(worked with Joseph Sullivan and Gahl Shemy)

Exercise XVIII.2 Let S_4 be the symmetric group of on 4 elements.

Lemma 1. Given a cycle $(i_1 \cdots i_m) \in S_n$ and $\sigma \in S_n$,

$$\sigma(i_1 \cdots i_m)\sigma^{-1} = (\sigma(i_1)) \cdots \sigma(i_m)).$$

Proof. For an index $i \in \{1, \ldots, n\}$, we have $i = \sigma(j)$ for some other index $j = \sigma^{-1}(i)$. Then

$$(\sigma(i_1 \cdots i_m)\sigma^{-1})(i) = (\sigma(i_1 \cdots i_m)\sigma^{-1})(\sigma(j)) = (\sigma(i_1 \cdots i_m))(j).$$

If $j \neq i_k$ for $k = 1, \ldots, m$, then

$$(\sigma(i_1 \cdots i_m))(j) = \sigma(j) = i.$$

If $j = i_k$ for some $k \in \{1, \ldots, m-1\}$, then

$$(\sigma(i_1 \cdots i_m))(j) = (\sigma(i_1 \cdots i_m))(i_k) = \sigma(i_{k+1}).$$

If $j = i_m$, then

$$(\sigma(i_1 \cdots i_m))(j) = (\sigma(i_1 \cdots i_m))(i_m) = \sigma(i_1).$$

Hence, we can write

$$\sigma(i_1 \cdots i_m)\sigma^{-1} = (\sigma(i_1)) \cdots \sigma(i_m)).$$

□

1 Exercise XVIII.2(a) Show that there are 5 conjugacy classes.

Proof. Every element $\sigma \in S_n$ has a decomposition into disjoint cycles $\sigma = \sigma_1 \cdots \sigma_m$. For any $\tau \in S_n$, we have

$$\tau\sigma\tau^{-1} = (\tau\sigma_1\tau^{-1}) \cdots (\tau\sigma_m\tau^{-1}),$$

where Lemma 1 tells us that $\tau\sigma_k\tau^{-1}$ is a cycle of the same length as σ_k. Then the cycle type of σ is the same as the cycle type of $\tau\sigma\tau^{-1}$.

Given cycles $(i_1 \cdots i_m)$ and $(j_1 \cdots j_m)$ in S_n of the same length, we can choose $\tau \in S_4$ which is the identity on $i \neq i_k$ and $\tau(i_k) = j_k$. Then applying Lemma 1,

$$\tau(i_1 \cdots i_m)\tau^{-1} = (\tau(i_1) \cdots \tau(i_m)) = (j_1 \cdots j_m).$$

In other words, all cycles of the same length are conjugate. Then given elements $\sigma, \tilde{\sigma} \in S_n$ of the same cycle type, we can factor each into products of disjoint cycles

$$\sigma = \sigma_1 \cdots \sigma_m, \quad \tilde{\sigma} = \tilde{\sigma}_1 \cdots \tilde{\sigma}_m,$$

where each σ_k and $\tilde{\sigma}_k$ are cycles of the same length. Moreover, as the cycles are disjoint, we can construct $\tau \in S_n$ such that $\tau\sigma_k\tau^{-1} = \tilde{\sigma}_k$ for all k, then $\tau\sigma\tau^{-1} = \tilde{\sigma}$.

We have shown that elements in S_n are conjugate if and only if they have the same cycle type. In particular, this implies that the number of conjugacy classes in S_n is the same as the number of cycle types, i.e., the number of partitions of n.

There are 5 partitions of 4:

$$1+1+1+1, \quad 1+1+2, \quad 2+2, \quad 1+3, \quad 4.$$

Hence, there are 5 conjugacy classes. □

2 Exercise XVIII.2(b) Show that A_4 has a unique subgroup of order 4, which is not cyclic, and which is normal in S_4. Show that the factor (quotient) group is isomorphic to S_3, so the representations of Exercise 1 give rise to representations of S_4.

Note that A_4 contains all the 3-cycles $(i\,j\,k) = (i\,j)(j\,k)$ where $\{i, j, k\} \subseteq \{1, 2, 3, 4\}$. There are 8 such elements, and each has order 3. The remaining elements of A_4 consist of the identity and the three even elements of order 2. So A_4 has a unique subgroup of order 4:

$$N = \{e, (1\,2)(3\,4), (1\,3)(2\,4), (1\,4)(2\,3)\}.$$

Since N contains the only elements of S_4 with cycle type $2 + 2$, and cycle type is invariant under conjugation, N must be a normal subgroup of S_4.

We now consider the quotient S_4/N. First, we see that

$$|S_4/N| = \frac{|S_4|}{|N|} = \frac{24}{4} = 6.$$

There are two groups of order 6: the cyclic group $\mathbb{Z}/6\mathbb{Z}$ and the symmetric group S_3. Note that $\mathbb{Z}/6\mathbb{Z}$ is generated by some element of order 6. However, every element of S_4 has order at most 4, so no element of S_4/N has order greater than 4. Therefore, S_4/N cannot be cyclic, so in fact $S_4/N \cong S_3$.

3 Extra Problem Show that the symmetric group S_4 has a representation on

$$V = \{(z_1, z_2, z_3, z_4) \mid z_1 + z_2 + z_3 + z_4 = 0\}$$

which permutes the coordinates. What is the dimension of this representation?

