1 Exercise 5.6.8 Let k be an algebraically closed field. An n-fold point (over k) is a scheme of the form $X = \operatorname{Spec} R$ such that X has only one point and R is a k-algebra of vector space dimension n over k (i.e., X has length n). Show that every double point is isomorphic to $\operatorname{Spec} k[x]/\langle x^2 \rangle$.

Proof. Suppose $X = \operatorname{Spec} R$ is a double point, i.e., R is a k-algebra with a unique prime ideal and $\dim_k R = 2$. Choose an element $y \in R$ such that $R = ky \oplus k$. Then we must have $y^2 = ay + b$ for some $a, b \in k$. Write $x = y - a/2$ and $c = a^2/4 + b$ for a change of variables, so that $R = kx \oplus k$ with $x^2 = c$. This gives a surjective k-algebra homomorphism $k[x]/\langle x^2 - c\rangle \to R$. In particular, this is a k-linear map and

$$\dim_k k[x]/\langle x^2 - c\rangle = 2 = \dim_k R,$$

so in fact $k[x]/\langle x^2 - c\rangle \cong R$ as k-algebras. Then the unique prime ideal of R corresponds to a unique prime ideal of $k[x]$ containing $\langle x^2 - c\rangle$. If $c \neq 0$, then because k is algebraically closed, we would have a pair of distinct prime ideals $\langle x \pm \sqrt{c}\rangle \trianglelefteq k[x]$ containing $\langle x^2 - c\rangle$. It follows that $c = 0$, and we obtain an isomorphism

$$\operatorname{Spec} k[x]/\langle x^2\rangle \cong \operatorname{Spec} R = X$$

as schemes over k. $\square$

On the other hand, find two non-isomorphic triple points over k, and describe them geometrically.

Use $R = k[x]/\langle x^3\rangle$ and $S = k[x,y]/\langle x^2, xy, y^2\rangle$.

Unique points are $\langle x\rangle$ and $\langle x, y\rangle$, respectively, but second isn't principal, so $R \not\cong S$.

If the double point is a point with a linear direction, we might consider $\operatorname{Spec} R$ to be a point with a quadratic direction.

And we might consider $\operatorname{Spec} S$ to be a point with two linear directions.

2 Exercise 5.6.9 Show that for a scheme X the following are equivalent:

(i) X is reduced, i.e., for every open subset $U \subseteq X$ the ring $\mathcal{O}_X(U)$ has no nilpotent elements.

(ii) For any open subset U_i of an open affine cover $\{U_i\}$ of X, the ring $\mathcal{O}_X(U_i)$ has no nilpotent elements.

(iii) For every point $P \in X$ the local ring $\mathcal{O}_{X,P}$ has no nilpotent elements.

Proof. Assuming (i) is true, (ii) follows trivially.

Assume (ii) is true and let $P \in X$. Choose an affine open cover $\{U_i = \operatorname{Spec} R_i\}$ of X and suppose $P \in U_i$. Then $\mathcal{O}_X(U_i) = R_i$ and

$$\mathcal{O}_{X,P} = \mathcal{O}_{U_i,P} = (R_i)_P$$

Assume for contradiction that $a/b \in (R_i)_P$ is nilpotent with $a^n/b^n = 0 \in (R_i)_P$. That is, $a, b \in R_i$ with $b \notin P$ and $ua^n = 0 \in R_i$ for some $u \in R \setminus P$. The fact that $a/b \neq 0 \in (R_i)_P$ implies $ua \neq 0 \in R_i$, however $(ua)^n = u^n a^n = 0 \in R_i$. That is, ua is a nilpotent element of R_i, which contradicts our assumption of (ii), hence (iii) is true.

Assume (iii) is true and let $U \subseteq X$ be open. Choose an affine open cover $\{U_i = \operatorname{Spec} R_i\}$ of U. Then for each i, the ringed space structure of X gives us a ring homomorphism $\mathcal{O}_X(U) \to \mathcal{O}_X(U_i) = R_i$. Moreover, there is a natural ring homomorphism $R_i \to (R_i)_P$ for each $P \in U_i$. Then an element $f \in \mathcal{O}_X(U)$ is nonzero only if its image in some R_i is nonzero. Now choose $P \in U_i \cap X_f$, so then $f \notin P$, telling us that f_P is a unit in $(R_i)_P$. In particular, f maps to a nonzero element under the ring homomorphism $\mathcal{O}_X(U) \to (R_i)_P = \mathcal{O}_{X,P}$. By assumption of (iii), $\mathcal{O}_{X,P}$ is reduced, so f_P is not nilpotent. It follows that f is not nilpotent (otherwise if $f^n = 0 \in \mathcal{O}_X(U)$ then it would map to $(f_P)^n = 0 \in (R_i)_P$), hence (i) is true. $\square$

3 Exercise 5.6.11 Let $X = Z(x_1^2 x_2 + x_1 x_2^2 x_3) \subseteq \mathbb{A}^3_{\mathbb{C}}$, and denote by π_i the projection to the ith coordinate. Compute the scheme-theoretic fibers $X_{x_i=a} = \pi_i^{-1}(a)$ for all $a \in \mathbb{C}$, and determine the set of isomorphism classes of these schemes.

