Homework 4

MATH CS 121 Intro to Probability

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Exercise 1

We consider $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mathbb{P})$ and define $F : \mathbb{R} \to [0,1]$ by $F(\alpha) := \mathbb{P}((-\infty, \alpha])$. Show that the following properties hold for F:

Exercise 1(i)

F is non-decreasing

Proof. Let $a, b \in \mathbb{R}$ with $a \le b$. Then $(-\infty, a] \subseteq (-\infty, b]$, and by the properties of the probability measure, we have

$$F(a) = \mathbb{P}((-\infty, a]) \le \mathbb{P}((-\infty, b]) = F(b).$$

Thus, F is non-decreasing.

$\square$

Exercise 1(ii)

F is right continuous

Proof. Let $\{a_n\}_{n\in\mathbb{N}} \subseteq \mathbb{R}$ be a sequence converging to a from the right, so $a \le a_n$ for all $n \in \mathbb{N}$. Now because F is non-decreasing, then for all $n \in \mathbb{N}$, we have

$$F(a) \le F(a_n).$$

We now define the following value for all $k \in \mathbb{N}$:

$$a'_n = \sup_{k \ge n} a_k.$$

This definition gives us $a_n \le a'_n$ for all $n \in \mathbb{N}$, and, moreover, that

$$F(a_n) \le F(a'_n)$$

for all $n \in \mathbb{N}$. Since $a_n \to a$, then by the definition of limit superior, we have

$$\lim_{n\to\infty} a_n' = \lim_{n\to\infty} \sup_{k\geq n} a_k = \limsup_{n\to\infty} a_n = \lim_{n\to\infty} a_n = a.$$

We now consider the limit

$$\lim_{n\to\infty} F(a_n') = \lim_{n\to\infty} \mathbb{P}((-\infty, a_n']).$$

Since $\{a_n'\}_{n\in\mathbb{N}}$ is a non-increasing sequence, then $\{(-\infty, a_n']\}_{n\in\mathbb{N}}$ is a non-increasing sequence with respect to inclusion. Then by the continuity of the probability measure, we have

$$\lim_{n\to\infty} F(a_n') = \mathbb{P}\left(\bigcap_{n=1}^{\infty}(-\infty, a_n']\right).$$

Now since $a \leq a_n \leq a_n'$ for all $n \in \mathbb{N}$ and $a_n' \to a$, then

$$\bigcap_{n=1}^{\infty}(-\infty, a_n'] = (-\infty, a].$$

Therefore, we have

$$\lim_{n\to\infty} F(a_n') = \mathbb{P}((-\infty, a]) = F(a).$$

Finally, since

$$F(a) \leq F(a_n) \leq F(a_n')$$

for all $n \in \mathbb{N}$, then by the squeeze theorem, we obtain

$$\lim_{n\to\infty} F(a_n) = F(a).$$

Since this is true of all sequences $\{a_n\}_{n\in\mathbb{N}} \subseteq \mathbb{R}$ converging to a from the right, we can conclude that

$$\lim_{x\to a^+} F(x) = F(a).$$

Thus, F is right-continuous.

$\square$

Exercise 1(iii)

$\boxed{\lim_{\alpha\to-\infty} F(\alpha) = 0 \text{ and } \lim_{\alpha\to\infty} F(\alpha) = 1.}$

Proof. Suppose $\{a_n\}_{n\in\mathbb{N}} \subseteq \mathbb{R}$ such that $a_n \to -\infty$. Without loss of generality, assume $\{a_n\}_{n\in\mathbb{N}}$ is non-increasing, since if it is not, we simply take a non-increasing subsequence. Then by the continuity of the the probability measure, we have

$$\begin{aligned}
\lim_{\alpha\to-\infty} F(\alpha) &= \lim_{n\to\infty} F(a_n) \\
&= \lim_{n\to\infty} \mathbb{P}((-\infty, a_n]) \\
&= \mathbb{P}\left(\bigcap_{n=1}^{\infty}(-\infty, a_n]\right) \\
&= \mathbb{P}(\varnothing) \\
&= 0.
\end{aligned}$$

