1 Let $f = a_0 + a_1 x + \cdots + a_n x^n \in R[x]$ be a polynomial.

Note. I use (b) in the proof of (a), but the proof of (b) does not rely on (a).

(a) Show that if f is a unit in $R[x]$, then a_0 is a unit in R and $a_1, \ldots, a_n$ are nilpotent.

Proof. Let $g = b_0 + b_1 x + \cdots + b_m x^m \in R[x]$ be any polynomial, then we have the product

$$fg = \sum_{d=0}^{n+m} c_d x^d \quad \text{where} \quad c_d = \sum_{i+j=d} a_i b_j.$$

Then $fg = 1$ if and only if $c_0 = 1$ and $c_d = 0$ for all $d > 0$. Assuming g is the inverse of f, this means that $1 = c_0 = a_0 b_0$, so indeed a_0 is a unit in R.

The next step is to show that the leading coefficient of f is nilpotent when $n \geq 1$. To do this, we first claim that $a_n^{k+1} b_{m-k} = 0$ for $k = 0, 1, \ldots, m$—we will prove this by induction on k. For the base case of $k = 0$, we immediately have

$$0 = c_{n+m} = a_n b_m.$$

Assuming the result holds for all indices less than some $k \geq 1$. Then

$$0 = c_{n+m-k} = a_n b_{m-k} + a_{n-1} b_{m-(k-1)} + \cdots + a_{n-k} b_m,$$

and multiplying by a_n^k gives

$$0 = a_n^{k+1} b_{m-k} + a_{n-1}(a_n^{k+1} b_{m-(k-1)}) + \cdots + a_{n-k}(a_n^{k+1} b_m) = a_n^{k+1} b_{m-k}.$$

This completes the induction. In particular, $k = m$ tells us that $a_n^{m+1} b_0 = 0$, and multiplying by a_0 gives $a_n^{m+1} = 0$, i.e., a_n is nilpotent in R.

Lastly, we perform induction on $n = \deg f$ to show that the coefficients of the remaining nonconstant terms are nonzero. For the base case of $n = 0$, $f = a_0$ has no nonconstant terms, so the result is vacuously true. Assume that the result holds for all unit polynomials of degree less than $n \geq 1$. Since a_n is nilpotent in R, we know that $a_n x^n$ is nilpotent in $R[x]$, so part (b) tells us that $f - a_n x^n$ is a unit in $R[x]$. But the degree of $f - a_n x^n$ is strictly less than n, so the inductive hypothesis tells us that all the coefficients of all of its nonconstant terms are nilpotent—these are precisely $a_1, \ldots, a_{n-1}$, so the induction is complete. $\square$

(b) Show that the sum of a unit and a nilpotent element is a unit.

Proof. We consider elements of an arbitrary ring.

If a is nilpotent with $a^n = 0$ then

$$(1 - a)(1 + a + \cdots + a^{n-1}) = 1 - a^n = 1.$$

In particular, $1 - a$ is a unit whenever a is nilpotent.

Let u be a unit and a be nilpotent. Then $-u^{-1}a$ is also nilpotent and the above result tells us that $1 + u^{-1}a$ is a unit. Multiplying by the unit u, we conclude that $u + a$ is a unit. $\square$

(c) Show that the converse of (a) also holds.

Proof. Since the set of nilpotents of $R[x]$ is an ideal (the nilradical), then any $R[x]$-linear combination of nilpotents $a_1, \dots, a_n \in R \subseteq R[x]$ is also nilpotent. In particular,

$$a_1 x + \cdots + a_n x^n$$

is nilpotent. Then if a_0 is a unit in R, it is still a unit in $R[x]$, so part (b) tells us that

$$f = a_0 + a_1 x + \cdots + a_n x^n$$

is a unit in $R[x]$. $\square$

2 Let M be an R-module, and $I \subset R$ an ideal. Show that if $M_{\mathfrak{m}} = \{0\}$ for all maximal ideals $\mathfrak{m} \subset R$ containing I, then $M = IM$.

Proof. Note that $M = IM$ if and only if $M/IM = 0$. Moreover, $M/IM = 0$ if and only if the localization $(M/IM)_{\mathfrak{m}} = 0$ at every maximal ideal $\mathfrak{m} \subseteq R$.

If $\mathfrak{m} \subseteq R$ is maximal ideal containing I, the quotient map followed by a canonical isomorphism gives us a surjection

$$0 = M_{\mathfrak{m}} \longrightarrow M_{\mathfrak{m}}/(IM)_{\mathfrak{m}} \cong (M/IM)_{\mathfrak{m}},$$

hence $(M/IM)_{\mathfrak{m}} = 0$.

