1 Exercise 1.1.7 Define $f : S^1 \times I \to S^1 \times I$ by $f(\theta, s) = (\theta + 2\pi s, s)$, so f restricts to the identity on the two boundary circles of $S^1 \times I$. Show that f is homotopic to the identity by a homotopy f_t that is stationary on one of the boundary circles. [Consider what f does to the path $s \mapsto (\theta_0, s)$ for a fixed $\theta_0 \in S^1$.]

Define the homotopy $f_t : S^1 \times I \to S^1 \times I$ by

$$f_t(\theta, s) = (\theta + 2\pi st, s).$$

Then $f_0 = \mathrm{id}_{S^1 \times I}$ and $f_t = f$. Moreover, for all points on the lower boundary circle $S^1 \times \{0\}$ we have

$$f_t(\theta, 0) = (\theta + 2\pi 0t, 0) = (\theta, 0).$$

That is, $f_t|_{S^1 \times \{0\}} = \mathrm{id}_{S^1 \times \{0\}}$, as desired.

2 Exercise 1.1.8 Does the Borsuk-Ulam theorem hold for the torus? In other words, for every map $f : S^1 \times S^1 \to \mathbb{R}^2$ must there exist $(x, y) \in S^1 \times S^1$ such that $f(x, y) = f(-x, -y)$?

No.

Proof. We represent the points in the circle S^1 as radians $\theta \in \mathbb{R}$, modulo 2π. With this representation, the antipodal point to $\theta \in S^1$ is $\theta + \pi \pmod{2\pi}$. Then there is a natural embedding of the circle into the real plane:

$$\begin{aligned} S^1 &\longrightarrow \mathbb{R}^2 \\ \theta &\longmapsto (\cos\theta, \sin\theta). \end{aligned}$$

We project the torus onto its first component, then embed the circle into the plane, giving us $f(x, y) = (\cos x, \sin x)$. The value at the antipodal point is

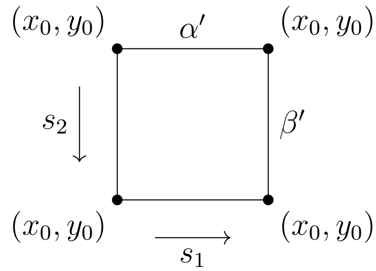
$$f(x + \pi, y + \pi) = (\cos(x + \pi), \sin(y + \pi)) = (-\cos x, -\sin y) = -f(x, y).$$

In particular, the Borsuk-Ulam theorem does not hold. $\square$

3 Exercise 1.1.10 From the isomorphism $\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0)$ it follows that loops in $X \times \{y_0\}$ and $\{x_0\} \times Y$ represent commuting elements of $\pi_1(X \times Y, (x_0, y_0))$. Construct an explicit homotopy demonstrating this.

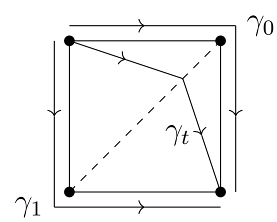
A loop $\alpha : (I, \partial I) \to (X, x_0)$ corresponds to a loop $\alpha' : (I, \partial I) \to X \times \{y_0\}$ defined by $\alpha'(s) = (\alpha(s), y_0)$. Similarly, a loop $\beta : (I, \partial I) \to (Y, y_0)$ corresponds to a loop $\beta' : (I, \partial I) \to \{x_0\} \times Y$ defined by $\beta'(s) = (x_0, \beta(s))$.

Denote the product map $H = \alpha \times \beta : I \times I \to X \times Y$, i.e., $H(s_1, s_2) = (\alpha(s_1), \beta(s_2))$. We can draw the parameter space of H as follows:



For a fixed s_2, the horizontal line $I \times \{s_2\}$ in this space corresponds to the the loop in $X \times \{\beta(s_2)\}$ following α in the X component. Similarly, for a fixed s_1, the vertical line $\{s_1\} \times I$ corresponds to the loop in $\{\alpha(s_1)\} \times Y$ following β in the Y component. In particular, note that the top and bottom edges both correspond to α', while the left and right edges both correspond to β'.

