



数值分析

1th Edition, 2021 年 12 月 19 日

数值分析

目录

1	第一章	1
1.1	误差	1
1.2	数值运算的误差估计	1
1.3	稳定性	1
1.4	习题	2
2	第二章	5
2.1	拉格朗日差值	5
2.2	牛顿差值	5
2.3	差值余项	5
2.4	插商	5
2.5	埃尔米特差值	6
2.6	习题	6
3	第三章	9
3.1	勒让德多项式	9
3.2	最佳平方逼近	9
3.3	正交函数族做最佳平方逼近	10
3.4	最小二乘拟合	10
3.5	习题	12
4	第四章	15
4.1	代数精度	15
4.2	复合型求积公式	16
4.3	高斯型求积公式	17
4.4	高斯——勒让德求积公式	17
4.5	习题	18
5	第七章	21
5.1	不动点迭代	21
5.2	不动点迭代的收敛性	21
5.3	牛顿迭代	22

5.4	牛顿迭代对重根的改进	22
5.5	习题	22
6	第九章	25
6.1	习题	25

Chapter 1

第一章

1.1 误差

- (1) 绝对误差: x 为准确值, x^* 为近似值, $e^* = x - x^*$ 为绝对误差 (有正负之分)。
- (2) 误差限: $|e^*| = |x^* - x| \leq \varepsilon^*, \varepsilon^*$ 为误差限。
- (3) 相对误差: $e_r^* = \frac{e^*}{x} = \frac{x^* - x}{x}$
- (4) 相对误差限: $\varepsilon_r^* = \frac{\varepsilon^*}{|x^*|}$

1.2 数值运算的误差估计

A 为函数值, $A = f(x_1, x_2, \dots, x_n), A^* = f(x_1^*, x_2^*, \dots, x_n^*)$

- (1) 误差限: $\varepsilon(A^*) \approx \sum_{k=1}^n \left| \left(\frac{\partial f}{\partial x_k} \right)^* \right| \varepsilon(x_k^*) \dots$
- (2) 相对误差线: $\varepsilon_r^* = \varepsilon_r(A^*) = \frac{\varepsilon(A^*)}{|A^*|} \approx \sum_{k=1}^n \left| \left(\frac{\partial f}{\partial x_k} \right)^* \right| \frac{\varepsilon(x_k^*)}{|A^*|}$

1.3 稳定性

Definition 1: 数值稳定

一个算法如果输入数据有误差, 而在计算中舍入误差不增长, 则称此算法是数值稳定的, 否则称此算法为不稳定的。

即在算法中 $E_n \leq E_0$ 则算法稳定。

1.4 习题

Exercise 1: 第 8 页例四

以侧得某场地长 l 的值为 $l^*=110\text{m}$, 宽 d 的值为 $d^*=80\text{m}$, 已知 $|l-l^*| \leq 0.2\text{m}$, $|d-d^*| \leq 0.1\text{m}$ 。试求面积 $s=ld$ 的绝对误差限与相对误差限。

Solution:

$$\text{因 } s=ld, \quad \frac{\partial s}{\partial l} = d, \quad \frac{\partial s}{\partial d} = l$$

$$\varepsilon(s^*) \approx \left| \left(\frac{\partial s}{\partial l} \right)^* \right| \varepsilon(l^*) + \left| \left(\frac{\partial s}{\partial d} \right)^* \right| \varepsilon(d^*)$$

$$\left(\frac{\partial s}{\partial l} \right)^* = d^* = 80 \text{ m}, \quad \left(\frac{\partial s}{\partial d} \right)^* = l^* = 110 \text{ m}$$

$$\varepsilon(l^*) = 0.2 \text{ m}, \quad \varepsilon(d^*) = 0.1 \text{ m}$$

$$\text{绝对误差限 } \varepsilon(s^*) \approx 80 \times (0.2) + 110 \times (0.1) = 27 (\text{ m}^2)$$

$$\text{相对误差限 } \varepsilon_r(s^*) = \frac{\varepsilon(s^*)}{|s^*|} = \frac{\varepsilon(s^*)}{l^*d^*} \approx \frac{27}{8800} = 0.31\%$$

Exercise 2: 20 页习题 4

下列各数都是经过四舍五入得到的近似数, 即误差限不超过最后一位的半个单位,

$$x_1^* = 1.1021, x_2^* = 0.031, x_3^* = 385.6, x_4^* = 56.430$$

利用数值计算的误差估计求下列各近似值的误差限

- $x_1^* + x_2^* + x_4^*$
- $x_1^* x_2^* x_3^*$
- x_2^* / x_4^*

Solution:

$$(1) f(x_1, x_2, x_4) = x_1 + x_2 + x_4$$

$$\frac{\partial f}{\partial x_1} = 1 \quad \frac{\partial f}{\partial x_2} = 1 \quad \frac{\partial f}{\partial x_4} = 1$$

$$\varepsilon(x_1^*) = 5 \times 10^{-5} \quad \varepsilon(x_2^*) = 5 \times 10^{-4} \quad \varepsilon(x_4^*) = 5 \times 10^{-4}$$

$$\varepsilon(A^*) \approx 1 \times \varepsilon(x_1^*) + 1 \times \varepsilon(x_2^*) + 1 \times \varepsilon(x_4^*) = 1.05 \times 10^{-3}$$

$$(2) f(x_1, x_2, x_3) = x_1 x_2 x_3$$

$$\frac{\partial f}{\partial x_1} = x_2 x_3, \quad \frac{\partial f}{\partial x_2} = x_1 x_3, \quad \frac{\partial f}{\partial x_3} = x_1 x_2$$

$$\varepsilon(x_1^*) = 5 \times 10^{-5} \quad \varepsilon(x_2^*) = 5 \times 10^{-4} \quad \varepsilon(x_3^*) = 5 \times 10^{-2}$$

$$\left(\frac{\partial f}{\partial x_1} \right)^* = x_2^* x_3^* = 11.9536 \quad \left(\frac{\partial f}{\partial x_2} \right)^* = x_1^* x_3^* = 424.96976 \quad \left(\frac{\partial f}{\partial x_3} \right)^* = x_1^* x_2^* = 0.0341651$$

$$\varepsilon(A^*) \approx \sum_{k=1}^3 \left| \left(\frac{\partial f}{\partial x_k} \right)^* \right| \varepsilon(x_k^*) \approx 0.215$$

$$(3) f(x_2, x_4) = \frac{x_2}{x_4}$$

$$\frac{\partial f}{\partial x_2} = \frac{1}{x_4} \quad \frac{\partial f}{\partial x_4} = -\frac{x_2}{x_4^2}$$

$$\begin{aligned}\varepsilon(x_2^*) &= 5 \times 10^* & \varepsilon(x_4^*) &= 5 \times 10^{-4} \\ \left(\frac{\partial f}{\partial x_2}\right)^* &\approx 0.0177 & \left(\frac{\partial f}{\partial x_4}\right)^* &\approx 1 \times 10^{-5} \\ \varepsilon(A^*) &\approx \left(\frac{\partial f}{\partial x_2}\right)^* \varepsilon(x_2^*) + \left(\frac{\partial f}{\partial x_4}\right)^* \varepsilon(x_4^*) \leq 10^{-5}\end{aligned}$$

Exercise 3: 20 页习题 6

设 $Y_0=28$, 按递推公式

$$Y_n = Y_{n-1} - \frac{1}{100} \sqrt{783}, n = 1, 2, \dots$$

计算到 Y_{100} 。若取 $\sqrt{783} \approx 27.982$ (5 位有效数字), 试问计算 Y_{100} 将有多大误差?