If $\mathfrak{m} \subseteq R$ is a maximal ideal not containing I, we can find a scalar $t \in I \setminus \mathfrak{m}$. Then for any element in the localization $\frac{m+IM}{s} \in (M/IM)_{\mathfrak{m}}$, we have $tm \in IM$ so

$$t(m + IM) = 0 \in M/IM \implies \tfrac{m+IM}{s} = 0 \in (M/IM)_{\mathfrak{m}}.$$

So again we deduce that $(M/IM)_{\mathfrak{m}} = 0$. $\square$

3 Let M be an R-module. Define the *support* of M to be

$$\mathrm{Supp}(M) := \{\text{all prime ideals } \mathfrak{p} \subset R \text{ such that } M_{\mathfrak{p}} \neq 0\},$$

and the *annihilator* of M to be

$$\mathrm{Ann}_R(M) := \{r \in R \text{ such that } rm = 0 \text{ for all } m \in M\}.$$

Show that if M is finitely generated over R, then $\mathrm{Supp}(M)$ is the same as the set of all prime ideals $\mathfrak{p} \subset R$ containing $\mathrm{Ann}_R(M)$.

Proof. Suppose $\mathfrak{p} \in \mathrm{Supp}(M)$. The fact that $M_{\mathfrak{p}} \neq 0$ means there is some $m \in M$ such that $tm \neq 0$ for all $t \in R \setminus \mathfrak{p}$. In particular, for all $a \in \mathrm{Ann}_R(M)$ we have $am = 0$, which implies $a \in \mathfrak{p}$. Hence, $\mathrm{Ann}_R(M) \subseteq \mathfrak{p}$.

Suppose $\mathfrak{p} \subseteq R$ is a prime ideal with $M_{\mathfrak{p}} = 0$, i.e., $\mathfrak{p} \notin \mathrm{Supp}(M)$. This means that for all $m \in M$ there is some $t \in R \setminus \mathfrak{p}$ such that $tm = 0$. If M is generated by $x_1, \ldots, x_n \in M$, we can choose scalars $t_i \in R \setminus \mathfrak{p}$ such that $t_i x_i = 0$. Since $\mathfrak{p}$ is prime, we know that the product $t = t_1 \cdots t_n$ is not in $\mathfrak{p}$. For any $m \in M$, write $m = \sum_{i=1}^n a_i x_i$ for some $a_i \in R$, then

$$tm = \sum_{i=1}^{n} a_i (t_1 \cdots t_i \cdots t_n) x_i = 0.$$

That is, $t \in \mathrm{Ann}_R(M)$. But since $t \notin \mathfrak{p}$, we conclude that $\mathrm{Ann}_R(M) \nsubseteq \mathfrak{p}$. $\square$

4 Let M be a nonzero module over a Noetherian ring R. We say that a prime ideal $\mathfrak{p} \subset R$ is *associated with* M if

$$\mathfrak{p} = \mathrm{Ann}_R(m)$$

for some $m \in M$, where $\mathrm{Ann}_R(m) := \{r \in R \colon rm = 0\}$.

Show that the set of prime ideals associated with M is nonempty.

(*Hint:* Consider a maximal element in the set $\{\mathrm{Ann}_R(m) \colon m \neq 0 \in M\}$.)

Proof. Note that $\mathrm{Ann}_R(m)$ is an ideal of R: given $a, b \in \mathrm{Ann}_R(m)$ and $r \in R$ we have

$$(ra + b)m = r(am) + bm = r \cdot 0 + 0 = 0,$$

hence $ra + b \in \mathrm{Ann}_R(m)$. Moreover, $\mathrm{Ann}_R(m)$ is a proper ideal if and only if $m \neq 0 \in M$, since both are equivalent to $1 \in \mathrm{Ann}_R(m)$.

Per the hint, consider the set $\mathcal{A} = \{\mathrm{Ann}_R(m) \mid m \neq 0 \in M\}$, partially ordered by inclusion. This is a set of proper ideals in R, which we know to be nonempty because M is nonzero. We will use Zorn's lemma to choose a maximal element of $\mathcal{A}$.

Suppose $\mathcal{C} \subseteq \mathcal{A}$ is a chain. Choose an arbitrary initial element $\mathfrak{a}_0 \in \mathcal{C}$ and inductively choose $\mathfrak{a}_i \in \mathcal{C}$ such that $\mathfrak{a}_i \subseteq \mathfrak{a}_{i+1}$, with strict inclusion whenever $\mathfrak{a}_i$ is not the maximum in $\mathcal{C}$. (It may be worth remarking on the choice of $\mathfrak{a}_i$'s here—to be completely formal, we are using dependent choice.) We now have an ascending sequence $\mathfrak{a}_0 \subseteq \mathfrak{a}_1 \subseteq \cdots$ of ideals in R. Since R is noetherian, this sequence eventually stabilizes, i.e., there is an index n such that $\mathfrak{a}_i = \mathfrak{a}_n$ for all $i \geq n$. However, by our choice of $\mathfrak{a}_i$'s, this is only possible if $\mathfrak{a}_n$ is the maximum of $\mathcal{C}$. In particular, $\mathfrak{a}_n \in \mathcal{A}$ is an upper bound for $\mathcal{C}$, so the condition for Zorn's lemma is satisfied.

