# Counting number of distinct real roots of Pham system (Assignment 1)

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## 1 Problem

**Definition 1 (Pham system)** *A system $f \in \mathbb{C}[x_1, \ldots, x_n]^n$ is called a Pham system if it has the following form:*

$$f_1 = x_1^{d_1} + \sum_{e_1 + \cdots + e_n < d_1} a_{1,e} x_1^{e_1} \cdots x_n^{e_n}$$

$$\vdots$$

$$f_n = x_1^{d_n} + \sum_{e_1 + \cdots + e_n < d_n} a_{n,e} x_1^{e_1} \cdots x_n^{e_n}$$

***Remark** 2* *The pham system has exactly $m = d_1 \cdots d_n$ many complex roots (counting multiplicies). Thus we will study the number of real roots of a real pham system.*

**Problem 3** *Devise an algorithm with the following specification:*

**In:** $f \in \mathbb{R}[x_1, \ldots, x_n]^n$, *Pham system*

**Out:** $N$, *the number of distinct real roots of $f$*

## 2 Theory

**Definition 4 (Discriminant matrix)** *Let $f \in \mathbb{C}[x_1, \ldots, x_n]^n$ be a Pham system of degrees $d_1, \ldots, d_n$. Let $m = d_1 \cdots d_n$. Let $z_1, \ldots, z_m \in \mathbb{C}^n$ be the complex roots of $f$. The discriminant matrix $Q$ of $f$ is defined by*

$$Q = VV^t$$

*where*

$$V = \begin{bmatrix} \omega_1(z_1) & \cdots & \omega_1(z_m) \\ \vdots & & \vdots \\ \omega_m(z_1) & \cdots & \omega_m(z_m) \end{bmatrix}$$

*where*

$$\omega = \{x_1^{e_1} \cdots x_n^{e_n} : e_1 < d_1, \ldots, e_n < d_n\}$$

**Theorem 5 (Multivariate Hermite)** *# of distinct real roots of $f = \sigma(Q)$*

**Proof.**

$$\begin{aligned}
Q_{ij} &= \sum_{k=1}^{m} \omega_i(z_k)\omega_j(z_k) \\
Q_{ij} &= \sum_{p=1}^{s} \omega(\alpha_p)\mu_p\omega_j(\alpha_p) + \sum_{q=1}^{t} v_q(\omega_i(\beta_q)\omega_j(\beta_q)) + \sum_{q=1}^{t} v_q(\overline{\omega_i(\beta_q)}\,\overline{\omega_j(\beta_q)}) \\
Q_{ij} &= \sum_{p=1}^{s} \omega_i(\alpha_p)\mu_p\omega_j(\alpha_p) + \sum_{q=1}^{t} v_q(\omega_i(\beta_q)\omega_j(\beta_q)\overline{\omega_i(\beta_q)}\,\overline{\omega_j(\beta_q)}) \\
Q_{ij} &= \sum_{p=1}^{s} \omega_i(\alpha_p)\mu_p\omega_j(\alpha_p) + \sum_{q=1}^{t} v_q(\omega_i(\beta_q)\omega_j(\beta_q)\overline{\omega_i(\beta_q)\omega_j(\beta_q)}) \\
Q_{ij} &= \sum_{p=1}^{s} \omega_i(\alpha_p)\mu_p\omega_j(\alpha_p) + \sum_{q=1}^{t} 2v_q((\mathrm{Re}(\omega_i(\beta_q))\mathrm{Re}(\omega_j(\beta_q) - \mathrm{Im}(\omega_i(\beta_q))\mathrm{Im}(\omega_j(\beta_q))) \\
Q_{ij} &= \sum_{p=1}^{s} \omega_i(\alpha_p)\mu_p\omega_j(\alpha_p) + \sum_{q=1}^{t} \mathrm{Re}(\omega_i(\beta_q))(2v_q)\mathrm{Re}(\omega_j(\beta_q) + \sum_{q=1}^{t} \mathrm{Im}(\omega_i(\beta_q))(-2v_q)\mathrm{Im}(\omega_j(\beta_q)
\end{aligned} \tag{1}$$

$$D = \begin{bmatrix} \mu_1 & & & & & & & & \\ & \ddots & & & & & & & \\ & & \mu_s & & & & & & \\ & & & 2v_1 & & & & & \\ & & & & \ddots & & & & \\ & & & & & 2v_t & & & \\ & & & & & & -2v_1 & & \\ & & & & & & & \ddots & \\ & & & & & & & & -2v_t \end{bmatrix}$$

$$T = \begin{bmatrix} \omega_0(\alpha_1) & \cdots & \omega_0(\alpha_s) & \mathrm{Re}(\omega_0(\beta_1)) & \cdots & \mathrm{Re}(\omega_0(\beta_t)) & \mathrm{Im}(\omega_0(\beta_1)) & \cdots & \mathrm{Im}(\omega_0(\beta_t)) \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \omega_m(\alpha_1) & \cdots & \omega_m(\alpha_s) & \mathrm{Re}(\omega_m(\beta_1)) & \cdots & \mathrm{Re}(\omega_m(\beta_t)) & \mathrm{Im}(\omega_m(\beta_1)) & \cdots & \mathrm{Im}(\omega_m(\beta_t)) \end{bmatrix}$$