Solution:

$$Y = \sqrt{783}, Y^* = 27.983$$

$$\delta = |Y - Y^*| \leq \frac{1}{2} \times 10^{-3}$$

$$Y_0 = 28 \quad Y_0^* = 28 \quad \delta_0 = |Y_0 - Y_0^*| = 0$$

$$|Y_1 - Y_1^*| = \left| 28 - \frac{1}{100} \cdot \sqrt{783} - 28 + \frac{1}{100} \times 27.483 \right| = \frac{1}{100} \delta$$

$$|Y_2 - Y_2^*| = \left| (Y_1 - Y_1^*) - \frac{1}{100} \sqrt{783} + \frac{1}{100} \times 27.983 \right| = \frac{2}{100} \delta$$

$$|Y_n - Y_n^*| = \frac{n}{100} \delta$$

$$\varepsilon^* = |Y_{100} - Y_{100}^*| = \frac{100}{100} \delta \leq \frac{1}{2} \times 10^{-3}$$

Exercise 4: 20 页习题 11

序列 y_n 满足递推关系

$$y_n = 10y_{n-1} - 1, n = 1, 2, \dots$$

若 $y_0 = \sqrt{2} \approx 1.41$ (3 位有效数字), 计算到 y_{10} 时误差有多大? 这个计算稳定吗?

Solution:

$$\delta = |y - y_0^*| \leq \frac{1}{2} \times 10^{-2}$$

$$|y_n - y_n^*| = |(10y_{n-1} - 1) - (10y_{n-1}^* - 1)|$$

$$= 10 |y_{n-1} - y_{n-1}^*|$$

$$= 10^n \delta > \delta \quad \text{不稳定}$$

$$|y_{10} - y_{10}^*| = 10^{10} \delta \leq \frac{1}{2} \times 10^8$$

Exercise 5: 20 页习题 13

$f(x) = \ln(x - \sqrt{x^2 - 1})$, 求 $f(30)$ 的值。若开平方用 6 位函数表, 问求对数时误差有多大? 若改用另一等价公式

$$\ln(x - \sqrt{x^2 - 1}) = -\ln(x + \sqrt{x^2 + 1})$$

计算, 求对数时误差有多大?

Solution:

来自高某人的修正:

$$\begin{aligned}
 u &= \sqrt{x^2 - 1}, f(u) = \ln(x - u) \\
 x &= 30, u = \sqrt{899}, u^* = 29.9833 \\
 \varepsilon(f(u^*)) &= |f'(u^*)| \varepsilon(u^*) = \left| \frac{1}{u^* - 30} \right| \times \frac{1}{2} \times 10^{-4} = 3 \times 10^{-3}
 \end{aligned}$$

$$\begin{aligned}
 u &= \sqrt{x^2 - 1}, f(u) = -\ln(x + u) \\
 x &= 30, u = \sqrt{899}, u^* = 29.9833 \\
 \varepsilon(f(u^*)) &= |f'(u^*)| \varepsilon(u^*) = \left| \frac{1}{u^* + 30} \right| \times \frac{1}{2} \times 10^{-4} = 8 \times 10^{-7}
 \end{aligned}$$

高江江的原稿:

$$\begin{aligned}
 y &= x - \sqrt{x^2 - 1} \quad f(x) = \ln y = g(y) \\
 x &= 30, \quad y = 30 - \sqrt{899}, \quad y^* = 30 - 29.9833 = 0.067 \\
 \varepsilon^* &= |g'(y^*)| \varepsilon(y^*) = \frac{|y - y^*|}{|y^*|} \approx 3 \times 10^{-3} \\
 \text{若使用等价公式} \\
 y &= x + \sqrt{x^2 - 1} \quad f(x) = -\ln y = g(y) \\
 x &= 30, \quad y = 30 + \sqrt{899}, \quad y^* = 59.9433 \\
 \varepsilon^* &= |g'(y^*)| \varepsilon(y^*) = \frac{|y - y^*|}{|y^*|} \approx 8 \times 10^{-7}
 \end{aligned}$$

Chapter 2

第二章

2.1 拉格朗日差值

$$l_k(x) = \frac{(x-x_0)\cdots(x-x_{k-1})(x-x_{k+1})\cdots(x-x_n)}{(x_k-x_0)\cdots(x_k-x_{k-1})(x_k-x_{k+1})\cdots(x_k-x_n)}$$

差值多项式为 $L_n(x) = \sum l_k(x)y_k$

如果令 $\omega_{n+1}(x) = (x-x_0)(x-x_1)\cdots(x-x_n)$

有 $\omega'_{n+1}(x_k) = (x_k-x_0)\cdots(x_k-x_{k-1})(x_k-x_{k+1})\cdots(x_k-x_n)$

$$L_n(x) = \sum_{k=0}^n y_k \frac{\omega_{n+1}(x)}{(x-x_k)\omega'_{n+1}(x_k)}$$

2.2 牛顿差值

$$P_n(x) = P_{n-1}(x) + a_n(x-x_0)\cdots(x-x_{n-1})$$

$$\text{其中 } a_n = \sum_{k=0}^n \frac{y_k}{(x-x_0)\cdots(x-x_{n-1})} = \sum_{k=0}^n \frac{y_k}{\omega_{n+1}(x_k)}$$

2.3 差值余项

$$R_n(x) = \frac{f^{(n+1)}(\eta)}{(n+1)!} \omega_{n+1}(x) \quad (\eta \in (a, b), \text{ 与 } x \text{ 无关})$$

