Math 118C
Spring 2021
Final Exam
Time Limit: 120 Minutes (or per DSP letter)

Name: ______________________
PERM Number: ______________________

- Submit your work on GradeScope within 20 minutes from the end of the exam time indicated above. If you have submitted a DSP letter, this submission window also extends accordingly for you.
- This is a closed-book exam, you should work independently. In particular, you should not discuss these questions with anyone nor seek help from any internet sources. Violations of academic integrity will be reported.
- If you want any clarification during the exam, ask the instructor over Zoom. Zoom ID: 426 530 1130.
- Organize your work, in a reasonably neat and coherent way. Work scattered all over a page without a clear ordering will receive very little credit.
- This exam contains 7 pages (including this cover page) and 6 problems.

1. (20 points) Define $r := \sqrt{x_1^2 + x_2^2 + x_3^2}$. Let $\omega := (\frac{1}{r})^3(x_3 dx_1 \wedge dx_2 - x_2 dx_1 \wedge dx_3 + x_1 dx_2 \wedge dx_3)$ be a 2-form on $\mathbb{R}^3 \setminus (0,0,0)$.

   (a) Show that $d\omega = 0$.

   (b) Let $B := \{(x_1, x_2, x_3) : (x_1 - 2)^2 + x_2^2 + x_3^2 = 3\}$ be a sphere in $\mathbb{R}^3$, find the integral $\int_B \omega$.

2. (15 points) Let $x_1, \ldots, x_n$ be the coordinates of $\mathbb{R}^n$, A be a $n \times n$ matrix of real numbers, and define $(y_1, \ldots, y_n) = (x_1, \ldots, x_n)A$. Using only the definitions of determinant and wedge product, prove that $dy_1 \wedge \ldots \wedge dy_n = det(A) dx_1 \wedge \ldots \wedge dx_n$.

3. (20 points) Let D be the closed unit disk in $\mathbb{R}^2$ and f be a continuous function on D. Show that for any $\epsilon > 0$, there exists a number n and functions $f_1, f_2, \ldots, f_n$ such that $f = f_1 + \ldots + f_n$ on D and the support of f_i has Lebesgue measure less than ϵ, for any $i = 1, \ldots, n$. State any theorem you use.

4. (20 points) Prove that a subset E of $\mathbb{R}^n$ is Lebesgue measurable if and only if for any $\epsilon > 0$, there exists an open set $U \subset \mathbb{R}^n$ such that $E \subset U$ and $m(U \setminus E) < \epsilon$.

5. (20 points) Let $\{f_n\}$ be a sequence of measurable functions and define $f := \liminf_n f_n$. Is f measurable? If yes, justify your answer. If no, give a counterexample.

6. (15 points) Let $\{f_n\}$ be a uniformly convergent and uniformly bounded sequence of Lebesgue integrable functions on $\mathbb{R}^1$ and let $f := \lim_n f_n$ be the limit. Is it true that

$$\lim_n \int_{\mathbb{R}^1} f_n dm = \int_{\mathbb{R}^1} f dm?$$

If yes, justify your answer. If no, give a counterexample. All integrals are Lebesgue integrals.