Assignment 5

MATH CS 117 Intro to Real Analysis

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Question 1

(a) Prove that any polynomial function is continuous on $\mathbb{R}$.

(b) Let p and q be polynomial functions. Let $\mathcal{Z} = \{x \in \mathbb{R} : q(x) = 0\}$. Prove that p/q is continuous on $\mathbb{R} \setminus \mathcal{Z}$.

(a)

Let $p : \mathbb{R} \to \mathbb{R}$ be a polynomial function and let $a \in \mathbb{R}$. Since p is a polynomial function, we have that $\lim_{x \to a} p(x) = p(a)$. This is equivalent to p being continuous at a. Thus f is continuous on $\mathbb{R}$.

(b)

Let $a \in \mathbb{R} \setminus \mathcal{Z}$. Since p and q are polynomial functions, we have

$$\lim_{x \to a} p(x) = p(a) \quad \text{and} \quad \lim_{x \to a} q(x) = q(a).$$

And since $q(a) \neq 0$, this gives us

$$\lim_{x \to a} (p/q)(x) = \lim_{x \to a} \frac{p(x)}{q(x)} = \frac{p(a)}{q(a)} = (p/q)(a).$$

Thus p/q is continuous at a, and therefore continuous on $\mathbb{R} \setminus \mathcal{Z}$

Question 2

Let $D \subset \mathbb{R}$. Prove that the set

$$X = \{f : D \to \mathbb{R} : \ f \text{ is continuous on } D\}$$

is a vector space over $\mathbb{R}$. (Define vector addition and scalar multiplication and show that X is closed under these operations. Define the zero vector. You don't need to verify all of the axioms.)

For any $\alpha \in \mathbb{R}$ and $f, g \in X$, we define the function $\alpha f + g$ for all $x \in D$ by

$$(\alpha f + g)(x) = \alpha f(x) + g(x).$$

Now let $a \in D$. We define the constant function $h : D \to \mathbb{R}$ by $h(x) = \alpha$ for all $x \in D$, so h is continuous at a. Since h and f are continuous at a, we have $hf = \alpha f$ continuous at a. And since g is also continuous at a, we have $\alpha f + g$ continuous at a. Thus $\alpha f + g$ is continuous on D, so X is closed under addition and scalar multiplication.

The zero vector is the zero function which maps all $x \in D$ to 0.

Question 3

Let $D \subset \mathbb{R}$. Suppose that $f : D \to \mathbb{R}$ and that there exists a constant $M > 0$ such that $|f(x) - f(y)| \leq M|x - y|$, for all $x, y \in D$. Prove that f is uniformly continuous on D. (Such a function is said to be Lipschitz continuous on D.)

Let $\varepsilon > 0$ be given. Define $\delta = \varepsilon/M$. Note that $\delta > 0$ since $\varepsilon, M > 0$. Now if $x, y \in D$ and $|x - y| < \delta$, then

$$|f(x) - f(y)| \leq M|x - y| < M\delta = \varepsilon.$$

Thus, f is uniform continuous on D.

Question 4

Suppose that $f : \mathbb{R} \to \mathbb{R}$ satisfies $|f(x) - f(y)| \le \frac{1}{2}|x - y|$, for all $x, y \in \mathbb{R}$.

(a) Given an arbitrary point $x_0 \in \mathbb{R}$, define a sequence $\{x_n\}_{n=1}^{\infty}$ recursively by $x_n = f(x_{n-1})$, $n \in \mathbb{N}$. Prove that $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence.

(b) Prove that there is a unique point $x \in \mathbb{R}$ such that $f(x) = x$.

(a)

We first prove by induction on k that for all $k \in \mathbb{N}$, $|x_{k+1} - x_k| \le \dfrac{1}{2^k}|x_1 - x_0|$. For the base case,

$$|x_2 - x_1| = |f(x_1) - f(x_0)| \le \frac{1}{2}|x_1 - x_0|.$$

For the inductive step, we assume that for some $k \in \mathbb{N}$ that

$$|x_{k+1} - x_k| \le \frac{1}{2^k}|x_1 - x_0|.$$

Then

$$|x_{k+2} - x_{k+1}| = |f(x_{k+1} - f(x_k)| \le \frac{1}{2}|x_{k+1} - x_k| \le \frac{1}{2^{k+1}}|x_1 - x_0|,$$

concluding the inductive step. Now let $\varepsilon > 0$ be given. We define $N \in \mathbb{N}$ such that

$$2^{N-1} > \frac{|x_1 - x_0|}{\varepsilon}.$$

So if $m > n \ge N$, then

$$\begin{aligned}
|x_m - x_n| &= |x_m - x_{m-1} + x_{m-1} - x_{m-2} + x_{m-2} - \cdots - x_{n+1} + x_{n+1} - x_n| \\
&\le |x_m - x_{m-1}| + |x_{m-1} - x_{m-2}| + \cdots + |x_{n+1} - x_n| \\
&\le \frac{1}{2^{m-1}}|x_1 - x_0| + \cdots + \frac{1}{2^n}|x_1 - x_0| \\
&= \frac{1}{2^{n-1}}|x_1 - x_0| \left(\frac{1}{2^{m-n}} + \cdots + \frac{1}{2} \right) \\
&\le \frac{1}{2^{n-1}}|x_1 - x_0| \cdot 1 \\
&\le \frac{1}{2^{N-1}}|x_1 - x_0| \\
&\le \frac{\varepsilon}{|x_1 - x_0|}|x_1 - x_0| \\
&= \varepsilon.
\end{aligned}$$

Thus $\{x_n\}_{n=1}^{\infty}$ is Cauchy.

