2 Let G and H be topological groups. Show that a group homomorphism $f : G \to H$ is continuous if and only if for every neighborhood V of the identity $e_h \in H$, there is a neighborhood U of the identity $e_g \in G$ such that $f(U) \subseteq V$.

Lemma 1. If G is a topological group and $x \in G$, then the map

$$\begin{aligned} G &\xrightarrow{m_x} G \\ y &\longmapsto xy \end{aligned}$$

is a homeomorphism of G.

Proof. Consider the subspace $\{x\} \times G$ of $G \times G$, with the subspace topology (which, trivially, agrees with its product topology). Then the inclusion $\{x\} \times G \hookrightarrow G \times G$ is continuous. Note that the projection to the second coordinate $\{x\} \times G \to G$ is injective. Since, in general, projections are continuous, open, and surjective, this projection is a homeomorphism.

Let $m : G \times G \to G$ denote the multiplication map, then $m_x = m(x, -)$ can be written as the following composition of continuous maps:

$$\begin{array}{ccccccc} G & \xrightarrow{\sim} & \{x\} \times G & \hookrightarrow & G \times G & \xrightarrow{m} & G \\ y & \longmapsto & & & & & xy \end{array}$$

Therefore, m_x is a continuous map, for all $x \in G$. Since G is a group, m_x is also bijective and has the continuous inverse $m_{x^{-1}}$, hence it is a homeomorphism.

□

Proof of Problem 2. If f is continuous, then any open neighborhood $V \subseteq H$ of e_h has an open preimage $f^{-1}(V) \subseteq G$. Because f is a group homomorphism, we have $f(e_g) = e_h \in V$, implying $e_g \in f^{-1}(V)$. Hence, $f^{-1}(V)$ is an open neighborhood of e_g, with $f(f^{-1}(V)) = V$.

Suppose f has the second property. Let $V \subseteq H$ be an open subset; we will prove $f^{-1}(V)$ is open in G by looking at a point. Let $x \in f^{-1}(V)$ and denote $y = f(x)$. Consider the shifted set $y^{-1}V \subseteq H$, which is open by Lemma 1 (can write $y^{-1}V = m_{y^{-1}}(V)$). Since $y \in V$,

$$e_h = y^{-1}y \in y^{-1}V.$$

That is, $y^{-1}V$ is an open neighborhood of e_h. Applying the assumed property of f, there is an open neighborhood $U \subseteq G$ of e_g, such that $f(U) \subseteq y^{-1}V$. Again applying Lemma 1, the shifted set xU is open in G. And $e_g \in U$ implies

$$x = xe_g \in xU.$$

That is, xU is an open neighborhood of x. Then

$$f(xU) = f(x)f(U) = yf(U) \subseteq y(y^{-1}V) = V,$$

which implies $xU \subseteq f^{-1}(V)$. Hence $f^{-1}(V)$ is open in G, so f is continuous.

□

4 A continuous map is called *proper* if the preimage of every compact set is compact. Show that there is no surjective proper map $\mathbb{R}^2 \to \mathbb{R}$.

Proof. Suppose, for contradiction, that we have a map $f : \mathbb{R}^2 \to \mathbb{R}$, which is continuous, surjective, and proper. Then

$$K = f^{-1}([-1, 1]) \subseteq \mathbb{R}^2$$

is compact, therefore bounded. Suppose K is contained in a ball of radius $R > 0$ around the origin. Define $B = B_R((0, 0)) \subseteq \mathbb{R}^2$, so $K \subseteq B$.

Since f is continuous and $\overline{B} \subseteq \mathbb{R}^2$ is compact, the image $f(\overline{B}) \subseteq \mathbb{R}$ is compact. So we can choose $M > 0$ such that $f(\overline{B}) \subseteq (-M, M)$. Since f is surjective, there are $a, b \in \mathbb{R}^2$ such that $f(a) = -M$ and $f(b) = M$. We know that $a, b \notin B$, because $\pm M \notin f(B)$.