Similarly, if $\{a_n\}_{n\in\mathbb{N}} \subseteq \mathbb{R}$ is a non-decreasing sequence such that $a_n \to \infty$, then by the continuity of the probability measure, we have

$$\begin{aligned}
\lim_{\alpha\to\infty} F(\alpha) &= \lim_{n\to\infty} F(a_n) \\
&= \lim_{n\to\infty} \mathbb{P}((-\infty, a_n]) \\
&= \mathbb{P}\left(\bigcup_{n=1}^{\infty}(-\infty, a_n]\right) \\
&= \mathbb{P}(\Omega) \\
&= 1.
\end{aligned}$$

□

Exercise 2

Let $\Omega = \{a, b, c\}$ and define three σ-algebras $\mathcal{F}_1 = \{\varnothing, \Omega\}$, $\mathcal{F}_2 = \{\{a, b, c\}, \{a, b\}, \{c\}, \varnothing\}$ and $\mathcal{F}_3 = 2^\Omega$. Define three functions $X, Y, Z : \Omega \to \mathbb{R}$ with the following properties:

Exercise 2(i)

X is $\mathcal{F}_3$-measurable but not $\mathcal{F}_2$-measurable,

Define the function X as follows:

$$\begin{aligned} X : \Omega &\to \mathbb{R} \\ a &\mapsto 1 \\ b &\mapsto 0 \\ c &\mapsto 1. \end{aligned}$$

Since $\mathcal{F}_3 = 2^\Omega$, then every function $\Omega \to \mathbb{R}$ will be $\mathcal{F}_3$-measurable, including X, in particular. However, $X^{-1}((-\infty, 0]) = \{b\} \notin \mathcal{F}_2$, so X is not $\mathcal{F}_2$-measurable.

Exercise 2(ii)

Y is $\mathcal{F}_2$-measurable but not $\mathcal{F}_1$-measurable,

Define the function Y as follows:

$$\begin{aligned} Y : \Omega &\to \mathbb{R} \\ a &\mapsto 0 \\ b &\mapsto 0 \\ c &\mapsto 1. \end{aligned}$$

Then we have

$$Y^{-1}((-\infty, \alpha]) = \begin{cases} \varnothing & \text{if } \alpha < 0, \\ \{a, b\} & \text{if } 0 \leq \alpha < 1, \\ \Omega & \text{if } 1 \leq \alpha. \end{cases}$$

So for all $\alpha \in \mathbb{R}$, we have $Y^{-1}((-\infty, \alpha]) \in \mathcal{F}_2$, so Y is $\mathcal{F}_2$-measurable. However, $Y^{-1}((-\infty, 0]) = \{a, b\} \notin \mathcal{F}_1$, so Y is not $\mathcal{F}_1$-measurable.

Exercise 2(iii)

Z is $\mathcal{F}_1$-measurable.

Define the function Z as follows:

$$\begin{aligned} Y : \Omega &\to \mathbb{R} \\ a &\mapsto 0 \\ b &\mapsto 0 \\ c &\mapsto 0. \end{aligned}$$

Then we have

$$Y^{-1}((-\infty, \alpha]) = \begin{cases} \varnothing & \text{if } \alpha < 0, \\ \Omega & \text{if } 0 \leq \alpha. \end{cases}$$

So for all $\alpha \in \mathbb{R}$, we have $Y^{-1}((-\infty, \alpha]) \in \mathcal{F}_1$, so Y is $\mathcal{F}_1$-measurable.

