

14.3 Analysis Model: Waves Under Boundary Conditions

In the preceding section, we discussed standing waves formed by identical waves moving in opposite directions in the same medium. One way to establish a standing wave on a string is to combine incoming and reflected waves from a rigid end. If a string is stretched between *two* rigid supports (Active Fig. 14.9a) and waves are established on the string, standing waves will be set up in the string by the continuous superposition of the waves incident on and reflected from the ends. This physical system is a model for the source of sound in any stringed instrument, such as the guitar, the violin, and the piano. The string has a number of natural patterns of oscillation, called **normal modes**, each of which has a characteristic frequency that is easily calculated.

This discussion is our first introduction to an important analysis model, the **wave under boundary conditions**. When boundary conditions are applied to a wave, we find very interesting behavior that has no analog in the physics of particles. The most prominent aspect of this behavior is **quantization**. We shall find that only certain waves—those that satisfy the boundary conditions—are allowed. The notion of quantization was introduced in Chapter 11 when we discussed the Bohr model of the atom. In that model, angular momentum was quantized. As we shall see in Chapter 29, this quantization is just an application of the wave under boundary conditions model.

In the standing wave pattern on a stretched string, the ends of the string must be nodes because these points are fixed, establishing the boundary condition on the waves. The rest of the pattern can be built from this boundary condition along with the requirement that nodes and antinodes are equally spaced and separated by one-fourth of a wavelength. The simplest pattern that satisfies these conditions has the required nodes at the ends of the string and an antinode at the center point (Active Fig. 14.9b). For this normal mode, the length of the string equals $\lambda/2$ (the distance between adjacent nodes):

$$L = \frac{\lambda_1}{2} \quad \text{or} \quad \lambda_1 = 2L$$

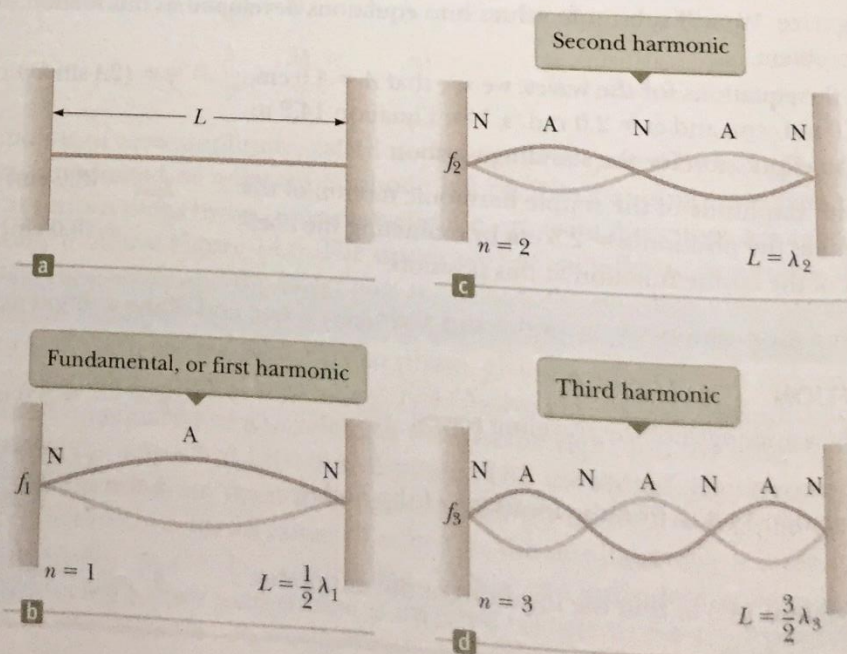
The next normal mode, of wavelength λ_2 (Active Fig. 14.9c), occurs when the length of the string equals one wavelength, that is, when $\lambda_2 = L$. In this mode, the two halves of the string are moving in opposite directions at a given instant, and we sometimes say that two *loops* occur. The third normal mode (Active Fig. 14.9d) corresponds to the case when the length equals $3\lambda/2$; therefore, $\lambda_3 = 2L/3$. In general, the wavelengths of the various normal modes can be conveniently expressed as

$$\lambda_n = \frac{2L}{n} \quad n = 1, 2, 3, \dots$$

14.5

► Wavelengths of normal modes

Active Figure 14.9 (a) A string of length L fixed at both ends. (b)–(d) The normal modes of vibration of the string in Active Figure 14.9a form a harmonic series. The string vibrates between the extremes shown.



