MA 402: Project 5

Instructions:

- Detailed instructions regarding submission are available on the class website[1].
- The zip file should contain three files hw5.pdf, hw5.tex, classnotes.sty.

1) (20 points) Consider the function $f(x) = x$ in the interval $[0, 2\pi)$.

(a) Derive the Fourier coefficients c_k for $k = 0, \pm 1, \pm 2, \dots$.

$$\begin{aligned} c_0 &= \frac{1}{2\pi}\int_0^{2\pi} x\ dx \\ &= \frac{1}{2\pi}\left[\frac{x^2}{2}\right]\Bigg|_0^{2\pi} \\ &= \frac{1}{2\pi}\frac{4\pi^2 - 0}{2} \\ &= \pi \end{aligned}$$

$$\begin{aligned} c_k &= \frac{1}{2\pi}\int_0^{2\pi} xe^{-ikx}\ dx,\ k = \pm 1,\ \pm 2, ... \\ &= \frac{1}{2\pi}\left[\frac{xe^{-ikx}}{(-ik)} - \frac{e^{-ikx}}{(-ik)^2}\right]\Bigg|_0^{2\pi} \\ &= \frac{1}{2\pi}\left(\frac{2\pi}{-ik} + \frac{1}{k^2} - \frac{1}{k^2}\right) \\ &= \frac{1}{-ik} \\ &= \frac{i^4}{i^3 k} \\ &= \frac{i}{k} \end{aligned}$$

[1] https://github.ncsu.edu/asaibab/ma402/blob/master/project.md

(b) Derive the Fourier coefficients a_0, a_k, b_k for $k = 1, 2, \ldots,$.

$$\begin{aligned} a_0 &= c_0 \\ &= \pi \end{aligned}$$

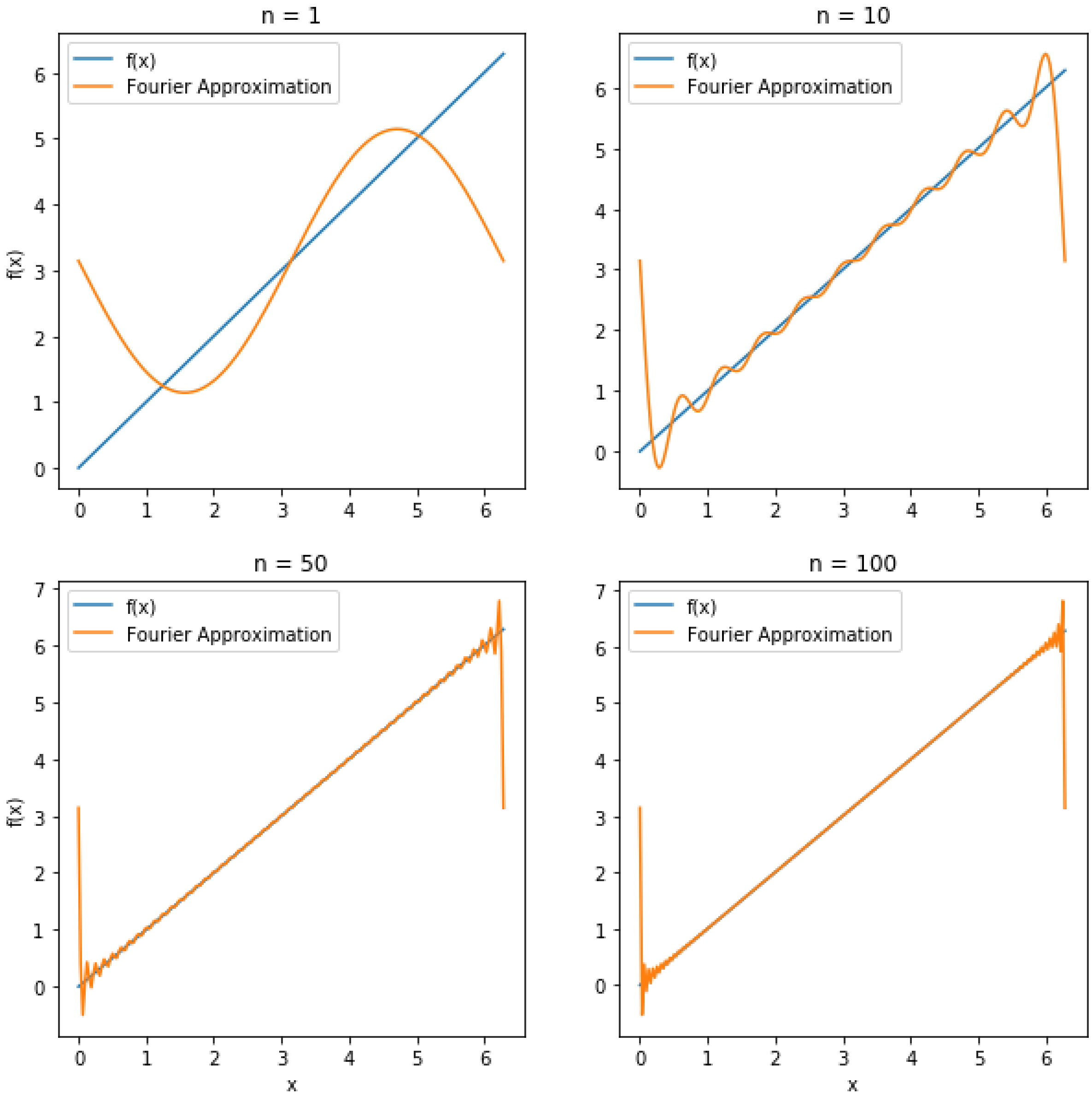
$$\begin{aligned} a_k &= c_k + c_{-k} \\ &= \frac{i}{k} + \frac{i}{-k} \\ &= 0 \end{aligned}$$

