1
(a) Show that every totally separated space has a continuous injection into $\{0,1\}^J$ for some index set J.

Proof. Let X be a totally separated space. For every pair of distinct points $x, y \in X$ there is a clopen subset $A \subseteq X$ separating x and y, i.e., $x \in A$ and $y \in A^c$. Let J be a separating collection of clopen subsets of X, i.e., every distinct pair of points in X is separated by some clopen set in J. (We could take J to be the collection of all clopen subsets of X, but this is not necessary.)

For each $A \in J$, consider the indicator/characteristic function $\chi_A : X \to \{0,1\}$ defined by $\chi_A(x) = 1$ if and only if $x \in A$. We check that χ_A is continuous; there are four preimages:

$$\chi_A^{-1}(\varnothing) = \varnothing, \quad \chi_A^{-1}(\{0\}) = A^c, \quad \chi_A^{-1}(\{1\}) = A, \quad \chi_A^{-1}(\{0,1\}) = X.$$

Since A is clopen, all four sets are open in X, so in fact χ_A is continuous.

By the universal property of the product, there is a unique continuous map $f : X \to \{0,1\}^J$ such that $\pi_A \circ f = \chi_A$ for all $A \in J$. By the assumption on J, given a pair of distinct points $x, y \in X$, there is a clopen set $A \in J$ separating x and y. Then $\chi_A(x) = 1$ and $\chi_A(y) = 0$, implying that $f(x) \neq f(y)$. Hence, f is injective. □

(b) Show that every second-countable totally separated space has a continuous injection into $\{0,1\}^{\mathbb{N}}$.

Proof. Let X be a totally separated space with countable basis $\mathcal{B}$. Define the set:

$$\mathcal{S} = \{(U,V) \in \mathcal{B}^2 : U \subseteq A \subseteq V^c \text{ for some clopen } A \subseteq X\}.$$

In other words, $\mathcal{S}$ is the pairs of basis elements which are separated by some clopen set. Note that $\mathcal{S}$ is countable, so we can enumerate its elements by $\mathcal{S} = \{(U_k, V_k)\}_{k\in\mathbb{N}}$. For each $k \in \mathbb{N}$, choose a representative clopen set $A_k \subseteq X$ separating U_k and V_k.

Then $J = \{A_k\}_{k\in\mathbb{N}}$ is a countable collection of clopen subsets of X. In order to apply part (a), we must show that J separates points in X. Given distinct points $x, y \in X$, there is a clopen set $A \subseteq X$ separating x and y. Since A and A^c are open neighborhoods of x and y, respectively, there are basis elements $U, V \in \mathcal{B}$ such that $x \in U \subseteq A$ and $y \in V \subseteq A^c$. In particular, A is a clopen set separating U and V, implying $(U, V) \in \mathcal{S}$. By construction, there is some $A_k \in J$ separating U_k and V_k. Hence, J is a countable collection of clopen sets which separates points in X.

By part (a), there is a continuous injection $X \to \{0,1\}^J$. Since J is countable, there is a natural homeomorphism $\{0,1\}^J \xrightarrow{\sim} \{0,1\}^{\mathbb{N}}$ by identifying the indexing elements $A_k \in J$ and $k \in \mathbb{N}$. Composing, we obtain a continuous injection $X \to \{0,1\}^{\mathbb{N}}$. □

(c) Show that $\{0,1\}^{\mathbb{N}}$ is homeomorphic to the *middle-thirds Cantor set*: the set of numbers in $[0,1]$ which have a base 3 expansion consisting only of 0's and 2's, with the subspace topology inherited from $\mathbb{R}$.

Proof. Let $C \subseteq [0,1]$ be the middle-thirds Cantor set. Define the map $f : \{0,1\}^{\mathbb{N}} \to C$ by

$$(a_n)_{n\in\mathbb{N}} \longmapsto \sum_{n\in\mathbb{N}} \frac{2a_n}{3^n}.$$

In other words, f maps a sequence of 0's and 1's to the corresponding base 3 decimal expansions of 0's and 2's. It is clear that f is bijective since elements of C are uniquely determined by their base 3 expansion and all base 3 expansions are described by a sequence of 0's and 1's.

