1 Let X be a nonempty topological space and let μ be a measure on X. Prove that if the functions $f_n : X \to [-\infty, +\infty]$ are μ-measurable for $n = 1, 2, \dots$, then the set

$$A = \{x \in X : \lim_{n\to\infty} f_n(x) \text{ exists}\}$$

is μ-measurable.

Proof. Consider the function $F : X \to [-\infty, +\infty]$ defined by

$$F(x) = \liminf_{n\to\infty} f_n(x) - \limsup_{n\to\infty} f_n(x).$$

(Note that this function is not well-defined when $\liminf f_n(x) = \limsup f_n(x) = \pm\infty$. The set of such points $x \in X$ can be described as the union of the intersections of the preimages of the closed sets $\{+\infty\}, \{-\infty\} \in [-\infty, +\infty]$ under the measurable functions $\liminf f_n(x)$ and $\limsup f_n(x)$. Then we can define F on the remaining points of X and the rest holds.)

Since each f_n is μ-measurable and the operations preserve measurability, F is μ-measurable. Note that the limit of $f_n(x)$ exists if and only if the limit infimum and limit supremum are equal, i.e., $A = F^{-1}(0)$. Since the singleton $\{0\} \in [-\infty, +\infty]$ is a closed—therefore Borel—set, its preimage is μ-measurable. $\square$

2 Prove that any Lebesgue-measurable function $f : \mathbb{R} \to \mathbb{R}$ that satisfies the relation

$$f(x + y) = f(x) + f(y) \quad \text{for all} \quad x, y \in \mathbb{R},$$

must be linear.

Proof. We will first prove that f is $\mathbb{Q}$-linear.

An inductive argument shows $f(a) = af(1)$ for all $a \in \mathbb{Z}_{>0}$. Then

$$f(0) = f(0 + 0) = f(0) + f(0),$$

which implies $f(0) = 0 = 0f(1)$. So

$$0 = f(0) = f(1 - 1) = f(1) + f(-1),$$

which implies $f(-1) = -f(1)$. This proves $f(a) = af(1)$ for all $a \in \mathbb{Z}$. For nonzero $b \in \mathbb{Z}$

$$f(1) = f(\tfrac{b}{b}) = bf(\tfrac{1}{b}),$$

which implies $f(\frac{1}{b}) = \frac{1}{b}f(1)$. It follows that $f(\frac{a}{b}) = \frac{a}{b}f(1)$ for all $\frac{a}{b} \in \mathbb{Q}$, i.e., f is $\mathbb{Q}$-linear.

Choose any $A \subseteq \mathbb{R}$ with $0 < \lambda(A) < \infty$. By Lusin's theorem, there is compact subset $K \subseteq A$ such that $\lambda(A \setminus K) < \lambda(A)$ and $f|_K$ is continuous. Then

$$\lambda(A) = \lambda(A \cap K) + \lambda(A \setminus K) < \lambda(K) + \lambda(A),$$

which implies $\lambda(K) > 0$. By Homework 3 Problem 4, there is an open interval $I \subseteq K - K$ containing 0.

We will prove that f is continuous at 0. Let $\{z_n\}_{n\in\mathbb{N}}$ be a sequence of points in I converging to zero. (This is equivalent to a sequence of points in $\mathbb{R}$ converging to zero, since every such sequence is eventually contained in I.) Choose $x_n, y_n \in K$ such that $z_n = x_n - y_n$ for all $n \in \mathbb{N}$. Since $f|_K$ is continuous and K is compact, it is uniformly continuous. So the convergence

$$|x_n - y_n| = |z_n| \longrightarrow 0$$

implies

$$|f(z_n)| = |f(x_n) - f(y_n)| \longrightarrow 0.$$

Hence, $f(z_n) \to f(0)$ so f is continuous at 0.

For any $r \in \mathbb{R}$, there is a sequence $\{q_n\}_{n\in\mathbb{N}}$ of rationals converging to r. In particular, this means we have convergence $(r - q_n) \to 0$. On one hand, the continuity of f at 0 implies $f(r - q_n) \to 0$. On the other hand, the additivity and $\mathbb{Q}$-linearity of f imply

$$f(r - q_n) = f(r) - f(q_n) = f(r) - q_n f(1).$$

Taking the limit of both sides, we obtain $f(r) = rf(1)$, i.e., f is $\mathbb{R}$-linear. □

3 Let $f : (0,1) \to \mathbb{R}$ be such that for every $x \in (0,1)$ there exists $\delta > 0$ and a Borel-measurable function $g : \mathbb{R} \to \mathbb{R}$ (both dependent on x), such that $f(y) = g(y)$ for all $y \in (x - \delta, x + \delta) \cap (0,1)$. Prove that f is Borel-measurable. (You can assume that $f(x) = 0$ outside the interval $(0,1)$).

