1 Exercise 0.2 Construct an explicit deformation retraction of $\mathbb{R}^n \setminus \{0\}$ onto S^{n-1}.

For $X = \mathbb{R}^n \setminus \{0\}$, define $F : X \times I \to X$ by

$$F(x,t) = (1-t)x + t\frac{x}{\|x\|}.$$

As the composition of continuous real functions, F is continuous, and therefore defines a homotopy between $F_0(x) = x$ (identity) and $F_1(x) = x/\|x\|$ (mapping x to its unit). Since $S^{n-1} \subseteq X$ is simply the set of unit vectors, we know that $x \in S^{n-1}$ if and only if $x = x/\|x\|$. So indeed $F_t|_{S^{n-1}} = \mathrm{id}_{S^{n-1}}$ and $F_1(X) \subseteq S^{n-1}$, hence F describes a deformation retraction of X onto S^{n-1}.

2 Exercise 0.3(c) Show that a map homotopic to a homotopy equivalence is a homotopy equivalence.

Proof. Suppose $F : X \times I \to Y$ is a homotopy $f_0 \simeq f_1$ such that $f_1 : X \to Y$ is a homotopy equivalence. Let $g : Y \to X$ be a homotopy inverse to f_1, i.e., $g \circ f_1 \simeq \mathrm{id}_X$ and $f_1 \circ g \simeq \mathrm{id}_Y$.

In other words, we have the following diagrams (not commuting, per se), where homotopies are represented by arrows '$\Rightarrow$' between maps:

$$\begin{tikzcd}
X \arrow[r, bend left=50, "f_0"] \arrow[r, phantom, shift left=2ex, "\Downarrow F"] \arrow[r, "f_1"'] \arrow[rr, bend right=40, "\mathrm{id}_X"'] \arrow[rr, phantom, bend right=15, "\Downarrow"] & Y \arrow[r, "g"] & X
\end{tikzcd}
\qquad
\begin{tikzcd}
Y \arrow[r, "g"] \arrow[rr, bend right=40, "\mathrm{id}_Y"'] \arrow[rr, phantom, bend right=15, "\Downarrow"] & X \arrow[r, bend left=50, "f_0"] \arrow[r, phantom, shift left=2ex, "\Downarrow F"] \arrow[r, "f_1"'] & Y
\end{tikzcd}$$

Intuitively, we will horizontally compose F with an "identity homotopy" on g to obtain homotopies from $g \circ f_0$ to $g \circ f_1$ and from $f_0 \circ g$ to $f_1 \circ g$.

With F and g continuous, their composition $gF := g \circ F : X \times I \to X$ is continuous and therefore describes a homotopy between $(gF)_0 = g \circ f_0$ and $(gF)_1 = g \circ f_1$. And since homotopy is an equivalence relation, we obtain

$$g \circ f_0 \simeq g \circ f_1 \simeq \mathrm{id}_X \,.$$

Similarly, with g and id_I continuous, their product $g \times \mathrm{id}_I : Y \times I \to X \times I$ is continuous. Subsequently, the composition $Fg := F \circ (g \times \mathrm{id}_I) : Y \times I \to Y$ is continuous and therefore describes a homotopy between $(Fg)_0 = f_0 \circ g$ and $(Fg)_1 = f_1 \circ g$, hence

$$f_0 \circ g \simeq f_1 \circ g \simeq \mathrm{id}_Y \,.$$

We conclude that g is a homotopy inverse for f_0, so in fact $f_0 : X \to Y$ is a homotopy equivalence. $\square$

3 Exercise 0.4

We first make explicit the notion of a "corestriction" in the context of topology.

If $f : X \to Y$ is a map and A is a subspace of X, then the restriction $f|_A : A \to Y$ is simply the composition of the inclusion $A \hookrightarrow X$ and f. In other words, $f|_A$ the unique map which makes the following diagram commute:

$$\begin{tikzcd} X \arrow[r, "f"] & Y \\ A \arrow[u, hook] \arrow[ur, dashed, "f|_A"'] & \end{tikzcd}$$

Similarly, if $f : X \to Y$ is a map and B is a subspace of Y such that $f(X) \subseteq B$, then the *corestriction* $f|^B$ is the unique map which makes the following diagram commute:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, dashed, "f|^B"'] & Y \\ & B \arrow[u, hook] \end{tikzcd}$$

Given a map $f : X \to Y$ and subspaces $A \hookrightarrow X$ and $B \hookrightarrow Y$ such that $f(A) \subseteq B$, we write $f|_A^B$ for the twice-restricted map $(f|_A)|^B : A \to B$. That is, $f|_A^B$ is the unique map which makes the following diagram commute:

$$\begin{tikzcd} X \arrow[r, "f"] & Y \\ A \arrow[u, hook] \arrow[ur] \arrow[r, dashed, "f|_A^B"'] & B \arrow[u, hook] \end{tikzcd}$$

A **deformation retraction in the weak sense** of a space X to a subspace A is a homotopy $f_t : X \to X$ such that $f_0 = \mathrm{id}_X$, $f_1(X) \subseteq A$, and $f_t(A) \subseteq A$ for all t. Show that if X deformation retracts to A in this weak sense, then the inclusion $A \hookrightarrow X$ is a homotopy equivalence.