$$\begin{aligned} b_k &= i(c_k - c_{-k}) \\ &= i\left(\frac{i}{k} - \frac{i}{-k}\right) \\ &= i\frac{2i}{k} \\ &= \frac{-2}{k} \end{aligned}$$

(c) Plot the partial Fourier series, along with the function f, by retaining $n = 1, 10, 50, 100$ terms in the summation (use the second form involving cosines and sines).

```
import numpy as np
import matplotlib.pyplot as plt

def problem1c(x, k):
  a0 = np.pi
  f = a0
  for i in range(1,k+1):
    bi = -2/i
    f += bi*np.sin(i*x)
  return f

resolution = 200
x = np.linspace(0,2*np.pi, resolution)

nrange = (1,10,50,100)

_, ((ax1,ax2),(ax3,ax4)) = plt.subplots(2,2,figsize=(10,10))
axes = (ax1,ax2,ax3,ax4)

for i in range(4):
  k = nrange[i]
  ax = axes[i]
  ax.plot(x,x,label='f(x)')
  ax.plot(x,problem1c(x,k),label='Fourier Approximation')
  ax.set_title('n = %d' % k)
  ax.legend()

ax1.set_ylabel('f(x)')
ax3.set_ylabel('f(x)')
ax3.set_xlabel('x')
```

```
ax4.set_xlabel('x')
plt.show()
```



(d) Comment on the convergence of the partial Fourier series. As n goes to ∞ the approximation converges to $f(x)$. The approximation becomes acceptable around $n = 10$ and is almost equal to $f(x)$ when $n = 100$.

Note: you should submit only 1 plot for this problem.

2) (15 points) (Denoising a signal) Consider the function $f(x)$ defined as

$$f(x) = -\frac{1}{5}\left(\frac{x(2\pi - x)}{10}\right)^5 (x+1.5)(x+2.5)(x-4) + 1.7 \qquad x \in [0, 2\pi).$$

(a) Sample this function at 512 evenly spaced points to obtain sample values $f_0, \ldots, f_{511}$. Add noise to this image as $\tilde{f}_j = f_j + \epsilon r_j$ where $r_j \sim \text{Normal}(0,1)$ and $\epsilon = 10^{-1}$. Plot the sampled function values alongside the noisy function values.