where the index n refers to the n th mode of oscillation. The natural frequencies associated with these modes are obtained from the relationship $f = v/\lambda$, where the wave speed v is determined by the tension T and linear mass density μ of the string and therefore is the same for all frequencies. Using Equation 14.5, we find that the frequencies of the normal modes are

$$f_n = \frac{v}{\lambda_n} = \frac{n}{2L} v \quad n = 1, 2, 3, \dots$$

14.6 ◀

► Frequencies of normal modes as functions of wave speed and length of string

Because $v = \sqrt{T/\mu}$ (Equation 13.21), we can express the natural frequencies of a stretched string as

$$f_n = \frac{n}{2L} \sqrt{\frac{T}{\mu}} \quad n = 1, 2, 3, \dots$$

14.7 ◀

► Frequencies of normal modes as functions of string tension and linear mass density

Equation 14.7 demonstrates the quantization that we mentioned as a feature of the wave under boundary conditions model. The frequencies are quantized because only certain frequencies of waves satisfy the boundary conditions and can exist on the string. The lowest frequency, corresponding to $n = 1$, is called the **fundamental frequency** f_1 and is

$$f_1 = \frac{1}{2L} \sqrt{\frac{T}{\mu}}$$

14.8 ◀

► Fundamental frequency of a taut string

The frequencies of the remaining normal modes are integer multiples of the fundamental frequency. Frequencies of normal modes that exhibit such an integer-multiple relationship form a **harmonic series**, and the normal modes are called **harmonics**. The fundamental frequency f_1 is the frequency of the first harmonic, the frequency $f_2 = 2f_1$ is the frequency of the second harmonic, and the frequency $f_n = nf_1$ is the frequency of the n th harmonic. Other oscillating systems, such as a drum-head, exhibit normal modes, but the frequencies are not related as integer multiples of a fundamental. Therefore, we do not use the term *harmonic* in association with those types of systems.

Let us examine further how the various harmonics are created in a string. To excite only a single harmonic, the string must be distorted into a shape that corresponds to that of the desired harmonic. After being released, the string vibrates at the frequency of that harmonic. This maneuver is difficult to perform, however, and is not how a string of a musical instrument is excited. If the string is distorted such that its shape is not that of just one harmonic, the resulting vibration includes a combination of various harmonics. Such a distortion occurs in musical instruments when the string is plucked (as in a guitar), bowed (as in a cello), or struck (as in a piano). When the string is distorted into a nonsinusoidal shape, only waves that satisfy the boundary conditions can persist on the string. These waves are the harmonics.

The frequency of a string that defines the musical note that it plays is that of the fundamental. The string's frequency can be varied by changing either the string's tension or its length. For example, the tension in guitar and violin strings is varied by a screw adjustment mechanism or by tuning pegs located on the neck of the instrument. As the tension is increased, the frequency of the normal modes increases in accordance with Equation 14.7. Once the instrument is "tuned," players vary the frequency by moving their fingers along the neck, thereby changing the length of the oscillating portion of the string. As the length is shortened, the frequency increases because, as Equation 14.7 specifies, the normal-mode frequencies are inversely proportional to string length.

Imagine that we have several strings of the same length under the same tension but varying linear mass density μ . The strings will have different wave speeds and therefore different fundamental frequencies. The linear mass density can be changed either by varying the diameter of the string or by wrapping extra mass around the string. Both of these possibilities can be seen on the guitar, on which the higher-frequency strings vary in diameter and the lower-frequency strings have additional wire wrapped around them.

QUICK QUIZ 14.3 When a standing wave is set up on a string fixed at both ends, which of the following statements is true? (a) The number of nodes is equal to the number of antinodes. (b) The wavelength is equal to the length of the string divided by an integer. (c) The frequency is equal to the number of nodes times the fundamental frequency. (d) The shape of the string at any instant shows a symmetry about the midpoint of the string.

Example 14.3 | Give Me a C Note!

The middle C string on a piano has a fundamental frequency of 262 Hz, and the string for the first A above middle C has a fundamental frequency of 440 Hz.

(A) Calculate the frequencies of the next two harmonics of the C string.

SOLUTION

Conceptualize Remember that the harmonics of a vibrating string have frequencies that are related by integer multiples of the fundamental.

Categorize This first part of the example is a simple substitution problem.

Knowing that the fundamental frequency is $f_1 = 262$ Hz, find the frequencies of the next harmonics by multiplying by integers:

$$f_2 = 2f_1 = 524 \text{ Hz}$$

$$f_3 = 3f_1 = 786 \text{ Hz}$$

(B) If the A and C strings have the same linear mass density μ and length L , determine the ratio of tensions in the two strings.

SOLUTION

Categorize This part of the example is more of an analysis problem than is part (A).

