(worked with Joseph Sullivan)

1 Vakil Exercise 1.2.C Let $(-)^{\vee\vee} : \mathsf{FinVec}_k \to \mathsf{FinVec}_k$ be the double dual functor from the category of finite-dimensional vector spaces over k to itself. Show that $(-)^{\vee\vee}$ is naturally isomorphic to the identity functor on FinVec_k. (Without the finite-dimensional hypothesis, we only get a natural transformation of functions from id to $(-)^{\vee\vee}$.)

Proof. We will construct a natural isomorphism:

$$\begin{tikzcd} \mathsf{FinVec}_k \arrow[r, bend left, "\mathrm{id}"] \arrow[r, bend right, "(-)^{\vee\vee}"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathsf{FinVec}_k \end{tikzcd}$$

For $V \in \mathsf{FinVec}_k$ define $\alpha_V : V \to V^{\vee\vee}$ by $\alpha_V(v) = \mathrm{eval}_v : V^\vee \to k$.

We check the linearity of α_V. We evaluate at elements $c \in k$; $u, v \in V$; and $\varphi \in V^\vee$:

$$\begin{aligned} \alpha_v(cu+v)(\varphi) &= \varphi(cu+v) \\ &= c\varphi(u) + \varphi(v) \\ &= c\alpha_v(u)(\varphi) + \alpha_v(v)(\varphi) \\ &= (c\alpha_v(u) + \alpha_v(v))(\varphi). \end{aligned}$$

hence $\alpha_V(cu+v) = a\alpha_V(u) + \alpha_V(v)$.

We check the injectivity of α_V. Suppose $v \in \ker \alpha_V$, i.e., $\alpha_V(v) : V^\vee \to k$ is the zero map. Assume—for contradiction—that $v \neq 0$. Then v can be extended to a basis $\{v, v_2, \ldots, v_n\}$ for V (where $n = \dim_k V$). Then there is a linear functional $\varphi : K \to k$ which maps $c_1 v + c_2 v_2 + \cdots + c_n v_n \mapsto c_1$ for all coefficients $c_i \in k$. Evaluating $\alpha_V(v)$ at φ, we find

$$\alpha_V(v)(\varphi) = \mathrm{eval}_v(\varphi) = \varphi(v) = 1 \neq 0.$$

This contradicts the assumption that $\alpha_V(v)$ is the zero map. Therefore, $v = 0$ and we conclude that α_V is injective.

Since α_V is an injective linear transformation and V is finite-dimensional, we deduce

$$\dim_k \operatorname{im} \alpha_V = \dim_k V = \dim_k V^\vee = \dim_k V^{\vee\vee}.$$

It follows that $\operatorname{im} \alpha_V = V^{\vee\vee}$, i.e., α_V is surjective—therefore an isomorphism.

We check the naturality of α, i.e., that the following diagram commutes for all $T \in \mathrm{Mor}(V, W)$:

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[d, "\alpha_V"'] & W \arrow[d, "\alpha_W"] \\ V^{\vee\vee} \arrow[r, "T^{\vee\vee}"'] & W^{\vee\vee} \end{tikzcd}$$

Given $v \in V$ consider the following elements of $W^{\vee\vee}$:

$$(T^{\vee\vee} \circ \alpha_V)(v) = \mathrm{eval}_v \circ T^{\vee}, \qquad (\alpha_W \circ T)(v) = \mathrm{eval}_{T(v)} .$$

For $\psi \in W^{\vee\vee}$ we evaluate

$$\mathrm{eval}_v(T^{\vee}(\psi)) = \mathrm{eval}_v(\psi \circ T) = \psi(T(v)) = \mathrm{eval}_{T(v)}(\psi),$$

hence α is a natural transformation. Since each α_V is an isomorphism, α is in fact a natural isomorphism. $\square$

2 Vakil Exercise 1.2.D Let $\mathcal{V}$ be the category whose objects are the k-vectors spaces k^n for each $n \geq 0$, and whose morphisms are linear transformations. The objects of $\mathcal{V}$ can be thought of as vector spaces with bases, and the morphisms as matrices. There is an obvious functor $\mathcal{V} \to \mathsf{FinVec}_k$, as each k^n *is* a finite-dimensional vector space.

Show that $\mathcal{V} \to \mathsf{FinVec}_k$ gives an equivalence of categories, by describing an "inverse" functor.

Proof. Let $J : \mathcal{V} \to \mathsf{FinVec}_k$ denote the described "inclusion" functor.