The balls $B_{2/3^k}(x)$—for $x \in \mathbb{R}$, $k \in \mathbb{N}$—generate the usual topology on $\mathbb{R}$. Therefore, it suffices to check the continuity of f on these balls. Fix a point $x \in C$ and define $a = f^{-1}(x)$, then we have the decimal expansion

$$x = \sum_{n\in\mathbb{N}} \frac{2a_n}{3^n}.$$

Fix some $k \in \mathbb{N}$ and consider the ball

$$B_{2/3^k}(x) = \{y \in C : |x-y| < 2/3^k\}.$$

Given $y \in C$ define $b = f^{-1}(y)$, then

$$|x-y| = \sum_{n\in\mathbb{N}} \frac{2|a_n - b_n|}{3^n}.$$

For $m \in \mathbb{N}$, we have

$$\sum_{n\geq m} \frac{2|a_n - b_n|}{3^n} \leq \sum_{n\geq m} \frac{2}{3^n} = \frac{1}{3^{m-1}}.$$

From this, we deduce that $|x-y| < 2/3^k$ if and only if $a_n = b_n$ for all $n \leq k$. In other words,

$$f^{-1}(B_{2/3^k}(x)) = \bigcap_{n=1}^{k} \pi_n^{-1}(a_n),$$

where $\pi_n : \{0,1\}^{\mathbb{N}} \to \{0,1\}$ is the projection map to the nth coordinate. As the finite intersection of subbasis elements, $f^{-1}(B_{2/3^k}(x))$ is open in $\{0,1\}^{\mathbb{N}}$.

The preimages $\pi_k^{-1}(t)$—for $t \in \{0,1\}$, $k \in \mathbb{N}$—generate the product topology on $\{0,1\}^{\mathbb{N}}$. Therefore, it suffices to check the continuity of f^{-1} on these sets. Fix values $t \in \{0,1\}$ and $k \in \mathbb{N}$, and consider the subbasis set

$$U = \pi_k^{-1}(t) = \{a \in \{0,1\}^{\mathbb{N}} : a_k = t\}.$$

Let $A \subseteq \{0,1\}^{\mathbb{N}}$ be a set containing a choice of representative $a \in \{0,1\}^{\mathbb{N}}$ for each combination of $a_1, \ldots, a_{k-1}$ and $a_k = t$. (Note that $|A| = 2^{k-1} < \infty$.) Then we can write

$$U = \bigcup_{a \in A} \bigcap_{n=1}^{k} \pi_n^{-1}(a_n) = \bigcup_{a \in A} f^{-1}(B_{2/3^k}(a)),$$

so

$$f(U) = \bigcup_{a \in A} B_{2/3^k}(a).$$

As the union of finitely many open balls, $f(U)$ is open is C. We conclude that f is an open mapping, hence a homeomorphism. $\square$

2 Let X be a locally compact Hausdorff space.

(a) Show that if K is compact in X and U is an open set containing K, then there is a function $f : X \to [0,1]$ which is supported on U and such that $f(K) = 1$.

Hint: First find a larger compact set whose interior contains K.

Lemma 1. Open subspaces of X are locally compact.

Proof. Let $V \subseteq X$ be an open subset and $x \in V$. (The result is trivial if V is empty.) Since X is locally compact, x has a compact neighborhood $L \subseteq X$. Since X is Hausdorff, L is closed in X, so $E = (X \setminus V) \cap L$ is also closed. Then E is compact as a closed subset of the compact set L.

(It is easy to show that disjoint compact subsets of a Hausdorff space are separated by disjoint open sets. Moreover, the condition that two sets A, B are separated is equivalent to the existence of an open set W such that $A \subseteq W$ and $\overline{W} \subseteq B^c$.)

Since X is Hausdorff, there is an closed set $F \subseteq X$ such that $x \in \operatorname{int} F$ and $F \subseteq X \setminus E$. Note

$$X \setminus E = (X \setminus (X \setminus V)) \cup (X \setminus L) = V \cup (X \setminus L),$$

so

$$F \cap L \subseteq (X \setminus E) \cap L = V.$$

Moreover,

$$x \in \operatorname{int} F \cap \operatorname{int} L = \operatorname{int}(F \cap L),$$

so $F \cap L$ is a compact neighborhood of x contained in V. □

We now prove the main result.

