Exercise 10.9 Define $(x,y) = T(r,\theta)$ on the rectangle

$$0 \leq r \leq a, \qquad 0 \leq \theta \leq 2\pi$$

by the equations

$$x = r\cos\theta, \qquad y = r\sin\theta.$$

Show

that T maps this rectangle onto the closed disc D with center at $(0,0)$ and radius a,

Proof. By definition, $T(r,\theta)$ is the point of $\mathbb{R}^2$ with magnitude r and angle θ to the positive x-axis. Since $0 \leq r \leq a$, then the maximum magnitude of $T(r,\theta)$ is a, i.e, the image of T is contained in D. Moreover, since the angle θ of any point can be chosen such that $0 \leq \theta \leq 2\pi$, then all possible points with radius at most a are attained by T. In other words, D is contained in the image of T. Hence, the image of T is precisely D. $\square$

that T is one-to-one in the interior of the rectangle,

Proof. Let (r_1,θ_1) and (r_2,θ_2) be two points in the interior of the rectangle such that $T(r_1,\theta_1) = T(r_2,\theta_2)$. Then

$$r_1 = |T(r_1,\theta_1)| = |T(r_2,\theta_2)| = r_2.$$

Since θ_1 and θ_2 refer to the same direction, then their difference must be some multiple of 2π. But since $0 < \theta_1 < 2\pi$ and $0 < \theta_2 < 2\pi$, then there difference must be zero, i.e., $\theta_1 = \theta_2$. Hence, $(r_1,\theta_1) = (r_2,\theta_2)$, so T is injective in the interior of the rectangle. $\square$

and that $J_T(r,\theta) = r$.

Proof.

$$J_T(r,\theta) = \begin{vmatrix} \frac{\partial}{\partial r} r\cos\theta & \frac{\partial}{\partial\theta} r\cos\theta \\ \frac{\partial}{\partial r} r\sin\theta & \frac{\partial}{\partial\theta} r\sin\theta \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r.$$

$\square$

If $f \in C(D)$, prove the formula for integration in polar coordinates:

$$\int_D f(x, y)\,\mathrm{d}x\,\mathrm{d}y = \int_0^a \int_0^{2\pi} f(T(r, \theta))r\,\mathrm{d}\theta\,\mathrm{d}r.$$

Hint: Let D_0 be the interior of D, minus the interval from $(0, 0)$ to $(0, a)$. As it stands, Theorem 10.9 applies to continuous functions f whose support lies in D_0. To remove this restriction, proceed as in Example 10.4.

Proof. Consider the complement of D_0, denoted by D_0^C. Define the distance function

$$h(\mathbf{x}) = d(\mathbf{x}, D_0^C) = \inf_{\mathbf{y} \in D_0^C} |\mathbf{x} - \mathbf{y}|,$$

which is continuous on $\mathbb{R}^2$ with $h(\mathbf{x}) = 0$ for all $\mathbf{x} \in D_0^C$. Now, for each $\delta > 0$, we define the piecewise function

$$\varphi_\delta(t) = \begin{cases} 1 & \text{if } t \geq \delta, \\ \dfrac{t}{\delta} & \text{otherwise,} \end{cases}$$

and

$$f_\delta(\mathbf{x}) = \varphi_\delta(h(\mathbf{x}))f(\mathbf{x}).$$

Then f_δ is a continuous function on $\mathbb{R}^2$, with support inside D_0. Moreover, $f_\delta(\mathbf{x}) = f(\mathbf{x})$ whenever $h(\mathbf{x}) \geq \delta$, and $|f_\delta(\mathbf{x})| \leq |f(\mathbf{x})|$ otherwise. Applying Theorem 10.9 to f_δ, we obtain

$$\int_{\mathbb{R}^2} f_\delta(\mathbf{x})\,\mathrm{d}\mathbf{x} = \int_{\mathbb{R}^2} f_\delta(T(\mathbf{x}))|J_T(\mathbf{x})|\,\mathrm{d}\mathbf{x} = \int_{\mathbb{R}^2} f_\delta(T(\mathbf{x}))r\,\mathrm{d}\mathbf{x} \tag{1}$$

