1 If (X, d) is a metric space, a map $f : X \to X$ is a *contraction* if there is a number $\alpha < 1$ such that

$$d(f(x), f(y)) \leq \alpha d(x, y).$$

Show that if f is a contraction of a complete metric space, then there is a unique point $x \in X$ such that $f(x) = x$.

Proof. We construct a sequence of points in X inductively: let $x_0 \in X$ be any arbitrary point and define $x_n = f(x_{n-1})$ for all $n \geq 1$. We will prove that this sequence is Cauchy, therefore convergent, and that the limit of the sequence in X is the unique fixed point.

For $n \in \mathbb{N}$, we use the fact that f is a contraction to compute

$$\begin{aligned} d(x_0, x_n) &= d(x_0, f^{n-1}(x_1)) \\ &\leq \sum_{k=0}^{n-1} d\big(f^k(x_0), f^k(x_1)\big) \\ &\leq \sum_{k=0}^{n-1} \alpha^k d(x_0, x_1) \\ &< \sum_{k=0}^{\infty} \alpha^k d(x_0, x_1) \\ &= \frac{d(x_0, x_1)}{1-\alpha}. \end{aligned}$$

Define $M = d(x_0, x_1)/(1-\alpha) \in \mathbb{R}_{\geq 0}$. For $n, m \in \mathbb{N}$, assuming $n \leq m$, we compute

$$d(x_n, x_m) = d(f^n(x_0), f^n(x_{m-n})) \leq \alpha^n d(x_0, x_{m-n}) < \alpha^n M.$$

Given $\varepsilon > 0$, we can choose $N \in \mathbb{N}$ such that $\alpha^N M < \varepsilon$. Then, if $n, m \geq N$,

$$d(x_n, x_m) < \alpha^{\min(n,m)} M \leq \alpha^N M < \varepsilon.$$

Hence, the sequence is Cauchy.

Since X is complete, the sequence converges to some $x \in X$; we will show that x is a fixed point of f. Let $\varepsilon > 0$ be given and choose $N \in \mathbb{N}$ such that $d(x, x_n) < \varepsilon$ for all $n \geq N$. Then

$$\begin{aligned} d(x, f(x)) &\leq d(x, x_{N+1}) + d(x_{N+1}, f(x)) \\ &= d(x, x_{N+1}) + d(f(x_N), f(x)) \\ &\leq d(x, x_{N+1}) + \alpha d(x_N, x) \\ &< \varepsilon + \alpha\varepsilon \\ &= (1+\alpha)\varepsilon. \end{aligned}$$

Since this holds for all $\varepsilon > 0$, we must have $d(x, f(x)) = 0$, implying that $x = f(x)$.

Lastly, we will prove that x is the unique fixed point of f. Suppose $y \in X$ is a fixed point of f, i.e., $f(y) = y$. Since $d(x, y) = d(f(x), f(y)) \leq \alpha d(x, y)$, we must have $d(x, y) = 0$, which means that $x = y$.

$\square$

2 In this question, we use the following definition of completion: the metric space Y is a completion of X if it contains an isometrically embedded copy of X whose closure is Y.

(a) A map $f : X \to Z$ between metric spaces is *Lipschitz* if there's a constant L such that

$$d(f(x), f(y)) \leq Ld(x, y).$$

Show that if Z is a complete metric space, then any Lipschitz map $f : X \to Z$ extends uniquely to the completion.

Proof. Let $X \hookrightarrow Y$ be the isometric inclusion of X into a completion Y; we identify X with its image in Y. We will construct a function $\tilde{f} : Y \to Z$ which is an extension of f.

For all $x \in X$, we define $\tilde{f}(x) = f(x)$, i.e., we manually enforce that $\tilde{f}|_X = f$.

Given $y \in Y \setminus X$, there is some sequence (x_n) in X, converging to y in Y. In particular, this is a Cauchy sequence in X. We check that $(f(x_n))$ is a Cauchy sequence in Z. Given $\varepsilon > 0$, we can choose $N \in \mathbb{N}$ such that $d(x_n, x_m) < \varepsilon/L$ for all $n, m \geq N$. Then, for any $n, m \geq N$,

$$d(f(x_n), f(x_m)) \leq Ld(x_n, x_m) < \varepsilon.$$

Hence, $(f(x_n))$ is a Cauchy sequence in the complete metric space Z and, therefore, converges to some point $z \in Z$. We define $\tilde{f}(y) = z$ (one can check that this is well-defined in the sense that it does not depend on the original choice of sequence (x_n)).

