Math 118C
Spring 2021
Midterm Exam
Time Limit: 75 Minutes (or extended per DSP letter)

Name: ____________________
PERM Number: ____________________

- Submit your work on GradeScope within 20 minutes from the end of the exam time indicated above. If you have submitted a DSP letter, this submission window also extends accordingly for you.
- This is a closed-book exam, you should work independently. In particular, you should not discuss these questions with anyone nor seek help from any internet sources. Violations of academic integrity will be reported.
- If you want any clarification during the exam, ask the instructor over Zoom. Zoom ID: 426 530 1130.
- Organize your work, in a reasonably neat and coherent way. Work scattered all over a page without a clear ordering will receive very little credit.
- This exam contains 7 pages (including this cover page) and 5 problems.

1. (25 points) Let f be a continuous function on $\mathbb{R}$ with period 2π. Suppose that all its Fourier coefficients are zero, i.e.

$$c_n := \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx = 0,$$

for all n. Prove that $f = 0$. State any result(s) you use.

2. (25 points) Let $F = (F_1, F_2) : \mathbb{R}^2 \to \mathbb{R}^2$ be the function defined by

$$F_1(x, y) = x^2 - y^2, \quad F_2(x, y) = 2xy.$$

Let $(x, y) \in \mathbb{R}^2$ be a point.

(a) Prove that if $(x, y) \neq (0, 0)$, there exists a neighborhood of (x, y) on which the map F is injective.

(b) Is there a neighborhood of $(0, 0)$ on which the map F is injective? Justify your answer.

3. (25 points) Let $F : B^n \to B^n$ be a continuously differentiable map, where

$$B^n := \{x \in \mathbb{R}^n : |x| \leq 1\}$$

is the closed unit ball in $\mathbb{R}^n$. Suppose that at every point $x \in B^n$, the norm of the derivative $||F'(x)|| < 1$. Show that there exists a point $x_0 \in B^n$ such that for $F(x_0) = x_0$.

4. (25 points) (a) (5 points) State the inverse function theorem in your own words.

(b) (5 points) State the implicit function theorem in your own words.

(c) (15 points) Prove the inverse function theorem using the implicit function theorem.

5. (25 points) Let n, m be two positive integers. Suppose that there exist two open subsets $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$, and two continuously differentiable maps $F : U \to V$ and $F^{-1} : V \to U$ that are inverse to each other. Prove that $n = m$.