1 Exercise 6.7.2 Compute the Hilbert function and the Hilbert polynomial of the "twisted cubic curve" $C = \{[s^3 : s^2t : st^2 : t^3] \mid [s:t] \in \mathbb{P}^1\} \subseteq \mathbb{P}^3$.

Denote $S = k[x, y, z, w]$, so $\mathbb{P}^3 = \operatorname{Proj} S$.

We compute

$$I(C) = \langle xw - yz, xz - y^2, yw - z^2 \rangle \trianglelefteq S.$$

Similar to the proof of Proposition 6.1.5, we consider a hyperplane $H = Z(w)$ and the following short exact sequence of graded k-vector spaces:

$$0 \longrightarrow S/I(C) \xrightarrow{\cdot w} S/I(C) \longrightarrow S/(I(C) + \langle w \rangle) \longrightarrow 0.$$

Note that

$$S(C) = S/I(C) \quad \text{and} \quad S(C \cap H) = S/(I(C) + \langle w \rangle),$$

so restricting to graded portions gives us another short exact sequence:

$$0 \longrightarrow S(C)^{(d-1)} \xrightarrow{\cdot w} S(C)^{(d)} \longrightarrow S(C \cap H)^{(d)} \longrightarrow 0.$$

Taking the dimension over k gives the following relation on Hilbert functions for $d \geq 1$:

$$h_C(d) = h_{C \cap H}(d) + h_C(d-1).$$

To compute $h_{C \cap H}(d)$, first write

$$I(C) + \langle w \rangle = \langle w, yz, xz - y^2, z^2 \rangle \trianglelefteq S.$$

Then the only distinct nonzero monomials of $S(C \cap H)^{(d)}$ are x^d, $x^{d-1}y$, and $x^{d-1}z$. So

$$h_{C \cap H}(d) = 3,$$

giving us

$$h_C(d) = h_C(d-1) + 3$$

for $d \geq 1$. And since $h_C(0) = 1$, we conclude that

$$h_C(d) = 3d + 1 = \chi_C(d).$$

2 Exercise 6.7.3 Let $X \subseteq \mathbb{P}^n$ be a projective scheme with Hilbert polynomial χ. As in example 6.1.10 define the *arithmetic genus* of X to be $g(X) = (-1)^{\dim X} \cdot (\chi(0) - 1)$.

(i) Show that $g(\mathbb{P}^n) = 0$.

We have seen that $h_{\mathbb{P}^n}(d) = \chi_{\mathbb{P}^n}(d) = \binom{d+n}{n}$, so then

$$g(\mathbb{P}^n) = (-1)^{\dim \mathbb{P}^n}(\chi_{\mathbb{P}^n}(0) - 1) = (-1)^n(1-1) = 0.$$

(ii) If X is a hypersurface of degree d in $\mathbb{P}^n$, show that $g(X) = \binom{d-1}{n}$. In particular, if $C \subseteq \mathbb{P}^2$ is a plane curve of degree d, then $g(C) = \frac{1}{2}(d-1)(d-2)$.

Proof. Suppose $X = Z(f) \subseteq \mathbb{P}^n$, where $f \in S = k[x_0, \ldots, x_n]$ is homogeneous of degree d. Consider the following short exact sequence of graded k-vector spaces:

$$0 \longrightarrow S \xrightarrow{\cdot f} S \longrightarrow S/\langle f \rangle \longrightarrow 0.$$

Note that

$$S(\mathbb{P}^n) = S \quad \text{and} \quad S(X) = S/\langle f \rangle,$$

so restricting to graded portions gives us another short exact sequence:

$$0 \longrightarrow S^{(s-d)} \xrightarrow{\cdot f} S^{(s)} \longrightarrow S(X)^{(d)} \longrightarrow 0.$$

Taking dimensions gives us the following relation on Hilbert functions for $s \geq d$:

$$h_X(s) = h_{\mathbb{P}^n}(s) - h_{\mathbb{P}^n}(s-d) = \binom{s+n}{n} - \binom{s-d+n}{n}.$$

Then the Hilbert polynomial of X is given by

$$\chi_X(s) = \frac{(s+n)\cdots(s+1)}{n!} - \frac{(s-d+n)\cdots(s-d+1)}{n!}.$$

Evaluating at zero, we obtain

$$\chi_X(0) = \frac{n!}{n!} - \frac{(-d+n)\cdots(-d+1)}{n!} = 1 - (-1)^n\binom{d-1}{n},$$

where the second equality applies the fact that $\binom{a}{b} = (-1)^b\binom{b-a+1}{b}$, for sufficiently robust definitions of the binomial coefficient. Hence, with $\dim X = n-1$, we find

$$g(X) = (-1)^{n-1}\left(\left(1 - (-1)^n\binom{d-1}{n}\right) - 1\right) = \binom{d-1}{n}.$$

$\square$

(iii) Compute the arithmetic genus of the union of the three coordinate axes

$$Z(x_1x_2, x_1x_3, x_2x_3) \subseteq \mathbb{P}^3.$$

Denote $S = k[w, x, y, z]$, so $\mathbb{P}^3 = \operatorname{Proj} S$, and $I = \langle xy, xz, yz \rangle \trianglelefteq S$, so $X = Z(I)$.

