1 Let $\{B_n\}$ be a nested sequence of closed balls in a complete normed space X, where

$$B_n = \bar{B}_{r_n}(x_n), \quad \text{with} \quad r_n \geq r > 0 \quad \text{for all} \quad n \in \mathbb{N}.$$

(a) Is it true that $\bigcap_{n=1}^{\infty} B_n \neq \varnothing$?

Yes.

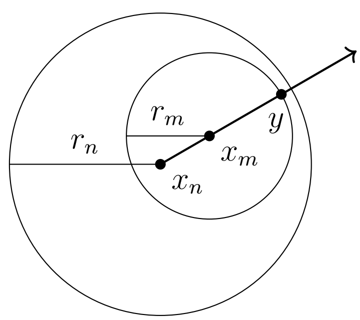
Proof. Define the radius $R = \inf r_n \geq r > 0$. Since the balls are nested, $\{r_n\}$ is a decreasing sequence converging to R.

We will prove that the sequence of center points $\{x_n\}$ is Cauchy.

Let $\varepsilon > 0$ be given and choose $N \in \mathbb{N}$ such that $r_N < R + \varepsilon$. Suppose $m \geq n \geq N$. If $x_n = x_m$, then trivially $\|x_n - x_m\| = 0 < \varepsilon$. Otherwise, if $x_n \neq x_m$, consider the ray in X starting from x_n in the direction of x_m. Let $y \in X$ be the point where this ray meets the boundary of B_m:

$$y = x_n + \frac{r_m}{\|x_m - x_n\|}(x_m - x_n).$$

Then x_n, x_m, y are all collinear and contained in B_n.



It follows that

$$\|x_n - x_m\| = \|x_n - y\| - \|y - x_m\| = \|x_n - y\| - r_m \leq r_n - r_m < \varepsilon.$$

As $\{x_n\}$ is a Cauchy sequence in the complete space X, we know that it converges to some point $x \in X$. For each $n \in \mathbb{N}$, all the points x_k for $k \geq n$ are contained in the closed ball B_n. In particular, B_n is sequentially closed so $x_k \to x$ implies $x \in B_n$. Hence, $x \in \bigcap_{n=1}^{\infty} B_n$. □

(b) Is it true $B \subseteq \bigcap_{n=1}^{\infty} B_n$ for some closed ball B with radius r?

Yes.

Proof. Let $y \in \bar{B}_r(x)$, where $x = \lim_{n\to\infty} x_n \in X$ as in the previous proof. For $n \in \mathbb{N}$ define the point $y_n = x_n + (y - x)$. Then $x_n \to x$ implies $y_n \to y$, since

$$\|y_n - y\| = \|x_n + (y - x) - y\| = \|x_n - x\| \xrightarrow{n\to\infty} 0.$$

On the other hand,

$$\|y_n - x_n\| = \|y - x\| \leq r \leq r_n,$$

which means $y_n \in B_n$, so $y \in B_k$ for all $k \leq n$. In other words, $y_k \in B_n$ for all $k \geq n$. Therefore, $y_k \to y$ in the closed ball B_n, so in fact $y \in B_n$. Hence, $\bar{B}_r(x) \subseteq \bigcap_{n=1}^{\infty} B_n$. □

2 Construct a Lebesgue-measurable set $A \subseteq [0,1]$ such that $m(A) = 1$ and A is of Baire first category in $[0,1]$.

For any $0 < \alpha \leq 1$, we can construct a "fat Cantor set" by removing the middle $\alpha/3$ of each interval (instead of the usual middle $1/3$). The resulting set $C_\alpha \subseteq [0,1]$ is homeomorphic to the usual cantor set by appropriate stretching and shrinking of the interval. In particular, the set is nowhere dense and has measure

$$m(C_\alpha) = 1 - \sum_{n=0}^{\infty} 2^n \left(\frac{\alpha}{3}\right)^{n+1} = 1 - \frac{\alpha}{3} \sum_{n=0}^{\infty} \left(\frac{2\alpha}{3}\right)^n = 1 - \frac{\alpha}{3} \cdot \frac{1}{1 - \frac{2\alpha}{3}} = \frac{3 - 3\alpha}{3 - 2\alpha}.$$

We now define the Baire first category set $A = \bigcup_{k=1}^{\infty} C_{1/k}$. Since $[0,1] \supseteq A \supseteq C_{1/k}$, we have

$$1 \geq m(A) \geq m(C_{1/k}) = \frac{3k - 3}{3k - 2}.$$

Taking the limit as $k \to \infty$, we obtain $m(A) = 1$.

3 Let $f : (0,1) \to \mathbb{R}$ be continuous. Prove that if $\lim_{n\to\infty} f(\frac{x}{n}) = 0$ for all $x \in (0,1)$, then $\lim_{x\to 0} f(x) = 0$.

4 Let $f \in C^\infty(\mathbb{R}, \mathbb{R})$ be such that for every $x \in \mathbb{R}$, there exists $N_x \in \mathbb{N}$ such that $f^{(n)}(x) = 0$ for all $n \geq N_x$. Prove that f is a polynomial on $\mathbb{R}$.