We will construct a functor $U : \mathsf{FinVec}_k \to \mathcal{V}$ which sends each n-dimensional vector space to the space $k^n \in \mathcal{V}$ by choosing some basis. (The simultaneous choice of basis for every object in the category is where we require a generalized set theory to be completely formal.) A linear map $T : V \to W$ in FinVec_k is sent to the linear map $U(T) : k^n \to k^m$ in $\mathcal{V}$, where the matrix of $U(T)$ in the standard bases of k^n and k^m is the same as the matrix of T in the bases of V and W chosen by U. The functorality of U follows immediately from its construction.

Assuming that the standard basis is chosen for each $k^n \in \mathsf{FinVec}_k$, U is in fact a left inverse of J in the sense that $U \circ J = \mathrm{id}_{\mathcal{V}}$. This means the identity on $\mathrm{id}_{\mathcal{V}}$ (i.e., the natural transformation $\mathrm{id}_{\mathrm{id}_{\mathcal{V}}} : \mathrm{id}_{\mathcal{V}} \Rightarrow \mathrm{id}_{\mathcal{V}}$) gives us a trivial natural isomorphism $U \circ J \cong \mathrm{id}_{\mathcal{V}}$.

We will construct a natural transformation

$$\begin{tikzcd} \mathsf{FinVec}_k \arrow[r, bend left, "J \circ U"] \arrow[r, bend right, "\mathrm{id}"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathsf{FinVec}_k \end{tikzcd}$$

For each n-dimensional $V \in \mathsf{FinVec}_k$ note that $J(U(V)) = k^n$ as an object of FinVec_k. We take the linear map $\alpha_V : k^n \to V$ with $e_i \mapsto v_i$, where $\{v_1, \ldots, v_n\}$ is the fixed basis of V. Clearly each α_V is an isomorphism, so α is the desired natural isomorphism.

Hence, U and J describe an equivalence of categories. $\square$

3 Vakil Exercise 1.3.B What are the initial and final objects (if they exist)?

Set

The empty set $\varnothing \in \mathsf{Set}$ is initial. For $S \in \mathsf{Set}$, a morphism $\varnothing \to S$ contains no information about mapping elements, so only the empty function is possible.

Any singleton $\{*\} \in \mathsf{Set}$ is terminal. For $S \in \mathsf{Set}$, a morphism $S \to \{*\}$ must map every element of S to the element $*$. Since functions are characterized by their behavior on elements, only the constant function ($x \mapsto *$ for all $x \in S$) is possible.

The category of sets has no zero object, since there are no functions to the empty set from any nonempty set, and there are multiple functions from a singleton to any set with at least two elements.

Ring

The ring of integers $\mathbb{Z} \in \mathsf{Ring}$ is initial (taking rings to have 1). For $R \in \mathsf{Ring}$, a morphism $\mathbb{Z} \to R$ must be a ring homomorphism sending $0_{\mathbb{Z}} \to 0_R$ and $1_{\mathbb{Z}} \to 1_R$, which completely characterizes the map for all elements of $\mathbb{Z}$.

The zero ring $0 \in \mathsf{Ring}$ is terminal (allowing $0 = 1$). A ring homomorphism is characterized by its corresponding function on the underlying sets of its domain and codomain. It follows that the there is at most one ring homomorphism $R \to 0$ for $R \in \mathsf{Ring}$, characterized by the zero function on the underlying sets. Since this is in fact a ring homomorphism, it is the unique morphism $R \to 0$.

Since ring homomorphisms must preserve 1, the zero ring—despite the name—is not a zero object of Ring. It is semistandard to give the name *rngs* to those things which are similar to rings but may lack 1. Then a *rng homomorphism* is similar to a ring homomorphism but need not preserve 1 when it happens to be present. In the category Rng of rngs, the zero rng is in fact a zero object.

Top

Since morphisms in Top are characterized by functions between the underlying sets, then the underlying set of any initial/terminal object in Top must itself be such an object in Set.

The empty space $\varnothing \in \mathsf{Top}$ is initial. For $X \in \mathsf{Top}$, the only possible morphism $\varnothing \to X$ is characterized by the empty function. Since the empty function is continuous, the morphism does exist and is therefore unique.

Any one-point space $\{*\} \in \mathsf{Top}$ is terminal. For $X \in \mathsf{Top}$, the only possible morphism $X \to \{*\}$ is characterized by the constant function. Since the constant function is continuous, the morphism does exist and is therefore unique.

For the same reason as Set, Top has no zero object.

If X is a set, then the subsets of X form a partially ordered set, where the order is given by inclusions. Informally, if $U \subseteq V$, then we have exactly one morphism $U \to V$ in the category (and otherwise none).

Let 2^X denote the poset of subsets of X considered as a category. This is a subcategory of Set with only the inclusion functions.

The empty set $\varnothing \in 2^X$ is initial. For $U \in 2^X$, we have $\varnothing \subseteq U$ so there is a morphism $\varnothing \to U$, which is unique by construction.

