Exercise 2.33 Let X be the set of all 2×3 matrices over a field K that have rank at most 1, considered as a subset of $\mathbb{A}^6 = \mathrm{Mat}(2 \times 3, K)$.

Show that X is an irreducible affine variety. What is its dimension.

Proof. Consider a 2×3 matrix with entries in K:

$$B = \begin{bmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \end{bmatrix}.$$

Then $\operatorname{rank} B \leq 1$ if and only if the dimension of the column space of B is at most 1, which is the case if and only if each pair of columns in B are linearly dependent. If each pair of columns is linearly dependent, then two of the columns each must be a scalar multiple of the third, implying that the dimension of the column space is 1. On the other hand, if some pair of columns of B is linearly independent, then the dimension of the column space must be at least 2, since it contains at least two linearly independent vectors. A given pair of columns of B is linearly independent if and only if the 2×2 minor of B containing those two columns has determinant equal to zero.

We define the following polynomials:

$$f_1 = \begin{vmatrix} x_2 & x_3 \\ x_5 & x_6 \end{vmatrix} = x_2 x_6 - x_3 x_5,$$

$$f_2 = \begin{vmatrix} x_1 & x_3 \\ x_4 & x_6 \end{vmatrix} = x_1 x_6 - x_3 x_4,$$

$$f_3 = \begin{vmatrix} x_1 & x_2 \\ x_4 & x_5 \end{vmatrix} = x_1 x_5 - x_2 x_4.$$

Then X is the affine variety $V(f_1, f_2, f_3)$.

□

Ran out of time. Not sure if the right way to go is trying to show that $I(X)$ is a prime ideal, maybe by showing $K[x_1, \ldots, x_6]/I(X)$ is an integral domain, but I wasn't able to work out either. Im pretty confident that the dimension is 4, though.

Exercise 2.40 Let $R = K[x_1, x_2, x_3, x_4]/\langle x_1x_4 - x_2x_3\rangle$. Show:

(a) R is an integral domain of dimension 3.

Proof. First, we see that $f = x_1x_4 - x_2x_3$ is irreducible in the ring $K[x_1, x_2, x_3, x_4]$. Suppose, to the contrary, that $f = pq$ for some non-units $p, q \in K[x_1, x_2, x_3, x_4]$. Since $\deg f = 2$, then it must be the case that $\deg p = \deg q = 1$. Let $a_0, \dots, a_4, b_0, \dots, b_4 \in K$, be the coefficients of p and q, respectively, so

$$f = (a_0 + a_1x_1 + \cdots + a_4x_4)(b_0 + b_1x_1 + \cdots + b_4x_4).$$

Since f has no constant term then either a_0 or b_0 is zero; without loss of generality, assume $b_0 = 0$. Then a_0q provides all the terms of degree 1. Since f has no degree 1 terms, then we must also have $a_0 = 0$, so

$$f = (a_1x_1 + \cdots + a_4x_4)(b_1x_1 + b_2x_2 + b_3x_3 + b_4x_4).$$

Since f has a nonzero term containing x_1, then either a_1 or b_1 is nonzero. Without loss of generality, assume $a_1 \neq 0$. Then f has no terms containing x_1^2, so we must have $b_1 = 0$. Since f has no terms containing x_1x_2 or x_1x_3, and q does not contain x_1, then we must also have $b_2 = b_3 = 0$, so

$$x_1x_4 - x_2x_3 = f = (a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4)(b_4x_4).$$

However, the right-hand side now lacks the term x_2x_3, which is a contradiction. Therefore f is irreducible in the unique factorization domain $K[x_1, x_2, x_3, x_4]$, implying that f is prime. Then $\langle f\rangle$ is a prime ideal, so the quotient ring $R = K[x_1, x_2, x_3, x_4]/\langle f\rangle$ is an integral domain.

