This exam concerns the cubic surface S in $\mathbb{P}^3$ with homogeneous equation

$$f = x^3 - xy^2 + z^3 - zt^2 = 0.$$

You may make any assumptions about the field k as you wish, but please be sure to state your assumptions.

Probably assuming algebraically closed and characteristic zero, i.e., just $\mathbb{C}$.

1 Verify that the cubic surface S is nonsingular, and that it contains the line $\ell = \{x = z = 0\}$.

By the Jacobi criterion, S is singular only where the rank of the following matrix is zero:

$$\begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} & \frac{\partial f}{\partial t} \end{bmatrix} = \begin{bmatrix} 3x^2 - y^2 & -2xy & 3z^2 - t^2 & -2zt \end{bmatrix}.$$

Assuming this matrix is zero, we deduce

$$-2xy = 0, \quad 3x^2 = y^2 \quad \implies \quad x = y = 0,$$

and the same implies $z = t = 0$. However, the homogeneous coordinate with all zeros is not a valid projective point, therefore S is nonsingular at every point.

Note $S = Z(f)$. We evaluate f at an arbitrary point $P = [0 : y : 0 : t]$ in the line ℓ:

$$f(P) = 0^3 - 0y^2 + 0^3 - 0t^2 = 0.$$

That is, $P \in Z(f)$, so in fact $\ell \subseteq S$.

2 Show that S contains 10 other lines ℓ_i, $i = 1, \ldots, 10$, none of which coincides with ℓ and each of which meets ℓ.

Hint: Consider the natural 1-parameter family of projective planes containing ℓ and find a condition on the parameter which guarantees that the intersection with the plane is singular.

A plane in $\mathbb{P}^3$ is given by $Z(g)$ for some homogeneous linear polynomial $g = ax + by + cz + dt$. In order for $\ell \subseteq Z(g)$, we must have $g(P) = 0$ for all $P = [0 : y : 0 : t] \in \ell$, i.e.,

$$0 = g(P) = a \cdot 0 + by + c \cdot 0 + dt = by + dt.$$

In particular,

$$0 = g(0, 1, 0, 0) = b \quad \text{and} \quad 0 = g(0, 0, 0, 1) = d.$$

So we can write $g = ax + cz$ for some $[a : c] \in \mathbb{P}^1$, where distinct choices of $[a : c]$ correspond to distinct planes $Z(g) \subseteq \mathbb{P}^3$ containing the line ℓ.

In the case that $c \neq 0$, we may assume $c = 1$ giving us the following isomorphism:

$$\begin{aligned} \mathbb{P}^2 &\longrightarrow Z(g) \subseteq \mathbb{P}^3 \\ [x : y : t] &\longmapsto [x : y : -ax : t] \end{aligned}$$

In this copy of $\mathbb{P}^2$, the variety corresponding to $Z(f, g) \subseteq \mathbb{P}^3$ is defined by the homogeneous cubic polynomial

$$F(x, y, t) = f(x, y, -ax, t) = x\left((1 - a^3)x^2 - y^2 + at^2\right).$$

The factor of x tells us that $Z(F) \subseteq \mathbb{P}^2$ contains the line $Z(x) \subseteq \mathbb{P}^2$, which corresponds to the line $\ell \subseteq \mathbb{P}^3$.

Let $G = (1 - a^3)x^2 - y^2 + at^2$ be the homogeneous degree 2 factor of F, so that $F = xG$ and $Z(F) = Z(x) \cup Z(G)$. From Homework 2 Problem 2, we know that $Z(G)$ is either a single line, two coincident lines, or a smooth quadratic curve. In particular, we require that $Z(G)$ is singular; by the Jacobi criterion, the following matrix must be rank zero (char $k \neq 2$):

$$\begin{bmatrix} \frac{\partial G}{\partial x} & \frac{\partial G}{\partial y} & \frac{\partial G}{\partial t} \end{bmatrix} = \begin{bmatrix} 2(1 - a^3)x & -2y & 2at \end{bmatrix}.$$

If $x \neq 0$ then we must have $a^3 = 1$, giving us three possible values for a: 1, ζ, and ζ^2, where $\zeta = e^{2\pi i/3}$ (i.e., some primitive third root of unity for k algebraically closed). In any of these cases, we obtain $G = -y^2 + at^2$, which has two distinct homogeneous linear factors, corresponding to the lines $Z(y \pm \sqrt{a}t) \subseteq \mathbb{P}^2$ (for some choice of square root). If $x = 0$ then we must have $t \neq 0$, which implies $a = 0$. In which case, $G = x^2 - y^2$ has homogeneous linear factors corresponding to lines $Z(x \pm y) \subseteq \mathbb{P}^2$.

In the case that $c = 0$, we may assume $a = 1$ giving us the following isomorphism:

$$\begin{aligned} \mathbb{P}^2 &\longrightarrow Z(g) \subseteq \mathbb{P}^3 \\ [y : z : t] &\longmapsto [0 : y : z : t] \end{aligned}$$

In this copy of $\mathbb{P}^2$, the variety corresponding to $Z(f,g) \subseteq \mathbb{P}^3$ is defined by the homogeneous cubic polynomial

$$F(y,z,t) = f(0,y,z,t) = z^3 - zt^2 = z(z+t)(z-t).$$

Similar to the previous case, the factor of z corresponds to the line $Z(z) \subseteq \mathbb{P}^2$ which, in turn, corresponds to the line $\ell \subseteq \mathbb{P}^3$. The other two homogeneous linear factors of F correspond to the lines $Z(z \pm t) \subseteq \mathbb{P}^2$.

Hence, for $g = ax + cz$ we have five possible values for $[a : c] \in \mathbb{P}^1$:

$$[1:0], \quad [0:1], \quad [1:1], \quad [\zeta:1], \quad [\zeta^2:1].$$

Each of these define a plane $Z(g) \subseteq \mathbb{P}^3$ such that the intersection with S is the union of three mutually coincident lines, one of which is ℓ. For distinct values of $[a,c]$ we get distinct planes, so we obtain ten lines in S intersecting ℓ:

$$\begin{aligned}
\ell_1 &= Z(x, z+t), & \ell_2 &= Z(x, z-t), \\
\ell_3 &= Z(z, x+y), & \ell_4 &= Z(z, x-y), \\
\ell_5 &= Z(x+z, y+t), & \ell_6 &= Z(x+z, y-t), \\
\ell_7 &= Z(\xi^2 x+z, y+\xi t), & \ell_8 &= Z(\xi^2 x+z, y-\xi t), \\
\ell_9 &= Z(\xi^4 x+z, y+\xi^2 t), & \ell_{10} &= Z(\xi^4 x+z, y-\xi^2 t),
\end{aligned}$$

where $\xi = \sqrt{\zeta} = e^{\pi i/3}$ is some primitive sixth root of unity.

3 Show that S contains 8 other lines ℓ_j, $j = 11, \ldots, 18$ none of which coincides with a previously constructed line, and each of which meets ℓ_1. (There are also two previously constructed lines which meet ℓ_1.)

We will use $\ell_1 = Z(x, z+t)$ from Problem 2. Again, we consider a plane $Z(g) \subseteq \mathbb{P}^3$ for some homogeneous linear polynomial $g = ax + by + cz + dt$. In order for $\ell_1 \subseteq Z(g)$, we must have $g(P) = 0$ for all $P = [0 : y : z : -z] \in \ell_1$, i.e.,

$$0 = g(P) = a \cdot 0 + by + cz + d(-z) = by + (c - d)z.$$

In particular,

$$0 = g(0, 1, 0, 0) = b \quad \text{and} \quad 0 = g(0, 0, 1, 0) = c - d.$$

So we can write $g = ax + c(z + t)$ for some $[a : c] \in \mathbb{P}^1$. Again, distinct choices of $[a : c]$ give distinct planes $Z(g) \subseteq \mathbb{P}^3$ containing ℓ_1.

In the case that $c \neq 0$, we may assume $c = 1$ giving us the following isomorphism:

$$\begin{aligned} \mathbb{P}^2 &\longrightarrow Z(g) \subseteq \mathbb{P}^3 \\ [x : y : z] &\longmapsto [x : y : z : -ax - z] \end{aligned}$$

In this copy of $\mathbb{P}^2$, the variety corresponding to $Z(f, g) \subseteq \mathbb{P}^3$ is defined by the homogeneous cubic polynomial

$$F(x, y, z) = f(x, y, z, -ax - z) = x(x^2 - y^2 - a^2xz - 2az^2).$$

The factor of x corresponds to the line $Z(x) \subseteq \mathbb{P}^2$ which, in turn, corresponds to the line $\ell_1 \subseteq \mathbb{P}^3$.

As in Problem 2, let $G = x^2 - y^2 - a^2xz - 2az^2$ be the homogeneous degree 2 factor of F, then $Z(G)$ is singular where the following matrix is rank zero:

$$\begin{bmatrix} \frac{\partial G}{\partial x} & \frac{\partial G}{\partial y} & \frac{\partial G}{\partial z} \end{bmatrix} = \begin{bmatrix} 2x - a^2z & -2y & -a^2x - 4az \end{bmatrix}.$$

The third entry tells us that either $a = 0$ or $z = -ax/4$. If $a = 0$ then $G = x^2 - y^2$, giving us the lines $Z(x \pm y) \subseteq \mathbb{P}^2$.

If $z = -ax/4$ then the first entry tells us $(8 + a^3)x = 0$. Then $y = 0$ and $a \neq 0$ implies both x and $z = -az/4$ are nonzero, so in fact $a^3 = -8$. Then we have three possibilities for a: -2, -2ζ, and $-2\zeta^2$, where $\zeta = e^{2\pi i/3}$ as before. Also like before, each choice of a gives a pair of lines $Z(x \pm y - az) \subseteq \mathbb{P}^2$ from the linear factors of G.

Lastly, when $c = 0$ we obtain the plane containing ℓ and ℓ_2, which we have already counted.

Hence, for $g = ax + c(z + t)$ we have four new possible values for $[a : c] \in \mathbb{P}^1$:

$$[0 : 1], \quad [-2 : 1], \quad [-2\zeta : 1], \quad [-2\zeta^2 : 1],$$

giving us eight new lines in S intersecting ℓ_1:

$$\begin{aligned} \ell_{11} &= Z(z + t, x + y), & \ell_{12} &= Z(z + t, x - y), \\ \ell_{13} &= Z(-2x + z + t, x + y + 2z), & \ell_{14} &= Z(-2x + z + t, x - y + 2z), \\ \ell_{15} &= Z(-2\zeta x + z + t, x + y + 2\zeta y), & \ell_{16} &= Z(-2\zeta x + z + t, x - y + 2\zeta y), \\ \ell_{17} &= Z(-2\zeta^2 x + z + t, x + y + 2\zeta^2 y), & \ell_{18} &= Z(-2\zeta^2 x + z + t, x - y + 2\zeta^2 y). \end{aligned}$$

4 Show that S contains 8 more lines ℓ_k, $k = 19, \ldots, 26$ none of which coincides with a previously constructed line, and each of which meets ℓ_{11}. (There are also two previously constructed lines which meet ℓ_{11}.)

We will use $\ell_{11} = Z(z+t, x+y)$ from Problem 3. For $g = ax + by + cx + dt$ we find $a = b$ and $c = d$ so we can write $g = a(x+y) + c(z+t)$.

If $a \neq 0$ we can assume $a = 1$ giving us the following isomorphism:

$$\begin{aligned} \mathbb{P}^2 &\longrightarrow Z(g) \subseteq \mathbb{P}^3 \\ [x : z : t] &\longmapsto [x : -c(z+t) - x : z : t] \end{aligned}$$

In this copy of $\mathbb{P}^2$, the variety corresponding to $Z(f, g) \subseteq \mathbb{P}^3$ is defined by the homogeneous cubic polynomial

$$F(x, z, t) = f(x, -c(z+t) - x, z, t) = (z+t)\big(-2cx^2 - c^2x(z+t) + z^2 - zt\big).$$

The factor of $z + t$ corresponds to the line $Z(z+t) \subseteq \mathbb{P}^2$ which, in turn, corresponds to the line $\ell_{11} \subseteq \mathbb{P}^3$.