Let $\mathrm{Ann}_R(m) \in \mathcal{A}$ be a maximal element. We already know that $\mathrm{Ann}_R(m)$ is an ideal of R, so it remains to prove it is prime. Suppose $r, s \in R$ with $rs \in \mathrm{Ann}_R(m)$. If $s \notin \mathrm{Ann}_R(m)$, then $sm \neq 0$ and we have

$$\mathrm{Ann}_R(m) \subseteq \mathrm{Ann}_R(sm) \in \mathcal{A}.$$

However, since $\mathrm{Ann}_R(m)$ is maximal in $\mathcal{A}$, we must have equality. And since $rsm = 0$, we conclude that

$$r \in \mathrm{Ann}_R(sm) = \mathrm{Ann}_R(m).$$

Hence, $\mathrm{Ann}_R(m)$ is a prime ideal associated with M. □

5 Let M be a flat R-module.

(a) Show that if R is an integral domain, then $M_{\text{tors}} = \{0\}$.

Proof. We check that M_{tors} is a submodule of M. Given $m, n \in M_{\text{tors}}$ and $r \in R$, there are nonzero scalars $s, t \in R$ such that $sm = tn = 0$. Then

$$st(rm + n) = rt(sm) + s(tn) = rt \cdot 0 + s \cdot 0 = 0,$$

hence $rm + n \in M_{\text{tors}}$. In particular, the inclusion map $M_{\text{tors}} \hookrightarrow M$ is an injective R-module homomorphism.

Let $F = \operatorname{Frac} R$ be the field of fractions of R. The inclusion $R \hookrightarrow F$ is an injective ring homomorphism, so it is also an injective R-module homomorphism. Since M is R-flat, the induced R-module homomorphism $R \otimes_R M \to F \otimes_R M$ is also injective. We now consider the following composition of injective R-module homomorphisms:

$$\begin{array}{ccccccc} M_{\text{tors}} & \hookrightarrow & M & \xrightarrow{\sim} & R \otimes_R M & \longrightarrow & F \otimes_R M \\ & & m & \longmapsto & 1 \otimes m & & \\ & & & & r \otimes m & \longmapsto & \frac{r}{1} \otimes m \end{array}$$

Given $m \in M_{\text{tors}}$ there is a nonzero scalar $r \in R$ such that $rm = 0 \in M$. Under the above map, m is sent to $1 \otimes m \in F \otimes_R M$. However, in $F \otimes_R M$, we have

$$1 \otimes m = \tfrac{r}{r} \otimes m = \tfrac{1}{r} \otimes rm = \tfrac{1}{r} \otimes 0 = 0.$$

In other words, $M_{\text{tors}} \to F \otimes_R M$ is the zero map. But because we already know this map to be injective, we must conclude that $M_{\text{tors}} = 0$. $\square$

(b) Show that for any ideal $I \subset R$ we have

$$I \otimes_R M \simeq IM$$

as R-modules.

Proof. The natural projection of R onto the quotient R/I can be used to construct the following short exact sequence of R-module homomorphisms:

$$0 \longrightarrow I \xhookrightarrow{\iota} R \xrightarrow{\pi} R/I \longrightarrow 0.$$

Since M is R-flat, there is an induced short exact sequence of R-module homomorphisms

$$0 \longrightarrow I \otimes_R M \xrightarrow{\iota\otimes\mathrm{id}_M} R \otimes_R M \xrightarrow{\pi\otimes\mathrm{id}_M} R/I \otimes_R M \longrightarrow 0.$$

By the universal property of the tensor product, these are in fact the unique maps such that the following diagram commutes with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & I \times M \arrow[r, "\iota\times\mathrm{id}_M"] \arrow[d] & R \times M \arrow[r, "\pi\times\mathrm{id}_M"] \arrow[d] & R/I \times M \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & I \otimes_R M \arrow[r, "\iota\otimes\mathrm{id}_M"] & R \otimes_R M \arrow[r, "\pi\otimes\mathrm{id}_M"] & R/I \otimes_R M \arrow[r] & 0
\end{tikzcd}$$

(The horizontal maps are R-linear while the vertical maps are R-bilinear.)

The natural projection of M onto the quotient M/IM can be used to construct the following short exact sequence of R-module homomorphisms:

$$0 \longrightarrow IM \hookrightarrow M \longrightarrow M/IM \longrightarrow 0.$$

Consider the following multiplication maps

$$\begin{aligned} I \times M &\longrightarrow IM \\ (a,m) &\longmapsto am \end{aligned} \qquad \begin{aligned} R \times M &\longrightarrow M \\ (r,m) &\longmapsto rm \end{aligned} \qquad \begin{aligned} R/I \times M &\longrightarrow M/IM \\ (r+I,m) &\longmapsto rm+IM \end{aligned}$$

Once again, these R-bilinear maps make the following diagram commute with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & I \times M \arrow[r] \arrow[d] & R \times M \arrow[r] \arrow[d] & R/I \times M \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & IM \arrow[r] & M \arrow[r] & M/IM \arrow[r] & 0
\end{tikzcd}$$

The universal property of the tensor product gives us unique R-linear maps α, β, γ such that the following diagram commutes with exact rows:

$$
\begin{tikzcd}
0 \arrow[rr] & & I \times M \arrow[rr] \arrow[dl] \arrow[dd] & & R \times M \arrow[rr] \arrow[dl] \arrow[dd] & & R/I \times M \arrow[r] \arrow[dl] \arrow[dd] & 0 \\
0 \arrow[r] & I \otimes_R M \arrow[rr] \arrow[dr, dashed, "\alpha"] & & R \otimes_R M \arrow[rr] \arrow[dr, dashed, "\beta"] & & R/I \otimes_R M \arrow[rr] \arrow[dr, dashed, "\gamma"] & & 0 \\
0 \arrow[rr] & & IM \arrow[rr] & & M \arrow[rr] & & M/IM \arrow[r] & 0
\end{tikzcd}
$$

Notice that β and γ are simply the canonical isomorphisms

$$
\begin{aligned}
R \otimes_R M &\longrightarrow M \\
r \otimes m &\longmapsto rm
\end{aligned}
\qquad\qquad
\begin{aligned}
R/I \otimes_R M &\longrightarrow M/IM \\
(r + I) \otimes m &\longmapsto rm + IM
\end{aligned}
$$

By the 5-lemma (proved below), α is an isomorphism $I \otimes_R M \cong IM$. $\square$

Lemma 1 (5-lemma)**.** Suppose the following diagram of R-module homomorphisms commutes with exact rows:

$$
\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, "\alpha"] & B \arrow[r] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & 0
\end{tikzcd}
$$

If two of the maps α, β, γ are isomorphisms then so is the third.

Proof. Note that a map $\varphi : G \to H$ is an isomorphism if and only if the sequence

$$0 \longrightarrow G \xrightarrow{\varphi} H \longrightarrow 0$$

is exact. That is, φ is an isomorphism if and only if $\ker \varphi = \operatorname{coker} \varphi = 0$.

By the snake lemma, there is an exact sequence

$$0 \to \ker \alpha \to \ker \beta \to \ker \gamma \to \operatorname{coker} \alpha \to \operatorname{coker} \beta \to \operatorname{coker} \gamma \to 0.$$

If α and β are isomorphisms then this becomes

$$0 \longrightarrow 0 \longrightarrow 0 \longrightarrow \ker \gamma \longrightarrow 0 \longrightarrow 0 \longrightarrow \operatorname{coker} \gamma \longrightarrow 0,$$

which means $\ker \gamma = \operatorname{coker} \gamma = 0$. The same argument shows that if α and γ are isomorphisms then $\ker \beta = \operatorname{coker} \beta = 0$, and if β and γ are isomorphisms then $\ker \alpha = \operatorname{coker} \alpha = 0$. $\square$

(c) Let $f : R \to S$ be a ring homomorphism. Show that the map

$$\begin{aligned} M &\to M \otimes_R S \\ m &\mapsto m \otimes 1 \end{aligned}$$

is injective if and only if $\ker(f) \subset \mathrm{Ann}_R(M)$.

(*Hint:* Consider the R-module exact sequence defined by f.)

Proof. Per the hint, there is an exact sequence of R-module homomorphisms

$$0 \longrightarrow \ker f \hookrightarrow R \xrightarrow{f} S.$$

Since M is R-flat, tensoring with M over R will produce another exact sequence. Additionally—similar to part (b)—we use canonical isomorphisms to obtain the following commutative diagram with exact rows:

$$\begin{tikzcd} 0 \arrow[r] & \ker f \otimes_R M \arrow[r] \arrow[d, "\wr"] & R \otimes_R M \arrow[r] \arrow[d, "\wr"] & S \otimes_R M \arrow[d, "\mathrm{id}"] \\ 0 \arrow[r] & (\ker f)M \arrow[r, hook] & M \arrow[r] & S \otimes_R M \end{tikzcd}$$

As the bottom row is exact, the map $M \to S \otimes_R M$ is injective if and only if $(\ker f)M = 0$. This is the case if and only if every element of $\ker f$ annihilates M, i.e., $\ker f \subseteq \mathrm{Ann}_R(M)$. $\square$