Exercise 3

Let's assume we have random variables $X_1, X_2, \dots$ such that for every $\omega \in \Omega$, the limit $\lim\limits_{n\to\infty} X_n(\omega)$ exists. Define a new function $X : \Omega \to \mathbb{R}$ by $X(\omega) := \lim\limits_{n\to\infty} X_n(\omega)$. Is X a random variable? (Hint: start with lim inf and lim sup.)

Proof. Let $\alpha \in \mathbb{R}$. We aim to prove that

$$X^{-1}((-\infty, \alpha]) \in \mathcal{F}.$$

For each $n \in \mathbb{N}$, we define the function $Y_n : \Omega \to \mathbb{R}$ such that $Y_n(\omega) = \inf\limits_{k\geq n} X_k(\omega)$. This definition of Y_n gives us

$$X(\omega) = \lim_{n\to\infty} X_n(\omega) = \sup_{n\in\mathbb{N}} \left(\inf_{k\geq n} X_k(\omega) \right) = \sup_{n\in\mathbb{N}} Y_n(\omega).$$

Moreover, we have

$$X^{-1}((\infty, \alpha]) = \{\omega \in \Omega : X(\omega) \leq \alpha\} = \{\omega \in \Omega : \sup_{n\in\mathbb{N}} Y_n(\omega) \leq \alpha\}.$$

By the definition of supremum, $\sup\limits_{n\in\mathbb{N}} Y_n(\omega) \leq \alpha$ if and only if $Y_n(\omega) \leq \alpha$ for all $n \in \mathbb{N}$. Therefore,

$$\begin{aligned} X^{-1}((\infty, \alpha]) &= \{\omega \in \Omega : Y_n(\omega) \leq \alpha, \text{ for all } n \in \mathbb{N}\} \\ &= \bigcap_{n=1}^{\infty} \{\omega \in \Omega : Y_n(\omega) \leq \alpha\} \\ &= \bigcap_{n=1}^{\infty} Y_n^{-1}((\infty, \alpha]). \end{aligned}$$

For a fixed $n \in \mathbb{N}$, we now consider the set $Y_n^{-1}((\infty, \alpha])$. By the definition of Y_n, we have

$$Y_n^{-1}((\infty, \alpha]) = \{\omega \in \Omega : Y_n(\omega) \leq \alpha\}.$$

For a given $\omega \in \Omega$, we will prove that $Y_n(\omega) \leq \alpha$ if and only if

$$\forall \varepsilon > 0,\ \exists \ell \geq n \text{ such that } X_\ell(\omega) < \alpha + \varepsilon.$$

First assume that $Y_n(\omega) \leq \alpha$. Because $Y_n(\omega) = \inf\limits_{k\geq n} X_k(\omega)$, we have that for every $\varepsilon > 0$, there exists some $\ell \geq n$ such that

$$X_\ell(\omega) < Y_n(\omega) + \varepsilon \leq \alpha + \varepsilon.$$

Now assume that $Y_n(\omega) \not\leq \alpha$, i.e., that $\alpha < Y_n(\omega)$, and let

$$\varepsilon = Y_n(\omega) - \alpha > 0.$$

Now for all $\ell \geq n$ we find that

$$\alpha + \varepsilon = \alpha + Y_n(\omega) - \alpha = Y_n(\omega) \leq X_\ell(\omega),$$

that is, $X_\ell(\omega) \not< \alpha + \varepsilon$. Thus, the equivalence is proven, and we can write

$$Y_n^{-1}((\infty, \alpha]) = \{\omega \in \Omega : \forall \varepsilon > 0, \exists \ell \geq n : X_\ell(\omega) < \alpha + \varepsilon\}.$$

Moreover, we write

$$Y_n^{-1}((\infty, \alpha]) = \{\omega \in \Omega : \forall m \in \mathbb{N}, \exists \ell \geq n : X_k(\omega) < \alpha + \tfrac{1}{m}\}.$$

The condition involving ε and the condition involving $\frac{1}{m}$ are equivalent since both are being used to express the notion inherent to an infimum that for any value which is strictly greater than the infimum by an arbitrarily small, yet nonzero, amount we can always find an element in the set which is strictly less than this value. Since ε and $\frac{1}{m}$ are nonzero and can be made arbitrarily small, they function equivalently, in this instance. We can now write

$$\begin{aligned} Y_n^{-1}((\infty, \alpha]) &= \bigcap_{m=1}^{\infty} \{\omega \in \Omega : \exists \ell \geq n : X_\ell(\omega) < \alpha + \tfrac{1}{m}\} \\ &= \bigcap_{m=1}^{\infty} \bigcup_{k=n}^{\infty} \{\omega \in \Omega : X_k(\omega) < \alpha + \tfrac{1}{m}\} \\ &= \bigcap_{m=1}^{\infty} \bigcup_{k=n}^{\infty} X_k^{-1}((-\infty, \alpha + \tfrac{1}{m})). \end{aligned}$$

For any $m \in \mathbb{N}$, note that

$$(\infty, \alpha + \tfrac{1}{m}) = \bigcup_{\ell=1}^{\infty} (-\infty, \alpha + \tfrac{1}{m} - \tfrac{1}{\ell}].$$

And since preimage preserves unions, then for any $k, m \in \mathbb{N}$, we have

$$X_k^{-1}((-\infty, \alpha + \tfrac{1}{m})) = X_k^{-1}\left(\bigcup_{\ell=1}^{\infty} (-\infty, \alpha + \tfrac{1}{m} - \tfrac{1}{\ell}]\right) = \bigcup_{\ell=1}^{\infty} X_k^{-1}((-\infty, \alpha + \tfrac{1}{m} - \tfrac{1}{\ell}]).$$

Therefore,

$$X^{-1}((-\infty, \alpha]) = \bigcap_{n=1}^{\infty} \bigcap_{m=1}^{\infty} \bigcup_{k=n}^{\infty} \bigcup_{\ell=1}^{\infty} X_k^{-1}((-\infty, \alpha + \tfrac{1}{m} - \tfrac{1}{\ell}]).$$

And since X_k is a random variable for all $k \in \mathbb{N}$, then for all $k, m, \ell \in \mathbb{N}$, we have

$$X_k^{-1}((-\infty, \alpha + \tfrac{1}{m} - \tfrac{1}{\ell}]) \in \mathcal{F}.$$

Now since $\mathcal{F}$ is a σ-algebra and is closed under countable unions and intersections, we have $X^{-1}((-\infty, \alpha]) \in \mathcal{F}$.

$\square$

Exercise 4

Use the informal definition of the integral/expectation I gave in class to derive the expected value of a Poisson distributed random variable with parameter $\lambda > 0$ and with probability mass function $p(n) = \mathbb{P}(\{n\}) = e^{-\lambda}\frac{\lambda^n}{n!}$.

Proof. We find the expected value of X by the following formula:

$$\mathbb{E}[X] = \sum_{\omega \in \Omega} X(\omega)\mathbb{P}(\{\omega\}).$$

Assuming $\Omega = \mathbb{N}$ and $X : \mathbb{N} \hookrightarrow \mathbb{R}$, then

$$\begin{aligned}\mathbb{E}[X] &= \sum_{n \in \mathbb{N}} n \cdot e^{-\lambda}\frac{\lambda^n}{n!} \\ &= \lambda e^{-\lambda} \sum_{n=1}^{\infty} \frac{\lambda^{n-1}}{(n-1)!} \\ &= \lambda e^{-\lambda} \sum_{n=0}^{\infty} \frac{\lambda^n}{n!}.\end{aligned}$$

And since the Taylor series of e^x is precisely

$$\sum_{n=0}^{\infty} \frac{x^n}{n!},$$

and converges everywhere; in particular, for $x > 0$. So

$$\mathbb{E}[X] = \lambda e^{-\lambda} e^{\lambda} = \lambda.$$

$\square$