We can extend f to a (not necessarily continuous) function on $\mathbb{R}^2$ by defining $f(\mathbf{x}) = 0$ for all $\mathbf{x} \in D^C$, so that its support remains in D. Let $I \subseteq \mathbb{R}^2$ be the 2-cell defined by $-a \leq x_i \leq a$ for $i = 1, 2$. Then the support of both f and f_δ is contained in I. We compare their integrals:

$$\begin{aligned} \left| \int_D f_\delta(\mathbf{x})\,\mathrm{d}\mathbf{x} - \int_D f(\mathbf{x})\,\mathrm{d}\mathbf{x} \right| &= \left| \int_I [f_\delta(\mathbf{x}) - f(\mathbf{x})]\,\mathrm{d}\mathbf{x} \right| \\ &= \left| \int_{-a}^a \int_{-a}^a [f_\delta(x, y) - f(x, y)]\,\mathrm{d}y\,\mathrm{d}x \right| \\ &\leq \int_{-a}^a \int_{-a}^a |f_\delta(x, y) - f(x, y)|\,\mathrm{d}y\,\mathrm{d}x \end{aligned}$$

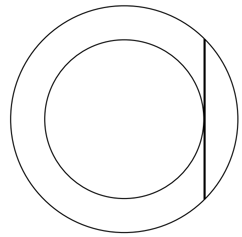
For a fixed $x \in [-a, a]$, the integrand is potentially nonzero only along the curve

$$I = \{y \in \mathbb{R} : 0 < h(x, y) < \delta\}(\cup\{0\} \text{ if } x \geq 0).$$

For $x < 0$, I can be described as the intersection of a vertical line with the annulus of $\mathbb{R}^2$ with inner radius $a - \delta$ and outer radius a. And if $x \geq 0$, then I also includes the segment from $(x, -\delta)$ to (x, δ). So the length of I is bound by 2δ plus the maximum length of a vertical line intersected with the described annulus, given by

$$2\sqrt{a^2 - (a - \delta)^2} = 2\sqrt{2a\delta - \delta^2} \leq C\sqrt{\delta},$$

for some constant $C > 0$. Geometrically, the maximum length of the intersection of a vertical line and an annulus is as follows:



Solving for the length of this segment gives the above formula. Hence, the length of I is bound by

$$2\delta + C\sqrt{\delta} = (2\sqrt{\delta} + C)\sqrt{\delta} \leq (2 + C)\sqrt{\delta},$$

assuming without loss of generality that $\delta < 1$. Therefore,

$$\int_{-a}^{a} |f_\delta(x,y) - f(x,y)| \,\mathrm{d}y = \int_I |f_\delta(x,y) - f(x,y)| \,\mathrm{d}y \leq \|f\|(2 + C)\sqrt{\delta},$$

so we obtain

$$\left| \int_D f_\delta(\mathbf{x}) \,\mathrm{d}\mathbf{x} - \int_D f(\mathbf{x}) \,\mathrm{d}\mathbf{x} \right| \leq \int_{-a}^{a} \|f\|(2 + C)\sqrt{\delta} \,\mathrm{d}x = 2a\|f\|(2 + C)\sqrt{\delta}. \tag{2}$$

We now compare the following integrals:

$$\begin{aligned}
\left| \int_{\mathbb{R}^2} f_\delta(T(\mathbf{x}))r \,\mathrm{d}\mathbf{x} - \int_{\mathbb{R}^2} f(T(\mathbf{x}))r \,\mathrm{d}\mathbf{x} \right| &= \left| \int_{\mathbb{R}^2} [f_\delta(T(\mathbf{x})) - f(T(\mathbf{x}))]r \,\mathrm{d}\mathbf{x} \right| \\
&= \left| \int_0^a \int_0^{2\pi} [f_\delta(T(r,\theta)) - f(T(r,\theta))]r \,\mathrm{d}\theta \,\mathrm{d}r \right| \\
&\leq \int_0^a \int_0^{2\pi} |f_\delta(T(r,\theta)) - f(T(r,\theta))|r \,\mathrm{d}\theta \,\mathrm{d}r.
\end{aligned}$$

We consider this integral in three parts:

$$\left(\int_0^{\sqrt{\delta}} + \int_{\sqrt{\delta}}^{a-\delta} + \int_{a-\delta}^{a} \right) \int_0^{2\pi} |f_\delta(T(r,\theta)) - f(T(r,\theta))|r \,\mathrm{d}\theta \,\mathrm{d}r,$$

