Exercise 5.23 Use diagonals to prove the following statements:

(a) The intersection of any two affine open subsets of a variety is again an affine open subset.

Proof. Let $U, V \subseteq X$ be affine open subsets of the variety X. We know $U \cap V$ is open in X, therefore an open subprevariety of X. It remains the prove affine. Since U and V are affine varieties, then their product $U \times V$ is again an affine variety. There is an inclusion morphism of prevarieties $\iota : U \times V \hookrightarrow X \times X$ which, in particular, is continuous. By definition, X being a variety means the diagonal Δ_X is closed in $X \times X$. Then the preimage $\iota^{-1}(\Delta_X)$ is closed in $U \times V$. As sets, we have

$$\iota^{-1}(\Delta_X) = (U \times V) \cap \Delta_X = \Delta_{U \cap V}.$$

As a closed subset of the affine variety $U \times V$, we deduce that $\Delta_{U\cap V}$ is an affine variety. There are inclusion morphisms of prevarieties, $U \cap V \hookrightarrow U$ and $U \cap V \hookrightarrow V$, which uniquely define the morphism of prevarieties

$$\begin{aligned} i : U \cap V &\to \Delta_{U\cap V} \\ x &\mapsto (x, x). \end{aligned}$$

Moreover, i is an isomorphism of prevarieties, with inverse given by the restriction to $\Delta_{U\cap V}$ of either the projection $\pi_U : U \times V \to U$ or the projection $\pi_V : U \times V \to V$, i.e.,

$$i^{-1} = \pi_U|_{\Delta_{U\cap V}} = \pi_V|_{\Delta_{U\cap V}}.$$

Hence, $U \cap V \cong \Delta_{U\cap V}$ as prevarieties. And since $\Delta_{U\cap V}$ is an affine variety, then so is $U \cap V$. $\square$

(b) If $X, Y \subseteq \mathbb{A}^n$ are two pure-dimensional affine varieties then every irreducible component of $X \cap Y$ has dimension at least $\dim X + \dim Y - n$.

Proof. There is an inclusion morphism $\iota : X \times Y \hookrightarrow \mathbb{A}^n \times \mathbb{A}^n$. Since $\mathbb{A}^n$ is an affine variety, then $\Delta_{\mathbb{A}^n}$ is closed in $\mathbb{A}^n \times \mathbb{A}^n$, with

$$\Delta_{\mathbb{A}^n} = V(x_1 - y_1, \ldots, x_n - y_n),$$

where each $x_j - y_j \in A(\mathbb{A}^n \times \mathbb{A}^n)$. Then, similar to (a), we find

$$\iota^{-1}(\Delta_{\mathbb{A}^n}) = (X \times Y) \cap \Delta_{\mathbb{A}^n} = \Delta_{X \cap Y},$$

with $X \cap Y \cong \Delta_{X \cap Y}$ as affine varieties. Then

$$\Delta_{X \cap Y} = V(x_1 - y_1, \ldots, x_n - y_n),$$

where each $x_j - y_j \in A(X \times Y)$. Then

$$I(\Delta_{X \cap Y}) = \sqrt{\langle x_1 - y_1, \ldots, x_n - y_n \rangle} \trianglelefteq A(X \times Y).$$

With the evaluation homomorphism

$$\begin{aligned} K[x_1, \ldots, x_n, y_1, \ldots, y_n] &\to K[x_1, \ldots, x_n] \\ p(x_1, \ldots, x_n, y_1, \ldots, y_n) &\mapsto p(x_1, \ldots, x_n, x_1, \ldots, x_n), \end{aligned}$$

we obtain

$$K[x_1, \ldots, x_n, y_1, \ldots, y_n]/\langle x_1 - y_1, \ldots, x_n - y_n \rangle \cong K[x_1, \ldots, x_n].$$

This is an integral domain, implying that $\langle x_1 - y_1, \ldots, x_n - y_n \rangle$ is a is prime ideal in $K[x_1, \ldots, x_n, y_1, \ldots, y_n]$. Therefore, its projection onto the quotient

$$A(X \times Y) = K[x_1, \ldots, x_n, y_1, \ldots, y_n]/I(X \times Y)$$

will again be a prime ideal, therefore a radical ideal. Then

$$I(\Delta_{X \cap Y}) = \langle x_1 - y_1, \ldots, x_n - y_n \rangle \trianglelefteq A(X \times Y).$$