Analyze Use Equation 14.8 to write expressions for the fundamental frequencies of the two strings:

$$f_{1A} = \frac{1}{2L} \sqrt{\frac{T_A}{\mu}} \quad \text{and} \quad f_{1C} = \frac{1}{2L} \sqrt{\frac{T_C}{\mu}}$$

Divide the first equation by the second and solve for the ratio of tensions:

$$\frac{f_{1A}}{f_{1C}} = \sqrt{\frac{T_A}{T_C}} \rightarrow \frac{T_A}{T_C} = \left(\frac{f_{1A}}{f_{1C}}\right)^2 = \left(\frac{440}{262}\right)^2 = 2.82$$

Finalize If the frequencies of piano strings were determined solely by tension, this result suggests that the ratio of tensions from the lowest string to the highest string on the piano would be enormous. Such large tensions would make it difficult to design a frame to support the strings. In reality, the frequencies of piano strings vary due to additional parameters, including the mass per unit length and the length of the string. The What If? below explores a variation in length.

What If? If you look inside a real piano, you'll see that the assumption made in part (B) is only partially true. The strings are not likely to have the same length. The string densities for the given notes might be equal, but suppose the length of the A string is only 64% of the length of the C string. What is the ratio of their tensions?

Answer Using Equation 14.8 again, we set up the ratio of frequencies:

$$\frac{f_{1A}}{f_{1C}} = \frac{L_C}{L_A} \sqrt{\frac{T_A}{T_C}} \rightarrow \frac{T_A}{T_C} = \left(\frac{L_A}{L_C}\right)^2 \left(\frac{f_{1A}}{f_{1C}}\right)^2$$

$$\frac{T_A}{T_C} = (0.64)^2 \left(\frac{440}{262}\right)^2 = 1.16$$

Notice that this result represents only a 16% increase in tension, compared with the 182% increase in part (B).

14.4 | Standing Waves in Air Columns

We have discussed musical instruments that use strings, which include guitars, violins, and pianos. What about instruments classified as brasses or woodwinds? These instruments produce music using a column of air. The waves under boundary conditions

model can be applied to sound waves in a column of air such as that inside an organ pipe or a clarinet. Standing waves are the result of interference between longitudinal sound waves traveling in opposite directions.

Whether a node or an antinode occurs at the end of an air column depends on whether that end is open or closed. The closed end of an air column is a **displacement node**, just as the fixed end of a vibrating string is a displacement node. Furthermore, because the pressure wave is 90° out of phase with the displacement wave (Section 13.6), the closed end of an air column corresponds to a **pressure antinode** (i.e., a point of maximum pressure variation). On the other hand, the open end of an air column is approximately a **displacement antinode** and a **pressure node**.

You may wonder how a sound wave can reflect from an open end because there may not appear to be a change in the medium at this point. It is indeed true that the medium through which the sound wave moves is air both inside and outside the pipe. Sound is a pressure wave, however, and a compression region of the sound wave is constrained by the sides of the pipe as long as the region is inside the pipe. As the compression region exits at the open end of the pipe, the constraint of the pipe is removed and the compressed air is free to expand into the atmosphere. Therefore, there is a change in the *character* of the medium between the inside of the pipe and the outside even though there is no change in the *material* of the medium. This change in character is sufficient to allow some reflection.¹

We can determine the modes of vibration of an air column by applying the appropriate boundary condition at the end of the column, along with the requirement that nodes and antinodes be separated by one fourth of a wavelength. We shall find that the frequency for sound waves in air columns is quantized, similar to the results found for waves on strings under boundary conditions.

The first three modes of vibration of a pipe that is open at both ends are shown in Figure 14.10a. Note that the ends are displacement antinodes (approximately). In