Now since $\langle f\rangle$ is a prime ideal in the coordinate ring of the affine space $\mathbb{A}^4$, then Remark 2.9 tells is that its zero locus $V(f)$ is an irreducible affine subvariety of $\mathbb{A}^4$. That is, $V(f)$ is an irreducible component of itself, in fact the only one. By Proposition 2.28(c),

$$\dim V(f) = \dim \mathbb{A}^4 - 1 = 3.$$

And by Lemma 2.27, the dimension of $V(f)$ is precisely the Krull dimension of its coordinate ring,

$$A(V(f)) = K[x_1, x_2, x_3, x_4]/I(V(f)).$$

Because f is prime,

$$I(V(f)) = \sqrt{\langle f\rangle} = \langle f\rangle,$$

so in fact

$$A(V(f)) = K[x_1, x_2, x_3, x_4]/\langle f\rangle = R.$$

Hence, the Krull dimension of R is 3.

□

(b) $x_1, \ldots, x_4$ are irreducible, but not prime in R. In particular, R is not a unique factorization domain.

Proof. Let $f = x_1x_4 - x_2x_3$, so $R = K[x_1, x_2, x_3, x_4]/\langle f \rangle$. Note that the quotient is symmetric with respect to the indeterminates, so the proof for each is identical. We first show that x_1 is irreducible in R. Suppose that $r, s \in R$ such that $x_1 + \langle f \rangle = rs$.

Any nonempty subset $S \subseteq K[x_1, x_2, x_3, x_4]$ of polynomials contains at least one polynomial whose degree is the minimum degree over all polynomials in S. This is because the set of degrees $\{\deg g \mid g \in S\}$ is a nonempty subset of nonnegative integers, meaning that the minimum degree is attained by some polynomial in S.

Suppose we have a polynomial $g \in K[x_1, x_2, x_3, x_4]$ of degree $n \geq 1$. For $j = 0, 1, \ldots, n$, define the polynomial g_j as the sum of the degree j terms of g (i.e., g_j is the degree j homogeneous component of g), then $g = g_0 + g_1 + \cdots + g_n$. If there is a polynomial $h \in K[x_1, x_2, x_3, x_4]$ such that $g_n = hf$, then the coset of g is given by

$$\begin{aligned} g + \langle f \rangle &= (g_0 + g_1 + \cdots + g_{n-1} + hf) + \langle f \rangle \\ &= (g_0 + g_1 + \cdots + g_{n-1}) + \langle f \rangle. \end{aligned}$$

In other words, $g_0 + g_1 + \cdots + g_{n-1}$ is a representative from the coset $g + \langle f \rangle$ with lesser degree than g. Importantly, this means that a minimum degree representative from a coset in R must have a leading homogeneous component not divisible by f.

We choose minimum degree representatives $p \in \pi^{-1}(r)$ and $q \in \pi^{-1}(s)$, so

$$x_1 + \langle f \rangle = (p + \langle f \rangle)(q + \langle f \rangle) = pq + \langle f \rangle.$$

Equivalently, $pq - x_1 \in \langle f \rangle$, meaning there is some polynomial $h \in K[x_1, x_2, x_3, x_4]$ such that in $K[x_1, x_2, x_3, x_4]$ we have

$$pq - x_1 = hf = h(x_1x_4 - x_2x_3).$$

Since f only contains terms of degree 2 (i.e., f is homogeneous of degree 2), then the nonzero terms of hf must be of degree at least 2. In particular, hf does not have x_1 as a term, so pq must have x_1 as a term. Without loss of generality, assume p has x_1 as a term and q has 1 as a term (It may be the case that one has ax_1 and the other has a^{-1} for some unit $a \in K$, but factoring out a from the former and multiplying the latter by a gives us x_1 and 1). Since hf contains no terms of degree 0 or 1, and q contains 1 as a term, then p contains no terms of degree 0 or 1, other than x_1.

Let $n = \deg p$, $m = \deg q$. p_j, $k = \deg h$. Assume, for contradiction, that $n \geq 2$, $m \geq 1$ and $h \neq 0$. Then x_1 is not a leading term of p and 1 is not a leading term of q, so we have leading terms

$$p_n q_m = h_k f,$$

where $n + m = k + 2$. Since f is irreducible in $K[x_1, x_2, x_3, x_4]$, then f must be a factor of either p_n or q_m. However, this is a contradiction as both p and q were chosen to be minimum degree representatives, meaning neither of their leading homogeneous components is divisible by f.

