Final

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1

Find the power series expansion at infinity of the function

$$f(z) = \frac{1+z^2}{1+z^4}.$$

We consider the function

$$g(z) = f\left(\tfrac{1}{z}\right) = \frac{1+\left(\frac{1}{z}\right)^2}{1+\left(\frac{1}{z}\right)^4} = \frac{z^4+z^2}{z^4+1}.$$

If $|z| < 1$, then $1+z^4 \neq 0$, so g is analytic on the open unit disc $\{|z| < 1\}$. The power series expansion of f at infinity will be derived from the power series expansion of g at zero. We rewrite g as

$$g(z) = z^2\left(\frac{1}{1+z^4} + z^2\frac{1}{1+z^4}\right).$$

Assuming $|z| < 1$, then we have the sum of a geometric series

$$\frac{1}{1+z^4} = \sum_{k=0}^{\infty}(-z^4)^k = \sum_{k=0}^{\infty}(-1)^k z^{4k}.$$

Then the value of $g(z)$ for $|z| < 1$ is given by

$$g(z) = z^2\left(\sum_{k=0}^{\infty}(-1)^k z^{4k} + \sum_{k=0}^{\infty}(-1)^k z^{4k+2}\right)$$

We now consider the two power series

$$\sum_{k=0}^{\infty}(-1)^k z^{4k} = z^0 - z^4 + z^8 - z^{12} + z^{16} - z^{20} + \cdots,$$

$$\sum_{k=0}^{\infty}(-1)^k z^{4k+2} = z^2 - z^6 + z^{10} - z^{14} + z^{18} - z^{22} + \cdots.$$

Both are alternating series. The first contains the even powers of z which are multiples of 4. The second contains the even powers of z which are not multiples of 4. Combining these, we obtain a series of all nonnegative even powers of z and alternates sign in consecutive pairs of terms. That is, their sum is

$$\sum_{n=0}^{\infty} a_n z^{2n} = z^0 + z^2 - z^4 - z^6 + z^8 + z^{10} - z^{12} - z^{14} + \cdots ,$$

where a_n can be defined by

$$a_n = \begin{cases} 1 & \text{if } n \bmod 4 \in \{0, 1\}, \\ -1 & \text{if } n \bmod 4 \in \{2, 3\}. \end{cases}$$

Substituting this back into the formula of g, we obtain the power series expansion of g at zero

$$g(z) = z^2 \sum_{n=0}^{\infty} a_n z^{2n} = \sum_{n=0}^{\infty} a_n z^{2n+2}.$$

Note that a_n are not precisely the coefficients given by the nth derivative of g at zero, as the exponent of z is $2n+2$. But one could redefine the series as $\sum b_k z^k$ where $b_{2n+2} = a_n$, and zero otherwise. Thus, the power series expansion of f at infinity is

$$f(z) = g\left(\tfrac{1}{z}\right) = \sum_{n=0}^{\infty} \frac{a_n}{z^{2n+2}},$$

where a_n is defined above.

2

> Let $f(z) : \mathbb{C} \to \mathbb{C}$ be an entire function such that for any $z \in \mathbb{C}$ there exists a natural number n such that $f^{(n)}(z) = 0$ (the value of n may vary for different values of z). Prove that $f(z)$ is a polynomial in z.

Lemma 1. *If a subset of $\mathbb{C}$ is uncountable, then it must have a non-isolated point.*

Proof. Suppose, for contradiction, that we have an uncountable subset $X \subseteq \mathbb{C}$ whose points are all isolated. That is, for each $z \in X$, there exists some $\delta > 0$ such that $|w - z| \geq \delta$ for all $w \in X$ with $w \neq z$. In particular, we define the distance

$$\delta(z) = \inf_{w \in X \setminus \{z\}} |w - z| > 0,$$

We will consider $\mathbb{Q}^2$ to be the set of rational complex numbers, i.e.,

$$\mathbb{Q}^2 = \{p + iq : p, q \in \mathbb{Q}\}.$$

Note that as $\mathbb{Q}$ is dense in $\mathbb{R}$, so too is $\mathbb{Q}^2$ dense in $\mathbb{C}$. So for each $z \in X$, there exist points in $\mathbb{Q}^2$ which are arbitrarily close to z. In particular, we choose some $q(z) \in \mathbb{Q}^2$ such that

$$|z - q(z)| < \frac{\delta(z)}{2}.$$

This defines a map $q : X \to \mathbb{Q}^2$. We now show that this map is injective. Suppose $z, w \in X$ such that $q(z) = q(w)$, then

$$|w - z| = |w - q(w) + q(z) - z| \leq |w - q(w)| + |z - q(z)| < \frac{\delta(w)}{2} + \frac{\delta(z)}{2}.$$

If $z \neq w$, then the definition of δ would imply

$$|w - z| < \frac{|z - w|}{2} + \frac{|w - z|}{2} = |w - z|,$$

so we must have $z = w$. Thus, $q : X \to \mathbb{Q}^2$ is an injection, which tells us that $|X| \leq |\mathbb{Q}^2|$. But, because X is uncountable and $\mathbb{Q}^2$ is countable, this is a contradiction.