The whole set $X \in 2^X$ is terminal. For $U \in 2^X$, we have $U \subseteq X$ so there is a morphism $U \to X$, which is unique by construction.

For the same reason as Set, 2^X has no zero object.

If X is a topological space, then the open sets form a partially ordered set, where the order is given by inclusion.

Let $O(X)$ denote the poset of open subsets of X considered as a category. This is a subcategory of 2^X.

The empty set $\varnothing \in O(X)$ is initial for the same reason it is initial in 2^X.

The whole space $X \in O(X)$ is terminal for the same reason it is terminal in 2^X.

For the same reason as Set and 2^X, $O(X)$ has no zero object.

4 Vakil Exercise 1.3.G Show that $\mathbb{Z}/10\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/12\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z}$.

Proof. Denote $M = \mathbb{Z}/10\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/12\mathbb{Z}$.

Since $\gcd(10, 12) = 2$, there exist $a, b \in \mathbb{Z}$ such that $10a + 12b = 2$. Then in M we have

$$\begin{aligned}
\overline{2} \otimes \overline{1} &= \overline{1} \otimes \overline{2} \\
&= \overline{1} \otimes \overline{10a + 12b} \\
&= \overline{1} \otimes \overline{10a} \\
&= \overline{10a} \otimes \overline{1} \\
&= \overline{0} \otimes \overline{1} \\
&= 0_M.
\end{aligned}$$

So all monomial generators of A can be written as $\overline{a} \otimes \overline{b}$ for some $a, b \in \{0, 1\}$. Note that

$$\overline{1} \otimes \overline{0} = \overline{0} \otimes \overline{1} = \overline{1} \otimes \overline{0} = 0_M$$

and

$$\overline{1} \otimes \overline{1} = 1_M.$$

So M consists only of the elements $0_M, 1_M$.

The multiplication map $\mathbb{Z}/10\mathbb{Z} \times \mathbb{Z}/12\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$ defined by $(\overline{a}, \overline{b}) \mapsto (\overline{ab})$ is surjective and $\mathbb{Z}$-bilinear (well-defined since 10 and 12 are both even). By the universal property of tensor products, the multiplication map factors through a surjective $\mathbb{Z}$-module homomorphism $M \to \mathbb{Z}/2\mathbb{Z}$ where $\overline{a} \otimes \overline{b} \mapsto \overline{ab}$. Since M has only two elements, this map is also injective—therefore an isomorphism. $\square$

5 Vakil Exercise 1.3.H Show that $- \otimes_R N$ gives a covariant functor $\mathsf{Mod}_R \to \mathsf{Mod}_R$. Show that $- \otimes_R N$ is a **right-exact functor**, i.e., if

$$A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

is an exact sequence of R-modules, then the sequence

$$A \otimes_R N \xrightarrow{f \otimes \mathrm{id}} B \otimes_R N \xrightarrow{g \otimes \mathrm{id}} C \otimes_R N \longrightarrow 0$$

is also exact.

Lemma 1. The sequence of R-modules

$$A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

is exact if and only if the sequence

$$0 \longrightarrow \mathrm{Hom}(C, N) \xrightarrow{g^*} \mathrm{Hom}(B, N) \xrightarrow{f^*} \mathrm{Hom}(A, N)$$

is exact for every R-module N.

Proof. Suppose the first sequence is exact and let N be an R-module.

Let $\varphi \in \ker g^*$, which means $\varphi \circ g = 0$. In other words $\varphi|_{\mathrm{im}\, g} = 0$. Since $\mathrm{im}\, g = C$, we in fact have $\varphi = 0$, hence $\ker g^* = 0$.

If $\varphi \in \mathrm{Hom}(C, N)$ then $f^* g^* \varphi = \varphi \circ (g \circ f) = \varphi \circ 0 = 0$, so $\mathrm{im}\, g^* \subseteq \ker f^*$.

Let $\varphi \in \ker f^*$. There is an isomorphism γ such that the following diagram commutes:

$$\begin{tikzcd} B \arrow[r, "g"] \arrow[d, "\pi"'] & \mathrm{im}\, g = C \\ B/\ker g \arrow[ur, dashed, "\gamma"'] & \end{tikzcd}$$

The fact that $\varphi \circ f = 0$ means $\mathrm{im}\, f \subseteq \ker \varphi$. Therefore, there is a homomorphism η such that the following diagram commutes:

$$\begin{tikzcd} B \arrow[r, "\varphi"] \arrow[d, "\pi'"'] & N \\ B/\mathrm{im}\, f \arrow[ur, dashed, "\eta"'] & \end{tikzcd}$$

Since $\mathrm{im}\, f = \ker g$, the projections π and π' in each diagram are the same. Let $K = \ker \pi$, then putting the two diagrams together gives us the following commutative diagram:

$$\begin{tikzcd}
 & B \arrow[dl, "g"'] \arrow[d, "\pi"] \arrow[dr, "\varphi"] & \\
C \arrow[r, "\gamma^{-1}"'] & B/K \arrow[r, "\eta"'] & N
\end{tikzcd}$$

In other words $g^*(\eta \circ \gamma^{-1}) = \varphi$, hence $\varphi \in \operatorname{im} g^*$. We conclude that $\operatorname{im} g^* = \ker f^*$.