2.4 插商

$$f[x_0, x_k] = \frac{f(x_k) - f(x_0)}{x_k - x_0}$$

$$f[x_0, x_1, x_k] = \frac{f[x_0, x_k] - f[x_0, x_1]}{x_k - x_1}$$

$$f[x_0, x_1, \dots, x_k] = \frac{f[x_0, \dots, x_{k-2}, x_k] - f[x_0, x_1, \dots, x_{k-1}]}{x_k - x_{k-1}}$$

$$= \sum \frac{f(x_k)}{\omega'_{n+1}(x_k)}$$

更一般的差商公式: $R_n(x) = f(x) - P_n(x) = f[x, x_0, \dots, x_n] \omega_{n+1}(x)$

一条重要的性质: $f[x_0, x_1, \dots, x_n] = \sum_{k=0}^n \frac{f(x_k)}{\omega_n(x_k)} = \frac{f^{(n)}(\xi)}{n!}$

Exercise 6: 32 页例 4

见课本, 很简单

2.5 埃尔米特差值

(1) 三点三次 (差值条件: 3 个点 + 中间点的导数)

$$P(x) = f(x_0) + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1) + A(x - x_0)(x - x_1)(x - x_2)$$

$$A = \frac{f'(x_1) - f[x_0, x_1] - (x_1 - x_0)f[x_0, x_1, x_2]}{(x_1 - x_0)(x_1 - x_2)}$$

$$\text{差值余项为 } R(x) = \frac{1}{4!} f^{(4)}(\eta) (x - x_0)(x - x_1)^2 (x - x_2)$$

(2) 两点三次 (差值条件: 2 个点 + 他俩的导数)

$$H_3 = \left(1 + 2 \frac{x - x_1}{x_2 - x_1}\right) \left(\frac{x - x_2}{x_1 - x_2}\right)^2 y_1 + \left(1 + 2 \frac{x - x_2}{x_1 - x_2}\right) \left(\frac{x - x_1}{x_2 - x_1}\right)^2 y_2 + (x - x_1) \left(\frac{x - x_2}{x_1 - x_2}\right)^2 y'_1 + (x - x_2) \left(\frac{x - x_1}{x_2 - x_1}\right)^2 y'_2$$

$$\text{差值余项为 } R(x) = \frac{1}{4!} f^{(4)}(\eta) (x - x_k)^2 (x - x_{k+1})^2$$

Exercise 7: 36 页例 6

给定 $f(x) = x^{\frac{3}{2}}$, $x_0 = \frac{1}{4}$, $x_1 = 1$, $x_2 = \frac{9}{4}$, 试求 $f(x)$ 在 $[\frac{1}{4}, \frac{9}{4}]$ 上的三次埃尔米特插值多项式 $P(x)$, 使它满足 $P(x_i) = f(x_i) (i = 0, 1, 2)$, $P'(x_1) = f'(x_1)$, 并写出余项表达式.

利用三个点的牛顿插值公式: $P = f(x_0) + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1) + A(x - x_0)(x - x_1)(x - x_2)$

x_i	f_i
$\frac{1}{4}$	$\frac{1}{8}$
1	1
$\frac{9}{4}$	$\frac{27}{8}$
	$\frac{19}{10}$
	$\frac{11}{30}$

由 $P'(x_1) = f'(x_1)$ 可解出 A

2.6 习题

Exercise 8: 49 页 12 题

若 $f(x) = a_0 + a_1x + \cdots + a_nx^n$ 有 n 个不同实根 x_1, x_2, \cdots, x_n , 证明:

$$\sum_{j=1}^n \frac{x_j^k}{f'(x_j)} = \begin{cases} 0, & 0 \leq k \leq n-2; \\ a_n^{-1}, & k = n-1. \end{cases}$$

Solution:

高某人的修改:

对 $f(x)$ 做代数变换 $f(x) = a_n(x - x_0) \cdots (x - x_{n-1}) = a_n\omega_n$

$$f'(x_j) = a_n\omega'_n(x_j)$$

$$\sum_{j=1}^n \frac{x_j^k}{f'(x_j)} = \sum_{j=1}^n \frac{x_j^k}{a_n\omega'_n(x_j)} = \frac{1}{a_n} \frac{(x_j^k)^{(n-1)}}{(n-1)!}$$

当 $0 \leq k \leq n-2$ 原式子为 0, 当 $k = n-1$ 时, 原式为 $\frac{1}{a_n}$

高江江的原稿:

$$f(x) = a_n(x - x_1)(x - x_2) \cdots (x - x_n)$$

对 $g(x) = x^k$ 上的 n 个节点差值, 插值节点为 $(x_1, x_1^k), (x_2, x_2^k) \cdots (x_n, x_n^k)$

$$\text{令 } b_i = a_n \sum_{j=1}^i \frac{x_j^k}{f'(x_j)} = \sum_{j=1}^i \frac{g(x_j)}{\omega'_n(x_j)}$$

有牛顿差值函数 $P_{n-1}(x) = b_1 + b_2(x - x_1) + \cdots + b_n(x - x_1)(x - x_2) \cdots (x - x_{n-1})$

因多项式差值对多项式函数的差值余项为 0, 即 $P_{n-1}(x) = g(x)$

当 $k \leq n-2$ 时, $g(x)$ 无 $n-1$ 次项, $P_{n-1}(x)$ 的 $n-1$ 次项系数 $b_n = 0$

$$\text{即 } \sum_{j=1}^i \frac{x_j^k}{f'(x_j)} = 0$$

当 $k = n-1$ 时, $g(x)$ 的 $n-1$ 次项系数为 1, $P_{n-1}(x)$ 的 $n-1$ 次项系数 $b_n = 1$

$$\text{即 } \sum_{j=1}^i \frac{x_j^k}{f'(x_j)} = \frac{1}{a_n}$$

Exercise 9: 49 页 13 题

求次数小于等于 3 的多项式 $P(x)$, 使满足条件

$$P(x_0) = f(x_0), P'(x_0) = f'(x_0), P''(x_0) = f''(x_0), P(x_1) = f(x_1)$$

Solution:

高某人的修改:

由牛顿插值会有: $P = f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2}f''(x_0)(x - x_0)^2 + A(x - x_0)^3$

再利用 $P(x_1) = f(x_1)$ 得出 A 的取值

Exercise 10: 49 页 14 题

求次数小于等于 3 的多项式 $P(x)$, 使满足条件

$$P(0) = 0, P'(0) = 1, P(1) = 1, P'(1) = 2$$

Solution:

直接带公式

Exercise 11: 49 页 16 题

求次数小于等于 4 的多项式 $P(x)$ ，使满足条件

$$P(0) = P'(0) = 0, P(1) = P'(1) = 1, P(2) = 1$$

Solution:

满足 $P(0) = P'(0) = 0, P(1) = P'(1) = 1$ 的埃尔米特差值可以直接带公式

$P(x) = H(x) + Ax^2(x-1)^2$ 满足 $P(2) = 1$ 解出 A 。

高某人的猜想: 用 $P(0) = 0, P(1) = 1, P'(1) = 1, P(2) = 1$ 的埃尔米特差值也可以求出来.

