1 For a function $f : [a, b] \to \mathbb{R}$ define for every $x \in [a, b)$

$$D^+ f(x) = \limsup_{h \to 0^+} \frac{f(x+h) - f(x)}{h}.$$

Prove that if $f : [a, b] \to \mathbb{R}$ is continuous and $D^+ f(x) \geq 0$ for all $x \in [a, b)$, then $f(b) \geq f(a)$.

Proof. First, suppose $D^+ f(x) > 0$ for all $x \in [a, b)$. Since f is continuous on the compact set $[a, b]$, it attains its supremum $M = \sup_{[a,b]} f$ at some point. Now define $c = \sup f^{-1}(M)$, which is on the interval $[a, b]$. Moreover, we can choose a sequence of points in $[a, b]$ approaching c from the left, and on which f equals M. With f continuous, it follows that $f(c) = M$.

If $c = b$, then $f(b) = M \geq f(a)$ and we are done. Assume for contradiction that $c < b$. With $D^+ f(c) > 0$, there must be some $h > 0$ such that

$$\frac{f(c+h) - f(c)}{h} > 0.$$

But this means $f(c) < f(c+h)$, which contradicts the maximality of $f(c) = M$.

Now suppose $D^+ f(x) \geq 0$ for all $x \in [a, b)$ and assume for contradiction that $f(b) < f(a)$. Then there is some $\varepsilon > 0$ such that $f(b) + (b - a)\varepsilon < f(a)$. Define the function

$$g(x) = f(x) + (x - a)\varepsilon,$$

which has

$$g(b) = f(b) + (b - a)\varepsilon < f(a) = g(a).$$

However,

$$\frac{g(x+h) - g(x)}{h} = \frac{f(x+h) - f(x)}{h} + \varepsilon \geq 0 + \varepsilon = \varepsilon,$$

so letting $h \to 0^+$, we obtain $D^+ g(x) \geq \varepsilon > 0$ for all $x \in [a, b)$. This contradicts the first result, which implies $g(b) \geq g(a)$. $\square$

2 Suppose $f_n : [0,1] \to [0,\infty)$ is a sequence of increasing and right-continuous function. Let

$$f(x) = \sum_{n=1}^{\infty} f_n(x), \quad x \in [0,1],$$

and assume that $f(1)$ is finite. Prove that

$$f'(x) = \sum_{n=1}^{\infty} f_n'(x)$$

for almost every $x \in [0,1]$ (in the sense of the Lebesgue measure).

Proof. For $x, y \in [0,1]$ with $x \le y$, we have $f_n(x) \le f_n(y)$ for all $n \in \mathbb{N}$. This gives us

$$f(x) = \sum_{n=1}^{\infty} f_n(x) \le \sum_{n=1}^{\infty} f_n(y) = f(y),$$

which means f is increasing, and therefore differentiable λ-a.e. in $[0,1]$.

For $k \in \mathbb{N}$, denote the partial sum $s_k = \sum_{n=1}^{k} f_n$ and remainder $r_k = \sum_{n=k+1}^{\infty} f_n$, which are both increasing and, therefore, differentiable λ-a.e. in $[0,1]$. In particular, if $x \in [0,1]$ is a point where the derivatives exist, we have

$$f'(x) = s_k'(x) + r_k'(x) = \sum_{n=1}^{k} f_n'(x) + r_k'(x).$$

Moreover, the derivatives of increasing functions are nonnegative, so

$$f'(x) \ge s_k'(x) = \sum_{n=1}^{k} f_n'(x).$$

Letting $k \to \infty$, we obtain

$$f'(x) \ge \sum_{n=1}^{\infty} f_n'(x) \qquad \text{for } \lambda\text{-a.e. } x \in [0,1].$$

We estimate

$$\int_0^1 f' \,\mathrm{d}x = \int_0^1 s_k' \,\mathrm{d}x + \int_0^1 r_k' \,\mathrm{d}x \le \int_0^1 \sum_{n=1}^{\infty} f_n' \,\mathrm{d}x + r_k(1) - r_k(0).$$

Since $\sum_{n=1}^{\infty} f_n'(x) \le f'(x)$ for λ-a.e. $x \in [0,1]$, we have

$$\int_0^1 \sum_{n=1}^{\infty} f_n' \,\mathrm{d}x \le \int_0^1 f' \,\mathrm{d}x.$$

These integrals are finite, so

$$0 \le \int_0^1 f' \, \mathrm{d}x - \int_0^1 \sum_{n=1}^{\infty} f_n' \, \mathrm{d}x = \int_0^1 \left(f' - \sum_{n=1}^{\infty} f_n' \right) \mathrm{d}x \le r_k(1) - r_k(0).$$

On one hand, we have

$$f(1) - f(0) = \lim_{k \to \infty} (s_k(1) - s_k(0)).$$

On the other hand, $f(0) \le f(1) < \infty$, so we can write

$$r_k(1) - r_k(0) = (f(1) - f(0)) - (s_k(1) - s_k(0)).$$

Taking the limit as $k \to \infty$, we obtain

$$\lim_{k \to \infty} (r_k(1) - s_k(0)) = 0.$$

Letting $k \to \infty$ in the above inequality, we deduce

$$\int_0^1 \left(f' - \sum_{n=1}^{\infty} f_n' \right) \mathrm{d}x = 0.$$

Since the integrand is nonnegative, we in fact must have

$$f'(x) = \sum_{n=1}^{\infty} f_n'(x) \qquad \text{for } \lambda\text{-a.e. } x \in [0, 1].$$

$\square$

3 Find an increasing function $f : [0,1] \to \mathbb{R}$ such that $f'(x) = 0$ a.e. in $[0,1]$, but f is not constant on any open subinterval of $[0,1]$.

For $n \in \mathbb{N}$, define a piecewise constant function $f_n : [0,1] \to \mathbb{R}$ by

$$f_n(x) = \frac{\lfloor 2^n x \rfloor}{2^{2n}}.$$

On each interval $[\frac{m}{2^n}, \frac{m+1}{2^n})$, this map is constantly $\frac{m}{2^{2n}}$. In particular, each f_n is increasing and right-continuous. For $x \in [0,1]$ define

$$f(x) = \sum_{n=1}^{\infty} f_n(x).$$

Then f is an increasing function and we we will check that it is finite:

$$f(1) = \sum_{n=1}^{\infty} f_n(1) = \sum_{n=1}^{\infty} \frac{\lfloor 2^n \rfloor}{2^{2n}} = \sum_{n=1}^{\infty} \frac{1}{2^n} = 1.$$

Then applying Problem 2, we find

$$f'(x) = \sum_{n=1}^{\infty} f_n'(x) = \sum_{n=1}^{\infty} 0 = 0$$

for λ-a.e. $x \in [0,1]$. Moreover, f is not constant on any open subinterval of $[0,1]$, since f_n ensures that points separated by at least a distance of $1/2^n$ will have distinct values (and $1/2^n$ is eventually small enough to affect any given open subinterval).