I worked with Joseph Sullivan and Gahl Shemy.

1 Exercise 1.1.9 Explicitly exhibit enough parameterizations to cover $S^1 \times S^1 \subseteq \mathbb{R}^4$.

The projection $(x, y) \mapsto x$ is a parameterization on both the open sets $\{(x, y) \in S^1 : y > 0\}$ and $\{(x, y) \in S^1 : y < 0\}$. Similarly, the projection $(x, y) \mapsto y$ is a parameterization on both the open sets $\{(x, y) \in S^1 : x > 0\}$ and $\{(x, y) \in S^1 : x < 0\}$. These four parameterizations cover S^1, denote them by $\varphi_i : U_i \to (-1, 1) \subseteq \mathbb{R}^1$, for $i = 1, 2, 3, 4$. Then we get a cover of $S^1 \times S^1$ by sixteen parameterizations

$$\varphi_i \times \varphi_j : U_i \times U_j \to (-1, 1) \times (-1, 1) \subseteq \mathbb{R}^2.$$

Indeed, this are smooth by Homework 2 Exercise 1.1.14, and have smooth inverses given by products of inverses, hence diffeomorphisms.

2 Exercise 1.1.15 Show that the projection map $X \times Y \to X$, carrying (x, y) to x, is smooth.

Proof. Denote the projection map by $f : X \times Y \to X$. Suppose $X \subseteq \mathbb{R}^N$ and $Y \subseteq \mathbb{R}^M$, then define $F : \mathbb{R}^{N+M} \to \mathbb{R}^N$ by

$$F(x_1, \ldots, x_N, y_1, \ldots, y_M) = (x_1, \ldots, x_N).$$

This map is linear, hence smooth. Moreover, $\mathbb{R}^N$ is an open neighborhood of X and $F|_X = f$. In other words, F is a smooth (global) extension of f, so f is smooth. □

3 Exercise 1.1.16 The *diagonal* Δ in $X \times X$ is the set of points of the form (x, x). Show that Δ is diffeomorphic to X, so Δ is a manifold if X is.

Proof. Let $f : X \to \Delta$ be the diagonal map $x \mapsto (x, x)$. Suppose $X \subseteq \mathbb{R}^N$ and define the diagonal map $F : \mathbb{R}^N \to \mathbb{R}^{2N}$ by

$$F(x_1, \ldots, x_N) = (x_1, \ldots, x_N, x_1, \ldots, x_N).$$

This map is linear, hence smooth. Moreover, it is a smooth extension of f, so f is smooth.

Note that $\Delta \subseteq \mathbb{R}^N$ Let $G : \mathbb{R}^{2N} \to \mathbb{R}^N$ be the projection map

$$G(x_1, \ldots, x_N, x'_1, \ldots, x'_N) = (x_1, \ldots, x_N).$$

This map is linear, hence smooth. Therefore, the restriction $g = G|_\Delta : \Delta \to X$ is smooth.

Lastly,

$$f(g(x, x)) = f(x) = (x, x) \quad \text{and} \quad g(f(x)) = g(x, x) = x,$$

so f and g are smooth inverses, hence diffeomorphisms.

Supposing X is a manifold, let $U \subseteq X$ be open and $\varphi : U \to V \subseteq \mathbb{R}^k$ be a smooth chart. Then $f(U) \subseteq \Delta$ is open and $\varphi \circ g : f(U) \to V$ is a smooth chart. An open cover of X is sent to an open cover of Δ under f, so the manifold structure of X corresponds to a manifold structure of Δ under f and g. □

4 Exercise 1.1.17 The *graph* of a map $f : X \to Y$ is the subset of $X \times Y$ defined by

$$\operatorname{graph}(f) = \{(x, f(x)) : x \in X\}.$$

Define $F : X \to \operatorname{graph}(f)$ by $F(x) = (x, f(x))$. Show that if f is smooth, F is a diffeomorphism; thus $\operatorname{graph}(f)$ is a manifold if X is.