Chapter 3

第三章

3.1 勒让德多项式

$$P_0(x) = 1, P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n$$

$$P_1(x) = x$$

$$P_2(x) = \frac{1}{2}(3x^2 - 1)$$

$$P_3(x) = \frac{5}{2}x^3 - \frac{3}{2}x$$

3.2 最佳平方逼近

逼近函数 $f(x) \in [0, 1]$ $d_i = \int_0^1 x^i f(x) dx \quad i = 0, 1, 2, \dots$

$$H = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \dots & \frac{1}{n+1} \\ \frac{1}{2} & \frac{1}{3} & \dots & \dots & \frac{1}{n+2} \\ \frac{1}{3} & \dots & \dots & \dots & \dots \\ \frac{1}{n+1} & \frac{1}{n+2} & \dots & \dots & \frac{1}{2n+1} \end{pmatrix}$$

$$D = (d_0, d_1, \dots, d_n)^\top$$

$$A = (a_0, a_1, \dots, a_n)^\top$$

$$HA = D \text{ 解出 } A,$$

逼近多项式为: $S_n^*(x) = a_0 + a_1x + a_2x^2 + \dots$

Exercise 12: 68 页例 6

设 $f(x) = \sqrt{1+x^2}$ 求 $[0,1]$ 上的一次最佳平方逼近多项式

Solution:

$$(f(x), \varphi_0(x)) = \int_0^1 1 \cdot \sqrt{1+x^2} dx = 1.147$$

$$(f(x), \varphi_1(x)) = \int_0^1 x \cdot \sqrt{1+x^2} dx = 0.609$$

$$\begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} = \begin{pmatrix} 1.147 \\ 0.609 \end{pmatrix}$$

$$S_1^* = 0.934 + 0.426x$$

3.3 正交函数族做最佳平方逼近

$$f(x) \in [-1, 1]$$

$$S_n^*(x) = a_0^* P_0(x) + a_1^* P_1(x) + \cdots + a_n^* P_n(x)$$

$$a_n^* = \frac{k+1}{2} \int_{-1}^1 f(x) P_k(x) dx$$

其中 $P(x)$ 为勒让德多项式对于区间不在 $[-1, 1]$ 的做变换 $x = \frac{b-a}{2}t + \frac{a+b}{2}$

Exercise 13: 71 页例 7

求 $f(x) = e^x$ 在 $[-1, 1]$ 上的三次最佳平方逼近多项式

Solution:

$$(f(x), P_0(x)) = \int_0^1 e^x dx = 2.3504$$

$$(f(x), P_1(x)) = \int_0^1 x e^x dx = 2.3504$$

$$(f(x), P_2(x)) = \int_0^1 \left(\frac{3}{2}x^2 - \frac{1}{2}\right) e^x dx = 2.3504$$

$$(f(x), P_3(x)) = \int_0^1 \left(\frac{5}{2}x^3 - \frac{3}{2}x\right) e^x dx = 2.3504$$

解出

$$a_0^* = 1.1752$$

$$a_1^* = 1.1036$$

$$a_2^* = 0.3578$$

$$a_3^* = 0.07046$$

$$S_3^*(x) = 0.9963 + 0.9976x + 0.5367x^2 + 0.1761x^3$$

3.4 最小二乘拟合

m 为数据点个数减一，n 为拟合曲线次数。一般 $\varphi_k = x^k, k = 0, 1, 2, \dots, n$

$$(\varphi_j, \varphi_k) = \sum_{i=0}^m \omega(x_i) \varphi_j(x_i) \varphi_k(x_i)$$

$$(f, \varphi_k) = \sum_{i=0}^m \omega(x_i) f(x_i) \varphi_k(x_i) = d_k$$

$$A = (a_0, a_1, \dots, a_n)^\top, D = (d_0, d_1, \dots, d_n)^\top$$

$$G = \begin{pmatrix} (\varphi_0, \varphi_0) & (\varphi_0, \varphi_1) \cdots & (\varphi_0, \varphi_n) \\ (\varphi_1, \varphi_1) & (\varphi_1, \varphi_1) \cdots & (\varphi_1, \varphi_n) \\ \vdots & & \vdots \\ (\varphi_n, \varphi_0) & (\varphi_n, \varphi_1) & \cdots (\varphi_n, \varphi_n) \end{pmatrix}$$

$GA = D$ 解出 A , 最小二乘多项式为:

$$S^*(x) = a_0 + a_1x + \cdots + a_nx^n$$

Exercise 14: 68 页例 6

已知一组实验数据如表 3-1 求它的拟合曲线.(用 $S_1(x) = a_0 + a_1x$)

x_i	1	2	3	4	5
f_i	4	4.5	6	8	8.5
ω_i	2	1	3	1	1

Solution:

$$(\varphi_0, \varphi_0) = \sum_{i=0}^4 \omega_i = 8$$

$$(\varphi_0, \varphi_1) = (\varphi_1, \varphi_0) = \sum_{i=0}^4 \omega_i x_i = 22$$

$$(\varphi_1, \varphi_1) = \sum_{i=0}^4 \omega_i x_i^2 = 74$$

$$(\varphi_0, f) = \sum_{i=0}^4 \omega_i f_i = 47$$

$$(\varphi_1, f) = \sum_{i=0}^4 \omega_i x_i f_i = 145.5$$

$$\begin{pmatrix} 8 & 22 \\ 22 & 74 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} = \begin{pmatrix} 47 \\ 145.5 \end{pmatrix}$$

$$S_1^* = 2.5648 + 1.2037x$$

3.5 习题

Exercise 15: 94 页习题 12

设 $f(x) = x^3 + 3x + 2, x \in [0, 1]$, 试求 $f(x)$ 在 $[0, 1]$ 上关于 $\rho(x) = 1, \phi = \text{span}\{1, x\}$ 的最佳平方逼近多项式。若取 $\phi = \text{span}\{1, x, x^2\}$, 那么最佳平方逼近多项式是什么?