$\square$

Proposition 1. *If $f(z) : \mathbb{C} \to \mathbb{C}$ is an entire function such that for any $z \in \mathbb{C}$ there exists a natural number n such that $f^{(n)}(z) = 0$, then $f(z)$ is a polynomial in z.*

Proof. For each $n \in \mathbb{N}$, we define a subset of the complex numbers

$$X_n = \{z \in \mathbb{C} : f^{(n)}(z) = 0\}.$$

For each $z \in \mathbb{C}$, since $f^{(n)}z = 0$ for some $n \in \mathbb{N}$, then we must have z in at least one of these subsets, namely X_n. In other words, the collection $\{X_n\}_{n\in\mathbb{N}}$ covers $\mathbb{C}$, so

$$\mathbb{C} = \bigcup_{n\in\mathbb{N}} X_n.$$

Since the countable union of X_n's is an uncountable set, i.e. $\mathbb{C}$, it follows that at least one subset in the collection, say X_N, is uncountable. By Lemma 1, we have that X_N contains at least one non-isolated point. And since $f^{(N)}$ is an entire function which is zero at all points in X_N, then $f^{(N)}$ must be identically zero on its domain $\mathbb{C}$. Moreover, since the derivative of the zero function is zero, then for each $k \geq N$, we have $f^{(k)}$ to be identically zero on $\mathbb{C}$.

We now consider the power series expansion of f around zero. Since f is entire, we have its power series expansion on all of $\mathbb{C}$ given by

$$f(z) = \sum_{k=0}^{\infty} a_k z^k, \quad a_k = \frac{f^{(k)}(0)}{k!}.$$

For all $k \geq N$, we have

$$a_k = \frac{f^{(k)}(0)}{k!} = \frac{0}{k!} = 0.$$

Therefore, all terms for $k \geq N$ are zero, so we have f to be the polynomial

$$f(z) = \sum_{k=0}^{N-1} a_k z^k = a_0 + a_1 z^1 + \cdots + a_{N-1} z^{N-1}.$$

□

3

Prove the following version of L'Hopital's rule. If $f(z)$ and $g(z)$ are analytic in $\mathbb{C}$, $z_0 \in \mathbb{C}$, and $f(z_0) = g(z_0) = 0$, and $g(z)$ is not identically zero, then

$$\lim_{z \to z_0} \frac{f(z)}{g(z)} = \lim_{z \to z_0} \frac{f'(z)}{g'(z)}.$$

Proof. Since both f and g are entire functions, then they have power series expansions on $\mathbb{C}$ given by

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k \quad \text{and} \quad g(z) = \sum_{k=0}^{\infty} b_k (z - z_0)^k,$$

where

$$a_k = \frac{f^{(k)}(z_0)}{k!} \quad \text{and} \quad b_k = \frac{g^{(k)}(z_0)}{k!}.$$

Since $f(z_0) = g(z_0) = 0$, then we know that

$$a_0 = f(z_0) = 0 \quad \text{and} \quad b_0 = g(z_0) = 0.$$

If f is identically zero, then the equality of the limits holds, trivially. Assuming both f and g are not identically zero, there there exist some $n, m \in \mathbb{N}$ such that

$$f^{(n)}(z_0) \neq 0 \quad \text{and} \quad g^{(m)}(z_0) \neq 0.$$

If such natural numbers did not exists, then the above power series expansions would be identically zero. In particular, we choose n and m to be the smallest such natural numbers. In other words, z_0 is a zero of order n for f and a zero of order m for g. Then we can rewrite the power series expansions as

$$f(z) = (z - z_0)^n \sum_{k=0}^{\infty} a_{n+k} (z - z_0)^k$$

and

$$g(z) = (z - z_0)^m \sum_{k=0}^{\infty} b_{m+k} (z - z_0)^k.$$

Then the limit of their quotient becomes

$$\frac{f(z)}{g(z)} = (z - z_0)^{n-m} \cdot \frac{a_n + \sum_{k=1}^{\infty} a_{n+k} (z - z_0)^k}{b_m + \sum_{k=1}^{\infty} b_{m+k} (z - z_0)^k}.$$

Notice that all the terms in the summations go to zero as $z \to z_0$, so we have the limit

$$\lim_{z \to z_0} \frac{a_n + \sum_{k=1}^{\infty} a_{n+k} (z - z_0)^k}{b_m + \sum_{k=1}^{\infty} b_{m+k} (z - z_0)^k} = \frac{a_n}{b_n}.$$

