

Example: Homology functors

$$\begin{array}{ccc}
 R\text{-Comp} & \xrightarrow{H_n} & \mathbb{Z}\text{-Mod} \\
 A_\bullet & \mapsto & \ker d_n / \operatorname{im} d_{n+1} \\
 f \downarrow & \mapsto & \downarrow \scriptstyle an + \operatorname{im} d_{n+1} \mapsto f_n(a_n) + \operatorname{im} d'_{n+1} \\
 A'_\bullet & \mapsto & \ker d'_n / \operatorname{im} d'_{n+1}
 \end{array}$$

A **pre-additive category** is a category \mathcal{C} such that for all $A, B \in \operatorname{Ob} \mathcal{C}$ the hom-set $\operatorname{Hom}_{\mathcal{C}}(A, B)$ is an (additive) abelian group such that

$$h \circ (f + g) = hf + hg \quad \text{and} \quad (f + g) \circ k = fk + gk$$

for all suitable $f, g, h, k \in \operatorname{Hom} \mathcal{C}$.

Let \mathcal{C}, \mathcal{D} be pre-additive categories.

A functor $F : \mathcal{C} \rightarrow \mathcal{D}$ is called **additive** if for all $A, B \in \operatorname{Ob} \mathcal{C}$ the map

$$F : \operatorname{Hom}_{\mathcal{C}}(A, B) \longrightarrow \operatorname{Hom}_{\mathcal{D}}(FA, FB)$$

is a homomorphism of abelian groups.

Let $F : R\text{-Mod} \rightarrow S\text{-Mod}$ be an additive functor. Then F induces a an additive functor

$$\begin{aligned}
 \tilde{F} : R\text{-Comp} &\longrightarrow S\text{-Comp}, \\
 A_\bullet &\longmapsto FA_\bullet.
 \end{aligned}$$

For $F : R\text{-Mod} \rightarrow S\text{-Mod}$, we care about e.g. $H_n \circ \tilde{F} : A_\bullet \mapsto H_n(FA_\bullet)$.

Important Example I: hom-functor

$$\begin{aligned}
 \operatorname{Hom}_R({}_R M_S, -) : R\text{-Mod} &\longrightarrow S\text{-Mod}, \\
 \operatorname{Hom}_R(-, {}_R N_T) : R\text{-Mod} &\longrightarrow \operatorname{Mod}\text{-}T.
 \end{aligned}$$

In general, $\operatorname{Hom}_R({}_R M_S, {}_R N_T)$ is an S - T -bimodule.

Important Example II: tensor-functor

Let $M \in \mathbf{Mod}\text{-}R$, $N \in R\text{-}\mathbf{Mod}$.

A **tensor product** of M and N consists of

- an abelian group T ,
- a \mathbb{Z} -bilinear and R -balanced map $\tau : M \times N \rightarrow T$,

satisfying the following universal property:

Whenever $\sigma : M \times N \rightarrow A$ is \mathbb{Z} -bilinear and R -balanced (A abelian group), there exists a unique \mathbb{Z} -linear map $\sigma' : T \rightarrow A$ such that the following diagram commutes:

$$\begin{array}{ccc} M \times N & \xrightarrow{\sigma} & A \\ \tau \downarrow & \nearrow \exists! \sigma' & \\ T & & \end{array}$$

Write $M \otimes_R N := T$ for a (the) tensor product.

Theorem 1. Let $M \in \mathbf{Mod}\text{-}R$ and $N \in R\text{-}\mathbf{Mod}$. There exists a unique up to isomorphism tensor product $M \otimes_R N$.

Proof. Let F be the free abelian group on basis $M \times N$, i.e., $F = \bigoplus_{m,n} \mathbb{Z}_{m,n}$.

Define $M \otimes_R N = F/U$ where U is the submodule generated by all elements of the form

$$\begin{aligned} &(m_1 + m_2, n) - (m_1, n) - (m_2, n), \\ &(m, n_1 + n_2) - (m, n_1) - (m, n_2), \\ &(mr, n) - (m, rn). \end{aligned}$$

Define $\tau : M \times N \rightarrow M \otimes_R N$ by $(m, n) \mapsto m \otimes n := (m, n) + U$. Then τ is \mathbb{Z} -bilinear and R -balanced.

Claim $M \otimes_R N$ with τ satisfied the universal property.

Given $\sigma : M \times N \rightarrow A$ an \mathbb{Z} -bilinear, R -balanced map to abelian group A . Define $\tilde{\sigma} : F \rightarrow A$ by $(m, n) \mapsto \sigma(m, n)$ and extend bilinearly.

By definition of U , $\tilde{\sigma}(U) = 0$.

Hence, there exists a \mathbb{Z} -linear map $\sigma' : F/U \rightarrow A$ with the property that

$$\sigma'(m, n) = \tilde{\sigma}((m, n) + U) = \tilde{\sigma}(m \otimes n).$$

Exercise: show σ' is unique. □

1/23/23 (missed)

define natural transformation, equivalence of categories, morita equivalence of rings, adjoint functors.

1/25/23

category theory is “the mathematician’s filing cabinet.” good tool for transporting information from one area of math to another area, when one has equivalence of categories or sufficiently good functors.

Let $F : \mathcal{C} \rightarrow \mathcal{D}$ and $G : \mathcal{D} \rightarrow \mathcal{C}$ be functors. We say that F and G are **adjoint**, written $F \dashv G$ if the bifunctors $\mathcal{C} \times \mathcal{D} \rightarrow \mathbf{Set}$ are naturally isomorphic:

$$\mathrm{Mor}_{\mathcal{D}}(F(-), -) \cong \mathrm{Mor}_{\mathcal{C}}(-, G(-)).$$

That is, the following diagram commutes:

$$\begin{array}{ccc} & \mathrm{Mor}_{\mathcal{D}}(F(-), -) & \\ \mathcal{C} \times \mathcal{D} & \begin{array}{c} \xrightarrow{\quad} \\ \Downarrow \\ \xrightarrow{\quad} \end{array} & \mathbf{Set} \\ & \mathrm{Mor}_{\mathcal{C}}(-, G(-)) & \end{array}$$

Examples

1.

- (a) $R \otimes_R - \cong \mathrm{id}_{R\text{-Mod}}$.
 $R \otimes_R - : R\text{-Mod} \rightarrow R\text{-Mod}$ well-defined since ${}_R R_R$ is a bimodule.
- (b) $\mathrm{Hom}_{R\text{-Mod}}({}_R R_R, -) \cong \mathrm{id}_{R\text{-Mod}}$

2. R morita equivalent to $M_n(R)$ for $n \in \mathbb{N}$.

Why? Let $F_R = R^n$ and $S = \mathrm{End}_R(F) \cong M_n(R)$ (ring iso).

Let ${}_S F_R$ is a bimodule.

Consider $F^* = {}_R (\mathrm{Hom}_R({}_S F_R, R))_S$.

Claim: The functors $\mathrm{Hom}_R(F, -) : \mathbf{Mod}\text{-}R \rightarrow \mathbf{Mod}\text{-}S$ and $\mathrm{Hom}_S(F^*, -) : \mathbf{Mod}\text{-}S \rightarrow \mathbf{Mod}\text{-}R$ are mutually inverse functors.

Want to show that for $M \in \mathbf{Mod}\text{-}R$, have iso $\mathrm{id}_{\mathbf{Mod}\text{-}R} \cong \mathrm{Hom}_S(F^*, \mathrm{Hom}_R(F, -))$. For $M \in \mathbf{Mod}\text{-}R$, have

$$\begin{aligned} \Phi(M) : M &\longrightarrow \mathrm{Hom}_S(F^*, \mathrm{Hom}_R(F, M)) \\ m &\longmapsto (f \mapsto (x \mapsto mf(x))) \end{aligned}$$

Check $\Phi(M)$ is an R -module homomorphism and in fact is an isomorphism of R -modules.

3. Let $R = K$ be a field. Then we have a duality $(-)^* : K\text{-mod} \rightarrow K\text{-mod}$, $V \mapsto V^*$ (dual vector space). Then there is a natural isomorphism $\Phi : \text{id}_{K\text{-mod}} \Rightarrow (-)^{**}$

$$\begin{aligned}\Phi(V) : V &\longrightarrow V^{**} = \text{Hom}_K(\text{Hom}_K(V, K), K), \\ x &\longmapsto \text{eval}_x = (f \mapsto f(x)).\end{aligned}$$

Then $(\Phi(V))_{V \in K\text{-mod}}$ is a duality $K\text{-mod} \rightarrow K\text{-mod}$.

We may extend Φ to a functor $K\text{-Mod} \rightarrow K\text{-Mod}$ contravariant.

$\Phi(V) : V \rightarrow V^{**}$ is not surjective if $\dim V = \infty$ (hw problem).

Example of adjoint pair.

slogan: “tensor functors are left-adjoint to hom-functors”

Let ${}_S B_R$ be an S - R -bimodule.

Then the functors

$$B \otimes_R - : R \longrightarrow R\text{-Mod}$$

is left-adjoint to

$$\text{Hom}_S(B, -) : S\text{-Mod} \longrightarrow R\text{-Mod}.$$

1/27/2023

An **additive category** is a pre-additive category \mathcal{C} together with the following additional data

- a **zero object** $0 \in \mathcal{C}$

such that

- for every $A \in \mathcal{C}$ the hom-sets $\text{Hom}(0, A)$ and $\text{Hom}(A, 0)$ are singletons;
- \mathcal{C} has all finite direct sums and finite direct products;

Additionally, say \mathcal{C} is **abelian** if

- every morphism in \mathcal{C} has a kernel and cokernel;
- every monomorphism is a kernel and every epimorphism is a cokernel.

Example

In **Ring**, there are categorical epimorphisms which fail to be surjective, e.g., $f : \mathbb{Z} \hookrightarrow \mathbb{Q}$.

Let $g, h \in \text{Hom}_{\text{Ring}}(\mathbb{Q}, R)$ be such that $gf = hf$, i.e., $g|_{\mathbb{Z}} = h|_{\mathbb{Z}}$. It follows that $g = h$. For $b \in \mathbb{Z} \setminus \{0\}$,

$$1_R = g(1_{\mathbb{Q}}) = g(b \cdot 1/b) = g(b)g(1/b) = g(1/b)g(b).$$

Hence, $g(b)$ is a unit in R and $g(1/b) = g(b)^{-1}$.

In $R\text{-Mod}$, $R\text{-Comp}$, monomorphisms/epimorphisms coincide with injective/surjective homomorphisms.

Example

abelian: $R\text{-Mod}$, in particular $\mathbb{Z}\text{-Mod} = \text{Ab}$.

$R\text{-Comp}$, let $\mathcal{T}\text{-Ab}$ be the full subcategory of Ab consisting of the torsion abelian groups ($A \in \text{Ab}$ such that for all $a \in A$ there exists $n \in \mathbb{Z} \setminus \{0\}$ such that $na = 0$).

What about the full subcategory $\mathcal{T}\text{-Ab}$ having objects as torsion-free abelian groups? No.

Further non-abelian: $R\text{-mod}$ is not abelian if R is left-noetherian

e.g., $R = K^{\mathbb{N}}$ with K a field. $I = K^{(\mathbb{N})}$ is not finitely generated as a left ideal, and canonical quotient map $\pi : R \twoheadrightarrow R/I$ in $R\text{-mod}$ does not have a kernel.

Note: $K^{\mathbb{N}} = \prod_{i \in \mathbb{N}} K_i$ with $K_i = K$ and $K^{(\mathbb{N})} = \{(k_i) \in K^{\mathbb{N}} \text{ with finite support}\}$.

Remark.

$$\begin{array}{cccc} R\text{-Mod} & \rightsquigarrow & R\text{-Comp} & \rightsquigarrow & H(R\text{-Comp}) & \rightsquigarrow & D(R\text{-Mod}) \\ \text{ab} & & \text{ab} & & \text{not ab} & & \text{not ab} \\ & & & & \text{in general} & & \text{derived category} \end{array}$$

On the road to derived functors.

Note: we will develop the theory for the abelian category $R\text{-Mod}$, but it easily adapts to arbitrary categories.

Let R and S be rings, $F : R\text{-Mod} \rightarrow S\text{-Mod}$ an additive functor (covariant or contravariant).

Say covariant F is **exact** if for all short exact sequences

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

in $R\text{-Mod}$, the image sequence

$$0 \longrightarrow FA \xrightarrow{Ff} FB \xrightarrow{Fg} FC \longrightarrow 0$$

is exact in $S\text{-Mod}$.

(In the case that F is contravariant, the definition is the same with respect to the with appropriate image sequence.)

Say covariant F is left-exact if for every exact sequence

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C$$

in $R\text{-Mod}$, (equivalent if add $C \rightarrow 0$) the image sequence

$$0 \longrightarrow FA \xrightarrow{Ff} FB \xrightarrow{Fg} FC$$

is exact in $S\text{-Mod}$.

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2/3/23

Recall: properties of groups

- torsion and torsion-free,
- divisible and reduced (no nonzero divisible subgroup).

Then

- $\text{Hom}_{\mathbb{Z}}(\text{torsion}, \text{torsion-free}) = 0$,
- $\text{Hom}_{\mathbb{Z}}(\text{divisible}, \text{reduced}) = 0$.

Examples showing that hom-functors fail to be right exact and tensor-functors fail to be left exact, in general.

1. $\text{Hom}_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, -)$ is not right exact, $n \geq 2$. Apply the canonical map $\mathbb{Z} \rightarrow \mathbb{Z}/n\mathbb{Z}$, get

$$\text{Hom}_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}) \longrightarrow \text{Hom}_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/n\mathbb{Z})$$

2. $F = \text{Hom}_{\mathbb{Z}}(\mathbb{Q}, -)$ is not right exact.

Take ${}_R M \in R\text{-mod}$, then

$$\begin{aligned} {}_R R^{(M)} &\longrightarrow {}_R M \\ 1_R &\longmapsto m \end{aligned}$$

Consider an epimorphism $g : \mathbb{Z}^{(\mathbb{N})} \rightarrow \mathbb{Q}$. Apply F to get

$$\text{Hom}_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z}^{(\mathbb{N})}) \longrightarrow \text{Hom}_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Q}).$$

\mathbb{Q} is divisible, $\mathbb{Z}^{\mathbb{N}}$ is reduced so left is zero, but right is nonzero.

3. $R = \mathbb{Z}$, $F = \mathbb{Z}/n\mathbb{Z} \otimes_{\mathbb{Z}} -$, $n \geq 2$. Apply F to embedding $\mathbb{Z} \hookrightarrow \mathbb{Q}$ to get

$$\mathbb{Z}/n\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z} \longrightarrow \mathbb{Z}/n\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q}.$$

Left is $\mathbb{Z}/n\mathbb{Z}$ but right is zero.

mnemonic torsion $\otimes_{\mathbb{Z}}$ divisible = 0.

Short-term program:

1. find exact functors
2. characterize exact sequences $A_{\bullet} \in R\text{-Mod}$ such that $F(A_{\bullet}) \in S\text{-Mod}$ is exact for any additive functor $F : R\text{-Mod} \rightarrow S\text{-Mod}$.

Theorem 2 (4). If $F : R\text{-Mod} \rightarrow S\text{-Mod}$ is an exact functor (sends SES's to SES's), then $F(A_{\bullet})$ is exact in $S\text{-Mod}$ whenever A_{\bullet} is exact in $R\text{-Mod}$.

Proof. Let $F : R\text{-Mod} \rightarrow S\text{-Mod}$ be exact and A_{\bullet} an exact sequence in $R\text{-Mod}$.

Then F sends A_{\bullet} to complex $F(A_{\bullet})$ in $S\text{-Mod}$.

Factor each $f_n : A_n \rightarrow A_{n-1}$ as

$$A_n \xrightarrow{\tilde{f}_n} \text{im } f_n \xrightarrow{\iota_n} A_{n-1}$$

Consider the short exact sequences for $n \in \mathbb{Z}$

$$0 \longrightarrow \text{im } \iota_{n+1} \xrightarrow{\iota_{n+1}} A_n \xrightarrow{\tilde{f}_n} \text{im } \tilde{f}_n = \text{im } f_n \longrightarrow 0.$$

If F is exact, we obtain a short exact sequence

$$0 \longrightarrow F(\text{im } \iota_{n+1}) \xrightarrow{F(\iota_{n+1})} F(A_n) \xrightarrow{F(\tilde{f}_n)} F(\text{im } \tilde{f}_n) \longrightarrow 0.$$

In particular,

$$\text{im } F(\iota_{n+1}) = \ker F(\tilde{f}_n)$$

for $n \in \mathbb{N}$.

Claim.

- (a) $\ker F(f_n) = \ker F(\tilde{f}_n)$,
- (b) $\text{im } F(f_{n+1}) = \text{im } F(\iota_{n+1})$.

Proof.

- (a) $f_n = \iota_n \circ \tilde{f}_n$, so $F(f_n) = F(\iota_n) \circ F(\tilde{f}_n)$.
 Since F is exact, $F(\iota_n)$ is a monomorphism.
 Hence, $\ker F(f_n) = \ker F(\tilde{f}_n)$.
- (b) $F(f_{n+1}) = F(\iota_{n+1}) \circ F(\tilde{f}_{n+1})$.
 Since F is exact, $F(\tilde{f}_{n+1})$ is an epimorphism.
 Hence, $\text{im } F(f_{n+1}) = \text{im } F(\iota_{n+1})$

Combining all the stuff completes the proof. \square

First installment of program part 1.