Solution:

当 $\phi = \text{span}\{1, x\}$ 时, $n=1$

$$d_0 = \int_0^1 x^2 + 3x + 2 dx = \frac{1}{3} + \frac{3}{2} + 2 = \frac{23}{6}$$

$$d_1 = \int_0^1 x(x^2 + 3x + 2) dx = \frac{1}{4} + 1 + 1 = \frac{9}{4}$$

$$H = \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{3} \end{pmatrix} \quad A = \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} \quad D = \begin{pmatrix} \frac{23}{6} \\ \frac{9}{4} \end{pmatrix}$$

$$HA = D \Rightarrow \begin{cases} a_0 + \frac{1}{2}a_1 = \frac{23}{6} \\ \frac{1}{2}a_0 + \frac{1}{3}a_1 = \frac{9}{4} \end{cases} \Rightarrow A = \begin{pmatrix} \frac{11}{6} \\ 4 \end{pmatrix}$$

$$S_n = \frac{11}{6} + 4x$$

当 $\phi = \text{span}\{1, x, x^2\}$ 时, $n=2$

$$d_2 = \int_0^1 x^2(x^2 + 3x + 2) dx = \frac{1}{5} + \frac{3}{4} + \frac{2}{3} = \frac{97}{60}$$

$$H = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \end{pmatrix} \quad A = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} \quad D = \begin{pmatrix} \frac{23}{6} \\ \frac{9}{4} \\ \frac{97}{60} \end{pmatrix}$$

$$HA = D \Rightarrow A = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$$

$S_n = 2 + 3x + x^2 = f(x)$ 从而最佳平方逼近为二次。

Exercise 16: 94 页练习题 13

求 $f(x) = x^3 \in [0, 1]$ 上关于 $\rho(x) = 1$ 的最佳平方逼近二次多项式。

Solution:

$$d_0 = \int_0^1 x^3 dx = \frac{1}{4}$$

$$d_1 = \int_0^1 x^4 dx = \frac{1}{5}$$

$$d_2 = \int_0^1 x^5 dx = \frac{1}{6}$$

$$H = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \end{pmatrix} \quad D = \begin{pmatrix} \frac{1}{4} \\ \frac{1}{5} \\ \frac{1}{6} \end{pmatrix} \quad A = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix}$$

$$HA = D \quad \text{解得} \quad A = \begin{pmatrix} 0.05 \\ -0.6 \\ 1.5 \end{pmatrix}$$

$$S_2 = 0.05 - 0.6x + 1.5x^2$$

Exercise 17: 94 页习题 14

求指数 $f(x)$ 在指定区间上对于 $\phi = \text{span}\{1, x\}$ 的最佳平方逼近:

- $f(x) = \frac{1}{x}, [1, 3]$
- $f(x) = e^x, [0, 1]$
- $f(x) = \cos \pi x, [0, 1]$
- $f(x) = \ln x, [1, 2]$

Solution:

参考上题自行解答

Exercise 18: 95 页习题 17

已知实验数据如下:

x_i	19	25	31	38	44
y_i	19.0	32.3	49.0	73.3	97.8

Solution:

$$\varphi_0 = 1, \quad \varphi_1 = x^2, \quad \omega(x) = 1$$

$$(\varphi_0, \varphi_0) = \sum_{i=0}^4 \omega(x_i) = 5$$

$$(\varphi_0, \varphi_1) = \sum_{i=0}^4 \omega(x_i) x_i^2 = 5327$$

$$(\varphi_1, \varphi_1) = \sum_{i=0}^4 \omega(x_i) x_i^4 = 7277699$$

$$(\varphi_0, y) = \sum_{i=0}^4 y_i = 271.4$$

$$(\varphi_1, y) = \sum_{i=0}^4 y_i x_i^2 = 369321.5$$

$$G = \begin{pmatrix} 5 & 5327 \\ 5327 & 7277699 \end{pmatrix} \quad A = \begin{pmatrix} a \\ b \end{pmatrix} \quad D = \begin{pmatrix} 271.4 \\ 369321.5 \end{pmatrix}$$

$$GA = D \Rightarrow A \approx \begin{pmatrix} 0.97 \\ 0.05 \end{pmatrix} \quad y = 0.97 + 0.05x^2$$

$$\delta = \sum_{i=0}^4 (y(x_i) - y_i)^2 \approx 0.1226$$

Chapter 4

第四章

4.1 代数精度

定义：如果求积公式

$$I(f) = \int_a^b f(x)dx \approx \sum A_k f(x_k)$$

对于次数不超过 m 的多项式均能准确地成立，但对于 $m+1$ 次多项式不准确成立，则称该求积公式具有 m 次代数精度。

有 m 阶代数精度满足：

$$\begin{cases} \sum A_k = b - a \\ \sum A_k x_k = \frac{1}{2}(b^2 - a^2) \\ \dots \\ \sum A_k x_k^m = \frac{1}{m+1}(b^{m+1} - a^{m+1}) \end{cases}$$

代数精度的余项：

$$R[f] = \int_a^b f(x)dx - \sum_{k=0}^n A_k f(x_k) = K f^{(m+1)}(\eta)$$

$$\begin{aligned} K &= \frac{1}{(m+1)!} \left[\int_a^b x^{m+1} dx - \sum_{k=0}^n A_k x_k^{m+1} \right] \\ &= \frac{1}{(m+1)!} \left[\frac{1}{m+2} (b^{m+2} - a^{m+2}) - \sum_{k=0}^n A_k x_k^{m+1} \right] \end{aligned}$$

Exercise 19: 100 页例 1

给定形如 $\int_0^1 f(x)dx = A_0 f(0) + A_1 f(1) + B_0 f'(0)$ 的求积公式，试确定系数 A_0, A_1, B_0 ，使得公式有尽可能高的代数精度

Solution: 当 $f(x) = 1$ 时, 得:

$$A_0 + A_1 = \int_0^1 1 dx = 1$$

当 $f(x) = x$ 时, 得:

$$A_1 + B_0 = \int_0^1 x dx = \frac{1}{2}$$

当 $f(x) = x^2$ 时, 得:

$$A_1 = \int_0^1 x^2 dx = \frac{1}{3}$$

解得: $A_1 = \frac{1}{3}, A_0 = \frac{2}{3}, B_0 = \frac{1}{6}$

Exercise 20: 102 页例 2

求例 1 中求积公式的余项

Solution: 代数精度为 2, 故 $R[f] = Kf'''(\eta), f(x) = x^3, f'''(\eta) = 3!$

$$K = \frac{1}{3!} [\int_0^1 x^3 dx - (\frac{2}{3}f(0) + \frac{1}{3}f(1) + \frac{1}{6}f'(0))] = -\frac{1}{72}$$

4.2 复合型求积公式

$$T_n = \frac{h}{2} [f(a) + 2 \sum_{k=1}^{n-1} f(x_k) + f(b)] \quad x_n = a + \frac{h}{n}k$$

$$\text{求积余项 } R_n(f) = \sum_{k=0}^{n-1} \left[-\frac{h^3}{12} f''(\eta_k) \right] \quad x_k = a + \frac{h}{n}k$$

$$S_n = \frac{h}{b} \left[f(a) + 4 \sum_{k=0}^{n-1} f\left(x_{k+\frac{1}{2}}\right) + 2 \sum_{k=1}^{n-1} f(x_k) + f(b) \right]$$

$$\text{求积余项 } R_n(f) = -\frac{b-a}{180} \left(\frac{h}{2}\right)^4 f^{(4)}(\eta)$$

Exercise 21: 109 页例 4

计算积分 $I = \int_0^1 e^x dx$, 若用复合梯形公式, 问区间 $[0,1]$ 应分多少等份才能使误差不超过 $\frac{1}{2} \times 10^{-5}$, 若改用复合 Simpson 公式, 要达到同样的精度, 区间 $[0,1]$ 应分为多少份?