We construct a homotopy of paths $\gamma_t : I \to I \times I$ in this space, from the path following the top and right edges to the path following the left and bottom edges:



Then the composition $H \circ \gamma_t$ is a homotopy of paths in $X \times Y$:

$$\alpha' \cdot \beta' = H \circ \gamma_0 \simeq H \circ \gamma_1 = \beta' \cdot \alpha'.$$

Then in $\pi_1(X \times Y, (x_0, y_0))$, we have

$$[\alpha'][\beta'] = [\alpha' \cdot \beta'] = [\beta' \cdot \alpha'] = [\beta'][\alpha'].$$

4 Exercise 1.1.12 Show that every homomorphism $\pi_1(S^1) \to \pi_1(S^1)$ can be realized as the induced homomorphism φ_* of a map $\varphi : S^1 \to S^1$.

Proof. Recall that $\pi_1(S^1) \cong \mathbb{Z}$, where the homotopy class of a loop in S^1 corresponds to its winding number. A homomorphism $\mathbb{Z} \to \mathbb{Z}$ must send 0 to itself and 1 to some $n \in \mathbb{Z}$—this information completely characterizes the homomorphism. We realize the circle as the quotient $q : I \to I/\partial I = S^1$. Define the map

$$\begin{aligned} \varphi' : I &\longrightarrow S^1, \\ s &\longmapsto ns \pmod 1. \end{aligned}$$

As $\varphi'(0) = 0 \equiv n = \varphi'(1) \pmod 1$, φ' is constant on ∂I and, therefore, factors through the quotient as follows:

$$\begin{tikzcd}
I \arrow[r, "\varphi'"] \arrow[d, "q"'] & S^1 \\
I/\partial I = S^1 \arrow[ur, dashed, "\exists!\varphi"']
\end{tikzcd}$$

We now compute the induced homomorphism $\varphi_* : \pi_1(S^1) \to \pi_1(S^1)$. Note that q can be interpreted as the loop in S^1 based at 0 which goes around the circle exactly once at a constant speed, which means that its homotopy class $[q] \in \pi_1(S^1)$ corresponds to $1 \in \mathbb{Z}$. Then we have

$$\varphi_*[q] = [\varphi \circ q] = [\varphi'].$$

By construction, φ' is the loop in S^1 which goes around the circle exactly n times (or $-n$ times backwards if n is negative) at a constant speed. Therefore, $\varphi_*[q] \in \pi_1(S^1)$ corresponds to $n \in \mathbb{Z}$, hence φ_* corresponds to the original homomorphism $\mathbb{Z} \to \mathbb{Z}$ sending 1 to n. □

5 Exercise 1.1.16 Show that there are not retractions $r : X \to A$ in the following cases:

Proposition 1.17 tells us that if such a retraction exists, then the inclusion $\iota : A \hookrightarrow X$ induces an injective homomorphism $\iota_* : \pi_1(A) \to \pi_1(X)$. We will compute the fundamental groups $\pi_1(X)$ and $\pi_1(A)$ to show that ι_* is not an injection. This, in turn, proves that no such retraction exists. (All spaces are path-connected, so any choice of base point suffices.)

(a) $X = \mathbb{R}^3$ with A any subspace homeomorphic to S^1.

The fundamental groups are

$$\pi_1(X) = \pi_1(\mathbb{R}^3) = 0$$

and

$$\pi_1(A) \cong \pi_1(S^1) \cong \mathbb{Z}.$$

There is no injection $\mathbb{Z} \to 0$.

(b) $X = S^1 \times D^2$ with A its boundary torus $S^1 \times S^1$.

The fundamental groups are

$$\pi_1(X) = \pi_1(S^1 \times D^2) \cong \pi_1(S^1) \times \pi_1(D^2) \cong \mathbb{Z} \times 0 = \mathbb{Z}$$

and

$$\pi_1(A) = \pi_1(S^1 \times S^1) \cong \pi_1(S^1) \times \pi_1(S^1) \cong \mathbb{Z} \times \mathbb{Z}.$$

Then ι_* corresponds to a homomorphism $\mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ which is simply the projection onto the first component; this map is not injective.

