MA 402: Project 4

Instructions:

- Detailed instructions regarding submission are available on the class website[1].
- The zip file should contain three files hw4.pdf, hw4.tex, classnotes.sty.

1) (10 points) The infamous RANDU generator was part of the Scientific Subroutine Package on IBM main-frame computers in the 1960s; the generator corresponds to

$$I_{n+1} \equiv (aI_n + c) \bmod m, \qquad n = 0, 1, \ldots$$

with $a = 65539$, $c = 0$ and $m = 2^{31}$.

(a) Show that

$$I_{n+2} - 6I_{n+1} + 9I_n \equiv 0 \bmod m.$$

Hint: Recall that $a \equiv b \bmod m$ means $a = km + b$ for some integer k. Another useful fact is that $a = 65539 = 2^{16} + 3$.

(b) Implement RANDU and verify graphically its severe lack of equi-distribution by creating a three dimensional plot of the $10,000$ points for some odd I_0 of your choice. More precisely, plot (in 3D)

$$\{(I_{n-1}, I_n, I_{n+1})/m\} \qquad n = 1, \ldots, 10,000.$$

(c) Plot $\{I_n/m\}$ for $n = 1, \ldots, 10000$ as a histogram with 30 bins.

2) (10 points) Certain Boeing 757 models are configured with 168 economy seats. Experience shows that only 90% of economy-class ticket holders actually show up to board the plane. If an airline sells 178 tickets, then what is the probability of overbooking, so that some passengers do not get the seat they paid for? Compute using

(a) the Binomial distribution;

(b) the normal approximation to the Binomial distribution.

You may use MATLAB/Python scripts to compute the necessary quantities.

3) (15 points) The county hospital is located at the center of a square whose sides are 3 miles wide. If an accident occurs within this square, then the hospital sends out an ambulance. The road network is rectangular, so the travel distance from the hospital, whose coordinates are $(0, 0)$, to the point (x_1, x_2) is $|x_1| + |x_2|$ (that is, the 1-norm or the Manhattan distance).

(a) (10 points) If an accident occurs at a point that is uniformly distributed in the square, find the expected travel time of the ambulance.

(b) (5 points) Compute this expectation using Monte Carlo integration. Report the results using $10, 100, 500$ samples.

[1] `https://github.ncsu.edu/asaibab/ma402/blob/master/project.md`

4) (15 points) Many people believe that the daily change of price of a companys stock on the stock market is a random variable with mean 0 and variance σ^2 . That is, if Y_n represents the price of the stock on the nth day, then

$$Y_n = Y_{n-1} + X_n \qquad n \geq 1$$

where $X_1, X_2, \ldots$ are independent and identically distributed random variables with mean 0 and variance σ^2. Suppose that the stock's price today is 100 and assume $\sigma^2 = 1$.

(a) Plot 20 different trajectories of the stock prices over the next 10 days.

(b) Using Monte Carlo, estimate the probability that the stocks price will exceed 105 after 10 days. Report the results using $10, 100, 500$ samples.

Hint: Let $X = Y_{10} = Y_0 + \sum_{i=1}^{10} X_i$. Use the fact that $P(X \geq 105) = E[I_{X \geq 105}]$, where $I_{X \geq 105}$ is an indicator random variable

$$I_{X \geq 105} = \begin{cases} 1 & X \geq 105 \\ 0 & \text{otherwise} \end{cases} .$$

(c) Compute this probability analytically. You can use the fact that if X_i's are independent normal random variables with mean μ and variance σ^2, then $\sum_{i=1}^{n} X_i$ is also a normal distribution with mean $n\mu$ and variance $n\sigma^2$.