1 Let the function $f : [a, b] \to \mathbb{R}$ be differentiable at every point $x \in [a, b]$. Is f necessarily absolutely continuous on $[a, b]$?

No.

Consider the function

$$f(x) = \begin{cases} x^2 \cos(2\pi/x^2) & \text{if } x \in (0, 1], \\ 0 & \text{if } x = 0, \end{cases}$$

on the interval $[0, 1]$. It is immediate that f is differentiable on $(0, 1]$, but we must check that is is also differentiable at 0 (from the right). For $h > 0$, we estimate

$$\left| \frac{f(0 + h) - f(0)}{h} \right| = \frac{|h^2 \cos(2\pi/h^2) - 0|}{h} \leq \frac{h^2}{h} = h.$$

Hence, f is differentiable at 0 (from the right) and

$$f'(0) = \lim_{h \to 0^+} \frac{f(0 + h) - f(0)}{h} = 0.$$

However, f is not absolutely continuous on $[0, 1]$. To see this, consider the collection of disjoint open intervals

$$(a_k, b_k) = \left(\tfrac{1}{\sqrt{k+1/4}}, \tfrac{1}{\sqrt{k}} \right) \subseteq [0, 1], \qquad k \in \mathbb{N}.$$

Since these are disjoint intervals in $[0, 1]$, we have

$$\sum_{k=1}^{\infty} (b_k - a_k) = \sum_{k=1}^{\infty} \lambda(a_k, b_k) = \lambda \left(\bigcup_{k=1}^{\infty} (a_k, b_k) \right) \leq \lambda[0, 1] = 1.$$

Since this is a positive summation, the tail must tend to zero, i.e.,

$$\lim_{N \to \infty} \sum_{k=N}^{\infty} (b_k - a_k) = 0.$$

This means that for every $\delta > 0$ there exists an $N \in \mathbb{N}$ such that $\{(a_k, b_k)\}_{k=N}^{\infty}$ is a collection of disjoint open intervals in $[0, 1]$ such that

$$\sum_{k=N}^{\infty} (b_k - a_k) < \delta.$$

However, we also have

$$\sum_{k=N}^{\infty} |f(b_k) - f(a_k)| = \sum_{k=N}^{\infty} \frac{1}{k} = +\infty.$$

Therefore, f cannot be absolutely continuous.

2 Let $A \subseteq [0,1]$ be a null set (a set that has zero Lebesgue measure). Construct an increasing and absolutely continuous function $f[0,1] \to \mathbb{R}$ such that

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = +\infty$$

for all $x \in A$.

Choose open sets $B_n \subseteq [0,1]$ containing A such that $\lambda(B_n) < 1/2^n$ and $B_n \supseteq B_{n+1}$.

Define

$$f(x) = \int_0^x \sum_{n=1}^{\infty} \chi_{B_n} \,\mathrm{d}\lambda.$$

We check that f is finite for all $x \in [0,1]$, i.e., that $\sum_{n=1}^{\infty} \chi_{B_n}$ is summable:

$$\begin{aligned} f(x) &= \int_0^x \sum_{n=1}^{\infty} \chi_{B_n} \,\mathrm{d}\lambda \\ &= \sum_{n=1}^{\infty} \int_0^x \chi_{B_n} \,\mathrm{d}\lambda \\ &\le \sum_{n=1}^{\infty} \int_0^1 \chi_{B_n} \,\mathrm{d}\lambda \\ &= \sum_{n=1}^{\infty} \lambda(B_n) \\ &\le \sum_{n=1}^{\infty} \frac{1}{2^n} \\ &= 1. \end{aligned}$$

As the integral of a nonnegative summable function, f is increasing and absolutely continuous.

We check that the derivative of f at each point $x \in A$ is unbounded. Assume $h > 0$, then

$$\begin{aligned} \frac{f(x+h)-f(x)}{h} &= \frac{1}{h}\left(\int_0^{x+h} \sum_{n=1}^{\infty} \chi_{B_n} \,\mathrm{d}\lambda - \int_0^x \sum_{n=1}^{\infty} \chi_{B_n} \,\mathrm{d}\lambda\right) \\ &= \frac{1}{h}\int_x^{x+h} \sum_{n=1}^{\infty} \chi_{B_n} \,\mathrm{d}\lambda \\ &= \frac{1}{h}\sum_{n=1}^{\infty} \int_x^{x+h} \chi_{B_n} \,\mathrm{d}\lambda \\ &= \frac{1}{h}\sum_{n=1}^{\infty} \lambda(B_n \cap (x, x+h)). \end{aligned}$$