Solution: 对于梯形公式

$$|R[f]| = \left| -\frac{b-a}{12} h^2 f''(\xi) \right| \leq \frac{1}{12} \left(\frac{1}{n}\right)^2 e \leq \frac{1}{2} \times 10^{-5}$$

得出 $n = 213$

对于 Simpson 公式

$$|R_n(f)| = \frac{b-a}{2880} h^4 |f^{(4)}(\xi)| \leq \frac{1}{2880} \left(\frac{1}{n}\right)^4 e \leq \frac{1}{2} \times 10^{-5}$$

得出 $n = 4$

4.3 高斯型求积公式

求积公式

$$\int_a^b f(x)\rho(x)dx \approx \sum_{k=0}^n A_k f(x_k)$$

有 $2n+1$ 次代数精度, 则称节点 x_k 为高斯点, 求积公式成为高斯求积公式。以这些点为零点的多项式

$$\omega_{n+1}(x) = (x - x_0)(x - x_1) \cdots (x - x_n)$$

与 $\rho(x)$ 正交。

$\omega_{n+1}(x)$ 的零点即为高斯求积公式的节点, 系数为

$$A_k = \int_a^b \rho(x) \frac{\omega_{n+1}^2(x)}{(x - x_k)^2 \omega_{n+1}'^2(x_k)} dx$$

Exercise 22: 120 页例 9

确定求积公式 $\int_0^1 \sqrt{x} f(x) dx = A_0 f(x_0) + A_1 f(x_1)$ 的系数 A_0, A_1 及节点 x_0, x_1 , 使得它具有最高代数精度

Solution: 求积节点为权函数的正交多项式零点, 设正交多项式为 $\omega(x) = (x - x_0)(x - x_1) = x^2 + bx + c$

该多项式与 $1, x$ 带权正交, 因此得到两个式子:

$$\int_0^1 \sqrt{x} \omega(x) dx = 0, \int_0^1 \sqrt{x} x \omega(x) dx = 0$$

$$\text{得到 } \omega(x) = x^2 - \frac{10}{9}x + \frac{5}{21}$$

$$\text{令 } \omega(x) = 0, x_0 = 0.289949, x_1 = 0.821162$$

具有三次精度

当 $f(x) = 1$ 时, 得:

$$A_0 + A_1 = \int_0^1 \sqrt{x} dx = \frac{2}{3}$$

当 $f(x) = x$ 时, 得:

$$A_0 x_0 + A_1 x_1 = \int_0^1 \sqrt{x} x dx = \frac{2}{5}$$

由此解出 $A_0 = 0.277556, A_1 = 0.389111$

4.4 高斯——勒让德求积公式

求积公式

$$\int_{-1}^1 f(x) dx \approx \sum_{k=0}^n f(x_k)$$

其中 x_k 为勒让德多项式的零点 (见 3.1), A_k 可由代数精度解出。

Exercise 23: 123 页例 10

用 4 点 ($n=3$) 的高斯-勒让德求积公式计算 $I = \int_0^{\frac{\pi}{2}} x^2 \cos x dx$

Exercise 23: 123 页例 10

Solution: 先将区间 $[0, \frac{\pi}{2}]$ 化为 $[-1, 1]$, $I = \int_{-1}^1 (\frac{\pi}{4})^3 (1+t)^2 \cos \frac{\pi}{4} (1+t) dt$

查表可得 $I = \sum_{k=0}^3 A_k f(x_k) = 0.467402$

4.5 习题

Exercise 24: 135 页习题 1

确定下列求积公式中的待定参数, 使其代数精度尽量高, 并指明所构造出的求积公式所具有的代数精度。

- $\int_{-h}^h f(x) dx \approx A_{-1}f(-h) + A_0f(0) + A_1f(h);$
- $\int_{-2h}^{2h} f(x) dx \approx A_{-1}f(-h) + A_0f(0) + A_1f(h);$
- $\int_{-1}^1 f(x) dx \approx [f(-1) + 2f(x_1) + 3f(x_2)] / 3;$
- $\int_0^h f(x) dx \approx h[f(0) + f(h)] / 2 + ah^2[f'(0) - f'(h)].$

Solution:

$$(1) \begin{cases} A_{-1} + A_0 + A_1 = 2h \\ -hA_{-1} + hA_1 = 0 \\ h^2A_{-1} + h^2A_1 = \frac{2}{3}h^3 \end{cases} \Rightarrow \begin{cases} A_{-1} = \frac{h}{3} \\ A_0 = \frac{4h}{3} \\ A_1 = \frac{h}{3} \end{cases}$$

求积公式有至少 2 次代数精度

$$-h^3A_{-1} + h^3A_1 = \frac{1}{4}(h^4 - (-h)^4)$$

$$h^4A_{-1} + h^4A_1 \neq \frac{1}{5}(h^5 - (-h)^5)$$

从而求积公式有 3 次代数精度

$$(2) \begin{cases} A_{-1} + A_0 + A_1 = 4h \\ -hA_{-1} + hA_1 = 0 \\ h^2A_{-1} + h^2A_1 = \frac{16}{3}h^3 \end{cases} \Rightarrow \begin{cases} A_{-1} = \frac{8h}{3} \\ A_0 = -\frac{4h}{3} \\ A_1 = \frac{8h}{3} \end{cases}$$