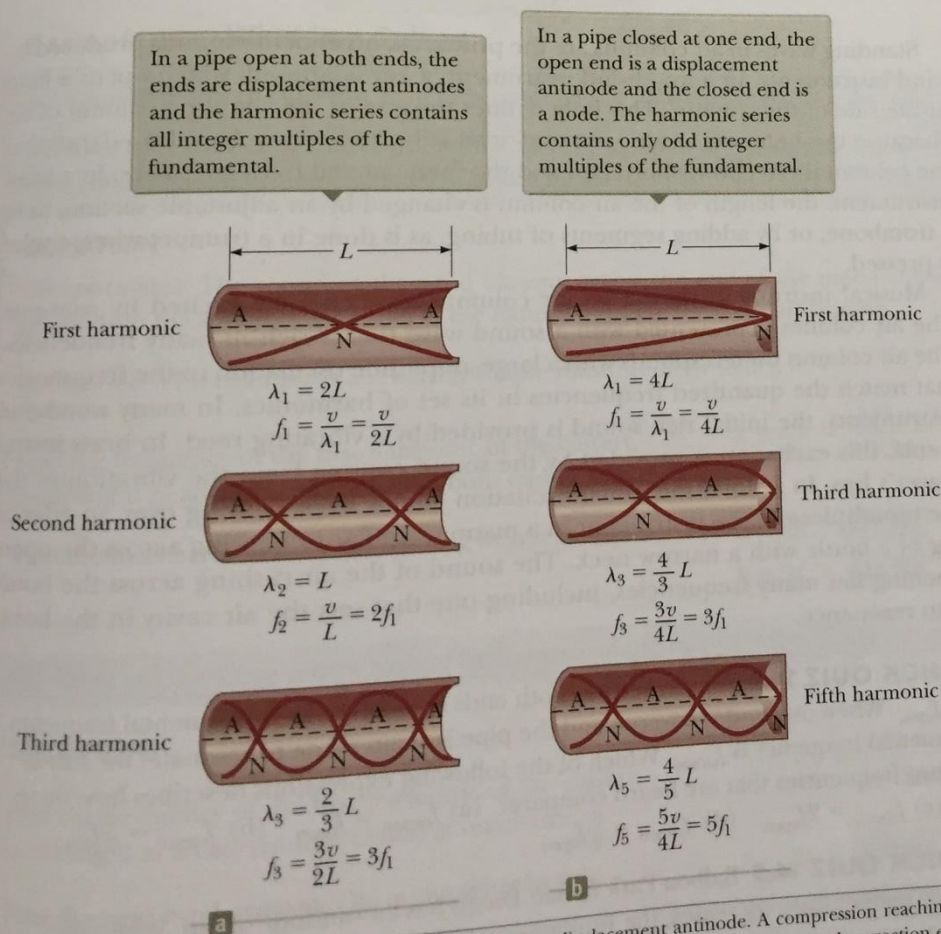


Figure 14.10 Graphical representations of the motion of elements of air in standing longitudinal waves in (a) a column open at both ends and (b) a column closed at one end.

Pitfall Prevention | 14.3 Sound Waves in Air Are Not Transverse

Note that the standing longitudinal waves are drawn as transverse waves in Figure 14.10. It is difficult to draw longitudinal displacements because they are in the same direction as the propagation. Therefore, it is best to interpret the curves in Figure 14.10 as a graphical representation of the waves (our diagrams of string waves are pictorial representations), with the vertical axis representing horizontal position of the elements of the medium.

¹Strictly speaking, the open end of an air column is not exactly a displacement antinode. A compression reaching an open end does not reflect until it passes beyond the end. For a tube of circular cross section, an end correction equal to approximately $0.6R$, where R is the tube's radius, must be added to the length of the air column. Hence, the effective length of the air column is longer than the true length L . We ignore this end correction in this discussion.

the fundamental mode, the standing wave extends between two adjacent antinodes, which is a distance of half a wavelength. Therefore, the wavelength is twice the length of the pipe, and the frequency of the fundamental f_1 is $v/2L$. As Figure 14.10a shows, the frequencies of the higher harmonics are $2f_1, 3f_1, \dots$. Therefore,

in a pipe open at both ends, the natural frequencies of vibration form a harmonic series that includes all integer multiples of the fundamental frequency.

Because all harmonics are present, we can express the natural frequencies of vibration as

$$f_n = n \frac{v}{2L} \quad n = 1, 2, 3, \dots$$

14.9

► Natural frequencies of a pipe open at both ends

where v is the speed of sound in air.

If a pipe is closed at one end and open at the other, the closed end is a displacement node and the open end is a displacement antinode (Fig. 14.10b). In this case, the wavelength for the fundamental mode is four times the length of the column. Hence, the fundamental frequency f_1 is equal to $v/4L$, and the frequencies of the higher harmonics are equal to $3f_1, 5f_1, \dots$. That is,

in a pipe that is closed at one end, the natural frequencies of oscillation form a harmonic series that includes only odd integer multiples of the fundamental frequency.

We express this result mathematically as

$$f_n = n \frac{v}{4L} \quad n = 1, 3, 5, \dots$$

14.10

► Natural frequencies of a pipe closed at one end and open at the other

Standing waves in air columns are the primary sources of the sounds produced by wind instruments. In a woodwind instrument, a key is pressed, which opens a hole in the side of the column. This hole defines the end of the vibrating column of air (because the hole acts as an open end at which pressure can be released), so that the column is effectively shortened and the fundamental frequency rises. In a brass instrument, the length of the air column is changed by an adjustable section, as in a trombone, or by adding segments of tubing, as is done in a trumpet when a valve is pressed.

