1 Let $\mathcal{F}$ be a collection of non-degenerate (not a point) closed balled in $\mathbb{R}^n$. A family $\mathcal{G} \subseteq \mathcal{F}$ of disjoint balls in $\mathcal{F}$ is called maximal if $\mathcal{G}$ consists of disjoint balls in $\mathcal{F}$ such that, for any ball $B \in \mathcal{F}$, there exists another ball $B' \in \mathcal{G}$ such that $B \cap B' \neq \varnothing$.

Prove that any such collection $\mathcal{F}$ contains a maximal family $\mathcal{G}$ of disjoint balls, if

(i) There exists two positive constants $d, D > 0$ such that any ball $B \in \mathcal{F}$ satisfies the lower and upper bounds
$$d \leq \operatorname{diam}(B) \leq D.$$

(ii) $\mathcal{F}$ is any collection of closed non-degenerate balls in $\mathbb{R}^n$.

Proof. Let $\mathcal{F}$ be any collection of closed non-degenerate balls in $\mathbb{R}^n$. Denote the collection of families of disjoint balls in $\mathcal{F}$ by
$$P = \{\mathcal{G} \subseteq \mathcal{F} : B \cap B' = \varnothing \text{ for all } B, B' \in \mathcal{G}\}.$$
We take P with inclusion "$\subseteq$" to be a partially ordered set, and we will apply Zorn's lemma to find a maximal family. Suppose $C \subseteq P$ is a chain, i.e., a totally ordered subset. Define the family
$$\mathcal{U} = \bigcup_{\mathcal{G} \in C} \mathcal{G}.$$
By construction, we have $\mathcal{G} \subseteq \mathcal{U}$ for all $\mathcal{G} \in \mathcal{U}$, i.e., $\mathcal{U}$ is an upper bound for C with respect to inclusion; we must check that $\mathcal{U} \in P$. Since $\mathcal{G} \subseteq \mathcal{F}$ for all $\mathcal{G} \in C$, then we have $\mathcal{U} \subseteq \mathcal{F}$. And for any balls $B, B' \in \mathcal{U}$, we must have $B \in \mathcal{G}$ and $B' \in \mathcal{G}'$ for some $\mathcal{G}, \mathcal{G}' \in C$. Since C is totally ordered, we can assume $\mathcal{G} \subseteq \mathcal{G}'$, so then $B \in \mathcal{G}'$. With $\mathcal{G}' \in P$ and $B, B' \in \mathcal{G}'$, we must have $B \cap B' = \varnothing$. Hence, $\mathcal{U}$ is a family of disjoint balls in $\mathcal{F}$, so indeed $\mathcal{U} \in P$.

As P is a partially ordered set such that every chain has an upper bound in P, Zorn's lemma tells us that there exists some $\mathcal{G} \in P$ which is maximal with respect to inclusion. It remains to prove that $\mathcal{G}$ is maximal in sense of the problem. Suppose $\mathcal{G}$ is not maximal in this sense, i.e., that there is some ball $B \in \mathcal{F}$ which is disjoint from all balls in $\mathcal{G}$. However, we could then take $\mathcal{G}' = \mathcal{G} \cup \{B\}$ to be another family of disjoint balls in $\mathcal{F}$ which strictly contains $\mathcal{G}$. That is, $\mathcal{G}' \in P$ but $\mathcal{G} \subsetneq \mathcal{G}'$, which contradicts the maximality of $\mathcal{G}$ in P with respect to inclusion. Hence, $\mathcal{G}$ is the desired family. $\square$

Is the family $\mathcal{G}$ unique in general?

No.

Consider the collection of closed balls
$$\mathcal{F} = \{\overline{B}_{1/2}(n) \subseteq \mathbb{R} : n \in \mathbb{Z}\}.$$
Then we have maximal families
$$\mathcal{G} = \{\overline{B}_{1/2}(2n) \subseteq \mathbb{R} : n \in \mathbb{Z}\} \quad \text{and} \quad \mathcal{G}' = \{\overline{B}_{1/2}(2n+1) \subseteq \mathbb{R} : n \in \mathbb{Z}\}.$$
It can be seen that these are maximal since each ball centered at an odd integer touches the two adjacent balls centered at even integers—and vice versa.

2 Let λ be Lebesgue measure on $\mathbb{R}$.

(a) Construct a Lebesgue-integrable function $f : \mathbb{R} \to [-\infty, \infty]$ for which there exists a Lebesgue-measurable set $A \subseteq \mathbb{R}$ such that $\lambda(A) > 0$ and for any $x \in A$ the limit

$$\lim_{r\to 0} \frac{1}{\lambda(B_r(x))} \int_{B_r(x)} f(y)\, \mathrm{d}y$$

exists but is different from $f(x)$.

(b) Prove that in fact for any $\varepsilon > 0$ one can reach $\lambda(\mathbb{R} \setminus A) < \varepsilon$ in the first part.

Let $\varepsilon > 0$ be given.

Let $\{q_k\}_{k=1}^{\infty}$ be an enumeration of the rationals. Consider the set

$$U = \bigcup_{k=1}^{\infty} B_{\varepsilon/2^{k+1}}(q_k),$$

which is an open neighborhood of $\mathbb{Q}$ with measure

$$\lambda(U) \le \sum_{k=1}^{\infty} \frac{\varepsilon}{2^k} = \varepsilon.$$

Define the function

$$f = \infty \cdot \chi_U.$$

As U is open, it is Lebesgue measurable, so f is Lebesgue-measurable. Moreover, f is nonnegative, so in fact f is Lebesgue-integrable.

