1. Choose the true statement in the following four statements on square matrices ( ).

(A) Eigenvectors must be nonzero vectors.

(B) Each eigenvalue of A is also an eigenvalue of A^3

(C) Eigenvalues must be nonzero.

(D) Two eigenvectors corresponding to the same eigenvalue are always linearly dependent.

2. Let $A=\begin{bmatrix}4 & 6 & 0\\ -3 & -5 & 0\\ -3 & -6 & 1\end{bmatrix}$. Choose an eigenvector of A corresponding to 1 ( ).

(A) $\begin{bmatrix}4\\1\\3\end{bmatrix}$ (B) $\begin{bmatrix}2\\1\\2\end{bmatrix}$ (C) $\begin{bmatrix}-5\\3\\2\end{bmatrix}$ (D) $\begin{bmatrix}2\\-1\\3\end{bmatrix}$

3. Choose the false statement in the following four statements on similar matrices ( ).

(A) Similar matrices always have exactly the same eigenvalues.

(B) Similar matrices always have exactly the same eigenvectors.

(C) If A and B are invertible $n \times n$ matrices, then AB is similar to BA.

(D) Similar matrices always have exactly the same determinant.

4. Let $A=\begin{bmatrix}0 & 1 & 1 & 1\\ 0 & 0 & 1 & 1\\ 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0\end{bmatrix}$. The dimensions of $ColA, NulA, ColA^2, NulA^2$ are ( ).

(A) 3, 1 , 3, 1 (B) 1, 3, 1, 3 (C) 3, 1, 2, 2 (D) $3,1,2,2$

5. Let $A=\begin{bmatrix}1 & 1\\ 0 & 2\end{bmatrix}$. Choose a matrix similar to A in the following four matrices ( ).

(A) $\begin{bmatrix}-1 & 0\\ 0 & -2\end{bmatrix}$ (B) $\begin{bmatrix}1 & 1\\ 2 & 2\end{bmatrix}$ (C) $\begin{bmatrix}1 & 0\\ 0 & 2\end{bmatrix}$ (D) $\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}$

6. The matrix $\begin{bmatrix}1 & 1 & 1\\ 1 & 2 & 3\\ 1 & 3 & k\end{bmatrix}$ is not invertible. Then $k =$_______.

7. Let $A=\begin{bmatrix}3&3&3\\3&3&3\\3&3&3\end{bmatrix}$ and $u=\begin{bmatrix}4\\k\\4\end{bmatrix}$ be an eigenvector of A. Then $k=$______.

8. Let A be a matrix such that $A\alpha=4\alpha$, where α is an eigenvector of 4. Then $A^2\alpha=$______α, $2A^2\alpha=$______α, and $(A^2+2022I)\alpha=$______α.

9. Let the quadratic form $Q(x)=2x_1^2+8x_1x_2+x_2^2+6x_1x_3+10x_2x_3+2x_3^2$ and the symmetric matrix A satisfy $Q(x)=x^TAx$. Then $A=$______.

10. Let $u=\begin{bmatrix}1\\0\\2\end{bmatrix}$ and $v=\begin{bmatrix}5\\3\\2\end{bmatrix}$. The inner product $u\cdot v=$______, the length $\|2u\|=$

______,

and the distance $\|u-v\|=$______.

三. Let $A=\begin{bmatrix}1&2&3\\0&-1&2\\1&2&4\end{bmatrix}$.

(1) Compute $detA$. (2) Compute A^{-1}.

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四. Let $A=\begin{bmatrix}1&-3&4&9\\-2&6&-6&-10\\-3&9&-6&-3\\3&-9&4&0\end{bmatrix}$.

(1) Please give an Echelon form of A. 4 2 2

(2) Please find ~~bases~~ for the row space $RowA$, the column space $ColA$, the null space $NulA$. 2

(3) Please find dimensions of $RowA$, $ColA$, $NulA$.

$\begin{pmatrix}3\\1\\0\\1\end{pmatrix}\ \begin{pmatrix}7\\0\\-6\\1\end{pmatrix}$

五. Solve the following the system of linear equations:

$x_1+2x_2+3x_3=2$

$0x_1-x_2+2x_3=4$

$x_1+2x_2+4x_3=8$

$x_3=6$

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六. Let $P_2=\{f(t):f(t)=a_0+a_1t+a_2t^2\}$ be the set of all the polynomials of degree at most 2. The sum of two elements of P_2 is defined as the sum of two polynomials. The scalar multiple $cf(t)$ is defined as the multiplication of a real number c and a polynomial $f(t)$.

(1) Prove that $\{1+t,-t,t^2+1\}$ is a basis of P_2. (2) Find $a,b,c\in\mathbb{R}$ such that

$$t^2+2t+3=a(1+t)+b(-t)+c(t^2+1)=(1+t,-t,t^2+1)\begin{bmatrix}a\\b\\c\end{bmatrix}.$$

七. Let $\alpha_1, \alpha_2, \alpha_3$ be three vectors in a linear space V over R. Vectors $\alpha_1, \alpha_2, \alpha_3$ are linearly independent. Let $\beta_1 = 2\alpha_1 + 2\alpha_2$, $\beta_2 = \alpha_2 + \alpha_3$, and $\beta_3 = 2\alpha_1 + 2\alpha_2 + 2\alpha_3$. Determine if $\beta_1, \beta_2, \beta_3$ are linearly dependent or linearly independent.

八. Let $\beta_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}, \beta_2 = \begin{bmatrix}2\\1\\0\end{bmatrix}, \beta_3 = \begin{bmatrix}3\\3\\1\end{bmatrix}, \alpha_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}, \alpha_2 = \begin{bmatrix}1\\1\\0\end{bmatrix}, \quad \alpha_3 = \begin{bmatrix}1\\1\\1\end{bmatrix}.$

(1) Please verify that $\{\beta_1, \beta_2, \beta_3\}$ is a basis of R^3 and $\{\alpha_1, \alpha_2, \alpha_3\}$ is another basis of R^3.

(2) Please find a matrix P such that $(\beta_1, \beta_2, \beta_3) = (\alpha_1, \alpha_2, \alpha_3)P$.

(3) Please find a, b, c such that $u = \begin{bmatrix}1\\2\\4\end{bmatrix} = a\alpha_1 + b\alpha_2 + c\alpha_3 = (\alpha_1, \alpha_2, \alpha_3)\begin{bmatrix}a\\b\\c\end{bmatrix}.$

九. Let $A = \begin{bmatrix}2 & 0 & 0\\0 & 3 & 2\\0 & 2 & 3\end{bmatrix}.$

(1) Please compute $det(A - \lambda I)$ and give all the eigenvalues of A.

(2) Determine if A is positive definite or not.

(3) Please give three linearly independent eigenvectors of A.

Orthogonally diagonalize the matrix A. Please give the orthonormal matrix Q and a diagonal matrix D such that $A = QDQ^{-1}$.

十. Let $A = I - \frac{2}{\alpha^T\alpha}\alpha\alpha^T$ be a 3×3 matrix, where I is the identity matrix and $\alpha \in \mathbb{R}^3$.

(1) If $\alpha = \begin{bmatrix}1\\0\\1\end{bmatrix}$, compute $\alpha^T\alpha$, $\alpha\alpha^T$, A and verify that $A^TA = I$ (i.e., A is a orthonormal matrix).

For any $\alpha \in \mathbb{R}^3$ and $\alpha \neq \mathbf{0}$, prove that $A^TA = I$.