Homework 5

MATH CS 121 Intro to Probability

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Exercise 1

Let $X \overset{d}{=} \mathrm{Bin}(4, 1/3)$ and $Y \overset{d}{=} \mathrm{Geom}(1/2)$. For each choice of Z, find the range (image) $R \subset \mathbb{R}$ of Z defined by

$$R \coloneqq \{x \in \mathbb{R} : \exists \omega \in \Omega \text{ s.t. } Z(\omega) = x\}$$

and calculate $\mathbb{E}[Z]$:

Exercise 1(a)

$Z = Y - X$.

Proposition 1. $Z(\Omega) = \{-3, -2, -1, 0\} \cup \mathbb{N}$.

Proof. We assume that X and Y are independent variables defined on the probability space Ω. By definition, $X(\Omega) = \{0, 1, 2, 3, 4\}$ and $Y(\Omega) = \mathbb{N}$. Then the independence of X and Y imply that for any $a \in \{0, 1, 2, 3, 4\}$ and $b \in \mathbb{N}$, we can find some $\omega \in \Omega$ such that $X(\omega) = a$ and $Y(\omega) = b$. Thus,

$$\begin{aligned}
Z(\Omega) &= \{Z(\omega) : \omega \in \Omega\} \\
&= \{Y(\omega) - X(\omega) : \omega \in \Omega\} \\
&= \{b - a : a \in \{0, 1, 2, 3, 4\},\, b \in \mathbb{N}\} \\
&= \{-3, -2, -1, 0\} \cup \mathbb{N}.
\end{aligned}$$

□

I know it probably isn't necessary, and there's probably a much more concise ways of proving it, but I went ahead and proved the expected values for general binomial and geometric distributions.

Lemma 1. *If* $X \stackrel{d}{=} \text{Bin}(n,p)$ *with* $n \in \mathbb{N}$ *and* $p \in (0,1)$*, then* $\mathbb{E}[X] = np$*.*

Proof. We define the following indicator for each $k \in \{0, \dots, n\}$ and $\omega \in \Omega$:

$$I_{X=k}(\omega) = \begin{cases} 1 & \text{if } X(\omega) = k, \\ 0 & \text{if } X(\omega) \neq k. \end{cases}$$

Then, X has the representation

$$X = \sum_{k=0}^{n} k \cdot I_{X=k}.$$

This implies that X is a simple random variable and, therefore, its expected value is given by

$$\mathbb{E}[X] = \sum_{k=0}^{n} k \cdot \mathbb{P}(X = k) = \sum_{k=0}^{n} k \binom{n}{k} p^k (1-p)^{n-k}.$$

Notice that when $k = 0$, then the summand is zero. Therefore, we can start the indexing of the summation at $k = 1$, giving us

$$\mathbb{E}[X] = \sum_{k=1}^{n} k \binom{n}{k} p^k (1-p)^{n-k}.$$

This is very similar to the binomial formula, but with an extra factor of k in each summand. Our goal, now, is to simplify $\mathbb{E}[X]$ by getting it in terms of the binomial formula. First, for a fixed $k \in \{1, \dots, n\}$, we look at the first two factors of the summand and expand the binomial coefficient; we derive the following equality:

$$\begin{aligned} k\binom{n}{k} &= k\frac{n!}{k!(n-k)!} \\ &= \frac{n!}{(k-1)!(n-k)!} \\ &= (n-k+1)\frac{n!}{(k-1)!(n-k+1)!} \\ &= (n-k+1)\binom{n}{k-1}. \end{aligned}$$

Then substituting this back into our equation for $\mathbb{E}[X]$ and obtain

$$\mathbb{E}[X] = \sum_{k=1}^{n} (n-k+1) \binom{n}{k-1} p^k (1-p)^{n-k}.$$

We re-index the summation to be from 0 to $n-1$ and replace k with $k+1$ in the summand to obtain

$$\mathbb{E}[X] = \sum_{k=0}^{n-1} (n-k) \binom{n}{k} p^{k+1} (1-p)^{n-k-1}.$$

To make the exponents of p and $(1-p)$ more favorable, we pull a common factor from all of the summands, so that

$$\mathbb{E}[X] = \frac{p}{1-p} \sum_{k=0}^{n-1} (n-k) \binom{n}{k} p^{k} (1-p)^{n-k}.$$