For $r \leq \sqrt{\delta}$ and $a - \delta \leq r \leq a$,

$$\int_0^{2\pi} |f_\delta(T(r,\theta)) - f(T(r,\theta))|r \,\mathrm{d}\theta \leq 2\pi\|f\|\delta,$$

then

$$\left(\int_0^{\sqrt{\delta}} + \int_{a-\delta}^{a} \right) 2\pi\|f\|\delta \,\mathrm{d}r = 2\pi\|f\|(\delta^{3/2} + \delta^2) \leq 4\pi\|f\|\delta.$$

For $\sqrt{\delta} < r < a - \delta$, the integrand is possibly nonzero only for values of θ such that $T(r, \theta)$ lies in the strip of width 2δ around the positive x-axis. In which case,

$$\sin\theta = \frac{\delta}{r} \leq \frac{\delta}{\sqrt{\delta}} = \sqrt{\delta}.$$

So the integrand is bound by

$$\int_{-\arcsin\sqrt{\delta}}^{\arcsin\sqrt{\delta}} |f_\delta(T(r,\theta)) - f(T(r,\theta))| r \, \mathrm{d}\theta \leq 2r\|f\| \arcsin\sqrt{\delta},$$

then

$$\int_{\sqrt{\delta}}^{a-\delta} 2r\|f\| \arcsin\sqrt{\delta} \, \mathrm{d}r \leq 2a^2\|f\| \arcsin\sqrt{\delta}.$$

Therefore, we obtain

$$\left| \int_{\mathbb{R}^2} f_\delta(T(\mathbf{x})) r \, \mathrm{d}\mathbf{x} - \int_{\mathbb{R}^2} f(T(\mathbf{x})) r \, \mathrm{d}\mathbf{x} \right| \leq 4\pi\|f\|\delta + 2a^2\|f\| \arcsin\sqrt{\delta}. \tag{3}$$

Combining (1), (2), and (3), we find

$$\begin{aligned} \left| \int_D f - \int_{\mathbb{R}^2} (f \circ T) r \right| &\leq \left| \int_D f - \int_D f_\delta \right| + \left| \int_D f_\delta - \int_{\mathbb{R}^2} (f_\delta \circ T) r \right| + \left| \int_{\mathbb{R}^2} (f_\delta \circ T) r - \int_{\mathbb{R}^2} (f \circ T) r \right| \\ &\leq 2a\|f\|(2 + C)\sqrt{\delta} + 0 + 4\pi\|f\|\delta + 2a^2\|f\| \arcsin\sqrt{\delta}. \end{aligned}$$

Since this holds for all $\delta > 0$, then letting $\delta \to 0$, we obtain

$$\int_D f = \int_{\mathbb{R}^2} (f \circ T) r.$$

This is precisely the desired result

$$\int_D f(x, y) \, \mathrm{d}x \, \mathrm{d}y = \int_0^a \int_0^{2\pi} f(T(r, \theta)) r \, \mathrm{d}\theta \, \mathrm{d}r.$$

$\square$

Exercise 10.10 Let $a \to \infty$ in Exercise 9 and prove that

$$\int_{\mathbb{R}^2} f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int_0^\infty \int_0^{2\pi} f(T(r,\theta))r\,\mathrm{d}\theta\,\mathrm{d}r,$$

for continuous functions f that decrease sufficiently rapidly as $|x|+|y| \to \infty$. (Find a more precise formulation.)

We prove the result for functions $f \in C(\mathbb{R}^2)$ for which there exist constants $b > 1$, $C, R > 0$ such that $|f(\mathbf{x})| \leq C|\mathbf{x}|^{-b}$ for all $|\mathbf{x}| \geq R$.