Proof. Let $F : X \times I \to X$ be a deformation retraction of X to A in the weak sense, i.e., $f_0 = \mathrm{id}_X$, $f_1(X) \subseteq A$, and $f_t(A) \subseteq A$. Denote the inclusion $\iota : A \hookrightarrow X$ and the corestriction $g = f_1|^A : X \to A$; we claim that these maps are homotopy inverses of each other.

By assumption, F gives a homotopy $\mathrm{id}_X \simeq f_1$. And the universal property of the corestriction gives us $f_1 = \iota \circ f_1|^A = \iota \circ g$, hence $\mathrm{id}_X \simeq \iota \circ g$. Since $F(A \times I) \subseteq A$, we may define the restricted map $G = F|_{A\times I}^A : A \times I \to A$. Then

$$g_0 = f_0|_A^A = \mathrm{id}_X|_A^A = \mathrm{id}_A$$

and

$$g_1 = f_1|_A^A = g|_A = g \circ \iota,$$

hence G describes a homotopy $\mathrm{id}_A \simeq g \circ \iota$, i.e., $\iota : A \hookrightarrow X$ is a homotopy equivalence. □

4 Exercise 0.5 Show that if a space X deformation retracts to a point $x \in X$, then for each neighborhood U of x in X there exists a neighborhood $V \subseteq U$ of x such that the inclusion $V \hookrightarrow U$ is nullhomotopic.

Proof. Let $F : X \times I \to X$ be a deformation retraction of X to $x \in X$, i.e., $f_0 = \mathrm{id}_X$, $f_1(X) \subseteq \{x\}$, and $f_t(x) = x$. Let $U \subseteq X$ be an open neighborhood of x. As F is continuous, the preimage $F^{-1}(U) \subseteq X \times I$ is an open set containing $\{x\} \times I$. For each $t \in I$, we can choose some open sets $V_t \subseteq X$ and $J_t \subseteq I$ such that

$$(x, t) \subseteq V_t \times J_t \subseteq F^{-1}(U).$$

(This is possible since $X \times I$ has the finite product (box) topology and $F^{-1}(U)$ is a open neighborhood of (x, t).) Then $\{J_t\}_{t \in I}$ is an open cover of the compact space I, so we may choose a finite subcover $\{J_{t_1}, \ldots, J_{t_n}\}$. Define $V = V_{t_1} \cap \cdots \cap V_{t_n}$, which is an open neighborhood of x in X. Moreover, we have

$$V \times I = \bigcup_{i=1}^{n} (V \times J_{t_i}) \subseteq \bigcup_{i=1}^{n} (V_{t_i} \times J_{t_i}) \subseteq F^{-1}(U).$$

Therefore, $F(V \times I) \subseteq U$; in particular, $V = f_0(V) \subseteq U$.

We may now define the restricted map $G = F|_{V \times I}^{U} : V \times I \to U$ (using the notation defined in the proof of Problem 3 to restrict both the domain and codomain). Denoting the inclusion $\iota : V \hookrightarrow U$, this gives us a homotopy between

$$g_0 = f_0|_V^U = \mathrm{id}_X|_V^U = \iota$$

and g_1. Note that for all $y \in V$, we have

$$g_1(y) = f_1(y) = x.$$

That is, g_1 is a constant map, so indeed ι is nullhomotopic. $\square$

5 Exercise 0.6

(a) Let X be the subspace of $\mathbb{R}^2$ consisting of the horizontal segment $[0,1]\times\{0\}$ together with all the vertical segments $\{r\}\times[0,1-r]$ for r a rational number in $[0,1]$. Show that X deformation retracts to any point in the segment $[0,1]\times\{0\}$, but not to any other point.

Proof. Assume for contradiction that X does deformation retract to a point $(r_0, y_0) \in X$ with $r_0 \in \mathbb{Q}$ and $y_0 \neq 0$. Since $X \subseteq \mathbb{R}^2$ has the subspace topology and the positive upper half plane $H = \{(x,y) \in \mathbb{R}^2 : y > 0\}$ is open in $\mathbb{R}^2$, then we can intersect H with X to obtain the set

$$U = H \cap X = \{(x,y) \in X : y > 0\},$$

which is an open neighborhood of (r_0, y_0) in X. By Problem 4, there is a neighborhood $V \subseteq U$ of (r_0, y_0) such that the inclusion $V \hookrightarrow U$ is nullhomotopic, i.e., there is a homotopy $F : V \times I \to U$ such that f_0 is the inclusion $V \hookrightarrow U$ and f_1 is a constant map.