We now notice that if we were to have $k = n$, then the first factor of $n-k$ would make the who term zero. Therefore, we can increase the indexing of summation up to n and not change its value, giving us

$$\mathbb{E}[X] = \frac{p}{1-p} \sum_{k=0}^{n} (n-k) \binom{n}{k} p^{k} (1-p)^{n-k}.$$

We now distribute over $(n-k)$ and split the summation into two recognizable expressions.

$$\begin{aligned}
\mathbb{E}[X] &= \frac{p}{1-p} \left(n \sum_{k=0}^{n} \binom{n}{k} p^{k} (1-p)^{n-k} - \sum_{k=0}^{n} k \binom{n}{k} p^{k} (1-p)^{n-k} \right) \\
&= \frac{p}{1-p} \left(n(p+1-p)^n - \mathbb{E}[X] \right) \\
&= \frac{p}{1-p} (n - \mathbb{E}[X]).
\end{aligned}$$

Finally, we solve for $\mathbb{E}[X]$ to complete the proof.

$$\begin{aligned}
(1-p)\mathbb{E}[X] &= np - p\mathbb{E}[X], \\
(1-p+p)\mathbb{E}[X] &= np, \\
\mathbb{E}[X] &= np.
\end{aligned}$$

$\square$

Lemma 2. *If* $Y \stackrel{d}{=} \mathrm{Geom}(p)$ *with* $n \in \mathbb{N}$ *and* $p \in (0,1]$*, then* $\mathbb{E}[Y] = \dfrac{1}{p}$.

Unlike Lemma 1, the proof of Lemma 2 will involve defining, specifically, the probability space Ω for which Y is a random variable. We do this under the assumption that the expected value of a geometric random variable is the same, regardless of the particular probability space.

Proof. We begin by defining a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. First, Ω will be the set of all infinite sequences in the set $\{0,1\}$, i.e.,

$$\Omega := \{\{\omega_n\}_{n\in\mathbb{N}} : \omega_k \in \{0,1\} \text{ for all } k \in \mathbb{N}\} = \{0,1\}^{\mathbb{N}}.$$

We could define $\mathcal{F}$ to be the power set of Ω, but because Ω is uncountable, then the power set is not the most convenient σ-algebra to define. Instead, since we are chiefly concerned with the probabilities of Y attaining certain values, it will be beneficial to have $\mathcal{F} = \sigma(Y)$. If we can do this, it will be easier to define $\mathbb{P} : \mathcal{F} \to [0,1]$ in such a way that makes finding $\mathbb{P}(Y = k)$, for any $k \in \mathbb{N}$, more natural.

First, for each $k \in \mathbb{N}$, we define

$$A_k := \{\{\omega_n\}_{n\in\mathbb{N}} \in \Omega : \omega_1 = \cdots = \omega_{k-1} = 0,\, \omega_k = 1\}.$$

Taking a zero to mean a failure and a one to mean a success, then A_k is precisely what we want $Y^{-1}(k)$ to be: the set of sequences of trials in which the first $k-1$ trials are failures and the kth trial is a success. Additionally, for reasons which will soon become clear, we define

$$A_\infty := \{\{\omega_n\}_{n\in\mathbb{N}} \in \Omega : \omega_k = 0 \text{ for all } k \in \mathbb{N}\} = \{\{0\}_{n\in\mathbb{N}}\},$$

which is the singleton of the sequence in Ω of all zeros. Note that this definition of the A_k's imply that they are all pairwise disjoint. Obviously, A_∞ is disjoint from A_k for all $k \in \mathbb{N}$. Now if $k, \ell \in \mathbb{N}$ with $k \neq \ell$, then without loss of generality we assume $k < \ell$. Then $\omega \in A_k$ implies that $\omega_k = 1$, so $\omega \notin A_\ell$. We now define $\mathcal{F}$ to be closure of the set

$$\{A_k : k \in \mathbb{N} \cup \{\infty\}\}$$

under countable unions. In other words

$$\mathcal{F} := \{\cup_{i\in I} A_i : I \subseteq \mathbb{N} \cup \{\infty\}\}.$$