Proof. For each $\sigma \in S_4$, define a map on V by

$$\rho(\sigma) : (z_1, z_2, z_3, z_4) \to (z_{\sigma(1)}, z_{\sigma(2)}, z_{\sigma(3)}, z_{\sigma(4)}).$$

It is clear that $\rho(\sigma)$ is a map from V to V, since the sum of the components of $\rho(\sigma)(z)$ is the same as the sum of the components of z for all $z \in V$. We check that $\rho(\sigma) \in \mathrm{GL}(V)$.

First, $\rho(\sigma)$ is linear, since for any $a \in \mathbb{C}$ and $z, w \in V$, we have

$$\rho(\sigma)(az + w) = a\rho(\sigma)(z) + \rho(\sigma)(w).$$

Moreover, $\rho(\sigma)$ is invertible, since for any $z \in V$, we have

$$\rho(\sigma^{-1})(\rho(\sigma)(z)) = z.$$

Hence, we have a map $\rho : S_4 \to \mathrm{GL}(V)$. We check that ρ is a group homomorphism. For $\sigma, \tau \in S_4$ and $z \in V$, we have

$$\rho(\sigma\tau)(z) = (z_{(\sigma\tau)(1)}, \ldots, z_{(\sigma\tau)(4)}) = \rho(\sigma)(z_{\tau(1)}, \ldots, z_{\tau(4)}) = \rho(\sigma)(\rho(\tau)(z)).$$

That is, $\rho(\sigma\tau) = \rho(\sigma) \circ \rho(\tau)$, so ρ is a group homomorphism, hence a representation.

To see that $\dim V = 3$, consider the map $T : \mathbb{C}^4 \to \mathbb{C}$ defined by $z \mapsto z_1 + z_2 + z_3 + z_4$. Notice that T is linear, surjective, and has kernel equal to V. Then we have

$$\dim V = \dim \ker T = \dim \mathbb{C}^4 - \dim \operatorname{im} T = 4 - 1 = 3.$$

$\square$

4 Exercise XVIII.2(e) Let ρ be the representation of [Extra Problem]. Define ρ' by

$$\rho'(\sigma) = \begin{cases} \rho(\sigma) & \text{if } \sigma \text{ is even,} \\ -\rho(\sigma) & \text{if } \sigma \text{ is odd.} \end{cases}$$

Show that ρ' is also irreducible and is non-isomorphic to ρ. This concludes the description of all irreducible representations of S_4.

Proof. We first show that ρ is irreducible.

Suppose $U \leq V$ is a nonzero invariant subspace; we claim that $U = V$. Note that it suffices to show U contains the point $(1, 0, 0, -1)$, since this vector and its images under $\rho((1\,2))$ and $\rho((1\,3))$ span all of V.

Choose any nonzero point $z = (z_1, z_2, z_3, z_4) \in U$, where we may assume $z_1 = 1$.

If it also happens that $z_2 = 1$, then $z = (1, 1, z_3, -2 - z_3)$ and we have the following element in U:

$$w = z + \rho((2\,3))z = (2, 2, -2, -2).$$

Then U also contains

$$u = \tfrac{1}{4}(w + \rho((2\,3))w) = (1, 0, 0, -1),$$

from which we obtain

$$\rho((1\,2))u = (0, 1, 0, -1) \quad \text{and} \quad \rho((1\,3))u = (0, 0, 1, -1).$$

These elements span V, so we must have $U = V$.

If $z_2 \neq 1$, we consider the following element of U:

$$w = z - z_2\rho((1\,2))z = (1 - z_2^2, 0, (1 - z_2)z_3, (1 - z_2)z_4).$$

$w = 0$, then $z_2 = -1$ and $z_3 = z_4 = 0$, so $z = (1, -1, 0, 0)$. Then U contains

$$\rho((2\,4))z = (1, 0, 0, -1),$$

implying $U = V$. If $w \neq 0$, then either $z_2 = -1$ or $1 - z_2^2 \neq 0$. In the first case, we have

$$\tfrac{1}{2}\rho((1\,3))w = (1, 0, 0, -1),$$

and we are done. In the second case, a rescaling of w gives us $u = (1, 0, w_3, -1 - w_3)$. Then U contains

$$v = \rho((1\,4))(-u - \rho((3\,4))u) = (1, 0, 1, -2)$$

and

$$\tfrac{1}{3}(2v + \rho((3\,4))v) = (1, 0, 0, -1).$$

Hence, $U = V$ in all cases and we conclude that ρ is an irreducible representation.

To show ρ' is also irreducible, we perform the same procedure, but multiplying by -1 as necessary to obtain the desired elements.

Note that ρ and ρ' are isomorphic if and only if there is a change of basis $T \in \mathrm{GL}(V)$ such that $T \circ \rho(\sigma) \circ T^{-1} = \rho'(\sigma)$ for all $\sigma \in S_4$. In particular, this would require that the characters of ρ and ρ' are the same. However

$$\chi_\rho((1\,2)) = \operatorname{tr} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 1,$$

but

$$\chi_{\rho'}((1\,2)) = \operatorname{tr} \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} = -1.$$

So ρ and ρ' are not isomorphic representations. $\square$