Notice that $\mathbb{R}^2 \setminus B$ is a path-connected set. From any point, one can draw the line towards the origin, until it hits the circle ∂B. Then, a path between any two points can be constructed by chaining each of their paths to the circle with an arc.

Let $\gamma : [0, 1] \to \mathbb{R}^2 \setminus B$ be a path from a to b outside of B, i.e., $\gamma(0) = a$ and $\gamma(1) = b$. Then $g = f \circ \gamma$ is a continuous function $[0, 1] \to \mathbb{R}$ with $g(0) = -M$ and $g(1) = M$. By the intermediate value theorem, there is some $t \in [0, 1]$ such that $0 = g(t) = f(\gamma(t))$. This means $\gamma(t) \in K \subseteq B$, which contradicts the choice of γ as a path outside B.

$\square$

5 A metric space is *proper* if every closed ball in it is compact.

(a) Show that every proper metric space is complete.

Proof. Let (X, d) be a proper metric space and (x_n) be a Cauchy sequence in X. For each $k \in \mathbb{N}$, choose $N_k \in \mathbb{N}$ such that

$$n, m \geq N_k \implies d(x_n, x_m) < \frac{1}{k}.$$

Define the closed ball

$$E_k = \overline{B_{1/k}(x_{N_k})}.$$

For all $n \geq N_k$, we have $d(x_n, x_{N_k}) < 1/k$, which tells us $x_n \in E_k$.

Define the set $E = \bigcap_{k \in \mathbb{N}} E_k$; we claim that E is a singleton. If E is nonempty, and $x, y \in E$, then $x, y \in E_k$ implies $d(x, y) \leq 2/k$, for all $k \in \mathbb{N}$. Letting $k \to \infty$, we obtain $d(x, y) = 0$, so $x = y$. Hence, E contains at most one point, and it remains to show E is nonempty.

Suppose, for contradiction, that E is empty, then

$$X = E^c = \bigcup_{k \in \mathbb{N}} E_k^c.$$

That is, the complements $\{E_k^c\}$ form an open cover of X. In particular, this is an open cover of the first closed ball E_1, which is compact since X is proper. Therefore, we can find a finite subcover

$$E_1 \subseteq \bigcup_{i=1}^{\ell} E_{k_i}^c.$$

Define $K = \max\{k_1, \ldots, k_\ell\}$, then $x_{N_K} \in E_k$ for all $k \leq K$. However, this means $x_{N_K} \in E_1$, but x_{N_K} is not in any $E_{k_1}^c, \ldots, E_{k_\ell}^c$, which is a contradiction.

It follows that $E = \{x\}$ for some $x \in X$. In fact, x is the limit of the sequence (x_n); we will verify this. Given $\varepsilon > 0$, choose $k \in \mathbb{N}$ such that $2/k < \varepsilon$. Then, for all $n \geq N_k$,

$$d(x_n, x) \leq d(x_n, x_{N_k}) + d(x_{N_k}, x) \leq \frac{1}{k} + \frac{1}{k} < \varepsilon,$$

hence $x_n \to x$. This proves X is complete. $\square$

(b) Show that every open set in a proper metric space is a union of a countable sequence $K_1 \subseteq K_2 \subseteq \cdots$ of compact sets. (Use Homework 2.)

Proof. Let (X, d) be a proper metric space. Let $V \subseteq X$ be an open set. For $n \in \mathbb{N}$, define the closed set

$$E_n = U(V^c, 1/n)^c = \{x \in X : B_{1/n}(x) \subseteq V\}.$$

In words, E_n is points of V which are at least a distance $1/n$ from its boundary. By construction, we have $E_n \subseteq E_{n+1} \subseteq V$. Fix a point $x_0 \in X$. For $n \in \mathbb{N}$, define the closed set

$$K_n = E_n \cap \overline{B_n(x_0)} = \{x \in E : d(x, x_0) \leq n\}.$$

Since X is proper, the closed ball is compact, implying the closed subset K_n is also compact. Like the E_n's, the balls are also nested, so we again have $K_n \subseteq K_{n+1} \subseteq V$.

