**1** Let $\{x_n\}$ be a Cauchy sequence of a metric space $(X, d)$. Suppose that for some subsequence $\{x_{n_k}\}$, $x_{n_k}$ converges to $x_0$ as $k \to \infty$, show that $x_n$ converges to $x_0$ too.

*Proof.* Let $\varepsilon > 0$ be given. Choose $N \in \mathbb{N}$ for the definitions of both the Cauchyness of $\{x_n\}$ and the convergence of the subsequence $x_{n_k} \to x_0$, with respect to $\varepsilon/2$. For any $n \geq N$, choose any $n_k \geq N$, then

$$\|x_n - x_0\| \leq \|x_n - x_{n_k}\| + \|x_{n_k} - x_0\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Hence, we indeed have convergence of the whole sequence $x_n \to x_0$. $\square$

**2** Construct a sequence $f_n(x)$ of bounded functions on $[0,1]$ converging to zero in $L^1$, but $f_n$ converges at no point in $[0,1]$.

For each $k \in \mathbb{N}$ and $i = 1, \dots, k$, define $f_{k,i} = \chi_{[\frac{i-1}{k}, \frac{i}{k}]}$, i.e., the characteristic function on the interval $[\frac{i-1}{k}, \frac{i}{k}] \subseteq [0,1]$. Choose an indexing $k_n, i_n \in \mathbb{N}$ such that the sequence $\{f_{k_n, i_n}\}$ progresses as follows: $f_{k,1}, f_{k,2}, \dots, f_{k,k}, f_{k+1,1}$. In particular, $k_n \to \infty$ as $n \to \infty$. It follows that the integral

$$\int_0^1 |f_{k,i}(x)| \mathrm{d}x = \int_0^1 \chi_{[\frac{i-1}{k}, \frac{i}{k}]}(x) \mathrm{d}x = \frac{1}{k}$$

converges to zero as $n \to \infty$ (with $k = k_n$). However, for each $x \in [0,1]$ and $k \in \mathbb{N}$, there is some $i \in \{1, \dots, k\}$ such that $x \in [\frac{i-1}{k}, \frac{i}{k}]$, which gives $f_{k,i}(x) = 1$. Since there are infinitely many such pairs of $k, i$ satisfying this, all of which appear in the sequence, the sequence does not converges pointwise at $x$.

**3** Let $(M, \mu)$ be a measure space, show that $L^\infty(M, \mu)$ is a Banach space.

*Proof.* We will take for granted that this is a normed vector space, as proving this primarily involves expanding definitions. It remains to prove that the space is complete.

Recall that the set $L^\infty(M, \mu)$ is a quotient—we denote the equivalence class of $f$ by $[f]$. Then $[f] = [g]$ whenever $\|[f] - [g]\|_\infty = \operatorname{ess\,sup}_M |f - g| = 0$.

Suppose $\{[f_n]\}$ is a Cauchy sequence in $L^\infty(M, \mu)$. By Problem 1, it suffices to show any subsequence converges. For each $k \in \mathbb{N}$, choose $n_k \in \mathbb{N}$ for the definition of Cauchyness with respect to $\varepsilon = 1/2^k$, and such that $n_k \geq n_{k-1}$. Replacing the original sequence with this subsequence, we have the property that

$$n \leq m \implies \|[f_n - f_m]\|_\infty = \|[f_n] - [f_m]\|_\infty < \frac{1}{2^n}.$$

For $n \leq m$, choose a null set $E_{n,m} \subseteq M$ (i.e., $\mu(E_{n,m}) = 0$) such that

$$\|[f_n - f_m]\|_\infty = \operatorname{ess\,sup}_M |f_n - f_m| \leq \sup_{M \setminus E_{n,m}} |f_n - f_m| < \frac{1}{2^n}.$$

Define the set

$$E = \bigcup_{n=1}^\infty \bigcup_{m=n}^\infty E_{n,m}.$$

By subadditivity, we have

$$\mu(E) \leq \sum_{n=1}^\infty \sum_{m=n}^\infty \mu(E_{n,n}) = \sum_{n=1}^\infty \sum_{m=n}^\infty 0 = 0.$$

Moreover, $E_{n,m} \subseteq E$ implies $M \setminus E \subseteq M \setminus E_{n,m}$, so

$$\sup_{M \setminus E} |f_n - f_m| \leq \sup_{M \setminus E_{n,m}} |f_n - f_m| < \frac{1}{2^n}.$$