Next, we check that $\tilde{f}$ is Lipschitz on Y. Suppose $x, y \in Y$, then there are sequences (x_n), (y_n) in X such that $x_n \to x$ and $y_n \to y$. For any $n \in \mathbb{N}$,

$$d(\tilde{f}(x), \tilde{f}(y)) \leq d(\tilde{f}(x), f(x_n)) + d(f(x_n), f(y_n)) + f(f(y_n), \tilde{f}(y)).$$

Let $\varepsilon > 0$ be given. Since $\tilde{f}(x)$ is defined as the limit of $f(x_n)$, there is some $N \in \mathbb{N}$ such that $d(\tilde{f}(x), f(x_n)) < \varepsilon$ for all $n \geq N$. (More precisely, this works when $x \in Y \setminus X$. In the case that $x \in X$, we may assume $x_n = x$ for all n. Then, trivially, $f(x_n) \to f(x)$.) For the same reason, we can assume N is chosen large enough that $d(\tilde{f}(y), f(y_n)) < \varepsilon$ for all $n \geq N$. Then, for $n \geq N$, we have

$$\begin{aligned} d(\tilde{f}(x), \tilde{f}(y)) &< 2\varepsilon + d(f(x_n), f(y_n)) \\ &\leq 2\varepsilon + Ld(x_n, y_n). \end{aligned}$$

Now, we examine

$$d(x_n, y_n) \leq d(x_n, x) + d(x, y) + d(y, y_n).$$

Since $x_n \to x$ and $y_n \to y$, we can also assume N is chosen large enough that $d(x, x_n)$ and $d(y, y_n)$ are both less than ε for all $n \geq N$. In which case,

$$d(x_n, y_n) < 2\varepsilon + d(x, y).$$

This now gives us

$$d(\tilde{f}(x), \tilde{f}(y)) < 2\varepsilon + L(2\varepsilon + d(x, y)).$$

Letting $\varepsilon \to 0$, we obtain $d(\tilde{f}(x), \tilde{f}(y)) \leq Ld(x,y)$, i.e., $\tilde{f}$ is Lipschitz, with the same Lipschitz constant as f.

It remains to prove that this Lipschitz extension of f is unique. Suppose $g : Y \to Z$ is a Lipschitz function such that $g|_X = f$. For any $x \in Y$, there is some sequence (x_n) in X converging to x. Then, for all $n \in \mathbb{N}$, we have

$$\begin{aligned} d(\tilde{f}(x), g(x)) &\leq d(\tilde{f}(x), \tilde{f}(x_n)) + d(\tilde{f}(x_n), g(x)) \\ &= d(\tilde{f}(x), \tilde{f}(x_n)) + d(g(x_n), g(x)) \\ &\leq Ld(x, x_n) + L'd(x_n, x) \\ &= (L + L')d(x_n, x), \end{aligned}$$

where L' is the Lipschitz constant for g. Letting $n \to \infty$, we obtain $d(\tilde{f}(x), g(x)) = 0$, so in fact $\tilde{f} = g$.

□

(b) Show that the completion of a metric space is unique. That is, if Y and Z are two completions of X, show that the identity map $X \to X$ extends to an isometry $Y \to Z$.

Proof. The isometric embedding $\mathrm{id}_X : X \hookrightarrow Y$ is, in particular, a Lipschitz map from X to a complete metric spaces. Therefore, by part (a), there exists a unique Lipschitz map $f : Y \to Y$ such that $f|_X = \mathrm{id}_X$. Since id_Y is a Lipschitz map on Y with $\mathrm{id}_Y|_X = \mathrm{id}_X$, then in fact $f = \mathrm{id}_Y$. In other words, the identity on Y is the only Lipschitz map on Y which restricts to the identity on X. The same is also true of Z.