Note that every point in V has a positive distance to the boundary and a finite distance to x_0, so is eventually in some K_n. Explicitly, for each $x \in V$, we have

$$d(x, V^c) > 0 \quad \text{and} \quad d(x, x_0) < \infty.$$

Therefore, we can choose $N_1, N_2 \in \mathbb{N}$ such that $d(x, V^c) < 1/N_1$ and $d(x, x_0) < N_2$. So if we define $N = \max\{N_1, N_2\}$, then we know $x \in K_N$.

Hence, we can write V as

$$V = \bigcup_{n \in \mathbb{N}} K_n,$$

which is a countable union of nested compact sets.

$\square$

6 Are the following subspaces closed? Prove it or give a counterexample.

(a) The set of *compactly supported* in $\mathcal{C}_B(\mathbb{R})$ with the sup norm.

No.

Let $X = \{f \in \mathcal{C}_B(\mathbb{R}) : f|_{\mathbb{R}\setminus K} = 0 \text{ for some compact set } K \subseteq \mathbb{R}\}$.

Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = e^{-x^2}.$$

We use the following facts from real analysis:

(i) f is continuous and positive on all of $\mathbb{R}$,

(ii) f is increasing on $(-\infty, 0]$ and decreasing on $[0, \infty)$,

(iii) $\lim_{|x|\to\infty} f(x) = 0$.

It follows from (i) and (ii) that $f \in \mathcal{C}_B(\mathbb{R}) \setminus X$. However, we claim that $f \in \overline{X}$.

Let $\varepsilon > 0$ and consider the open ball

$$B_\varepsilon(f) = \{g \in \mathcal{C}_B(\mathbb{R}) : \|f - g\|_\infty < \varepsilon\}.$$

Consider the function $g : \mathbb{R} \to \mathbb{R}$ defined by

$$g(x) = \max\{f(x) - \varepsilon/2,\ 0\}.$$

Since g is continuous and $0 \leq g(x) < f(x)$ for all $x \in \mathbb{R}$, we have $g \in \mathcal{C}_B(\mathbb{R})$. Moreover, since $f(x) - \varepsilon < g(x) < f(x)$ for all $x \in \mathbb{R}$, we have $\|f - g\|_\infty < \varepsilon$, i.e., $g \in B_\varepsilon(f)$.

By (iii), there is some $M \in \mathbb{R}$ such that $f(x) < \varepsilon/2$ whenever $|x| \geq M$. Then $K = [-M, M]$ is a compact set with $g|_{\mathbb{R}\setminus K} = 0$, hence $g \in X$.

We have shown that every open ball around f has a nonempty intersection with X, so in fact $f \in \overline{X}$. But since $f \notin X$, this implies $X \neq \overline{X}$, i.e., X is not closed.

(b) The set of functions in $\mathcal{C}(\mathbb{R})$ with the compact-open topology which are zero on the set $[0,1]$.

Yes.

Proof. Let $X = \{f \in \mathcal{C}(\mathbb{R}) : f|_{[0,1]} = 0\}$; we will show that $\mathcal{C}(\mathbb{R}) \setminus X$ is open.

Let $f \in \mathcal{C}(\mathbb{R}) \setminus X$, so there is some $x \in [0,1]$ such that $f(x) \neq 0$; denote $a = f(x)$.

Consider the compact-open topology subbasis set

$$U = S(\{x\}, B_{|a|}(a)) = \{g \in \mathcal{C}(\mathbb{R}) : g(x) \in B_{|a|}(a)\}.$$

Then U is an open neighborhood of f, since $f(x) = a \in B_{|a|}(a)$.

Any $g \in U$ must have $|a - g(x)| < |a|$; in particular, $g(x) \neq 0$. Therefore, $g \notin X$, and we conclude that $U \subseteq \mathcal{C}(\mathbb{R}) \setminus X$. Hence, $\mathcal{C}(\mathbb{R}) \setminus X$ is open, so X is closed.

□