1 Exercise 7.8.3 Let $f : \mathcal{F} \to \mathcal{G}$ be a morphism of locally free sheaves on a scheme X over a field k. Let $P \in X$ be a point, and denote by $\mathcal{F}(P)$ (resp. $\mathcal{G}(P)$) the fiber of the vector bundle $\mathcal{F}$ (resp. $\mathcal{G}$) over P, which is a k-vector space. Are the following statements true or false?

If $\iota : P \to X$ is the inclusion map, then the fiber of $\mathcal{F}$ over P is defined as the pullback

$$\mathcal{F}(P) = (\iota^*\mathcal{F})(P) = (\iota^{-1}\mathcal{F})(P) \otimes_{(\iota^{-1}\mathcal{O}_X)(P)} \mathcal{O}_P(P) = \mathcal{F}_P \otimes_{\mathcal{O}_{X,P}} k(P).$$

Given $f : \mathcal{F} \to \mathcal{G}$, there is an induced morphism of stalks $f_P : \mathcal{F}_P \to \mathcal{G}_P$. We have seen that f is injective/surjective if and only if f_P is injective/surjective at all points $P \in X$. There is an induced morphism of $\mathcal{O}_{X,P}$-modules, given by

$$f_P \otimes \mathrm{id}_{k(P)} : \mathcal{F}_P \otimes_{\mathcal{O}_{X,P}} k(P) \longrightarrow \mathcal{G}_P \otimes_{\mathcal{O}_{X,P}} k(P).$$

This is the induced map of fibers $\mathcal{F}(P) \to \mathcal{G}(P)$.

(i) If $f : \mathcal{F} \to \mathcal{G}$ is injective then the induced map $\mathcal{F}(P) \to \mathcal{G}(P)$ is injective for all $P \in X$.

No.

Consider $X = \mathbb{P}^n$, with locally free sheaves $\mathcal{F} = \mathcal{O}_{\mathbb{P}^n}(-1)$ and $\mathcal{G} = \mathcal{O}_{\mathbb{P}^n}$. Then we have an exact sequence of $\mathcal{O}_{\mathbb{P}^n}$-modules

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-1) \xrightarrow{\cdot x_0} \mathcal{O}_{\mathbb{P}^n}.$$

In other words, $f = \cdot x_0$ is an injective morphism of sheaves. Consider the induced map on fibers:

$$(\mathcal{O}_{\mathbb{P}^n}(-1))_P \otimes_{\mathcal{O}_{\mathbb{P}^n,P}} k(P) \longrightarrow \mathcal{O}_{\mathbb{P}^n,P} \otimes_{\mathcal{O}_{\mathbb{P}^n,P}} k(P) = k(P)$$

Note that the right side (the residue field $k(P)$) has the structure of an $\mathcal{O}_{\mathbb{P}^n,P}$ module by the evaluation map $\varphi \mapsto \varphi(P) \in k(P)$. So if we take $P = [0 : 1 : \cdots] \in \mathbb{P}^n$, then any $\frac{g}{h} \in (\mathcal{O}_{\mathbb{P}^n}(-1))_P$ will have $\frac{x_0 g}{h} \in \mathcal{O}_{\mathbb{P}^n,P}$, which is then zero in $k(P)$. That is, the induced map on fibers is trivial. But since the left side is not trivial, the map is not injective.

(ii) If $f : \mathcal{F} \to \mathcal{G}$ is surjective then the induced map $\mathcal{F}(P) \to \mathcal{G}(P)$ is surjective for all $P \in X$.

Yes.

If f is surjective, then the map on stalks f_P is surjective, which implies that the map on fibers $f_P \otimes \mathrm{id}_{k(P)}$ is surjective. This follows from the fact that tensoring is right-exact.

2 Exercise 7.8.4 Prove the following generalizations of example 7.1.16: If X is a smooth curve over some field k, $\mathcal{L}$ a line bundle on X, and $P \in X$ a point, then there is an exact sequence

$$0 \longrightarrow \mathcal{L}(-P) \longrightarrow \mathcal{L} \longrightarrow k_p \longrightarrow 0,$$

where k_P denotes the "skyscraper sheaf"

$$k_P(U) = \begin{cases} k & \text{if } P \in U, \\ 0 & \text{if } P \notin U. \end{cases}$$

Consider the case that $\mathcal{L} = \mathcal{O}_X$. By definition, if $U \subseteq X$ is an open neighborhood of P, we have

$$\mathcal{O}_X(-P)(U) = \{\varphi \in K(X) \mid (\varphi) - P \geq 0 \text{ on } U\}.$$

This means $\varphi \in \mathcal{O}_X(-P)(U)$ if and only if $\varphi \in \mathcal{O}_X(U)$ with $\varphi(P) = 0$. And if $U \subseteq X$ is open not containing P, then $\mathcal{O}_X(-P)(U) = \mathcal{O}_X(U)$. Suppose s_0 is a rational section on $\mathcal{O}_X(-P)$, so that $\mathcal{O}_X(-P) = \mathcal{O}_X((s_0))$. Then we can also write

$$\mathcal{O}_X(-P)(U) = \mathcal{O}_X((s_0))(U) = \{\varphi \in K(X) \mid (\varphi) + (s_0) \geq 0 \text{ on } U\}.$$

Then we have a sequence of $\mathcal{O}_X$-modules

$$0 \longrightarrow \mathcal{O}_X(-P) \xrightarrow{\cdot s_0} \mathcal{O}_X \xrightarrow{\mathrm{eval}_P} k_P \longrightarrow 0.$$

By construction, this sequence is exact on open sets $U \subseteq X$ containing P, since $\mathcal{O}_X(-P)(U)$ is precisely the sections in $\mathcal{O}_X(U)$ which evaluate to zero at P. And when U does not contain P, the skyscraper sheaf is zero, and the sequence describes an automorphism of $\mathcal{O}_X(U)$.

For a general line bundle $\mathcal{L}$, we choose a rational section ℓ on $\mathcal{L}$, so that $\mathcal{L} \cong \mathcal{O}_X((\ell))$. Then the desired short exact sequence is obtained via the first case.

3 Exercise 7.8.5 If X is an *affine* variety over a field k and $\mathcal{F}$ a locally free sheaf of rank r on X, is then necessarily $\mathcal{F} \cong \mathcal{O}_X^{\oplus r}$?

4 Exercise 7.8.6 Let X be a scheme, and let $\mathcal{F}$ be a locally free sheaf on X. Show that $(\mathcal{F}^\vee)^\vee \cong \mathcal{F}$. Show by example that this statement is in general false if $\mathcal{F}$ is only quasi-coherent but not locally free.