The inclusion $X \hookrightarrow Z$ uniquely extends to a Lipschitz map $f : Y \to Z$, which restricts to the identity on X. We claim that f is surjective. Let $z \in Z$, then there is a sequence (x_n) in X converging to z (in Z). In particular, the sequence is Cauchy in X, so has some limit $y \in Y$. Since f is continuous, $f(x_n) \to f(y)$. And since $f|_X = \mathrm{id}_X$, we also have $f(x_n) = x_n \to z$. Hence, $f(y) = z$, so f is surjective.

Similarly, the inclusion $X \hookrightarrow Y$ uniquely extends to a Lipschitz map $g : Z \to Y$, which restricts to the identity on X. Moreover, by the same argument, g is surjective. Then $g \circ f : Y \to Y$ and $f \circ g : Z \to Z$ are Lipschitz maps which restrict to the identity on X. Since the identities on Y and Z, respectively, are the unique such maps, then f and g are inverses.

Lastly, we check that f is an isometry. In part (a), we showed that the extension has the same Lipschitz constant as the original map, which is 1 for id_X, id_Y, implying the same for f and g. So, for all $x, y \in Y$,

$$d(f(x), f(y)) \leq d(x,y) = d(g(f(x)), g(f(y))) \leq d(f(x), f(y)).$$

Hence $d(f(x), f(y)) = d(x,y)$, so f is an isometry $Y \to Z$.

□

3 Show that the closed unit ball in $C_B([0,1])$ with the norm topology is not compact.

Proof. For $n \in \mathbb{N}$, define the function $f_n \in C_B([0,1])$ by $f_n(x) = x^n$. Pointwise, this sequence converges to the function

$$f(x) = \begin{cases} 0 & \text{if } 0 \leq x < 0, \\ 1 & \text{if } x = 1. \end{cases}$$

Clearly, f is not continuous, so $f \notin C_B([0,1])$. Therefore, the sequence (f_n) does not converge in $C_B([0,1])$, with the norm topology. This is because converging in the norm topology means uniform convergence in $[0,1]$, and if a sequence of functions uniformly converges, it must also converges pointwise to the same limit. But (f_n) converges pointwise to $f \notin C_B([0,1])$, i.e., it does not converge uniformly to a function in $C_B([0,1])$.

In particular, (f_n) is a sequence in $C_B([0,1])$ with no convergent subsequence, implying that $C_B([0,1])$ cannot be compact.

□

4 Recall that we say that a sequence (v_n) in a topological vector space X is *Cauchy* if for every neighborhood U of 0 there is an N such that for $m, n \geq N$, $v_m - v_n \in U$. The topological vector space X is *complete* if every Cauchy sequence converges.

(a) Show that if X is a *locally compact* space (every point has a compact neighborhood) then $C(X)$ with the compact-open topology is complete.

For $K \subseteq X$ compact and $U \subseteq \mathbb{R}$ open, denote

$$S(K, U) = \{f \in C(x) : f(K) \subseteq U\}.$$

The collection of all such $S(K, U)$ is a subbasis for the compact-open topology on $C(X)$, meaning that the collection of all finite intersections of such sets is a basis.

Proof. Suppose (f_n) is Cauchy sequence in $C(X)$.

For any $x \in X$, there is a sequence $(f_n(x))$ in $\mathbb{R}$, which we claim to be Cauchy. Let $\varepsilon > 0$ be given. Let $K \subseteq X$ be a compact neighborhood of x and define $U = S(K, B_\varepsilon(0))$, which is an open neighborhood of $0 \in C(X)$. Since (f_n) is Cauchy in $C(X)$, there is some $N \in \mathbb{N}$ such that, for all $n, m \geq N$, $f_n - f_m \in U$. Then, for $m, n \geq N$, we have

$$|f_n(x) - f_m(x)| \leq \|f_n - f_m\|_K < \varepsilon.$$

That is, $(f_n(x))$ is Cauchy in $\mathbb{R}$. Since $\mathbb{R}$ is complete, we may define a function $f : X \to \mathbb{R}$ by

$$f(x) = \lim_{n \to \infty} f_n(x).$$

By definition, $f_n \to f$ pointwise in X. We claim that this convergence is uniform in every compact subset of X. Let $K \subseteq X$ be compact and $\varepsilon > 0$ be given. As before, we choose $N \in \mathbb{N}$ such that $f_n - f_m \in S(K, B_\varepsilon(0))$, for all $n, m \geq N$. So, for all $x \in K$ and $n, m \geq N$,

$$|f_n(x) - f_m(x)| \leq \|f_n - f_m\|_K < \varepsilon.$$

Letting $m \to \infty$ in this inequality, we obtain $|f_n(x) - f(x)| \leq \varepsilon$ for all $x \in K$. Hence, we have uniform convergence $f_n \to f$ in every compact subset of X.