(c) $X = S^1 \times D^2$ with A the circle shown in the figure.

It can be seen that a path in A is nullhomotopic in X, even relative to any basepoint in A. This means that ι_* must be the trivial map. However, $\pi_1(A) \cong (S^1) \cong \mathbb{Z}$ is not trivial, so ι_* is not injective.

(d) $X = D^2 \vee D^2$ with A its boundary $S^1 \vee S^1$.

The fundamental groups are

$$\pi_1(X) = \pi_1(D^2 \vee D^2) \cong \pi_1(D^2) * \pi_1(D^2) = 0 * 0 = 0$$

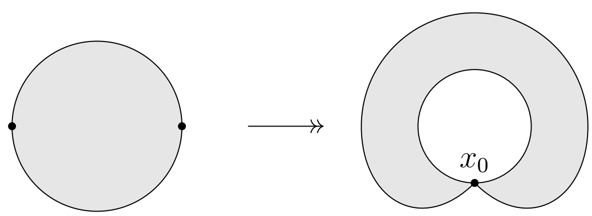
and

$$\pi_1(A) = \pi_1(S^1 \vee S^1) \cong \pi_1(S^1) * \pi_1(S^1) \cong \mathbb{Z} * \mathbb{Z}.$$

As $\mathbb{Z} * \mathbb{Z}$ is nontrivial, there is no injection $\mathbb{Z} * \mathbb{Z} \to 0$.

(e) X a disc with two points on its boundary identified and A its boundary $S^1 \vee S^1$.

We are given X as the following quotient of the disc D^2:



There is a deformation retraction of X onto the inner boundary circle, giving us a homotopy equivalence $X \simeq S^1$ and fundamental group

$$\pi_1(X) = \pi_1(X, x_0) \cong \pi_1(S^1) \cong \mathbb{Z}.$$

This can been seen intuitively in the drawing of X, as any loop based at x_0 (in particular, those in the outer boundary circle) can be homotoped to a loop contained in the inner boundary circle. The fundamental group of A is

$$\pi_1(A) = \pi_1(A, x_0) \cong \pi_1(S^1 \vee S^1) \cong \pi_1(S^1) * \pi_1(S^1) \cong \mathbb{Z} * \mathbb{Z}.$$

Then ι_* corresponds to a homomorphism $\mathbb{Z} * \mathbb{Z} \to \mathbb{Z}$ which acts as the identity on the first copy of $\mathbb{Z}$. All other elements of the free product must also be sent somewhere in $\mathbb{Z}$, so this is not an injection.

(f) X the Möbius band and A its boundary circle.

Note that X deformation retracts to the circle through its middle. This gives us the fundamental group

$$\pi_1(X) \cong \pi_1(S^1) \cong \mathbb{Z},$$

where a loop in X is corresponds to its winding number around the Möbius band. In particular, if f is the loop in X following the boundary circle, its homotopy class in $\pi_1(X)$ corresponds to $2 \in \mathbb{Z}$ (or -2 depending on orientation). In other words, if g is the loop in X following the middle circle, its homotopy class in $\pi_1(X)$ corresponds to $1 \in \mathbb{Z}$, which means

$$[f] = [g][g] = [g]^2 \in \pi_1(X).$$

The fundamental group of A is

$$\pi_1(A) \cong \pi_1(S^1) \cong \mathbb{Z},$$

where f corresponds to $1 \in \mathbb{Z}$.

Assume, for contradiction, a retraction $r : X \to A$ exists. By definition, $r \circ \iota = \mathrm{id}_A$, which implies the induced homomorphism is

$$r_* \circ \iota_* = (r \circ \iota)_* = \mathrm{id}_{\pi_1(A)}.$$

So in $\pi_1(A)$, we have

$$[f] = r_*(\iota_*[f]) = r_*([g]^2) = (r_*[g])^2.$$

This is a contradiction, as $[f]$ is a generator of the cyclic group $\pi_1(A) \cong \mathbb{Z}$, so it is not the square of any element.