Let $\mathbb{A}^3_{\mathbb{C}} = \operatorname{Spec} \mathbb{C}[x, y, z]$.

Ler $R = \mathbb{C}[x, y, z]/\langle x^2 y + xy^2 z\rangle$, so $X = \operatorname{Spec} R$.

For the fiber $X_{x=a}$, we have the following corresponding commutative diagram of rings:

$$\begin{tikzcd} R \otimes_{\mathbb{C}[x]} \mathbb{C} & \mathbb{C} \arrow[l, hook'] \\ R \arrow[u, hook] & \mathbb{C}[x] \arrow[l, "\mathrm{eval}_x"] \arrow[u, "\mathrm{eval}_a"'] \end{tikzcd}$$

This means that in $R \otimes_{\mathbb{C}[x]} \mathbb{C}$, we have

$$x \otimes 1 = x(1 \otimes 1) = (1 \otimes a) = a \otimes 1.$$

In other words,

$$R \otimes_{\mathbb{C}[x]} \mathbb{C} = R/\langle x - a\rangle = \mathbb{C}[y, z]/\langle a^2 y + ay^2 z\rangle,$$

so

$$X_{x=a} = \begin{cases} \operatorname{Spec} \mathbb{C}[y, z] & \text{if } a = 0, \\ \operatorname{Spec} \mathbb{C}[y, z]/\langle y(a + yz)\rangle & \text{if } a \neq 0. \end{cases}$$

Similarly, for the fiber $X_{y=a}$, we have the following commutative diagram of rings:

$$\begin{tikzcd} R \otimes_{\mathbb{C}[y]} \mathbb{C} & \mathbb{C} \arrow[l, hook'] \\ R \arrow[u, hook] & \mathbb{C}[y] \arrow[l, "\mathrm{eval}_y"] \arrow[u, "\mathrm{eval}_a"'] \end{tikzcd}$$

This gives us

$$X_{y=a} = \begin{cases} \operatorname{Spec} \mathbb{C}[x, z] & \text{if } a = 0, \\ \operatorname{Spec} \mathbb{C}[x, z]/\langle x(x + az)\rangle & \text{if } a \neq 0. \end{cases}$$

Lastly,

$$X_{z=a} = \begin{cases} \operatorname{Spec} \mathbb{C}[x, y]/\langle x^2 y\rangle & \text{if } a = 0, \\ \operatorname{Spec} \mathbb{C}[x, y]/\langle xy(x + ay)\rangle & \text{if } a \neq 0. \end{cases}$$

We now have the following isomorphism classes for $a \neq 0$:

(i) $X_{x=0} \cong X_{y=0} \cong \mathbb{A}^2$,

(ii) $X_{x=a}$ is the union of a line and a hyperbola,

(iii) $X_{y=a}$ is the union of two lines through the origin,

(iv) $X_{z=0}$ is the union of a two lines through the origin, one having multiplicity 2,

(v) $X_{z=a}$ is the union of three lines through the origin.

4 Exercise 5.6.12 Let X be a prevariety over an algebraically closed field k, and let $P \in X$ be a (closed) point of X. Let $D = \operatorname{Spec} k[x]/\langle x^2\rangle$ be the "double point." Show that the tangent space $T_{X,P}$ to X at P can be canonically identified with the set of morphisms $D \to X$ that map the unique point of D to P.

Proof. Since the tangent space is a local construction, an affine neighborhood of P would have the same tangent space. Without loss of generality, we may assume X is an affine variety, i.e., $X = \operatorname{Spec} R$ where R is a finitely generated reduced k-algebra. Then $\mathfrak{m} = P$ is a maximal ideal of R and the tangent space $T_{X,P}$ is the dual of the k-vector space $\mathfrak{m}/\mathfrak{m}^2$.

(Additionally, the unique point of D is the maximal ideal $\langle \overline{x}\rangle = \langle x\rangle/\langle x^2\rangle$.)

Consider a morphism $f : D \to X$ such that $f(\langle x\rangle) = P$. Since f is a morphism of locally ringed spaces, the induced morphism of rings

$$f^* : R \longrightarrow k[x]/\langle x^2\rangle$$

has $(f^*)^{-1}(\langle \overline{x}\rangle) = \mathfrak{m}$, which implies $\mathfrak{m}^2$ is mapped into $\langle \overline{x}\rangle^2 = \langle 0\rangle$. By the universal property of quotients, f^* uniquely factors to a morphism

$$\overline{f^*} : R/\mathfrak{m}^2 \longrightarrow k[x]/\langle x^2\rangle,$$

which restricts to a k-linear map

$$\overline{f^*}|_{\mathfrak{m}/\mathfrak{m}^2} : \mathfrak{m}/\mathfrak{m}^2 \longrightarrow \langle \overline{x}\rangle = \langle x\rangle/\langle x^2\rangle.$$

Evaluation at 1 is a k-linear map $\operatorname{eval}_1 : \langle \overline{x}\rangle \to k$ (defined by $\overline{x} \mapsto 1$). Composition gives us a k-linear functional

$$\operatorname{eval}_1 \circ \overline{f^*}|_{\mathfrak{m}/\mathfrak{m}^2} : \mathfrak{m}/\mathfrak{m}^2 \longrightarrow k,$$

which is an element of $T_{X,P}$.