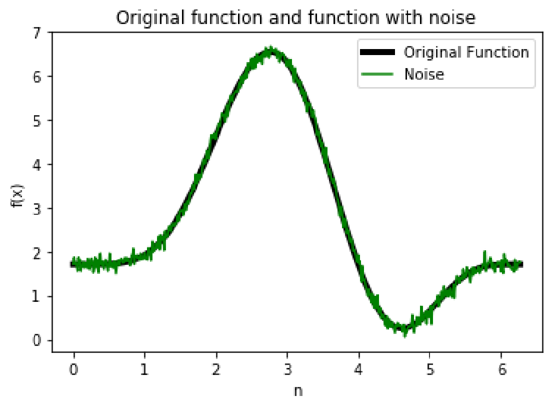
(b) Denoise the signal as follows: set all the Fourier coefficients c_k to be zero except for lowest 4 frequencies. Plot the denoised signal with the original function.

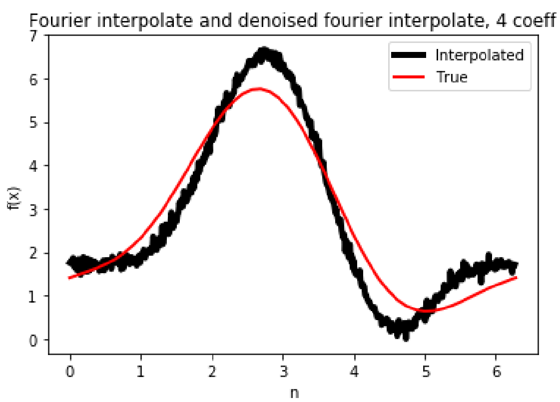
(c) Repeat the previous part, but this time keeping only the lowest 10 frequencies.

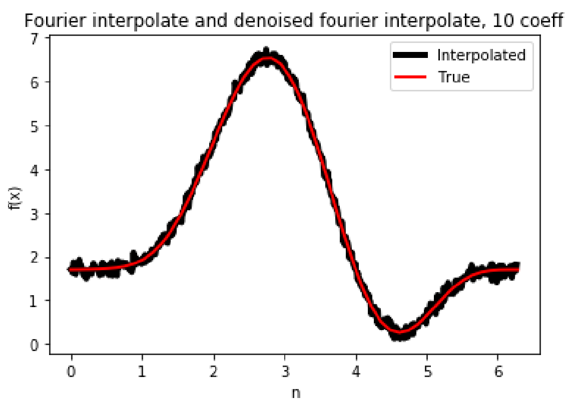
```
# Function
f = lambda x: (-1/5)*(((x * (2 * math.pi - x))/(10))**5) * (x + 1.5) * (x + 2.5) * (x -

# Number of points
n = 512
# Number of coeff
nSmall = 4

# Random noise
noise = np.random.normal(0,1,n)
noise = 0.1 * noise

# Set of evenly spaced points
xs = np.arange(n)*(2*np.pi/n)

# Function values
fs = f(xs) + noise

# Fourier stuff
fk = np.fft.fft(fs)/n
ck = np.fft.fftshift(fk)
k  = np.arange(-n/2,n/2)

# Getting smallest values and update ck
index = heapq.nsmallest(nSmall, enumerate(ck), key=lambda x: x[1])

ckNew = np.zeros(len(ck),dtype=complex)

#ckNew[252] = ck[252]
#ckNew[253] = ck[253]
ckNew[254] = ck[254]
ckNew[255] = ck[255]
ckNew[256] = ck[256]
ckNew[257] = ck[257]
#ckNew[258] = ck[258]
#ckNew[259] = ck[259]
#ckNew[260] = ck[260]
#ckNew[261] = ck[261]


# Fourier stuff
```

```
x = np.linspace(0,2*np.pi)
fint = 0*1j*x
for j in range(n):
  fint += ckNew[j]*np.exp(1j*k[j]*x)

# Plots
fig1, ax1 = plt.subplots()

ax1.plot(xs, f(xs), 'k', linewidth=4)
ax1.plot(xs, f(xs) + noise, 'g-')
ax1.legend({'Original Function', 'Noise'})
ax1.set_xlabel('n')
ax1.set_ylabel('f(x)')
ax1.set_title('Original function and function with noise')

fig2, ax2 = plt.subplots()

ax2.plot(xs, f(xs) + noise, 'k', linewidth = 4)
ax2.plot(x, np.real(fint), 'r', linewidth = 2)
ax2.legend({'True','Interpolated'})
ax2.set_xlabel('n')
ax2.set_ylabel('f(x)')
ax2.set_title('Fourier interpolate and denoised fourier interpolate, %s coeff' % nSmall)
```

Note: you should submit 3 plots for this problem.



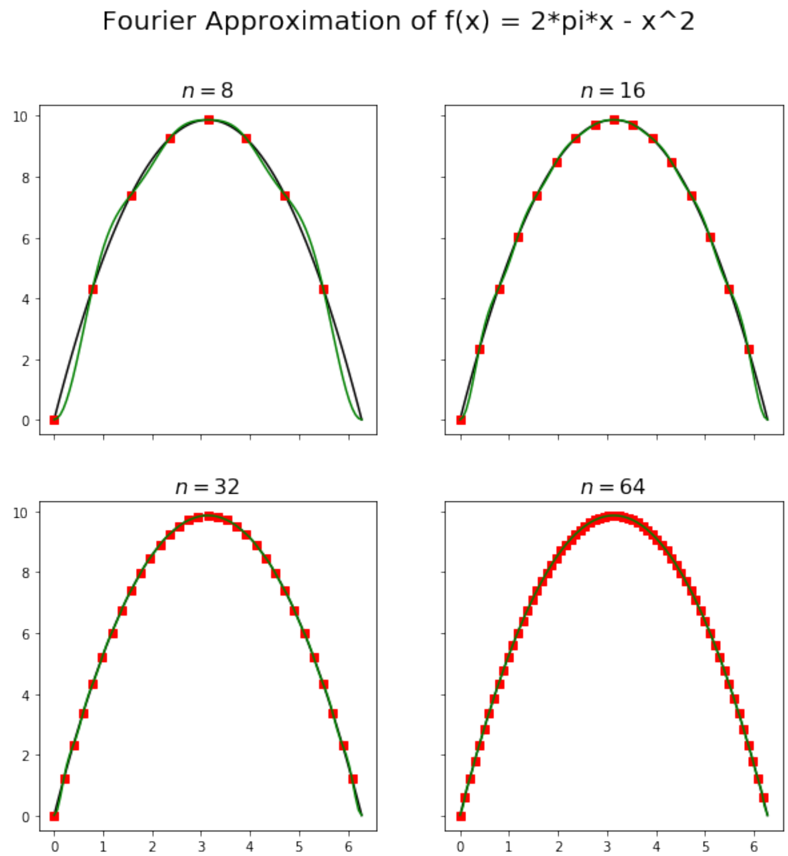




3) (15 points) Consider the function $f(x)$ defined as

$$f(x) = 2\pi x - x^2 \qquad x \in [0, 2\pi).$$

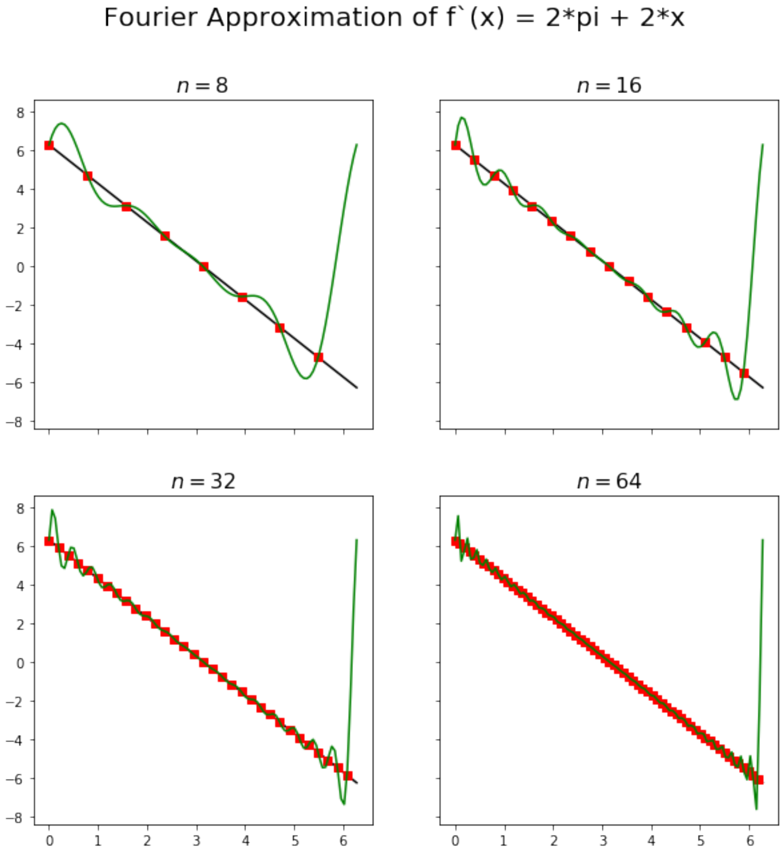
(a) Compute the Fourier interpolant $p(x) = \sum_{k=-n/2}^{n/2-1} c_k e^{ikx}$. Plot the interpolant with the original function for $n = 8, 16, 32, 64$ points.




```python
import numpy as np
import pylab as p


x = np.linspace(0, 2*np.pi, 100)

def f(x):
  y = 2*np.pi*x - x**2
  return y

fig, axarray = p.subplots(2,2, sharex = True, sharey = True, figsize = (10,10))
fig.suptitle('Fourier Approximation of f(x) = 2*pi*x - x^2', fontsize = 20)

nlst = [8, 16, 32, 64]
for n, ax in zip(nlst, axarray.flatten()):
  xs = np.arange(n)*(2*np.pi/n)

  fk = np.fft.fft(f(xs))/float(n)
  ck = np.fft.fftshift(fk)
```

```python
k  = np.arange(-n/2,n/2)

fint = 0*1j*x ;
for i in range(n):
    fint += ck[i]*np.exp(1j*k[i]*x)

ax.plot(x, f(x), 'k-')
ax.plot(xs, f(xs), 'rs')
ax.plot(x, np.real(fint), 'g-')
ax.set_title('$n = $' + str(n), fontsize = 16)
```

