Yoneda Lemma For any functor $F : \mathcal{C} \to \mathsf{Set}$, whose domain $\mathcal{C}$ is locally small and any object $c \in \mathcal{C}$, there is a bijection

$$\mathrm{Nat}(\mathrm{Mor}_{\mathcal{C}}(c, -), F) \cong Fc$$

that associates a natural transformation $\alpha : \mathrm{Mor}_{\mathcal{C}}(c, -) \Rightarrow F$ with the element $\alpha_c(1_c) \in Fc$. Moreover, this correspondence is natural in both c and F.

Yoneda Lemma' For any functor $F : \mathcal{C} \to \mathsf{Set}$, whose domain $\mathcal{C}$ is locally small and any object $c \in \mathcal{C}$, there is a bijection

$$\mathrm{Mor}_{[\mathcal{C},\mathsf{Set}]}(h_c, F) \cong Fc$$

that associates a natural transformation $\alpha : h_c \to F$ with the element $\alpha_c(1_c) \in Fc$. Moreover, this correspondence is natural in both c and F.

Proof. The map $\Phi : \mathrm{Nat}(\mathrm{Mor}_{\mathcal{C}}(c, -), F) \to Fc$ is easy to construct. Given a natural transformation $\alpha : \mathrm{Mor}_{\mathcal{C}}(C, -) \Rightarrow F$, we simply define

$$\Phi(\alpha) := \alpha_c(1_c),$$

where $\alpha_c : \mathrm{Mor}_{\mathcal{C}}(c, c) \to Fc$ is the component of α at c.

We now wish to construct Φ's inverse, $\Psi : Fc \to \mathrm{Nat}(\mathrm{Mor}_{\mathcal{C}}(c, -), F)$. That is, given an element $x \in Fc$, we must construct a natural transformation $\Psi(x) : \mathrm{Mor}_{\mathcal{C}}(c, -) \Rightarrow F$. To do this, we will construct its components $\Psi(x)_a : \mathrm{Mor}_{\mathcal{C}}(c, a) \to Fa$ for each object $a \in \mathcal{C}$. Moreover, this construction must adhere to the naturality condition, i.e., for all morphisms $f : a \to b$ in $\mathcal{C}$, the following diagram must commute:

$$\begin{tikzcd}
\mathrm{Mor}_{\mathcal{C}}(c, a) \arrow[r, "\Psi(x)_a"] \arrow[d, "f_*"'] & Fa \arrow[d, "Ff"] \\
\mathrm{Mor}_{\mathcal{C}}(c, b) \arrow[r, "\Psi(x)_b"'] & Fb
\end{tikzcd}$$

Here, $f_* = \mathrm{Mor}_{\mathcal{C}}(c, f)$ is the function which takes a morphism $g : c \to a$ and sends it to the composition $f_* g = f \circ g : c \to b$.

Let us look at what this diagram is saying in the particular case of $f : c \to a$. The diagram looks like this:

$$\begin{tikzcd}
\mathrm{Mor}_{\mathcal{C}}(c, c) \arrow[r, "\Psi(x)_c"] \arrow[d, "f_*"'] & Fc \arrow[d, "Ff"] \\
\mathrm{Mor}_{\mathcal{C}}(c, a) \arrow[r, "\Psi(x)_a"'] & Fa
\end{tikzcd}$$

Consider the identity 1_c in the upper left corner. Following the left side of the square, we obtain

$$\Psi(x)_a(f_*1_c) = \Psi(x)_a(f \circ 1_c) = \Psi(x)_a(f).$$

Since we eventually want to this square to commute, this must be equal to the result of following the right side, i.e., we must have

$$\Psi(x)_a(f) = Ff(\Psi(x)_c(1_c)).$$

In other words, it would suffice to define $\Psi(x)_c(1_c)$.

Recall our definition of Φ, for which we want Ψ to be an inverse. Plugging in $\Psi(x)$ for α,

$$\Psi(x)_c(1_c) = \Phi(\Psi(x)) = x.$$

Our hand is now forced to define

$$\Psi(x)_a(f) := Ff(x).$$

We can be reasonably confident that because we made only the "obvious" choices that Ψ is correct, but there are a few things we must check to be sure.

First, we check the naturality of $\Psi(x)$, i.e., that the diagram from earlier commutes for all $f : a \to b$ in $\mathcal{C}$. For a morphism $g : c \to a$ in $\mathcal{C}$, we find

$$\begin{aligned}
\Psi(x)_b(f_*g) &= \Psi(x)_b(f \circ g) && \text{def of } f_* \\
&= F(f \circ g)(x) && \text{def of } \Psi \\
&= (Ff \circ Fg)(x) && \text{functorality of } F \\
&= Ff(Fg(x)) && \text{def of } \circ \\
&= Ff(\Psi(x)_a(g)). && \text{def of } \Psi
\end{aligned}$$

This tells us that $\Psi(x)$ is indeed a natural transformation $\mathrm{Mor}_{\mathcal{C}}(c, -) \Rightarrow F$, so Ψ is a well-defined map.

Lastly, we check that Φ and Ψ are inverses.

For $x \in Fc$, we apply definitions to obtain

$$\Phi(\Psi(x)) = \Psi(x)_c(1_c) = F1_c(x) = 1_{Fc}(x) = x.$$

For a natural transformation $\alpha : \mathrm{Mor}_C\, C(c, -) \Rightarrow F$, we consider the natural transformation $\Psi(\Phi(\alpha))$ at a morphism $f \in \mathrm{Mor}_{\mathcal{C}}(c, b)$.

$$\begin{aligned}
\Psi(\Phi(\alpha))_b(f) &= Ff(\alpha_c(1_c)) && \text{defs of } \Phi \text{ and } \Psi \\
&= (Ff \circ \alpha_c)(1_c) \\
&= (\alpha_b \circ f_*)(1_c) && \text{naturality of } \alpha \\
&= \alpha_b(f_*1_c) \\
&= \alpha_b(f)
\end{aligned}$$

This shows that $\Psi(\Phi(\alpha)) = \alpha$.

We conclude that Φ and Ψ are inverses. $\square$

remains to prove naturality in c and F

then do embedding