The only distinct nonzero monomials of $S(X)^{(d)} = (S/I)$ are $x^a w^{d-a}, y^a w^{d-a}, z^a w^{d-a}$, w, for $a = 1, \ldots, d$. Hence, we have the Hilbert function/polynomial

$$h_X(d) = 3d + 1 = \chi_X(d).$$

Then the arithmetic genus is

$$g(X) = (-1)^{\dim X}(\chi_X(0) - 1) = (-1)^1(1 - 1) = 0.$$

3 Exercise 6.7.4 For $N = (n+1)(m+1) - 1$ let $Z \subseteq \mathbb{P}^N$ be the image of the Segre embedding $\mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^N$. Show that the degree of Z is $\binom{n+m}{n}$.

Proof. Denote $X = \mathbb{P}^n = \operatorname{Proj} k[x_0, \ldots, x_n]$ and $Y = \mathbb{P}^m = \operatorname{Proj} k[y_0, \ldots, y_m]$.

Denote the Segre embedding of $X \times Y$ by $Z = \operatorname{Proj} k[z_{00}, \ldots, z_{nm}]/I$, where I is the homogeneous ideal $\langle z_{ij}z_{k\ell} - z_{i\ell}z_{kj} \rangle$.

We define a k-algebra homomorphism

$$\begin{aligned} \varphi : k[z_{00}, \ldots, z_{nm}] &\longrightarrow k[x_0, \ldots, x_n, y_0, \ldots, y_m] = S(X) \otimes_k S(Y), \\ z_{ij} &\longmapsto x_i y_j. \end{aligned}$$

Under φ, the generators of I are mapped as follows:

$$z_{ij}z_{k\ell} - z_{i\ell}z_{kj} \longmapsto x_i y_j x_k y_\ell - x_i y_\ell x_k y_j = 0.$$

Hence, $I \subseteq \ker \varphi$, so φ factors through the quotient by I, giving us a well-defined map

$$\begin{aligned} k[z_{00}, \ldots, z_{nm}]/I = S(Z) &\longrightarrow S(X) \otimes_k S(Y), \\ z_{ij} &\longmapsto x_i y_j. \end{aligned}$$

In particular, this map is an injective morphism of graded k-vector spaces. The image of the monomials of degree d under this map are precisely the monomials with total degree d with respect to $x_0, \ldots, x_n$ and total degree d with respect to $y_0, \ldots, y_m$. That is, this map induces an isomorphism of graded k-vector spaces

$$S(Z)^{(d)} \cong S(X)^{(d)} \otimes_k S(Y)^{(d)}.$$

Taking the dimension, we obtain the hilbert function/polynomial

$$h_Z(d) = h_X(d) \cdot h_Y(d) = \binom{d+n}{n} \cdot \binom{d+m}{m} = \chi_Z(d),$$

with leading coefficient $1/(n!m!)$. Then the degree of Z is

$$\frac{(\dim Z)!}{n!m!} = \frac{(n+m)!}{n!m!} = \binom{n+m}{n}.$$

□

4 Exercise 6.7.6 Let $C \subseteq \mathbb{P}^n$ be an irreducible curve of degree d. Show that C is contained in a linear subspace of $\mathbb{P}^n$ of dimension d.

Proof. If $d \geq n$, then trivially $C \subseteq \mathbb{P}^n$.

If $d < n$, then we claim that C is contained in some hyperplane of $\mathbb{P}^n$. Let $H \subseteq \mathbb{P}^n$ be a hyperplane, i.e., $H = Z(\ell)$ for some homogeneous linear $\ell \in k[x_0, \ldots, x_n]$. Since C is irreducible, either $C \subseteq H$ or $C \cap H$ is 0-dimensional. In the latter case, we apply Bézout's theorem to find

$$\deg(C \cap H) = \deg C \cdot \deg \ell = d.$$

This gives us the length of $C \cap H$, i.e., the number of points in the intersection, counted with multiplicity. In particular, $C \cap H$ contains at most d points. Since we are assuming $C \nsubseteq H$, we can pick another point $P \in C \setminus H$. Then since $d + 1 \leq n$, we can choose a new hyperplane $H' \subseteq \mathbb{P}^n$ containing the (at most) d points in $C \cap H$ and the point P. In which case, $C \cap H'$ will have at least $d + 1$ points, counted with multiplicity.

If $\dim(C \cap H') = 1$, then the intersection is precisely C, meaning $C \subseteq H'$. If $\dim(C \cap H') = 0$, then its degree equals the number of points with multiplicity, which is at least $d + 1$. The contrapositive of Bézouts theorem then implies that some component of C is contained in H'. But since C is irreducible, we in fact have $C \subseteq H'$ (which is also a contradiction, so we must have $\dim(C \cap H') = 1$).

We conclude that when $C \subseteq \mathbb{P}^n$ for any $n > d$, we can find a hyperplane of $\mathbb{P}^n$ containing C. This hyperplane is isomorphic to $\mathbb{P}^{n-1}$, and we can continue inductively until C is contained in a linear subspace of $\mathbb{P}^n$ isomorphic to $\mathbb{P}^d$, i.e., of dimension d. □

5 Exercise 6.7.7 Let X and Y be subvarieties of $\mathbb{P}^n_k$ that lie in disjoint linear subspaces of $\mathbb{P}^n_k$. Recall from exercises 3.5.7 and 4.6.1 that the join $J(X,Y) \subseteq \mathbb{P}^n_k$ of X and Y is defined to be the union of all lines $\overline{PQ}$ with $P \in X$ and $Q \in Y$.

(i) Show that $S(J(X,Y))^{(d)} \cong \bigoplus_{i+j=d} S(X)^{(i)} \otimes_k S(Y)^{(j)}$.

(ii) Show that $\deg J(X,Y) = \deg X \cdot \deg Y$.