Proof. Applying Exercise 9, we find

$$\begin{aligned}\int_{\mathbb{R}^2} f(\mathbf{x})\,\mathrm{d}\mathbf{x} &= \lim_{R\to\infty} \int_{|\mathbf{x}|\leq R} f(\mathbf{x})\,\mathrm{d}\mathbf{x} \\ &= \lim_{R\to\infty} \int_0^R \int_0^{2\pi} f(T(r,\theta))r\,\mathrm{d}\theta\,\mathrm{d}r \\ &= \int_0^\infty \int_0^{2\pi} f(T(r,\theta))r\,\mathrm{d}\theta\,\mathrm{d}r.\end{aligned}$$

□

Apply this to

$$f(x,y) = \exp(-x^2 - y^2)$$

to derive formula (101) of Chap. 8, i.e.

$$\int_{-\infty}^\infty e^{-s^2}\,\mathrm{d}s = \sqrt{\pi}.$$

Proof. Given that f decreases sufficiently rapidly, we apply the above result to obtain

$$\begin{aligned}\int_{\mathbb{R}^2} f(x,y)\,\mathrm{d}x\,\mathrm{d}y &= \int_0^\infty \int_0^{2\pi} f(T(r,\theta))r\,\mathrm{d}\theta\,\mathrm{d}r \\ \int_{-\infty}^\infty \int_{-\infty}^\infty e^{-x^2-y^2}\,\mathrm{d}x\,\mathrm{d}y &= \int_0^\infty \int_0^{2\pi} e^{-r^2\cos^2\theta - r^2\sin^2\theta} r\,\mathrm{d}\theta\,\mathrm{d}r \\ \int_{-\infty}^\infty e^{-x^2}\,\mathrm{d}x \int_{-\infty}^\infty e^{-y^2}\,\mathrm{d}y &= \int_0^\infty \int_0^{2\pi} re^{-r^2}\,\mathrm{d}\theta\,\mathrm{d}r \\ \left(\int_{-\infty}^\infty e^{-s^2}\,\mathrm{d}s\right)^2 &= 2\pi \int_0^\infty re^{-r^2}\,\mathrm{d}r.\end{aligned}$$

Take $u = e^{-r^2}$ for change of variables, then $\mathrm{d}u = -2re^{-r^2}\,\mathrm{d}r$, so

$$\left(\int_{-\infty}^\infty e^{-s^2}\,\mathrm{d}s\right)^2 = 2\pi \int_1^0 \frac{-1}{2}\,\mathrm{d}u = \pi.$$

Hence,

$$\int_{-\infty}^{\infty} e^{-s^2}\,\mathrm{d}s = \sqrt{\pi}.$$

$\square$

Exercise 10.14 Prove formula (46), i.e.

$$\varepsilon(j_1, \ldots, j_k) = s(j_1, \ldots, j_k),$$

where s is as in Definition 9.33, i.e.

$$s(j_1, \ldots, j_n) = \prod_{p<q} \operatorname{sgn}(j_q - j_p).$$

Proof. Let J be the k tuple with the entries $j_1, \ldots, j_k$ in increasing order. The k-form

$$\mathrm{d}x_{j_1} \wedge \cdots \wedge \mathrm{d}x_{j_k}$$

can be obtained from the basic k-form $\mathrm{d}x_J$ after a finite number of interchanges of adjacent pairs. We perform induction on the number of adjacent interchanges n, to prove that

$$\varepsilon(j_1, \ldots, j_k) = s(j_1, \ldots, j_k) = (-1)^n.$$

If $j_1 < j_2 < \cdots < j_k$, then $J = (j_1, \ldots, j_k)$ corresponds to the basic k-form

$$\mathrm{d}x_J = \mathrm{d}x_{j_1} \wedge \cdots \wedge \mathrm{d}x_{j_k}.$$

That is, $\varepsilon(j_1, \ldots, j_k) = 1$. Moreover, since $j_p < j_q$ for all $p < q$, then

$$s(j_1, \ldots, j_n) = \prod_{p<q} \operatorname{sgn}(j_q - j_p) = \prod_{p<q} 1 = 1.$$

This proves the base case of zero interchanges with $1 = (-1)^0$.

For the inductive hypothesis, assume that the result holds for any k-form obtained from $\mathrm{d}x_J$ by $n-1$ or fewer adjacent interchanges. Then given a k-form $\mathrm{d}x_{j_1} \wedge \cdots \wedge \mathrm{d}x_{j_k}$ obtained from the basic k-form $\mathrm{d}x_J$ after n adjacent interchanges, then there is some k-form

$$\mathrm{d}x_{j_1} \wedge \cdots \wedge \mathrm{d}x_{j_{i+1}} \wedge \mathrm{d}x_{j_i} \wedge \cdots \wedge \mathrm{d}x_{j_k}$$

which can be obtained from $\mathrm{d}x_J$ by $n-1$ adjacent interchanges. Then

$$\begin{aligned}
\varepsilon(j_1, \ldots, j_k)\,\mathrm{d}x_J &= \mathrm{d}x_{j_1} \wedge \cdots \wedge \mathrm{d}x_{j_k} \\
&= -(\mathrm{d}x_{j_1} \wedge \cdots \wedge \mathrm{d}x_{j_{i+1}} \wedge \mathrm{d}x_{j_i} \wedge \cdots \wedge \mathrm{d}x_{j_k}) \\
&= -\varepsilon(j_1, \ldots, j_{i+1}, j_i, \ldots, j_k)\,\mathrm{d}x_J.
\end{aligned}$$

Applying the inductive hypothesis, we obtain

$$\varepsilon(j_1, \ldots, j_k) = -\varepsilon(j_1, \ldots, j_{i+1}, j_i, \ldots, j_k) = -s(j_1, \ldots, j_{i+1}, j_i, \ldots, j_k) = -(-1)^{n-1}.$$

The only difference in the expressions for $s(j_1, \ldots, j_k)$ and $s(j_1, \ldots, j_{i+1}, j_i, \ldots, j_k)$ is that the former includes the factor $\operatorname{sgn}(j_{i+1} - j_i)$, while the latter includes the factor $\operatorname{sgn}(j_i - j_{i+1})$. That is,

$$s(j_1, \ldots, j_{i+1}, j_i, \ldots, j_k) = \frac{\operatorname{sgn}(j_i - j_{i+1})}{\operatorname{sgn}(j_{i+1} - j_i)} s(j_1, \ldots, j_k) = -s(j_1, \ldots, j_k),$$

so in fact $\varepsilon(j_1, \ldots, j_k) = s(j_1, \ldots, j_k) = (-1)^n$. This completes the induction.

$\square$

Exercise 10.15 If ω and λ are $k-$ and $m-$forms, respectively, prove that

$$\omega \wedge \lambda = (-1)^{km} \lambda \wedge \omega.$$

Proof. If ω and λ are basic forms.

$$\omega \wedge \lambda = \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_k} \wedge \mathrm{d}x_{j_1} \wedge \cdots \wedge \mathrm{d}x_{j_m}.$$

We can perform k adjacent interchanges to move $\mathrm{d}x_{j_1}$ to the front. From Exercise 10.14, this gives us

$$(-1)^k \omega \wedge \lambda = \mathrm{d}x_{j_1} \wedge \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_k} \wedge \mathrm{d}x_{j_2} \wedge \cdots \wedge \mathrm{d}x_{j_m}.$$

Repeating this for all m of the 1-form components of λ, we obtain

$$(-1)^{km} \omega \wedge \lambda = \mathrm{d}x_{j_1} \wedge \cdots \wedge \mathrm{d}x_{j_m} \wedge \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_k} = \lambda \wedge \omega.$$

For general k- and m-forms, we simply apply this result termwise.

□

Exercise 10.22 As in Example 10.37, define ζ in $\mathbb{R}^3 \setminus \{0\}$ by

$$\zeta = \frac{x\,\mathrm{d}y \wedge \mathrm{d}z + y\,\mathrm{d}z \wedge \mathrm{d}x + z\,\mathrm{d}x + \mathrm{d}y}{r^3}$$

where $r = (x^2 + y^2 + z^2)^{1/2}$, let D be the rectangle given by $0 \leq u \leq \pi$, $0 \leq v \leq 2\pi$, and let Σ be the 2-surface in $\mathbb{R}^3$, with parameter domain D, given by

$$x = \sin u \cos v, \qquad y = \sin u \sin v, \qquad z = \cos u.$$

(a) Prove that $\mathrm{d}\zeta = 0$ in $\mathbb{R}^3 \setminus \{0\}$.