Proposition 1 (5). Let I be any indexing set and consider the functors

$$\begin{array}{ll}
 (R\text{-Mod})^I \longrightarrow R\text{-Mod} & (R\text{-Mod})^{(I)} \longrightarrow R\text{-Mod} \\
 (M_i)_{i \in I} \longmapsto \prod_{i \in I} M_i & (M_i)_{i \in I} \longmapsto \bigoplus_{i \in I} M_i \\
 (f_i)_{i \in I} \longmapsto \prod_{i \in I} f_i & (f_i)_{i \in I} \longmapsto \bigoplus_{i \in I} f_i.
 \end{array}$$

Both functors are exact.

Proof. obvious \square

First installment of program part 2.

Let $A, B \in R\text{-Mod}$ and $f \in \text{Hom}_R(A, B)$.

Say f is **split** if $\ker f$ is a direct summand of A and $\text{im } f$ is a direct summand of B , i.e., there exist $A', B' \in R\text{-Mod}$ with $A = \ker f \oplus A'$ and $B = \text{im } f \oplus B'$.

In Birge notation, f split if $\ker f \subseteq^\oplus A$ and $\text{im } f \subseteq^\oplus B$.

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Q: Is the following true or false? $P \in R\text{-Mod}$ finitely generated projective, then there exists $n \in \mathbb{N}$ with P isomorphic to a direct summand of ${}_R R^n$.

True.

Up to isomorphism, $P \subseteq^\oplus {}_R R^{(I)} = \bigoplus_{i \in I} R_i$ with $R_i = R$.

With $P = Rx_1 + \cdots + Rx_m$.

Pick finite $I' \subseteq I$ with $x_k \in \bigoplus_{i \in I'} R_i$ for all $k \leq m$.

Then $P \subseteq \bigoplus_{i \in I'} R_i \cong R^n$ with $n = |I'|$ and hence $P \subseteq^\oplus R^n$.

In general, ${}_R V \subseteq U \subseteq {}_R M$, if $V \subseteq^\oplus M$ then $V \subseteq^\oplus U$. Look up the “modular law.”

Theorem 3 (9). For $P \in R\text{-Mod}$, the following are equivalent

- (1) ${}_R P$ is projective (direct summand of free left R -module);
- (2) $\text{Hom}_R(P, -) : R\text{-Mod} \rightarrow \mathbb{Z}\text{-Mod}$ is exact;
- (3) whenever $f \in \text{Hom}_R(B, C)$ is an epimorphism and $g \in \text{Hom}_R(P, C)$, there exists $\varphi \in \text{Hom}_R(P, B)$ with $f \circ \varphi = g$, i.e., each diagram of the following format

$$\begin{array}{ccccc} & & P & & \\ & \swarrow \varphi & \downarrow g & & \\ B & \xrightarrow{f} & C & \longrightarrow & 0 \end{array}$$

can be supplemented to a commutative triangle;

- (4) every epimorphism onto M is split.

Remark. Will write $[M, -] := \text{Hom}_R(M, -)$.

Proof. Follow sequence (1) \implies (3) \implies (2) \implies (4) \implies (1).

(1) \implies (3).

have $M \leq^\oplus {}_R F$ with F free on basis $\{x_i\}_{i \in I}$, $F = M \oplus N$. Let f and g be as under (3)

$$\begin{array}{ccccc} B & \xrightarrow{f} & C & \longrightarrow & 0 \\ & \nwarrow \psi & \uparrow g & & \\ & & M & & \\ & & \uparrow \pi & & \\ & & F & & \end{array}$$

where $\varphi : F \rightarrow M$ is the projection along N and $\iota : M \hookrightarrow F$ to be the embedding. Define $\psi \in \text{Hom}_R(F, B)$ via $\psi(x_i) = b_i$ if $f(b_i) = g\pi(x_i)$. (This is how it has to be.) This is well-defined because F is free, so we can choose any images of the basis elements x_i ; simply choosing any $b_i \in f^{-1}g\pi(x_i)$, which exist because f is an epimorphism.

Then $f \circ \psi = g \circ \pi$ and thus $g \circ (\psi \circ \iota) = g \circ \pi \circ \iota = g$.

Hence, define $\varphi = \psi \circ \iota$ is as required.

$$\begin{array}{ccccc}
B & \xrightarrow{f} & C & \longrightarrow & 0 \\
& \nwarrow \varphi & \uparrow g & & \\
& & M & & \\
& \swarrow \psi & \uparrow \pi & & \\
& & F & &
\end{array}$$

(3) \implies (2).

Since $[M, -]$ is left exact, it suffices to show that $[M, -]$ takes epis to epis.

Let $B \xrightarrow{f} C \rightarrow 0$ be exact.

To see that $[M, f] : [M, B] \rightarrow [M, C]$ is an epi, let $g \in [M, C]$.

By (3), there exists $\varphi \in [M, B]$ with $[M, f](\varphi) = f \circ \varphi = g$.

(2) \implies (4).

Let $f : N \rightarrow M$ be an epimorphism.

Then by (2), the map $[M, f] : [M, N] \rightarrow [M, M]$ is an epimorphism.

In particular, $\text{id}_M \in \text{im}[M, f]$ —there exists $\varphi \in [M, N]$ with $f \circ \varphi = \text{id}_M$, i.e.,

$$\begin{array}{ccc}
M & \xrightarrow{\varphi} & N \\
\cong \parallel & \swarrow f & \\
M & &
\end{array}$$

Hence, $N = \text{im } \varphi \oplus \ker f$, so $\ker f \leq^\oplus N$ and indeed f splits.

(4) \implies (1).

If $\{m_i\}_{i \in I}$ is a generating set for M , then $F = R^{(I)} \rightarrow M$ sending $(r_i) \mapsto \sum_{\text{finite}} r_i m_i$ is an epimorphism. By (4), f splits, i.e., $F = \ker f \oplus N$. Thus $N \cong F / \ker f \cong M$, i.e., (1) holds. \square

Examples of projective modules and structure results.

1. R a ring. All $M \in R\text{-Mod}$ are projective if and only if every submodule U of any left R module N is a direct summand $N = U \oplus V$.

(Every $R\text{-Mod}$ is projective iff every $R\text{-Mod}$ is semisimple.)

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$M \in R\text{-Mod}$.

Recall. A projective resolution of M is any exact sequence

$$\cdots \longrightarrow P_2 \xrightarrow{f_2} P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} M \longrightarrow 0$$

with all P_i projective.

Example

$R = \mathbb{Z}/p^n\mathbb{Z}$, p prime, $n \geq 1$.

Write $\bar{x} = x + p^n\mathbb{Z}$ and consider $M = R\bar{p}^{n-1}$.

A projective resolution of M is

$$\begin{array}{ccccccc} \cdots & \rightarrow & R & \xrightarrow{f_2} & R & \xrightarrow{f_1} & R \xrightarrow{f_0} M = R\bar{p}^{n-1} \rightarrow 0 \\ & & \searrow r \mapsto r\bar{p}^{n-1} & & \searrow r \mapsto r\bar{p}^n & & \searrow r \mapsto r\bar{p}^{n-1} \\ & & \ker f_1 = R\bar{p}^{n-1} & & \ker f_0 = R\bar{p} & & \end{array}$$

All kernels fail to be free.

R is a local ring: $R \geq R\bar{p} \geq R\bar{p}^2 \geq \cdots \geq R\bar{p}^n = 0$.

These are all ideals, so $R\bar{p}$ is the only maximal ideal. Since projectives over a local ring are free, the modules $R\bar{p}$ and $R\bar{p}^{n-1}$ are not projective.

Let $M \in R\text{-Mod}$ and

$$\cdots \longrightarrow P_2 \xrightarrow{f_2} P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} M \longrightarrow 0$$

a projective resolution. Then $\text{im } f_n = \ker f_{n-1}$ is called an **n th syzygy** of M .

(“syzygy” from astronomy term “syn-zygon” (yoked together).)

Remark. syzygies are not unique up to isomorphism.

e.g., $R\bar{p}$ is a first syzygy of $M = R\bar{p}^{n-1}$.

But consider

$$\begin{array}{ccc} R \oplus R & \xrightarrow{g_0} & M \longrightarrow 0 \\ (x, y) & \longmapsto & x\bar{p}^{n-1} \end{array}$$

Then $\ker g_0 = R\bar{p} \oplus R$, also a first syzygy of M .

Lemma 1 (15 Schanuel). Let $f : P \rightarrow M$, $g : Q \rightarrow M$ be epimorphisms in $R\text{-Mod}$ with P and Q projective. Then $\ker f \oplus Q \cong \ker g \oplus P$.

“First syzygies of a module are unique up to projective summands.”

Proof. Consider the following commutative diagram:

$$\begin{array}{ccccccc}
 & & & & 0 & & \\
 & & & & \downarrow & & \\
 & & & & \ker g & & \\
 & & & & \downarrow & & \\
 & & & & Q & & \\
 & & & & \downarrow g & & \\
 0 & \longrightarrow & \ker f & \hookrightarrow & P & \xrightarrow{f} & M \longrightarrow 0 \\
 & & & & & & \downarrow \\
 & & & & & & 0
 \end{array}$$

Let $L = P \times_M Q$ be the pullback of $f : P \rightarrow M$ and $g : Q \rightarrow M$ with projections $\pi_P : L \rightarrow P$ and $\pi_Q : L \rightarrow Q$:

$$\begin{array}{ccccccc}
 & & & & 0 & & \\
 & & & & \downarrow & & \\
 & & & & \ker g & & \\
 & & & & \downarrow & & \\
 & & & & Q & & \\
 & & L & \xrightarrow{\pi_Q} & \downarrow g & & \\
 & & \downarrow \pi_P & & M & & \\
 0 & \longrightarrow & \ker f & \hookrightarrow & P & \xrightarrow{f} & M \longrightarrow 0 \\
 & & & & & & \downarrow \\
 & & & & & & 0
 \end{array}$$

By the universal property of the pullback, we have unique maps i and j satisfying

$$\begin{array}{ccc}
 \ker f & \xrightarrow{0} & \ker g \\
 \downarrow i & & \downarrow j \\
 L & \xrightarrow{\pi_Q} & Q \\
 \downarrow \pi_P & & \downarrow g \\
 P & \xrightarrow{f} & M
 \end{array}$$

Note that the perimeter of these diagrams commute because the composition of a map and the inclusion of its kernel is always zero. Combining these into our original diagram, we get

$$\begin{array}{ccccccc}
& & & & 0 & & \\
& & & & \downarrow & & \\
& & & \ker g & \xlongequal{\quad} & \ker g & \\
& & & \downarrow j & & \downarrow & \\
& \ker f & \xrightarrow{i} & L & \xrightarrow{\pi_Q} & Q & \\
& \parallel & & \downarrow \pi_P & & \downarrow g & \\
0 & \longrightarrow & \ker f & \hookrightarrow & P & \xrightarrow{f} & M \longrightarrow 0 \\
& & & & \downarrow & & \\
& & & & 0 & &
\end{array}$$

Add zeros in places to get following diagram:

$$\begin{array}{ccccccc}
& & & 0 & & 0 & \\
& & & \downarrow & & \downarrow & \\
& & & \ker g & \xlongequal{\quad} & \ker g & \\
& & & \downarrow j & & \downarrow & \\
0 & \longrightarrow & \ker f & \xrightarrow{i} & L & \xrightarrow{\pi_Q} & Q \longrightarrow 0 \\
& & \parallel & & \downarrow \pi_P & & \downarrow g \\
0 & \longrightarrow & \ker f & \hookrightarrow & P & \xrightarrow{f} & M \longrightarrow 0 \\
& & & & \downarrow & & \downarrow \\
& & & & 0 & & 0
\end{array}$$

Claim that center row and column are exact.

Easy to check that i and j are monomorphisms.

By construction, $\pi_Q \circ i = 0$ and $\pi_P \circ j = 0$.

Say $\ell \in \ker \pi_Q$, then

$$f(\pi_P(\ell)) = g(\pi_Q(\ell)) = g(0) = 0.$$

Therefore, $\pi_P(\ell) \in \ker f$, but then $\ell = i(\pi_P(\ell)) \in \operatorname{im} i$. Hence, $\operatorname{im} i = \ker \pi_Q$.

Similarly, $\operatorname{im} j = \ker \pi_P$.

Given $y \in Q$, since f is an epimorphism there exists $x \in P$ such that $f(x) = g(y)$. With a concrete construction of the pullback, it is clear that (x, y) should be an element of L . In which case, $\pi_Q(x, y) = y$, so π_Q is an epimorphism.

Since P and Q are projective, both π_P and π_Q split, hence

$$P \oplus \ker g \cong L \cong Q \oplus \ker f.$$

□

Lemma 2 (16). Suppose U_n and V_n are n th syzygies of M in $R\text{-Mod}$. Then there exists projective modules X_n, Y_n with $U_n \oplus X_n \cong V_n \oplus Y_n$.

“All the n th syzygies are unique up to projective summands.”

Proof. Schanuel + induction [Gauchospace]

□

Theorem 4 (17).

(1) For $M \in R\text{-Mod}$, the following are equivalent:

- (a) The projective dimension $\text{pdim } M \leq n < \infty$;
- (b) There exists a projective resolution

$$0 \longrightarrow P_n \xrightarrow{f_n} P_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_0} M \longrightarrow 0$$

such that $\text{im } f_n = \ker f_{n-1}$ (an n th syzygy) is projective;

- (c) In all projective resolutions of M , the n th syzygy is projective.

(2) For $M \in R\text{-Mod}$, the following are equivalent:

- (a) $\text{pdim } M = \infty$;
- (b) There exists a non-projective resolution of M with a projective n th syzygy;
- (c) In all projective resolutions of M , the n th syzygy is non-projective.

Proof. (1)

(a \implies b)

Suppose $\text{pdim } M \leq n$, i.e., there is a projective resolution

$$0 \longrightarrow P_n \xrightarrow{f_n} P_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_0} M \longrightarrow 0$$

Then $\text{im } f_n \cong P_n$ is projective.

(b \implies c)

Suppose (b) holds, and let

$$\cdots \longrightarrow Q_2 \xrightarrow{g_2} Q_1 \xrightarrow{g_1} Q_0 \xrightarrow{g_0} M \longrightarrow 0$$

be another projective resolution of M . By Lemma 16, $\text{im } g_n = \ker g_{n-1}$ differs only by projective summands from $\text{im } f_n = \ker f_{n-1}$. Therefore, since $\ker f_{n-1}$ is projective, so is $\ker g_{n-1}$.

(c \implies a)

Assume (c) and let

$$\cdots \longrightarrow P_{n+1} \xrightarrow{f_{n+1}} P_n \xrightarrow{f_n} P_{n-1} \xrightarrow{f_{n-1}} \cdots \longrightarrow P_0 \xrightarrow{f_0} M \longrightarrow 0$$

be a projective resolution of M . Then $\text{im } f_n = \ker f_{n-1}$ is projective by hypothesis. So consider the sequence

$$0 \longrightarrow \ker f_{n-1} \hookrightarrow P_{n-1} \xrightarrow{f_{n-1}} P_{n-2} \xrightarrow{f_{n-2}} \cdots \xrightarrow{f_0} M \longrightarrow 0$$

This is a projective resolution of length $\leq n$. □

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$M \in R\text{-Mod}$ is **injective** if and only if $\text{Hom}_R(-, M) : R\text{-Mod} \rightarrow \mathbb{Z}\text{-Mod}$ is exact.

Theorem 5 (18). For $E \in R\text{-Mod}$, the following are equivalent:

- (1) ${}_R E$ is injective;
- (2) whenever $f : X \rightarrow Y$ is an monomorphism and $g : X \rightarrow E$ is any, there exists ψ with $\psi \circ f = g$, i.e., each diagram of the format

$$\begin{array}{ccc} 0 & \longrightarrow & X \xrightarrow{f} Y \\ & & \downarrow g \quad \swarrow \psi \\ & & E \end{array}$$

can be supplement to a commutative triangle;

- (3) every monomorphism with domain M is split.

Remarks.

- (a) $M_i \in R\text{-Mod}$ for $i \in I$, then $\coprod M_i$ injective iff all M_i injective.
- (b) All left R -modules injective iff R semisimple iff all right R -modules injective.
- (c) \mathbb{Z} , $\mathbb{Z}/m\mathbb{Z}$ ($m \geq 2$) not injective

Lemma 3 (19 Baer's Criterion). Let $M \in R\text{-Mod}$. Then the following are equivalent:

- (1) M is injective;
- (2) For every diagram with $I \trianglelefteq R$ left ideal

$$\begin{array}{ccc} I & \hookrightarrow & R \\ \downarrow \varphi & & \swarrow \Phi \\ M & & \end{array}$$

there exists Φ with $\Phi|_I = \varphi$

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Proof. (2 \implies 1)

Without loss of generality, $X \leq Y$ is a submodule and f the embedding. To find $\psi \in \text{Hom}(Y, M)$ with $g = \psi \circ f$, we picked a maximal element (Y_0, g_0) in the poset

$$\mathcal{P} = \{(Y', g') \mid X \leq Y' \leq Y, g'|_X = g\}.$$

Assume $Y_0 \subset Y$ and pick $\hat{y} \in Y \setminus Y_0$. Define

$$I = \{r \in R \mid r\hat{y} \in Y_0\} \subseteq {}_R R$$

and $\varphi \in \text{Hom}_R(I, M)$ with $\varphi(r) = g_0(r\hat{y})$.

By (2), there exists $\Phi \in \text{Hom}_R(R, M)$ with $\Phi|_I = \varphi$.

Set $Y_1 = Y_0 + R\hat{y}$ and define $g_1 : Y_1 \rightarrow M$ with $y + r\hat{y} \mapsto g_0(y) + \Phi(r)$ if $y \in Y_0$ and $r \in R$.