By definition, $\mathcal{F}$ is closed under countable unions, so to show that $\mathcal{F}$ is a σ-algebra, we need only show that $\varnothing, \Omega \in \mathcal{F}$ and that $\mathcal{F}$ is closed under complements. We can either add $\varnothing$ to the definition of $\mathcal{F}$ or consider it to be the result of an empty union; either way, we have $\varnothing \in \mathcal{F}$. To show that $\Omega \in \mathcal{F}$, we claim that

$$\Omega = \bigcup_{k\in\mathbb{N}\cup\{\infty\}} A_k,$$

which is clearly in $\mathcal{F}$. The inclusion of the union in Ω is trivial, as each A_k is defined to be a subset of Ω. For the opposite inclusion, suppose $\omega = \{\omega_n\}_{n\in\mathbb{N}} \in \Omega$. It is either the case that ω is a sequence of all zeros or has a one at at least one index. If ω is all zeros, then $\omega = \{0\}_{n\in\mathbb{N}} \in A_\infty$. Otherwise, ω has some ones and we define $k = \min\{j \in \mathbb{N} : \omega_j = 1\}$. Then we have

$$\omega_1 = \cdots \omega_{k-1} = 0 \quad \text{and} \quad \omega_k = 1,$$

which implies that $\omega \in A_k$. Thus, the equality is proven and we have $\Omega \in \mathcal{F}$. Along with the fact that the A_k's are disjoint, this implies that the set of A_k's are a partition of Ω. Now to show that $\mathcal{F}$ is closed under complements, let $\cup_{i\in I} A_i \in \mathcal{F}$ and consider the complement

$$\left(\bigcup_{i\in I} A_i\right)^C.$$

Since Ω is equal to the union of all A_k's, then

$$\left(\bigcup_{i\in I} A_i\right)^C = \left(\bigcup_{k\in\mathbb{N}\cup\{\infty\}} A_k\right) \setminus \left(\bigcup_{i\in I} A_i\right) = \bigcup_{\substack{k\in\mathbb{N}\cup\{\infty\}\\ k\notin I}} A_k.$$

The second equality follows from the fact that the A_k's are pairwise disjoint. Then if we define $J = (N \cup \{\infty\}) \setminus I \subseteq \mathbb{N} \cup \{\infty\}$, then

$$\left(\bigcup_{i\in I} A_i\right)^C = \bigcup_{i\in J} A_i \in \mathcal{F}.$$

Thus, $\mathcal{F}$ is closed under complements and is, therefore, a σ-algebra.

We now define the probability measure $\mathbb{P} : \mathcal{F} \to [0, 1]$ on each of the A_k's by

$$\mathbb{P}(A_k) := \begin{cases} (1-p)^{k-1}p & \text{if } k \in \mathbb{N}, \\ 0 & \text{if } k = \infty. \end{cases}$$

Then for each $\cup_{i\in I} A_i \in \mathcal{F}$, we define

$$\mathbb{P}(\cup_{i\in I} A_i) := \sum_{i\in I} \mathbb{P}(A_i).$$

This map is well-defined since each element of $\mathcal{F}$ only has one representation as the union of A_k's. Moreover, it is non-negative and for all $\cup_{i\in I} A_i \in \mathcal{F}$, we have

$$\begin{aligned}
\mathbb{P}(\cup_{i\in I} A_i) &= \sum_{i\in I} \mathbb{P}(A_i) \\
&\le \sum_{k\in\mathbb{N}\cup\{\infty\}} \mathbb{P}(A_k) \\
&= \mathbb{P}(A_\infty) + \sum_{k\in\mathbb{N}} \mathbb{P}(A_k) \\
&= 0 + \sum_{k\in\mathbb{N}} (1-p)^{k-1}p \\
&= \sum_{k=0}^{\infty} (1-p)^k p.
\end{aligned}$$

This is a geometric series with common ratio $1-p$. And since $p \in (0,1]$, then $1-p \in [0,1)$, so

$$\mathbb{P}(\cup_{i\in I} A_i) \leq \sum_{k=0}^{\infty}(1-p)^k p = \frac{p}{1-(1-p)} = \frac{p}{p} = 1.$$

Thus, $\mathbb{P}(\cup_{i\in I} A_i) \in [0,1]$. This also shows that $\mathbb{P}(\Omega) = 1$ since

$$\mathbb{P}(\Omega) = \mathbb{P}\left(\bigcup_{k\in\mathbb{N}\cup\{\infty\}} A_k\right) = \sum_{k\in\mathbb{N}\cup\{\infty\}} \mathbb{P}(A_k) = 1.$$

And by definition, the probability $\mathbb{P}$ of a union of disjoint sets in $\mathcal{F}$ is the sum of the probabilities of the sets. Thus, $\mathbb{P}$ is a probability measure on $\mathcal{F}$.