6 Exercise 1.1.18 Using the technique in the proof of Proposition 1.14, show that if a space X is obtained from a path-connected subspace A by attaching a cell e^n with $n \geq 2$, then the inclusion $A \hookrightarrow X$ induces a surjection $\pi_1(A) \to \pi_1(X)$.

Proof. Let $f : I \to X$ be a loop based at a point in A. We will find a loop g homotopic to f which does not pass through a given point $x \in e^n$.

The preimage $f^{-1}(e^n)$ is an open subset of the real interval $(0, 1)$ and, therefore, can be written as the countable union of disjoint open intervals: $f^{-1}(e^n) = \bigcup_i (a_i, b_i)$. Then $f^{-1}(x)$ is a compact set covered by these intervals, so it must be covered by finitely many—say (a_i, b_i) for $i = 1, \ldots, k$. The restricted path $f_i = f|_{[a_i,b_i]}$ is contained in the closure $\overline{e^n} = D^n$, with its endpoints $f(a_i)$ and $f(b_i)$ in the boundary $\partial e^n = S^{n-1}$. As S^{n-1} is path-connected for $n \geq 2$, we can choose a path g_i between these endpoints contained in S^{n-1}—in particular, g_i does not pass through x. Since D^n is simply-connected, f_i and g_i are homotopic as paths (i.e., relative the endpoints). Then we may homotope f by deforming f_i to g_i for $i = 1, \ldots, k$, with the resultant loop g having $g(I)$ disjoint from x.

With D^n homeomorphic to a convex subset of $\mathbb{R}^n$, we can take $x \in e^n$ as the focus of a radial projection $D^n \setminus \{x\} \to S^{n-1}$. Take $h_t : D^n \setminus \{x\} \to D^n \setminus \{x\}$ to be the straight line homotopy between the identity and the radial projection. In other words, h_t describes a deformation retraction of $D^n \setminus \{x\}$ onto its boundary. Gluing this homotopy to the identity on A gives us a deformation retraction of $X \setminus \{x\}$ onto A. Then the composition $h_t \circ g$ gives a homotopy between $h_0 \circ g = g$ and a loop $h = h_1 \circ g$ entirely contained in A. Hence, f and h are homotopic loops, so $[h] = [f] \in \pi_1(X)$. ☐

Apply this to show:

(a) The wedge sum $S^1 \vee S^2$ has fundamental group $\mathbb{Z}$.

We can consider $S^1 \vee S^2$ as the result of attaching e^2 to S^1 with a constant attaching map $\partial e^n \to S^1$. Therefore, the inclusion $S^1 \hookrightarrow S^1 \vee S^2$ induces a surjection of fundamental groups:

$$\mathbb{Z} \cong \pi_1(S^1) \longrightarrow \pi_1(S^1 \vee S^2).$$

This is also an injection by Proposition 1.17, as there is a retraction $S^1 \vee S^2 \to S^1$ which acts as the identity on S^1 and sends all of S^2 to the distinguished point. Hence, we have an isomorphism $\pi_1(S^1 \vee S^2) \cong \mathbb{Z}$.

(b) For a path-connected CW complex X the inclusion map $X^1 \hookrightarrow X$ of its 1-skeleton induces a surjection $\pi_1(X^1) \to \pi_1(X)$. [For the case that X has infinitely many cells, see Proposition A.1 in the Appendix.]

Proof. By Proposition A.1, a given loop in X is contained in a finite subcomplex Y of X. Then the subcomplex $X^1 \cup Y$ can be constructed by sequentially attaching finitely many cells (of dimension at least 2) to the 1-skeleton. Applying the result at each step of this construction, we deduce that the inclusion $X^1 \hookrightarrow X^1 \cup Y$ induces a surjective homomorphism $\pi_1(X^1) \to \pi_1(X^1 \cup Y)$, which is then included into $\pi_1(X)$. Applying this to all loops, we conclude that the inclusion $X^1 \hookrightarrow X$ induces a surjection $\pi_1(X^1) \to \pi_1(X)$. □