That is, $I(\Delta_{X \cap Y})$ is an ideal of the Noetherian ring $A(X \times Y)$, generated by n elements. By Krull's height theorem, the minimal prime ideals of $A(X \times Y)$ containing $I(\Delta_{X \cap Y})$ have height at most n. This corresponds to the irreducible components of $\Delta_{X \cap Y}$ having codimension in $X \times Y$ at most n. Each irreducible component of $X \cap Y$ is isomorphic to an irreducible component Z of $\Delta_{X \cap Y}$, of the same dimension and for which we have

$$\dim(X \times Y) = \dim Z + \operatorname{codim}_{X \times Y} Z \leq \dim Z + n,$$

so

$$\dim X + \dim Y - n \leq \dim Z.$$

□

Exercise 6.13 Let $a \in \mathbb{P}^n$ be a point. Show that the one-point set $\{a\}$ is a projective variety, and compute explicit generators for the ideal $I_{\mathrm{p}}(\{a\}) \trianglelefteq K[x_0, \dots, x_n]$.

Proof. Suppose $a = [a_0, a_1, \dots, a_n]$. We know that not all a_j's are zero, otherwise a would not describe a linear subspace of $\mathbb{A}^{n+1}$. Without loss of generality, assume $a_0 \neq 0$ (otherwise, we replace a_0 with a nonzero a_j in the following argument), then

$$a = [a_0, a_1, \dots, a_n] = \left[1, \tfrac{a_1}{a_0}, \dots, \tfrac{a_n}{a_0}\right].$$

For any $x \in \mathbb{P}^n$, we have $x = a$ if and only if $x = [x_0, x_1, \dots, x_n]$ with $x_0 \neq 0$ and

$$\left[1, \tfrac{x_1}{x_0}, \dots, \tfrac{x_n}{x_0}\right] = \left[1, \tfrac{a_1}{a_0}, \dots, \tfrac{a_n}{a_0}\right].$$

This last equality holds if and only if, for $j = 1, \dots, n$, we have $\frac{x_j}{x_0} = \frac{a_j}{a_0}$, which is equivalent to $a_0 x_j - a_j x_0 = 0$. Therefore, the singleton $\{a\}$ is given as the zero locus

$$\{a\} = V_{\mathrm{p}}\left(x_1 - \tfrac{a_1}{a_0} x_0, \dots, x_n - \tfrac{a_n}{a_0} x_0\right)$$

of homogeneous polynomials. So in fact, $\{a\}$ is a projective variety.

□

Let

$$J = \left\langle x_1 - \tfrac{a_1}{a_0} x_0, \dots, x_n - \tfrac{a_n}{a_0} x_0 \right\rangle \trianglelefteq K[x_0, \dots, x_n]$$

be the ideal generated by homogeneous polynomials; we claim $I_{\mathrm{p}}(\{a\}) = J$.

By the projective Nullstellensatz, we have

$$I_{\mathrm{p}}(\{a\}) = I_{\mathrm{p}}(V_{\mathrm{p}}(J)) = \sqrt{J}.$$

To show that J is a radical ideal, we will show that it is a prime ideal, by showing that the quotient ring $K[x_0, \dots, x_n]/J$ is an integral domain. Consider the evaluation homomorphism of rings

$$\begin{aligned} \varphi : K[x_0, \dots, x_n] &\to K[x] \\ p(x_0, x_1, \dots, x_n) &\mapsto p\left(x, \tfrac{a_1}{a_0} x, \dots, \tfrac{a_n}{a_0} x\right). \end{aligned}$$

This map is surjective, as the restriction to $K[x_0]$ is simply the isomorphism $K[x_0] \to K[x]$ by $x_0 \mapsto x$. Moreover, its kernel is precisely the ideal J, so

$$K[x_0, \dots, x_n]/J = K[x_0, \dots, x_n]/\ker\varphi \cong K[x].$$

Since $K[x]$ is an integral domain, this proves J is a prime, therefore radical, ideal. Hence,

$$I_{\mathrm{p}}(\{a\}) = \sqrt{J} = J = \left\langle x_1 - \tfrac{a_1}{a_0} x_0, \dots, x_n - \tfrac{a_n}{a_0} x_0 \right\rangle.$$