Y \arrow[r] & Y \times_Z Y
\end{tikzcd}$$

Give the following names to relevant maps:

$$\begin{tikzcd}
X_1 \times_Y X_2 \arrow[dr, dashed, "\exists! f"] \arrow[drr, bend left, "p_2"] \arrow[ddr, bend right, "p_1"'] & & \\
& X_1 \times_Z X_2 \arrow[r, "q_2"'] \arrow[d, "q_1"] & X_2 \arrow[d, "\beta"] & \\
& X_1 \arrow[r, "\alpha"'] & Y \arrow[dr, "\gamma"] & \\
& & & Z
\end{tikzcd}$$

By the universal property of the fiber product $Y \times_Z Y$, we have the following commutative diagram:

$$
\begin{tikzcd}
Y \arrow[dr, dashed, "\exists! i"] \arrow[drr, bend left, "1_Y"] \arrow[ddr, bend right, "1_Y"'] & & \\
& Y \times_Z Y \arrow[r, "r_2"'] \arrow[d, "r_1"] & Y \arrow[d, "\gamma"] \\
& Y \arrow[r, "\gamma"'] & Z
\end{tikzcd}
$$