Proof. Notice that F can be written as the following composition:

$$\begin{array}{ccccc} X & \longrightarrow & \Delta & \xrightarrow{\operatorname{id}_X \times f} & \operatorname{graph}(f) \\ x & \longmapsto & (x, x) & \longmapsto & (x, f(x)) \end{array}$$

The first map (the diagonal map) is smooth by the previous problem (Exercise 1.1.16), and the latter is smooth as the product of smooth maps. The projection map $X \times Y \to X$ restricted to $\operatorname{graph}(f)$ is inverse to F and smooth by Problem 2 (Exercise 1.1.15), hence F is a diffeomorphism.

Similar to the previous problem, when X is a manifold, its charts can be turned into charts on $\operatorname{graph}(f)$, giving $\operatorname{graph}(f)$ a manifold structure. $\square$

5 Exercise 1.1.18
(a) An extremely useful function $f : \mathbb{R}^1 \to \mathbb{R}^1$ is

$$f(x) = \begin{cases} e^{-1/x^2} & x > 0, \\ 0 & x \leq 0. \end{cases}$$

Prove that f is smooth.

Proof. We need only check that f is smooth at 0. From Homework 1 Problem 6, we know that the function

$$\hat{f}(x) = \begin{cases} e^{-1/x^2} & x \neq 0, \\ 0 & x = 0, \end{cases}$$

is smooth. Moreover, $\hat{f}^{(n)}(0) = 0$ for all n, which implies that derivatives of all orders of f at 0 from the right are all zero. Since the the derivatives of all orders of f at 0 from the left are also all zero, we conclude that f is smooth at 0 with $f^{(n)}(0) = 0$ for all n. $\square$

(b) Show that $g(x) = f(x-a)f(b-x)$ is a smooth function, positive on (a, b), and zero elsewhere. Then

$$h(x) = \frac{\int_{-\infty}^{x} g \, \mathrm{d}x}{\int_{-\infty}^{\infty} g \, \mathrm{d}x}$$

is a smooth function satisfying $h(x) = 0$ for $x \leq a$, $h(x) = 1$ for $x \geq b$ and $0 < h(x) < 1$ for $x \in (a, b)$.

Proof. We can write g as the composition

$$
\begin{tikzcd}
\mathbb{R}^1 \arrow[r] & \mathbb{R}^2 \arrow[r] & \mathbb{R}^2 \arrow[r] & \mathbb{R}^2 \arrow[r] & \mathbb{R}^1 \\
x \arrow[r, mapsto] & (x,x) & (x,y) \arrow[r, mapsto] & (f(x),f(y)) & \\
& (x,y) \arrow[r, mapsto] & (x-a,b-y) & (x,y) \arrow[r, mapsto] & xy
\end{tikzcd}
$$

The first map is the diagonal, the second map is linear, the third map is $f \times f$, and the last map is multiplication. We know all of these to be smooth, so g is smooth as their composition.

If $x \leq a$, then $x - a \leq 0$ so $f(x-a) = 0$. If $x \geq b$, then $b - x \geq 0$ so $f(b-x) = 0$. In either case, $g(x) = 0$.

If $a < x < b$ then $x - a > 0$ and $b - x > 0$, so $f(x-a)$ and $f(b-x)$ are positive. In which case $g(x)$ is positive.

By the fundamental theorem of calculus, $G(x) = \int_{-\infty}^{x} g \,\mathrm{d}y$ is differentiable, with $G' = g$. Since g is smooth, this implies G is smooth. Multiplying by the constant $1/\int_{-\infty}^{\infty} g \,\mathrm{d}y$ gives us the smooth function h.

Given $x \leq a$, we know $g(y) = 0$ for all $y \leq x$, so

$$h(x) = \frac{\int_{-\infty}^{x} g \,\mathrm{d}y}{\int_{-\infty}^{\infty} g \,\mathrm{d}y} = \frac{\int_{-\infty}^{x} 0 \,\mathrm{d}y}{\int_{-\infty}^{\infty} g \,\mathrm{d}y} = \frac{0}{\int_{-\infty}^{\infty} g \,\mathrm{d}y} = 0.$$

Given $x \geq b$, we know $g(y) = 0$ for all $y \geq x$, so

$$h(x) = \frac{\int_{-\infty}^{x} g \,\mathrm{d}y}{\int_{-\infty}^{\infty} g \,\mathrm{d}y} = \frac{\int_{-\infty}^{b} g \,\mathrm{d}y + \int_{b}^{x} g \,\mathrm{d}y}{\int_{-\infty}^{b} g \,\mathrm{d}y + \int_{b}^{\infty} g \,\mathrm{d}y} = \frac{\int_{-\infty}^{b} g \,\mathrm{d}y + 0}{\int_{-\infty}^{b} g \,\mathrm{d}y + 0} = 1.$$

Since $g(x) > 0$ for all $x \in (a,b)$, we know that G is strictly increasing on (a,b). The same is true of h, as a positive scalar multiple of G, so

$$0 = h(a) < h(x) < h(b) = 1$$

for all $x \in (a,b)$. $\square$

(c) Now construct a smooth function on $\mathbb{R}^k$ that equals 1 on the ball of radius a, zero outside the ball of radius b, and is strictly between 0 and 1 at intermediate points.

Proof. For $x < y$, let $h_x^y : \mathbb{R}^1 \to [0,1]$ be constructed as h in part (b) with $a = x$ and $b = y$.

Let $F : \mathbb{R}^k \to \mathbb{R}^1$ be given by the following composition:

$$
\begin{tikzcd}
\mathbb{R}^k \arrow[r, "\|-\|^2"] & \mathbb{R}^1 \arrow[r, "h_{a^2}^{b^2}"] & \mathbb{R}^1 \arrow[r] & \mathbb{R}^1 \\
(x_1, \ldots, x_k) \arrow[r, mapsto] & \sum x_i^2 & x \arrow[r, mapsto] & 1 - x \\
& x \arrow[r, mapsto] & h_{a^2}^{b^2}(x). &
\end{tikzcd}
$$

The first map is a polynomial, the second map is $h_{a^2}^{b^2}$, and the last map is linear. We know that each of these maps is smooth, therefore F is smooth as their composition.

For $x \in \mathbb{R}^k$ with $\|x\| \leq a$, we have $\|x\|^2 \leq a^2$. This implies $h_{a^2}^{b^2}(\|x\|^2) = 0$, so $F(x) = 1$.

For $x \in \mathbb{R}^k$ with $\|x\| \geq b$, we have $\|x\|^2 \geq b^2$. This implies $h_{a^2}^{b^2}(\|x\|^2) = 1$, so $F(x) = 0$.

For $a < \|x\| < b^2$, we have $a^2 < \|x\|^2 < b^2$. This implies $0 < h_{a^2}^{b^2}(\|x\|^2) < 1$, so $0 < F(x) < 1$.

□

6 Exercise 1.2.2 If U is an open subset of the manifold X, check that

$$T_x(U) = T_x(X) \quad \text{for} \quad x \in U.$$

Proof. Let $V \subseteq X$ be an open neighborhood of x with smooth parameterization $\varphi : W \to V$ with $W \subseteq \mathbb{R}^k$ open and $\varphi(0) = x$. Then $U \cap V$ is an open neighborhood of x contained in U, and the restriction of φ to the open set $\varphi^{-1}(U \cap V) \subseteq W$ is a smooth parameterization of $U \cap V$. Hence, we have the tangent spaces

$$T_x(X) = \mathrm{d}\varphi_0(\mathbb{R}^k) = \mathrm{d}(\varphi|_{\varphi^{-1}(U\cap V)})_0(\mathbb{R}^k) = T_x(U).$$

□

7 Exercise 1.2.3 Let V be a vector subspace of $\mathbb{R}^N$. Show that $T_x(V) = V$ if $x \in V$.

Proof. Suppose $v_1, \ldots, v_k \in V$ form a basis for V. The linear map $L : \mathbb{R}^k \to V$ defined by $e_i \to v_i$ is an isomorphism of vector spaces. Fixing $x \in V$, the map $\varphi : \mathbb{R}^k \to V$ sending $y \mapsto \mathrm{L}(y) + x$ is a smooth parameterization with $\varphi(0) = x$. Then the tangent space is given by

$$T_x(V) = \mathrm{d}\varphi_0(\mathbb{R}^k) = \mathrm{d}(L + x)_0(\mathbb{R}^k) = \mathrm{d}L_0(\mathbb{R}^k) = L(\mathbb{R}^k) = V.$$

□

8 Exercise 1.2.4 Suppose that $f : X \to Y$ is a diffeomorphism, and prove that at each x its derivative $\mathrm{d}f_x$ is an isomorphism of tangent spaces.

Proof. Let $g : Y \to X$ be a smooth inverse of f, i.e., $f \circ g = \mathrm{id}_Y$ and $g \circ f = \mathrm{id}_X$. Their derivatives give linear maps $\mathrm{d}f_x : T_x(X) \to T_{f(x)}(Y)$ and $\mathrm{d}g_{f(x)} : T_{f(x)}(Y) \to T_x(X)$. The chain rule lets us compute

$$\mathrm{d}(f \circ g)_{f(x)} = \mathrm{d}f_{g(f(x))} \circ \mathrm{d}g_{f(x)} = \mathrm{d}f_x \circ \mathrm{d}g_{f(x)}$$

and

$$\mathrm{d}(g \circ f)_x = \mathrm{d}g_{f(x)} \circ \mathrm{d}f_x.$$

On the other hand,

$$\mathrm{d}f \circ g_{f(x)} = \mathrm{did}_{y\,f(x)} = \mathrm{id}_Y$$

and

$$\mathrm{d}g \circ f_x = \mathrm{did}_{X\,x} = \mathrm{id}_X\,.$$

Hence $\mathrm{d}f_x$ and $\mathrm{d}g_{f(x)}$ are linear inverses. In particular, $\mathrm{d}f_x$ is an isomorphism. □

9 Exercise 1.2.9

(a) Show that for any manifolds X and Y,

$$T_{(x,y)}(X \times Y) = T_x(X) \times T_y(Y).$$

Proof. Suppose we have smooth local parameterizations $\varphi : U \to X$ and $\psi : V \to Y$ at the points $x \in X$ and $y \in Y$, respectively, with $U \subseteq \mathbb{R}^k$ and $V \subseteq \mathbb{R}^\ell$ open sets. Additionally, assume $\varphi(0) = x$ and $\psi(0) = y$. Then their product $\varphi \times \psi : U \times V \to X \times Y$ is a smooth local parameterization at $(x, y) \in X \times Y$, with $(\varphi \times \psi)(0) = (\varphi(0), \psi(0)) = (x, y)$. Thus, we compute the tangent space

$$T_{(x,y)}(X \times Y) = \mathrm{d}(\varphi \times \psi)_{(0,0)}(\mathbb{R}^{k+\ell}) = \mathrm{d}\varphi_0(\mathbb{R}^k) \times \mathrm{d}\psi_0(\mathbb{R}^\ell) = T_x(X) \times T_y(Y).$$

□

(b) Let $f : X \times Y \to X$ be the projection map $(x, y) \mapsto x$. Show that

$$\mathrm{d}f_{(x,y)} : T_x(X) \times T_y(Y) \to T_x(X)$$

is the analogous projection $(v, w) \mapsto v$.

Proof. Let $f : X \times Y \to X$ be the projection map. Let $\varphi : U \to X$ and $\psi : V \to Y$ be smooth local parameterizations with $\varphi(0) = x$ and $\psi(0) = y$, then there is a commutative diagram

$$\begin{tikzcd} X \times Y \arrow[r, "f"] & X \\ U \times V \arrow[u, "\varphi\times\psi"] \arrow[r, dashed, "h"'] & U \arrow[u, "\varphi"'] \end{tikzcd}$$

Note that $h : U \times V \to U$ is simply a restriction of the the linear projection $L : \mathbb{R}^k \times \mathbb{R}^\ell \to \mathbb{R}^k$. Taking derivatives gives us

$$\begin{tikzcd} T_{(x,y)}(X \times Y) \arrow[r, "\mathrm{d}f_{(x,y)}"] & T_x(X) \\ \mathbb{R}^k \times \mathbb{R}^\ell \arrow[u, "\mathrm{d}(\varphi\times\psi)_0"] \arrow[r, "\mathrm{d}h_0 = L"'] & \mathbb{R}^k \arrow[u, "\mathrm{d}\varphi_0"'] \end{tikzcd}$$

For $(v, w) \in T_x(X) \times T_y(Y) = T_{(x,y)}(X \times Y)$, we compute

$$\begin{aligned}
\mathrm{d}f_{(x,y)}(v, w) &= (\mathrm{d}\varphi_0 \circ \mathrm{d}h_0 \circ \mathrm{d}(\varphi \times \psi)_0^{-1})(v, w) \\
&= (\mathrm{d}\varphi_0 \circ L \circ (\mathrm{d}\varphi_0^{-1} \times \mathrm{d}\psi_0^{-1}))(v, w) \\
&= \mathrm{d}\varphi_0(L(\mathrm{d}\varphi_0^{-1}(v), \mathrm{d}\psi_0^{-1}(w))) \\
&= \mathrm{d}\varphi_0(\mathrm{d}\varphi_0^{-1}(v)) \\
&= v.
\end{aligned}$$

□

(c) Fixing any $y \in Y$ gives an injection mapping $f : X \to X \times Y$ by $f(x) = (x, y)$. Show that $\mathrm{d}f_x(v) = (v, 0)$.

Proof. □

(d) Let $f : X \to X'$, $g : Y \to Y'$ be any smooth maps. Prove that

$$\mathrm{d}(f \times g)_{(x,y)} = \mathrm{d}f_x \times \mathrm{d}g_y.$$

Proof. Consider parameterizations described by the following (commutative) diagrams:

$$\begin{tikzcd}
X \arrow[r, "f"] & X' \\
U \arrow[u, "\varphi"] \arrow[r, dashed, "h"'] & U' \arrow[u, "\varphi'"']
\end{tikzcd}
\qquad
\begin{tikzcd}
Y \arrow[r, "g"] & Y' \\
V \arrow[u, "\psi"] \arrow[r, dashed, "k"'] & V' \arrow[u, "\psi'"']
\end{tikzcd}$$

Taking derivatives gives us the following commutative diagrams:

$$\begin{tikzcd}
T_x(X) \arrow[r, "\mathrm{d}f_x"] & T_{f(x)}(X') \\
\mathbb{R}^n \arrow[u, "\mathrm{d}\varphi_0"] \arrow[r, "\mathrm{d}h_0"'] & \mathbb{R}^{n'} \arrow[u, "\mathrm{d}\varphi'_0"']
\end{tikzcd}
\qquad
\begin{tikzcd}
T_y(Y) \arrow[r, "\mathrm{d}g_y"] & T_{g(y)}(Y') \\
\mathbb{R}^m \arrow[u, "\mathrm{d}\psi_0"] \arrow[r, "\mathrm{d}k_0"'] & \mathbb{R}^{m'} \arrow[u, "\mathrm{d}\psi'_0"']
\end{tikzcd}$$

Taking the products of these pairs of diagrams gives us parameterizations

$$\begin{tikzcd}
X \times Y \arrow[r, "f \times g"] & X' \times Y' \\
U \times V \arrow[u, "\varphi \times \psi"] \arrow[r, "h \times k"'] & U' \times V' \arrow[u, "\varphi' \times \psi'"']
\end{tikzcd}
\qquad
\begin{tikzcd}
T_x(X) \times T_y(Y) \arrow[r, "\mathrm{d}f_x \times \mathrm{d}g_y"] & T_{f(x)}(X') \times T_{g(y)}(Y') \\
\mathbb{R}^n \times \mathbb{R}^m \arrow[u, "\mathrm{d}\varphi_0 \times \mathrm{d}\psi_0"] \arrow[r, "\mathrm{d}h_0 \times \mathrm{d}k_0"'] & \mathbb{R}^{n'} \times \mathbb{R}^{m'} \arrow[u, "\mathrm{d}\varphi'_0 \times \mathrm{d}\psi'_0"']
\end{tikzcd}$$