Well-definedness of g_1 . Suppose $y_1 + r_1\hat{y} = y_2 + r_2\hat{y}$ with $y_i \in Y_0$ and $r_i \in R$. Then

$$\begin{aligned} 0 &= g_0(0) \\ &= g_0((y_1 + r_1\hat{y}) - (y_2 + r_2\hat{y})) \\ &= g_0((y_1 - y_2) + (r_1 - r_2)\hat{y}) \\ &= g_0(y_1 - y_2) + g_0((r_1 - r_2)\hat{y}) \\ &= g_0(y_1) - g_0(y_2) + \varphi(r_1 - r_2) \\ &= g_0(y_1) - g_0(y_2) + \Phi(r_1 - r_2) \\ &= (g_0(y_1) + \Phi(r_1)) - (g_0(y_2) + \Phi(r_2)) \\ &= g_1(y_1 + r_1\hat{y}) - g_1(y_2 + r_2\hat{y}). \end{aligned}$$

□

Theorem 6 (20). Let R be a PID and $M \in R\text{-Mod}$, then FTAE:

- (1) M is injective;
- (2) M is divisible, i.e., $aM = M$ for all $a \in R \setminus \{0\}$;

Proof. (1 \implies 2)

Assume (1), and let $a \in R \setminus \{0\}$. Fix $x \in M$ and consider

$$\begin{array}{ccc} ra & & I = Ra \hookrightarrow R \\ \downarrow & & \downarrow g \quad \swarrow \exists \psi \\ rx & & M \end{array}$$

So $g(a) = g(a \cdot 1) = \psi(a \cdot 1) = a\psi(1) \in aM$.

(2 \implies 1)

Assume (2). We will apply Baer, so let ${}_RI \hookrightarrow {}_RR$ and $\varphi \in \text{Hom}_R(I, M)$.

Know $I = Ra$ wlog $a \neq 0$. By hypothesis, there exists $x \in M$ with $\varphi(a) = a \cdot x$. Define $\Phi \in \text{Hom}(R, M)$ via $r \mapsto rx$. Check that $\Phi|_I = \varphi$. \square

Supplement: The divisible abelian groups.

Examples.

1. \mathbb{Q}
2. $\mathbb{Z}(p^\infty)$ “Prüfer groups” for prime p , where

$$\mathbb{Z}(p^\infty) := \bigoplus_{i \in \mathbb{N}} \mathbb{Z}x_i / U(p)$$

where $U(p)$ is the subgroup of $\bigoplus_{i \in \mathbb{N}} \mathbb{Z}x_i$, generated by $px_1, px_{i+1} - x_i$ for all $i \in \mathbb{N}$.

Find that $\mathbb{Z}\bar{x}_i \cong \mathbb{Z}/p^i\mathbb{Z}$. Get embeddings

$$\begin{aligned} \mathbb{Z}\bar{x}_i \cong \mathbb{Z}/p^i\mathbb{Z} &\hookrightarrow \mathbb{Z}/p^{i+1}\mathbb{Z} \\ z + p^i\mathbb{Z} &\longmapsto pz + p^{i+1}\mathbb{Z} \end{aligned}$$

With $\mathbb{Z}\bar{x}_i \cong \mathbb{Z}/p^i\mathbb{Z}$, can visualize

$$\mathbb{Z}\bar{x}_1 \xhookrightarrow{p} \mathbb{Z}\bar{x}_2 \xhookrightarrow{p} \mathbb{Z}\bar{x}_3 \xhookrightarrow{p} \mathbb{Z}\bar{x}_4 \xhookrightarrow{p} \dots \hookrightarrow \mathbb{Z}(p^\infty)$$

Alternate descriptions:

$$\mathbb{Z}(p^\infty) = T_p(\mathbb{Q}/\mathbb{Z}) = \{x + \mathbb{Z} \mid p^n x \in \mathbb{Z} \text{ for some } n \in \mathbb{N}\}.$$

Reminder: T a torsion abelian group. Then

$$T = \bigoplus_{p \text{ prime}} T_p \quad \text{where} \quad T_p = \{x \in T \mid p^n x = 0 \text{ for some } n \in \mathbb{N}\}.$$

Also

$$\mathbb{Z}(p^\infty) \cong \{x \in \mathbb{C} \mid x^{p^n} = 1 \text{ for some } n \in \mathbb{N}\} \quad (\text{multiplicative group}).$$

Theorem 7 (21). $A \in \mathbb{Z}\text{-Mod}$ is divisible iff

$$A \cong \mathbb{Q}^{(I)} \oplus \bigoplus_{p \text{ prime}} (\mathbb{Z}(p^\infty))^{(I_p)}$$

Proof. Exercise. □

Injective Resolutions.

Theorem 8 (22 Echmann). Every left R -module is a submodule of (or embeds into) an injective module.

Lemma 4 (23). If $M \in \mathbb{Z}\text{-Mod}$, then there exists a divisible $D \in \mathbb{Z}\text{-Mod}$ with $M \xrightarrow{\cong} D$.

Every abelian group can be embedded as a subgroup of some divisible (injective) abelian group.

Proof. We know $M \cong \mathbb{Z}^{(I)}/K \subseteq \mathbb{Q}^{(I)}/K$ (divisible) for some subgroup $K \leq \mathbb{Z}^{(I)}$. □

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A right R -module M is **flat** in case the tensor functor $M \otimes_R - : R\text{-Mod} \rightarrow \mathbb{Z}\text{-Mod}$ is exact. Flatness of a left R -module is defined symmetrically.

Remarks.

- (1) Any tensor functor $M \otimes_R -$ is right exact, hence M_R is flat precisely when $M \otimes_R -$ takes monomorphisms to monomorphisms.
 - (2) Since tensor products commute with arbitrary direct sums, a direct sum $\bigoplus_{i \in I} M_i$ is flat if and only if each M_i is flat.
 - (3) The regular R -module R_R is flat, because $R \otimes_R - \cong \text{id}_{R\text{-Mod}}$. In light of the preceding remark, we find that every free R -module is flat, and hence so is every projective R -module. In particular, if R is semisimple, then all R -modules are flat.
Caution: Here the converse does not hold. In fact, all right R -modules are flat if and only if R is von Neumann regular, meaning that for any $r \in R$ there exists $s \in R$ with $rsr = r$. For instance: the endomorphism rings of infinite dimensional vector spaces are von Neumann regular but not semisimple.
 - (4) Using the adjointness of tensor and hom functors, one obtains: If ${}_S M_R$ is a bimodule such that M_R is flat and $N \in S\text{-Mod}$ is injective, then ${}_R \text{Hom}_S(M, N)$ is injective in $R\text{-Mod}$. The proof is a carbon copy of that for injectivity of ${}_R \text{Hom}_{\mathbb{Z}}(R, D)$ in case D is injective in $\mathbb{Z}\text{-Mod}$.
-

Lemma 5. If D is a divisible abelian group, then $\text{Hom}_{\mathbb{Z}}(R, D)$ is an injective left R -module.

Generalize: ${}_S M_R$ bimodule with M_R flat and ${}_S N$ injective, then $R(\text{Hom}_S(M, N))$ is injective R -module.

Proof. Consider a short exact sequence of left R -modules:

$$0 \longrightarrow U \longrightarrow V \longrightarrow W \longrightarrow 0$$

Recall that $R \otimes_R - \cong \text{id}_{R\text{-Mod}}$ is an exact functor, so we get an exact sequence of left R -modules:

$$0 \longrightarrow {}_R(RR_R \otimes_R {}_R U) \longrightarrow {}_R(RR_R \otimes_R {}_R V) \longrightarrow {}_R(RR_R \otimes_R {}_R W) \longrightarrow 0$$

The forgetful functor $R\text{-Mod} \rightarrow \mathbb{Z}\text{-Mod} = \mathbf{Ab}$ is exact so we also have an exact sequence of abelian groups:

$$0 \longrightarrow {}_{\mathbb{Z}}({}_{\mathbb{Z}}R_R \otimes_R {}_R U) \longrightarrow {}_{\mathbb{Z}}({}_{\mathbb{Z}}R_R \otimes_R {}_R V) \longrightarrow {}_{\mathbb{Z}}({}_{\mathbb{Z}}R_R \otimes_R {}_R W) \longrightarrow 0$$

Then if ${}_{\mathbb{Z}}D \in \mathbb{Z}\text{-Mod}$ is divisible, it must also be injective by some result. Then by definition the contravariant hom functor $\text{Hom}_{\mathbb{Z}}(-, D) : \mathbb{Z}\text{-Mod} \rightarrow \mathbb{Z}\text{-Mod}$ is exact, so we get an exact sequence of abelian groups:

$$0 \longrightarrow \text{Hom}_{\mathbb{Z}}(R \otimes_R U, D) \longrightarrow \text{Hom}_{\mathbb{Z}}(R \otimes_R V, D) \longrightarrow \text{Hom}_{\mathbb{Z}}(R \otimes_R W, D) \longrightarrow 0$$

For a left R -module X we have the hom-tensor adjunction:

$${}_{\mathbb{Z}}(\text{Hom}_{\mathbb{Z}}({}_{\mathbb{Z}}({}_{\mathbb{Z}}R_R \otimes_R {}_R X), {}_{\mathbb{Z}}D)) \cong {}_{\mathbb{Z}}(\text{Hom}_R({}_R X_{\mathbb{Z}}, {}_R(\text{Hom}_{\mathbb{Z}}({}_{\mathbb{Z}}R_R, {}_{\mathbb{Z}}D))))$$

$$\begin{aligned} \text{Hom}_{\mathbb{Z}}(R \otimes_R X, D) &= {}_{\mathbb{Z}}(\text{Hom}_{\mathbb{Z}}({}_{\mathbb{Z}}({}_{\mathbb{Z}}R_R \otimes_R {}_R X), {}_{\mathbb{Z}}D)) \\ &\cong {}_{\mathbb{Z}}(\text{Hom}_R({}_R X_{\mathbb{Z}}, {}_R(\text{Hom}_{\mathbb{Z}}({}_{\mathbb{Z}}R_R, {}_{\mathbb{Z}}D)))) \\ &= \text{Hom}_R(X, \text{Hom}_{\mathbb{Z}}(R, D)) \end{aligned}$$

Hence, we obtain an exact sequence of abelian groups:

$$0 \rightarrow \text{Hom}_R(U, \text{Hom}_{\mathbb{Z}}(R, D)) \rightarrow \text{Hom}_R(V, \text{Hom}_{\mathbb{Z}}(R, D)) \rightarrow \text{Hom}_R(W, \text{Hom}_{\mathbb{Z}}(R, D)) \rightarrow 0$$

In other words, $\text{Hom}_R(-, \text{Hom}_{\mathbb{Z}}(R, D)) : R\text{-Mod} \rightarrow \mathbb{Z}\text{-Mod}$ is an exact functor, so by definition, $\text{Hom}_{\mathbb{Z}}(R, D)$ is an injective left R -module. \square

Theorem 9. Let R be any ring. Every left R -module can be embedded as a submodule of an injective left R -module.

Proof. Let ${}_R M \in R\text{-Mod}$. There is a homomorphism of abelian groups defined as follows:

$$\begin{aligned} \varphi : {}_{\mathbb{Z}}M &\longrightarrow \text{Hom}_{\mathbb{Z}}(R, M) \\ m &\longmapsto (\varphi_m : r \mapsto rm) \end{aligned}$$

(Probably easy to check well-defined, i.e., that φ_m is \mathbb{Z} -linear for all $m \in M$, and that φ itself is \mathbb{Z} -linear.)

We check that φ is injective. Suppose $\varphi_m = 0 \in \text{Hom}_{\mathbb{Z}}(R, M)$, i.e., that $\varphi_m(r) = 0$ for all $r \in R$. Then in particular, $m = 1 \cdot m = \varphi_m(1) = 0$, so $\ker \varphi = 0$.

By Lemma 23, there is an embedding $\iota : {}_{\mathbb{Z}}M \hookrightarrow {}_{\mathbb{Z}}D$ with D an injective abelian group.

Left exactness of the hom-functor gives us an injection $\iota_* : \text{Hom}_{\mathbb{Z}}(R, M) \rightarrow \text{Hom}_{\mathbb{Z}}(R, D)$ of abelian groups.

Hence, we have an injection of abelian group $\Phi := \iota_* \circ \varphi : {}_{\mathbb{Z}}M \rightarrow \text{Hom}_{\mathbb{Z}}(R, D)$.

Then Φ acts by $\Phi(m) = \iota_* \varphi_m = \iota \circ \varphi_m$ with $(\iota \circ \varphi_m)(x) = \iota(\varphi_m(x)) = \iota(xm) = xm$ in D .

Recall that $\text{Hom}_{\mathbb{Z}}(R, D)$ has a left R -module structure $r \cdot f = f(- \cdot r)$, i.e., $(r \cdot f)(x) = f(xr)$ (where xr is simply multiplication in R) for all $r, x \in R$ and $f \in \text{Hom}_{\mathbb{Z}}(R, D)$.

We check that Φ is R -linear. For $r \in R$ and $m \in M$,

$$\Phi(rm) = \iota_* \varphi_{rm} = \iota \circ \varphi_{rm}.$$

Applying this to $x \in R$ we have the following in D :

$$(\iota \circ \varphi_{rm})(x) = \iota_* \varphi_{rm}(x) = x \cdot rm = xr \cdot m = \iota_* \varphi_m(xr) = (r \cdot \iota_* \varphi_m)(x) = (r \cdot \Phi(m))(x)$$

Hence, $\Phi(rm) = r \cdot \Phi(m)$, so indeed Φ is R -linear.

In other words, Φ corresponds to a left R -module homomorphism $\tilde{\Phi} : {}_R M \rightarrow {}_R(\text{Hom}_{\mathbb{Z}}(R, D))$ which is defined the same as Φ set-theoretically. Since Φ is injective, so is $\tilde{\Phi}$.

Thus, we have embedded ${}_R M$ into $\text{Hom}_{\mathbb{Z}}(R, D)$, which is an injective left R -module by the previous lemma. \square

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Derived Functors

Theorem 10 (31 Long Exact Homology Sequence). Suppose \mathcal{A} is SES in $R\text{-Comp}$:

$$0 \longrightarrow A_{\bullet} \xrightarrow{f} A'_{\bullet} \xrightarrow{f'} A''_{\bullet} \longrightarrow 0$$

Then for each $n \in \mathbb{Z}$, there exists $\partial_n \in \text{Hom}_R(H_n(A''_{\bullet}), H_{n-1}(A_{\bullet}))$ such that the following sequence is exact in $R\text{-Mod}$:

$$\cdots \longrightarrow H_n(A_{\bullet}) \xrightarrow{H_n(f)} H_n(A'_{\bullet}) \xrightarrow{H_n(f')} H_n(A''_{\bullet}) \xrightarrow{\partial_n} H_{n-1}(A_{\bullet}) \longrightarrow \cdots$$

In addition, we have naturality in the following sense: For any \mathcal{A} as above, the family $(\partial_n)_{n \in \mathbb{Z}} = (\partial_n^{\mathcal{A}})_{n \in \mathbb{Z}}$ satisfies the following condition: Whenever $h : \mathcal{A} \rightarrow \mathcal{B}$ is a morphism of SES's, i.e., the following diagram commutes:

$$\begin{array}{ccccccc} 0 & \longrightarrow & A_{\bullet} & \xrightarrow{f} & A'_{\bullet} & \xrightarrow{f'} & A''_{\bullet} \longrightarrow 0 \\ & & \downarrow h & & \downarrow h' & & \downarrow h'' \\ 0 & \longrightarrow & B_{\bullet} & \xrightarrow{g} & B'_{\bullet} & \xrightarrow{g'} & B''_{\bullet} \longrightarrow 0 \end{array}$$

then the following diagram commutes:

$$\begin{array}{ccc} H_n(A''_\bullet) & \xrightarrow{\partial_n^A} & H_{n-1}(A_\bullet) \\ H_n(h'') \downarrow & & \downarrow H_{n-1}(h) \\ H_n(B''_\bullet) & \xrightarrow{\partial_n^B} & H_{n-1}(B_\bullet) \end{array}$$

Let $u : A_\bullet \rightarrow A'_\bullet$ be a morphism in $R\text{-Comp}$. Say $u = (u_n)_{n \in \mathbb{Z}}$.

Call u **nullhomotopic** if there exists $s_n \in \text{Hom}_R(A_n, A'_{n+1})$ such that

$$u_n = d'_{n+1} \circ s_n + s_{n-1} \circ d_n.$$

Looks like

$$\begin{array}{ccccc} A_{n+1} & \xrightarrow{d_{n+1}} & A_n & \xrightarrow{d_n} & A_{n-1} \\ u_{n+1} \downarrow & \swarrow s_n & \downarrow u_n & \swarrow s_{n-1} & \downarrow u_{n-1} \\ A'_{n+1} & \xrightarrow{d'_{n+1}} & A'_n & \xrightarrow{d'_n} & A'_{n-1} \end{array}$$

Call $(s_n)_{n \in \mathbb{Z}}$ chain maps of degree $+1$.

Proposition 2 (32). Suppose $u \in \text{Hom}_{R\text{-Comp}}(A_\bullet, A'_\bullet)$ is nullhomotopic. Then $H_n(u) = 0$ for all $n \in \mathbb{Z}$.

Let $f, g : A_\bullet \rightarrow A'_\bullet$ be morphisms in $R\text{-Comp}$.

Say f and g are **homotopic**, denoted $f \simeq g$, if $f - g$ is nullhomotopic.