We now define the random variable $Y : \Omega \to \mathbb{N}\cup\{\infty\}$ such that

$$Y(\omega) := k, \quad \omega \in A_k, \quad k \in \mathbb{N}\cup\{\infty\}.$$

This is defined for all $\omega \in \Omega$ since the set of A_k's is a partition of Ω. Our definition of $\mathcal{F}$ makes it clear that Y is $\mathcal{F}$-measurable, since $Y^{-1}(k) = A_k \in \mathcal{F}$ for all $k \in \mathbb{N}\cup\{\infty\}$. Moreover, $\mathbb{P}(Y = k) = \mathbb{P}(Y^{-1}(k)) = \mathbb{P}(A_k)$ for all $k \in \mathbb{N}\cup\{\infty\}$. Now even though $\infty \notin \mathbb{R}$, we still consider Y to be a function from Ω to $\mathbb{R}$ since $\mathbb{P}(Y = \infty) = \mathbb{P}(A_\infty) = 0$. This means that the range of Y is, effectively, $\mathbb{N} \subseteq \mathbb{R}$, since the only thing Y maps to ∞ is the all-zero sequence $\{0\}_{n\in\mathbb{N}}$. And for all $k \in \mathbb{N}$, we have

$$\mathbb{P}(Y = k) = \mathbb{P}(A_k) = (1-p)^{k-1}p.$$

Thus, we have that $Y \stackrel{d}{=} \mathrm{Geom}(p)$ is a random variable on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

We now construct a sequence of simple random variables $\{Y_n\}_{n\in\mathbb{N}}$ converging to Y in the following way:

$$Y_n = \sum_{k=1}^{n} kI_{A_k}.$$

Each Y_n is $\mathcal{F}$-measurable since $Y_n^{-1}(k) = A_k \in \mathcal{F}$ for each $k \in \{1,\ldots,n\}$, where $\{1,\ldots,n\}$ is the range of Y_n. One can see that $Y_n \to Y$ as $n \to \infty$, since for each $\omega \in \Omega$,

$$\lim_{n\to\infty} Y_n(\omega) = \sum_{k=1}^{\infty} kI_{A_k}(\omega).$$

Recall that the set of A_k's is a partition of Ω, so ω is in some A_ℓ, in which case

$$\lim_{n\to\infty} Y_n(\omega) = \ell I_{A_\ell}(\omega) = \ell = Y(\omega).$$

Now for some $n \in \mathbb{N}$, Y_n is a simple random variable, so we have the expected value

$$\mathbb{E}[Y_n] = \sum_{k=1}^{n} k\mathbb{P}(A_k) = \sum_{k=1}^{n} k(1-p)^{k-1}p = p\sum_{k=1}^{n} k(1-p)^{k-1}.$$

Now since $Y_n \to Y$, we have the

$$\mathbb{E}[Y] = \lim_{n\to\infty} p \sum_{k=1}^{n} k(1-p)^{k-1} = p \sum_{k=1}^{\infty} k(1-p)^{k-1}.$$

We now define the power series

$$F(z) = \sum_{k=1}^{\infty} z^k.$$

Taking the derivative of F, we find

$$F'(z) = \frac{\mathrm{d}}{\mathrm{d}z} \sum_{k=1}^{\infty} z^k = \sum_{k=1}^{\infty} \frac{\mathrm{d}}{\mathrm{d}z} z^k = \sum_{k=1}^{\infty} k z^{k-1}.$$

Therefore,

$$\mathbb{E}[Y] = pF'(1-p).$$

Moreover, if $|z| < 1$, then $F(z)$ is a geometric series with common ratio z and

$$F(z) = \frac{1}{1-z}.$$

Taking the derivative, we find

$$F'(z) = \frac{\mathrm{d}}{\mathrm{d}z} \frac{1}{1-z} = \frac{1}{(1-z)^2}.$$

Now since $p \in (0,1]$, then $1-p \in [0,1)$, so $|1-p| < 1$. Thus,

$$\mathbb{E}[Y] = pF'(1-p) = p \cdot \frac{1}{(1-(1-p))^2} = \frac{p}{p^2} = \frac{1}{p}.$$