Suppose the second sequence is exact for every R-module N.

The exactness at $\operatorname{Hom}(B, N)$ implies $f^* g^* \varphi = 0$ for all $\varphi \in \operatorname{Hom}(C, N)$. For $N = C$ this tells us $g \circ f = f^* g^* \operatorname{id}_C = 0$, i.e., $\operatorname{im} f \subseteq \ker g$.

If $\pi : B \to B/\operatorname{im} f$ is the natural projection, then $\pi \circ f = 0$. For $N = B/\operatorname{im} f$ the exactness at $\operatorname{Hom}(B, N)$ tells us $\pi \in \ker f^* = \operatorname{im} g^*$, implying $\pi = \varphi \circ g$ for some $\varphi \in \operatorname{Hom}(C, N)$. In particular, we have $\ker g \subseteq \ker \pi = \operatorname{im} f$. We conclude that $\operatorname{im} f = \ker g$.

If $\pi : C \to C/\operatorname{im} g$ is the natural projection, then $\pi \circ g = 0$. For $N = C/\operatorname{im} g$ the exactness at $\operatorname{Hom}(C, N)$ tells us $\pi \in \ker g^* = 0$, implying $\pi = 0$. In other words $C/\operatorname{im} g = \operatorname{im} \pi = 0$, so in fact $\operatorname{im} g = C$. $\square$

Lemma 2. For any R-modules A, B, and C there is a natural isomorphism

$$\operatorname{Hom}(A, \operatorname{Hom}(B, C)) \cong \operatorname{Hom}(A \otimes_R B, C).$$

Proof. Given $f : A \to \operatorname{Hom}(B, C)$ there is a map $A \times B \to C$ defined by $(a, b) \mapsto f(a)(b)$. We check that this map is bilinear. Since f is an R-module homomorphism, we have

$$f(ra_1 + a_2)(b) = (rf(a_1) + f(a_2))(b) = rf(a_1)(b) + f(a_2)(b).$$

Since $f(a)$ is an R-module homomorphism, we have

$$f(a)(rb_1 + b_2) = rf(a)(b_1) + f(a)(b_2).$$

Hence, the described map is R-bilinear and therefore factors through an R-module homomorphism $F(f) : A \otimes_R B \to C$, i.e., $F(f)(a \otimes b) = f(a)(b)$.

This gives us a map

$$F : \operatorname{Hom}(A, \operatorname{Hom}(B, C)) \longrightarrow \operatorname{Hom}(A \otimes_R B, C).$$

For $f, g \in \operatorname{Hom}(A, \operatorname{Hom}(B, C))$ and $x \in A \otimes_R B$ we have

$$F(rf + g)(x) = (rf + g)(x) = rf(x) + g(x) = rF(f)(x) + F(g)(x).$$

That is, F is an R-module homomorphism.

Given $f : A \otimes_R B \to C$ we define a map $G(f) : A \to \operatorname{Hom}(B, C)$ by $a \mapsto f(a \otimes -)$, where $f(a \otimes -)(b) = f(a \otimes b)$.

(I skip some details here because it is just more tedious linearity checking.)

Hence, we have an R-module homomorphism

$$G : \mathrm{Hom}(A \otimes_R B, C) \longrightarrow \mathrm{Hom}(A, \mathrm{Hom}(B, C)).$$

Then F and G are inverses, describing the isomorphism in question. $\square$

We now prove the main result.

Proof. By Lemma 1, the exactness of the sequence

$$A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

is equivalent to the exactness of the sequence

$$0 \longrightarrow \mathrm{Hom}(A, \mathrm{Hom}(N, M)) \longrightarrow \mathrm{Hom}(B, \mathrm{Hom}(N, M)) \longrightarrow \mathrm{Hom}(C, \mathrm{Hom}(N, M))$$

for all R-modules N and M. By Lemma 2, this gives us the exact sequence

$$0 \longrightarrow \mathrm{Hom}(A \otimes_R N, M) \longrightarrow \mathrm{Hom}(B \otimes_R N, M) \longrightarrow \mathrm{Hom}(C \otimes_R N, M).$$

Again applying Lemma 1, we obtain the exact sequence

$$A \otimes_R N \longrightarrow B \otimes_R N \longrightarrow C \otimes_R N \longrightarrow 0.$$

$\square$