(a) Let $\mathcal{A}$ be an abelian category satisfying AB3. An object P of $\mathcal{A}$ is said to be **strictly projective** if the functor $h' : X \mapsto \mathrm{Hom}_{\mathcal{A}}(P, X)$ from $\mathcal{A}$ to Ab is exact (projectivity of P), strict (i.e., $h'(X) = 0$ implies $X = 0$), and commutes with direct sums. For such an object P we set $R = \mathrm{Hom}_{\mathcal{A}}(P, P)$. Prove that h' determines an equivalence of $\mathcal{A}$ and Mod-R.

Proof. We prove that h' is an equivalence by showing that it is full, faithful, and essentially surjective on objects.

First, we show that h' is faithful. Let $f : X \to Y$ be a morphism in $\mathcal{A}$ such that $h'(f) = 0$. Decompose f into its image, with morphisms

$$\begin{tikzcd} X \arrow[r, two heads, "i"] \arrow[rr, bend right, "f"'] & I \arrow[r, hook, "j"] & Y \end{tikzcd}$$

Since h' is exact, it sends this to a similar diagram in Mod-R:

$$\begin{tikzcd} h'(X) \arrow[r, two heads, "h'(i)"] \arrow[rr, bend right, "h'(f)=0"'] & h'(I) \arrow[r, hook, "h'(j)"] & h'(Y) \end{tikzcd}$$

Since $h'(j)$ is a monomorphism, $h'(j) \circ h'(i) = 0$ implies that $h'(i) = 0$. But since $h'(i)$ is an epimorphism, we must have $h'(I) = 0$. Then strictly projective gives us $I = 0$. Now since f factors through the zero object, it must be zero.

We now show full, i.e., that h' gives a surjection $\mathrm{Hom}_{\mathcal{A}}(X, Y) \to \mathrm{Hom}_R(h'(X), h'(Y))$. First, consider the case when $X = P^{(I)}$. Consider a morphism

$$\oplus_{\alpha \in I} R = h'(\oplus_{\alpha \in I} P) \xrightarrow{f} h'(Y).$$

For each $\alpha \in I$, let $\iota_\alpha : R \hookrightarrow \oplus_{\alpha \in I} R$ be the canonical inclusion and define $f_\alpha = f \circ \iota_\alpha : R \to h'(Y)$. Then take $\varphi_\alpha = f_\alpha(\mathrm{id}_P) \in h'(Y) = \mathrm{Hom}_\alpha(P, Y)$. By the universal property of the direct sum, there is a morphism $\varphi : \bigoplus_{\alpha \in I} P \to Y$ such that for all $\alpha \in I$ the following diagram commutes in $\mathcal{A}$:

$$\begin{tikzcd} P \arrow[r, hook, "\iota_\alpha"] \arrow[rr, bend right, "\varphi_\alpha"'] & \bigoplus_{\alpha \in I} P \arrow[r, dashed, "\varphi"] & Y \end{tikzcd}$$

Then h' sends this diagram to the following diagram Mod-R:

$$\begin{tikzcd} P \arrow[r, hook, "\iota_\alpha"] \arrow[rr, bend right, "f_\alpha"'] & \bigoplus_{\alpha \in I} P \arrow[r, "h'(\varphi)"] & Y \end{tikzcd}$$

But by uniqueness in the universal property of the direct sum, we must have $h'(\varphi) = f$.

We now consider the general case: X is arbitrary. We want to find something like a free resolution for X in the category $\mathcal{A}$. Consider the direct sum $\bigoplus_{\alpha\in h'(X)} P$, where the indexing set is $h'(X) = \mathrm{Hom}_{\mathcal{A}}(P, X)$. Then each $\alpha \in h'(X)$ is a morphism $\alpha : P \to X$. So by the universal property of the direct sum, there is a unique morphism $\varphi : \bigoplus_{\alpha\in h'(X)} P \to X$ such that for all $\alpha \in h'(X)$ the following diagram commutes in $\mathcal{A}$:

$$\begin{tikzcd}
P \arrow[r, hook, "\iota_\alpha"] \arrow[rr, bend right, "\alpha"'] & \bigoplus_{\alpha\in h'(X)} P \arrow[r, dashed, "\varphi"] & X
\end{tikzcd}$$