$\square$

Proposition 2. *For* $X \stackrel{d}{=} \mathrm{Bin}(4, 1/3)$, $Y \stackrel{d}{=} \mathrm{Geom}(1/2)$, *and* $Z = Y - X$, $\mathbb{E}[Z] = 2/3$

Proof. By the linearity of $\mathbb{E}$ proved in Exercise 2, we have

$$\mathbb{E}[Z] = \mathbb{E}[Y - X] = \mathbb{E}[Y] - \mathbb{E}[X].$$

Using the expected values for X and Y found in Lemmas 1 and 2, respectively, we have

$$\mathbb{E}[Z] = \frac{1}{1/2} - 4 \cdot \frac{1}{3} = \frac{2}{3}.$$

$\square$

Exercise 1(b)

$Z = X^2 + 3Y$.

Proposition 3. $Z(\Omega) = \{3k : k \in \mathbb{N}\} \cup \mathbb{P}\{3k+1 : k \in \mathbb{N}\}$.

Proof. Recall that $X(\Omega) = \{0, 1, 2, 3, 4\}$ and $Y(\Omega) = \mathbb{N}$. Then

$$X^2(\Omega) = \{k^2 : k \in X(\Omega)\} = \{0, 1, 4, 9, 16\},$$

and

$$3Y(\Omega) = \{3k : k \in Y(\Omega)\} = \{3k : k \in \mathbb{N}\}.$$

Now,

$$\begin{aligned} Z(\Omega) &= \{Z(\omega) : \omega \in \Omega\} \\ &= \{X^2(\omega) + 3Y : \omega \in \Omega\} \\ &= \{a + b : a \in \{0, 1, 4, 9, 16\},\, b \in \{3k : k \in \mathbb{N}\}. \end{aligned}$$

Now since 0 and 9 are both multiples of 3 while 1, 4, and 16 are all one more than a multiple of 3, this set can be equivalently written as

$$Z(\Omega) = \{a + b : a \in \{0, 1\},\, b \in \{3k : k \in \mathbb{N}\} = \{3k : k \in \mathbb{N}\} \cup \{3k+1 : k \in \mathbb{N}\}.$$

□

Proposition 4. $\mathbb{E}[Z] = 26/3$.

Proof. We first calculate $\mathbb{E}[X^2]$.

$$\begin{aligned} \mathbb{E}[X^2] &= \sum_{k \in X^2(\Omega)} k\mathbb{P}(X^2 = k) \\ &= \sum_{k \in X(\Omega)} k^2 \mathbb{P}(X^2 = k^2) \\ &= \sum_{k=0}^{4} k^2 \mathbb{P}(X = k) \\ &= \sum_{k=0}^{4} k^2 \binom{4}{k} \left(\frac{1}{3}\right)^k \left(\frac{2}{3}\right)^{4-k} \\ &= \frac{8}{3}. \end{aligned}$$

Then

$$\mathbb{E}[Z] = \mathbb{E}[X^2 + 3Y] = \mathbb{E}[X^2] + 3\mathbb{E}[Y] = \frac{8}{3} + 3\frac{1}{1/2} = \frac{26}{3}.$$

□

Exercise 2

Show that the integral we defined for simple random variables is linear, i.e., for simple random variables X, Y we get that $\mathbb{E}[aX + bY] = a\mathbb{E}[X] + b\mathbb{E}[Y]$ (this of course extends to general random variables).

Proposition 5. *For simple random variables X, Y we get that $\mathbb{E}[aX+bY] = a\mathbb{E}[X]+b\mathbb{E}[Y]$.*

Proof. Suppose X and Y are simple random variables with

$$X = \sum_{i=1}^{n} a_i I_{A_i} \quad \text{and} \quad Y = \sum_{j=1}^{m} b_j I_{B_j},$$

where $\{A_i\}$ and $\{B_j\}$ are partitions of Ω. Now let $a, b \in \mathbb{R}$ and consider the random variable $aX + bY$. For $i = 1, \ldots, n$ and $j = 1, \ldots, m$, we define

$$C_{ij} := A_i \cap B_j.$$

Now for any $\omega \in \Omega$, we have $\omega \in A_i$ and $\omega \in B_j$ for some i, j. This implies that

$$\omega \in A_i \cap B_j = C_{ij}.$$

Now if $ij \neq k\ell$, then

$$C_{ij} \cap C_{k\ell} = (A_i \cap B_j) \cap (A_k \cap B_k) = (A_i \cap A_k) \cap (B_j \cap B_ell).$$