Musical instruments based on air columns are generally excited by *resonance*. The air column is presented with a sound wave that is rich in many frequencies. The air column then responds with a large-amplitude oscillation to the frequencies that match the quantized frequencies in its set of harmonics. In many woodwind instruments, the initial rich sound is provided by a vibrating reed. In brass instruments, this excitation is provided by the sound coming from the vibration of the player's lips. In a flute, the initial excitation comes from blowing over an edge at the mouthpiece of the instrument in a manner similar to blowing across the opening of a bottle with a narrow neck. The sound of the air rushing across the bottle into resonance.

QUICK QUIZ 14.4 A pipe open at both ends resonates at a fundamental frequency f_{open} . When one end is covered and the pipe is again made to resonate, the fundamental frequency is f_{closed} . Which of the following expressions describes how these two frequencies that are heard compare? (a) $f_{\text{closed}} = f_{\text{open}}$ (b) $f_{\text{closed}} = \frac{1}{2}f_{\text{open}}$ (c) $f_{\text{closed}} = 2f_{\text{open}}$ (d) $f_{\text{closed}} = \frac{3}{2}f_{\text{open}}$

QUICK QUIZ 14.5 Balboa Park in San Diego has an outdoor organ. When the air temperature increases, the fundamental frequency of one of the organ pipes (a) stays the same, (b) goes down, (c) goes up, or (d) is impossible to determine.

THINKING PHYSICS 14.2

A bugle has no valves, keys, slides, or finger holes. How can it play a song?

Reasoning Songs for the bugle are limited to harmonics of the fundamental frequency because the bugle has no control over frequencies by means of valves, keys, slides, or finger holes. The player obtains different notes by changing the tension in the lips as the bugle is played to excite different harmonics. The normal

playing range of a bugle is among the third, fourth, fifth, and sixth harmonics of the fundamental. As examples, "Reveille" is played with just the three notes D (294 Hz), G (392 Hz), and B (490 Hz), and "Taps" is played with these same three notes and the D one octave above the lower D (588 Hz). Note that the frequencies of these four notes are, respectively, three, four, five, and six times the fundamental of 98 Hz. ◀

THINKING PHYSICS 14.3

If an orchestra doesn't warm up before a performance, the strings go flat and the wind instruments go sharp during the performance. Why?

Reasoning Without warming up, all the instruments will be at room temperature at the beginning of the concert. As the wind instruments are played, they fill with warm air from the player's exhalation. The increase in temperature of the air in the instrument causes an increase in the speed of sound, which raises

the fundamental frequencies of the air columns. As a result, the wind instruments go sharp. The strings on the stringed instruments also increase in temperature due to the friction of rubbing with the bow. This increase in temperature results in thermal expansion, which causes a decrease in the tension in the strings. (We will study thermal expansion in Chapter 16.) With a decrease in tension, the wave speed on the strings drops and the fundamental frequencies decrease. Therefore, the stringed instruments go flat. ◀

Example 14.4 | Wind in a Culvert

A section of drainage culvert 1.23 m in length makes a howling noise when the wind blows across its open ends.

(A) Determine the frequencies of the first three harmonics of the culvert if it is cylindrical in shape and open at both ends. Take $v = 343$ m/s as the speed of sound in air.

SOLUTION

Conceptualize The sound of the wind blowing across the end of the pipe contains many frequencies, and the culvert responds to the sound by vibrating at the natural frequencies of the air column.

Categorize This example is a relatively simple substitution problem.

Find the frequency of the first harmonic of the culvert, modeling it as an air column open at both ends:

$$f_1 = \frac{v}{2L} = \frac{343 \text{ m/s}}{2(1.23 \text{ m})} = 139 \text{ Hz}$$

Find the next harmonics by multiplying by integers:

$$f_2 = 2f_1 = 279 \text{ Hz}$$

$$f_3 = 3f_1 = 418 \text{ Hz}$$

(B) What are the three lowest natural frequencies of the culvert if it is blocked at one end?

SOLUTION

Find the frequency of the first harmonic of the culvert, modeling it as an air column closed at one end:

$$f_1 = \frac{v}{4L} = \frac{343 \text{ m/s}}{4(1.23 \text{ m})} = 69.7 \text{ Hz}$$

$$f_3 = 3f_1 = 209 \text{ Hz}$$

$$f_5 = 5f_1 = 349 \text{ Hz}$$

Find the next two harmonics by multiplying by odd integers: