Ecercise 4.12 (Affine conics). An irreducible quadric curve in $\mathbb{A}^2$ is also called an *affine conic*. Show that every affine conic over a field of characteristic not equal to 2 is isomorphic to exactly one of the varieties $X_1 = V(x_2 - x_1^2)$ and $X_2 = V(x_1 x_2 - 1)$, with an isomorphism given by a linear transformation followed by a translation.

Proof. Suppose $X \subseteq \mathbb{A}^2$ is an affine conic, then Remark 2.38 tells us that its ideal is given by $I(X) = \langle f \rangle$ for some irreducible polynomial $f \in K[x, y]$ with $\deg f = 2$. Then we can write

$$f = a_1 x^2 + a_2 xy + a_3 y^2 + a_4 x + a_5 y + a_6$$

for some $a_1, \ldots, a_6 \in K$. At least one of a_1, a_2, a_3 must be nonzero. If $a_1 = a_3 = 0$, then we can divide f by the nonzero coefficient of xy, and the zero locus remains the same. We can then assume

$$f = xy + a_4 x + a_5 y + a_6 = (x + a_5)(y + a_4) - c,$$

where $c = a_4 a_5 - a_6 \in K$. We now define a map $V(f) \to V(xy - c)$, componentwise by $g = (x + a_5, y + a_4)$. We verify the codomain of g. If $(x, y) \in V(f)$, then evaluating $xy - c$ at $g(x, y)$ results in $f(x, y) = 0$, so $f(x, y) \in V(xy - c)$. The inverse of g is given by $g^{-1} = (x - a_5, y - a_4)$. By Proposition 4.7, both g and g^{-1} are morphisms, hence isomorphisms.

We now have $X \cong V(xy - c)$. Notice that $c \neq 0$, otherwise $X \cong V(xy) = V(x) \cup V(y)$ is a decomposition into proper closed subsets, but X is irreducible. Then $c \in K^\times$, so $V(xy - c) = V(c^{-1}xy - 1)$. We now define a morphism $V(c^{-1}xy - 1) \to V(xy - 1)$ by $h = (x, c^{-1}y)$, which has the inverse morphism $h^{-1} = (x, cy)$. Then the composition $h \circ g : V(f) \to V(xy - 1)$ is an isomorphism of affine varieties, and can be written as

$$h \circ g = (x + a_5, c^{-1}(y + a_4)) = (x, c^{-1}y) + (a_5, c^{-1}a_4),$$

which proves the case when $a_1 = a_3 = 0$.

Now suppose one of a_1 or a_3 is nonzero. We will assume a_1 is nonzero, as the argument for a_3 nonzero is symmetric, by swapping x and y. Then $a_1 \in K^\times$, and we can divide f by a_1 and the zero locus remains the same. We can then assume

$$f = x^2 + a_2 xy + a_3 y^2 + a_4 x + a_5 y + a_6.$$

Since K is algebraically closed, then

$$x^2 + a_2 x + a_3 = (x - a)(x - b),$$

for some $a, b \in K$, so

$$f = (x - ay)(x - by) + d(x - cy) + e,$$

for some $c, d, e \in K$. If $a = b = c$, then

$$f = (x - ay)^2 + d(x - ay) + e,$$

which can be factored in the same way as $x^2 + dx + e$ in $K[x]$, because K is algebraically closed. Since f is irreducible, this cannot be the case.

If $a = b \neq c$, then

$$f = (x - ay)^2 + d(x - cy) + e,$$

and we define a morphism $V(f) \to V(x^2 + dy + c)$ by $g = (x - ay, x - cy)$, which has the inverse $g^{-1} = \left(\frac{ay-cx}{a-c}, \frac{x-y}{a-c}\right)$. Then we define a morphism to $V(x^2 - y)$ by $h = (x, dy + c)$ which has inverse $h^{-1} = (x, d^{-1}(y - c))$. Then

$$V(f) \cong V(x^2 - y) = V(y - x^2),$$

by the isomorphism

$$h \circ g = (x - ad^{-1}y,\ x - cd^{-1}y) + (acd^{-1},\ c^2d^{-1}),$$

which proves the case for $a = b \neq c$.

Lastly, if $a \neq b$, then $x + ay$ and $x + by$ are a basis for the k-vector space $Kx + Ky$ of homogenous degree 1 polynomials in $K[x, y]$. That is, there is some $c_1, c_2 \in K$ such that

$$c_1(x + ay) + c_2(x + by) = x + cy,$$

so

$$f = (x + ay)(x + by) + c_1(x + ay) + c_2(x + by) + e.$$

Then we define a morphism $V(f) \to V(xy + c_1x + c_2y + e)$ by $g = (x + ay, x + by)$, which has inverse $g^{-1} = \left(\frac{ay-bx}{a-b}, \frac{x-y}{a-b}\right)$. And we have shown (in the first case of $a_1 = a_3 = 0$) that the zero locus of a polynomial of this form is isomorphic to $V(xy - 1)$ by some $h : V(xy+c_1x+c_2y+e) \to V(xy-1)$, which is a linear transformation followed by a translation. Since g is a linear transformation, then the composition $h \circ g : V(f) \to V(xy - 1)$ is an isomorphism by a linear transformation followed by a translation.

We have shown that $V(f)$ is isomorphic to one of $V(y - x^2)$ or $V(xy - 1)$. To show that $V(f)$ is isomorphic to exactly one of these affine varieties, it remains to show that $V(y - x^2) \ncong V(xy - 1)$. The affine varieties are isomorphic if and only if their coordinate rings,

$$K[x, y]/\langle y - x^2\rangle \quad \text{and} \quad K[x, y]/\langle xy - 1\rangle,$$

are isomorphic as K-algebras. For the former, we consider the K-algebra homomorphism

$$\begin{aligned} K[x, y] &\to K[x] \\ p(x, y) &\mapsto p(x, x^2), \end{aligned}$$

i.e., the map determined by $y \mapsto x^2$. This map is the identity on $K[x]$, so it is surjective. Moreover, its kernel is precisely the ideal $\langle y - x^2\rangle$, so

$$K[x, y]/\langle y - x^2\rangle \cong K[x].$$

On the other hand, we have a K-algebra homomorphism

$$\begin{aligned} K[x, y] &\to K[x, x^{-1}] \\ p(x, y) &\mapsto p(x, x^{-1}), \end{aligned}$$

i.e., the map determined by $y \mapsto x^{-1}$. This map is surjective, as any polynomial in $K[x, x^{-1}]$ could be mapped to a polynomial in $K[x, y]$, by $x^{-1} \mapsto y$, whose image is the original polynomial in $K[x, x^{-1}]$. Moreover, the kernel of this map is precisely the ideal $\langle xy - 1\rangle$, so

$$K[x, y]/\langle xy - 1\rangle \cong K[x, x^{-1}],$$

as K-algebras.

However, $K[x]$ and $K[x, x^{-1}]$ are not isomorphic as K-algebras. If there were a K-algebra isomorphism $\varphi : K[x] \to K[x, x^{-1}]$, then the units of both rings would correspond under φ. However, $K[x]^{\times} = K^{\times}$ and $\varphi(K^{\times}) = K^{\times}$, but x is a unit of $K[x, x^{-1}]$, which is not in $K^{\times}$. Thus,

$$K[x, y]/\langle y - x^2\rangle \cong K[x] \ncong K[x, x^{-1}] \cong K[x, y]/\langle xy - 1\rangle,$$

as K-algebras, which implies $V(y - x^2) \ncong V(xy - 1)$ as affine varieties.

$\square$

Exercise 5.8(a) Show that every isomorphism $f : \mathbb{P}^1 \to \mathbb{P}^1$ is of the form $f(x) = \frac{ax+b}{cx+d}$ for some $a, b, c, d \in K$, where x is an affine coordinate on $\mathbb{A}^1 \subseteq \mathbb{P}^1$.

Lemma 1. A fractional linear transformation $f : \mathbb{P}^1 \to \mathbb{P}^1$ given by $f(x) = \frac{ax+b}{cx+d}$ is a morphism.

Proof. For $a, b \in K$, we claim the dilation and translation map $f : \mathbb{P}^1 \to \mathbb{P}^1$ given by $x \mapsto ax + b$ is a morphism. For an open subset $U \subseteq \mathbb{P}^1$ and regular function $\varphi \in \mathscr{O}_{\mathbb{P}^1}(U)$, we consider the pullback $f^*\varphi$. To see that $f^*\varphi$ is regular on $f^{-1}(U)$, we consider a point $x_0 \in f^{-1}(U)$. Then $\varphi|_V = \frac{g}{h}$ for some open neighborhood $V \subseteq U$ of $f(x_0)$ and polynomials $g, h \in A(\mathbb{A}^1) = K[x]$. Then

$$f^*\varphi|_{f^{-1}(V)} = \frac{g \circ f}{h \circ f} = \frac{g'}{h'},$$

where $g'(x) = g(ax + b)$ and $h'(x) = h(ax + b)$ are polynomials in $K[x]$. Hence, $f^*\varphi \in \mathscr{O}_{\mathbb{P}^1}(f^{-1}(U))$, so f is a morphism.

As noted in Example 5.5(a), the map $\mathbb{P}^1 \to \mathbb{P}^1$ given by the inversion $x \mapsto \frac{1}{x}$ is a morphism. Therefore, the composition of dilations, translations, and inversions are morphisms $\mathbb{P}^1 \to \mathbb{P}^1$. It is straightforward to check that any such compositions yield fractional linear transformations.

Moreover, when $ad - bc \neq 0$, the fractional linear transformation $\frac{ax+b}{cx+d}$ is invertible, and the inverse is again a fractional linear transformation. Therefore, such a map is in fact an isomorphism $\mathbb{P}^1 \to \mathbb{P}^1$.

□

Lemma 2. Given three distinct points $a_1, a_2, a_3 \in \mathbb{P}^1$ and three distinct points $b_1, b_2, b_3 \in \mathbb{P}^1$, there exists a unique fractional linear transformation $f : \mathbb{P}^1 \to \mathbb{P}^2$ such that $f(a_i) = b_i$ for $i = 1, 2, 3$.

Proof. We first define the fractional linear transformation

$$f = \frac{(x - a_1)(a_2 - a_3)}{(x - a_3)(a_2 - a_1)} = \frac{(a_2 - a_3)x - a_1(a_2 - a_3)}{(a_2 - a_1)x - a_3(a_2 - a_1)},$$

which maps

$$a_1 \mapsto 0 \quad a_2 \mapsto 1 \quad a_3 \mapsto \infty.$$

Then, similarly, we define

$$g = \frac{(x - b_1)(b_2 - b_3)}{(x - b_3)(b_2 - b_1)} = \frac{(b_2 - b_3)x - b_1(b_2 - b_3)}{(b_2 - b_1)x - b_3(b_2 - b_1)},$$

which maps

$$b_1 \mapsto 0 \quad b_2 \mapsto 1 \quad b_3 \mapsto \infty.$$

Then the condition for the invertibility of g is

$$(b_1 - b_2)(b_2 - b_3)(b_1 - b_3) \neq 0,$$

which is true since b_1, b_2, b_3 are distinct and K is an integral domain. Then the composition $g^{-1} \circ f$ is a fractional linear transformation which maps $a_i \mapsto b_i$ for $i = 1, 2, 3$.

To see uniqueness, we suppose that

$$\frac{a_1 x + b_1}{c_1 x + d_1} = \frac{a_2 x + b_2}{c_2 x + d_2}$$

for all $x \in \mathbb{P}^1$. Evaluating at the points $0, 1, \infty, -d_1/c_1$ gives us the conditions

$$\frac{b_1}{d_1} = \frac{b_2}{d_2}, \quad \frac{a_1 + b_1}{c_1 + d_1} = \frac{a_2 + b_2}{c_2 + d_2} \quad \frac{a_1}{b_1} = \frac{a_2}{b_2}, \quad \frac{d_1}{c_1} = \frac{d_2}{c_2}.$$

From these we can deduce that they represent the same fractional linear transformation, up to scaling the numerator and denominator by the same unit.

□

Proposition 1. Every isomorphism $f : \mathbb{P}^1 \to \mathbb{P}^1$ is of the form $f(x) = \frac{ax+b}{cx+d}$ for some $a, b, c, d \in K$, where x is an affine coordinate on $\mathbb{A}^1 \subseteq \mathbb{P}^1$.

Proof. Let $f : \mathbb{P}^1 \to \mathbb{P}^1$ be an isomorphism. We define a fractional linear transformation $g : \mathbb{P}^1 \to \mathbb{P}^1$ such that $g(f(0)) = 0$, $g(f(1)) = 1$, and $g(f(\infty)) = \infty$. Since f is a bijection, then the points $f(0), f(1), f(\infty)$ are distinct, so g is an isomorphism. Therefore, the composition $h = g \circ f : \mathbb{P}^1 \to \mathbb{P}^1$ is an isomorphism which is the identity on $0, 1, \infty$.

Considering $\mathbb{A}^1$ and an subset of $\mathbb{P}^1$, we have the identity polynomial $x \in K[x] = \mathscr{O}_{\mathbb{P}^1}(\mathbb{A}^1)$. Then since h is a morphism, the pullback $h^* x = h$ is a regular function on $h^{-1}(\mathbb{A}^1) = \mathbb{A}^1$. That is, h is a polynomial on $\mathbb{A}^1$. Since K is algebraically closed, then h splits into

$$h = a(x - \alpha_1) \cdots (x - \alpha_n),$$

where $\alpha_1, \ldots, \alpha_n$ are the roots of h in K. Since h is injective and $h(0) = 0$, then 0 is the only root of h, so $h = ax^n$. Since $h(1) = 1$, then $a = 1$. If we assume K to be of characteristic 0, then it contains the algebraic numbers $\overline{\mathbb{Q}}$ as a subfield. In particular, it contains exactly n distinct nth roots of units, each of which evaluate to 1 under h. However, since h is injective, then there can only be one root of unity, so $h = x$. That is, h is the identity polynomial on $\mathbb{A}^1$.

Extending h to the point at ∞, we must have h to be the identity on all of $\mathbb{P}^1$. The identity is a fractional linear transformation given by

$$\frac{1x + 0}{0x + 1} = h = g \circ f.$$

Recall that g is invertible, with fractional linear inverse g^{-1}. Then the composition $g^{-1} = g^{-1} \circ g \circ f = f$ is a fractional linear transformation.

□

Exercise 5.8(b) Given three distinct points $a_1, a_2, a_3 \in \mathbb{P}^1$ and three distinct points $b_1, b_2, b_3 \in \mathbb{P}^1$, show that there is a unique isomorphism $f : \mathbb{P}^1 \to \mathbb{P}^1$ such that $f(a_i) = b_i$ for $i = 1, 2, 3$.

Proof. By Lemma 1 and Proposition 1, the isomorphisms $\mathbb{P}^1 \to \mathbb{P}^1$ are precisely the fractional linear transformations $\mathbb{P}^1 \to \mathbb{P}^1$. By Lemma 2, there exists a unique fractional linear transformations $f : \mathbb{P}^1 \to \mathbb{P}^1$ satisfying $f(a_i) = b_i$ for $i = 1, 2, 3$, which must be the unique such isomorphism.

$\square$