求积公式有至少 2 次代数精度

$$-h^3A_{-1} + h^3A_1 = \frac{1}{4}((2h)^4 - (-2h)^4)$$

$$h^4A_{-1} + h^4A_1 \neq \frac{1}{5}((2h)^5 - (-2h)^5)$$

从而求积公式有 3 次代数精度

$$(3) \begin{cases} \frac{1}{3} + \frac{2}{3} + \frac{3}{3} = 2 \\ -\frac{1}{3} + \frac{2}{3}x_1 + x_2 = 0 \\ \frac{1}{3} + \frac{2}{3}x_1^2 + x_2^2 = \frac{2}{3} \end{cases} \Rightarrow \begin{cases} x_1 = \frac{\sqrt{6}+1}{5} \\ x_2 = -\frac{2\sqrt{6}}{15} + \frac{1}{5} \end{cases} \quad \text{or} \quad \begin{cases} x_1 = \frac{1-\sqrt{6}}{5} \\ x_2 = \frac{2\sqrt{6}}{15} + \frac{1}{5} \end{cases}$$

$$2x_1 + 3x_2 = 1$$

$$2x_1^2 + 3x_2^2 = 1$$

$$H_0x_0^3 + A_1x_1^3 + A_2x_2^3 \neq \frac{1}{4}(a^2 - b^2)$$

从而求积公式有 2 次代数精度

(4) 用代数精度的定义

令 $f(x) = 1, x, x^2$

$$\begin{cases} \int_0^h 1 dx = \frac{h}{2}(1+1) + 0 = h. \\ \int_0^h x dx = \frac{h}{2}(0+h) + ah^2(1-1) = \frac{h^2}{2} \Rightarrow a = \frac{1}{12} \\ \int_0^h x^2 dx = \frac{h}{2}(0+h^2) + ah^2(0-2h) = \frac{h^3}{3} \end{cases}$$

将 $f(x) = x^3, x^4$ 代入求积公式

$$\int_0^h x^3 dx = \frac{h}{2}(0+h^3) + \frac{h^2}{12}(0-3h^2)$$

$$\int_0^h x^4 dx \neq \frac{h}{2}(0+h^4) + \frac{h^2}{12}(0-4h^3)$$

从而求积公式有 3 次代数精度

Exercise 25: 136 页习题 10

试构造高斯型求积公式

$$\int_0^1 \frac{1}{\sqrt{x}} f(x) dx \approx A_0 f(x_0) + A_1 f(x_1)$$

Solution:

$$\omega(x) = (x - x_0)(x - x_1) = x^2 + bx + c$$

$$\begin{cases} \int_0^1 \rho(x) \omega(x) dx = \int_0^1 \sqrt{x} \omega(x) dx = 0 \\ \int_0^1 x \rho(x) \omega(x) dx = \int_0^1 x \sqrt{x} \omega(x) dx = 0 \end{cases} \Rightarrow \begin{cases} \frac{2}{5} + \frac{2}{3}b + 2c = 0 \\ \frac{2}{7} + \frac{2}{5}b + \frac{2}{3}c = 0 \end{cases} = \begin{cases} b = -\frac{6}{7} \\ c = \frac{3}{35} \end{cases}$$

$$\text{得 } \omega(x) = x^2 - \frac{6}{7}x + \frac{3}{35}$$

$$\text{零点为 } \begin{cases} x_0 = -\frac{2\sqrt{30}}{35} + \frac{3}{7} \approx 0.115587 \\ x_1 = \frac{2\sqrt{30}}{35} + \frac{3}{7} \approx 0.741556 \end{cases}$$

$$\text{根据代数精度的定义, 当 } f(x) = 1 \text{ 时 } A_0 + A_1 = \int_0^1 \frac{1}{\sqrt{x}} dx = 2$$

$$\text{当 } f(x) = x \text{ 时 } A_0 x_0 + A_1 x_1 = \int_0^1 \frac{1}{\sqrt{x}} \cdot x dx = \frac{2}{3}$$

$$\text{得 } \begin{cases} A_0 = 1.30429 \\ A_1 = 0.69571 \end{cases}$$

Exercise 26

用 $n=2,3$ 的高斯-勒让德公式计算积分

$$\int_1^3 e^x \sin x dx$$

Solution:

$$x = \frac{3-1}{2}t + \frac{1+3}{2} = t + 2$$

$$\int_1^3 e^x \sin x dx = \int_{-1}^1 e^{t+2} \sin(t+2) dt$$

当 $n=2$ 时

$$\int_{-1}^1 e^{t+2} \sin(t+2) dt \approx \frac{5}{9} f\left(-\frac{\sqrt{15}}{5}\right) + \frac{8}{9} f(0) + \frac{5}{9} f\left(\frac{\sqrt{15}}{5}\right).$$

$$\approx 10.948402566$$

当 $n = 2$ 时

$$\begin{aligned} & \int_{-1}^1 e^{t+2} \sin(t+2) dt \\ & \approx 0.347654 \times [f(0.8611363) + f(-0.8611363)] + 0.6521452 \times [f(0.339981) + f(-0.3399810)] \\ & = 10.9501401 \end{aligned}$$

Chapter 5

第七章

5.1 不动点迭代

将 $f(x) = 0$ 改写成 $x = \varphi(x)$ 的形式, 求 $f(x)$ 的零点即求 $\varphi(x)$ 的不动点。选择一个初始近似值 x_0 , 令 $x_1 = \varphi(x_0)$

迭代公式为 $x_{k+1} = \varphi(x_k) \quad k = 0, 1, 2, \dots$

5.2 不动点迭代的收敛性

(1) 设 x^* 为 $\varphi(x)$ 的不动点, $\varphi'(x)$ 在 x^* 的某个邻域连续, 且 $|\varphi'(x^*)| < 1$, 则不动点迭代公式收敛。 (x^*) 为根的准确值。

(2) 如果

$$\varphi'(x^*) = \varphi''(x^*) = \dots = \varphi^{(p-1)}(x^*) = 0$$

$$\varphi^{(p)} \neq 0$$

则此迭代是 p 阶收敛的。

$p=1$ 又叫线性收敛, $p=2$ 又叫平方收敛, $p>1$ 又叫超线性收敛。

迭代误差 $e_k = x_k - x^*$ 满足

$$\frac{e_{k+1}}{e_k^p} \rightarrow \frac{\varphi^{(p)}(x^*)}{p!}$$

Exercise 27: 218 页例 4

见课本, 很简单

5.3 牛顿迭代

求 $f(x) = 0$, 给定初值 x_0

迭代公式

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

牛顿迭代是二阶收敛的

其中 $\varphi''(x^*) = \frac{f''(x^*)}{f'(x^*)}$ $\lim_{k \rightarrow \infty} \frac{x_{k+1} - x^*}{(x_k - x^*)^2} = \frac{f''(x^*)}{2f'(x^*)}$