Proof. Write

$$\zeta = \frac{x}{r^3}\,\mathrm{d}y \wedge \mathrm{d}z + \frac{y}{r^3}\,\mathrm{d}z \wedge \mathrm{d}x + \frac{z}{r^3}\,\mathrm{d}x \wedge \mathrm{d}y.$$

Taking the derivative, each term of ζ will be the exterior product of the sum of three 1-forms and a 2-form. Only the 1 form which is distinct from both 1-forms in the 2-form will produce a nonzero term. So

$$\begin{aligned}
\mathrm{d}\zeta &= D_x\frac{x}{r^3}\,\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z + D_y\frac{y}{r^3}\,\mathrm{d}y \wedge \mathrm{d}z \wedge \mathrm{d}x + D_z\frac{z}{r^3}\,\mathrm{d}z \wedge \mathrm{d}x \wedge \mathrm{d}y \\
&= \left(D_x\frac{x}{r^3} + D_y\frac{y}{r^3} + D_z\frac{z}{r^3}\right)\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z \\
&= \left(\frac{-2x^2 + y^2 + z^2}{r^5} + \frac{x^2 - 2y^2 + z^2}{r^5} + \frac{x^2 + y^2 - 2z^2}{r^5}\right)\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z \\
&= 0\,\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z \\
&= 0.
\end{aligned}$$

□

(b) Let S denote the restriction of Σ to a parameter domain $E \subseteq D$. Prove that

$$\int_S \zeta = \int_E \sin u \,\mathrm{d}u\,\mathrm{d}v = A(S),$$

where A denotes area, as in Sec. 10.43. Note that this contains (115) as a special case.

Proof.

$$\begin{aligned}
\int_S \zeta &= \int_E \left(\frac{x}{r^3}\frac{\partial(y,z)}{\partial(u,v)} + \frac{y}{r^3}\frac{\partial(z,x)}{\partial(u,v)} + \frac{z}{r^3}\frac{\partial(x,y)}{\partial(u,v)} \right) \mathrm{d}u\,\mathrm{d}v \\
&= \int_E \left(\frac{x}{r^3}(\sin^2 u \cos v) + \frac{y}{r^3}(\sin^2 u \sin v) + \frac{z}{r^3}(\cos u \sin u) \right) \mathrm{d}u\,\mathrm{d}v \\
&= \int_E \frac{\sin^3 u \cos^2 v + \sin^3 u \sin^2 v + \cos^2 u \sin u}{(\sin^2 u \cos^2 v + \sin^2 u \sin^2 v + \cos^2 u)^{3/2}} \,\mathrm{d}u\,\mathrm{d}v \\
&= \int_E \frac{\sin^3 u + \cos^2 u \sin u}{(\sin^2 u + \cos^2 u)^{3/2}} \,\mathrm{d}u\,\mathrm{d}v \\
&= \int_E \sin u \,\mathrm{d}u\,\mathrm{d}v
\end{aligned}$$

□

(c) Suppose $g, h_1, h_2, h_3 \in C^2([0,1])$ with $g > 0$. Let $(x,y,z) = \Phi(s,t)$ define a 2-surface Φ, with parameter domain I^2, by

$$x = g(t)h_1(s), \qquad y = g(t)h_2(s), \qquad z = g(t)h_3(s).$$

Prove that

$$\int_\Phi \zeta = 0,$$

directly from (35).
Note the shape of the range of Φ: For fixed s, $\Phi(s,t)$ runs over an interval on a line through 0. The range of Φ thus lies in a "cone" with vertex at the origin.

(d) Let E be a closed rectangle in D, with edges parallel to those of D. Suppose $f \in C^2(D), f > 0$. Let Ω be the 2-surface with parameter domain E, defined by

$$\Omega(u, v) = f(u, v)\Sigma(u, v).$$

Define S as in (b) and prove that

$$\int_{\Omega} \zeta = \int_{S} \zeta = A(S).$$

(Since S is the "radial projection" of Ω into the unit sphere, this result makes it reasonable to call $\int_{\Omega} \zeta$ the "solid angle" subtended by the range of Ω at the origin.)
Hint: Consider the 3-surface Ψ given by

$$\Psi(t, u, v) = [1 - t + tf(u, v)]\Sigma(u, v),$$

where $(u, v) \in E, 0 \leq t \leq 1$. For fixed v, the mapping $(t, u) \to \Psi(t, u, v)$ is a 2-surface Φ to which (c) can be applied to show that $\int_{\Phi} \zeta = 0$. The same thing holds when u is fixed. By (a) and Stokes' theorem,

$$\int_{\partial\Psi} \zeta = \int_{\Phi} \mathrm{d}\zeta = 0.$$

Deduce the Stokes and divergence theorems in calculus from the general stokes theorem.