Q1 Let $K_1, K_2, \ldots, K_n$ be subfields of K. The composite field of $K_1, K_2, \ldots, K_n$, denoted $K_1K_2, \cdots K_n$, is defined to be the smallest subfield of K containing $K_1, K_2, \ldots, K_n$.

(a) Suppose that $K_j = F(S_j)$ for some $S_j \subseteq K$, $1 \leq j \leq n$. Show that $K_1K_2 \cdots K_n = F(S_1 \cup S_2 \cup \cdots \cup S_n)$.

(b) Let $K \subseteq \overline{F}$ be a finite separable field extension of F and $L \subseteq \overline{F}$ be the Galois closure of K over F. Suppose that $\mathrm{Gal}(L/F) = \{\sigma_1, \ldots, \sigma_n\}$. Show that $L = \sigma_1(K)\sigma_2(K) \cdots \sigma_n(K)$.

Q2 §14.4 Problem 5.

Q3 §14.4 Problem 9.

Q4 §14.7 Problem 12.
(Hint: One can use Cauchy's Theorem: If G is a finite group, p is a prime number and $p \mid |G|$, then G has a subgroup of order p.)

Q5 Let F be a field and n be a positive integer. Suppose that $\mathrm{ch}(F) = 0$ or $\mathrm{ch}(F) \nmid n$ and $x^n - 1$ splits completely over F. Denote by $\sqrt[n]{a}$ a root in $\overline{F}$ of $x^n - a \in F[x]$. Let $m = [F(\sqrt[n]{a}), F]$. Show that m is the smallest positive integer such that $(\sqrt[n]{a})^m \in F$.