(worked with Joseph Sullivan, Gahl Shemy)

1 Exercise I.9

(a) Let G be a group and H a subgroup of finite index. Show that there exists a normal subgroup N of G contained in H and also of finite index. [*Hint*: If $(G : H) = n$, find a homomorphism of G into S_n whose kernel is contained in H.]

Proof. Consider the map $\varphi : G \to \operatorname{Perm}(G/H)$, where $\varphi(g) = \varphi_g : xH \mapsto gxH$. For $g, h \in G$,

$$\varphi_{gh}(xH) = ghxH = g\varphi_h(xH) = \varphi_g(\varphi_h(xH)) = (\varphi_g \circ \varphi_h)(xH),$$

so φ is in fact a group homomorphism. Then, $N = \ker \varphi$ is a normal subgroup of G, with

$$G/N \cong \operatorname{im} \varphi \leq \operatorname{Perm}(G/H).$$

Since $|G/H| = [G : H]$ is finite, so is $[G : N] = |G/N| \leq |\operatorname{Perm}(G/H)|$. If $n \in N$, then we have $nH = \varphi_n(eH) = H$, implying that $n \in H$. Hence, $N \subseteq H$, so N is as desired.

□

(b) Let G be a group and let H_1, H_2 be subgroups of finite index. Prove that $H_1 \cap H_2$ has finite index.

Proof. By part (a), there are normal subgroups of finite index $N_1, N_2 \trianglelefteq G$ such that $N_1 \subseteq H_1$ and $N_2 \subseteq H_2$, then $N_1 \cap N_2$ is a normal subgroup of G contained in $H_1 \cap H_2$. By [some] isomorphism theorem,

$$N_1/(N_1 \cap N_2) \cong (N_1N_2)/N_2,$$

so we deduce

$$\begin{aligned}[G : N_1 \cap N_2] &= [G : N_1][N_1 : N_1 \cap N_2] \\ &= [G : N_1][N_1N_2 : N_2] \\ &\leq [G : N_1][G : N_2].\end{aligned}$$

In particular, $N_1 \cap N_2$ is of finite index in G. Since $H_1 \cap H_2$ is contained in $N_1 \cap N_2$, we conclude $[G : H_1 \cap H_2] \leq [G : N_1 \cap N_2] < \infty$.

□

2 Exercise I.14 Let G be a finite group and let N be a normal subgroup such that N and G/N have relatively prime orders.

(a) Let H be a subgroup of G having the same order as G/N. Prove that $G = HN$.

Proof. Since $H, N \leq HN \leq G$, then $|H|$ and $|N|$ divide $|HN|$, which divides $|G|$. Moreover, the least common multiple of $|H|$ and $|N|$, which is $|H||N|$ since they are coprime, must divide $|HN|$. Then

$$|H||N| \leq |HN| \leq |G| = |G/N||N| = |H||N|,$$

which implies $|HN| = |G|$. Since all orders are finite, we conclude that $HN = G$.

□

(b) Let g be an automorphism of G. Prove that $g(N) = N$.

Proof. By the diamond isomorphism theorem,

$$g(N)/(g(N) \cap N) \cong g(N)N/N \leq G/N.$$

In particular, $|g(N)N/N|$ divides $|G/N|$. Additionally,

$$|N| = |g(N)| = |g(N)N/N||g(N) \cap N|,$$

which means that $|g(N)N/N|$ also divides $|N|$. Since $|N|$ and $|G/N|$ are coprime, we must have $|g(N)N/N| = 1$. Hence, $g(N)N = N$, implying that $g(N) = N$.

□

3 Exercise I.15 Let G be a finite group operating on a finite set S with $\#(S) \geq 2$. Assume that there is only one orbit. Prove that there exists an element $x \in G$ which has no fixed point, i.e. $xs \neq s$ for all $s \in S$.

Proof. For all $s \in S$, we have $G \cdot s = S$. Then

$$|S| = |G \cdot s| = |G : G_s| = \frac{|G|}{|G_s|},$$

so

$$|G| = |G| \sum_{s \in S} \frac{1}{|S|} = \sum_{s \in S} |G_s|.$$

Let $C = \{(x, s) \in G \times S \mid xs = s\}$. Then, applying Exercise I.17 (the next problem) both ways, we obtain

$$\sum_{x \in G} |S^x| = |C| = \sum_{s \in S} |G_s|,$$

where S^x is the subset of S fixed by g. So, we have found $|G| = \sum_{x \in G} S^x$. Since $S^e = S$, then $|S^e| = |S| \geq 2$. We deduce that there is some $x \in G$ with $|S^x| = 0$, i.e., x has no fixed points in S.

$\square$

4 Exercise I.17 Let X, Y be finite sets and let C be a subset of $X \times Y$. For $x \in X$ let $\varphi(x) =$ number of elements $y \in Y$ such that $(x, y) \in C$. Verify that

$$\#(C) = \sum_{x \in X} \varphi(x).$$

Proof. For $x \in X$, let C_x be the subset of C with the first component equal to x. Then, C can be written as the disjoint union $C = \bigsqcup_{x \in X} C_x$. By construction, $|C_x| = \varphi(x)$, so

$$|C| = \left| \bigsqcup_{x \in X} C_x \right| = \sum_{x \in X} |C_x| = \sum_{x \in X} \varphi(x).$$

□