Let $G = -2cx^2 - c^2x(z+t) + z^2 - zt$ be the homogeneous degree 2 factor of F, then $Z(G)$ is singular where the following matrix is rank zero:

$$\begin{bmatrix} \frac{\partial G}{\partial x} & \frac{\partial G}{\partial z} & \frac{\partial G}{\partial t} \end{bmatrix} = \begin{bmatrix} -4cx - c^2(z+t) & 2z - c^2x - t & -c^2x - t \end{bmatrix}.$$

The first entry tells us that either $c = 0$ or $4x + c(z+t) = 0$. If $c = 0$ then $G = z^2 - zt$, giving us the lines $Z(z), Z(z-t) \subseteq \mathbb{P}^2$. However, $Z(z)$ corresponds to ℓ_3 which we have already counted.

Otherwise, the second and third entries imply that $z = 0$ so the first entry tells us $4x + ct = 0$. The third entry also tells us $t = -c^2x$, so in fact we have $(4 - c^3)x = 0$. And since $z = 0$ and $c \neq 0$ then x and $t = -c^2x$ are both nonzero, so we deduce that $c^3 = 4$. So we have three possibilities for c: $\sqrt[3]{4}$, $\sqrt[3]{4}\zeta$, and $\sqrt[3]{4}\zeta^2$. Each choice of c gives a pair of lines corresponding to the linear factors of G (I could not find these factors)

Lastly, when $a = 0$ we may assume $c = 1$ giving us the following isomorphism:

$$\begin{aligned} \mathbb{P}^2 &\longrightarrow Z(g) \subseteq \mathbb{P}^3 \\ [x : y : z] &\longmapsto [x : y : z : -z] \end{aligned}$$

In this copy of $\mathbb{P}^2$, the variety corresponding to $Z(f, g) \subseteq \mathbb{P}^3$ is defined by the homogeneous cubic polynomial

$$F(x, y, z) = f(x, y, z, -z) = x(x+y)(x-y).$$

The linear factors correspond to ℓ_1, ℓ_{11}, and ℓ_{12}, respectively.

Hence, for $g = a(x+y) + c(z+t)$ we have four new possible values for $[a : c] \in \mathbb{P}^1$:

$$[1:0], \quad [1:\sqrt[3]{4}], \quad [1:\sqrt[3]{4}\zeta], \quad [1:\sqrt[3]{4}\zeta^2],$$

giving us eight new lines in S intersecting ℓ_{11}:

$$\begin{aligned}
\ell_{19} &= Z(???), & \ell_{20} &= Z(x+y, z-t), \\
\ell_{21} &= Z(x+y+\sqrt[3]{4}(z+t), ???), & \ell_{22} &= Z(x+y+\sqrt[3]{4}(z+t), ???), \\
\ell_{23} &= Z(x+y+\sqrt[3]{4}\zeta(z+t), ???), & \ell_{24} &= Z(x+y+\sqrt[3]{4}\zeta(z+t), ???), \\
\ell_{25} &= Z(x+y+\sqrt[3]{4}\zeta^2(z+t), ???), & \ell_{26} &= Z(x+y+\sqrt[3]{4}\zeta^2(z+t), ???).
\end{aligned}$$

Note: As mentioned above, I seem to have found ℓ_3 again instead of a new line. Just from guessing, I believe the missing line is

$$\ell_{19} = Z(x-y, z-t),$$

but this line does not intersect ℓ_{11} so I am not exactly sure what happened. It's possible I did not choose the correct lines to go off of.

Additionally, I did not find the second equations for $\ell_{21}, \ldots, \ell_{26}$ since G was too hard to factor in this case.

Conclude that S contains at least 27 different lines.

We conclude that S contains at least 27 different lines, namely $\ell, \ell_1, \ldots, \ell_{26}$.