Since $ij \neq k\ell$, then either $i \neq k$, in which case $A_i \cap A_k = \varnothing$, or $j \neq \ell$, in which case $B_j \cap B_\ell = \varnothing$. Either way, we have

$$C_{ij} \cap C_{k\ell} = (A_i \cap A_k) \cap (B_j \cap B_\ell) = \varnothing.$$

Thus, $\{C_{ij}\}$ is a partition of Ω. Similarly, for a fixed i, $\{C_{ij}\}$ is a partition of A_i and for a fixed j, $\{C_{ij}\}$ is a partition of B_j. Therefore,

$$I_{A_i} = \sum_{j=1}^{m} I_{C_{ij}} \quad \text{and} \quad I_{B_j} = \sum_{i=1}^{n} I_{C_{ij}},$$

which gives us

$$X = \sum_{i=1}^{n} a_i \sum_{j=1}^{m} I_{C_{ij}} = \sum_{i=1}^{n} \sum_{j=1}^{m} a_i I_{C_{ij}}$$

and

$$Y = \sum_{j=1}^{m} b_j \sum_{i=1}^{n} I_{C_{ij}} = \sum_{j=1}^{m} \sum_{i=1}^{n} b_j I_{C_{ij}}.$$

Since the expected value of a simple random variable is independent of its representation, then

$$\mathbb{E}[X] = \sum_{i=1}^{n} \sum_{j=1}^{m} a_i \mathbb{P}(C_{ij}) \quad \text{and} \quad \mathbb{E}[Y] = \sum_{j=1}^{m} \sum_{i=1}^{n} b_j \mathbb{P}(C_{ij}).$$

Now,

$$\begin{aligned}
aX + bY &= a\sum_{i=1}^{n}\sum_{j=1}^{m} a_i I_{C_{ij}} + b\sum_{j=1}^{m}\sum_{i=1}^{n} b_j I_{C_{ij}} \\
&= \sum_{i=1}^{n}\sum_{j=1}^{m} aa_i I_{C_{ij}} + \sum_{j=1}^{m}\sum_{i=1}^{n} bb_j I_{C_{ij}} \\
&= \sum_{i=1}^{n}\sum_{j=1}^{m} (aa_i + bb_j) I_{C_{ij}}.
\end{aligned}$$

Thus, $aX + bY$ is a simple random variable, with the above representation. Then

$$\begin{aligned}
\mathbb{E}[aX + bY] &= \sum_{i=1}^{n}\sum_{j=1}^{m} (aa_i + bb_j)\mathbb{P}(C_{ij}) \\
&= \sum_{i=1}^{n}\sum_{j=1}^{m} aa_i \mathbb{P}(C_{ij}) + \sum_{j=1}^{m}\sum_{i=1}^{n} bb_j \mathbb{P}(C_{ij}) \\
&= a\sum_{i=1}^{n}\sum_{j=1}^{m} a_i \mathbb{P}(C_{ij}) + b\sum_{j=1}^{m}\sum_{i=1}^{n} b_j \mathbb{P}(C_{ij}) \\
&= a\mathbb{E}[X] + b\mathbb{E}[Y].
\end{aligned}$$

$\square$

Exercise 3

Suppose we play the following game based in tosses of a fair coin. You pay me \$10, and I agree to pay you $\$n^2$ if heads comes up first on the nth toss. If we play this game repeatedly, how much money do you expect to win or lose per game over the long run?

Proposition 6. *We expect to lose \$4.*

Proof. Let $X \overset{d}{=} \mathrm{Geom}(1/2)$ be the random variable denoting the index of the first occurrence of heads after an arbitrary number of tosses of a fair coin. Then the random variable denoting the amount of money we win in this scenario is given by

$$X^2 - 10.$$

Then the amount of money we expect to win, in dollars, is given by

$$\mathbb{E}[X^2 - 10] = \mathbb{E}[X^2] - 10.$$

To find $\mathbb{E}[X^2]$, we use that fact that $X \overset{d}{=} \mathrm{Geom}(p)$ implies that