(b)

Since $\{x_n\}_{n=1}^{\infty}$ is Cauchy, it converges to a limit x. Additonally, since $x_{n-1} \to x$ and $x_n = f(x_{n-1})$, then $x_n = f(x_{n-1}) \to f(x)$. So since $x_n \to x$ and $x_n \to f(x)$, we have $f(x) = x$.

Question 5

Let Λ be an arbitrary nonempty set. Let $\{G_\lambda\}_{\lambda \in \Lambda}$ be a family of open sets in $\mathbb{R}$ indexed by Λ. Prove that $\bigcup_{\lambda \in \Lambda} G_\lambda$ is an open set.

Let $x \in \bigcup_{\lambda \in \Lambda} G_\lambda$. Then there is some $\lambda_x \in \Lambda$ such that $x \in G_{\lambda_x}$. Since G_{λ_x} is an open set, there is some neighborhood U of x such that $U \subseteq G_{\lambda_x} \subseteq \bigcup_{\lambda \in \Lambda} G_\lambda$. Therefore, $\bigcup_{\lambda \in \Lambda} G_\lambda$ contains a neighborhood of each of its points, so it is an open set.

Question 6

Show that $\emptyset$ and $\mathbb{R}$ are open and closed.

Let $x \in \mathbb{R}$, then for any neighborhood U of x, $U \subseteq \mathbb{R}$. So $\mathbb{R}$ is open, and since $\mathbb{R}' = \mathbb{R} \subseteq \mathbb{R}$, it is also closed. Since $\emptyset = \mathbb{R} \setminus \mathbb{R}$ and $\mathbb{R}$ is both open and closed, then $\emptyset$ is also both open and closed.

Question 7

(a) Let $D \subset \mathbb{R}$. Suppose that $f : D \to \mathbb{R}$ and $g : D \to \mathbb{R}$ are bounded and uniformly continuous on D. Prove that $f \cdot g$ is uniformly continuous on D.

(b) Give a counterexample showing that boundedness is necessary in part (**??**).

(a)

Since f and g are bounded, then for all $x \in D$, $|f(x)| < M_1$ and $|g(x)| < M_2$ for some $M_1, M_2 > 0$. Define $M = \max\{M_1, M_2\}$. Let $\varepsilon > 0$ be given. Since f and g are uniformly continuous, choose $\delta_1, \delta_2 > 0$ such that for all $x, y \in D$,

$$|x - y| < \delta_1 \implies |f(x) - f(y)| < \frac{\varepsilon}{2M},$$

$$|x - y| < \delta_2 \implies |g(x) - g(y)| < \frac{\varepsilon}{2M}.$$

Then define $\delta = \min\{\delta_1, \delta_2\}$. So if $x, y \in D$ and $|x - y| < \delta$, then

$$\begin{aligned}
|(f \cdot g)(x) - (f \cdot g)(y)| &= |f(x)g(x) - f(y)g(y)| \\
&= |f(x)(g(x) - g(y) + g(y)) + (f(x) - f(y) - f(x))g(y)| \\
&= |f(x)(g(x) - g(y)) + f(x)g(y) + (f(x) - f(y))g(y) - f(x)g(y)| \\
&= |f(x)(g(x) - g(y)) + (f(x) - f(y))g(y)| \\
&\leq |f(x)||g(x) - g(y)| + |f(x) - f(y)||g(y)| \\
&< M \cdot \frac{\varepsilon}{2M} + \frac{\varepsilon}{2M} \cdot M \\
&= \varepsilon.
\end{aligned}$$

Thus, $f \cdot g$ is uniformly continuous.

(b)

Let $f, g : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = g(x) = x$. Both are unbounded and uniformly continuous since for any $\varepsilon > 0$, we have that $|x-y| < \varepsilon$ implies $|f(x)-f(y)| = |g(x)-g(y)| = |x - y| < \varepsilon$ for all $x, y \in \mathbb{R}$. However, $f \cdot g$ is not uniformly continuous since $(f \cdot g)(x) = x^2$ which is not uniformly continuous.

Question 8

Prove that $K \subset \mathbb{R}$ is compact if and only if every infinite subset in K has an accumulation point in K.

Suppose K is compact, and therefore closed and bounded. Let $E \subseteq K$ be an infinite subset. Since K is bounded, E is also bounded. So since E is a bounded infinite set, it has an accumulation point, which is in K since K is closed. Therefore every infinite subset in K has an accumulation point in K.

Suppose that every infinite subset of K has an accumulation point in K. Let $x \in K'$. Pick a sequence $\{x_n\}_{n=1}^{\infty}$ in $K \setminus \{x\}$ which converges to x. Then the set $\{x_n : n \in \mathbb{N}\} \subseteq K$ has its only accumulation point at x. And since it is an infinite subset of K, then it has an accumulation point in K. Therefore $x \in K$, and K is closed. To prove K is bounded, suppose to the contrary that K is unbounded. Then we pick a point $x_0 \in K$ and define a sequence $\{x_n\}_{n=1}^{\infty}$ in K by $|x_n| > |x_{n-1}| + 1$. The set $\{x_n : n \in \mathbb{N}\} \subseteq K$ is infinite, since all terms of the sequence are distinct. But it has no accumulation points since for each x_n, the neighborhood $(x_n - \frac{1}{2}, x_n + \frac{1}{2})$ contains no points in the set $\{x_n : n \in \mathbb{N}\}$. However, since every infinite subset of K has an accumulation point, this is a contradiction, so K must be bounded. Since K is closed and bounded, it is compact.