Then either $n = 1$, $m = 0$, or $h = 0$. In the first case, $p = x_1$, implying that $q + \langle f\rangle = 1 + \langle f\rangle$ is a unit in R. In the second case, $q = 1$, whose coset is the unit in R . And in the third case, $pq = x_1$, implying that either p or q is a unit in K, which is a unit in the quotient R. Hence, x_1 is irreducible in R.

We now show that x_1 is not prime in R, by showing that $R/\langle x_1 + \langle f\rangle\rangle$ is not an integral domain. Simplifying the quotient, in particular using the third isomorphism theorem for rings, we find

$$\begin{aligned} R/\langle x_1 + \langle f\rangle\rangle &\cong K[x_1, x_2, x_3, x_4]/\langle x_1,\ x_1x_4 - x_2x_3\rangle \\ &= K[x_1, x_2, x_3, x_4]/\langle x_1,\ x_2x_3\rangle \\ &\cong (K[x_1, x_2, x_3, x_4]/\langle x_1\rangle)/(\langle x_1,\ x_2x_3\rangle/\langle x_1\rangle) \\ &\cong K[x_2, x_3, x_4]/\langle x_2x_3\rangle. \end{aligned}$$

In the last quotient ring, the cosets of x_2 and x_3 are both nonzero elements, but their product is zero. Thus, this is not an integral domain, so x_1 is not prime in R.

We conclude that R is not a unique factorization domain, since the elements x_1, x_2, x_3, x_4 are irreducible but not prime.

□

(c) x_1x_4 and x_2x_3 are two decompositions of the same element of R into irreducible elements that do not agree up to units.

Proof. We have $x_1x_4 - x_2x_3 \in \langle f\rangle$, which translates in R to

$$\begin{aligned} x_1x_4 + \langle f\rangle &= x_2x_3 + \langle f\rangle \\ (x_1 + \langle f\rangle)(x_4 + \langle f\rangle) &= (x_2 + \langle f\rangle)(x_3 + \langle f\rangle). \end{aligned}$$

We know that the indeterminates are irreducible, so these are both irreducible decompositions.

□

Ran out of time to show indeterminates non-associate in R, but I'm pretty sure you could just pull the associate from R back into the polynomial ring to get a contradiction.

(d) $\langle x_1, x_2\rangle$ is a prime ideal of codimension 1 in R that is not principal.

Proof. We find

$$\begin{aligned} R/\langle x_1, x_2\rangle &\cong K[x_1, x_2, x_3, x_4]/\langle x_1, x_2,\ x_1x_4 - x_2x_3\rangle \\ &= K[x_1, x_2, x_3, x_4]/\langle x_1, x_2\rangle \\ &\cong K[x_3, x_4]. \end{aligned}$$

This is an integral domain so $\langle x_1, x_2\rangle$ is a prime ideal of R. Then $V(x_1, x_2)$ is an irreducible affine subvariety of $V(R)$ with coordinate ring

$$A(V(x_1, x_2)) = R/\langle x_1, x_2\rangle \cong K[x_3, x_4] \cong K[x_1, x_2] = A(\mathbb{A}^2).$$

Then the codimension of the prime ideal $\langle x_1, x_2\rangle$ is given by

$$\begin{aligned} \operatorname{codim}_R\langle x_1, x_2\rangle &= \operatorname{codim}_{V(R)} V(x_1, x_2) \\ &= \dim V(R) - \dim V(x_1, x_2) \\ &= \dim R - \dim \mathbb{A}^2 \\ &= 3 - 2 \\ &= 1. \end{aligned}$$

□

I'm not confident this shows that the ideal isn't principal, since I didn't really finish (c):

Suppose $\langle p\rangle = \langle x_1, x_2\rangle$, then p contains no nonzero terms of degree less than 1. Since both x_1 and x_2 are degree 1, then we would have $x_1 = ap$ and $x_2 = bp$ a for some nonzero $a, b \in K$. But then $x_1 = ab^{-1}x_2$, which is a contradiction.