For each $x \in M \setminus E$ and $m \geq n$, we have

$$|f_n(x) - f_m(x)| \leq \sup_{M \setminus E} |f_n - f_m| < \frac{1}{2^n}.$$

This means that the sequence $\{f_n(x)\}$ of complex values is Cauchy: given $\varepsilon > 0$, one can choose $N \in \mathbb{N}$ such that $1/2^N < \varepsilon$. Since $\mathbb{C}$ is complete, the sequence converges to some value in $\mathbb{C}$, which we will denote by $f(x)$. This gives us a function $f : M \setminus E \to \mathbb{C}$, to which the sequence $\{f_n|_{M \setminus E}\}$ converges pointwise. Letting $m \to \infty$ in the previous inequality yields

$$|f_n(x) - f(x)| \leq \frac{1}{2^n}.$$

Taking the supremum over all $x \in M \setminus E$ gives us

$$\sup_{M \setminus E} |f_n - f| \leq \frac{1}{2^n}.$$

In particular, this value is finite. Since $[f_1] \in L^\infty(M,\mu)$, we know $\|[f_1]\|_\infty < \infty$. Choose a null set $E' \subseteq M$ such that $\sup_{M\setminus E'} |f| < \infty$. Then $E'' = E \cap E'$ is also a null set and

$$\sup_{M\setminus E''} |f| = \sup_{M\setminus E''} |f - f_1 + f_1| \le \sup_{M\setminus E'} |f_1 - f| + \sup_{M\setminus E'} |f_1| < \infty.$$

We extend $f$ to $M$ by defining $f|_E = 0$. Then

$$\|[f]\|_\infty = \operatorname{ess\,sup}_M |f| \le \sup_{M\setminus E''} |f| < \infty,$$

which means $[f] \in L^1(M,\mu)$. Moreover,

$$\|[f_n] - [f]\|_\infty = \|[f_n - f]\|_\infty \le \sup_{M\setminus E} |f_n - f| \le \frac{1}{2^n},$$

which converges to zero as $n \to \infty$, hence $[f_n] \to [f]$ in $L^1(M,\mu)$. $\square$

**4** Let $X$ be Banach space and $Y$ be a closed subspace of $X$. Prove that the quotient space $X/Y$ is a Banach space. Note that if $[x] = [y]$ in $X/Y$, then $x - y \in Y$, $\|[x]\| = \inf_{y \in [x]} \|y\|$.

*Proof.* The quotient of a vector space by a linear subspace is always a vector space, but we must check that the proposed norm is in fact a norm.

Nonnegativity follows immediately from the nonnegativity of the norm in $X$.

Since $0 \in [0]$, then $\|[0]\| \leq \|0\| = 0$, so indeed $\|[0]\| = 0$.

Suppose $\|[x]\| = 0$. For each $n \in \mathbb{N}$, choose $y_n \in [x]$ such that $\|y_n\| \leq 1/2^n$. In other words, $\{y_n\}$ is a sequence in $[x]$ converging to zero. Define $z_n = x - y_n \in Y$, then

$$\|x - z_n\| = \| - y_n\| \leq \frac{1}{2^n}.$$

This means that $\{z_n\}$ is a sequence in $Y$ converging to $x$. Since $Y$ is closed, this implies $x \in Y$, so $[x] = 0$.

For all nonzero scalars $a$, we have $y \in [x]$ if and only if $ay \in [ax]$. And in which case, $\|ay\| = |a|\|y\|$, hence we have homogeneity

$$\|[ax]\| = \inf_{ay \in [ax]} \|ay\| = |a| \inf_{y \in [x]} \|y\| = |a|\|[x]\|.$$

Lastly, the triangle inequality:

$$\begin{aligned}
\|[x] + [y]\| &= \inf_{z \in [x+y]} \|z\| \\
&= \inf_{u \in [x], v \in [y]} \|u + v\| \\
&\leq \inf_{u \in [x], v \in [y]} (\|u\| + \|v\|) \\
&\leq \inf_{u \in [x]} \|u\| + \inf_{v \in [y]} \|v\| \\
&= \|[x]\| + \|[y]\|.
\end{aligned}$$

We conclude that $X/Y$ is a normed space with the stated norm.

Finally, we show that the space is complete. Let $\{[x_n]\}$ be a Cauchy sequence in $X/Y$. Our goal is to choose a representative sequence in $X$ which is also Cauchy. By Problem 1, and similar to the proof of Problem 2, we may assume without loss of generality that for all $n \in \mathbb{N}$ and $m \geq n$,

$$\|[x_n] - [x_m]\| < \frac{1}{2^n}.$$

(If this is not the case, use Cauchyness to choose a subsequence such that it is true, and re-index.) For each $n \in \mathbb{N}$, choose $y_n \in [x_n - x_{n+1}]$ such that

$$\|[x_n] - [x_{n+1}]\| \leq \|y_n\| < \frac{1}{2^n}.$$

Set $x'_1 = x_1$ and $x'_{n+1} = x'_n - y_n$ for all $n \in \mathbb{N}$. This inductive definition gives us a new representative sequence $\{x'_n\}$ in $X$ with $[x'_n] = [x_n]$. Moreover,

$$\|x'_n - x'_{n+1}\| = \|x'_n - (x'_n - y_n)\| = \|y_n\| < \frac{1}{2^n},$$

so $\{x'_n\}$ is a Cauchy sequence in $X$ and, therefore, converges to some $x \in X$. Then

$$\|[x_n] - [x]\| = \|[x'_n] - [x]\| = \|[x'_n - x]\| \leq \|x'_n - x\|,$$

which converges to zero as $n \to \infty$. So $[x_n] \to [x]$ in $X/Y$, which is therefore is complete. $\square$