$$\mathrm{Var}(X) = \frac{1-p}{p^2}.$$

Now using the definition of variance, we find

$$\begin{aligned}
\mathrm{Var}(X) &= \mathbb{E}[(X - \mathbb{E}[X])^2] \\
&= \mathbb{E}[X^2 - 2\mathbb{E}[X]X + \mathbb{E}[X]^2] \\
&= \mathbb{E}[X^2] - 2\mathbb{E}[X]\mathbb{E}[X] + \mathbb{E}[X]^2 \\
&= \mathbb{E}[X^2] - \mathbb{E}[X]^2.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathbb{E}[X^2] &= \mathrm{Var}(X) + \mathbb{E}[X]^2 \\
&= \frac{1-1/2}{(1/2)^2} + \left(\frac{1}{1/2}\right)^2 \\
&= \frac{1-1/2}{(1/2)^2} + \left(\frac{1}{1/2}\right)^2 \\
&= \frac{1/2}{1/4} + 4 \\
&= 2 + 4 \\
&= 6.
\end{aligned}$$

Thus,

$$\mathbb{E}[X^2 - 10] = 6 - 10 = -4,$$

meaning we expect to lose \$4 dollars.

$\square$

Exercise 4

If $\mathbb{E}[X] = 1$ and $\text{Var}(X) = 5$, find

Exercise 4(a)

$\mathbb{E}[(2 + X)^2]$,

First, we use the linearity of expected value.

$$\begin{aligned}
\mathbb{E}[(2+X)^2] &= \mathbb{E}[4 + 4X + X^2] \\
&= 4 + 4\mathbb{E}[X] + \mathbb{E}[X^2] \\
&= 4 + 4 \cdot 1 + \mathbb{E}[X^2] \\
&= 8 + \mathbb{E}[X^2].
\end{aligned}$$

Now since $\text{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2$, then

$$\begin{aligned}
\mathbb{E}[(2+X)^2] &= 8 + \text{Var}(X) + \mathbb{E}[X]^2 \\
&= 8 + 5 + 1^2 \\
&= 14.
\end{aligned}$$

Exercise 4(b)

$\text{Var}(4 + 3X)$.

We use the definition of variance and the linearity of expected value.

$$\begin{aligned}
\text{Var}(4+3X) &= \mathbb{E}[(4 + 3X - \mathbb{E}[4 + 3X])^2] \\
&= \mathbb{E}[(4 + 3X - 4 - 3\mathbb{E}[X])^2] \\
&= \mathbb{E}[(3X - 3\mathbb{E}[X])^2] \\
&= \mathbb{E}[9(X - \mathbb{E}[X])^2] \\
&= 9\mathbb{E}[(X - \mathbb{E}[X])^2] \\
&= 9\,\text{Var}(X) \\
&= 9 \cdot 5 \\
&= 45.
\end{aligned}$$

Exercise 5

Geno (the basketball coach for UConn) goes to the grocery store and finds that Wheaties is placing pictures of UConn basketball players inside its cereal boxes. A total of 6 players are featured, each appearing with equal probability. Find the expected number of boxes Geno needs to buy in order to obtain pictures of 3 different players.
Hint: Let X be the total number of cereal boxes that Geno buys and let N_i be the number of boxes that Geno buys to get the ith player after obtaining the $(i-1)$th player. Then, $X = N_1 + N_2 + N_3$. What type of random variable is each N_i? Use the fact that the expected value is linear!

Assuming that each box contains a picture of some player, then the first box Geno buys will have a picture of a player; $N_1 = 1$. Then N_2 is the number of boxes Geno must buy to obtain a picture of a player other than the first player. In other words, $N_2 \overset{d}{=} \text{Geom}(5/6)$, since every box bought after the first has a probability of $5/6$ of having a picture different from that of the first box. After the box containing the second unique player picture has been purchased, N_3 is the number of boxes Geno must buy to obtain a picture of a player different from both the first and second player. That is, $N_3 \overset{d}{=} \text{Geom}(4/6)$, since every box after the (N_1+N_2)th box has a probability of $4/6$ of containing a picture of a player different from the first two. Then by the linearity of the expected value,

$$\mathbb{E}[X] = \mathbb{E}[N_1 + N_2 + N_3] = \mathbb{E}[N_1] + \mathbb{E}[N_2] + \mathbb{E}[N_3] = 1 + \frac{1}{5/6} + \frac{1}{4/6} = \frac{37}{10}.$$