(worked with Joseph Sullivan and Gahl Shemy)

All the problems in this problem set refer to the group G with the identity element $1 \in G$ and presentation:

$$G = \langle -1, x, y \mid (-1)^2 = 1, x^2 = y^2 = (xy)^2 = -1 \rangle.$$

1 Show that G has order 8, and show that x generates a normal subgroup of G.

Proof. We first see that -1 commutes with both x and y:

$$(-1)x = x^2x = xx^2 = x(-1) \quad \text{and} \quad (-1)y = y^2y = yy^2 = y(-1).$$

Therefore, it makes sense to write $-x = (-1)x = x(-1)$ and $-y = (-1)y = y(-1)$. Then we find the inverses of x and y:

$$x^4 = (-1)^2 = 1 \quad \text{and} \quad y^4 = (-1)^2 = 1,$$

so $x^{-1} = x^3 = -x$ and $y^{-1} = y^3 - y$. We compute yx:

$$-1 = xyxy \implies yx = (-x)(-1)(-y) = -xy.$$

Hence, we know how to commute all pairs of generators so every element of G can be written in the form $(-1)^a x^b y^c$ where $a, b, c \in \{0, 1\}$. In particular, we deduce that $|G| \leq 8$.

Define the set of 8 elements $H = \{(-1)^a x^b y^c \mid a, b, c \in \{0, 1\}\}$. One can define the multiplication $(-1)^a x^b y^c (-1)^\alpha x^\beta y^\gamma$ in H in a way that agrees with the relations on G. Under this multiplication, H is a group of order 8 and there is an obvious surjective group homomorphism $G \to H$. This implies $|G| \geq 8$.

Since the order of x is 4, the subgroup $\langle x \rangle$ is index 2 in G which implies $\langle x \rangle \trianglelefteq G$. $\square$

2 Calculate $xyx^{-1}y^{-1}$, and find the center of G.

We compute

$$xyx^{-1}y^{-1} = xy(-x)(-y) = (-1)^2 xyxy = (xy)^2 = -1.$$

This implies $x, y \notin Z(G)$. Similarly,

$$(-x)(-y)(-x)^{-1}(-y)^{-1} = (-1)^2 xyxy = -1,$$

so $-x, -y \notin Z(G)$. Then

$$(xy)y(xy)^{-1}y^{-1} = xy^2y^{-1}x^{-1}(-y) = x(-1)(-y)(-x)(-y) = xyxy = -1,$$

so $\pm xy \notin Z(G)$. Hence, $Z(G) = \{\pm 1\}$.

3 Find four distinct one-dimensional representations of G, and one irreducible two-dimensional representation.

We find the conjugacy classes of G; let $\overline{a}$ denote the conjugacy class of $a \in G$. Clearly, $\overline{1} = \{1\}$. We conjugate -1:

$$x(-1)x^{-1} = (-1)^2x^2 = -1 \quad \text{and} \quad y(-1)y^{-1} = (-1)^2y^2 = -1,$$

hence $\overline{-1} = \{-1\}$. Then

$$(-1)x(-1)^{-1} = (-1)^2x = x \quad \text{and} \quad yxy^{-1} = -xy(-y) = xy^2 = -x,$$

so $\overline{x} = \{\pm x\}$. It is the same to see $[y] = \{\pm y\}$. Then

$$x(xy)x^{-1} = -y(-x) = yx = -xy,$$

so $\overline{xy} = \{\pm xy\}$.

Hence, we have five conjugacy classes: $\overline{1}, \overline{-1}, \overline{x}, \overline{y}, \overline{xy}$.

There is always the trivial representation which maps all elements to $[1] \in \mathrm{GL}_1(\mathbb{C})$.

Note that $\langle y\rangle$ and $\langle xy\rangle$ are normal subgroups of order 4 by the same argument as $\langle x\rangle$. So for $a = x, y, xy$ we have $G/\langle a\rangle \cong \mathbb{Z}/2\mathbb{Z}$ and there is a representation sending $\langle a\rangle \mapsto [1]$ and all other elements to $[-1]$.

There is an irreducible representation $G \to \mathrm{GL}_2(\mathbb{C})$ given by

$$-1 \mapsto \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \quad x \mapsto \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \quad y \mapsto \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad xy \mapsto \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix},$$

4 Find as much of the character table of G as you can.

	$\overline{1}$	$\overline{-1}$	$\overline{x}$	$\overline{y}$	$\overline{xy}$
triv	1	1	1	1	1
$G/\langle x\rangle$	1	1	1	-1	-1
$G/\langle y\rangle$	1	1	-1	1	-1
$G/\langle xy\rangle$	1	1	-1	-1	1
2d	2	-2	0	0	0