Since B_n is an open neighborhood of x, there is an open ball of radius $r_n > 0$ around x contained in B_n. For $0 < h < r_1$, define

$$N_h = \sup\{n \in \mathbb{N} : (x, x+h) \subseteq B_n\},$$

then $N_h \to \infty$ as $h \to 0^+$. Then

$$\begin{aligned}
\frac{f(x+h) - f(x)}{h} &\geq \frac{1}{h} \sum_{n=1}^{N_h} \lambda(B_n \cap (x, x+h)) \\
&= \frac{1}{h} \sum_{n=1}^{N_h} \lambda(x, x+h) \\
&= \frac{1}{h} \sum_{n=1}^{N_h} h \\
&= \sum_{n=1}^{N_h} 1 \\
&= N_h.
\end{aligned}$$

The same holds for $h < 0$, so we conclude that

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \geq \lim_{n \to \infty} N_h = +\infty.$$

3 Let $f, g \in L^1(\mathbb{R})$. Prove that the function

$$\varphi(t) = \int_{\mathbb{R}} |f(x) + tg(x)| \, \mathrm{d}x$$

is well-defined in $\mathbb{R}$, finite for all $t \in \mathbb{R}$, and is differentiable a.e. in $\mathbb{R}$.

Proof. Since f and g are summable, so is $f + tg$ for all $t \in \mathbb{R}$, with

$$\int_{\mathbb{R}} |f(x) + tg(x)| \, \mathrm{d}x \leq \int_{\mathbb{R}} |f(x)| \, \mathrm{d}x + |t| \int_{\mathbb{R}} |g(x)| \, \mathrm{d}x < \infty.$$

That is, φ is well-defined and finite in $\mathbb{R}$.

Moreover, φ is Lipchitz, with constant $M = \int_{\mathbb{R}} |g(x)| \, \mathrm{d}x < \infty$. For $x, y \in \mathbb{R}$, we have

$$\begin{aligned}
|\varphi(x) - \varphi(y)| &= \left| \int_{\mathbb{R}} |f(t) + xg(t)| \, \mathrm{d}t - \int_{\mathbb{R}} |f(t) + yg(t)| \, \mathrm{d}t \right| \\
&\leq \int_{\mathbb{R}} \big| |f(t) + xg(t)| - |f(t) + yg(t)| \big| \, \mathrm{d}t \\
&\leq \int_{\mathbb{R}} \big| f(t) + xg(t) - f(t) - yg(t) \big| \, \mathrm{d}t \\
&= \int_{\mathbb{R}} |(x - y)g(t)| \, \mathrm{d}t \\
&= M|x - y|.
\end{aligned}$$

In particular, for all $n \in \mathbb{N}$, we have $\varphi \in \mathrm{Lip}[-n, n] \subseteq \mathrm{BV}[-n, n]$, so φ is differentiable almost everywhere in $[-n, n]$. Say φ is not differentiable only in the set $A_n \subseteq [-n, n]$, then $\lambda(A_n) = 0$. It follows that $\bigcup_{n=1}^{\infty} A_n$ is the set of points in $\bigcup_{n=1}^{\infty} [-n, n] = \mathbb{R}$ at which φ is not differentiable. But

$$\lambda\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} \lambda(A_n) = \sum_{n=1}^{\infty} 0 = 0,$$

so indeed φ is differentiable almost everywhere in $\mathbb{R}$. $\square$

4 Suppose $f : [0,1] \to \mathbb{R}$ is continuous and absolutely continuous in $[a,1]$ for all $a \in (0,1)$. Is f necessarily absolutely continuous on $[0,1]$?

No.

Consider the same function as in Problem 1:

$$f(x) = \begin{cases} x^2\cos(2\pi/x^2) & \text{if } x \in (0,1], \\ 0 & \text{if } x = 0. \end{cases}$$

For all $a \in (0,1)$ we have $f \in C^1[a,1] \subseteq \mathrm{AC}[a,1]$, but $f \notin \mathrm{AC}[0,1]$. (Note that while f is differentiable on all of $[0,1]$, its derivative is not continuous at 0.)

If f is in addition of bounded variation on $[0,1]$ is it necessarily absolutely continuous on $[0,1]$?

Yes.

Proof. We prove the contrapositive. Suppose $f \in \mathrm{AC}[a,1]$ for all $a \in (0,1)$ but $f \notin \mathrm{AC}[0,1]$; we claim that $f \notin \mathrm{BV}[0,1]$. To prove this, we will construct a family of partitions of $[0,1]$ over which the variation of f is unbounded.

Since f is not absolutely continuous on $[0,1]$, by definition there is some $M > 0$ such that for all $\delta > 0$ there is a collection of disjoint open intervals $\{(x_i, y_i)\}_{i=1}^{\infty}$ in $[0,1]$ with

$$\sum_{i=1}^{\infty}(y_i - x_i) < \delta \quad \text{and} \quad \sum_{i=1}^{\infty}|f(y_i) - f(x_i)| \geq M.$$

We will use this fact to inductively construct a countable collection of disjoint open intervals in $[0,1]$ with unbounded variation. We will take finite subcollections of these intervals and use their endpoints to define our partitions of $[0,1]$.

Choose any $a_0 \in (0,1)$, e.g., $a_0 = 1/2$. Take $\delta > 0$ for the absolute continuity of f on $[a_0,1]$ with $\varepsilon = M/2$. As $f \notin \mathrm{AC}[0,1]$, we can find a collection of disjoint open intervals $\{(x_i^{(0)}, y_i^{(0)})\}_{i=1}^{\infty}$ in $[0,1]$ with

$$\sum_{i=1}^{\infty}(y_i^{(0)} - x_i^{(0)}) < \delta \quad \text{and} \quad \sum_{i=1}^{\infty}|f(y_i^{(0)}) - f(x_i^{(0)})| \geq M.$$

Since the intervals $(x_i^{(0)}, y_i^{(0)})$ are disjoint, most are either entirely contained in $[0, a_0)$ or $(a_0, 1]$; at most one interval is cut in half by a_0. If this occurs—if $a_0 \in (x_i^{(0)}, y_i^{(0)})$—then we can replace this interval with two new intervals: $(x_i^{(0)}, a_0)$ and $(a_0, y_i^{(0)})$. Then the total measure of the collection of intervals is still less than δ and

$$|f(y_i^{(0)}) - f(a_0)| + |f(a_0) - f(x_i^{(0)})| \geq |f(y_i^{(0)}) - f(x_i^{(0)})|,$$

so the sum of differences in f is still at least M. Without loss of generality, we may assume that each interval $(x_i^{(0)}, y_i^{(0)})$ is entirely contained in either $[0, a_0)$ or $(a_0, 1]$. In particular, we have

$$\sum_j (y_j^{(0)} - x_j^{(0)}) < \delta,$$

where the sum is taken over all j such that $(x_j^{(0)}, y_j^{(0)})$ is contained in $(a_0, 1]$. By the absolute continuity of f on $[a_0, 1]$, we have

$$\sum_j |f(y_j^{(0)}) - f(x_j^{(0)})| < \frac{M}{2},$$

where the sum is taken over the same j as the previous sum. It follows that

$$\sum_k |f(y_k^{(0)}) - f(x_k^{(0)})| > \frac{M}{2},$$

where the sum is taken over all k such that $(x_k^{(0)}, y_k^{(0)})$ is contained in $[0, a_0)$. Since the inequality is strict, we can choose finitely many terms in k which still sum to more than $M/2$. After reindexing, we can assume

$$\sum_{k=1}^{n_0} |f(y_k^{(0)}) - f(x_k^{(0)})| > \frac{M}{2}$$

and $x_k^{(0)} < x_{k+1}^{(0)}$ for $k = 1, \ldots, n_0 - 1$.

We hope that $x_1^{(0)} > 0$, but if this is not the case, we can make a slight modification to the value of $x_1^{(0)}$. If $x_1^{(0)} = 0$ then, because f is continuous, there is some $c \in (0, y_1^{(0)})$ such that $f(c)$ is very close to $f(0)$. We replace $x_1^{(0)}$ with c close enough so that

$$\sum_{k=1}^{n_0} |f(y_k^{(0)}) - f(x_k^{(0)})| > \frac{M}{2}$$

and

$$0 < x_1^{(0)} < y_1^{(0)} < x_2^{(0)} < \cdots < x_{n_0}^{(0)} < y_{n_0}^{(0)} \leq a_0.$$

Lastly, define $a_1 = x_1^{(0)} < a_0$.

For each $\ell \geq 1$, repeat the above process with a_ℓ to find points

$$0 < a_{\ell+1} = x_1^{(\ell)} < y_1^{(\ell)} < x_2^{(\ell)} < \cdots < x_{n_0}^{(\ell)} < y_{n_0}^{(\ell)} \leq a_\ell$$

such that

$$\sum_{k=1}^{n_\ell} |f(y_k^{(\ell)}) - f(x_k^{(\ell)})| > \frac{M}{2}.$$

For $N \in \mathbb{N}$, we build the following partition by enumerating the endpoint of the intervals we have constructed for $\ell = 1, \ldots, N$:

$$P_N = \{t_i\}_{i=1}^n = \bigcup_{\ell=0}^{N} \bigcup_{k=1}^{n_\ell} \{x_k^{(\ell)}, y_k^{(\ell)}\}$$

such that $t_i < t_{i+1}$ for $i = 1, \ldots, n$. This forms a partition of the interval $[a_N, 1]$ with variational sum

$$\begin{aligned} V(f, P_N) &= \sum_{i=1}^{n} |f(t_i) - f(t_{i+1})| \\ &\geq \sum_{\ell=0}^{N} \sum_{k=1}^{n_\ell} |f(y_k^{(\ell)}) - f(x_k^{(\ell)})| \\ &\geq \sum_{\ell=0}^{N} \frac{M}{2} \\ &= \frac{(N+1)M}{2}. \end{aligned}$$

Then we can bound below the total variation of f by

$$V_0^1(f) \geq \sup_{N \in \mathbb{N}} V(f, P_N) = +\infty.$$

In other words, f is not of bounded variation on $[0, 1]$. $\square$