$\sigma(D) = (s + t) - t = s$ =number of distinct real roots
By Sly. Law of Inertia, $\sigma(Q) = \sigma(D)$, hence number of distinct real roots of $f = \sigma(D) = s$ ■

**Remark 6** *Hermite's theorem is "useless" because*

1. *Computation of $Q$ requires computing the roots of $f$.*
2. *Computation of $\sigma(Q)$ requires computing the roots of the characteristic polynomial of $Q$.*

*Hence, to make Hermite's theorem useful, we need to find ways to*

1. *Compute $Q$ **without** computing the roots of $f$.*
2. *Compute $\sigma(Q)$**without** computing the roots of the characteristic polynomial of $Q$.*

*In the following, we will tackle the challenges one by one.*

**Definition 7 (Mutipilcation matrix)** *Let $g \in \mathbb{C}[x_1, \ldots, x_n]$. Then the multiplication matrix $M_g$ for $g$ is defined by*

$$g \begin{bmatrix} \omega_1 \\ \vdots \\ \omega_m \end{bmatrix} \equiv_f M_g \begin{bmatrix} \omega_1 \\ \vdots \\ \omega_m \end{bmatrix}$$

**Theorem 8** *We have $Q_{ij} = trM_{\omega_i \omega_j}$.*

**Proof.**

1. Let $f(z) = 0$
   $$g(z) \begin{bmatrix} \omega_1(z) \\ \vdots \\ \omega_m(z) \end{bmatrix} = M_g \begin{bmatrix} \omega_1(z) \\ \vdots \\ \omega_m(z) \end{bmatrix}$$
   $$Q_{ij} = \sum_{k=1}^{m} \omega_i(z_k)\omega_j(z_k)$$
2. $$Q_{ij} = \sum_{k=1}^{m} (\omega_i\omega_j)(z_k)$$
   $$Q_{ij} = \sum_{k=1}^{m} g(z_k)$$
   $$Q_{ij} = trM_{\omega_i \omega_j}$$

■

**Theorem 9** *We have*

$$Q_{ij} = \sum_{\mu\nu} M_{i\mu\nu} M_{j\nu\mu}$$

*where*

$$\omega_i \omega_j \equiv_f \sum_k M_{ijk}\omega_k$$

**Proof.**

$$\begin{aligned} Q_{ij} &= \sum_{\mu} (M_{\omega_i \omega_j})_{\mu\mu} \\ &= \sum_{\mu} (M_{\omega_i} M_{\omega_j})_{\mu\mu} \end{aligned} \tag{2}$$

■

# 3 Algorithm

**Algorithm 10 (DiscriminantMatrix)**

**In:** $f \in \mathbb{R}[x_1, \ldots, x_n]^n$, *Pham system*

**Out:** $Q$, *the discriminant matrix of $f$*

1. *$\omega =$ Monomial basis of function $f$*
2. *$p =$ dimension of matrix*

3. $Q_{ij}$ = *trace of matrix* $M$
4. $Q_{ij} = trM_{\omega_i\omega_j}$
5. *Construct* $Q$ *with dimension mxm from* $Q_{ij}$
6. *Return* $Q$

**Remark 11** *Recall that Descartes' theorem is exact when all the roots are real.*

**Algorithm 12 (Signature)**

**In:** $M \in \mathbb{R}^{m\times m}$, *symmetric*

**Out:** $S = \sigma(M)$

1. $C = |\lambda I - M|$, *the characteristic polynomial of* $Q$
2. $m_+$ = *the sign variation count of the coefficients of* $C$
3. $m_0$ = *the least exponent in* $C$
4. $m_- = m - m_+ - m_0$
5. $S = m_+ - m_-$
6. *Return* $S$

**Algorithm 13 (NumberDistinctRealRoots)**

**In:** $f \in \mathbb{R}[x_1, \ldots, x_n]^n$, *Pham system*

**Out:** $N$, *the number of distinct real roots of* $f$

1. $Q = DiscriminantMatrix\,(f)$
2. $N = Signature\,(Q)$
3. *Return* $N$