Two complexes are **homotopy equivalent**, $A_\bullet \simeq A'_\bullet$, if there exist $f : A_\bullet \rightarrow A'_\bullet$ and $f' : A'_\bullet \rightarrow A_\bullet$ such that $f' \circ f \simeq \text{id}_{A_\bullet}$ and $f \circ f' \simeq \text{id}_{A'_\bullet}$.

Corollary 1. If $f \simeq g$ then $H_n(f) = H_n(g)$ for all $n \in \mathbb{Z}$.

Deleted Resolution

Let $M \in R\text{-Mod}$ and

$$\cdots \longrightarrow P_n \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

be a projective resolution of M .

The **deleted projective resolution** is the complex P_\bullet :

$$\cdots \longrightarrow P_n \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow 0$$

(maybe not exact at P_0)

Given injective resolution

$$0 \longrightarrow M \longrightarrow E_0 \longrightarrow E_1 \longrightarrow \cdots \longrightarrow E_n \longrightarrow \cdots$$

the **deleted injective resolution** is complex E_\bullet :

$$0 \longrightarrow E_0 \longrightarrow E_1 \longrightarrow \cdots \longrightarrow E_n \longrightarrow \cdots$$

Plan: study model problem.

Look at $\mathcal{F} = X \otimes_R -$. For SES

$$0 \longrightarrow M \xrightarrow{f} M' \xrightarrow{f'} M'' \longrightarrow 0$$

in $R\text{-Mod}$, we want some information about $\ker(X \otimes_R f : X \otimes_R M \rightarrow X \otimes_R M')$.

Step 1.

One can construct projective resolutions of M, M', M'' such we get a SES of deleted projective resolutions $0 \rightarrow P_\bullet \rightarrow P'_\bullet \rightarrow P''_\bullet \rightarrow 0$.

In particular, if $\bar{f} = (f_n)_{n \in \mathbb{Z}}$ and

some more stuff...

Lemma 6 (33 Comparison). Let $M, M' \in R\text{-Mod}$, $f \in \text{Hom}_R(M, M')$. Let P_\bullet, P'_\bullet be deleted projective resolutions. Then there exists $\bar{f} = (f_n)_{n \geq 0} \in \text{Hom}_R(P_\bullet, P'_\bullet)$ such that the following diagram commutes:

$$\begin{array}{ccccccc} \cdots & \longrightarrow & P_1 & \xrightarrow{d_1} & P_0 & \xrightarrow{d_0} & M \longrightarrow 0 \\ & & \downarrow f_1 & & \downarrow f_0 & & \downarrow f \\ \cdots & \longrightarrow & P'_1 & \xrightarrow{d'_1} & P'_0 & \xrightarrow{d'_0} & M' \longrightarrow 0 \end{array}$$

Moreover, given another chain map $\bar{g} \in \text{Hom}_R(P_\bullet, P'_\bullet)$, we obtain $\bar{f} \simeq \bar{g}$ (as chain maps of deleted projective resolutions).

Corollary 2. Any two deleted projective resolutions are homotopy equivalent.

Lemma 7 (Horseshoe). Given a SES

$$0 \longrightarrow M \xrightarrow{f} M' \xrightarrow{f'} M'' \longrightarrow 0$$

and projective resolutions $P_\bullet \rightarrow M$ and $P'_\bullet \rightarrow M''$, there is a projective resolution $P'_\bullet \rightarrow M'$ and chain maps \bar{f} and \bar{f}' such that stuff commute good:

$$\begin{array}{ccccccc}
0 & \longrightarrow & P_{\bullet} & \xrightarrow{\bar{f}} & P'_{\bullet} & \xrightarrow{\bar{f}'} & P''_{\bullet} \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & M & \xrightarrow{f} & M' & \xrightarrow{f'} & M'' \longrightarrow 0
\end{array}$$

Left Derived Functors

Let $T : R\text{-Mod} \rightarrow S\text{-Mod}$ be a covariant additive functor.

Want to extract functors $L_n T$ for $n \geq 0$ which sends a SES to a LES.

Also if T is right exact, want $L_0(T) \cong T$.

Pick and fix a (deleted) projective resolution $\mathbb{P}_M \rightarrow M$ for each $M \in R\text{-Mod}$.

Define $L_n T$ on objects $M \in R\text{-Mod}$ by

$$L_n T(M) = H_n(T(\mathbb{P}_M)).$$

Define $L_n T$ on morphisms $f \in \text{Hom}_R(M, M')$.

By Lemma 33, choose and fix $\bar{f} \in \text{Hom}_{R\text{-Comp}}(\mathbb{P}_M, \mathbb{P}_{M'})$ lying over f . Then define

$$L_n T(f) = H_n(T(\bar{f})).$$

Theorem 11 (36). $L_n T$ is a well-defined covariant functor $R\text{-Mod} \rightarrow S\text{-Mod}$.

Moreover, if $\hat{\mathbb{P}}_M$ is a different choice of deleted projective resolution for each $M \in R\text{-Mod}$, then the functor $\hat{L}_n T$ defined with respect to the \mathbb{P}_M 's satisfies $\hat{L}_n T \cong L_n T$.

Theorem 12 (37). All notations as before. Suppose

$$0 \longrightarrow M \xrightarrow{f} M' \xrightarrow{f'} M'' \longrightarrow 0$$

is a SES in $R\text{-Mod}$. Then there exists an exact sequence

$$\cdots \longrightarrow L_{n+1} T(M'') \xrightarrow{\partial_{n+1}} L_n T(M) \xrightarrow{L_n T(f)} L_n T(M') \xrightarrow{L_n T(f')} L_n T(M'') \xrightarrow{\partial_n} \cdots$$

$$L_1(M'') \xrightarrow{\partial_1} L_0 T(M) \xrightarrow{L_0 T(f)} L_0 T(M') \xrightarrow{L_0 T(f')} L_0 T(M'') \longrightarrow 0$$

(In particular, $L_0 T$ is always right exact.)

Moreover:

- (a) If T is right-exact then $L_0 T \cong T$;
- (b) If M is projective then $L_n T(M) = 0$ for $n \geq 1$;
- (c) All ∂_n for $n \geq 1$ are natural in the sense specified in Theorem 31.

Theorem 13 (38 Dimension Shifting). Notation as before. Then

$$L_n T(M) \cong L_{n-1} T(K_1) \cong L_{n-1}(K_2) \cong \cdots \cong L_1 T(K_{n-1})$$

where K_i is an i th syzygy of M .

Other derived functors in a nutshell, i.e., left/right derived functors of co-/contravariant functors.

mnemonic:

left derived fill in the left, come from left ∞

right derived fill in the right, come from right ∞

$T : R\text{-Mod} \rightarrow S\text{-Mod}$, $M \in R\text{-Mod}$, get injective resolution $M \rightarrow \mathbb{E}_M$. Apply T , take n th homology, define $R^n T(M) = H_n(T(\mathbb{E}_M))$.

From $0 \rightarrow M \rightarrow M' \rightarrow M'' \rightarrow 0$ get LES

$$0 \longrightarrow R^0 T(M) \longrightarrow R^0 T(M') \longrightarrow R^0 T(M'') \xrightarrow{\partial_1} R_1 T(M) \longrightarrow \cdots$$

If T left exact then $R^0 T \cong T$.

If M injective then $R^n T(M) = 0$ for $n \geq 1$.

For contravariant $T : R\text{-Mod}^{\text{op}} \rightarrow S\text{-Mod}$, right used \mathbb{P}_M and left uses \mathbb{E}_M .

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Change of index position in right derived functors.

Indexing going down from left to right A_\bullet :

$$\cdots \longrightarrow A_{n+1} \xrightarrow{d_{n+1}} A_n \xrightarrow{d_n} A_{n-1} \xrightarrow{d_{n-1}} \cdots$$

Indexing going up from left to right B_\bullet :

$$\cdots \longrightarrow B_{n-1} \xrightarrow{d_n} B_n \xrightarrow{d_{n+1}} B_{n+1} \longrightarrow \cdots$$

(Some use indexing with superscripts)

n th homology of A_\bullet is $H_n(A_\bullet) = \ker d_n / \text{im } d_{n+1}$.

n th homology of B_\bullet is $H^n(B_\bullet) = \ker d_{n+1} / \text{im } d_n$.

“ n th homology is the homology in the position which is labeled n ”

n th right derived functor: homology of a complex of type B_\bullet , write R^n instead of R_n .

The Functors Tor and Ext.

Let ${}_S X_R$ be a bimodule, $T = X \otimes_R - : R\text{-Mod} \rightarrow S\text{-Mod}$.

We know T is right exact but not left exact in general.

Define left derived functor $\text{Tor}_n^R(X, Y) = L_n T(Y) = H_n(X \otimes_R \mathbb{P}_Y)$ with \mathbb{P}_Y a deleted projective resolution of Y .

For $\text{Tor}_n^R(X, f) = L_n T(f)$ for $f \in \text{Hom}_R(Y, Y')$.

Comments.

1. T is right exact, hence $\text{Tor}_0^R(X, Y) = X \otimes_R Y$.

2. If

$$0 \longrightarrow Y \xrightarrow{f} Y' \xrightarrow{f'} Y'' \longrightarrow 0$$

is exact in $R\text{-Mod}$, then we get an exact sequence in $S\text{-Mod}$:

$$\cdots \xrightarrow{\partial_2} \text{Tor}_1^R(X, Y) \rightarrow \text{Tor}_1^R(X, Y') \rightarrow \text{Tor}_1^R(X, Y'') \xrightarrow{\partial_1} X \otimes_R Y \rightarrow X \otimes_R Y' \rightarrow X \otimes_R Y'' \rightarrow 0$$

3. Let $Y \cong P/K$, P projective, i.e.,

$$0 \longrightarrow K \hookrightarrow P \xrightarrow{\text{can}} Y \longrightarrow 0$$

is exact, then $\text{Tor}_n^R(X, P) = 0$ for all $n \geq 1$. So

$$0 \longrightarrow \text{Tor}_1^R(X, P/K) \longrightarrow X \otimes_R K \xrightarrow{X \otimes \iota} X \otimes_R P \longrightarrow \cdots$$

is exact. So $\text{Tor}_1^R(X, P/K) = \ker(X \otimes \iota)$

Define analogously

$$\text{Tor}_n^R(-, Y)(X) = H_n(\mathbb{P}_X \otimes_R Y) = L_n \tilde{T}(X).$$

Then $\text{Tor}_n^R(X, Y)$ has two definitions.

How are $H_n(X \otimes_R \mathbb{P}_Y)$ and $H_n(\mathbb{P}_X \otimes_R Y)$ related?

Theorem 14 (39 Balancedness of arguments of Tor). For $X \in \text{Mod-}R$, $Y \in R\text{-Mod}$,

$$H_n(X \otimes_R \mathbb{P}_Y) \cong H_n(\mathbb{P}_X \otimes_R Y)$$

in $S\text{-Mod}$. (Both defined as $\text{Tor}_n^R(X, Y)$ up to isomorphism.)

Proof. See Rotman □

Comment.

1. The tor functors commute with arbitrary direct sums.

Reason: tensor and homology commutes with arbitrary direct sums, and tor is just the composition of these two devices.

2. The tor functors also commute with direct limits extending over directed partially ordered sets (not with arbitrary colimits nor limits).

A poset (S, \leq) is **directed** if for all $x, y \in S$, there exists $z \in S$ with $x \leq z$ and $y \leq z$.

Theorem 15 (40).

(A) For $X \in \mathbf{Mod}\text{-}R$, TFAE:

- (i) X_R is flat,
- (ii) $\mathrm{Tor}_n^R(X, Y) = 0$ for all $n \geq 1$ and $Y \in R\text{-}\mathbf{Mod}$,
- (iii) $\mathrm{Tor}_1^R(X, Y) = 0$ for all $Y \in R\text{-}\mathbf{Mod}$.

(B) Analogous statement for ${}_R Y$.

Proof. (i \implies ii)

Assume (i) and let $Y \in R\text{-}\mathbf{Mod}$, $\cdots \rightarrow P_1 \rightarrow P_0 \rightarrow Y \rightarrow 0$ a projective resolution.

By flatness, get exact $\cdots \rightarrow X \otimes P_1 \rightarrow X \otimes P_0 \rightarrow X \otimes Y \rightarrow 0$.

Since X is flat,

$$\cdots \longrightarrow X \otimes P_4 \longrightarrow X \otimes P_3 \longrightarrow X \otimes P_2 \longrightarrow X \otimes P_1 \longrightarrow X \otimes P_0$$

is an exact portion of $X \otimes \mathbb{P}_Y$.

Somehow get $H_n(X \otimes \mathbb{P}_Y) = 0$ for all $n \geq 1$.

(ii \implies iii) trivial

(iii \implies i)

Assume (iii) and let $0 \rightarrow Y \rightarrow Y' \rightarrow Y'' \rightarrow 0$ be an exact sequence in $R\text{-}\mathbf{Mod}$. Then by long exact homology sequence get

$$\mathrm{Tor}_1^R(X, Y'') \longrightarrow X \otimes Y \longrightarrow X \otimes Y' \longrightarrow X \otimes Y'' \longrightarrow 0$$

is exact. But $\mathrm{Tor}_1^R(X, Y'')$ is zero by hypothesis, i.e., $X \otimes_R -$ is exact. □

Tor for $R = \mathbb{Z}$. ('tor' is short for 'torsion,' comes from \mathbb{Z}) [Analogous for R a PID.]

If R is commutative, then $X \otimes_R Y \cong Y \otimes_R X$, and hence $\mathrm{Tor}_n^R(X, Y) \cong \mathrm{Tor}_n^R(Y, X)$.

Proposition 3 (41). If $X, Y \in \mathbb{Z}\text{-}\mathbf{Mod}$, then $\mathrm{Tor}_n^{\mathbb{Z}}(X, Y) = 0$ for $n \geq 2$.

Proof. Know $\text{pdim}_{\mathbb{Z}} Y = 1$, i.e., any first syzygy K of Y is projective.

So by dimension-shifting, $\text{Tor}_2^{\mathbb{Z}}(X, Y) \cong \text{Tor}_1^{\mathbb{Z}}(X, K_1) = 0$ since K_1 is projective and hence flat.

Rest by induction. For $n \geq 3$, get $\text{Tor}_n^{\mathbb{Z}}(X, Y) \cong \text{Tor}_{n-1}^{\mathbb{Z}}(X, K_1) = 0$ since K_1 projective. \square

Example.

$\text{Tor}_1^{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z}/6\mathbb{Z}) = 0$ because \mathbb{Q} is torsionfree hence flat.

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Proposition 4 (42).

- (1) $\text{Tor}_1^{\mathbb{Z}}(X, Y) \cong \text{Tor}_1^{\mathbb{Z}}(T(X), T(Y))$ where $T(X)$ is torsion subgroup of X . (Tor only sees torsion part.)
- (2) $\text{Tor}_1^{\mathbb{Z}}(X, \mathbb{Z}/n\mathbb{Z}) = \{x \in X \mid nx = 0\} =: X[n]$.
- (3) $\text{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) \cong \mathbb{Z}/d\mathbb{Z}$ where $d = \gcd(m, n)$.
- (4) $\text{Tor}_1^{\mathbb{Z}}(X, Y)$ is torsion for all X and Y .
- (5) $\text{Tor}_1^{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z}, Y) \cong T(Y)$ for all Y .

Proof. (1) Consider the exact sequence $0 \rightarrow T(Y) \rightarrow Y \rightarrow Y/T(Y) \rightarrow 0$. Then

$$\text{Tor}_2^{\mathbb{Z}}(X, Y/T(Y)) \longrightarrow \text{Tor}_1^{\mathbb{Z}}(X, T(Y)) \longrightarrow \text{Tor}_1^{\mathbb{Z}}(X, Y) \longrightarrow \text{Tor}_1^{\mathbb{Z}}(X, Y/T(Y))$$

is exact.

But $\text{Tor}_2^{\mathbb{Z}}(X, Y/T(Y)) = 0$ and $\text{Tor}_1^{\mathbb{Z}}(X, Y/T(Y)) = 0$ because $n \geq 2$ and $Y/T(Y)$ is torsion-free, respectively.

Hence, $\text{Tor}_1^{\mathbb{Z}}(X, Y) \cong \text{Tor}_1^{\mathbb{Z}}(X, T(Y)) \cong \text{Tor}_1^{\mathbb{Z}}(T(X), T(Y))$.

(2) Consider

$$0 \longrightarrow n\mathbb{Z} \xhookrightarrow{\iota} \mathbb{Z} \xrightarrow{\text{can}} \mathbb{Z}/n\mathbb{Z} \longrightarrow 0$$

Then get

$$\begin{array}{ccccccc} \text{Tor}_1^{\mathbb{Z}}(X, \mathbb{Z}) & \longrightarrow & \text{Tor}_1^{\mathbb{Z}}(X, \mathbb{Z}/n\mathbb{Z}) & \xrightarrow{\partial_1} & X \otimes_{\mathbb{Z}} n\mathbb{Z} & \longrightarrow & X \otimes_{\mathbb{Z}} \mathbb{Z} \\ \parallel & & & & & & \\ 0 & & & & & & \end{array}$$

This

$$\begin{aligned}\mathrm{Tor}_1^{\mathbb{Z}}(X, \mathbb{Z}/n\mathbb{Z}) &\cong \ker(X \otimes \iota) \\ &= \{x \otimes n \in X \otimes_{\mathbb{Z}} n\mathbb{Z} \mid x \otimes n = 0 \in X \otimes_{\mathbb{Z}} \mathbb{Z}\}.\end{aligned}$$

Note $x \otimes n = nx \otimes 1$ in $X \otimes_{\mathbb{Z}} \mathbb{Z}$ but not in $X \otimes_{\mathbb{Z}} n\mathbb{Z}$. In particular, $X \otimes_{\mathbb{Z}} \mathbb{Z} \cong X$ via $X \otimes 1 \mapsto x$.

$$\begin{aligned}&= \{x \otimes n \in X \otimes_{\mathbb{Z}} n\mathbb{Z} \mid xn = 0 \in X\} \\ &\cong \{x \in X \mid xn = 0\} \\ &= X[n].\end{aligned}$$

Why is isomorphism true? Have map

$$\begin{array}{ccc}X \otimes n\mathbb{Z} & \xrightarrow{\cong} & X \otimes \mathbb{Z} \xrightarrow{\cong} X \\ x \otimes n & \longmapsto & x \otimes 1 \\ & & x \otimes z \longmapsto zx\end{array}$$

(3) Have

$$\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) \cong \mathbb{Z}/m\mathbb{Z}[n] = \frac{m}{d}\mathbb{Z}/m\mathbb{Z} \cong \mathbb{Z}/d\mathbb{Z}$$

(Last is cyclic of order d .)