Then the set $A = \mathbb{R} \setminus U$ is Lebesgue-measurable with

$$\lambda(A) = \lambda(\mathbb{R}) - \lambda(U) \ge \infty - \varepsilon = \infty.$$

For all $x \in A$, we have $f(x) = 0$. However, for every $r > 0$ there is some rational $q_k \in B_r(x)$. Then $B_r(x) \cap U$ is an open neighborhood of q_k contained in $B_r(x)$. In particular, we know $\lambda(B_r(x) \cap U) > 0$, so

$$\begin{aligned}
\int_{B_r(x)} f(y)\, \mathrm{d}y &= \int_{B_r(x)\cap U} f(y)\, \mathrm{d}y + \int_{B_r(x)\setminus U} f(y)\, \mathrm{d}y \\
&= \int_{B_r(x)\cap U} \infty\, \mathrm{d}y + \int_{B_r(x)\setminus U} 0\, \mathrm{d}y \\
&= \infty \lambda(B_r(x) \cap U) \\
&= \infty.
\end{aligned}$$

Hence, for all $x \in A$ we have

$$\lim_{r\to 0} \frac{1}{\lambda(B_r(x))} \int_{B_r(x)} f(y)\, \mathrm{d}y = \infty \ne 0 = f(x).$$

3 Let $\alpha \in (0,1)$ and let λ be the Lebesgue measure on $\mathbb{R}$. Construct a Borel set $E \subseteq [-1,1]$ such that
$$\lim_{r\to 0} \frac{\lambda(E \cap [-r,r])}{2r} = \alpha.$$

For each $k \in \mathbb{N}$, will construct I_k to be an interval contained in $[\frac{1}{k+1}, \frac{1}{k})$, such that
$$\lambda(I_k) = \alpha\lambda\left[\tfrac{1}{k+1}, \tfrac{1}{k}\right) = \alpha\left(\tfrac{1}{k} - \tfrac{1}{k+1}\right) = \tfrac{\alpha}{k(k+1)}.$$
Explicitly, we write
$$I_k = \left[\tfrac{1}{k+1}, \tfrac{1+\alpha/k}{k+1}\right] \subseteq \left[\tfrac{1}{k+1}, \tfrac{1}{k}\right),$$
which has the desired measure.

Now define
$$E' = \bigcup_{k=1}^{\infty} I_k \subseteq (0,1).$$
For $r \in (0,1)$, consider the intersection
$$E' \cap [0,r] = \bigcup_{k=1}^{\infty} I_k \cap [0,r] = (I_n \cap [\tfrac{1}{n+1}, r]) \cup \bigcup_{k=n+1}^{\infty} I_k,$$
where $n \in \mathbb{N}$ is such that $r \in [\frac{1}{n+1}, \frac{1}{n})$. We compute the measure of this set using the fact that the I_k's are disjoint and Lebesgue-measurable:
$$\lambda(E' \cap [0,r]) = \lambda(I_n \cap [\tfrac{1}{n+1}, r]) + \sum_{k=n+1}^{\infty} \frac{\alpha}{k(k+1)} = \lambda(I_n \cap [\tfrac{1}{n+1}, r]) + \frac{\alpha}{n+1}.$$
We bound the measure
$$0 \leq \lambda(I_n \cap [\tfrac{1}{n+1}, r]) \leq \lambda(I_n) = \frac{\alpha}{n(n+1)}.$$
Then
$$\frac{\alpha}{r(n+1)} \leq \frac{\lambda(E' \cap [0,r])}{r} \leq \frac{\alpha}{r(n+1)}\left(1 + \tfrac{1}{n}\right).$$
Since $\frac{1}{n+1} \leq r < \frac{1}{n}$, we have
$$\frac{n}{n+1} < \frac{1}{r(n+1)} \leq \frac{n+1}{n+1} = 1.$$
Note that $n \to \infty$ as $r \to 0$, so this gives us
$$\lim_{r\to 0} \frac{1}{r(n+1)} = 1,$$
and therefore
$$\lim_{r\to 0} \frac{\lambda(E' \cap [0,r])}{r} = \alpha.$$
Then the desired subset of $[-1,1]$ is given by
$$E = \pm E' = \{\pm x : x \in E\}.$$

4 Let the function $f : \mathbb{R} \to \mathbb{R}$ be increasing and right-continuous. Define the function $\nu : 2^{\mathbb{R}} \to [0, \infty]$ as follows:

1. For any open set $U = \bigcup_{i=1}^{\infty} (a_i, b_i)$ where (a_i, b_i) are disjoint, set

$$\nu(U) = \sum_{i=1}^{\infty} (f(b_i) - f(a_i)).$$

   If U is a finite union of disjoint open intervals, then ν is defined as a finite sum accordingly.

2. For any $A \subseteq \mathbb{R}$ define

$$\nu(A) = \inf\{\nu(U) : A \subseteq U, U \text{ open}\}.$$

   Prove that ν is a Borel measure on $\mathbb{R}$.