**5**

**(a)** Prove that in every inner product space $X$, the following identity (parallelogram law) holds:

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2, \quad \forall x, y \in X.$$

*Proof.* Applying the definition of the induced norm and the bilinearity (with respect to real scalars), we calculate

$$\begin{aligned}
\|x+y\|^2 + \|x-y\|^2 &= \langle x+y, x+y\rangle + \langle x-y, x-y\rangle \\
&= \langle x, x\rangle + \langle x, y\rangle + \langle y, x\rangle + \langle y, y\rangle \\
&\quad + \langle x, x\rangle - \langle x, y\rangle - \langle y, x\rangle + \langle y, y\rangle \\
&= 2\|x\|^2 + 2\|y\|^2.
\end{aligned}$$

□

**(b)** Prove that the parallelogram law characterizes inner product spaces. Namely, suppose that the parallelogram law holds on a normed linear space $X$, then one can define an inner product $\langle -, - \rangle$ in $X$ in such a way that $\|x\| = \langle x, x \rangle^{1/2}$ for all $x \in X$. (Hint: consider the polarization identity

$$\langle x, y \rangle = \frac{1}{4}\left( (\|x+y\|^2 - \|x-y\|^2) - i(\|x+iy\|^2 - \|x-iy\|^2) \right).$$

*Proof.* We first check positive definite:

$$\begin{aligned}
\langle x, x \rangle &= \frac{1}{4}\left( (\|x+x\|^2 - \|x-x\|^2) - i(\|x+ix\|^2 - \|x-ix\|^2) \right) \\
&= \frac{1}{4}\left( (\|2x\|^2 - 0) - i(\|x+ix\|^2 - (|-i|\|ix+x\|)^2) \right) \\
&= \frac{1}{4}\left( 4\|x\|^2 - i(\|x+ix\|^2 - \|x+ix\|^2) \right) \\
&= \|x\|^2 \\
&\geq 0,
\end{aligned}$$

with equality if and only if $x = 0$, by definition of the norm.

We now check conjugate symmetry:

$$\begin{aligned}
\overline{\langle y, x \rangle} &= \frac{1}{4}\left( (\|y+x\|^2 - \|y-x\|^2) + i(\|y+ix\|^2 - \|y-ix\|^2) \right) \\
&= \frac{1}{4}\left( (\|x+y\|^2 - \|x-y\|^2) + i((|i|\|x-iy\|)^2 - (|-i|\|x+iy\|)^2) \right) \\
&= \frac{1}{4}\left( (\|x+y\|^2 - \|x-y\|^2) - i(\|x+iy\|^2 - \|x-iy\|^2) \right).
\end{aligned}$$

From the parallelogram law, we deduce

$$\|x+x'+y\|^2 = 2\|x+y\|^2 + 2\|x'\|^2 - \|x-x'+y\|^2$$

and

$$\|x+x'+y\|^2 = 2\|x'+y\|^2 + 2\|x\|^2 - \|x'-x+y\|^2.$$

Then

$$\|x+x'+y\|^2 = \|x\|^2 + \|x'\|^2 + \|x+y\|^2 + \|x'+y\|^2 - \|x-x'+y\|^2 - \|x'-x+y\|^2$$

and, similarly,

$$\|x+x'-y\|^2 = \|x\|^2 + \|x'\|^2 + \|x-y\|^2 + \|x'-y\|^2 - \|x-x'-y\|^2 - \|x'-x-y\|^2.$$

Therefore,

$$\|x+x'+y\|^2 - \|x+x'-y\|^2 = \|x+y\|^2 - \|x-y\| + \|x'+y\| - \|x'-y\|$$

and, similarly,

$$\|x + x' + iy\|^2 - \|x + x' - iy\|^2 = \|x + iy\|^2 - \|x - iy\| + \|x' + iy\| - \|x' - iy\|.$$

Substituting into the definition of the inner product gives us

$$\langle x + x', y\rangle = \langle x, y\rangle + \langle x', y\rangle.$$

It follows that the inner product is $\mathbb{Z}$-linear and therefore $\mathbb{Q}$-linear in the first argument. Then

$$i\langle x, y\rangle =$$

$$\begin{aligned}
i\langle x, y\rangle &= i\frac{1}{4}\left((\|x + y\|^2 - \|x - y\|^2) - i(\|x + iy\|^2 - \|x - iy\|^2)\right) \\
&= \frac{1}{4}\left(i((|-i|\|ix + iy\|)^2 - (|-i|\|ix - iy\|)^2) + ((|-i|\|ix - y\|)^2 - (|-i|\|ix + y\|)^2)\right) \\
&= \frac{1}{4}\left((\|ix + y\|^2 - \|ix - y\|^2) - i(\|ix + iy\|^2 - \|ix - iy\|^2)\right) \\
&= \langle ix, y\rangle.
\end{aligned}$$

Hence, the inner product is $\mathbb{Q}(i)$-linear in the first argument. Since $\mathbb{Q}(i)$ is dense in $\mathbb{C}$ and the inner product is continuous, we conclude that it must also be $\mathbb{C}$-linear. $\square$