(4) (a) For Y finitely generated torsion, have $T(Y) = \bigoplus_{\text{finite}} C_i$ with $C_i \cong \mathbb{Z}/p_i^{m_i}\mathbb{Z}$.

Hence,

$$\mathrm{Tor}_1^{\mathbb{Z}}(X, Y) \cong \mathrm{Tor}_1^{\mathbb{Z}}(X, T(Y)) \cong \bigoplus_{\text{finite}} \mathrm{Tor}_1^{\mathbb{Z}}(X, C_i) \cong \bigoplus_{\text{finite}} X[p_i^{m_i}].$$

Each $X[p_i^{m_i}]$ is obviously torsion by definition, so direct sum is torsion.

(b) Y arbitrary. Then $Y = \bigcup_{i \in I} Y_i$ with Y_i finitely generated. In other words, Y is the direct limit of all of its finitely generated subgroups? That is, $Y = \varinjlim_{i \in I} Y_i$.

Then

$$\mathrm{Tor}_1^{\mathbb{Z}}(X, Y) \cong \varinjlim_{i \in I} \mathrm{Tor}_1^{\mathbb{Z}}(X, Y_i)$$

Each $\mathrm{Tor}_1^{\mathbb{Z}}(X, Y_i)$ is torsion, so the limit is torsion.

(5) Consider the following exact sequence:

$$\begin{array}{ccccccc}\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Q}, Y) & \longrightarrow & \mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z}, Y) & \longrightarrow & \mathbb{Z} \otimes_{\mathbb{Z}} Y & \longrightarrow & \mathbb{Q} \otimes_{\mathbb{Z}} Y \\ \parallel & & & & \downarrow \cong & \nearrow & \\ 0 & & & & Y & \xrightarrow{\psi=1 \otimes -} & \end{array}$$

First since \mathbb{Q} torsionfree.

Hence

$$\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z}, Y) \cong \ker(\iota \otimes Y) \cong \ker \psi \leq Y.$$

Since $\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z}, Y)$ is torsion, then $\ker \psi$ is torsion and sitting inside Y , so $\ker \psi \subseteq T(Y)$.

To see that $T(Y) \subseteq \ker \psi$, let $y \in T(Y)$ and $z \in \mathbb{Z} \setminus \{0\}$ with $zy = 0$.

Then $\psi(y) = 1 \otimes y \in \mathbb{Q} \otimes_{\mathbb{Z}} Y$. Here,

$$1 \otimes y = \frac{z}{z} \otimes y = \frac{1}{z} \otimes zy = 0.$$

□

last class or two of A

4/3/23

starting 236B

(Re)introduce additive and abelian categories

Prototypical example: sheaves of abelian groups on a topological space.

Will define derived categories (objects: complexes of objects in an abelian category. morphism: want such that quasi-isomorphisms of complexes become isomorphisms of objects.)

Then get a natural way of talking about derived functors.

will find a few functors f_* , f^* , $f_!$, $f^!$, \mathbb{D} , $\underline{\text{Hom}}$ for sheaves.

will define triangulated category

will define t -structures (extra data on triangulated category)

example is perverse sheaves

textbook: *Methods of Homological Algebra* by Gelfand and Manin

Examples

- (a) **Ab** is abelian groups with group homomorphisms
 - (b) **$R\text{-Mod}$** is (left) R -modules with (left) R -module homomorphisms
 - (c) **SAb** is sheaves of abelian groups
 PAb is presheaves of abelian groups
 - (d) (quasi-)coherent sheaves of modules over a ringed space
-

Let \mathcal{C} be a (locally-small) category.

A1. Each set $\text{Mor}_{\mathcal{C}}(X, Y)$ is an abelian group $(+)$, the composition of morphism is bi-additive. In which case, usually write $\text{Hom}_{\mathcal{C}}$ in place of $\text{Mor}_{\mathcal{C}}$.

(note that it is ambiguous whether A1 is a structure or a property.)

In particular, $\text{Hom}_{\mathcal{C}}$ is a functor $\mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \mathbf{Ab}$.

And automatically get a zero morphism $0 \in \text{Hom}_{\mathcal{C}}(X, Y)$.

A2. There exists a zero object $0 \in \mathcal{C}$, i.e., an object such that $\text{Hom}_{\mathcal{C}}(0, 0) = 0$.

This gives $\text{Hom}_{\mathcal{C}}(0, X) = 0$ and $\text{Hom}_{\mathcal{C}}(X, 0) = 0$ for all $X \in \mathcal{C}$.

Always have identity $\text{id}_0 : 0 \rightarrow 0$, but then $\text{id}_0 = 0$. So then for any $f : 0 \rightarrow X$, must have $f = f \circ \text{id}_0 = f \circ 0 = 0$, using A1.

Exercise: any two zero objects are isomorphic.

A3. For any $X_1, X_2 \in \mathcal{C}$, there exists an object $Y \in \mathcal{C}$ any morphisms

$$X \xrightleftharpoons[p_1]{i_1} Y \quad X_2 \xrightleftharpoons[p_2]{i_2} Y$$

such that $p_1 i_1 = \text{id}_{X_1}$, $p_2 i_2 = \text{id}_{X_2}$, $i_1 p_1 + i_2 p_2 = \text{id}_Y$, and $p_2 i_1 = p_1 i_2 = 0$.

Lemma 8. We have a cartesian diagram

$$\begin{array}{ccc} Y & \xrightarrow{p_1} & X_1 \\ p_2 \downarrow & & \downarrow \\ X_2 & \longrightarrow & 0 \end{array}$$

That is, for all Z with $Z \rightarrow X_i$ making the diagram commute, there is a unique $Z \rightarrow Y$ which completes the diagram. and a co-cartesian diagram

$$\begin{array}{ccc} Y & \xleftarrow{i_1} & X_1 \\ i_2 \uparrow & & \uparrow \\ X_2 & \xleftarrow{\quad} & 0 \end{array}$$

Proof. Suppose we have commutative diagram

$$\begin{array}{ccccc} Z & & & & \\ & \searrow \varphi & & \searrow q_1 & \\ & & Y & \xrightarrow{p_1} & X_1 \\ & \searrow q_2 & \downarrow p_2 & & \downarrow \\ & & X_2 & \longrightarrow & 0 \end{array}$$

Need to construct $\varphi : Z \rightarrow Y$ such that $q_i = p_i \circ \varphi$.

Take $\varphi = i_1 q_1 + i_2 q_2$, then

$$p_1 \circ \varphi = p_1 i_1 q_1 + p_1 i_2 q_2 = \text{id}_{X_1} q_1 + 0 q_2 = q_1.$$

Similarly, $p_2 \circ \varphi = q_2$.

Verify uniqueness of φ ...

□

An **additive category** is a category satisfying A1, A2, A3.

To state A4, need more notation

Let \mathcal{C} be a category satisfying A1 and A2.

Let $\varphi : X \rightarrow Y$ be a morphism in \mathcal{C} .

A **kernel** of φ is a morphism $i : Z \rightarrow X$ such that

- (a) $\varphi \circ i = 0$
- (b) for any $i' : Z' \rightarrow X$ with $\varphi \circ i' = 0$, there is a unique $g : Z' \rightarrow Z$ such that $i' = i \circ g$.
(that is, i is unique up to unique isomorphism with respect to (a)).

A **cokernel** is the dual notion.

Exercise: this definition of kernel is equivalent to the following: for all $Z' \in \mathcal{C}$, the sequence

$$0 \longrightarrow \text{Hom}(Z', Z) \xrightarrow{i_*} \text{Hom}(Z', X) \xrightarrow{\varphi_*} \text{Hom}(Z', X)$$

is exact. Similar for cokernel.

A4. For any morphism $\varphi : X \rightarrow Y$, there is a sequence of morphisms

$$K \xrightarrow{k} X \xrightarrow{i} I \xrightarrow{j} Y \xrightarrow{c} K'$$

such that

- (a) $j \circ i = \varphi$
- (b) $K = \ker \varphi$ and $K' = \text{coker } \varphi$
- (c) $I = \text{coker } k = \ker c$

An **abelian category** is a category satisfying A1-4.

4/5/23

II Sheaves

Examples from complex analysis

(a) Holomorphic functions on the Riemann sphere $\mathbb{P} = \mathbb{C} \cup \{\infty\}$.

For each open subset $U \subseteq \mathbb{P}$, we consider the ring of holomorphic functions $f : U \rightarrow \mathbb{C}$.

There is a sheaf \mathcal{O} of holomorphic functions on \mathbb{P} is given by the data

$$\{(f, V) \mid V \subseteq \mathbb{P} \text{ open}, f : V \rightarrow \mathbb{C} \text{ holomorphic}\}.$$

Note that V need not be the maximal open set on which f is defined.

(b) The sheaf of solutions of a linear ODE.

Let $U \subseteq \mathbb{P}$ be open, $a_i(z) \in \Gamma(U, \mathcal{O}) := \{f : U \rightarrow \mathbb{C} \text{ holomorphic}\}$ for $i = 0, 1, \dots, n-1$.

Denote by S the collection of all (V, f) such that $V \subseteq U$ is open and f is holomorphic on V such that

$$Lf := \frac{d^n f}{dz^n} + \sum_{i=0}^n a_i(z) \frac{d^i f}{dz^i} = 0.$$

When V is connected and simply connected, basic result of ODE gives

$$\Gamma(V) = \{f : V \rightarrow \mathbb{C} \text{ holomorphic with } Lf = 0\} \cong \mathbb{C}^n.$$

In general, the solution may depend on the topology of V .

e.g., $U = \mathbb{C} \setminus \{0\}$, $L = \frac{d^2}{dz^2} + \frac{1}{z} \frac{d}{dz}$. The solutions are $c_1 \log z + c_2$ for any “branch” of \log , but $\Gamma(U) = \{\text{const}\}$.

(Riemann-Hilbert correspondence)

A **presheaf (of sets)** \mathcal{F} on a topological space Y consists of the following data:

- a set $\mathcal{F}(U)$ for each open subset $U \subseteq Y$,
- a (restriction) map $r_{UV} : \mathcal{F}(U) \rightarrow \mathcal{F}(V)$ for each pair of open sets $V \subseteq U$,

satisfying

- $r_{UU} = \text{id}_{\mathcal{F}(U)}$,
- $r_{VW} \circ r_{UV} = r_{UW}$ whenever $W \subseteq V \subseteq U$.

A **sheaf** is a presheaf \mathcal{F} satisfying:

- for any open covering $U = \bigcup_{i \in I} U_i$ and $s_i \in \mathcal{F}(U_i)$ such that for all $i, j \in I$ with

$$r_{U_i, U_i \cap U_j}(s_i) = r_{U_j, U_i \cap U_j}(s_j)$$

then there exists a unique $s \in \mathcal{F}(U)$ with $s_i = r_{U, U_i}(s)$ for all $i \in I$.

A **morphism of presheaves on Y** $f : \mathcal{F} \rightarrow \mathcal{G}$ consists of the following data:

- a family of maps $f(U) : \mathcal{F}(U) \rightarrow \mathcal{G}(U)$ for each open subset $U \subseteq Y$

satisfying

- for each pair of open sets $V \subseteq U$, the following diagram commutes:

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{f(U)} & \mathcal{G}(U) \\ r_{UV}^{\mathcal{F}} \downarrow & & \downarrow r_{UV}^{\mathcal{G}} \\ \mathcal{F}(V) & \xrightarrow{f(V)} & \mathcal{G}(V) \end{array}$$

A **morphism of sheaves** is a morphism of the underlying presheaves.

A **presheaf of groups/rings/vector spaces** is a presheaf \mathcal{F} of sets such that each $\mathcal{F}(U)$ is a group/ring/vector space and each restriction map is a morphism in the corresponding category.

sheaves and morphisms defined in the natural way.

A **abelian presheaf** is a presheaf of abelian groups. (this course only, not standard language)

Let $f : \mathcal{F} \rightarrow \mathcal{G}$ be a morphism of abelian presheaves.

Recall $f(U) : \mathcal{F}(U) \rightarrow \mathcal{G}(U)$.

Define a kernel $K(U) := \ker f(U)$ and a cokernel $C(U) := \operatorname{coker} f(U)$.

With natural restrictions, K and C are presheaves. (exercise)

A sequence of presheaves

$$\mathcal{F} \xrightarrow{f} \mathcal{G} \xrightarrow{g} \mathcal{H}$$

is exact if for all U , the sequence

$$\mathcal{F}(U) \xrightarrow{f(U)} \mathcal{G}(U) \xrightarrow{g(U)} \mathcal{H}(U)$$

is exact.

This implies that \mathbf{PAb} is an abelian category.

What about abelian sheaves?

Let $f : \mathcal{F} \rightarrow \mathcal{G}$ be a morphism of abelian sheaves.

Can try to define kernel $K(U) = \ker f(U)$ and cokernel $C(U) := \operatorname{coker} f(U)$. Know that both are presheaves, but are they sheaves?

Proposition 5.

- (a) The kernel K is an abelian sheaf.
- (b) The cokernel presheaf C may not be a sheaf.

4/10/23

Functors Between Abelian Categories.

Let \mathcal{C} and \mathcal{C}' be additive categories.

A functor $F : \mathcal{C} \rightarrow \mathcal{C}'$ is **additive** if all maps

$$F : \operatorname{Hom}_{\mathcal{C}}(X, Y) \rightarrow \operatorname{Hom}_{\mathcal{C}'}(FX, FY)$$

are homomorphisms of abelian groups.

A **complex in \mathcal{C}** , denoted by X^\bullet , is a sequence of objects and morphisms:

$$\dots \xrightarrow{d^{n-1}} X^n \xrightarrow{d^n} X^{n+1} \xrightarrow{d^{n+1}} \dots$$

with $d^n \circ d^{n-1} = 0$.

Assume \mathcal{C} and \mathcal{C}' are abelian. Have diagram

$$\begin{array}{ccccc}
 & & \text{coker } d^n & & \\
 & & \uparrow & \searrow^{b^{n+1}} & \\
 X^n & \xrightarrow{d^n} & X^{n+1} & \xrightarrow{d^{n+1}} & X^{n+1} \\
 & \searrow^{a^n} & \uparrow & & \\
 & & \text{ker } d^{n+1} & &
 \end{array}$$

The $(n+1)$ -**cohomology** of X^\bullet is

$$H^{n+1}(X^\bullet) := \text{coker } a^n = \text{ker } b^{n+1}.$$

X^\bullet is **acyclic** at X^n if $H^n(X^\bullet) = 0$.

X^\bullet is **exact/acyclic** if it is acyclic at X^n for all n .

An additive functor $F : \mathcal{C} \rightarrow \mathcal{C}'$ is **exact** if it sends SES $0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$ in \mathcal{C} to SES $0 \rightarrow FX \rightarrow FY \rightarrow FZ \rightarrow 0$ in \mathcal{C}' .

left exact in case each $0 \rightarrow FX \rightarrow FY \rightarrow FZ$ is exact.

right exact in case each $FX \rightarrow FY \rightarrow FZ \rightarrow 0$ is exact.

Examples.

1. \mathcal{C} abelian. Then $\text{Hom}_{\mathcal{C}}(Y, -) : \mathcal{C} \rightarrow \mathbf{Ab}$ and $\text{Hom}_{\mathcal{C}}(-, Y) : \mathcal{C}^{\text{op}} \rightarrow \mathbf{Ab}$ are left exact.
2. Consider $R\text{-Mod}$ and a right R -module Y_R . Then $Y \otimes_R - : R\text{-Mod} \rightarrow \mathbf{Ab}$ is right exact.

Proposition 6. X a topological space, $U \subseteq X$ open, consider the category \mathbf{SAb}_X of sheaves of abelian groups on X . The functor $\mathbf{SAb} \rightarrow \mathbf{Ab}, \mathcal{F} \mapsto \mathcal{F}(U)$, is left exact.

Proof. The inclusion functor $\iota : \mathbf{SAb} \hookrightarrow \mathbf{PAb}$ is left exact: kernel does not need sheafification.

The functor $\mathbf{PAb} \rightarrow \mathbf{Ab}, \mathcal{F} \mapsto \mathcal{F}(U)$ is exact by definition.

Composing an exact functor with a left exact functor gives a left exact functor. □

Let \mathcal{C} be an abelian category.

An object Y is **projective** if $\text{Hom}_{\mathcal{C}}(Y, -)$ is exact.

An object X is **injective** if $\text{Hom}_{\mathcal{C}}(-, X)$ is exact.

A right R -module Y_R is **flat** if $Y \otimes_R -$ is exact.

Let $f : M \rightarrow N$ be a continuous map of topological spaces, \mathcal{F} a sheaf on M .

The **direct image** or **pushforward** $f_*\mathcal{F}$ is a sheaf defined by

$$f_*\mathcal{F}(U) := \mathcal{F}(f^{-1}(U)).$$

and restriction for $V \subseteq U$ induced from $r_{f^{-1}(U), f^{-1}(V)}$.

Exercise: This is a sheaf.

Proposition 7.

- (a) Let $F : M \rightarrow \text{pt}$, then $f_*\mathcal{F} = \Gamma(M, \mathcal{F}) = \mathcal{F}(M)$.
- (b) $i : M \rightarrow N$ closed embedding, then have stalk

$$(i_*\mathcal{F})_x = \begin{cases} \mathcal{F}_x & x \in M \\ 0 & x \notin M \end{cases}$$

Called “extension by zero.”

(Exercise: $i : M \hookrightarrow N$ open embedding; $i_*\mathcal{F}$ may have nonzero stalk at $x \in N \setminus M$)

- (c) $f_* : \mathbf{SAb}_M \rightarrow \mathbf{SAb}_N$ is a functor: $(fg)_* = f_* \circ g_*$ and $\text{id}_* = \text{id}$ (not sure right id condition here).
-

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Let $f : M \rightarrow N$ be a continuous map.

Let $\mathcal{F} \in \mathbf{SAb}_N$ be a sheaf on N .

Define $f_p^*\mathcal{F}$ as a presheaf

$$f_p^*\mathcal{F}(U) = \mathcal{F}(f(U)) := \varinjlim_{f(U) \subseteq V \subseteq N} \mathcal{F}(V)$$

Then take sheafification $f^*\mathcal{F} = s(f_p^*\mathcal{F})$, called the **inverse image** or **pullback**.

Exercise: for all $x \in M$, have stalk $(f^*\mathcal{F})_x = \mathcal{F}_{f(x)}$.

Proposition 8. Adjunction $f^* \dashv f_*$.

$$\text{Hom}_{\mathbf{SAb}_M}(f^*\mathcal{F}, \mathcal{G}) \cong \text{Hom}_{\mathbf{SAb}_N}(\mathcal{F}, f_*\mathcal{G})$$

Proof. As $s \dashv \iota$, we only need to show

$$\text{Hom}_{\mathbf{PAb}_M}(f_p^*\mathcal{F}, \mathcal{G}) \cong \text{Hom}_{\mathbf{PAb}_N}(\mathcal{F}, f_*\mathcal{G}).$$

Want to establish a functorial morphism $\mathcal{F} \rightarrow f_* f_p^* \mathcal{F}$.

(Remark: Why should we expect this? In the original statement, take $\mathcal{G} = f_p^* \mathcal{F}$, then should have $\text{Hom}(f_p^* \mathcal{F}, f_p^* \mathcal{F}) \cong \text{Hom}(\mathcal{F}, f_* f_p^* \mathcal{F})$. Identity id on the left should give something on the right, which is the thing we are talking about.)

Note, for $V \subseteq N$ open, need to define morphism $\mathcal{F}(V) \rightarrow f_* f_p^* \mathcal{F}(V)$. Have

$$f_* f_p^* \mathcal{F}(V) = (f_p^* \mathcal{F})(f^{-1}(V)) = \varinjlim_{f(f^{-1}(V)) \subseteq U \subseteq N} \mathcal{F}(U)$$

But V is an open neighborhood of $f(f^{-1}(V))$, so there is a canonical morphism $\mathcal{F}(V) \rightarrow \varinjlim \mathcal{F}(U)$.

This is compatible with restrictions and gives us a presheaf morphism $i_{\mathcal{F}} : \mathcal{F} \rightarrow f_* f_p^* \mathcal{F}$.

And this induces the arrow in the statement: given $\psi \in \text{Hom}_{\mathbf{PAb}_M}(f_p^* \mathcal{F}, \mathcal{G})$, send to the composition

$$\mathcal{F} \xrightarrow{i_{\mathcal{F}}} f_* f_p^* \mathcal{F} \xrightarrow{f_* \psi} f_* \mathcal{G}$$

The other direction uses $f_p^* f_* \mathcal{G} \rightarrow \mathcal{G}$.

Exercise: construct this and check it is the desired inverse. □

(Think about adjunction $f^* \dashv f_*$ as a pair of natural transformations $\text{id} \rightarrow f_* f^*$ and $f^* f_* \rightarrow \text{id}$.)

Proposition 9. $\mathcal{C} \xrightleftharpoons[G]{F} \mathcal{D}$ and $F \dashv G$. Then F is right exact and G is left exact.

Proof. We prove the second part, i.e., that G is left exact. Given SES

$$0 \longrightarrow Y' \xrightarrow{f} Y \xrightarrow{g} Y'' \longrightarrow 0$$

in \mathcal{D} . Apply left exact functor $\text{Hom}_{\mathcal{D}}(FX, -)$ for all $X \in \mathcal{C}$. Gives a

$$0 \longrightarrow \text{Hom}_{\mathcal{D}}(FX, Y') \longrightarrow \text{Hom}_{\mathcal{D}}(FX, Y) \longrightarrow \text{Hom}_{\mathcal{D}}(FX, Y'')$$

This is really the same as

$$0 \longrightarrow \text{Hom}_{\mathcal{C}}(X, GY') \longrightarrow \text{Hom}_{\mathcal{C}}(X, GY) \longrightarrow \text{Hom}_{\mathcal{C}}(X, GY'')$$

This is true for any $X \in \mathcal{C}$, which is equivalent to

$$0 \longrightarrow GY' \longrightarrow GY \longrightarrow GY''$$

being exact. □

Proposition 10. In \mathbf{SAb} , f_* is left exact and f^* is exact.

Proof. By earlier exercise, f^* is exact on stalks. □

Direct Images with Compact Support. (III.8.7-10)

Assume all topological spaces are

1. locally compact: for any $x \in X$ there exists an open set U and compact set K such that $x \in U \subseteq K$;
2. first-countable: every point has a countable local basis;
3. (often) Hausdorff.

A morphism between topological spaces is **proper** if the preimages of compact sets are compact. (Sort of like a morphism with compact fibers.)

Let $f : X \rightarrow Y$ be continuous, \mathcal{F} a sheaf on X , $U \subseteq Y$ open. Define the **direct image with compact support** sheaf $f_!\mathcal{F}$ on Y by

$$f_!\mathcal{F}(U) = \{s \in \Gamma(f^{-1}(U), \mathcal{F}) \mid f : \text{supp}(s) \rightarrow U \text{ is proper}\}.$$

Let $s \in \Gamma(V, \mathcal{G})$, the **support** of s is

$$\text{supp}(s) := \overline{\{x \in V \mid \bar{s} \neq 0 \in \mathcal{G}_x\}}.$$

where $\Gamma(V, \mathcal{G}) \rightarrow \mathcal{G}_x$, $s \mapsto \bar{s}$, takes the germ of s at x . In other words, the closure of the set of points where the germ of s is nonzero.

Theorem 16.

- (a) $f_!\mathcal{F}$ is a subsheaf of $f_*\mathcal{F}$;
- (b) $f_! : \mathbf{SAb}_X \rightarrow \mathbf{SAb}_Y$ is a left exact functor

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Proof. (a) It is clear that $f_!\mathcal{F}$ is a subpresheaf of $f_*\mathcal{F}$, since $f_!\mathcal{F}(U) \subseteq f_*\mathcal{F}(U)$ by definition. So the main thing to check is that $f_!\mathcal{F}$ is a sheaf.

Any set of compatible sections of $f_!\mathcal{F}$ glue uniquely to a section of $f_*\mathcal{F}$. We want this glued section to be a section of $f_!\mathcal{F}$, i.e., that it satisfies the relevant condition.

This comes down to a topological statement.

Exercise: given a collection of proper maps $V_i \rightarrow U_i$, the glued map $\bigcup V_i \rightarrow \bigcup U_i$ is proper.

(b) follows from the fact that $f_!$ is a subsheaf? □

Consider the special case $f : X \rightarrow \text{pt}$. Then $f_! \mathcal{F} \approx f_! \mathcal{F}(\text{pt})$ is the sections $s \in \mathcal{F}(X)$ such that $\text{supp}(s)$ is compact, denote this by $\Gamma_c(X, \mathcal{F})$.

(Remark: category of abelian sheaves over a point is equivalent (isomorphic?) to the category of abelian groups.)

Now consider a single point $y \in Y$. Have the following diagram:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ i \uparrow & & \uparrow j \\ f^{-1}(y) & \longrightarrow & y \end{array}$$

Proposition 11. For a point $y \in Y$, we have the stalk

$$(f_! \mathcal{F})_y = \Gamma_c(f^{-1}(y), \mathcal{F}|_{f^{-1}(y)}) = i^* \mathcal{F}.$$

Proof. First want to construct the morphism $\varphi : (f_! \mathcal{F})_y \rightarrow \Gamma_c(f^{-1}(y), \mathcal{F}|_{f^{-1}(y)})$.

Let $s \in (f_! \mathcal{F})_y$. Choose a representative $\tilde{s} \in \Gamma(f^{-1}(U), \mathcal{F})$ with U an open neighborhood of y and $\text{supp}(\tilde{s}) \rightarrow U$ proper.

Then $\tilde{s}|_{f^{-1}(y)}$ is in $\Gamma_c(f^{-1}(y), \mathcal{F}|_{f^{-1}(y)})$, since $\text{supp}(\tilde{s}|_{f^{-1}(y)}) = \text{supp}(\tilde{s}) \cap f^{-1}(y)$.

Exercise: $\varphi(s) := \tilde{s}|_{f^{-1}(y)}$ only depends on s .

Now show φ is injective. Suppose $\varphi(s) = 0$. Then induced section $\tilde{s} = 0$, which says that $\text{supp}(\tilde{s})$ is disjoint from $f^{-1}(y)$. In particular, $y \notin f(\text{supp} \tilde{s})$. Note that the support is closed and f is proper (also need space is locally compact), so $f(\text{supp} \tilde{s})$ is closed. So there is a neighborhood of y where a representative of s is zero, hence $s = 0$.

Now show φ is surjective. Choose a local basis $\{U_i\}$ around y such that $\bigcap U_i = y$. Then $f^{-1}(y) = \bigcap f^{-1}(U_i)$.

Exercise: locally compact implies $\Gamma_c(f^{-1}(y), \mathcal{F}|_{f^{-1}(y)}) = \varinjlim A_i$, where

$$A_i = \{t \in \Gamma(f^{-1}(U_i), \mathcal{F}) \mid \text{supp } t = K \cap f^{-1}(U_i) \text{ for some compact } K \subseteq X\}.$$

(This is annoying to verify.)

□

Example

1. open immersion $U \hookrightarrow X$, \mathcal{F} a sheaf on U . Look at stalk:

$$(i_! \mathcal{F})_x = \begin{cases} \mathcal{F}_x & x \in U \\ 0 & x \notin U \end{cases}$$

For points in the complement, map is never proper. Called “extension by zero.”

2. $j : E \rightarrow X$ proper (in particular, closed embedding). Then $j_! \mathcal{G} = j_* \mathcal{G}$. The closed image of a compact set is also compact, so every map is proper.

Derived Categories via Localizations. (More classical approach)

Let $f : K^\bullet \rightarrow L^\bullet$ be a morphism of complexes in an abelian category \mathcal{A} .

Say f is a **quasi-isomorphism** if the induced morphisms $H^n(f) : H^n(K^\bullet) \rightarrow H^n(L^\bullet)$ are all isomorphisms.

Define $\mathbf{Kom}(\mathcal{A})$ to be the category of complexes in \mathcal{A} .

There exists a category $D(\mathcal{A})$ and a functor $Q : \mathbf{Kom}(\mathcal{A}) \rightarrow D(\mathcal{A})$ such that

1. $Q(f)$ is an isomorphism for any quasi-isomorphism f ;
2. Q is universal with respect to all such categories:

$$\begin{array}{ccc} \mathbf{Kom}(\mathcal{A}) & \xrightarrow{\quad} & \mathcal{D}' \\ & \searrow & \nearrow \exists! \\ & D(\mathcal{A}) & \end{array}$$

Call $D(\mathcal{A})$ the **derived category** of \mathcal{A} .

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Remark: Consider full subcategory $\mathbf{Kom}^+(\mathcal{A})$ of $\mathbf{Kom}(\mathcal{A})$ consisting of all complexes K^\bullet with $K^i = 0$ for $i \ll 0$, i.e., K^\bullet is bounded from below. Also have $\mathbf{Kom}^-(\mathcal{A})$ of complexes bounded above, and $\mathbf{Kom}^b(\mathcal{A})$ of complexes bounded from both sides. These induce bounded derived categories $D^+(\mathcal{A})$, $D^-(\mathcal{A})$, and $D^b(\mathcal{A})$, respectively, which are full subcategories of $D(\mathcal{A})$.

Construction of $D(\mathcal{A})$.

Think about it like “localization of noncommutative rings.”

A class of morphisms $S \subseteq \text{Mor } \mathcal{B}$ is said to be **localizing** if

- (a) S is closed under composition: $id_X \in S$ for all $X \in \mathcal{B}$ and $s \circ t \in S$ for all $s, t \in S$ for which the composition is defined;
- (b) extension: for all $f \in \text{Mor } \mathcal{B}$ and $s \in S$, there exists $g \in \text{Mor } \mathcal{B}$ and $t \in S$ such that the following diagram commutes:

$$\begin{array}{ccc} W & \xrightarrow{\quad g \quad} & Z \\ \downarrow t & & \downarrow s \\ X & \xrightarrow{\quad f \quad} & Y \end{array} \quad \text{or} \quad \begin{array}{ccc} W & \xleftarrow{\quad g \quad} & Z \\ \uparrow t & & \uparrow s \\ X & \xleftarrow{\quad f \quad} & Y \end{array}$$

- (c) $f, g \in \text{Mor}(X, Y)$, the existence of $s \in S$ with $sf = sg$ is equivalence to the existence of $t \in S$ with $ft = gt$.

The **localization** of \mathcal{B} at S is the category $\mathcal{B}[S^{-1}]$ with $\text{Ob } \mathcal{B}[S^{-1}] = \text{Ob } \mathcal{B}$. A morphism $X \rightarrow Y$ in $\mathcal{B}[S^{-1}]$ is the equivalence class of **roofs** of the form

$$\begin{array}{ccc} & X' & \\ s \swarrow & & \searrow f \\ X & & Y \end{array}$$

with $s \in S$ and $f \in \text{Mor } \mathcal{B}$. Two roofs are equivalence if there exists a third roof on top, which makes the following diagram commute:

$$\begin{array}{ccccc} & & X''' & & \\ & r \swarrow & & \searrow h & \\ X' & & & & X'' \\ s \downarrow & f \searrow & & t \swarrow & \downarrow g \\ X & & & & Y \end{array}$$

The identity id_X is the class of

$$\begin{array}{ccc} & X & \\ \text{id}_X \swarrow & & \searrow \text{id}_X \\ X & & X \end{array}$$

The composition of roofs (s, f) and (t, g) is the class of (st', gf') where

$$\begin{array}{ccccccc} & & & X'' & & & \\ & & t' \swarrow & & \searrow f' & & \\ & X' & & & & Y' & \\ s \swarrow & & f \searrow & & t \swarrow & g \searrow & \\ X & & & Y & & & Z \end{array}$$

Extension condition gives existence of square, so

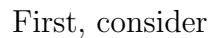
$$gt^{-1}fs^{-1} = gf't'^{-1}s^{-1} = (gf')(st')^{-1}.$$

Lemma 9.

- (a) This is an equivalence relation on roofs,
- (b) composition is well-defined.

Proof. (a) Reflexive and symmetric easy.

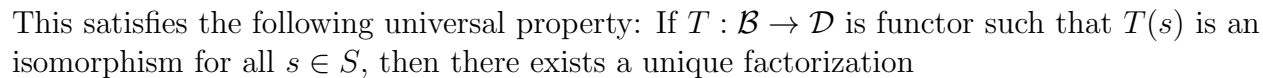
Suppose $(s, f) \sim (t, g)$ and $(t, g) \sim (u, e)$. So we have



Then $srv = tpk$. By first diagram, have $srv = thv$. So by (c), there exists $w : Z''' \rightarrow W$ in S such that $hvw = pkw$. Now



Construct morphism $F : \mathcal{B} \rightarrow \mathcal{B}[S^{-1}]$ which is the identity on objects and sends a morphism $f \in \text{Mor}_{\mathcal{B}}(X, Y)$ to the equivalence class of the roof



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Proposition 12. Let $T : \mathcal{B} \rightarrow \mathcal{D}$ be a functor such that $T(s)$ is an isomorphism for any s in a localizing class S of \mathcal{B} . Then T uniquely factors through $\mathcal{B}[S^{-1}]$:

$$\begin{array}{ccc}
\mathcal{B} & \xrightarrow{T} & \mathcal{D} \\
& \searrow F & \nearrow \exists! G \\
& \mathcal{B}[S^{-1}] &
\end{array}$$

Proof. To construct $G : \mathcal{B}[S^{-1}] \rightarrow \mathcal{D}$ such that $T = G \circ F$, must have

$$G(X \in \mathcal{B}[S^{-1}]) = T(X \in \mathcal{B})$$

and

$$G([s, f]) = T(f) \circ T(s)^{-1}.$$

Exercise: well-defined and unique. □

Issue: The quasi-isomorphisms in $\mathbf{Kom}(\mathcal{A})$ do not form a localizing class.

Fix n . For $K^\bullet = (K^\bullet, d_K^\bullet)$, define a complex $K[n]^\bullet$ by

$$(K[n])^i := K^{n+i} \quad \text{and} \quad d_{K[n]} := (-1)^n d_K.$$

(A shift to the left by n .)

$$\dots K^i \xrightarrow{d_K^i} K^{i+1} \xrightarrow{d_K^{i+1}} \dots$$

For $f : K^\bullet \rightarrow L^\bullet$, define $f[n] : K[n]^\bullet \rightarrow L[n]^\bullet$ to coincide with f componentwise.

This describes a **translation functor** $T^n : \mathbf{Kom}(\mathcal{A}) \rightarrow \mathbf{Kom}(\mathcal{A})$, which is an autoequivalence.

Let $f : K^\bullet \rightarrow L^\bullet$ be a morphism of complexes, the **cone** of f is the complex $c(f)^\bullet$ with

$$c(f)^i = K[1]^i \oplus L^i \quad \text{and} \quad d_{c(f)}(K^{i+1}, L_i) = (-d_K K^{i+1}, f(K^{i+1}) + d_L(L^i))$$

$$\begin{array}{ccc}
K^{i+1} & \xrightarrow{-d_K} & K^{i+1} \\
\oplus & \searrow f^{i+1} & \oplus \\
L_i & \xrightarrow{d_L} & L^{i+1}
\end{array}$$

Check $d_{c(f)}^2 = 0$.

Example: f is a morphism of a “0-complex,” i.e., $f : K^0 \rightarrow L^0$ where K^0 and L^0 have at most one nonzero entry, in the 0th position. Then $c(f)$ is the complex

$$\begin{array}{ccccccc}
\dots & \longrightarrow & 0 & \longrightarrow & K^0 & \xrightarrow{f} & L^0 & \longrightarrow & 0 & \longrightarrow & \dots \\
& & & & \uparrow & & \uparrow & & & & \\
& & & & \text{deg } -1 & & \text{deg } 0 & & & &
\end{array}$$

In particular, $H^{-1}(c(f)) = \ker f$ and $H^0(c(f)) = \operatorname{coker} f$.

The **cylinder** $\operatorname{Cyl}(f)$ is a complex

$$\operatorname{Cyl}(f)^\bullet = K^\bullet \oplus K[1]^\bullet \oplus L^\bullet$$

and $d_{\operatorname{Cyl}(f)}^i$ given by

$$\begin{array}{ccc} K^i & \xrightarrow{d_K} & K^{i+1} \\ & \searrow -\operatorname{id} & \nearrow \\ K^{i+1} & \xrightarrow{-d_K} & K^{i+2} \\ & \searrow f & \nearrow \\ L_i & \xrightarrow{d_L} & L^{i+1} \end{array}$$

Lemma 10. The following diagram commutes with exact rows:

$$\begin{array}{ccccccc} 0 & \longrightarrow & L^\bullet & \longrightarrow & c(f) & \longrightarrow & K[1]^\bullet \longrightarrow 0 \\ & & \downarrow \alpha & & \parallel & & \\ 0 & \longrightarrow & K^\bullet & \longrightarrow & \operatorname{Cyl}(f) & \longrightarrow & c(f) \longrightarrow 0 \\ & & \parallel & & \downarrow \beta & & \\ & & K^\bullet & \xrightarrow{f} & L^\bullet & & \end{array}$$

It's functorial in f . The morphisms α and β are quasi-isomorphisms: $\beta\alpha = \operatorname{id}_L$ and $\alpha\beta$ is homotopic to $\operatorname{id}_{\operatorname{Cyl}(f)}$.

Say $f, g : K^\bullet \rightarrow L^\bullet$ are **homotopic**, written $f \sim g$ if there exists k such that $f - g = kd + dk$ where

$$\begin{array}{ccccc} K^{i-1} & \longrightarrow & K^i & \longrightarrow & K^{i+1} \\ & \swarrow k^i & \downarrow f-g & \swarrow k^{i+1} & \\ L^{i-1} & \longrightarrow & L^i & \longrightarrow & L^{i+1} \end{array}$$

Proof. α is the “inclusion.”

And β maps

$$\begin{array}{ccc} (K^i, K^{i+1}, L^i) & \xrightarrow{d_{\operatorname{Cyl}(f)}} & (\quad) \\ \downarrow \beta_i & & \downarrow \beta^{i+1} \\ (f(K^i) + L^i) & \xrightarrow{d_L} & (\quad) \end{array}$$

Consider map

$$\begin{aligned} \mathrm{Cyl}(f)^i &\xrightarrow{k^i} \mathrm{Cyl}(f)^{i-1} \\ (K^i, K^{i+1}, L^i) &\longmapsto (0, K^i, 0) \end{aligned}$$

Then $\alpha\beta = \mathrm{id} + dk + kd$. □

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Let \mathcal{A} be an abelian category.

The **homotopy category** $K(\mathcal{A})$ is

- $\mathrm{Ob} K(\mathcal{A}) = \mathrm{Ob} \mathbf{Kom}(\mathcal{A})$;
- $\mathrm{Mor} K(\mathcal{A}) = \mathrm{Mor} \mathbf{Kom}(\mathcal{A}) / \{\text{homotopy equivalence}\}$

Recall that homotopy equivalence on morphisms induces the same morphisms on cohomology, so it makes sense to talk about quasi-isomorphisms inside $K(\mathcal{A})$.

Theorem 17 (III.4.4). The class of quasi-isomorphisms in $K(\mathcal{A})$ is localizing.

Will give a more conceptual proof later, based on two more results:

- $K(\mathcal{A})$ is a triangulated category;
- quasi-isomorphisms are “obtained by cohomological functors.”

Taste of proof of localization:

obviously closed under composition

Want to complete the following diagram:

$$\begin{array}{ccc} N^\bullet & \xrightarrow[\text{q-iso}]{k} & M^\bullet \\ \downarrow h & & \downarrow g \\ K^\bullet & \xrightarrow[\text{q-iso}]{f} & L^\bullet \end{array}$$

This comes from the following diagram:

$$\begin{array}{ccccccc} c(\pi g)^\bullet[-1] & \xrightarrow{k} & M^\bullet & \xrightarrow{\pi g} & c(f)^\bullet & \longrightarrow & c(\pi g)^\bullet \\ \downarrow \exists h & & \downarrow g & & \parallel & & \downarrow h[1] \\ K^\bullet & \xrightarrow{f} & L^\bullet & \xrightarrow{\pi} & c(f)^\bullet & \longrightarrow & K^\bullet[1] \end{array}$$

This diagram commutes up to homotopy.

$c(f)$ is cyclic (no cohomology), will imply something about M^\bullet .

Bottom row is common thing.

Can now define derived category as $D(\mathcal{A}) := K(\mathcal{A})[\{\text{q-iso}\}^{-1}]$.

$D(\mathcal{A})$ is an additive category.

For two morphisms

$$\begin{array}{ccc} & Z & \\ s \swarrow & & \searrow f \\ X & & Y \end{array} \quad \begin{array}{ccc} & Z' & \\ s' \swarrow & & \searrow f' \\ X & & Y \end{array}$$

take

$$\begin{array}{ccc} U & \xrightarrow{r'} & Z' \\ \downarrow r & & \downarrow s' \\ Z & \xrightarrow{s} & X \end{array}$$

Both r and r' are quasi-isomorphisms.

$$\begin{array}{ccccc} & & U & & \\ & r \swarrow & & \searrow r' & \\ Z & & & & Z' \\ s \downarrow & f \searrow & s' \swarrow & & \downarrow f' \\ X & & & & Y \end{array}$$

Take

$$(s, f) + (s', f') = (sr = s'r', fr + f'r')$$

Exercise: this is an additive structure.

Let \mathcal{D} be an additive category.

A **triangle** in \mathcal{D} is a sequence

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$$

and a morphism of triangles is a diagram

$$\begin{array}{ccccccc} X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & X[1] \\ \downarrow f & & \downarrow g & & \downarrow k & & \downarrow f[1] \\ X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & X'[1] \end{array}$$

A **Δ -category structure** on \mathcal{D} consists of the data

- (a) Additive autoequivalence $T : \mathcal{D} \rightarrow \mathcal{D}$ called the translation, denote $(X[n] := T^n X$ and $f[n] := T^n f)$;

(b) a class of *distinguished triangles*

satisfying

TR1. (1) Every triangle of the form

$$X \xrightarrow{\text{id}} X \longrightarrow 0 \longrightarrow X[1]$$

is distinguished,

(2) any triangle isomorphic to a distinguished triangle is distinguished,

(3) any morphism $u : X \rightarrow Y$ can be completed to a distinguished triangle.

TR2. A triangle

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$$

is distinguished if and only if

$$Y \xrightarrow{v} Z \xrightarrow{w} X[1] \xrightarrow{-u[1]} Y[1]$$

is distinguished.

TR3. Assume we have two distinguished triangles and morphisms f and g , then there exists h (not necessarily unique) which completes the following diagram:

$$\begin{array}{ccccccc} X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & X[1] \\ \downarrow f & & \downarrow g & & \downarrow \exists h & & \downarrow f[1] \\ X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & X[1] \end{array}$$

TR4. Starting with commutative square

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ g \downarrow & & \downarrow w \\ X' & \xrightarrow{h} & Y' \end{array}$$

Then there is a (unique?) way to complete it to a diagram:

$$\begin{array}{ccccc} X & \xrightarrow{f} & Y & \longrightarrow & Z \\ g \downarrow & & \downarrow w & & \downarrow \\ X' & \xrightarrow{h} & Y' & \longrightarrow & Z' \\ \downarrow & & \downarrow & & \downarrow \\ X'' & \longrightarrow & Y'' & \longrightarrow & Z'' \end{array}$$

(Unknown if TR4 follows from the previous axioms)

generalized by stable ∞ -categories.

Octahedron: Given three triangles of the form

$$X \xrightarrow{f} Y \xrightarrow{h} Y/X \longrightarrow X[1]$$

$$Y \xrightarrow{g} Z \xrightarrow{k} Z/Y \longrightarrow Y[1]$$

$$X \xrightarrow{g \circ f} Z \xrightarrow{\ell} Z/X \longrightarrow X[1]$$

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Proposition 13. Let \mathcal{D} be a triangulated category, and

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$$

Then there exists $U \in \text{Ob } \mathcal{D}$ such that the following sequences are exact

$$\begin{aligned} \cdots \longrightarrow \text{Hom}(U, X[i]) \xrightarrow{(u[i])_*} \text{Hom}(U, Y[i]) \xrightarrow{(v[i])_*} \text{Hom}(U, Z[i]) \\ \xrightarrow{(w[i])_*} \text{Hom}(U, X[i+1]) \longrightarrow \cdots \end{aligned}$$

and

$$\cdots \longrightarrow \text{Hom}(X[i+1], U) \xrightarrow{(w[i])^*} \text{Hom}(Z[i], U) \longrightarrow \cdots$$

Proof. By TR2, we only need to check exactness at $\text{Hom}(U, Y)$ (i.e., we can shift to get Lemma applies elsewhere).

First, we show that $vu = 0$. By TR3 (applied to TR1 and assumption)

$$\begin{array}{ccccccc} X & \xrightarrow{\text{id}} & X & \xrightarrow{0} & 0 & \longrightarrow & X[1] \\ \downarrow \text{id} & & \downarrow u & & \downarrow h & & \downarrow \text{id} \\ X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & X[1] \end{array}$$

The only option for h is 0 so $0 = vu$. Thus $v_*u_* = 0$ so the hom sequence is a complex.

Now, we show exactness. Let $f : U \rightarrow Y$ be such that $vf = 0$. ($f \in \ker v_*$, show $f \in \text{im } u_*$)
Need to find $g : U \rightarrow X$ such that $f = ug$. By TR2 and TR3

$$\begin{array}{ccccccc} U & \xrightarrow{\text{id}} & U & \longrightarrow & 0 & \longrightarrow & U[1] \\ \downarrow g & & \downarrow f & & \downarrow & & \downarrow g[1] \\ X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \longrightarrow & X[1] \end{array}$$

So $f = ug$, so exact at $\text{Hom}(U, Y)$ so exact. □

Corollary 3. If f and g in TR3 are isomorphisms, then h is also an isomorphism. In particular, distinguished triangles in TR1c is unique up to (non-unique) isomorphism.

$$\begin{array}{ccccc} & X & \longrightarrow & Y & \longrightarrow & Z \\ \text{Proof.} & \downarrow f & & \downarrow g & & \downarrow h \\ & X' & \longrightarrow & Y' & \longrightarrow & Z' \end{array}$$

□

Apply $\text{Hom}(Z', -)$ to get

$$\begin{array}{ccccccccc} \text{Hom}(Z', X) & \rightarrow & \text{Hom}(Z', Y) & \rightarrow & \text{Hom}(Z', Z) & \rightarrow & \text{Hom}(Z', X[1]) & \rightarrow & \text{Hom}(Z', Y[1]) \\ \cong \downarrow f_* & & \cong \downarrow g_* & & \downarrow h_* & & \cong \downarrow f[1]_* & & \cong \downarrow g[1]_* \\ \text{Hom}(Z', X') & \rightarrow & \text{Hom}(Z', Y') & \rightarrow & \text{Hom}(Z', Z') & \rightarrow & \text{Hom}(Z', X'[1]) & \rightarrow & \text{Hom}(Z', Y'[1]) \end{array}$$

So by the five lemma, h_* is an isomorphism of abelian groups, so there exists $\varphi : Z' \rightarrow Z$ such that $h\varphi = \text{id}_{Z'}$.

By applying $\text{Hom}(-, Z)$, get left inverse of h , which implies h is an isomorphism.

Corollary 4. Diagram

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & X[1] \\ \downarrow f & & \downarrow g & & \downarrow h & & \downarrow f[1] \\ Z'[-1] & \longrightarrow & X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z' & \xrightarrow{w'} & X'[1] \end{array}$$

There exists f, h if $v'gu = 0$.

If, moreover, $\text{Hom}(X, Z'[-1]) = 0$, then this morphism of distinguished triangles is unique.

By TR1c and corollary, any $u : X \rightarrow Y$ determines an object $C(u)$ (unique up to isomorphism) in distinguished triangle

$$X \xrightarrow{u} Y \longrightarrow C(u) \longrightarrow X[1]$$

This object is called the **cone** of u .

Recall TR4:

$$\begin{array}{ccccc} X & \xrightarrow{u} & Y & \longrightarrow & C(u) \\ \parallel & & \downarrow v & & \downarrow \\ X & \xrightarrow{vy} & Z & \longrightarrow & C(vu) \\ \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & C(v) & \longrightarrow & C(v) \end{array}$$

third column is important distinguished triangle, like an isomorphism theorem

$$0 \longrightarrow Y/X \longrightarrow Z/X \longrightarrow Z/Y \longrightarrow 0$$

A triangle in $\mathbf{Kom}(\mathcal{A})/K(\mathcal{A})$ is distinguished if it is

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Recall $K(\mathcal{A})$ with autoequivalence $T = [1]$.

A distinguished triangle is a triangle isomorphic to

$$K^\bullet \xrightarrow{\bar{f}} \mathrm{Cyl}(f) \xrightarrow{\pi} c(f) \xrightarrow{\delta} K[1]^\bullet$$

for some $f : K^\bullet \rightarrow L^\bullet$.

Theorem 18. This gives $K(\mathcal{A})$ the structure of a triangulated category.

Proof. Must check TR1-4.

Only show TR1.

Must construct

$$\begin{array}{ccccccc} X & \xrightarrow{\mathrm{id}} & X & \longrightarrow & 0 & \longrightarrow & X[1] \\ \parallel & & \parallel & & \downarrow & & \parallel \\ X & \xrightarrow{\mathrm{id}} & X & \longrightarrow & c(\mathrm{id}) & \longrightarrow & X[1] \end{array}$$

This is to show $0 : c(\mathrm{id}) \rightarrow c(\mathrm{id})$ is homotopic to $\mathrm{id} : c(\mathrm{id}) \rightarrow c(\mathrm{id})$.

Note: $c(\mathrm{id}) = X[1] \oplus X$ by definition. The differential of this cone is

$$\begin{array}{ccc} X^{i+1} & \xrightarrow{-d^{i+1}} & X^{i+2} \\ & \searrow \mathrm{id} & \\ X^i & \xrightarrow{d^i} & X^{i+1} \end{array}$$

Write

$$h = \begin{bmatrix} X^{i+1} & X^i \end{bmatrix} \begin{bmatrix} 0 & 0 \\ \mathrm{id} & 0 \end{bmatrix} = \begin{bmatrix} X^i & 0 \end{bmatrix}$$

so

$$\mathrm{id}_{c(\mathrm{id})} = h \circ d + d \circ h$$

Then

$$\begin{array}{ccccc}
& & X^1 \oplus X^0 & \longrightarrow & X^2 \oplus X^1 \\
& \swarrow h & \downarrow \text{id} & \nwarrow h & \\
XX^0 \oplus X^{-1} & \longrightarrow & X^1 \oplus X^0 & &
\end{array}$$

with

$$\begin{array}{ccccc}
& & (a, b) & \longrightarrow & (-da, a + db) \\
& \swarrow h & \downarrow \text{id} & \nwarrow h & \\
X(b, 0) & \longrightarrow & (-db, b) & &
\end{array}$$

□

Let \mathcal{D} be a triangulated category and \mathcal{A} an abelian category.

A functor $H : \mathcal{D} \rightarrow \mathcal{A}$ is **cohomological** if its additive and for any distinguished triangle

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$$

in \mathcal{D} the sequence

$$H(X) \xrightarrow{H(u)} H(Y) \xrightarrow{H(v)} H(Z)$$

in \mathcal{A} is exact.

Then TR2 gives us a long exact sequence

$$\cdots \longrightarrow H(X[i]) \longrightarrow H(Y[i]) \longrightarrow H(Z[i]) \longrightarrow H(X[i+1]) \longrightarrow \cdots$$

Examples

1. $\mathcal{D} = K(\mathcal{A}) \rightarrow \mathcal{A}, C^\bullet \mapsto H^0(C^\bullet)$.
2. $\mathcal{D} \rightarrow \mathbf{Ab}, X \mapsto \text{Hom}_{\mathcal{D}}(U, X)$ for some fixed $U \in \mathcal{D}$.

Proposition 14. Let $H : \mathcal{D} \rightarrow \mathcal{A}$ be a cohomological functor. Consider \mathcal{S} , the class of morphisms $s \in \text{Hom } \mathcal{D}$ such that $H^i(s) := H(s[i])$ is an isomorphism in \mathcal{A} for any $i \in \mathbb{Z}$. Then \mathcal{S} is a localizing class (a-c). Moreover, \mathcal{S} is compatible with \mathcal{D} , i.e.,

- (d) $s \in \mathcal{S}$ iff $T(s) \in \mathcal{S}$;
- (d) In TR3, if $f, g \in \mathcal{S}$ then there exists $h \in \mathcal{S}$ completing the diagram.

Example: If $s : C^\bullet \rightarrow D^\bullet$, then $s[i] : C^\bullet[i] \rightarrow D^\bullet[i]$. Need $H(s[i]) : H^i(C^\bullet) \rightarrow H^i(D^\bullet)$ to be an isomorphism. Then \mathcal{S} is the quasi-isomorphisms.

Proof. (a) ok

(b) Let $f : X \rightarrow Y$ and $s : Z \rightarrow Y$. Take distinguished triangle

$$\begin{array}{ccccccc} & & X & & & & \\ & & \downarrow f & & & & \\ Z & \xrightarrow{s} & Y & \xrightarrow{u} & C & \longrightarrow & Z[1] \end{array}$$

Then make square

$$\begin{array}{ccccccc} & & X & \xrightarrow{uf} & C & & \\ & & \downarrow f & & \parallel & & \\ Z & \xrightarrow{s} & Y & \xrightarrow{u} & C & \longrightarrow & Z[1] \end{array}$$

Complete top row to a triangle

$$\begin{array}{ccccccc} W & \xrightarrow{t} & X & \xrightarrow{uf} & C & \longrightarrow & W[1] \\ g \downarrow & & \downarrow f & & \parallel & & \downarrow g[1] \\ Z & \xrightarrow{s} & Y & \xrightarrow{u} & C & \longrightarrow & Z[1] \end{array}$$

with $t \in \mathcal{S}$ Then

$$H(Z[i]) \xrightarrow{\cong} H(Y[i]) \longrightarrow H(C[i]) = 0$$

(c) Consider $f, g \in \text{Mor}(X, Y)$. If there exists $s : Y \rightarrow Y'$ such that $s(f - g) = 0$, consider distinguished triangle

$$Z \xrightarrow{u} Y \xrightarrow{s} Y' \xrightarrow{\delta} Z[1]$$

Then

$$\begin{array}{ccccc} \text{Hom}(X, Z) & \xrightarrow{u_*} & \text{Hom}(X, Y) & \xrightarrow{s_*} & \text{Hom}(X, Y') \\ v \longmapsto & & f - g & \longmapsto & 0 \end{array}$$

There exists $v : X \rightarrow Z$ such that $f - g = uv$.

Consider distinguished triangle

$$X' \xrightarrow{t} X \xrightarrow{v} Z \longrightarrow X'[1]$$

Then $vt = 0$ implies $(f - g)t = uvt = 0$.

Hve $s \in \mathcal{S}$, implies $H^*(Z) = 0$ implies $t \in \mathcal{S}$.

(d) ok

(e) consequence of 5-Lemma. □

Theorem 19. Let \mathcal{D} be a triangulated category and \mathcal{S} a localizing class compatible with \mathcal{D} . Then $\mathcal{D}[S^{-1}]$ is a triangulated category with a natural translation T , distinguished triangles defined to be the triangles isomorphic to the images of distinguished triangles in \mathcal{D} under the canonical functor $\mathcal{D} \rightarrow \mathcal{D}[S^{-1}]$.

Example: $D(\mathcal{A}) = K(\mathcal{A})[\{\text{q-iso}\}^{-1}]$ is a triangulated category.

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Proof. Only need to show how to complete $a : X \rightarrow Y$ in $\mathcal{D}[S^{-1}]$ to a distinguished triangle. By definition, a is represented by a roof

$$\begin{array}{ccc} & Y' & \\ s \swarrow & & \searrow f \\ X & & Y \end{array}$$

with $s \in \mathcal{S}$. Then there exists

$$\begin{array}{ccccc} & & Y' & & \\ & s \swarrow & & \searrow f & \\ X & & & & Y \\ & \swarrow f' \text{ (dashed)} & & \nwarrow s' \text{ (dashed)} & \\ & & Y'' & & \end{array}$$

In \mathcal{D} , take distinguished triangle

$$X \xrightarrow{f'} Y'' \xrightarrow{g} Z \xrightarrow{h} X[1]$$

This gives a distinguished triangle in $\mathcal{D}[S^{-1}]$

$$\begin{array}{ccccc} X & \xrightarrow{a} & Y & \xrightarrow{gs'} & Z & \xrightarrow{h} & X[1] \\ & & \searrow s & & \nearrow g & & \\ & & & Z'' & & & \end{array}$$

Corollary 5. $D(\mathcal{A})$ is a triangulated category.

Proposition 15. Consider the functor $Q : \mathcal{A} \rightarrow D(\mathcal{A})$ sending an object E to the complex

$$\begin{array}{ccccccc} \cdots & \longrightarrow & 0 & \longrightarrow & E & \longrightarrow & 0 \longrightarrow \cdots \\ & & & & \uparrow & & \\ & & & & \text{deg } 0 & & \end{array}$$

and sending morphisms in the obvious unique way. This yields an equivalence of \mathcal{A} with the subcategory of $D(\mathcal{A})$ formed by H^0 -complexes (i.e., complexes K^\bullet such that $H^i(K^\bullet) = 0$ for $i \neq 0$).

Proof. Have sequence

$$\mathcal{A} \longrightarrow K(\mathcal{A}) \longrightarrow D(\mathcal{A})$$

First functor is fully faithful.

Let $X, Y \in \mathcal{A}$. Need to show that $\text{Hom}_{\mathcal{A}}(E, F) = \text{Hom}_{K(\mathcal{A})}(E, F)$ is equal (isomorphic) to $\text{Hom}_{D(\mathcal{A})}(QE, QF)$. Take map

$$\begin{array}{ccc} \text{Hom}_{\mathcal{A}}(E, F) & \xrightarrow{a} & \text{Hom}_{D(\mathcal{A})}(E, F) \\ f & \longmapsto & [(\text{id}_X, f)] \end{array}$$

The inverse b is induced by $H^0 : D(\mathcal{A}) \rightarrow \mathcal{A}$. By definition $b \circ a = \text{id}$.

To check $a \circ b = \text{id}$, say $\tilde{f} : X \rightarrow F$ (between 0-complexes) in $D(\mathcal{A})$ is represented by

$$\begin{array}{ccc} & Z & \\ s \swarrow & & \searrow f \\ X & & Y \end{array}$$

As s is a quasi-isomorphism, $b(\tilde{f})$ is

$$\begin{array}{ccc} & X & \\ \text{id}_X \swarrow & & \searrow g \\ X & & Y \end{array}$$

where $g = H^0(f) \circ H^0(s)^{-1}$

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Natural morphism $K^+(I) \rightarrow D^+(\mathcal{A})$

If there exists enough injectives in \mathcal{A} , then this is an equivalence.

Concretely, for any $C^\bullet \in D^+(\mathcal{A})$ there exists a quasi-isomorphism $C^\bullet \rightarrow I^\bullet$ with $I^\bullet \in K^+(I)$.

In the following, we will not use “injective” property. Will only use: for all $C \in \mathcal{A}$ there exists $I \in \mathcal{I}$ with $C \hookrightarrow I$.

suppose we have a complex

$$\begin{array}{ccccccc} \cdots & \longrightarrow & 0 & \longrightarrow & C^0 & \xrightarrow{d_C^0} & C^1 & \xrightarrow{d_C^1} & C^2 & \longrightarrow & \cdots \\ & & & & \downarrow t^0 & & \downarrow t^1 & & \downarrow t^2 & & \\ & & 0 & \longrightarrow & I^0 & \xrightarrow{d_I^0} & I^1 & \xrightarrow{d_I^1} & I^2 & \longrightarrow & \cdots \end{array}$$

with t^i quasi-isomorphisms and $I^i \in \mathcal{I}$.

A **cocartesian product** (fibered coproduct) is the completion of the following diagram:

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ g \downarrow & & \downarrow \\ C & \dashrightarrow & B \sqcup_A C \end{array}$$

In abelian category,

$$A \xrightarrow{(f,g)} B \oplus C \longrightarrow B \sqcup_A C \longrightarrow 0$$

Now take

$$\begin{array}{ccccc} C^0 & \longrightarrow & C^1 & & \\ \downarrow t^0 & & \downarrow & \searrow t^1 & \\ I^0 & \longrightarrow & I^0 \sqcup_{C^0} C^1 & \hookrightarrow & I^1 \\ & \searrow & \uparrow & \nearrow & \\ & & d_I^0 & & \end{array}$$

Can check $t^0 : \ker d_I^0 \rightarrow \ker d_C^0$ is an isomorphism—basically by definition of \sqcup .

Next, consider

$$\begin{array}{ccccccc}
& & C^1 & \xrightarrow{\quad} & C^2 & & \\
& \swarrow t^1 & \downarrow & & \downarrow & \searrow t^2 & \\
I^1 & \xrightarrow{\quad} & \text{coker } d_I^0 & \xrightarrow{\quad} & C^2 \sqcup_{C^1} \text{coker } d_I^0 & \xrightarrow{\quad} & I^2 \\
& \searrow & & & & \swarrow & \\
& & & & d_I^1 & &
\end{array}$$

Can check this is quasi-isomorphism at degree 1.

All the rest will be essentially the same story.

Details in III.5.25.

Derived Functors.

Let $F : \mathcal{A} \rightarrow \mathcal{B}$ be an additive functor between abelian categories. This induces a functor $\text{Kom}(\mathcal{A}) \rightarrow \text{Kom}(\mathcal{B})$, by applying F termwise. This clearly preserves homotopy, so it gives a well-defined functor $K(F) : K(\mathcal{A}) \rightarrow K(\mathcal{B})$.

Issue: $K(F)$ preserves quasi-isomorphisms only when F is exact. But we want to define functors $D(\mathcal{A}) \rightarrow D(\mathcal{B})$ for left/right exact functors F .

Assume: F is left exact and we work with Kom^+ , K^+ , D^+ .

Recall functor $Q_A : K^+(\mathcal{A}) \rightarrow D^+(\mathcal{A})$.

Theorem 20. There exists a **derived functor** $RF : D^+(\mathcal{A}) \rightarrow D^+(\mathcal{B})$, together with a natural transformation

$$Q_B \circ K^+(F) \xrightarrow{\varepsilon_F} RF \circ Q_A$$

such that

- (1) RF is exact: $T_B \circ RF = RF \circ T_A$ and it sends distinguished triangles to distinguished triangles;
- (2) universal property: given $G : D^+(\mathcal{A}) \rightarrow D^+(\mathcal{B})$ and $\varepsilon_G : Q_B \circ K^+(F) \Rightarrow G \circ Q_A$, then there exists a unique $\eta : RF \Rightarrow G$ such that $\varepsilon_G = \eta \circ \varepsilon_F$.

A class of objects R in \mathcal{A} is **adapted** to F if

- (1) R is closed under finite direct sum.
- (2) For every $X^\bullet \in \text{Kom}^+(\mathcal{A})$ acyclic complex of objects in R , we have $F(X^\bullet)$ is also acyclic.
- (3) There exists enough objects in R such that for any $M \in \mathcal{A}$, there exists an injection $M \hookrightarrow B$ for some $B \in R$.

View R as an additive full subcategory of \mathcal{A} , us convention $\text{Kom}^+(R)$ and $K^+(R)$ as subcategories of $\text{Kom}^+(\mathcal{A})$ and $K^+(\mathcal{A})$, respectively, as these don't necessarily require the abelian structure.

Lemma 11.

- For every $X^\bullet \in \mathbf{Kom}^+(\mathcal{A})$, there exists a quasi-isomorphism $X^\bullet \rightarrow M^\bullet$ with $M^\bullet \in \mathbf{Kom}^+(R)$.
- The class of quasi-isomorphisms in $K^+(R)$ is localizing.
- $\mathcal{G} := K^+(R)[\{\text{q-iso}\}^{-1}] \rightarrow D^+(\mathcal{A})$ is an equivalence

Now have diagram

$$\begin{array}{ccccc}
 & & K^+(\mathcal{A}) & \xrightarrow{K^+(F)} & K^+(\mathcal{B}) \\
 & \nearrow & & & \downarrow Q_B \\
 K^+(R) & & & & D^+(\mathcal{B}) \\
 \downarrow & \nearrow \text{dashed} & & & \uparrow RF \\
 K^+(R)[\{\text{q-iso}\}^{-1}] & \xrightarrow{\cong} & & & D^+(\mathcal{A})
 \end{array}$$

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Note that a morphism u in $K^+(R)$ is a quasi-isomorphism iff $c(u)$ is acyclic. Then by (2) of adapted class, acyclic $c(u) \in K^+(R)$ tells us that $F(c(u)) = C(F(u))$ is acyclic (strictly speaking, this is $K^+(F)(i(c(u)))$). Then $F(u)$ is quasi-isomorphic.

Quasi-isomorphism in $K^+(R)$ sent to isomorphism in $D^+(\mathcal{B})$ by top path in diagram. This map factors through the localization, equivalent to $D^+(\mathcal{A})$, so we get the functor $RF : D^+(\mathcal{A}) \rightarrow D^+(\mathcal{B})$.

Explicitly, for any $X^\bullet \in D^+(\mathcal{A})$, we take a quasi-isomorphism $X^\bullet \rightarrow M^\bullet \in \mathbf{Kom}^+(R)$ and $RF(X^\bullet) = F(M^\bullet)$

Proposition 16. This is a derived functor RF with $\varepsilon_F(X^\bullet) : F(X^\bullet) \rightarrow F(M^\bullet)$ induced by $X^\bullet \rightarrow M^\bullet$, giving natural transformation $Q_B \circ F \Rightarrow RF \circ Q_A$.

Proof. Exercise □

Exercise: If \mathcal{A} has enough injectives, then the class of injective objects is adapted to any left exact functor. (Only need to verify (2): exercise from last time.)

Theorem 21. Let $F : \mathcal{A} \rightarrow \mathcal{B}$ and $G : \mathcal{B} \rightarrow \mathcal{C}$ be left exact functors. Suppose we have adapted classed R_A for F and R_B for G such that $F(M) \in R_B$ if $M \in R_A$. Then there is a natural isomorphism $RG \circ RF \Rightarrow R(G \circ F)$.

Proof. By definition, replace X^\bullet with some M^\bullet and map to $FM^\bullet \in \mathbf{Kom}^+(R_B)$ then get $G(FM^\bullet)$. □

Want to define \mathbf{RHom} .

Let \mathcal{A} be an abelian category with enough injectives.

Let $X^\bullet \in \mathbf{Kom}(\mathcal{A})$ and $Y^\bullet \in \mathbf{Kom}^+(\mathcal{A})$.

Define **hom complex**

$$\mathrm{Hom}_{\mathcal{A}}^n(X^\bullet, Y^\bullet) := \prod_{i \in \mathbb{Z}} \mathrm{Hom}_{\mathcal{A}}(X^i, Y^{n+i})$$

with differentials

$$d((f_i)_{i \in \mathbb{Z}}) := (d_Y \circ f_i + (-1)^{n+1} f_{i+1} \circ d_X)_{i \in \mathbb{Z}}$$

Key Observation: $Z^0(\mathrm{Hom}^\bullet) = \mathrm{Hom}_{\mathbf{Kom}(\mathcal{A})}(X^\bullet, Y^\bullet)$.

$B^0(\mathrm{Hom}^\bullet) = \{f \text{ homotopic to } 0\}$.

So $H^0(\mathrm{Hom}^\bullet(X^\bullet, Y^\bullet)) = \mathrm{Hom}_{K(\mathcal{A})}(X^\bullet, Y^\bullet)$.

Choose a quasi-isomorphism $Y^\bullet \rightarrow I^\bullet \in \mathbf{Kom}^+(\mathcal{I})$, where \mathcal{I} is class of injectives in \mathcal{A} . Define

$$\mathrm{RHom}_{\mathcal{A}}(X^\bullet, Y^\bullet) := \mathrm{Hom}^\bullet(X^\bullet, I^\bullet) \in D(\mathbf{Ab}).$$

Won't write dot, so remember $X, Y \in \mathcal{A}$.

Note

$$\begin{aligned} \mathrm{Ext}_{\mathcal{A}}^i(X, Y) &:= \mathrm{Hom}_{D(\mathcal{A})}(X[0], Y[i]) \\ &= \mathrm{Hom}_{D(\mathcal{A})}(X[0], I^\bullet[i]) \\ &= \mathrm{Hom}_{K(\mathcal{A})}(X[0], I^\bullet[i]) \\ &= H^i(\mathrm{RHom}_{\mathcal{A}}(X, Y)) \end{aligned}$$

where $Y \rightarrow I^\bullet$ is some quasi-isomorphism.