Flower Snark

MATH CS 120FG Graph Theory I

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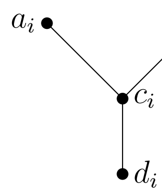
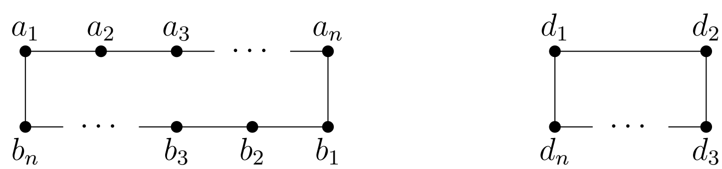
Prove that the flower snarks J_n defined in class do have chromatic index 4.

Let J_n with $n \geq 3$ odd be the flower snark with n claws of the form



and the edges



We will show by contradiction that $\chi'(J_n) = \Delta J_n + 1 = 4$. Suppose, to the contrary, that J_n has a proper 3-edge-coloring, and let P be such a coloring. We define the ordered tuple

$$Y_i = (P(a_i c_i), P(b_i c_i), P(d_i c_i)),$$

for each i, with $1 \leq i \leq n$, and where $P(e)$ is the color assigned to the edge e under P. This Y_i represents the colors assigned to the three edges of each claw. Since P is a proper edge-coloring, and the three edges of a claw are all incident to each other, the three values of Y_i are distinct.

We define now the ordered tuple

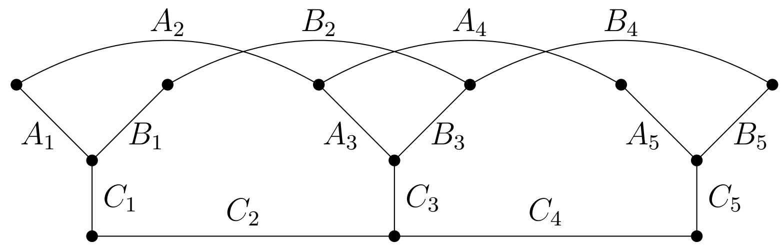
$$E_i = (P(a_i a_{i+1}), P(b_i b_{i+1}), P(c_i c_{i+1})),$$

for each i, with $1 \leq i \leq n-1$, as well as

$$E_0 = (P(b_n a_1), P(a_n, b_1), P(c_n c_1)).$$

These tuples represent the colors assigned to the three edges between each claw.

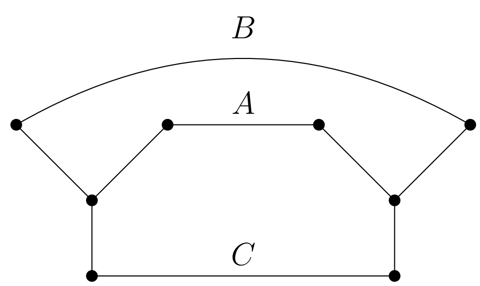
To better illustrate the meaning of E and Y, we now consider three consecutive claws and the colors given to edges by P:



If the leftmost claw is the ith claw, then we find

$$\begin{aligned} Y_i &= (A_1, \quad B_1, \quad C_1), \\ E_i &= (A_2, \quad B_2, \quad C_2), \\ Y_{i+1} &= (A_3, \quad B_3, \quad C_3), \\ E_{i+1} &= (A_4, \quad B_4, \quad C_4), \\ Y_{i+2} &= (A_5, \quad B_5, \quad C_5). \end{aligned}$$

Notice that any sequence of E_i, Y_{i+1}, E_{i+1} cannot have any equal place values, since this would correspond to adjacent edges being of the same color. This means that if we know any two tuples in such a sequence, we can determine the third. Note, however, that the same cannot be said for a sequence Y_i, E_i, Y_{i+1}, since Y_i and Y_{i+1} to not correspond to colors of adjacent edges. Consider now the adjacency between the first and nth claws:



This gives us $E_0 = (A, B, C)$, which acts the same as other E_i towards the right. However, from the left, it functions as if A and B were swapped. We will define $E'_0 = (B, A, C)$ to account for this. So, the coloring of P is described by the sequence

$$E_0, Y_1, E_1, Y_2, E_2, \dots, Y_n, E'_0.$$

If the colors of P are given by the set $\{\alpha, \beta, \gamma\}$, then each E_i is one of 27 ordered 3-tuples of these colors. We either have all three values of E_i the same, two the same and one different, or all three different. If all are the same (e.g. $E_i = \{\alpha, \alpha, \alpha\}$), then Y_{i+1} would have to have one of each color but not have any equal place values to E_i, which is not possible. So no E_i has all three values the same.

We consider now the case where all three are different. Suppose some $E_i = (\alpha, \beta, \gamma)$, then we have either

$$Y_{i+1} = (\beta, \gamma, \alpha) \text{ or } Y_{i+1} = (\gamma, \alpha, \beta).$$

Since E_{i+1} does not share any place values with E_i or Y_{i+1},

$$E_{i+1} = (\gamma, \alpha, \beta) \text{ or } E_{i+1} = (\beta, \gamma, \alpha).$$

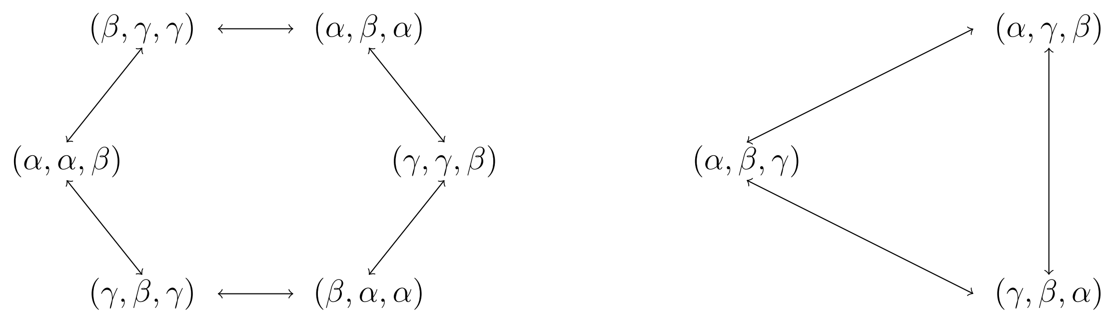
For the third case, suppose now that some $E_i = (\alpha, \alpha, \beta)$, then

$$Y_{i+1} = (\beta, \gamma, \alpha) \text{ or } Y_{i+1} = (\gamma, \beta, \alpha),$$

which implies

$$E_{i+1} = (\gamma, \beta, \gamma) \text{ or } E_{i+1} = (\beta, \gamma, \gamma).$$

Performing a similar process for all distinct values of E_i, we obtain the following graph with vertices as possible values of E_i and directed edges as choices for Y_i, yielding E_{i+1}:



A walk along this graph is a sequence of E's and Y's. The coloring P of J_n is captured by the walk

$$W_P = E_0, Y_1, E_1, Y_2, E_2, \ldots, Y_n, E_0'$$

in this graph. This walk uses an odd number n of edges. However, any choice of E_0 in this graph results in an E_0' which is an even number of edges away or unreachable. For example, if $E_0 = (\alpha, \beta, \alpha)$, then $E_0' = (\beta, \alpha, \alpha$. So W_P is an odd (E_0, E_0')-walk, however, it is clear that the only walks between these two vertices in the above graph are even. Similarly, if $E_0 = (\alpha, \beta, \gamma)$, then $E_0' = (\beta, \alpha, \gamma$. However, there is no walk between these two vertices, so W_P is impossible.

Therefore, W_P is impossible, so $\chi'(J_n) = 4$.