For any compact subset $K \subseteq X$, $(f_n|_K)$ is a sequence of continuous functions on K, converging uniformly to $f|_K$. This implies that $f|_K$ is a continuous function, i.e., f is continuous on each compact subset of X.

Each point of X has a compact neighborhood, which contains an open neighborhood. Then f is continuous on each of the compact neighborhoods, so it is also continuous on each of the open neighborhoods. These open neighborhoods form an open cover of X, so we conclude that f is continuous on all of X, hence $f \in C(X)$.

In remains to prove that (f_n) converges to f in $C(X)$ (with the compact-open topology). Let $\varepsilon > 0$ be given. Suppose $S(K, U)$ is an open neighborhood of f in the subbasis, i.e., $K \subseteq X$ compact, $U \subseteq \mathbb{R}$ open, and $f(K) \subseteq U$. Since $f(K)$ is a compact subset of the metric space $\mathbb{R}$, then by Homework 2 Problem 3(d), we can assume ε is small enough so that $U(f(K), \varepsilon) \subseteq U$, where

$$U(f(K), \varepsilon) = \{a \in \mathbb{R} : d(a, f(K)) < \varepsilon\}.$$

Since $f_n \to f$ uniformly in K, there is some $N \in \mathbb{N}$ such that $\|f_n - f\|_K < \varepsilon$, for $n \geq N$. In which case,

$$f_n(K) \subseteq U(f(K), \varepsilon) \subseteq U,$$

implying $f_n \in S(K, U)$. Now, suppose $B = \bigcap_{i=1}^{r} S(K_i, U_i)$ is an arbitrary set in the basis for the compact-open topology on $C(X)$. For $i = 1, \ldots, r$, we have just shown that there is some $N_i \in \mathbb{N}$ such that $f_n \in S(K_i, U_i)$, for all $n \geq N_i$. Define $N = \max\{N_1, \ldots, N_r\}$. Then, for all $n \geq N$, $f_n \in S(K_i, U_i)$ for $i = 1, \ldots, r$, implying that $f_n \in B$. Since every open neighborhood of f contains a neighborhood in the basis, this proves that $f_n \to f$ in the compact-open topology on $C(X)$.

□

(b) Show that $C_B(\mathbb{R})$ is not complete when given the compact-open topology.

Proof. We construct a sequence which is Cauchy in the compact-open topology on $C_B(\mathbb{R})$, but does not converge. For $n \in \mathbb{N}$, define the function $f_n : \mathbb{R} \to \mathbb{R}$ by

$$f_n(x) = \begin{cases} -n & \text{if } x < -n, \\ x & \text{if } -n \leq x \leq n, \\ n & \text{if } x > n. \end{cases}$$

Then $f_n \in C_B(\mathbb{R})$ and the sequence (f_n) converges pointwise to $f = \mathrm{id}_{\mathbb{R}} \notin C_B(\mathbb{R})$. This is a Cauchy sequence in the compact-open topology on $C_B(\mathbb{R})$, since it converges uniformly on any compact subset of $\mathbb{R}$. However, the sequence does not converge in the compact-open topology on $C_B(\mathbb{R})$, since the limit there would be the same as the pointwise limit, which is not in $C_B(\mathbb{R})$.

□

(c) (Optional, only if you're familiar with Lebesgue integration) Show that $C_B([0,1])$ is not complete when given the *weak-* topology*, which is generated by inverse images of open sets under maps $\int_M : C_B([0,1]) \to \mathbb{R}$ which send f to its integral over a measurable set M. (The completion is called $L^\infty([0,1])$.)