We claim that φ is an epimorphism. Let $C = \operatorname{coker}\varphi$, then h' sends the above diagram to the following diagram in $\mathsf{Mod}\text{-}R$:

$$\begin{tikzcd}
R \arrow[r, hook, "\iota_\alpha"] \arrow[rr, bend right, "h'(\alpha)"'] & \bigoplus_{\alpha\in h'(X)} R \arrow[r, "h'(\varphi)"] & h'(X) \arrow[r] & h'(C)
\end{tikzcd}$$

Since h' is exact, it preserves kernels and cokernels, so $h'(C)$ is the relevant cokernel. But for any $\alpha \in h'(X)$, we have $\mathrm{id}_P \in R$ and $h'(\alpha)(\mathrm{id}_P) = \alpha$. By commutativity, this means α is in the image of $h'(\varphi)$, so $h'(\varphi)$ is surjective, so the cokernel is zero. Since h' is strict, this means $C = 0$, so φ is an epimorphism.

We can now construct the following exact sequence in $\mathcal{A}$ (something like a free presentation):

$$\bigoplus\nolimits_J P \xrightarrow{\varphi_1} \bigoplus\nolimits_I P \xrightarrow{\varphi_0} X \longrightarrow 0$$

Here, φ_0 is the epimorphism constructed above, and φ_1 uses the same construction applied to the kernel of φ_0. We now apply the contravariant functors $\mathrm{Hom}_{\mathcal{A}}(-, Y)$ and $\mathrm{Hom}_R(h'(-), h'(Y))$ to get exact sequences of abelian groups:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_{\mathcal{A}}(X,Y) \arrow[r] \arrow[d, "h'"] & \mathrm{Hom}_{\mathcal{A}}(\bigoplus_I P, Y) \arrow[r] \arrow[d, "h'"] & \mathrm{Hom}_{\mathcal{A}}(\bigoplus_J P, Y) \arrow[d, "h'"] \\
0 \arrow[r] & \mathrm{Hom}_R(h'(X), h'(Y)) \arrow[r] & \mathrm{Hom}_R(\bigoplus_I R, h'(Y)) \arrow[r] & \mathrm{Hom}_R(\bigoplus_J R, h'(Y))
\end{tikzcd}$$

The vertical arrows are h' mapping morphisms. The last two vertical morphisms are surjective from the special case of fullness of h' and injective from the faithfulness of h'. Adding an extra zero to the left of each row allows us to apply the 5-Lemma, which tells us that the first vertical arrow is an isomorphism. In particular, this arrow is surjective, so h' is full for arbitrary X.

Lastly, we check essentially surjective on objects. Let M be a right R-module and consider a free presentation

$$\bigoplus_J R \xrightarrow{f_1} \bigoplus_I R \xrightarrow{f_0} M \longrightarrow 0$$

Here, $M = \operatorname{coker} \varphi_1$. Clearly, the first two terms are in the image of h', since we can just take the same size direct sum of copies of P. Then the fullness and faithfulness of h' tells us there is a unique $\varphi \in \operatorname{Hom}_{\mathcal{A}}(\bigoplus_J P, \bigoplus_I P)$ such that $h'(\varphi) = f_1$. Taking the cokernel of φ, we get an exact sequence in $\mathcal{A}$:

$$\bigoplus_J P \xrightarrow{\varphi} \bigoplus_I P \twoheadrightarrow \operatorname{coker} \varphi \longrightarrow 0$$

Since h' is exact, it sends this diagram to an exact sequence in $\mathsf{Mod}\text{-}R$:

$$\bigoplus_J R \xrightarrow{f_1} \bigoplus_I R \twoheadrightarrow h'(\operatorname{coker} \varphi) \longrightarrow 0$$

In particular, $h'(\operatorname{coker} \varphi)$ is the cokernel of f_1. Since the cokernel is unique up to isomorphism, we conclude that $M \cong h'(\operatorname{coker} \varphi)$. $\square$

(b) Let $\mathcal{A}$ be a Noetherian category (i.e., any increasing chain of subobjects stabilizes), and let P be a projective object in $\mathcal{A}$ such that h' is a strict functor. Then the ring $R = \operatorname{Hom}_{\mathcal{A}}(P, P)$ is right Noetherian and h' determines an equivalence between $\mathcal{A}$ and the category of finitely generated R-modules.