Proof. We claim that for any closed interval $[a,b] \subseteq (0,1)$, we can find a Borel-measurable function $g : \mathbb{R} \to \mathbb{R}$ such that $g(x) = f(x)$ for all $x \in [a,b]$. For each $x \in [a,b]$ we can choose a value $\delta_x > 0$ and a Borel-measurable function $g_x : \mathbb{R} \to \mathbb{R}$ be such that $B_{\delta_x}(x) \subseteq (0,1)$ and $g_x(y) = f(y)$ for all $y \in B_{\delta_x}(x)$. The collection $\{B_{\delta_x}(x)\}_{x\in[a,b]}$ forms an open cover of the compact interval $[a,b]$, so there is a finite subcover denoted by $B_{\delta_{x_i}}(x_i)$ for $i = 1, \ldots, m$.

Define the initial set $A_1 = B_{\delta_{x_1}}(x_1)$ and for $k = 2, \ldots, m$, define the sets

$$A_i = B_{\delta_{x_i}}(x_i) \setminus \bigcup_{j=1}^{i-1} A_j.$$

Then the A_k's are mutually disjoint Borel-measurable subsets of $(0,1)$ such that

$$[a,b] \subseteq \bigcup_{i=1}^{m} B_{\delta_{x_i}}(x_i) = \bigcup_{i=1}^{m} A_i.$$

Additionally, $g_{x_i}(x) = f(x)$ for all $x \in A_i$. We now define the function

$$g = \sum_{i=1}^{m} \chi_{A_i} g_{x_i}.$$

As the sum of products of Borel-measurable functions, g is also Borel-measurable. Every point $x \in [a, b]$ is contained in exactly one A_i. If $x \in A_k$, then $A_k \subseteq B_{\delta_{x_k}}(x_k)$, so

$$g(x) = \sum_{i=1}^{m} \chi_{A_i}(x) g_{x_i}(x) = g_{x_k}(x) = f(x).$$

Hence, for every closed interval $[a, b] \subseteq (0, 1)$, there is a Borel-measurable function $g : \mathbb{R} \to \mathbb{R}$ that agrees with f on $[a, b]$ and is zero outside $(0, 1)$.

For each $n \in \mathbb{N}$ (for $n \geq 3$), we consider the closed interval $I_n = [\frac{1}{n}, 1 - \frac{1}{n}] \subseteq (0, 1)$. By the above result, there is a Borel-measurable function $f_n : \mathbb{R} \to \mathbb{R}$ that agrees with f on I_n and is zero outside $(0, 1)$. Then f can be written as limit of Borel-measurable functions

$$f(x) = \lim_{n\to\infty} f_n(x).$$

Hence, f is Borel-measurable. $\square$

4 Give an example of a collection of Lebesgue-measurable nonnegative functions $\{f_\alpha\}_{\alpha \in A}$ ($f_\alpha : \mathbb{R} \to \mathbb{R}$) such that the function

$$g(x) = \sup_{\alpha \in A} f_\alpha(x), \quad x \in \mathbb{R}$$

is finite for all $x \in \mathbb{R}$ but g is not Lebesgue-measurable. Here A is a nonempty indexing set.

Let $V \subseteq \mathbb{R}$ be a Vitali set. For each $v \in V$, the characteristic function $\chi_{\{v\}}$ is Lebesgue-measurable and nonnegative. Then for all $x \in \mathbb{R}$,

$$\sup_{v \in V} \chi_{\{v\}}(x) = \chi_V(x)$$

is clearly finite. However, $\{1\} \subseteq \mathbb{R}$ is a Borel set with preimage

$$\chi_V^{-1}(\{1\}) = V,$$

which is not Lebesgue-measurable.

5 A function $f : \mathbb{R}^n \to \mathbb{R}$ is called lower semi-continuous at the point $x \in \mathbb{R}^n$ if, for any sequence $x_k \in \mathbb{R}^n$ with $x_k \to x$, one has

$$\liminf_{k\to\infty} f(x_k) \geq f(x).$$

Prove that any lower semi-continuous function on $\mathbb{R}^n$ is Borel-measurable.

Proof. Let $a \in \mathbb{R}$ and consider the set $A = f^{-1}((a, +\infty)) \subseteq \mathbb{R}^n$. To show f is Borel-measurable, it suffices to check that A is Borel-measurable. Fix a point $x \in A$ and choose $0 < \varepsilon < f(x) - a$. Then the lower semi-continuity of f tells us that there is some $\delta > 0$ such that $B_\delta(x) \subseteq A$, hence A is open—therefore Borel-measurable. $\square$