A **graph** Γ is characterized by the following information:

- a collection V (or Γ_0) of **vertices** (sing. vertex) (or **nodes** or **points**);
- a collection E (or Γ_1) of **edges** (or **arcs** or **lines**);
- a rule that associates each edge with two vertices (not necessarily distinct) called its **endpoints**;

A **loop** is an edge whose endpoints are the same vertex.

Two different edges are **parallel** is they have the same endpoints.

A graph is **simple** if it has no loops or parallel edges.

When two vertices are the endpoints of an edge, they are said to be **adjacent** or **neighbors**.

A graph is **finite** if it has finitely many vertices and edges.

The **null graph** is the graph with no vertices and no edges. It is stupid and we ignore it, I think.

Let $\mathbb{2} = \{0, 1\}$ be a two-element set with distinguished elements.

A **simple graph** is implemented with the following data:

- a set V;
- an anti-reflexive symmetric relation $\leftrightarrow$ on V called the **adjacency** relation, i.e.,
  - $v \not\leftrightarrow v$ for all $v \in V$,
  - $u \leftrightarrow v$ iff $v \leftrightarrow u$ for all $u, v \in V$;

A simple graph is an implementation of a graph as follows:

- vertices V;
- edges $E = \{\{u, v\} \subseteq V \mid u \leftrightarrow v\}$;
- endpoints of $e = \{u, v\} \in E$ are its elements, u and v.

By this implementation, a simple graph is finite if and only if both its vertex set V and edge set E have finite cardinality.

Because the adjacency relation is anti-reflexive, each edge always has two distinct vertices. Moreover, if two edges $e_1, e_2 \in E$ have the same endpoints if and only if they are the same set—hence there are no parallel edges. Therefore, a "simple graph" is indeed a graph which is simple in the above sense.

A **graph** Γ is given by the following data:

- a collection of vertices V;
- a relation $\sim$ on V called the **adjacency** relation.

A **directed graph** is a type of graph in which edges have a certain orientation/direction. We might think of the edges as arrows.

Given by the following data:

- a collection of vertices V;
- a collection of edges E;
- for each edge $e \in E$ two vertices $\mathrm{source}(e), \mathrm{target}(e) \in V$.

- simple graph: injective function $E \to \binom{V}{2}$; (or symmetric irreflexive relation on V)
- multigraph: arbitrary function $E \to \binom{V}{2}$;
- loop graph: injective function $E \to \binom{V}{2} \cup \binom{V}{1}$; (or symmetric relation on V)
- pseudograph: arbitrary function $E \to \binom{V}{2} \cup \binom{V}{1}$;

directed variants

- directed graph: injective function $E \to V^2 \setminus \Delta_V$; (or irreflexive relation on V)
- directed multigraph: arbitrary function $E \to V^2 \setminus \Delta_V$;
- directed loop graph: injective function $E \to V^2$; (or any relation on V)
- directed pseudograph (quiver): arbitrary function $E \to V^2$;

not commutative:

$$
\begin{tikzcd}
\text{di(simple)} \arrow[rrr, hook, shift right] \arrow[rd, two heads, "\pi"'] \arrow[ddd, hook, shift right] & & & \text{dimulti} \arrow[lll, two heads, shift right, "\pi"'] \arrow[ld, two heads, "\pi"'] \arrow[ddd, hook, shift right] \\
& \text{simple} \arrow[r, hook, shift right] \arrow[lu, "\delta"'] \arrow[d, hook, shift right] & \text{multi} \arrow[l, two heads, shift right, "\pi"'] \arrow[ru, "\delta"'] \arrow[d, hook, shift right] & \\
& \text{loop} \arrow[r, hook, shift right] \arrow[u, two heads, shift right, "\lambda"'] \arrow[ld, "\delta"'] & \text{pseudo} \arrow[l, two heads, shift right, "\pi"'] \arrow[u, two heads, shift right, "\lambda"'] \arrow[rd, "\delta"'] & \\
\text{diloop} \arrow[rrr, hook, shift right] \arrow[ru, two heads, "\pi"'] \arrow[uuu, two heads, shift right, "\lambda"'] & & & \text{dipseudo} \arrow[lll, two heads, shift right, "\pi"'] \arrow[lu, two heads, "\pi"'] \arrow[uuu, two heads, shift right, "\lambda"']
\end{tikzcd}
$$

An **isomorphism** of graphs $f : \Gamma \to \Gamma'$ is consists of the following data:

- a bijection $f : V \to V'$;
- a bijection $f : E \to E'$;

such that

- $\text{source}(f(e)) = f(\text{source}(e))$ for all $e \in E$;
- $\text{target}(f(e)) = f(\text{target}(e))$ for all $e \in E$;
- equiv $d(e) = (u, v)$ implies $d(f(e)) = (f(u), f(v))$ for all $e \in E$.
- map $f_{u,v} : \Gamma(u, v) \to \Gamma'(f(u), f(v))$ for all $u, v \in V$.

An **automorphism** of a graph Γ is an isomorphism $\Gamma \to \Gamma$.

Denote the set of such automorphisms by $\text{Aut}(\Gamma)$.

Note that $\text{Aut}(\Gamma)$ is "naturally" a group under composition.

Let G be a group and Γ be a graph.

An **acton** of G on Γ consists of the following data:

- a group homomorphism $G \to \text{Aut}(\Gamma)$, denoted $g \mapsto (g \cdot -)$.

Equivalently, an action consists of a function $A : G \times \Gamma \to \Gamma$, denoted $(g, x) \mapsto g \cdot x$ for x in V or E, such that

- $1_G \cdot x = x$ for all $x \in \Gamma$;
- $g \cdot (h \cdot x) = (gh) \cdot x$ for all $x \in \Gamma$;
- $g \cdot (e : u \to v) = g \cdot e : g \cdot u \to g \cdot v$ for all $e \in E$;

Let G be a group generated by $S \subseteq G$.

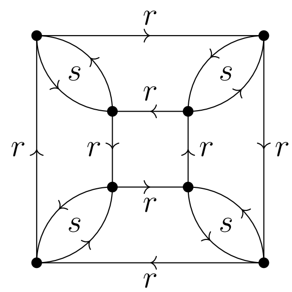
We construct a directed graph $\text{Cay}(G, S)$, called the **Cayley graph** of G with respect to S, as follows:

- vertices of $\text{Cay}(G, S)$ are the elements of G;
- edges of $\text{Cay}(G, S)$ are $g \to gs$ for all $g \in G$ and $s \in S$.

There is a path $(s_1, \ldots, s_n) : g \mapsto h$ in $\text{Cay}(G, S)$ if and only $gs_1 \cdots s_n = h \in G$.

Let $D_n = \langle s, r \mid s^2 = r^n = (sr)^2 = 1 \rangle$

Then $\text{Cay}(D_4, \{s, r\})$ is as follows:



There is a natural action of G on $\mathrm{Cay}(G, S)$ by $g \mapsto \varphi_g \in \mathrm{Aut}(\mathrm{Cay}(G, S))$, where

$$\mathrm{Cay}(G,S) \xrightarrow{\varphi_g} \mathrm{Cay}(G,S)$$

$$\begin{array}{ccc} v & \longmapsto & gv \\ {\scriptstyle s}\big\downarrow & \longmapsto & \big\downarrow{\scriptstyle s} \\ vs & \longmapsto & gvs \end{array}$$

The map $\varphi : G \to \mathrm{Aut}(\mathrm{Cay}(G, S))$ is an isomorphism of groups.