Proof. Applying Lemma 1 to each $x \in K \subseteq U$, there is a compact set $L_x \subseteq X$ such that $x \in \operatorname{int} L_x$ and $L_x \subseteq U$. Then the collection of interiors $\{\operatorname{int} L_x\}_{x \in K}$ is an open cover of the compact set K, so there is a finite subcover $\{\operatorname{int} L_{x_i}\}_{i=1}^n$. Then the union $L = \bigcup_{i=1}^n L_{x_i}$ is a compact subset of U with

$$K \subseteq \bigcup_{i=1}^{n} \operatorname{int} L_{x_i} \subseteq \operatorname{int} L.$$

Now L is compact and Hausdorff—therefore normal. By Urysohn's lemma, there is a continuous function $f : L \to [0,1]$ such that $f|_K = 1$ and $f|_{L \setminus \operatorname{int} L} = 0$. We can extend f to a function $X \to [0,1]$ by defining $f|_{X \setminus L} = 0$.

By construction, the support of f is contained in $\operatorname{int} L \subseteq U$. We have that $f|_L$ is continuous, but we must check that f is continuous on all of X. Given an open set $U \subseteq [0,1]$ there are two cases we must consider: $0 \in U$ and $0 \notin U$.

If $0 \notin U$, then $f^{-1}(U) \subseteq \operatorname{int} L$. Since $f|_{\operatorname{int} L}$ is continuous, $f^{-1}(U)$ is an open subset of the open subspace $\operatorname{int} L \subseteq X$ and—therefore—also an open subset of X.

If $0 \in U$, then we write

$$f^{-1}(U) = f|_L^{-1}(U) \cup (X \setminus L).$$

Since $f|_L$ is continuous, $f|_L^{-1}(U)$ is open in the subspace $L \subseteq X$, i.e., there is an open set $V \subseteq X$ such that $f|_L^{-1}(U) = V \cap L$. Then we have

$$f^{-1}(U) = (V \cap L) \cup (X \setminus L) = V \cup (X \setminus L).$$

Since L is compact and X is Hausdorff, L is closed in X. Therefore $X \setminus L$ is open in X, implying $f^{-1}(U)$ is also open in X. Hence f is continuous. □

(b) Define the subspaces $C_c(X) \subseteq C_0(X) \subseteq C_B(X)$ as follows:

- A function is in $C_c(X)$ if it is *compactly supported*, i.e. it is zero outside a compact set.
- A function $f \in C_0(X)$ if it "vanishes at infinity", i.e. for every $\varepsilon > 0$, $\{x \in X : |f(x)| \geq \varepsilon\}$ is compact.

Show that $C_0(X)$ is the closure of $C_c(X)$ in the sup norm topology.

Proof. Let $f \in C_0(X)$. For $\varepsilon > 0$ consider the compactly supported supremum norm ball

$$B_\varepsilon(f) = \{g \in C_0(X) : \|f - g\|_\infty < \varepsilon\}.$$

We want to show that $B_\varepsilon(f) \cap C_c(X) \neq \varnothing$.

Define $\delta = \varepsilon/2$ and the function $g : X \to \mathbb{R}$ by

$$g(x) = \max\{0, f(x) - \delta\} + \min\{0, f(x) + \delta\}.$$

The operations preserve continuity and boundedness, so $g \in C_B(X)$. Moreover, this definition gives us $\|f - g\|_\infty \leq \delta < \varepsilon$, implying $g \in B_\varepsilon(f)$.

Since $f \in C_0(X)$, the set $K = \{x \in X : |f(x)| \geq \delta\}$ is compact. For every $x \in X \setminus K$, we have $|f(x)| < \delta$. Note that $f(x) \leq 0$ implies $f(x) > -\delta$, and $f(x) \geq 0$ implies $f(x) < \delta$. In either case, $g(x) = 0$. Hence, the support of g is contained in K, so in fact $g \in C_c(X)$. □

3 A set $A \subset X$ is a *retract* if there is a continuous map $X \to A$ which is the identity on A.

- Convince yourself that $\{0, 1\}$ is not a retract of $[0, 1]$.
- Convince yourself that the two obvious circles are (separately) retracts of the torus.