(b) Compute the derivative of the interpolant $p'(x)$. Plot this derivative against the true derivative for $n = 8, 16, 32, 64$ points.




```
import numpy as np
import pylab as p

x = np.linspace(0, 2*np.pi, 100)

def f(x):
  y = 2*np.pi - 2*x
  return y

fig, axarray = p.subplots(2,2, sharex = True, sharey = True, figsize = (10,10))
fig.suptitle('Fourier Approximation of f`(x) = 2*pi - 2*x', fontsize = 20)

nlst = [8, 16, 32, 64]
for n, ax in zip(nlst, axarray.flatten()):
  xs = np.arange(n)*(2*np.pi/n)

  fk = np.fft.fft(f(xs))/float(n)
  ck = np.fft.fftshift(fk)
  k  = np.arange(-n/2,n/2)

  fint = 0*1j*x ;
  for i in range(n):
      fint += ck[i]*np.exp(1j*k[i]*x)
```

```
ax.plot(x, f(x), 'k-')
ax.plot(xs, f(xs), 'rs')
ax.plot(x, np.real(fint), 'g-')
ax.set_title('$n = $' + str(n), fontsize = 16)
```

(c) Comment on the accuracy of the Fourier series in parts (a) and (b).

The approximation for part (a) works pretty well even at small n's due to the nature of the continuity of the f. In part (b) the approximation is a little weird due to the discontinuity of f'. They both have issues at the edges of the functions but that is to be expected.

Note: you should submit 2 plots for this problem.