(worked with Joseph Sullivan, Gahl Shemy)

1 Exercise I.19 Let G be a finite group operating on a finite set S.

(a) For each $s \in S$ show that
$$\sum_{t \in G \cdot s} \frac{1}{|G \cdot t|} = 1.$$

Proof. Note that for all $t \in G \cdot s$, we have $G \cdot t = G \cdot s$, since the orbits of G in S form a partition of S. So
$$\sum_{t \in G \cdot s} \frac{1}{|G \cdot t|} = \sum_{t \in G \cdot s} \frac{1}{|G \cdot s|} = |G \cdot s| \frac{1}{|G \cdot s|} = 1.$$
□

(b) For each $x \in G$ define $f(x) =$ number of elements $s \in S$ such that $xs = s$. Prove that the number of orbits of G in S is equal to
$$\frac{1}{|G|} \sum_{x \in G} f(x).$$

Proof. By the orbit-stabilizer theorem,
$$|G \cdot s| = [G : G_s] = \frac{|G|}{|G_s|}.$$

Let S/G be the set of orbits of G in S. Then, with (a), the number of orbits of G in S is
$$|S/G| = \sum_{O \in S/G} 1 = \sum_{O \in S/G} \sum_{s \in O} \frac{1}{|G \cdot s|} = \sum_{s \in S} \frac{1}{|G \cdot s|} = \frac{1}{|G|} \sum_{s \in S} |G_s|.$$

Let $C = \{(x, s) \in G \times S \mid xs = s\}$. Then, by Homework 2 Problem 4,
$$\sum_{x \in G} f(x) = |C| = \sum_{s \in S} |G_s|,$$
so indeed
$$|S/G| = \frac{1}{|G|} \sum_{x \in G} f(x).$$
□

2 Exercise I.21 Let G be a finite group and H a subgroup. Let P_H be a p-Sylow subgroup of H. Prove that there exists a p-Sylow subgroup P of G such that $P_H = P \cap H$.

Proof. We have $|P_H| = p^k$, for some $k \in \mathbb{Z}_{\geq 0}$; in particular, P_H is a p-subgroup of G. So there is a p-Sylow subgroup $P \leq G$ containing P_H. We claim that $P_H = P \cap H$.

Since $P_H \leq P$ and $P_H \leq H$, it is evident that $P_H \leq P \cap H$. Moreover, this implies that $|P_H| = p^k$ divides $|P \cap H|$. And since $P \cap H \leq P$, we also have $|P \cap H|$ dividing $|P|$, so $|P \cap H|$ is a power of p. Since P_H is a p-Sylow subgroup of H, then p^k is the maximum power of p that divides the order of H. So in fact, $|P \cap H| = p^k = |P_H|$. Therefore, $P \cap H$ contains P and has the same order, so they must be equal.

□

3 Exercise I.22 Let H be a normal subgroup of a finite group G and assume that $|H| = p$. Prove that H is contained in every p-Sylow subgroup of G.

Proof. Since H is a p-subgroup of G, it is contained in some p-Sylow subgroup $P \leq G$. For any other p-Sylow subgroup $P' \leq G$, we have $P' = gPg^{-1}$ for some $g \in G$. Then, since H is normal in G, we have

$$H = gHg^{-1} \subseteq gPg^{-1} = P'.$$

$\square$

4 Exercise I.23 Let P, P' be p-Sylow subgroups of a finite group G.

(a) If $P' \subseteq N(P)$ (normalizer of P), then $P' = P$.

Proof. Consider PP', which the set of all elements xy for $x \in P$ and $y \in P'$. We claim that it is subgroup of G. Let $x_1y_1, x_2y_2 \in PP'$; we want to show that the product is still in G. Since $y_1 \in P' \subseteq N(P)$, then $y_1x_2y_1^{-1} \in P$, so

$$(x_1y_1)(x_2y_2) = x_1y_1x_2(y_1^{-1}y_1)y_2 = (x_1y_1x_2y_1^{-1})(y_1y_2) \in PP'.$$

Additionally, for $xy \in PP'$, we require $(xy)^{-1}$ in PP'. Since $y^{-1} \in P' \subseteq N(P)$, we know that $y^{-1}P = Py^{-1}$. Hence,

$$(xy)^{-1} = y^{-1}x^{-1} \in y^{-1}P = Py^{-1} \subseteq PP'.$$

We conclude that PP' is a subgroup of G containing P and P'.

By the diamond isomorphism theorem (listed in Lang as one of the canonical isomorphisms), $P' \leq N(P)$ implies $PP'/P' \cong P/(P \cap P')$, so

$$|PP'| = \frac{|P||P'|}{|P \cap P'|}.$$

We deduce that $|PP'|$ is a power of p, as it divides $|P||P'|$, which is a power of p. Since PP' contains P as a subgroup, then $|PP'|$ is at least $|P|$, the maximum power of p dividing $|G|$. So in fact, $P = PP' = P'$.

□

(b) If $N(P') = N(P)$, then $P' = P$.

Proof. Since $P' \subseteq N(P') = N(P)$, then by (a), we have $P' = P$.

□

(c) We have $N(N(P)) = N(P)$.

Proof. As always, $N(P) \subseteq N(N(P))$, so we prove the opposite inclusion. Let $x \in N(N(P))$, meaning $xN(P)x^{-1} = N(P)$. Since $P \subseteq N(P)$,

$$xPx^{-1} \subseteq xN(P)x^{-1} = N(P).$$

Since xPx^{-1} is a p-Sylow subgroup of G, (a) implies $xPx^{-1} = P$. Hence, $x \in N(P)$.

□