(a) Show that if A is a retract of a Hausdorff space X, then A is closed in X.

Proof. Let $r : X \to A$ be a retraction map, i.e., r is continuous and $r|_A = \mathrm{id}_A$. We will show that A^c is open in X. Given a point $x \in A^c$, we know $x \neq r(x)$. Since X is Hausdorff, there are disjoint open sets $U, V \subseteq X$ such that $x \in U$ and $r(x) \in V$. Then $V \cap A$ is open in the subspace $A \subseteq X$. Since r is continuous, the preimage $r^{-1}(V \cap A)$ open in X. Then the intersection $W = U \cap r^{-1}(V \cap A)$ is an open neighborhood of x.

Since r restricts to the identity on A, we know that

$$r^{-1}(V \cap A) \cap A = V \cap A.$$

Since U is disjoint from V, it is also disjoint from $V \cap A$. Therefore,

$$W \cap A = U \cap r^{-1}(V \cap A) \cap A = U \cap V \cap A = \varnothing,$$

so $W \subseteq A^c$, hence A^c is open in X. $\square$

(b) A space Y is an *absolute retract* if whenever X is a normal space which contains a closed set Y_0 homeomorphic to Y, then X retracts to Y_0. Show that $\mathbb{R}^J$ is an absolute retract, for any index set J.

Proof. Let X be a normal space with a closed subspace Y homeomorphic to $\mathbb{R}^J$ for some index set J. For $j \in J$, let $\pi_j : \mathbb{R}^J \to \mathbb{R}$ be the projection to the jth coordinate. The homeomorphism $f : Y \xrightarrow{\sim} \mathbb{R}^J$ is characterized by its components $f_j = \pi_j \circ f : Y \to \mathbb{R}$.

By Tietze, there is a continuous extension $F_j : X \to \mathbb{R}$ such that $F_j|_Y = f_j$. By the universal property of the topological product, there is a unique continuous map $F : X \to \mathbb{R}^J$ such that $F_j = \pi_j \circ F$. Moreover, this construction gives us $F|_Y = f$.

Define the continuous map $r = f^{-1} \circ F : X \to Y$. Then $r|_Y = f^{-1} \circ F|_Y = \mathrm{id}_Y$, so r describes a retraction of X onto Y. $\square$

4 Show that a metric space X is compact if and only if every continuous function $f : X \to \mathbb{R}$ is bounded.

Proof. If X is compact and $f : X \to \mathbb{R}$ is continuous, then the image $f(X) \subseteq \mathbb{R}$ is compact. In particular, $f(X)$ is a bounded subset of $\mathbb{R}$, implying that f is a bounded function.

If X is not compact, then it is not sequentially compact, i.e., we can find a sequence $(x_n)_{n\in\mathbb{N}}$ of points in X with no convergent subsequence. Without loss of generality, we may assume that the x_n's are distinct (since there must be no infinitely recurring terms, otherwise they would form a trivially convergent subsequence).

In particular, each x_n is not a limit point of the image of the sequence, so we can find a radius $r_n > 0$ such that the ball $B_{r_n}(x_n)$ contains no other points of the sequence. Define the smaller ball $B_n = B_{\varepsilon_n/2}(x_n)$, which is an open neighborhood of x_n. For $n \neq m$, we know that $d(x_n, x_m) < \max\{r_n, r_m\}$, which implies that $B_n \cap B_m = \varnothing$.

For each $n \in \mathbb{N}$, we use Urysohn's lemma to construct a continuous map $f_n : X \to [0, 1]$ such that $f_n(x_n) = 1$ and $f_n|_{B_n^c} = 0$. Define $f = \sum_{n\in\mathbb{N}} n f_n$; since the supports of the f_n's are disjoint, this is sum is well-defined. However, $f(x_n) = n$ for all $n \in \mathbb{N}$, so this function is unbounded. □