Therefore, we have

$$\lim_{z\to z_0}\frac{f(z)}{g(z)}=\frac{a_n}{b_m}\cdot\lim_{z\to z_0}(z-z_0)^{n-m}.$$

Depending on the relation between n and m, this limit can take on three different values, namely

$$\lim_{z\to z_0}\frac{f(z)}{g(z)}=\begin{cases}\frac{a_n}{b_m} & \text{if } n=m,\\ 0 & \text{if } n>m,\\ \infty & \text{if } n<m.\end{cases} \tag{1}$$

We now return to the power series expansions of f and g, and take their derivatives:

$$f'(z)=\sum_{k=1}^{\infty}ka_k(z-z_0)^{k-1}$$

and

$$g'(z)=\sum_{k=1}^{\infty}kb_k(z-z_0)^{k-1}.$$

Recall. that z_0 is a zero of order n for f and a zero of order m for g. We factor these as

$$f'(z)=(z-z_0)^{n-1}\sum_{k=0}^{\infty}(n+k)a_{n+k}(z-z_0)^k$$

and

$$g'(z)=(z-z_0)^{m-1}\sum_{k=0}^{\infty}(m+k)b_{m+k}(z-z_0)^k.$$

Like before, we find their quotient to be

$$\frac{f'(z)}{g'(z)}=(z-z_0)^{n-m}\cdot\frac{na_n+\displaystyle\sum_{k=1}^{\infty}(n+k)a_{n+k}(z-z_0)^k}{mb_m+\displaystyle\sum_{k=1}^{\infty}(m+k)b_{m+k}(z-z_0)^k}.$$

Also like before, the terms in the summation go to zero, and we have

$$\lim_{z\to z_0}\frac{na_n+\displaystyle\sum_{k=1}^{\infty}(n+k)a_{n+k}(z-z_0)^k}{mb_m+\displaystyle\sum_{k=1}^{\infty}(m+k)b_{m+k}(z-z_0)^k}=\frac{na_n}{mb_m}.$$

Thus, we obtain the limit

$$\lim_{z\to z_0}\frac{f'(z)}{g'(z)}=\frac{na_n}{mb_m}\cdot\lim_{z\to z_0}(z-z_0)^{n-m}.$$

Similar to (1), this limit is zero when $n > m$ and infinity when $n < m$. And when $n = m$, this limit simplifies to

$$\frac{na_n}{mb_m} = \frac{a_n}{b_m}.$$

Thus, we obtain the equality

$$\lim_{z \to z_0} \frac{f(z)}{g(z)} = \lim_{z \to z_0} \frac{f'(z)}{g'(z)}.$$

□

4

Let $D \subset \mathbb{R}^2$ be a domain and let the function $u : D \to \mathbb{R}$ be a harmonic function that is infinitely many times differentiable in D. Let $(x_0, y_0) \in D$. Prove that if

$$\frac{\partial^n u}{\partial x^n}(x_0, y_0) = \frac{\partial^n u}{\partial y^n}(x_0, y_0) = 0, \quad \text{for all } n = 1, 2, \dots,$$

then u is constant in D.

Proof. Since u is harmonic in D, then there exists a harmonic conjugate $v : D \to \mathbb{R}$ such that $f = u + iv : D \to \mathbb{C}$ is analytic in D. Then f has a power series expansion around $z_0 = x_0 + iy_0$, given by

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k,$$

converging for $|z - z_0| < R$, where $R > 0$. For all $k \in \mathbb{N}$, we have

$$a_k = \frac{f^{(k)}(z_0)}{k!}.$$

Since f is analytic, the kth derivative of f is given by

$$f^{(k)} = \frac{\partial^k u}{\partial x^k} + i \frac{\partial^k v}{\partial x^k},$$

and the Cauchy-Riemann equations give us

$$\frac{\partial^k v}{\partial x^k} = \frac{\partial^k v}{\partial x^{k-1} \partial x} = -\frac{\partial^k u}{\partial x^{k-1} \partial y}.$$

Now since all partial derivatives of u are zero at (x_0, y_0), then

$$f^{(k)}(z_0) = \frac{\partial^k u}{\partial x^k}(x_0, y_0) - i \frac{\partial^k u}{\partial x^{k-1} \partial y}(x_0, y_0) = 0.$$

Thus, for all $z \in D$ with $|z - z_0| < R$, we have

$$f(z) = a_0 = f(z_0).$$

Since f is constant on the disc $\{|z - z_0| < R\} \subseteq D$, which contains non-isolated points, then f must be constant on D. Since f is constant on D, then, in particular, $u = \operatorname{Re} f$ is constant on D.

$\square$