Suppose $a, b \in V$ are any two points. We can fix an argument of F to obtain continuous maps $\alpha, \beta : I \to U$ where $\alpha(t) = F(a,t)$ and $\beta(t) = F(b,t)$ for each $t \in I$. In other words, α and β are paths in U which start at

$$\alpha(0) = f_0(a) = a \quad \text{and} \quad \beta(0) = f_0(b) = b$$

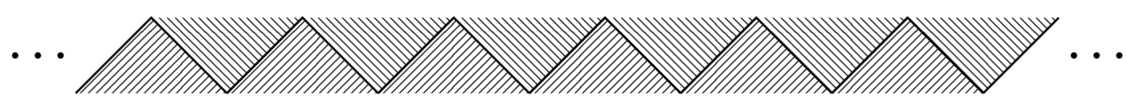
and end at

$$\alpha(1) = f_1(a) = f_1(b) = \beta(1).$$

Following α forwards then β backwards gives us a path from a to b in U. As $a, b \in V$ were arbitrary, this shows that V is path-connected via paths in U.

Note that U is a subspace of $(I \cap \mathbb{Q}) \times I$, so V contains a box $(J \times \mathbb{Q}) \times K$ for some nontrivial open intervals $J, K \subseteq \mathbb{R}$. However, after projecting U onto $\mathbb{Q}$, we obtain a nonempty interval of $\mathbb{Q}$ which is path connected inside of $\mathbb{Q}$. Since $\mathbb{Q}$ is totally path-disconnected, so this is a contradiction. □

(b) Let Y be the subspace of $\mathbb{R}^2$ that is the union of an infinite number of copies of X arranged as in the figure below. Show that Y is contractible but does not deformation retract onto any point.



We build a contraction of Y out of two parts. First, take the weak deformation retraction of X onto Z from part (c). Second, take a deformation retraction of Z onto some point $z \in Z$. For each point in Z, we scale the distance to z along Z by $(1-t)$; this is a continuous map $Z \times I \to Z$ fixing z with the map at $t = 1$ constantly mapping to z. We combine these homotopies by adjusting the time interval of the first to fit within $[0, 1/2]$ and the second to fit within $[1/2, 1]$. Overall, this gives a contraction of Y to the point z.

No deformation retraction of Y to a point exists for the same reason that no deformation retraction of of X exists to a point on an attached interval. That is, such a deformation retraction of Y would induce such a deformation retraction of X.

(c) Let Z be the zigzag subspace of Y homeomorphic to $\mathbb{R}$ indicated by the heavier line. Show there is a deformation retraction in the weak sense of Y onto Z, but no true deformation retraction.

Intuitively, we imagine Z as a piece of string which can be bent, but not stretched or compressed along its length. Each smaller line segment of Y attached to Z is made of the same type of string, with one end tied to Z. We construct a weak deformation retraction of X onto Z where at each time $t \in I$, we pull Z to the right a distance of t units along its length. Any given point in Y moves at a constant speed with respect to t, first moving down its segment onto Z, then following Z to the right for the remainder of the time interval. After Z has been pulled a distance of 1 unit, then the longest segments (which are a length of 1 unit) will have been completely pulled onto Z. Moreover, every point staring in Z remains in Z for the duration of the time interval. Hence, this defines a weak deformation retraction of Y onto Z.

No true deformation retraction of Y onto Z exists for the same reason no true deformation retraction of X onto a point in an attached segment exists. In particular, a true deformation retraction of Y onto Z would induce a true deformation retraction of X onto the lower and left segments.

6 Lecture 3 Given a subspace $A \subseteq X$ with quotient map $q : X \to X/A$ and homotopy $f_t : X \times I \to X$ such that $f_0 = \mathrm{id}_X$ and $f_1(A) = \{a\}$ for some $a \in A$, consider the family of functions $\overline{f}_t$ which make the following diagrams commute:

$$\begin{tikzcd} X \arrow[r, "f_t"] \arrow[d, "q"] & X \arrow[r, "q"] & X/A \\ X/A \arrow[urr, dashed, "\overline{f}_t"'] & & \end{tikzcd}$$

Show that each function $\overline{f}_t : X/A \to X/A$ is continuous.

Proof. Let $V \subseteq X/A$ be an open set and consider the preimage $\overline{f}_t^{-1}(V) \subseteq X/A$. By definition of the quotient topology on X/A, we know that a subset of X/A is open if and only if its preimage under q is open in X. We have the preimage

$$q^{-1}(\overline{f}_t^{-1}(V)) = (\overline{f}_t \circ q)^{-1}(V) = (q \circ f_t)^{-1}(V).$$

We know that $q \circ f_t$ is continuous, so this set is open in X. Hence, $\overline{f}_t^{-1}(V)$ is open in X/A and we conclude that $\overline{f}_t$ is continuous. $\square$