On the other hand, given $v \in T_{X,P}$, we can define a k-linear map

$$\begin{aligned} \tilde{v} : \mathfrak{m}/\mathfrak{m}^2 &\longrightarrow \langle \overline{x}\rangle, \\ a &\longmapsto v(a)\overline{x}. \end{aligned}$$

If $\pi : R \to R/\mathfrak{m}^2$ is the quotient map, we obtain a k-linear map

$$\tilde{v} \circ \pi|_{\mathfrak{m}} : \mathfrak{m} \longrightarrow \langle \overline{x}\rangle.$$

Note that this map also agrees with multiplication. (Though a proper ideal is not a ring, we sometimes consider it to be a *rng*—a ring lacking identity. In this sense, the map is a morphism of rngs.)

Since $\mathfrak{m} \trianglelefteq R$ is maximal and k is algebraically closed, we have $R/\mathfrak{m} = k$, so $R = \mathfrak{m} + k$. By the same argument, $k[x]/\langle x^2\rangle = \langle \overline{x}\rangle + k$. We can therefore uniquely extend the map to a k-algebra homomorphism

$$\varphi : R \longrightarrow k[x]/\langle x^2\rangle,$$

where $\varphi|_{\mathfrak{m}} = \tilde{v} \circ \pi|_{\mathfrak{m}}$. This corresponds to a morphism of varieties $D \to X$ that maps the unique point of D to P.

These constructions are inverse to each other, and thus describe the desired identification. $\square$

5 Exercise 6.7.1 Let X be a collection of four distinct points in some $\mathbb{P}^n$. What are the possible Hilbert functions h_X?

Note that the four points must live inside a copy of $\mathbb{P}^3$. A linear change of basis gives

$$X \subseteq \mathbb{P}^3 = Z(x_4, \ldots, x_n) \subseteq \mathbb{P}^n.$$

In which case, $I(X) \subseteq \langle x_4, \ldots, x_n \rangle \trianglelefteq k[x_0, \ldots, x_n]$, meaning that $S(X)$ has no monomials containing $x_4, \ldots, x_n$, so we can consider it to simply be a quotient of $k[x_0, \ldots, x_3]$. In other words, for computing the Hilbert functions, we can assume $X \subseteq \mathbb{P}^3$.

(i) If the four points of X are noncoplanar, then we can perform a linear change of basis to write the points as

$$[1:0:0:0], [0:1:0:0], [0:0:1:0], [0:0:0:1] \in \mathbb{P}^3,$$

then

$$I(X) = \langle x_0^{d_0} \cdots x_3^{d_3} \mid 0 \le d_i \le 2, \textstyle\sum d_i = 3 \rangle \trianglelefteq S = k[x_0, \ldots, x_3].$$

As this is ideal is generated by degree 3 monomials, no smaller monomials are eliminated in $S(X) = S/I(X)$. And for $d \ge 3$, this ideal eliminates all monomials except those of the form x_i^d. That is,

$$S(X)^{(d)} = \begin{cases} S^{(d)} & \text{if } d \le 2, \\ k \cdot \{x_0^d, \ldots, x_3^d\} & \text{if } d \ge 3. \end{cases}$$

Hence,

$$h_X(s) = \begin{cases} \binom{s+3}{s} & \text{if } s \le 2, \\ 4 & \text{if } s \ge 3. \end{cases}$$

(ii) If the four points of X are coplanar, but not colinear, we can perform a linear change of basis to write the points as

$$[1:0:0], [0:1:0], [0:0:1], [a:b:c] \in \mathbb{P}^2,$$

where a and b are nonzero. Then

$$I(X) = \left\langle \begin{matrix} axyz = cx^2y = bx^2z, & bx^2y = cxy^2, \\ bxyz = cxy^2 = ay^2z, & cx^2z = axz^2, \\ cxyz = bxz^2 = ayz^2, & cy^2z = byz^2 \end{matrix} \right\rangle \trianglelefteq S = k[x, y, z].$$

Using these relations, we can rewrite a monomial in $S(X)$ of degree at least 3 and containing at least two variables as a monomial (times a scalar) containing only the variables x and y. Moreover, we can rewrite a monomial of degree at least 3 containing both x and y to have the degree of y be 1. That is,

$$S(X)^{(d)} = \begin{cases} S^{(d)} & \text{if } d \le 2, \\ k \cdot \{x^d, y^d, z^d, x^{d-1}y\} & \text{if } d \ge 3. \end{cases}$$

Hence,

$$h_X(s) = \begin{cases} \binom{s+2}{s} & \text{if } s \le 2, \\ 4 & \text{if } s \ge 3. \end{cases}$$

(iii) If the four points are colinear, we can perform a linear change of basis to write the points as

$$[1 : 0], [0 : 1], [1 : 1], [a : b] \in \mathbb{P}^1,$$