Applying the definition of the derivative for maps of manifolds and the product result for the usual derivative (i.e., from analysis), we obtain

$$\begin{aligned}
\mathrm{d}(f \times g)_{(x,y)} &= \mathrm{d}(\varphi' \times \psi')_0 \circ \mathrm{d}(h \times k)_0 \circ \mathrm{d}(\varphi' \times \psi')_0^{-1} \\
&= (\mathrm{d}\varphi'_0 \times \mathrm{d}\psi'_0) \circ (\mathrm{d}h_0 \times \mathrm{d}k_0) \circ (\mathrm{d}\varphi_0^{-1} \times \mathrm{d}\psi_0^{-1}) \\
&= (\mathrm{d}\varphi'_0 \circ \mathrm{d}h_0 \circ \mathrm{d}\varphi_0^{-1}) \times (\mathrm{d}\psi'_0 \circ \mathrm{d}k_0 \circ \mathrm{d}\psi_0^{-1}) \\
&= \mathrm{d}f_x \times \mathrm{d}g_y.
\end{aligned}$$

□

10 Exercise 1.2.11
(a) Suppose that $f : X \to Y$ is a smooth map, and let $F : X \to X \times Y$ be $F(x) = (x, f(x))$. Show that

$$\mathrm{d}F_x(v) = (v, \mathrm{d}f_x(v)).$$

Proof. Let $\Delta : X \to \Delta_X \subseteq X \times X$ be the diagonal map. Then (similar to Exercise 1.1.17 above) F can be written as the composition

$$X \xrightarrow{\Delta} \Delta_X \xrightarrow{\mathrm{id}_X \times f} \mathrm{graph}(f) \subseteq X \times Y.$$

Then

$$\mathrm{d}F_x = \mathrm{d}((\mathrm{id}_X \times f) \circ \Delta)_x = \mathrm{d}(\mathrm{id}_X \times f)_{(x,x)} \circ \mathrm{d}\Delta_x = (\mathrm{id}_{T_x(X)} \times \mathrm{d}f_x) \circ \Delta,$$

so

$$\mathrm{d}F_x(v) = (\mathrm{id}_{T_x(X)}\, \mathrm{d}f_x)(\Delta(v)) = (\mathrm{id}_{T_x(X)}\, \mathrm{d}f_x)(v, v) = (v, \mathrm{d}f_x(v)).$$

□

(b) Prove that the tangent space to graph(f) at the point $(x, f(x))$ is the graph of $\mathrm{d}f_x : T_x(X) \to T_{f(x)}(Y)$.

Proof. Since $F : X \to \mathrm{graph}(f)$ is a diffeomorphism, the derivative is an isomorphism. In particular, it is surjective, and by part (a) its image is precisely the graph of $\mathrm{d}f_x$. □

11 Exercise 1.3.2
(a) If X is compact and Y connected, show every submersion $f : X \to Y$ is surjective.

(b) Show that there exist no submersions of compact manifolds into Euclidean spaces.

12 Exercise 1.3.3 Show that the curve $t \mapsto (t, t^2 . t^3)$ embeds $\mathbb{R}^1$ into $\mathbb{R}^3$. Find two independent functions that globally define the image. Are your functions independent on all of $\mathbb{R}^3$, or just on an open neighborhood of the image?

13 Exercise 1.3.4 Prove the following extension of Converse 2. Suppose that $Z \subseteq X \subseteq Y$ are manifolds, and $z \in Z$. Then there exist independent functions $g_1, \ldots, g_\ell$ on a neighborhood W of z in Y such that

$$Z \cap W = \{y \in W : g_1(y) = \cdots = g_\ell(y) = 0\}$$

and

$$X \cap W = \{y \in W : g_1(y) = \cdots = g_m(y) = 0\},$$

where $\ell = m$ is the codimension of Z in X.

14 Exercise 1.3.6 More generally, let p be any homogeneous degree m polynomial in k variables. Prove that the set of points x, where $p(x) = a$, is a $(k-1)$-dimensional submanifold of $\mathbb{R}^k$, provided that $a \neq 0$. Show that the manifolds obtained with $a > 0$ are all diffeomorphic, as are those with $a < 0$. [Hint: Use Euler's identity for homogeneous polynomials

$$\sum_{i=1}^{k} x \frac{\partial p}{\partial x_i} = m \cdot p$$

to prove that 0 is the only critical value of p.]