5.4 牛顿迭代对重根的改进

(1) 知道根的重数 m

$$x_{k+1} = x_k - m \frac{f(x_k)}{f'(x_k)} \quad (\text{二阶收敛})$$

(2) 不知道根的重数

$$x_{k+1} = x_k - \frac{f(x_k)f'(x_k)}{[f'(x_k)]^2 - f(x_k)f''(x_k)}$$

Exercise 28: 227 页例 9

见课本, 很简单

5.5 习题

Exercise 29: 238 页习题 2

为求方程 $x^3 - x^2 - 1 = 0$ 在 $x_0 = 1.5$ 附近的一个根, 设将方程改写成一下等价形式, 并建立相应的迭代公式。

- $x = 1 + 1/x^2$, 迭代公式 $x_{k+1} = 1 + 1/x_k^2$
- $x^3 = 1 + x^2$, 迭代公式 $x_{k+1} = \sqrt[3]{1 + x_k^2}$
- $x^2 = \frac{1}{x-1}$, 迭代公式 $x_{k+1} = 1/\sqrt{x_k - 1}$

试分析每种迭代公式的收敛性, 并选取一种公式求出具有四位有效数字的近似根。

Solution:

(1) $\varphi(x) = 1 + \frac{1}{x^2}$ $|\varphi'(x_0)| = \left| \frac{-2}{x_0^3} \right| < 1$

而 $\varphi'(x) \neq 0$, 故迭代是一阶收敛。

$$(2) \varphi(x) = \sqrt[3]{1+x^2} \quad |\varphi'(x_0)| = \left| \frac{2}{3} x_0 (1+x_0^2)^{-\frac{2}{3}} \right| \approx \frac{1}{2.19} < 1$$

而 $x^* \neq 0$, 故 $\varphi'(x^*) \neq 0$ 故迭代是一阶收敛。

$$(3) \varphi(x) = (x-1)^{-\frac{1}{2}} \quad |\varphi'(x)| = \left| -\frac{1}{2} (x_0-1)^{-\frac{3}{2}} \right| \approx 0.79 < 1$$

而 $\varphi'(x) \neq 0$, 故迭代是一阶收敛。

Exercise 30: 239 页习题 9

研究求 \sqrt{a} 的牛顿公式

$$x_{k+1} = \frac{1}{2} \left(x_k + \frac{a}{x_k} \right), x_0 > 0$$

证明对一切 $k = 1, 2, \dots, x_k \geq \sqrt{a}$ 且序列 x_1, x_2, \dots 是递减的。

Solution:

因 $x_0 > 0$, 由基本不等式得 $x_k + \frac{a}{x_k} \geq 2\sqrt{x_k \cdot \frac{a}{x_k}} = 2\sqrt{a}$

$$x_{k+1} = \frac{1}{2} \left(x_k + \frac{a}{x_k} \right) \geq \sqrt{a}$$

再证 $x_{k+1} < x_k$

$$\frac{x_{k+1}}{x_k} = \frac{\frac{1}{2} \left(x_k + \frac{a}{x_k} \right)}{x_k} = \frac{1}{2} \left(1 + \frac{a}{x_k^2} \right) \leq \frac{1}{2} \left(1 + \frac{a}{a} \right) = 1$$

故 $x_{k+1} < x_k$

Exercise 31: 239 页习题 12

应用牛顿法于方程 $x^3 - a$, 导出求立方根 $\sqrt[3]{a}$ 的迭代公式, 并讨论其收敛性。

Solution:

$$f(x) = x^3 - a, \quad f'(x) = 3x^2$$

$$x_{k+1} = x_k - \frac{x_k^3 - a}{3x_k^2} = \frac{2x_k^3 - a}{3x_k^2}$$

当 $a \neq 0$ 时, $\sqrt[3]{a}$ 为 $f(x)$ 的单根, 二阶收敛

当 $a = 0$ 时 $x_{k+1} = \frac{2}{3}x_k \quad \lim_{k \rightarrow \infty} x_k = 0$ 收敛

Exercise 32: 239 页习题 14

应用牛顿法于方程 $f(x) = x^n - a = 0$ 和 $f(x) = 1 - \frac{a}{x^n} = 0$, 分别导出求 $\sqrt[n]{a}$ 的迭代公式, 并讨论其收敛性。并求

$$\lim_{k \rightarrow \infty} (\sqrt[n]{a} - x_{k+1}) / (\sqrt[n]{a} - x_k)^2$$

Solution:

$$f(x) = x^n - a, \quad f'(x) = nx^{n-1}, \quad f''(x) = n(n-1)x^{n-2}$$

$$\text{迭代公式为: } x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} = \frac{(n-1)x_k^n + a}{nx_k^{n-1}}$$

$$\lim_{k \rightarrow \infty} (\sqrt[n]{a} - x_{k+1}) / (\sqrt[n]{a} - x_k)^2 = -\frac{f''(\sqrt[n]{a})}{2f'(\sqrt[n]{a})} = -\frac{n-1}{2\sqrt[n]{a}}$$

$$f = 1 - \frac{a}{x^n}, \quad f'(x) = \frac{an}{x^{n+1}}, \quad f''(x) = -\frac{an(n+1)}{x^{n+2}}$$

$$\text{迭代公式为: } x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} = \frac{(an+a)x_k - x_k^{n+1}}{an}$$

$$\lim_{k \rightarrow \infty} (\sqrt[n]{a} - x_{k+1}) / (\sqrt[n]{a} - x_k)^2 = -\frac{f''(\sqrt[n]{a})}{2f'(\sqrt[n]{a})} = \frac{n+1}{2\sqrt[n]{a}}$$

Exercice 33: 239 页习题 15

证明迭代公式

$$x_{k+1} = \frac{x_k(x_k^2 + 3a)}{3x_k^2 + a}$$

是计算 \sqrt{a} 的三阶方法。假定初值 x_0 充分靠近根 x^* , 求

$$\lim_{k \rightarrow \infty} (\sqrt{a} - x_{k+1}) / (\sqrt{a} - x_k)^3$$

Solution:

迭代公式 $x = \varphi(x) = \frac{x(x^2 + 3a)}{3x^2 + a}$, 其中 \sqrt{a} 为不动点。

证此迭代为 3 阶的, 即证 $\varphi'(\sqrt{a}) = \varphi''(\sqrt{a}) = 0, \varphi'''(\sqrt{a}) \neq 0$

$$\lim_{k \rightarrow \infty} (\sqrt{a} - x_{k+1}) / (\sqrt{a} - x_k)^3 = \frac{\varphi^{(3)}(x^*)}{3!}$$

Chapter 6

第九章

Definition 2: 局部截断误差

$$T_{n+1} = y(x_{n+1}) - y_{n+1} =$$

6.1 习题

Exercise 34

证明对任意参数 t ，下列龙格-库塔公式是二阶的：

$$\begin{cases} y_{n+1} = y_n + \frac{h}{2}(K_2 + K_3) \\ K_1 = f(x_n, y_n) \\ K_2 = f(x_n + th, y_n + thK_1) \\ K_3 = f(x_n + (1-t)h, y_n + (1-t)hK_1) \end{cases}$$

Solution: