

## **Phys 229ab Advanced Mathematical Methods: Conformal Field Theory**

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Notes for Physics 229, 2017-2018. These notes are in progress. They were last updated on April 3, 2018. In a few places, there are reminders [in blue](#) to the author to add additional discussion/comments. There may also be empty citations and missing figures. Some of the discussion is not original and is borrowed from various sources. I have tried to indicate with a footnote when a section closely follows a particular reference.

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## Resources

These introductory notes are heavily based on Silviu Pufu’s Bootstrap 2017 lectures [1], John Cardy’s book *Scaling and Renormalization in Statistical Physics* [2], Slava Rychkov’s EPFL lectures on CFT [3], and John McGreevy’s lectures on QFT [4].

## 1. Introduction

### 1.1. QFT and emergent symmetry

Quantum Field Theory is a universal language for theoretical physics. It shows up in many different settings, for example

- statistical physics,
- condensed matter physics,
- particle physics (SM and beyond),
- string theory/holography.

The microscopic physics in all of these settings can be quite complicated. However, often the macroscopic physics displays extra “emergent” symmetries that can help us do computations.

For example, in condensed matter physics, we are interested in describing a material made up of atoms with some lattice spacing  $a$ .<sup>1</sup> We refer to the detailed lattice system as the “microscopic theory.” Quantum field theory is a good description at distances much larger than the lattice spacing  $x \gg a$ , or equivalently energy/momenta much lower than the UV cutoff  $\Lambda_{UV} = 1/a$ .<sup>2</sup> At these large scales, the discrete translation symmetry of the lattice becomes a continuous symmetry.

Similarly, in statistical physics, particle physics, and string theory, QFT can describe distance scales much larger than the characteristic scales of the microscopic theory.

Statistical systems are described by QFTs in Euclidean signature, e.g. on  $\mathbb{R}^d$ . Such QFTs capture properties of the equilibrium state. By contrast, condensed matter and particle systems are described by QFTs in Lorentzian signature, e.g. on  $\mathbb{R}^{d-1,1}$ . Such QFTs encode time-dependent quantum dynamics.

We will be interested in QFTs with rotational symmetry, by which we mean  $\text{SO}(d)$  symmetry in Euclidean signature and  $\text{SO}(d-1, 1)$  symmetry in

<sup>1</sup>We ignore the possibility of lattice defects for the moment.

<sup>2</sup>QFTs that are valid below a UV cutoff are often called “effective field theories” (EFTs).

Lorentzian signature. In particle physics and string theory, this symmetry is built into the microscopic theory. However in lattice systems, rotational symmetry must be emergent. This means that correlation functions become rotationally invariant in the limit of large distances, even though microscopic correlation functions are not rotationally-invariant.<sup>3</sup> In particular, for condensed matter systems, the effective “speed of light” associated with  $\text{SO}(d-1, 1)$ -invariance is an emergent property and has nothing to do with the speed of actual light. (We will see some explicit examples later.)

Under general conditions,  $\text{SO}(d)$ -invariant Euclidean QFTs are in one-to-one correspondence with  $\text{SO}(d-1, 1)$ -invariant Lorentzian QFTs. The map between them is called Wick rotation, and we will discuss it in detail. Because of this correspondence, we can focus mostly on Euclidean QFTs, and later understand Lorentzian QFTs by Wick rotating what we learned in Euclidean signature.

### 1.2. The mass gap and critical points

So far, we are interested in theories with Poincare symmetry

$$G_{\text{Poincare}} = \mathbb{R}^d \rtimes \text{SO}(d). \quad (1)$$

From the point of view of long-distance physics, the most important property of a Poincare-invariant theory is its mass gap

$$m_{\text{gap}} = E_1 - E_0, \quad (2)$$

where  $E_0, E_1$  are the energies of the ground state and first excited state, respectively.<sup>4</sup> Theories with  $m_{\text{gap}} > 0$  are called “gapped.”

To understand why  $m_{\text{gap}}$  is important, let us study a two-point function of a scalar local operator  $\phi(x) = \phi(x^0, \mathbf{x})$ . We demand that  $\phi(0)$  have vanishing vacuum expectation value by subtracting off an appropriate multiple of the unit operator. By Poincare invariance, it suffices to consider  $\langle 0 | \phi(x^0, \mathbf{0}) \phi(0) | 0 \rangle$  with  $x^0 > 0$ . The two-point function is then given by

$$\begin{aligned} \langle 0 | \phi(x^0, \mathbf{0}) \phi(0) | 0 \rangle &= \langle 0 | \phi(0) e^{-Hx^0} \phi(0) | 0 \rangle \\ &= \sum_{\psi} |\langle 0 | \phi(0) | \psi \rangle|^2 e^{-E_{\psi} x^0}. \end{aligned} \quad (3)$$

<sup>3</sup>Emergent rotational symmetry is very familiar: we often cannot determine the orientation of a microscopic lattice using macroscopic observations. Some examples of materials *without* emergent  $\text{SO}(d)$  symmetry are crystals like salt. A very exotic example is the Haah code [1].

<sup>4</sup>Some QFTs with topological order can have multiple degenerate ground states.

Here,  $H$  is the Hamiltonian with the vacuum energy subtracted off, and  $\psi$  runs over an orthonormal basis of eigenstates of  $H$ . To get the right-hand side, we have used  $\phi(x^0, \mathbf{0}) = e^{Hx^0} \phi(0, \mathbf{0}) e^{-Hx^0}$  and  $H|0\rangle = 0$ .<sup>5</sup>

The key point is that the operator  $e^{-Hx^0}$  exponentially damps states with energy  $E_\psi \gg 1/x^0$ . At large  $x^0$ , the correlator is dominated by  $\psi$  with the smallest nonzero eigenvalue of  $H$ , which is  $m_{\text{gap}}$ .<sup>6</sup>

Thus, when  $m_{\text{gap}}$  is nonzero, correlation functions of local operators fall off at least as fast as  $e^{-|x|/\xi}$ , where  $\xi \equiv 1/m_{\text{gap}}$  is called the “correlation length.” Generic statistical and condensed matter systems have microscopic correlation lengths  $\xi \sim a$ , or equivalently  $m_{\text{gap}} \sim \Lambda_{UV}$ . At long distances, they are described by QFTs whose local correlation functions vanish, called topological quantum field theories (TQFTs).

However, sometimes by tuning parameters in the microscopic Hamiltonian, we can make  $m_{\text{gap}}$  much smaller than  $\Lambda_{UV}$ , and even arrange for  $m_{\text{gap}}$  to vanish. Points in parameter space where  $m_{\text{gap}} = 0$  are called critical points. At a critical point, the system experiences a phase transition, and develops nonzero correlations at arbitrarily long distances.<sup>7</sup>

Long-distance correlation functions at critical points have no intrinsic length-scale because all memory of dimensionful microscopic quantities (like the lattice spacing  $a$ ) disappears when distances become arbitrarily large. For example, critical two-point functions behave as pure power laws

$$\langle \phi(x) \phi(0) \rangle = \frac{C}{|x|^{2\Delta}} \quad (\text{critical point, } x \gg a), \quad (4)$$

where  $C$  and  $\Delta$  are constants depending on  $\phi$ . The quantity  $\Delta$  is called the scaling dimension of  $\phi$ .

### 1.3. Scaling and conformal symmetry

A more precise way to state the lack of an intrinsic length scale is to say that theories with  $m_{\text{gap}} = 0$  have an emergent symmetry under rescaling

$$x^\mu \rightarrow \lambda x^\mu \quad (\lambda > 0). \quad (5)$$

<sup>5</sup>Note that the Euclidean time-evolution operator is  $e^{-Hx^0}$  as opposed to the familiar  $e^{-iHt}$  in Lorentzian signature. They are related by Wick rotation  $x^0 = it$ . We will discuss this in much more detail in later sections.

<sup>6</sup>Note that the vacuum does not contribute as an intermediate state because we have demanded  $\langle 0|\phi(0)|0\rangle = 0$ .

<sup>7</sup>If the Standard Model were like a generic condensed matter system, we might expect  $m_{\text{gap}}$  to be close to the UV cutoff, which is perhaps the GUT scale  $10^{15}\text{GeV}$  or Planck scale  $10^{18}\text{GeV}$ . The hierarchy problem is the problem of explaining why the Standard Model is so close to a critical point.

Under very general conditions (that we will discuss), critical points also display less obvious emergent symmetries called conformal transformations. A conformal transformation  $x \rightarrow x'(x)$  is a map that looks like a rotation and rescaling near each point,

$$\frac{\partial x'^\mu}{\partial x^\nu} = \Omega(x) R^\mu{}_\nu(x), \quad R^\mu{}_\nu \in \text{SO}(d). \quad (6)$$

An example is a special conformal transformation,

$$x^\mu \rightarrow \frac{x^\mu - b^\mu x^2}{1 - 2b \cdot x + b^2 x^2} \quad (b \in \mathbb{R}^d). \quad (7)$$

In 2-dimensions, there are more exotic examples, like the one pictured in figure ??.<sup>8</sup>

Here is some rough intuition for why critical points display conformal symmetry. We know that a critical theory is invariant under rescalings and rotations. If the theory is also local, in the sense that degrees of freedom at a point only interact directly with other degrees of freedom at nearby points, then the theory should also be invariant under transformations that locally look like a rescaling and rotation. This is the defining property of a conformal transformation. Turning this rough intuition into a theorem is a difficult problem (that we will discuss in more detail later). However, it seems to be true in a wide class of systems.

QFTs that are invariant under conformal symmetry are called conformal field theories (CFTs). To summarize, CFTs describe critical points where  $m_{\text{gap}} = 0$ . One can also understand the neighborhood of a critical point where  $m_{\text{gap}}$  is nonzero (but still  $m_{\text{gap}} \ll \Lambda_{UV}$ ) by studying perturbations of the associated CFT.

#### 1.4. Examples of critical points<sup>9</sup>

So far our discussion has been very abstract, so let us introduce some examples. One of our goals will be to infer from examples a set of axioms that CFTs should satisfy. We will study these axioms in the next part of the course. Another goal will be to be more precise about how and why statistical and condensed matter systems are described by quantum field theory, and how critical points come about.

<sup>8</sup>We will give a precise definition for what it means for a theory to be invariant under transformations like (5) and (7) later in the course.

<sup>9</sup>Sources for this section: Pufu [1], Cardy [2].

#### 1.4.1. Magnets

Our first examples of critical points occur in magnets. Given a magnet with temperature  $T$ , we can apply a magnetic field  $\vec{H}$  and measure the magnetization  $\vec{M}$ . There are three main types of magnets in 3-dimensions, which are distinguished by their symmetries:<sup>10</sup>

- *Uni-axial magnet*: individual magnetic moments  $\vec{\mu}$  are confined to lie along a fixed axis. Uni-axial magnets have an  $O(1) = \mathbb{Z}_2$  symmetry under which  $\vec{H} \rightarrow -\vec{H}$  and  $\vec{M} \rightarrow -\vec{M}$ .
- *XY magnet*: magnetic moments  $\vec{\mu}$  are oriented in a plane. Such magnets have an  $O(2)$  symmetry under which  $\vec{H}$  and  $\vec{M}$  rotate in the plane.
- *Heisenberg magnet*: magnetic moments  $\vec{\mu}$  are unconstrained. Such magnets have  $O(3)$  symmetry under which  $\vec{H}$  and  $\vec{M}$  transform in the vector representation.

For the moment, we will focus on the simplest case of uni-axial magnets. We denote the projections of  $\vec{H}, \vec{M}$  onto the appropriate axis by  $H, M$ .

In experiments, we observe the following. There exists a “critical temperature”  $T_c$ , such that

- For  $T < T_c$ , the preferred state of the magnet has nonzero magnetization  $M \neq 0$  when  $H = 0$ . In other words, the  $\mathbb{Z}_2$  symmetry is spontaneously broken.
- For  $T > T_c$ , the magnet has  $M = 0$  when  $H = 0$ , i.e. the  $\mathbb{Z}_2$  symmetry is unbroken.

The corresponding phase diagram is pictured in figure ???. The point

$$H = 0, T = T_c \tag{8}$$

is a critical point, and is described by a CFT at long distances. To reach the critical point, we must tune two parameters:  $H$  and  $T$ . Tuning  $H = 0$  is easy because that is where the microscopic theory has  $\mathbb{Z}_2$  symmetry. However, the value of  $T_c$  depends on the specific material.

In more detail, the behavior of the magnetization in different phases is shown in figure ??. Close to  $T_c$ , observables exhibit so-called “scaling” behavior, characterized by various critical exponents. Let us define the

<sup>10</sup>Here, we mean non-spacetime symmetries, usually called “global” or “flavor” symmetries. The emergent spacetime symmetry group is still the Poincare group, or the conformal group at the critical point.



dimensionless couplings

$$t \equiv \frac{T - T_c}{T_c}, \quad h = \frac{H}{k_B T}. \quad (9)$$

Some examples of critical exponents are

- $\alpha$ : the heat capacity at  $h = 0$  behaves as

$$C = \frac{\partial^2 F}{\partial T^2} \propto |t|^{-\alpha}. \quad (10)$$

(Here  $F$  is the free-energy.)

- $\beta$ : the spontaneous magnetization behaves as

$$\lim_{H \rightarrow 0^+} M \propto (-t)^\beta. \quad (11)$$

- $\gamma$ : the zero-field susceptibility behaves as

$$\chi = \left. \frac{\partial M}{\partial H} \right|_{H=0} \propto |t|^{-\gamma}. \quad (12)$$

- $\delta$ : the magnetization at  $T = T_c$  behaves as

$$|M| \propto |h|^{1/\delta}. \quad (13)$$

- $\nu$  and  $\eta$ : the correlation length  $\xi$  can be measured by studying a two-point correlation functions of spins

$$G(x) = \langle s(x)s(0) \rangle - \langle s(0) \rangle^2. \quad (14)$$

Away from the critical point,  $G(x) \sim e^{-|x|/\xi}$  decays exponentially. However, as  $t \rightarrow 0$ , the correlation length diverges as

$$\xi \propto |t|^{-\nu}. \quad (15)$$

Equivalently, the mass-gap goes to zero as  $m_{\text{gap}} \propto |t|^\nu$ . Precisely at  $t = 0$ , the two-point function takes the form

$$G(x) \propto \frac{1}{|x|^{d-2+\eta}}, \quad (16)$$

i.e. the spin operator has dimension  $\Delta_s = \frac{d-2}{2} + \frac{\eta}{2}$ .

Don't worry, I can't keep track of all these critical exponents either. We will see shortly that all of this behavior can be explained using effective field theory, scaling symmetry, and dimensional analysis.

Now, here is an amazing fact:

**Claim 1.** *We find the same critical exponents in many different uni-axial magnets, regardless of what material they're made of.*

In fact, critical uni-axial magnets are all described by the same scale-invariant QFT at long distances. This phenomenon is called “critical universality.”

#### 1.4.2. Liquid-vapor transitions

Other critical points appear in liquid-vapor transitions. For example, the phase-diagram of water is pictured in figure ?? . Near room temperatures and pressures, there is a sharp distinction between the liquid and gas phases. However, at higher temperatures and pressures, the distinction between liquid and gas disappears at a critical point  $(T_c, P_c)$ . For example, in water  $T_c = 647 \text{ K}$ ,  $P_c = 374 \text{ Atm}$ .

Note that the critical points of magnets and water are both obtained by tuning two parameters. Comparing neighborhoods of the critical points in figures ?? and ?? , we can make the following rough analogy between water and magnets:

$$\begin{aligned} P - P_c &\sim H, \\ \rho - \rho_c &\sim M. \end{aligned} \tag{17}$$

where  $\rho$  is the density and  $\rho_c$  is the critical density.<sup>11</sup>

In measurements of critical water, we again observe scaling behavior. For example, the heat capacity behaves as

$$C \sim |t|^{-\alpha}, \tag{18}$$

where  $t$  is again given by (9). Additionally, the difference in density between the liquid and gas phases behaves as

$$\rho_{\text{liquid}} - \rho_{\text{gas}} \sim (-t)^\beta. \tag{19}$$

Amazingly,

**Claim 2.** *Water and other liquid-vapor transitions have precisely the same critical exponents as uni-axial magnets.*

We say that liquid-vapor transitions are in the same “universality class” as uni-axial magnets.

*End of lecture 1*

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<sup>11</sup>Water does not have a microscopic  $\mathbb{Z}_2$  symmetry, but it turns out that one emerges near the critical point. Roughly speaking, the  $\mathbb{Z}_2$  switches the liquid (high-density) and gas (low-density) phases. To make a more precise analogy, we should identify  $M$  with the combination of  $\rho$  and  $P$  that flips sign under the emergent  $\mathbb{Z}_2$ .

### 1.4.3. The Ising model

The Ising model is a simplified model of a magnet that is still rich enough to capture its critical behavior. Its degrees of freedom are classical spins taking values  $\pm 1$ , with one spin for each site on a cubic lattice<sup>12</sup>

$$s_i \in \{\pm 1\}, \quad i \in \mathbb{Z}^d. \quad (20)$$

The partition function is a sum over all configurations of spins, with Boltzmann weights that mimic the interactions between physical spins in a magnet,

$$\begin{aligned} Z(K, h) &= \sum_{\{s_i\}} e^{-S[s]}, \\ S[s] &= -K \sum_{\langle ij \rangle} s_i s_j - h \sum_j s_j. \end{aligned} \quad (21)$$

The notation  $\langle ij \rangle$  means that  $i, j$  are neighboring lattice sites. (You can think of  $K$  as the product  $\beta J$  where  $\beta$  is the inverse temperature and  $J$  is the spin-spin interaction.) We can also compute correlation functions by inserting spins into the sum

$$\langle s_{i_1} \cdots s_{i_n} \rangle = \frac{1}{Z} \sum_{\{s_i\}} s_{i_1} \cdots s_{i_n} e^{-S[s]}. \quad (22)$$

The “magnetization” is proportional to the one-point function  $\langle s_i \rangle$  (which is independent of  $i$ ).

When  $K$  is positive, the term in  $S[s]$  proportional to  $K$  makes spins want to align. This term has a  $\mathbb{Z}_2$  symmetry under which  $s_i \rightarrow -s_i$  for all  $i$ . The term proportional to  $h$  breaks this  $\mathbb{Z}_2$  symmetry and causes spins to preferentially have the same sign as  $h$ . Both of these effects compete against statistical fluctuations.

In dimension  $d \geq 2$ , the Ising model famously exhibits a critical point at a special value  $K = K_c$  and  $h = 0$ . For  $K > K_c$ , spins spontaneously align and break the  $\mathbb{Z}_2$  symmetry. For  $K < K_c$ , statistical fluctuations cause the spins to randomize and the  $\mathbb{Z}_2$  is unbroken.

The Ising critical point displays precisely the same long-distance correlation functions and critical exponents as real uni-axial magnets and liquid-vapor systems. In fact, for our purposes, we can think of the Ising model as yet another physical system in the same universality class as these other systems. Its main distinguishing feature is that it is much simpler than e.g.

<sup>12</sup>The Ising model can also be formulated on other lattices, and in many cases these models lie in the same universality class.

actual water, and thus much easier to study.<sup>13</sup> By studying the mathematics of the Ising model, we will be able to understand gaplessness and a few QFT/CFT axioms in a simple way. By the power of critical universality, quantities we compute in the abstract Ising model at  $K = K_c$  agree exactly with interesting quantities in real physical systems.

#### 1.4.4. Continuum $\phi^4$ theory

The Ising lattice model is a good starting point for understanding why classical statistical systems are described by QFT at long distances. We can think of each configuration of spins as a map

$$s : \mathbb{Z}^d \rightarrow \{\pm 1\}, \quad (23)$$

and the partition function is a sum over all such maps. This is a discrete version of a path integral.

Typically, in QFT we integrate over continuous functions on a continuous space, e.g.

$$\phi : \mathbb{R}^d \rightarrow \mathbb{R}. \quad (24)$$

If we study correlation functions of the Ising model at long distances (much larger than the lattice spacing), it's not hard to imagine that we can approximate the lattice as continuous  $i \in \mathbb{Z}^d \rightarrow x \in \mathbb{R}^d$ . Similarly, we can replace individual spins  $s_i$  with average densities of spins  $\phi(x)$  in small neighborhoods, so that the effective spin at a point becomes a real number.

This suggests that the partition sum of the Ising model might be related to the path integral for a scalar field on  $\mathbb{R}^d$ , at least at long distances. That is, perhaps we can take

$$\begin{aligned} \mathbb{Z}^d &\rightarrow \mathbb{R}^d \\ s_i &\rightarrow \phi(x) \\ \sum_{\{s_i\}} &\rightarrow \int D\phi \\ e^{-S[s]} &\rightarrow e^{-S[\phi]}, \end{aligned} \quad (25)$$

for some continuum action  $S[\phi]$ , without changing the long-distance behavior of correlation functions too much.

<sup>13</sup>There are many other abstract statistical lattice models in the same universality class as the Ising model, for example compass models [arXiv:1303.5922](#) and the Blume-Capel model [arXiv:1711.10946](#). The Ising model on a cubic lattice is arguably the simplest, however the Blume-Capel model has some features that make it even better for simulation.

There is a non-rigorous procedure called “taking the continuum limit” of a lattice model that tries to justify these replacements. However, it involves several uncontrolled approximations, and it is perhaps more honest to just study a continuum scalar theory and compare its behavior with the Ising model.

For concreteness, let us focus on 3-dimensions. The simplest interacting theory of a scalar field in 3d has Euclidean action

$$S[\phi] = \int d^3x \left( \frac{1}{2}(\partial\phi)^2 + \frac{1}{2}m^2\phi^2 + g\phi^4 \right). \quad (26)$$

Note that  $\phi$  has mass-dimension  $1/2$ , so that  $m$  and  $g$  both have mass-dimension 1. Thus,  $\phi^2$  and  $\phi^4$  are relevant interactions, and we can think of this theory as an RG flow from the free-boson at high energies to something else at low energies.

Let us map out the phase diagram of this theory. Because  $m$  and  $g$  are the only dimensionful parameters, only their ratio matters.<sup>14</sup> Suppose first that  $m^2/g^2 \gg 0$  is large and positive. In this limit, the theory has a single massive vacuum with mass approximately  $m_{\text{gap}} \approx m$ . Now suppose  $m^2/g^2 \ll 0$  is very negative. In this case, the theory has two very massive vacua that spontaneously break the  $\mathbb{Z}_2$  symmetry, with  $m_{\text{gap}} \approx \sqrt{-2m^2}$ .

If we start at very large  $|m^2|$  (in either phase) and decrease  $|m^2|$ ,  $m_{\text{gap}}$  decreases too. There exists a critical ratio  $m^2/g^2 = r_c$  where  $m_{\text{gap}} \rightarrow 0$  and the theory is described by a nontrivial CFT. One can justify this claim using the  $\epsilon$ -expansion and other perturbative techniques, as we will discuss, or from numerical simulations.

The phase diagram we have just described is the same as the Ising phase diagram. Indeed  $\phi^4$ -theory is also in the Ising universality class.

Near the critical point, the  $\phi^4$  interaction cannot be treated as a small perturbation. More explicitly, because the coupling constant  $g$  has mass-dimension 1, perturbation theory is really an expansion in  $gx$ , where  $x$  is the characteristic length scale of the observable we are computing. In the long-distance limit  $x \rightarrow \infty$  (where the CFT emerges), this expansion breaks down. Thus, the Ising CFT cannot be studied with traditional perturbation theory. We will later discuss two different perturbative (but uncontrolled)

<sup>14</sup>To be more precise, there is also a UV cutoff  $\Lambda$  that depends on how the theory is regularized. The quartic term gives rise to a linear divergence proportional to the  $\phi^2$  term, so physical masses are given by  $m_{\text{phys}}^2 = m^2 + \alpha g \Lambda$ , with  $\alpha$  a dimensionless number. (In some regularization schemes, like dimensional regularization, the linear divergence vanishes.) However, shifting  $m^2$  is the only way the cutoff can enter a physical observable. Thus, our discussion becomes correct after the replacement  $m^2 \rightarrow m_{\text{phys}}^2$ .

expansions for the Ising CFT: the  $\epsilon$ -expansion and the large- $N$  expansion.

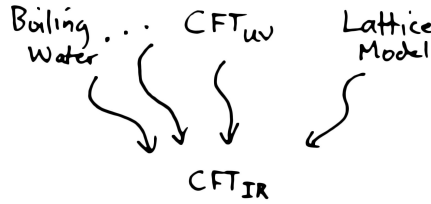


Fig. 1. Many microscopic theories can flow to the same IR CFT. We say that the theories are IR equivalent, or IR dual.

The theory with action (26) and  $m^2/g^2 = r_c$  is equivalent to the free boson at short distances (high energies/UV) and the Ising CFT at long distances (low energies/IR). The free boson is itself a CFT, so we have an RG flow from one CFT to another (figure 1). This construction, where we start with a UV CFT and perturb it so that it flows to an IR CFT, is one possible definition of a general QFT. In this definition, we must allow for the possibility that the IR theory is gapped, in which case  $\text{CFT}_{\text{IR}}$  is a TQFT which is technically a special case of a CFT (where all local correlation functions are zero).

An RG flow from the free boson is perhaps the cleanest theoretical construction of the Ising CFT. It shows that the Ising CFT inherits properties of a continuum quantum field theory. For example,  $\phi^4$ -theory has rotational invariance and  $\mathbb{Z}_2$  symmetry, so the Ising CFT does too. The Ising lattice model is easier to simulate, but it does not have microscopic rotational symmetry. Meanwhile, water has rotational symmetry but no microscopic  $\mathbb{Z}_2$  symmetry. We will learn a lot about the Ising CFT by going back and forth between different microscopic realizations.

#### 1.4.5. Other universality classes

We have seen several different systems that fall into the Ising universality class. However, not every critical point is described by the Ising CFT. Another important class of 3d CFTs are the  $O(N)$  models. These can be described as the critical point of a theory of  $N$  bosons  $\phi_i$  ( $i = 1, \dots, N$ )

with a quartic interaction that respects an  $O(N)$  global symmetry

$$S = \int d^3x \left( \frac{1}{2} \sum_i \partial_\mu \phi_i \partial^\mu \phi_i + \frac{1}{2} m^2 \sum_i \phi_i \phi_i + g \left( \sum_i \phi_i \phi_i \right)^2 \right). \quad (27)$$

XY magnets are in the same universality class of the  $O(2)$  model and Heisenberg magnets are in the same universality class of the  $O(3)$  model.

[Discuss non-uniqueness of UV Lagrangian description. An example is particle-vortex duality between the gauged  \$O\(2\)\$  model and the  \$O\(2\)\$  model. Distinguish between IR duality and exact duality.](#)

But this is just the tip of the iceberg — there is a huge zoo of different types of critical points that we can construct in a variety of ways. One of our goals will be to better understand this zoo.

## 2. Path integrals, quantization, and the quantum-classical correspondence<sup>15</sup>

In this section, we explain in more detail why classical statistical systems are described by Euclidean quantum field theory, and also how they are related to quantum condensed-matter systems via Wick rotation. Along the way, we will introduce some important technical concepts in QFT, like cutting-and-gluing rules for path integrals and the idea of quantization.

### 2.1. The 1d Ising model and the transfer matrix

Let us start with the Ising lattice model in 1-dimension. For concreteness, we will study the theory on a periodic lattice with length  $M$ , so the spins  $s_i$  are labeled by  $i \in \mathbb{Z}_M$ . The partition function is given by

$$Z_1 = \sum_{\{s_i = \pm 1\}} e^{-S[s]}$$

$$S[s] = -K \sum_{i=1}^M s_i s_{i+1} - h \sum_{i=1}^M s_i. \quad (28)$$

We refer to  $S[s]$  as the “action,” even though it is equal to  $\beta H$ , where  $H$  is the classical Hamiltonian. This is because we would like to reserve the word Hamiltonian for a completely different object that will appear shortly.

As mentioned in the introduction, the partition sum should be thought of as a discrete version of a 1-dimensional path-integral. This 1-dimensional

<sup>15</sup>Sources for this section: McGreevy [4].

path integral can be computed by relating it to a 0-dimensional quantum theory. This is an example of the notion of *quantization*.

The key idea is to build up the partition sum by moving along the lattice site-by-site. Forget about periodicity for the moment, and consider the contribution to the partition function from spins  $j < i$  for some fixed  $i$ ,

$$Z_{\text{partial}}(i, s_i) = \sum_{\{s_j: j < i\}} e^{K \sum_{j < i} s_j s_{j+1} + h \sum_{j < i} s_j}. \quad (29)$$

Because of the interaction term  $s_{i-1} s_i$ , we cannot do the sum over  $\{s_1, \dots, s_{i-1}\}$  without specifying the spin  $s_i$ . Thus, we have a function of  $s_i$ . In short,  $Z_{\text{partial}}(i, s)$  is the partition function of the theory on the lattice  $1 \dots i$ , with fixed boundary condition  $s$  at site  $i$ .<sup>16</sup>

Note that  $Z_{\text{partial}}(i+1, s_{i+1})$  can be related to  $Z_{\text{partial}}(i, s_i)$  by inserting the remaining Boltzmann weights that depend on  $s_i$  and performing the sum over  $s_i = \pm 1$ ,

$$Z_{\text{partial}}(i+1, s_{i+1}) = \sum_{s_i = \pm 1} T(s_{i+1}, s_i) Z_{\text{partial}}(i, s_i), \quad (30)$$

where

$$T(s_{i+1}, s_i) \equiv e^{K s_i s_{i+1} + h s_i}. \quad (31)$$

The key step is to recognize (30) as a discrete version of the Schrodinger equation in a 2-dimensional Hilbert space  $\mathcal{H}$ . This Hilbert space has basis  $|s\rangle = |\pm 1\rangle$ . The  $T(s', s)$ 's are elements of a  $2 \times 2$  matrix  $\hat{T}$  acting on  $\mathcal{H}$

$$T(s', s) = \langle s' | \hat{T} | s \rangle, \quad \hat{T} = \begin{pmatrix} e^{K+h} & e^{-K-h} \\ e^{-K+h} & e^{K-h} \end{pmatrix}, \quad (32)$$

and  $Z_{\text{partial}}(i, s)$  are the components of a vector  $|\Psi_i\rangle \in \mathcal{H}$ ,

$$Z_{\text{partial}}(i, s) = \langle s | \Psi_i \rangle. \quad (33)$$

In this notation, (30) becomes

$$|\Psi_{i+1}\rangle = \hat{T} |\Psi_i\rangle. \quad (34)$$

The matrix  $\hat{T}$  is called the “transfer matrix”, and it plays the role of a discrete time-translation operator. Here,  $i$  should be thought of as a discrete Euclidean time coordinate.

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<sup>16</sup>We must also impose some boundary condition at site 1. The precise choice is not important for this discussion, so we have left it implicit.



To be explicit, the (integrated) Schrodinger equation in a quantum theory in Euclidean time is

$$|\Psi(\tau + \Delta\tau)\rangle = e^{-\Delta\tau \hat{H}} |\Psi(\tau)\rangle, \quad (35)$$

where  $\tau$  is the Euclidean time coordinate,  $\Delta\tau$  is some time-step, and  $\hat{H}$  is the quantum Hamiltonian. Thus, the 1-dimensional Ising lattice model is equivalent to a 2-state quantum theory with Hamiltonian

$$\hat{H} = -\frac{1}{\Delta\tau} \log \hat{T}. \quad (36)$$

When the lattice is periodic, the partition function is related to the transfer matrix by

$$\begin{aligned} Z_1 &= \sum_{\{s_i\}} \langle s_M | \hat{T} | s_{M-1} \rangle \langle s_{M-1} | \hat{T} | s_{M-2} \rangle \cdots \langle s_1 | \hat{T} | s_M \rangle \\ &= \text{Tr}(\hat{T}^M). \end{aligned} \quad (37)$$

This is now easy to evaluate by diagonalizing  $\hat{T}$ ,

$$\text{Tr}(\hat{T}^M) = \lambda_+^M + \lambda_-^M, \quad (38)$$

where

$$\begin{aligned} \lambda_{\pm} &= e^K \cosh h \pm \sqrt{e^{2K} \sinh^2 h + e^{-2K}} \\ &\rightarrow \begin{cases} 2 \cosh K \\ 2 \sinh K \end{cases} \quad (\text{when } h = 0). \end{aligned} \quad (39)$$

In the thermodynamic limit  $M \rightarrow \infty$ , the partition function is dominated by the larger eigenvalue

$$Z_1 = \lambda_+^M \left( 1 + \left( \frac{\lambda_-}{\lambda_+} \right)^M \right) \approx \lambda_+^M. \quad (40)$$

In quantum mechanical language, the state with the largest eigenvalue of  $\hat{T}$  has the smallest eigenvalue of  $\hat{H}$  — i.e. it is the ground state, and we should call it  $|0\rangle$ . We have shown that the ground state dominates the thermodynamic limit. Contributions from the excited state are exponentially suppressed in the energy gap times the size of the system

$$\begin{aligned} \left( \frac{\lambda_-}{\lambda_+} \right)^M &= e^{-(M\Delta\tau)m_{\text{gap}}}, \\ m_{\text{gap}} &\equiv -\frac{1}{\Delta\tau} \log(\lambda_-/\lambda_+). \end{aligned} \quad (41)$$

*End of lecture 2*

We can also use the transfer matrix to compute correlation functions. For example, consider the two-point function  $\langle s_{i_1} s_{i_2} \rangle_{\mathbb{Z}_M}$  where the subscript  $M$  indicates that we are on a periodic lattice with length  $M$ . Suppose  $i_1 > i_2$ . We have

$$\begin{aligned} \langle s_{i_1} s_{i_2} \rangle_{\mathbb{Z}_M} &= \frac{1}{Z_1} \sum_{\{s_i\}} \langle s_M | \hat{T} | s_{M-1} \rangle \cdots \langle s_{i_1+1} | \hat{T} | s_{i_1} \rangle s_{i_1} \langle s_{i_1} | \hat{T} | s_{i_1-1} \rangle \cdots \\ &\quad \times \cdots \langle s_{i_2+1} | \hat{T} | s_{i_2} \rangle s_{i_2} \langle s_{i_2} | \hat{T} | s_{i_2-1} \rangle \cdots \langle s_1 | \hat{T} | s_M \rangle \\ &= \frac{1}{Z_1} \text{Tr}(\hat{T}^{M-i_1} \hat{\sigma}^z \hat{T}^{i_1-i_2} \sigma^z \hat{T}^{i_2}) \quad (i_1 > i_2). \end{aligned} \quad (42)$$

Here, we introduced the Pauli spin operator  $\sigma^z$  that measures the spin of a state

$$\sigma^z |s\rangle = s |s\rangle. \quad (43)$$

It is easy to compute the correlation function (42) by expressing  $\hat{s}$  in the eigenbasis of  $\hat{T}$ .

**Exercise 2.1.** Show that in the limit of large  $M$  and large “distance”  $i_1 - i_2$ , the correlator factorizes into a product of expectation values  $\langle 0 | \sigma^z | 0 \rangle$ , plus exponential corrections from the excited state

$$\langle s_{i_1} s_{i_2} \rangle_{\mathbb{Z}_M} = \langle 0 | \sigma^z | 0 \rangle^2 + O(e^{-m_{\text{gap}} \tau}, e^{-m_{\text{gap}}(L-\tau)}), \quad (44)$$

where  $\tau \equiv (i_1 - i_2) \Delta \tau$ ,  $L \equiv M \Delta \tau$ .

Let us write (42) in a slightly different way by introducing “Heisenberg picture” operators

$$\hat{s}(i) = \hat{T}^{-i} \sigma^z \hat{T}^i. \quad (45)$$

Equation (42) is equivalent to

$$\langle s_{i_1} s_{i_2} \rangle_{\mathbb{Z}_M} = \frac{1}{Z_1} \text{Tr}(\hat{s}(i_1) \hat{s}(i_2) \hat{T}^M) \quad (i_1 > i_2). \quad (46)$$

Note that in deriving (42, 46), we used that  $i_1 > i_2$ . If instead  $i_2 > i_1$ , the path integral would give a product of operators in the opposite order

$$\langle s_{i_1} s_{i_2} \rangle_{\mathbb{Z}_M} = \frac{1}{Z_1} \text{Tr}(\hat{s}(i_2) \hat{s}(i_1) \hat{T}^M) \quad (i_2 > i_1). \quad (47)$$

The general statement is that the path integral becomes a time-ordered product of quantum operators<sup>17</sup>

$$\langle s_{i_1} \cdots s_{i_n} \rangle_{\mathbb{Z}_M} = \frac{1}{Z_1} \text{Tr}(T\{\hat{s}(i_1) \cdots \hat{s}(i_n)\} \hat{T}^M), \quad (48)$$

where the definition of the time-ordering symbol is

$$T\{\hat{s}(i_1) \cdots \hat{s}(i_n)\} \equiv \hat{s}(i_1) \cdots \hat{s}(i_n) \theta(i_1 > \cdots > i_n) + \text{permutations}. \quad (49)$$

Here  $\theta(i_1 > \cdots > i_n)$  is 1 if the  $i_k$  are in the specified order and zero otherwise.

Finally, in the limit of large- $M$ , the factor  $\frac{\hat{T}^M}{\lambda_+^M}$  projects onto the ground state, so a path integral on the full lattice  $\mathbb{Z}$  becomes a vacuum expectation value of a time-ordered product

$$\langle s_{i_1} \cdots s_{i_n} \rangle_{\mathbb{Z}} = \langle 0 | T\{\hat{s}(i_1) \cdots \hat{s}(i_n)\} | 0 \rangle. \quad (50)$$

**Exercise 2.2.** Consider adding a next-to-nearest neighbor interaction to the 1d Ising model

$$S[s] = \sum_i K s_i s_{i+1} + K' s_i s_{i+2}. \quad (51)$$

Find the associated quantum Hilbert space and transfer matrix, and compute the partition function on  $\mathbb{Z}_M$  in the limit of large  $M$ .

## 2.2. Quantization in quantum mechanics

The procedure of turning a path integral into a product of quantum operators is called quantization. It is an extremely general procedure that ultimately stems from “cutting and gluing” rules of the path integral.

For a more familiar example, let us review quantization for the path integral of a quantum particle on the real line. This theory has Euclidean action

$$S[x] = \int d\tau \left( \frac{1}{2} \dot{x}^2 + V(x) \right), \quad (52)$$

where  $x(\tau)$  is a map from Euclidean time  $\tau$  to  $\mathbb{R}$ . Consider the path integral on the interval  $\tau \in [\tau_a, \tau_b]$  with fixed boundary conditions

$$U(x_b, \tau_b; x_a, \tau_a) \equiv \int_{\substack{x(\tau_b)=x_b \\ x(\tau_a)=x_a}} Dx(\tau) e^{-S[x]}. \quad (53)$$

<sup>17</sup>We hope that the time-ordering symbol  $T\{\cdots\}$  will not be confused with the transfer matrix.

By grouping paths according to their positions at a fixed time  $\tau_c \in (\tau_a, \tau_b)$ , we obtain a simple “gluing” rule for  $U$ ,

$$U(x_b, \tau_b; x_a, \tau_a) = \int_{-\infty}^{\infty} dx_c U(x_b, \tau_b; x_c, \tau_c) U(x_c, \tau_c; x_a, \tau_a). \quad (54)$$

To quantize the theory, we build up the path integral in small time-increments using the gluing rule. Suppose we have already computed the path integral  $U(x_0, \tau_0; x_a, \tau_a)$  on the interval  $[\tau_a, \tau_0]$ . To extend  $\tau_0 \rightarrow \tau_0 + \epsilon$ , we write

$$U(x_1, \tau_0 + \epsilon; x_a, \tau_a) = \int dx_0 U(x_1, \tau_0 + \epsilon; x_0, \tau_0) U(x_0, \tau_0; x_a, \tau_a). \quad (55)$$

The quantity

$$U(x_1, \tau_0 + \epsilon; x_0, \tau_0) = U(x_1, \epsilon; x_0, 0). \quad (56)$$

plays the role of a transfer matrix, and  $U(x_0, \tau_0; x_a, \tau_a)$  plays the role of a state at time  $\tau_0$ . To see this, we introduce a Hilbert space

$$\mathcal{H} = \text{Span}\{|x\rangle : x \in \mathbb{R}\} \quad (57)$$

with inner product  $\langle x'|x\rangle = \delta(x - x')$ . The  $|x\rangle$  are analogs of the basis states  $|s = \pm 1\rangle$  in the Ising model. Defining  $\widehat{T}_\epsilon$  and  $|\Psi(\tau_0)\rangle$  by

$$\begin{aligned} \langle x_1 | \widehat{T}_\epsilon | x_0 \rangle &= U(x_1, \epsilon; x_0, 0), \\ \langle x_0 | \Psi(\tau_0) \rangle &= U(x_0, \tau_0; x_a, \tau_a), \end{aligned} \quad (58)$$

equation (55) becomes

$$|\Psi(\tau_0 + \epsilon)\rangle = \widehat{T}_\epsilon |\Psi(\tau_0)\rangle. \quad (59)$$

To recover the Schrodinger equation, we must show that

$$\widehat{T}_\epsilon = 1 - \epsilon \widehat{H} + O(\epsilon^2), \quad (60)$$

where

$$\widehat{H} = -\frac{1}{2} \frac{\partial^2}{\partial x^2} + V(x) \quad (61)$$

is the usual quantum-mechanical Hamiltonian. This is a standard argument, and we give a quick version here for completeness. We have

$$\begin{aligned} \langle x_1 | \widehat{T}_\epsilon | x_0 \rangle &= \int_{\substack{x(\epsilon)=x_1 \\ x(0)=x_0}} Dx(\tau) e^{-S[x]}, \quad (\epsilon \ll 1) \\ S[x] &= \int_0^\epsilon d\tau \left( \frac{1}{2} \dot{x}^2 + V(x) \right). \end{aligned} \quad (62)$$

Because the time interval is so short, if  $|x(\tau) - x_0|$  ever becomes larger than  $O(\epsilon)$ , the kinetic term will cause the amplitude to be highly suppressed. Thus, let us assume  $|x(\tau) - x_0|$  is of order  $\epsilon$ . This means we can replace  $V(x) \rightarrow V(x_0)$  (up to subleading corrections in  $\epsilon$ ), so the potential factors out of the integrand

$$e^{-S[x]} = e^{-\epsilon V(x_0)} \exp\left(\int_0^\epsilon d\tau \frac{\dot{x}^2}{2}\right). \quad (63)$$

We can now split  $x(\tau)$  into a classical term and a fluctuation term  $\zeta(\tau)$  with boundary conditions  $\zeta(0) = \zeta(\epsilon) = 0$ ,

$$\begin{aligned} x(\tau) &= x_0 \left(1 - \frac{\tau}{\epsilon}\right) + x_1 \frac{\tau}{\epsilon} + \zeta(\tau), \\ \int_0^\epsilon d\tau \frac{\dot{x}^2}{2} &= \frac{(x_1 - x_0)^2}{2\epsilon} + \int_0^\epsilon d\tau \frac{\dot{\zeta}^2}{2}. \end{aligned} \quad (64)$$

The path integral over  $\zeta$  contributes a constant  $A_\epsilon$  that depends on how the theory is regulated.<sup>18</sup> We thus find

$$\langle x_1 | \hat{T} | x_0 \rangle = A_\epsilon e^{-\epsilon V(x_0)} e^{-\frac{(x_1 - x_0)^2}{2\epsilon}} (1 + O(\epsilon)). \quad (65)$$

Finally, the Gaussian factor can be expanded in  $\epsilon$  using

$$e^{-\frac{x^2}{2\epsilon}} = \sqrt{2\pi\epsilon} \left( \delta(x) + \frac{\epsilon}{2} \delta''(x) + O(\epsilon^2) \right), \quad (66)$$

which gives

$$\begin{aligned} \langle x_1 | \hat{T} | x_0 \rangle &= A_\epsilon \sqrt{2\pi\epsilon} \left( \delta(x_1 - x_0) - \epsilon \left( -\frac{1}{2} \delta''(x_1 - x_0) + V(x_0) \delta(x_1 - x_0) \right) + O(\epsilon^2) \right) \\ &= A_\epsilon \sqrt{2\pi\epsilon} \langle x_1 | 1 - \epsilon \hat{H} + O(\epsilon^2) | x_0 \rangle. \end{aligned} \quad (67)$$

The prefactor  $A_\epsilon \sqrt{2\pi\epsilon}$  is an overall regulator-dependent constant that can be renormalized away by adding a “cosmological constant” term to the action.

Thus, in the limit  $\epsilon \rightarrow 0$ , equation (59) becomes a Euclidean version of the Schrodinger equation

$$\frac{d}{d\tau} |\Psi(\tau)\rangle = -\hat{H} |\Psi(\tau)\rangle. \quad (68)$$

Integrating using the initial condition  $U(x', 0, x, 0) = \delta(x - x')$ , we find

$$U(x_b, \tau_b; x_a, \tau_a) = \langle x_b | e^{-\hat{H}(\tau_b - \tau_a)} | x_a \rangle. \quad (69)$$

<sup>18</sup>A very simple regulator is to approximate  $x(\tau)$  as a piecewise linear path with segments of length  $\epsilon$ . This is equivalent to simply setting  $\zeta = 0$  and not doing the integral.

Path integrals on different geometries correspond to different quantum mechanical observables. For example, the path integral on a circle  $S^1_\beta$  of length  $\beta$  with periodic boundary conditions is equal to

$$\int_{-\infty}^{\infty} dx U(x, \beta; x, 0) = \text{Tr}(e^{-\beta \hat{H}}), \quad (70)$$

which is the quantum partition function at inverse temperature  $\beta$ . This is the continuum analog of equation (37). Inserting observables into the path integral gives a time-ordered product of the associated quantum operators. For example, with periodic boundary conditions,

$$\begin{aligned} \langle x(\tau_1) \cdots x(\tau_n) \rangle_{S^1_\beta} &= \int_{x(\beta)=x(0)} Dx x(\tau_1) \cdots x(\tau_n) e^{-S[x]} \\ &= \text{Tr}(T\{\hat{x}(\tau_1) \cdots \hat{x}(\tau_n)\} e^{-\beta \hat{H}}), \end{aligned} \quad (71)$$

where we introduced the operator

$$\begin{aligned} \hat{x}(\tau) &= e^{\tau \hat{H}} \hat{x} e^{-\tau \hat{H}}, \\ \hat{x}|x\rangle &= x|x\rangle. \end{aligned} \quad (72)$$

To derive (71), we cut the interval  $[0, \beta]$  into segments  $[0, \tau_1] \cup [\tau_1, \tau_2] \cup \cdots \cup [\tau_n, \beta]$  (assuming  $\tau_n > \cdots > \tau_1$ ) and use (69) for each segment. The operator  $\hat{x}$  is the analog of  $\hat{s}$  and equation (71) is the analog of (48).

Finally, in the limit  $\beta \rightarrow \infty$ , the factor  $e^{-\beta \hat{H}}$  projects onto the ground state. Thus the Euclidean path integral on  $\mathbb{R}$  is equal to a time-ordered correlation function in the ground state,

$$\langle x(\tau_1) \cdots x(\tau_n) \rangle_{\mathbb{R}} = \langle 0 | T\{\hat{x}(\tau_1) \cdots \hat{x}(\tau_n)\} | 0 \rangle. \quad (73)$$

This is the continuum analog of (50).

*End of lecture 3*

### 2.3. The 2d Ising model

Let us now consider a slightly more complicated case: the 2d Ising model. This will be a good toy example for how cutting, gluing, and quantization work in a general QFT. For simplicity, we set  $h = 0$ . We consider the partition function on the doubly-periodic lattice  $\mathbb{Z}_M \times \mathbb{Z}_N$  and label spins  $s_{i,j}$  by a pair  $(i, j) \in \mathbb{Z}_M \times \mathbb{Z}_N$ .

The action is given by

$$\begin{aligned}
S[s] &= -K \sum_{i,j} (s_{i,j} s_{i+1,j} + s_{i,j} s_{i,j+1}) \\
&= K \sum_{i,j} \left( \frac{1}{2} (s_{i,j+1} - s_{i,j})^2 - 1 \right) - K \sum_{i,j} s_{i,j} s_{i+1,j} \\
&= \text{const.} + \sum_{j=1}^N L(\mathbf{s}_{j+1}, \mathbf{s}_j).
\end{aligned} \tag{74}$$

In the last line, we split the action into contributions from pairs of neighboring rows. The notation  $s_j$  represents the configuration of spins in the  $j$ -th row,

$$(\mathbf{s}_j)_i = s_{i,j}. \tag{75}$$

The action associated with a pair of neighboring rows is given by

$$L(\mathbf{s}', \mathbf{s}) = \frac{1}{2} K \sum_{i=1}^M (\mathbf{s}'_i - \mathbf{s}_i)^2 - \frac{1}{2} K \sum_{i=1}^M (\mathbf{s}_{i+1} \mathbf{s}_i + \mathbf{s}'_{i+1} \mathbf{s}'_i). \tag{76}$$

(The constant in (74) gives an unimportant multiplicative constant in the path integral that will disappear in normalized correlation functions, so we will ignore it.)

To quantize the theory, we can think of the  $j$  direction as time, so  $\mathbf{s}_j$  is interpreted as a classical configuration on a fixed time-slice. The Hilbert space has an orthonormal basis vector for each such configuration,

$$\begin{aligned}
\mathcal{H}_M &= \text{Span} \{ |\pm 1, \pm 1, \dots, \pm 1\rangle \} \\
&= \bigotimes_{i=1}^M \mathcal{H}_i,
\end{aligned} \tag{77}$$

where  $\mathcal{H}_i$  is a 1-qubit Hilbert space for each site  $i$ .  $\mathcal{H}_M$  is the quantum Hilbert space of  $M$  qubits, and is  $2^M$ -dimensional.

The transfer matrix between successive time slices is a  $2^M \times 2^M$  matrix with entries

$$\langle \mathbf{s}' | \hat{T} | \mathbf{s} \rangle = e^{-L(\mathbf{s}', \mathbf{s})}. \tag{78}$$

The partition function on  $\mathbb{Z}_M \times \mathbb{Z}_N$  is then

$$Z(\mathbb{Z}_M \times \mathbb{Z}_N) = \text{Tr}_{\mathcal{H}_M}(\hat{T}^N). \tag{79}$$

To compute correlation functions, we need an operator that measures the spin at site  $i$ . This is simply the Pauli spin matrix  $\sigma_i^z$  associated with the  $i$ -th site

$$\sigma_i^z |s_1, \dots, s_j, \dots, s_M\rangle = s_i |s_1, \dots, s_i, \dots, s_M\rangle. \quad (80)$$

We also have Heisenberg picture operators

$$\hat{s}_{i,j} = \hat{T}^{-j} \sigma_i^z \hat{T}^j. \quad (81)$$

Correlation functions become traces of time-ordered products, e.g.

$$\begin{aligned} \langle s_{i_1, j_1} s_{i_2, j_2} \rangle &= \text{Tr}_{\mathcal{H}_M} (\hat{T}^{N+j_2-j_1} \sigma_{i_1}^z \hat{T}_2^{j_1-j_2} \sigma_{i_2}^z) \theta(j_1 - j_2) + (1 \leftrightarrow 2) \\ &= \text{Tr}_{\mathcal{H}_M} (T \{ \hat{s}_{i_1, j_1} \hat{s}_{i_2, j_2} \} \hat{T}^N). \end{aligned} \quad (82)$$

Let us now emphasize an important new ingredient in the 2-dimensional case compared to the 1-dimensional case. To arrive at (79), we had to choose  $j$  as the time direction. We then cut the path integral along rows of constant  $j$ . However, we could just as well have chosen  $i$  as the time direction and cut the path integral along columns of constant  $i$ . This would give a different Hilbert space  $\mathcal{H}_N$  with dimension  $2^N$ , a new transfer matrix  $\hat{T}'$  (acting on  $\mathcal{H}_N$ ), and a different formula for the same path integral,

$$Z(\mathbb{Z}_M \times \mathbb{Z}_N) = \text{Tr}_{\mathcal{H}_N} (\hat{T}'^M) = \text{Tr}_{\mathcal{H}_M} (\hat{T}^N). \quad (83)$$

In this new quantization, an insertion of  $s_{i,j}$  in the path integral becomes an insertion of

$$s_{i,j} \rightarrow \hat{T}'^{-i} \sigma_j^z \hat{T}'^i \quad (84)$$

in a time-ordered product of quantum operators. Let us emphasize that the operators (81) and (84) truly are different, even though they represent the same path integral variable. They even act on different-dimensional Hilbert spaces ( $2^M$  vs.  $2^N$ )! Furthermore, even the notion of “time”-ordering is different in this different quantization: in one case, operators are ordered according to their  $i$  coordinates, and in the other they are ordered according to their  $j$  coordinates.

**Exercise 2.3.** Show how to quantize the 2d Ising model in yet another way, using  $i + j$  as the time coordinate and  $i - j$  as space coordinate. What is the Hilbert space? What is the transfer matrix?



#### 2.4. Atiyah-Segal axioms

We are now ready to understand some axioms of continuum QFT. We will simultaneously state them and give examples in a Lagrangian theory of a real scalar field  $\phi$ . A version of these axioms in TQFTs is due to Atiyah and a version in 2d CFTs is due to Segal. We will be interested in QFTs that depend on the geometry of the space in which they live, so when we say “manifold” below, we mean “Riemannian manifold with metric.”

Consider a  $d$ -dimensional QFT  $Q$ .

- (1) To every  $(d - 1)$ -manifold  $N$  without boundary,  $Q$  assigns a Hilbert space  $\mathcal{H}_N$  called the space of states on  $N$ .

For example, in theories with an explicit path integral over some fields (“Lagrangian theories”),  $\mathcal{H}_N$  has an orthonormal basis state for every field configuration on  $N$ . In scalar field theory,

$$\mathcal{H}_N = \text{Span}\{|\phi_b\rangle : \phi_b \in C(N, \mathbb{R})\}. \quad (85)$$

with inner product  $\langle \phi_{b1} | \phi_{b2} \rangle = \delta(\phi_{b1} - \phi_{b2})$  (a functional  $\delta$ -function). Here  $C(N, \mathbb{R})$  is a space of real-valued functions on  $N$ .<sup>19</sup> More formally, we can write

$$\mathcal{H}_N = L^2(C(N, \mathbb{R}), \mathbb{C}), \quad (86)$$

where  $L^2(X)$  denotes the square-integrable complex functions on  $X$ . The space  $\mathcal{H}_N$  is the analog of the Hilbert spaces (57) and (77) for quantum mechanics and the 2d Ising model. Let us discuss the example of quantum mechanics more explicitly. Quantum mechanics is a 1-dimensional QFT. In this case  $d - 1 = 0$ , and the only possible 0-manifolds are disjoint unions of points. The space of field configurations on a point is  $C(\bullet, \mathbb{R}) = \mathbb{R}$ , so the Hilbert space associated to a point is  $\mathcal{H}_\bullet = L^2(C(\bullet, \mathbb{R}), \mathbb{C}) = L^2(\mathbb{R}, \mathbb{C})$ . This is spanned by the usual basis states  $|x\rangle$  with inner product  $\delta(x - x')$ .<sup>20</sup>

- (2) The space of states on a disconnected union is a tensor product

$$\mathcal{H}_{N_1 \sqcup N_2} = \mathcal{H}_{N_1} \otimes \mathcal{H}_{N_2}. \quad (87)$$

<sup>19</sup>We are being deliberately vague about *which* space of functions on  $N$ , e.g.  $C^1(N)$ ,  $C^\infty(N)$ , etc.. In reality, the path integral is defined by introducing a regulator and taking a limit as the regulator is removed. Different regulators will result in different spaces of functions, and we prefer to be agnostic about which regulator is being used.

<sup>20</sup>Note in particular that  $L^2(C(\bullet, \mathbb{R}), \mathbb{C})$  is totally different from  $L^2(\bullet, \mathbb{C}) = \mathbb{C}$ . In general, don’t confuse the Hilbert space  $\mathcal{H}_N$  with the space of square-integrable functions  $L^2(N, \mathbb{C})$ .

This should be clear in scalar theory from the definition (85). It will be useful to assign a Hilbert space to the empty manifold, and the only reasonable choice consistent with (87) is

$$\mathcal{H}_\emptyset = \mathbb{C}. \quad (88)$$

- (3) To every  $d$ -manifold  $M$  with incoming boundary  $N$  and outgoing boundary  $N'$ ,  $Q$  assigns a transition amplitude<sup>21</sup>

$$Z_M : \mathcal{H}_N \rightarrow \mathcal{H}_{N'}. \quad (89)$$

In Lagrangian theories,  $Z_M$  is obtained by fixing boundary conditions on  $N, N'$  and performing the path integral over the interior. For example in a scalar theory, the matrix elements of  $Z_M$  are given by

$$\langle \phi'_b | Z_M | \phi_b \rangle = \int_{\substack{\phi|_N = \phi_b \\ \phi|_{N'} = \phi'_b}} D\phi(x) e^{-S[\phi]}. \quad (90)$$

Here, the notation  $\phi|_N$  means the restriction of  $\phi$  to  $N$ .

In the special case where  $M$  is a closed manifold (without boundary),  $Z_M$  is a linear map

$$Z_M : \mathbb{C} \rightarrow \mathbb{C}. \quad (91)$$

A linear map  $\mathbb{C} \rightarrow \mathbb{C}$  is simply multiplication by complex number. That number is the “partition function” of the QFT on  $M$ .

Another important case is where  $M$  has boundaries  $(\emptyset, N)$ . (The first element of the ordered pair is the incoming boundary and the second is the outgoing boundary.) In this case,  $Z_M$  gives a linear map

$$Z_M : \mathbb{C} \rightarrow \mathcal{H}_N. \quad (92)$$

Any such map is uniquely determined by its action on  $1 \in \mathbb{C}$ , so this is equivalent to a state  $|\Psi\rangle = Z_M(1) \in \mathcal{H}_N$ . Thus, we can “prepare a state” by performing the path integral with fixed boundary conditions. We saw several examples in the previous sections, for example the definition of  $Z_{\text{partial}}(i, s)$  in the 1d Ising model.

Finally, consider the case where  $M$  has boundaries  $(N, \emptyset)$ . Then we have a linear map

$$Z_M : \mathcal{H}_N \rightarrow \mathbb{C}, \quad (93)$$

which is equivalent to a bra-state  $\langle \Psi | \in \mathcal{H}_N^*$ .

The usefulness of assigning  $\mathcal{H}_\emptyset = \mathbb{C}$  is that all these special cases (91, 92, 93) are covered by one axiom (89).

<sup>21</sup>By incoming vs. outgoing boundary, we assume that each boundary component  $N$  comes equipped with a nonzero infinitesimal vector field pointing into or out of  $M$ .

- (4) Suppose  $M_1$  has boundaries  $(N, N')$  and  $M_2$  has boundaries  $(N', N'')$ . We can form a new  $d$ -manifold  $M_1 \cup_{N'} M_2$  by gluing  $M_1$  and  $M_2$  along  $N'$ . This manifold has boundaries  $(N, N'')$ . The total transition amplitude is the composition

$$Z_{M_1 \cup_{N'} M_2} = Z_{M_2} Z_{M_1} : \mathcal{H}_N \rightarrow \mathcal{H}_{N''}. \quad (94)$$

In scalar field theory, this follows by cutting the path integral along  $N'$  by fixing boundary conditions  $\phi'_b \in C(N', \mathbb{R})$ , and then finally performing the  $(d-1)$ -dimensional path integral over  $\phi'_b$ ,

$$\begin{aligned} & \int_{\substack{\phi|_N = \phi_b \\ \phi|_{N''} = \phi'_b}} D\phi(x) e^{-S[\phi]} \\ &= \int_{y \in N'} D\phi'_b(y) \int_{\substack{\phi_2|_{N'} = \phi'_b \\ \phi_1|_{N''} = \phi'_b \\ x \in M_2}} D\phi_2(x) e^{-S[\phi_2]} \int_{\substack{\phi_1|_N = \phi_b \\ \phi_1|_{N'} = \phi'_b \\ x \in M_1}} D\phi_1(x) e^{-S[\phi_1]}. \end{aligned} \quad (95)$$

This is the field-theory analog of our gluing rule for transition amplitudes in quantum mechanics (54). In the notation (90), we have simply inserted a complete set of states on  $N'$ ,

$$\langle \phi''_b | Z_{M_2} Z_{M_1} | \phi_b \rangle = \int_{y \in N'} D\phi'_b(y) \langle \phi''_b | Z_{M_2} | \phi'_b \rangle \langle \phi'_b | Z_{M_1} | \phi_b \rangle. \quad (96)$$

- (5) Describe generalization to correlation functions.  
 (6) Discuss orientation, orientation reversal, conjugation.

Discuss the possibility of giving the manifolds different amounts of structure, for instance topological structure, a Riemannian metric, etc. In these notes, we will initially be interested in manifolds with a metric. Later, we will see that in certain cases we can relax this structure to only a conformal class of metrics.

Although we have given explicit prescriptions in the case of a continuum theory, all of these axioms have analogs in the case of the 2d Ising model. It is not hard to imagine how the continuum axioms might arise in the continuum limit of the Ising model, at least in the case where the “manifolds”  $M, N, \dots$  are regions of flat space.

*End of lecture 4*

## 2.5. Quantization in continuum QFT

Let us use the above axioms to describe the quantization procedure one more time, now in continuum QFT. Consider the  $d$ -manifold  $M_L = I_L \times N$ ,

where  $I_L = [0, L]$  is an interval of length  $L$  and  $N$  is a  $(d - 1)$ -manifold, and suppose  $M_L$  is endowed with the product metric. The composition rule (94) implies that

$$Z_{M_L} = e^{-L\hat{H}} \quad (97)$$

for some operator

$$\hat{H} : \mathcal{H}_N \rightarrow \mathcal{H}_N, \quad (98)$$

namely the Hamiltonian.

Consider a correlation function on  $S_\beta^1 \times N$

$$\langle \phi(\tau_1, y_1) \cdots \phi(\tau_n, y_n) \rangle_{S_\beta^1 \times N} = \int_{\phi(\beta, y) = \phi(0, y)} D\phi(x) \phi(\tau_1, y_1) \cdots \phi(\tau_n, y_n) e^{-S[\phi]}. \quad (99)$$

Here,  $\tau, y$  are coordinates on  $S_\beta^1, N$ , respectively. As before, we introduce operators  $\hat{\phi}(y)$  that measure the field configuration on a time-slice

$$\hat{\phi}(y)|\phi_b\rangle = \phi_b(y)|\phi_b\rangle. \quad (100)$$

To compute (99), we cut the path integral at times  $\tau_1, \dots, \tau_n$  and repeatedly use (97). By the usual logic, this gives

$$\langle \phi(\tau_1, y_1) \cdots \phi(\tau_n, y_n) \rangle_{S_\beta^1 \times N} = \text{Tr}_{\mathcal{H}_N} (e^{-\beta\hat{H}} T\{\hat{\phi}(\tau_1, y_1) \cdots \hat{\phi}(\tau_n, y_n)\}), \quad (101)$$

where

$$\hat{\phi}(\tau, y) \equiv e^{\tau\hat{H}} \hat{\phi}(y) e^{-\tau\hat{H}}, \quad (102)$$

and  $T\{\cdots\}$  represents time ordering in  $\tau$ . Taking the limit of a long interval, the factor  $e^{-\beta\hat{H}}$  projects onto the ground state, and we obtain<sup>22</sup>

$$\langle \phi(\tau_1, y_1) \cdots \phi(\tau_n, y_n) \rangle_{\mathbb{R} \times N} = \langle 0 | T\{\hat{\phi}(\tau_1, y_1) \cdots \hat{\phi}(\tau_n, y_n)\} | 0 \rangle. \quad (103)$$

For a QFT in  $\mathbb{R}^d$ , we can quantize the theory in many different ways by choosing different directions to play the role of “time.” In every quantization of the theory, we have

$$\langle \phi(x_1) \cdots \phi(x_n) \rangle_{\mathbb{R}^d} = \langle 0 | T\{\hat{\phi}(x_1) \cdots \hat{\phi}(x_n)\} | 0 \rangle. \quad (104)$$

However, in different quantizations, the objects appearing on the right-hand side are different. The Hilbert spaces are different (though isomorphic if the

<sup>22</sup>In (103), we are assuming that the theory has a unique ground state on  $N$ . More generally, there might be multiple ground states on  $N$ , in which case the limit  $\beta \rightarrow \infty$  becomes a trace in the space of ground states.

theory is  $\text{SO}(d)$ -invariant), the ground states  $|0\rangle$  are different, the quantum operators  $\hat{\phi}(x)$  are different, and the time-ordering symbols means different things. However, because they originate from a single path integral, the resulting expectation values are the same. It is sometimes useful to think of expressions like the RHS above as “interpretations” of one underlying path integral.

Discuss boundary conditions and whether we should think of  $\mathbb{R}^d$  as having a boundary or not.

### 2.6. The quantum transverse-field Ising model

After all this abstract nonsense, let’s return to more concrete things and try to understand the critical point of the 2d Ising model. The transfer matrix  $\hat{T}$  of the 2d Ising lattice model was diagonalized in 1944 by Onsager. Unfortunately, we won’t have time to describe his solution. Instead, we will study a closely related model in the same universality class as the lattice model.

Recall that the transfer matrix has matrix elements  $\langle \mathbf{s}' | \hat{T} | \mathbf{s} \rangle = e^{-L(\mathbf{s}', \mathbf{s})}$ . Let us write  $\hat{T}$  in a more familiar way as an operator on a spin chain. First split  $L$  into contributions from horizontal and vertical bonds

$$\begin{aligned} L(\mathbf{s}', \mathbf{s}) &= L_h(\mathbf{s}') + L_h(\mathbf{s}) + L_v(\mathbf{s}', \mathbf{s}). \\ L_h(\mathbf{s}) &= -\frac{1}{2}K \sum_i \mathbf{s}_{i+1} \mathbf{s}_i, \\ L_v(\mathbf{s}', \mathbf{s}) &= \sum_i \frac{1}{2}K(\mathbf{s}'_i - \mathbf{s}_i)^2. \end{aligned} \tag{105}$$

Note that

$$e^{\frac{1}{2}K \sum_i \sigma_i^z \sigma_{i+1}^z} |\mathbf{s}\rangle = e^{-L_h(\mathbf{s})} |\mathbf{s}\rangle. \tag{106}$$

Meanwhile,  $L_v$  only involves spins at a single site, so let us imagine that we have only one site. Note that

$$\langle s' | (1 + e^{-2K} \sigma^x) | s \rangle = e^{-\frac{1}{2}K(s' - s)^2}. \tag{107}$$

We also have

$$\begin{aligned} 1 + e^{-2K} \sigma^x &= e^{A + K' \sigma^x}, \quad \text{where} \\ \tanh K' &= e^{-2K}, \\ e^A &= \sqrt{1 - e^{-4K}}, \end{aligned} \tag{108}$$

which follows by expanding out the Taylor series for  $e^{A+K'\sigma^x}$  and matching the coefficients of  $1, \sigma^x$ . Thus,

$$e^{AM} \langle \mathbf{s}' | e^{K' \sum_i \sigma_i^x} | \mathbf{s} \rangle = e^{-L_v(\mathbf{s}', \mathbf{s})}. \quad (109)$$

The constant  $e^{AM}$  will cancel in correlation functions, so we will ignore it. Putting everything together, we find

$$\hat{T} \propto \exp \left( \frac{1}{2} K \sum_i \sigma_i^z \sigma_{i+1}^z \right) \exp \left( K' \sum_i \sigma_i^x \right) \exp \left( \frac{1}{2} K \sum_i \sigma_i^z \sigma_{i+1}^z \right). \quad (110)$$

In a quantum-mechanical interpretation, we would write  $\hat{T} = e^{-\Delta\tau \hat{H}}$ , but the resulting  $\hat{H}$  would be very complicated.

To get the theory we'll actually study, we first allow  $K', K$  to be independent parameters, and then take a limit where  $K, K'$  become small with fixed ratio  $K'/K = g$ . These modifications may seem somewhat drastic, but it turns out that, at the critical point, they can be compensated by simply rescaling the time coordinate. (We will not prove this, unfortunately.) This results in

$$\hat{T} \rightarrow e^{-\Delta\tau \hat{H}}, \quad (111)$$

where  $\Delta\tau$  is small and the Hamiltonian  $\hat{H}$  is relatively simple,

$$\hat{H} = \sum_i \sigma_i^z \sigma_{i+1}^z + g \sum_i \sigma_i^x. \quad (112)$$

This is the Hamiltonian of the “quantum transverse-field Ising model” (TFIM). It describes a 1-dimensional chain of quantum spins with a nearest-neighbor interaction in the  $z$ -direction and an applied “transverse” magnetic field in the  $x$ -direction.<sup>23,24</sup>

### 2.6.1. Solution via the Jordan-Wigner transformation

In the remainder of this section, let us solve the TFIM by diagonalizing  $\hat{H}$ . Since our Hilbert space is a tensor product of two possible states for each

<sup>23</sup>Don't confuse the transverse field in the quantum Ising model with the applied magnetic field  $h$  in the thermodynamic Ising model. At the beginning of this discussion, we set  $h = 0$ .

<sup>24</sup>A similar procedure starting from the  $d$ -dimensional classical Ising model gives the quantum TFIM on a  $(d-1)$ -dimensional spatial lattice, which is again in the same universality class.

site, it's tempting to think of it as a fermionic Fock space, with creation and annihilation operators

$$\sigma_n^\pm \equiv \frac{1}{2}(\sigma_n^y \pm i\sigma_n^z). \quad (113)$$

This is not correct, since these operators commute rather than anticommute at different sites. However, this can be fixed with a classic trick called the Jordan-Wigner transformation. We define fermionic creation and annihilation operators

$$c_n^\dagger = \left( \prod_{i=1}^{n-1} \sigma_i^x \right) \sigma_n^+, \quad c_n = \left( \prod_{i=1}^{n-1} \sigma_i^x \right) \sigma_n^-, \quad (114)$$

which now satisfy canonical anticommutation relations

$$\{c_n^\dagger, c_m\} = \delta_{nm}, \quad \{c_n, c_m\} = \{c_n^\dagger, c_m^\dagger\} = 0. \quad (115)$$

These new creation and annihilation operators are *nonlocal* on the spin chain. We can think of them as creating and destroying fermionic solitons.

**Exercise 2.4.** Verify the anticommutation relations (115).

It's perhaps not surprising that we can write nonlocal variables that behave like fermions. The surprise is that the Hamiltonian in these new variables is still local, and actually quite simple

$$H = \sum_{n=1}^{M-1} (c_n^\dagger + c_n)(c_{n+1}^\dagger - c_{n+1}) - P(c_M^\dagger + c_M)(c_1^\dagger - c_1) + g \sum_{n=1}^M (2c_n^\dagger c_n - 1) \quad (116)$$

where  $P = \prod_{n=1}^M \sigma_n^x = (-1)^F$  is the parity operator. Within each parity eigenspace, the Hamiltonian becomes simply

$$H = \sum_{n=1}^N (c_n^\dagger + c_n)(c_{n+1}^\dagger - c_{n+1}) + g \sum_{n=1}^N (2c_n^\dagger c_n - 1), \quad (117)$$

where when  $P = -1$  we must impose periodic boundary conditions  $c_{M+1} = c_1$ , and when  $P = 1$  we must impose antiperiodic boundary conditions  $c_{M+1} = -c_1$ .<sup>25</sup> Our Hamiltonian is now translationally invariant and

<sup>25</sup>The choice of which eigenspace  $P = \pm 1$  to consider is equivalent to a choice of spin structure. The TFIM is a bosonic theory and can be formulated without choosing a spin structure. However, once we introduce fermions, we are required to choose a spin structure.

quadratic in creation and annihilation operators, so it can be diagonalized via a Fourier transform and Bogoliubov transformation:

$$H = \sum_k \left[ -(2 \cos k - 2g) c_k^\dagger c_k - i \sin k (c_{-k}^\dagger c_k^\dagger + c_{-k} c_k) \right] - Mg \quad (118)$$

$$= \sum_k \epsilon(k) \left( b_k^\dagger b_k - \frac{1}{2} \right), \quad (119)$$

where  $b_k^\dagger$  and  $b_k$  are new canonically normalized creation and annihilation operators, and

$$\epsilon(k) = 2\sqrt{g^2 - 2g \cos(k) + 1} \quad (120)$$

is the dispersion relation for the free fermionic quasiparticles created by  $b_k^\dagger$ . Because of the boundary conditions imposed by parity, the quasimomenta must take the values

$$k = \begin{cases} \frac{2m\pi}{M} & P = -1, \\ \frac{(2m+1)\pi}{M} & P = 1. \end{cases} \quad (121)$$

for  $m = 0, 1, \dots, M-1$ .

[Write out the Bogoliubov transformation in more detail.](#)

**Exercise 2.5.** Derive expressions (116), (118), and (119).

In the continuum limit  $M \rightarrow \infty$ ,  $k$  becomes continuous. Correlation functions are dominated by the smallest values of  $\epsilon(k)$ , which (assuming  $g > 0$ ) occur near  $k = 0$ . Expanding around this point, we have

$$\epsilon(k)^2 = 4(1 - g)^2 + 4gk^2 + O(k^4). \quad (122)$$

Thus, the mass gap is given by  $m_{\text{gap}} = |g - 1|$ . At the special point  $g = g_c = 1$ , the mass gap goes to zero and we have a critical point. Long-distance correlation functions are dominated by states with arbitrarily low energy, which requires  $k \rightarrow 0$ . For such states, we can drop the  $O(k^4)$  term, and we obtain a relativistic dispersion relation<sup>26</sup>

$$\epsilon(k)^2 = 4k^2 \quad (g = g_c, \ k \ll 1). \quad (123)$$

To summarize, in the continuum limit, at the critical coupling, and at long distances, the quantum TFIM becomes a free relativistic fermion. This is yet another example of IR equivalence/duality. The IR theory has an emergent  $\text{SO}(1, 1)$  Lorentz symmetry that was not present in the original

<sup>26</sup>To obtain the usual dispersion relation  $\epsilon = |k|$ , we need to redefine either time or space by a factor of 2.



spin system. In fact, it also has emergent conformal symmetry, as we'll see later.

The fact that the critical point of the 1+1d quantum TFIM (and consequently other theories in the same universality class, like the 2d Ising lattice model) is equivalent to a free theory is very special to 2-dimensions. Our derivation relied in a fundamental way on the Jordan-Wigner transformation and does not work, for example in 2+1d.<sup>27</sup>

*End of lecture 5*

### 3. CFTs in perturbation theory

#### 3.1. *Scaling and renormalization*<sup>28</sup>

In the quantum TFIM, we found  $m_{\text{gap}} = |g - g_c|$ , which means that the correlation length critical exponent  $\nu$  (defined in equation (15)) is  $\nu = 1$  for this theory. In this section, we explain how critical exponents are related to operator dimensions in the scale-invariant QFT (SFT) that describes the critical point.

Consider a microscopic system that has a long-distance description in terms of a quantum field theory  $\text{QFT}_{\text{IR}}$ . We will think of  $\text{QFT}_{\text{IR}}$  in the language of effective field theory: it has a length cutoff  $a$  and a set of coupling constants  $g_A$  determined by running and matching.

**Matching:** First, we choose a cutoff  $a \sim a_{\text{UV}}$  near the characteristic length scales of the microscopic theory. We fix the coupling constants  $g_A(a)$  at this scale by computing physical observables in both the UV and IR theories at distances  $x \gtrsim a$  and demanding that they agree term-by-term in a power series in  $a/x$ . To match more terms, we must adjust more coupling constants.

An important point is that this matching procedure is analytic in the UV parameters. Specifically, if the action of the UV theory has a parameter  $t$ , then  $g_A(a)$  will generically have a regular power-series expansion in  $t$

$$g_A(a) = \gamma_{A0} + \gamma_{A1}t + \gamma_{A2}t^2 + \dots \quad (124)$$

<sup>27</sup>There are newly-discovered versions of the Jordan-Wigner transformation in higher dimensions ([arXiv:1711.00515](#) — one of your classmates is a coauthor!), but they do not relate the 2+1d TFIM to a free theory.

<sup>28</sup>Sources for this section: Pufu [1], Cardy [2].

Roughly speaking this is because the matching happens at a fixed length scale, while nonanalyticities come from summing up contributions over many scales, as we will see shortly.<sup>29</sup>

**Running:** Now we perform a “coarse-graining” operation where we rescale the cutoff  $a \rightarrow ba$  with  $b = 1 + \delta\ell$ ,  $\delta\ell \ll 1$  and adjust the coupling constants  $g_A$  to leave the long-distance physics invariant.<sup>30</sup> The adjustment takes the form

$$a \frac{dg_A}{da} = -\beta_A(g), \quad (125)$$

where  $\beta_A(g)$  is a vector-field in the space of couplings, called the “beta function.” The trajectories defined by (125) are “renormalization group (RG) flows.”<sup>31</sup>

An example RG flow is plotted in figure ???. We will be interested in fixed-points where  $\beta_A(g_*) = 0$ . At a fixed-point, observables become scale-invariant. To see this explicitly, note that by dimensional analysis, any dimensionless observable (for example a dimensionless ratio of correlation functions) must have the form

$$G\left(\frac{x_{ij}}{a}; g_A\right), \quad (126)$$

where  $x_{ij}$  are various distances and  $g_A$  are coupling constants. The  $\beta$ -function is defined by demanding that physical observables are invariant under simultaneously rescaling  $a$  and modifying the  $g_A$ , i.e.

$$\left(a \frac{\partial}{\partial a} + \beta_A(g) \frac{\partial}{\partial g_A}\right) G\left(\frac{x_{ij}}{a}; g_A\right) = 0. \quad (127)$$

<sup>29</sup>Put another way, the matching computation depends on degrees of freedom at some high energy scale. These degrees of freedom don’t care about whether the theory will eventually reach a fixed-point at low energies (that is a low-energy question). From the point of view of the UV degrees of freedom, the point  $t = 0$  is just some generic point in the space of couplings and there is no reason to have a singularity there.

<sup>30</sup>In practice, we usually use a modified version of this condition, where we keep the physics invariant up to corrections of the form  $(a/x)^n$  for some power  $n$  that depends on the desired accuracy. This goes hand-in-hand with the matching procedure: first we decide on how many power-law corrections  $(a/x)^n$  we want to keep, we then match coupling constants up to this order and find RG equations that preserve observables up to this order. A common simplification is to throw away all corrections  $(a/x)^n$  with  $n > 0$  — i.e. take the limit as the cutoff is completely removed  $a \rightarrow 0$ . This is the version of RG you find in most textbooks.

<sup>31</sup>The beta function  $\beta_A$  is analytic in  $g_A$  by a similar argument to the one above: it is computed by comparing two nearby length/energy scales.

Evaluating this equation at a fixed point  $g_*$ , we find

$$a \frac{\partial}{\partial a} G\left(\frac{x_{ij}}{a}; g_*\right) = 0 \quad \implies \quad G\left(\frac{x_{ij}}{a}; g_*\right) = f\left(\frac{x_{ij}}{x_{kl}}\right), \quad (128)$$

i.e.  $G$  is a function of ratios of distances alone.

A fixed-point of the RG flow must be a critical point. The reason is that the scaling behavior (128) is inconsistent with the long-distance behavior

$$\langle \mathcal{O}(x) \mathcal{O}(y) \rangle \sim e^{-m_{\text{gap}}|x-y|} \quad (|x-y| \gg a). \quad (129)$$

The only exceptions are  $m_{\text{gap}} = 0$  (so that we have a nontrivial scale-invariant theory) or  $m_{\text{gap}} = \infty$  (so that we have a TQFT).

Very close to a fixed point,  $\beta_A$  takes the form

$$\beta_A = - \sum_B Y_{AB} u_B + O(u^2), \quad (130)$$

where  $u = g - g_*$ . Redefining the  $u_A$  to diagonalize  $Y_{AB}$ ,<sup>32</sup> we have

$$\beta_A \approx -y_A u_A, \quad (131)$$

which integrates to

$$u_A \sim b^{y_A}. \quad (132)$$

We call the  $u_A$  “scaling variables.”

### 3.1.1. Relevant, marginal, and irrelevant variables

From here, we can distinguish three cases

- $y_A > 0$ : the coupling  $u_A$  is relevant and grows at long distances. An RG flow in the  $u_A$ -direction takes us away from the fixed-point.
- $y_A = 0$ : the coupling  $u_A$  is marginal. The fate of an RG flow in the  $u_A$ -direction depends on higher-order terms in the  $\beta$ -function.
- $y_A < 0$ : the coupling  $u_A$  is irrelevant. An RG flow in the  $u_A$ -direction takes us towards the fixed-point.

Relevant couplings must be tuned to stay close to the fixed point. From our discussion of the phase diagram of the Ising model, the fixed-point of the critical Ising model must have two relevant couplings: a thermal scaling variable  $u_t$  and a magnetic scaling variable  $u_h$ . In addition, there

<sup>32</sup>It is possible to have situations where  $Y_{AB}$  cannot be diagonalized, but rather has a nontrivial Jordan block form. This occurs, for example in logarithmic CFTs, which we may discuss briefly later. However, logarithmic CFTs are not unitary. We will prove later that in unitary CFTs, the matrix  $Y_{AB}$  is diagonalizable. For now, let us assume it.

are an infinite number of irrelevant variables  $u_3, \dots$ .<sup>33</sup> By our discussion of matching, the scaling variables  $u_t, u_h$  should be analytic in the parameters of the UV theory. They must also vanish when  $t = h = 0$  and be consistent with  $\mathbb{Z}_2$  symmetry. Thus,

$$\begin{aligned} u_t &= t/t_0 + O(t^2, h^2) \\ u_h &= h/h_0 + O(th), \end{aligned} \quad (133)$$

where  $t_0$  and  $h_0$  are non-universal constants.<sup>34</sup>

In the action, scaling variables  $u_A$  multiply local operators  $\Phi_A$  that transform in a simple way under coarse-graining. The action linearized around the fixed-point is given by

$$S = S_0 + \int \frac{d^d x}{a^d} \sum_A u_A \Phi_A(x). \quad (134)$$

Here, factors of the length-cutoff  $a^{-d}$  are needed to make  $S$  dimensionless. For example, if the UV theory is a lattice model,  $a^{-d}$  comes from writing the sum over lattice sites as an integral over space

$$\sum_{i \in \mathbb{Z}^d} \rightarrow \int \frac{d^d x}{a^d}. \quad (135)$$

In our conventions, the  $\Phi_A$  have classical dimension zero. (The word “classical” is to distinguish from another notion of dimension that we will introduce shortly.)

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<sup>33</sup>When relevant variables respect a symmetry but irrelevant variables do not, that symmetry will be emergent at long-distances. The irrelevant variables that are initially nonzero depend on the microscopic theory. For example, in magnets with no applied magnetic field, all  $\mathbb{Z}_2$ -odd irrelevant variables will vanish by symmetry. However, in liquids the  $\mathbb{Z}_2$ -symmetry is emergent, so there will in general be nonzero  $\mathbb{Z}_2$ -odd irrelevant variables. In systems where rotational invariance is emergent, there will also be irrelevant variables that transform nontrivially under  $\text{SO}(d)$ , e.g. a tensor  $u_{\mu_1 \dots \mu_j}$ .

<sup>34</sup>Generically, the map between UV and IR parameters is non-degenerate (i.e. differentiable with invertible derivative) at the fixed-point, so e.g.  $t_0, h_0$  are not 0 or  $\infty$ . To get a nondegenerate map, we might need to find the right parameters in the microscopic theory to wiggle. For example, if we had a magnet such that  $u_t \sim t^2$  (I don’t know of any case where this actually happens), then in addition to dialing the temperature and magnetic field, we could also compress the magnet with some pressure  $P$ . The three-dimensional map between the UV parameters  $(T, H, P)$  and scaling variables  $u_t, u_h, u_3$  will then generically be nondegenerate. If it still isn’t, we can add an additional parameter, etc..

In order for the action to be invariant under coarse-graining, we must change the operators according to

$$\begin{aligned} a &\rightarrow ba, \\ u_A &\rightarrow b^{y_A} u_A, \\ \Phi_A &\rightarrow b^{\Delta_A} \Phi_A, \end{aligned} \tag{136}$$

where

$$y_A + \Delta_A = d. \tag{137}$$

The number  $\Delta_A$  is called the “scaling dimension” of  $\Phi_A$ .

At the fixed-point  $u_A = 0$ , our coarse-graining rules fix the two-point function

$$\langle \Phi_A(x_1) \Phi_A(x_2) \rangle \propto \frac{a^{2\Delta_A}}{|x_1 - x_2|^{2\Delta_A}}. \tag{138}$$

The right hand side is the only quantity consistent with Poincare invariance, with having classical dimension zero, and with the transformations (136).

We can simplify this argument by introducing a version of dimensional analysis specialized to our fixed-point. We define operators and coupling constants with nontrivial length dimension by

$$\begin{aligned} \mathcal{O}_A &= a^{-\Delta_A} \Phi_A, \\ \lambda_A &= a^{-y_A} u_A = a^{\Delta_A - d} u_A. \end{aligned} \tag{139}$$

These combinations are invariant under the transformation (136). Invariance under coarse-graining then means that correlators of  $\mathcal{O}_A$ ’s can be functions of  $\lambda_A$  (and kinematic variables like distances), but are independent of  $a$ . The tradeoff is that correlators must now also be consistent with dimensional analysis under which  $\mathcal{O}_A$  has length-dimension  $-\Delta_A$  and  $\lambda_A$  has length-dimension  $-y_A = \Delta_A - d$ .

For example, at the fixed-point  $\lambda_A = 0$ , a two-point function of  $\mathcal{O}_A$  must be given by

$$\langle \mathcal{O}_A(x_1) \mathcal{O}_A(x_2) \rangle \propto \frac{1}{|x_1 - x_2|^{2\Delta_A}}. \tag{140}$$

The right-hand side is the only quantity consistent with Poincare symmetry and dimensional analysis. (Of course, it also follows trivially from (138) and (139).)<sup>35</sup>

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<sup>35</sup>Note that (140) determines the critical exponent  $\eta = 2\Delta_h + 2 - d$ .

We can now analyze the action using this specially-adapted version of dimensional analysis. We have

$$S = S_0 + \int d^d x \sum_A \lambda_A \mathcal{O}_A(x). \quad (141)$$

Let us split the sum into relevant and irrelevant contributions,

$$S = \left( S_0 + \int d^d x \sum_{\Delta_A < d} \lambda_A \mathcal{O}_A(x) \right) + \int d^d x \sum_{\Delta_A > d} \lambda_A \mathcal{O}_A(x). \quad (142)$$

(For simplicity we assume there are no marginal operators.) The irrelevant interactions scale towards the fixed-point, so at low energies we will assume they can be treated as small perturbations of the action in parentheses.<sup>36</sup>

For concreteness, consider the Ising model. Let us set the irrelevant  $\lambda$ 's to zero and tune  $\lambda_h = 0$  as well. There is now exactly one nonzero RG-invariant dimensionful quantity, namely  $\lambda_t$ . We can associate a mass scale to it given by

$$m_t = |\lambda_t|^{1/y_t} = |u_t|^{1/y_t} a^{-1}. \quad (143)$$

By dimensional analysis, any other dimensionful quantity must be proportional to  $m_t$  to the appropriate power. (Remember that we cannot use the UV cutoff anymore because it is not RG-invariant.) For example, the mass gap of the theory in the IR must be

$$m_{\text{gap}} \propto m_t \propto |u_t|^{1/y_t} \propto |t|^{1/y_t}. \quad (144)$$

This determines the critical exponent<sup>37</sup>

$$\nu = \frac{1}{y_t} = \frac{1}{d - \Delta_t}. \quad (145)$$

Note that the constant of proportionality in (144) can depend on the sign of  $t$ . The reason is that, depending on the sign of  $t$ , the theory may flow to different theories in the IR (figure ??). For example, in the Ising model, when  $t > 0$  the theory flows to a gapped phase where  $\mathbb{Z}_2$  is unbroken, while when  $t < 0$ , the theory flows to a gapped phase that spontaneously breaks  $\mathbb{Z}_2$ . No matter where the theory flows,  $m_t$  is still the only mass-scale around, so  $m_{\text{gap}}$  must be proportional to it. However, the actual dynamics

<sup>36</sup>This assumption is not always correct — it can happen that the effects of an irrelevant interaction initially decrease near the fixed-point, but begin to grow when the effect of relevant interactions gets strong. This is called a “dangerously-irrelevant” operator.

<sup>37</sup>Our calculation of  $\nu$  in the 2d Ising model shows that  $\Delta_t = 1$  for that theory. The corresponding  $\mathcal{O}_t$  is the mass of a 2d free fermion, which indeed has dimension 1.

that determines  $m_{\text{gap}}$  can be arbitrarily complicated and generally depends in a nontrivial way on the actual IR theory.

*End of lecture 6*

### 3.1.2. The cosmological constant

An important observable is the free energy density, defined by

$$f = - \lim_{V \rightarrow \infty} \frac{1}{V} \log Z_{M_V}, \quad (146)$$

where  $Z_{M_V}$  is the partition function in a space of volume  $V$  (for example a  $d$ -torus with side lengths  $V^{1/d}$ ). Understanding  $f$  involves an extra subtlety that we glossed over in the previous discussion.

An additional operator that is always present in our effective field theory is the unit operator  $\mathbf{1}$ . This corresponds to a “cosmological constant” term in the action

$$S \supset \int d^d x \lambda_{\mathbf{1}}. \quad (147)$$

The coefficient  $\lambda_{\mathbf{1}}$  must be matched to the UV theory just like all the others. It encodes contributions to the free energy from UV degrees of freedom not included in the IR effective field theory.

Strictly speaking  $\lambda_{\mathbf{1}}$  is another relevant coupling. However, we have ignored it so far because its only physical effect is to shift the free energy density — it cancels out of all correlation functions.

For concreteness, consider again the Ising model with no irrelevant couplings and with  $\lambda_h$  set to zero. The only mass scale besides  $\lambda_{\mathbf{1}}$  is  $m_t$ . Thus, by dimensional analysis, the free-energy density must take the form

$$f = \lambda_{\mathbf{1}} + A_{\text{sign}(t)} m_t^d, \quad (148)$$

where  $A_{\text{sign}(t)}$  is a constant depending only on the sign of  $t$  (i.e. which IR theory we flow to). We cannot have a more complicated combination of  $\lambda_{\mathbf{1}}$  and  $\lambda_t$  because shifting  $\lambda_{\mathbf{1}}$  can only shift  $f$ . Thus, only the *singular* part of the free energy density displays homogeneous scaling behavior,

$$f_{\text{singular}} = A_{\text{sign}(t)} m_t^d \propto |t|^{d/y_t}. \quad (149)$$

### 3.1.3. Other critical exponents

We are now ready to compute more critical exponents.

- $\alpha$ : The value of  $\alpha$  follows immediately from (148, 149):

$$C \propto \frac{\partial^2 f}{\partial t^2} \propto |t|^{d/y_t-2} = |t|^{-\alpha}. \quad (150)$$

Thus,

$$\alpha = 2 - \frac{d}{y_t} = \frac{d - 2\Delta_t}{d - \Delta_t}. \quad (151)$$

- $\beta$ : The magnetization is proportional to  $\frac{\partial f}{\partial h} \propto \frac{\partial f}{\partial \lambda_h}$ . By dimensional analysis, this must be

$$M \propto \frac{\partial f}{\partial \lambda_h} \propto m_t^{d-y_h} \propto |t|^{\frac{d-y_h}{y_t}} = |t|^\beta. \quad (152)$$

Thus,

$$\beta = \frac{d - y_h}{y_t} = \frac{\Delta_h}{d - \Delta_t}. \quad (153)$$

This is a clear case where the constant of proportionality depends on the sign of  $t$ . When  $t$  is positive, the theory is in the  $\mathbb{Z}_2$ -unbroken phase, so the constant of proportionality is zero. When  $t$  is negative,  $\mathbb{Z}_2$  is broken and the constant of proportionality is nonzero.

- $\gamma$ : The zero-field susceptibility involves an additional derivative with respect to  $\lambda_h$ ,

$$\chi \propto \frac{\partial^2 f}{\partial \lambda_h^2} \propto m_t^{d-2y_h} \propto |t|^{\frac{d-2y_h}{y_t}} = |t|^{-\gamma}. \quad (154)$$

Thus

$$\gamma = \frac{2y_h - d}{y_t} = \frac{d - 2\Delta_h}{d - \Delta_t}. \quad (155)$$

- $\delta$ : To compute the magnetization at  $T = T_c$ , we must consider the case  $\lambda_t = 0$  and  $\lambda_h \neq 0$ . Once again, there is only one mass scale given by

$$m_h = |\lambda_h|^{1/y_h} \propto |h|^{1/y_h}. \quad (156)$$

This time, dimensional analysis gives

$$\frac{\partial f}{\partial \lambda_h} \propto m_h^{d-y_h} \propto |h|^{\frac{d-y_h}{y_h}} = |h|^{1/\delta}. \quad (157)$$

Thus

$$\delta = \frac{y_h}{d - y_h} = \frac{d - \Delta_h}{\Delta_h}. \quad (158)$$



Because all of these exponents depend only on two quantities  $\Delta_h, \Delta_t$  (or  $y_h, y_t$ ), they satisfy various “scaling relations.” For example,

$$\begin{aligned}\alpha + 2\beta + \gamma &= 2 \\ \alpha + \beta(1 + \delta) &= 2.\end{aligned}\tag{159}$$

Historically, these scaling relations played an important role in verifying the above picture of the renormalization group. For us, the important point is that scaling behavior follows from the underlying assumption of having a fixed-point of the renormalization group, and the various critical exponents are determined in terms of the more fundamental operator dimensions  $\Delta_A$ .

### 3.2. The Wilson-Fisher theory and the $\epsilon$ -expansion<sup>38</sup>

As we discussed in section ??, the 3d Ising model arises as the long-distance limit of  $\phi^4$ -theory in 3-dimensions (suitably tuned to reach a critical point). Unfortunately the corresponding fixed-point is strongly-coupled, so it is difficult to explore the above ideas or do calculations.

However, we can obtain a related, weakly-coupled fixed-point by instead considering  $\phi^4$ -theory in  $(4 - \epsilon)$ -dimensions with  $\epsilon \ll 1$ . This is the Wilson-Fisher (WF) theory. Observables in the WF theory can be calculated in a systematic perturbative expansion in  $\epsilon$ . In principle, the 3d Ising model is obtained by resumming the  $\epsilon$ -expansion and taking  $\epsilon \rightarrow 1$ .

There are several issues with this idea, such as

- Does it even make sense to have a QFT in non-integer-dimensions?
- Does it matter that the theory is necessarily non-unitary if  $d$  is not an integer ([arXiv:](#))?
- The perturbative expansion in  $\epsilon$  is generally only asymptotic. Can one compute all the nonperturbative corrections needed to go from an asymptotic expansion in  $\epsilon$  to an actual function of  $\epsilon$ ?

(I’m sure you can think of more.) However, none of these issues will prevent us from doing perturbation theory. Furthermore, amazingly, if we simply set  $\epsilon = 1$  in the resulting perturbative series, truncated to low orders, we find results that aren’t terrible! For these reasons, the  $\epsilon$ -expansion has been one of the most popular tools for studying the 3d Ising model, even though it is on conceptually shaky ground. For us, it is useful because it illustrates many of the ideas of scaling and renormalization in a perturbative setting.

<sup>38</sup>Sources for this section: Pufu [1].

So let us study  $\phi^4$ -theory in  $d$ -dimensions. The theory has length-cutoff  $a$  and action

$$S = \int \frac{d^d x}{a^d} \left( \frac{1}{2} a^2 \partial_\mu \phi \partial^\mu \phi + g_1 \phi + \frac{1}{2} g_2 \phi^2 + \frac{1}{4!} g_4 \phi^4 \right). \quad (160)$$

(A cubic term can be removed by a field-redefinition  $\phi \rightarrow \phi + c$ .) Here, we are departing from the usual textbook conventions and assuming that  $\phi, g_i$  are classically dimensionless. The textbook conventions will follow by analyzing RG transformations near the Gaussian fixed point  $g_i = 0$ . However, there will also be another fixed-point and our starting point is more democratic between the two.

To find scaling dimensions at the Gaussian fixed-point, we should consider infinitesimal  $g_i$ , so that loop effects are highly suppressed. Thus, we can treat the interactions classically and simply find a coarse-graining transformation that leaves (160) invariant. This is<sup>39</sup>

$$\begin{aligned} a &\rightarrow ba, \\ \phi &\rightarrow b^{\frac{d-2}{2}} \phi, \\ g_1 &\rightarrow b^{\frac{d}{2}+1} g_1, \\ g_2 &\rightarrow b^2 g_2, \\ g_4 &\rightarrow b^{4-d} g_4. \end{aligned} \quad (161)$$

It follows that

$$\begin{aligned} y_1 &= \frac{d}{2} + 1 > 0 && \text{(relevant)} \\ y_2 &= 2 > 0 && \text{(relevant)} \\ y_4 &= 4 - d \begin{cases} > 0 & (d < 4) & \text{(relevant)} \\ = 0 & (d = 4) & \text{(marginal)} \\ < 0 & (d > 4) & \text{(irrelevant)}. \end{cases} \end{aligned} \quad (162)$$

Thus, when  $d < 4$ , the quartic coupling is relevant and causes the theory to flow away from the Gaussian fixed-point. When  $d = 4$ , the quartic coupling is marginal. A more careful calculation (done in any QFT textbook) shows that in  $d = 4$ ,  $g_4$  flows logarithmically to zero in the infrared (we say  $g_4$  is “marginally irrelevant”). Finally, when  $d > 4$ , the quartic coupling is

<sup>39</sup>It may look unfamiliar that coupling constants like  $g_2$  depend on RG scale. This is because they differ from the usual textbook conventions by explicit powers of  $a$ . We can define an RG-invariant dimensionful coupling by  $\lambda_2^{\text{Gaussian}} = a^{-2} g_2$ . This is what is usually denoted as  $m^2$  in textbooks.

irrelevant and the theory flows back to the Gaussian fixed-point in the IR. It turns out that the Ising model in  $d > 4$  is not described by the Gaussian fixed-point, so we will focus on the case  $d < 4$ .

To understand what happens further from the Gaussian fixed-point, we need the  $\beta$ -function. To quadratic order in the coupling-constants, this is given by

$$\begin{aligned} a \frac{dg_1}{da} &= -\beta_1(g) = \frac{d+2}{2}g_1 - \frac{g_1g_2}{d-2} + \dots \\ a \frac{dg_2}{da} &= -\beta_2(g) = 2g_2 - S_d g_1^2 - \frac{g_2^2}{d-2} - \frac{g_2g_4}{2(d-2)^2 S_d} - \frac{g_4^2}{6(d-2)^3 S_d^2} + \dots \\ a \frac{dg_4}{da} &= -\beta_4(g) = (4-d)g_4 - 3S_d g_2^2 - \frac{4g_2g_4}{d-2} - \frac{3g_4^2}{2(d-2)^2 S_d} + \dots, \end{aligned} \quad (163)$$

where

$$S_d = \frac{2\pi^{d/2}}{\Gamma(d/2)} \quad (164)$$

is the volume of the unit sphere in  $d$ -dimensions.<sup>40</sup> Let us make some observations. Firstly, note that  $g_1 = g_2 = g_4 = 0$  is indeed a fixed-point of these equations. The  $\beta$ -functions above are good close to this fixed-point. Note also that the linear terms follow from scaling at the Gaussian fixed-point, i.e. the linear term in  $-\beta_i$  is  $y_i g_i$ .

When  $\epsilon = 4 - d$  is small, the linear term in  $\beta_4$  is small, and it's possible for it to be balanced against the quadratic terms while still staying within the regime of validity of perturbation theory. This gives another fixed-point:

$$g_{1*} = 0, \quad g_{2*} = 0, \quad g_{4*} = \frac{16\pi^2}{3}\epsilon + O(\epsilon^2) \quad (166)$$

This is the fixed-point of the Wilson-Fisher theory. The above approximation for  $g_*$  is well-controlled because the fixed-point occurs at small values of the couplings, where we can trust the  $\beta$ -functions (163). An RG flow diagram is shown in figure ??.

<sup>40</sup> $\beta$ -functions themselves are not physical observables — they depend on a choice of *scheme*, which consists of a regulator and a definition of coupling constants. The  $\beta$ -function here comes from using conformal perturbation theory with a cutoff regulator. We will derive it later in the course. The general formula in this scheme is

$$a \frac{dg_A}{da} = (d - \Delta_A)g_A - \frac{S_d}{2} \sum_{B,C} f_{BCA} g_B g_C, \quad (165)$$

where  $f_{BCA}$  is the OPE coefficient in  $\mathcal{O}_B \mathcal{O}_C \sim f_{BCA} \mathcal{O}_A$  and  $\Delta_A$  is the scaling dimension of  $\mathcal{O}_A$  at the fixed-point we are perturbing around. Here we are doing conformal perturbation theory around the Gaussian fixed-point.

Linearizing the  $\beta$ -functions around the Wilson-Fisher point, we find

$$\begin{aligned} a \frac{dg_1}{da} &= \left(3 - \frac{\epsilon}{2}\right) g_1 + \dots \\ a \frac{dg_2}{da} &= \left(2 - \frac{\epsilon}{3}\right) g_2 + \dots \\ a \frac{dg_4}{da} &= -\epsilon (g_4 - g_{4*}) + \dots, \end{aligned} \quad (167)$$

where we have dropped terms of higher order in the couplings and in  $\epsilon$ . Thus, at the Wilson-Fisher point we have

$$\begin{aligned} y_1 &= 3 - \frac{\epsilon}{2} + O(\epsilon^2), \quad y_2 = 2 - \frac{\epsilon}{3} + O(\epsilon^2), \quad y_4 = -\epsilon + O(\epsilon^2) \\ \Delta_\phi &= 1 - \frac{\epsilon}{2} + O(\epsilon^2), \quad \Delta_{\phi^2} = 2 - \frac{2\epsilon}{3} + O(\epsilon^2), \quad \Delta_{\phi^4} = 4 + O(\epsilon^2). \end{aligned} \quad (168)$$

Note in particular that while  $\phi^4$  is relevant at the Gaussian fixed-point, it becomes irrelevant at the Wilson-Fisher point. Thus the Wilson-Fisher point has two relevant parameters, which is consistent with experimental observations of the Ising universality class. Plugging in  $\epsilon = 1$ , we find  $\Delta_\phi \sim 0.5$  and  $\Delta_{\phi^2} \sim 1.33$  in 3d, which is not too far from the correct answers  $\Delta_\phi \sim 0.518$ ,  $\Delta_{\phi^2} \sim 1.412$ . This is nice, but not very rigorous.

Finally, let us discuss scaling variables and dimensionful couplings in this context. Consider the flow between the Gaussian point and the Wilson-Fisher point, where  $g_1 = g_2 = 0$  and  $g_4 = g(a)$  is a function of  $a$ . The  $\beta$ -function equation is (ignoring cubic and higher terms in  $g_4$ )

$$a \frac{dg}{da} = \epsilon g(1 - g/g_*), \quad (169)$$

which has solution

$$g(a) = \frac{1}{\frac{1}{g_*} + \left(\frac{a_0}{a}\right)^\epsilon} \quad (170)$$

with some dimensionful integration constant  $a_0$ .

When  $a$  is small, (170) behaves like  $g(a) \sim a^\epsilon = a^{y_4^{\text{Gauss}}}$ , as expected. We can define an RG-invariant dimensionful coupling associated with the Gaussian fixed-point by

$$\lambda_4^{\text{Gauss}} = a^{-y_4^{\text{Gauss}}} g(a) \Big|_{a \rightarrow 0} = \frac{1}{a_0^\epsilon}. \quad (171)$$

Thus,  $|\lambda_4^{\text{Gauss}}|^{-1/\epsilon} = a_0$  represents the distance scale at which the flow starts in the UV.

When  $a$  is large, the solution behaves as  $g(a) - g_* \sim a^{-\epsilon} = a^{y_4^{\text{WF}}}$ . The RG-invariant dimensionful coupling associated to the Wilson-Fisher point is then given by

$$\lambda_4^{\text{WF}} = a^{-y_4^{\text{WF}}} (g(a) - g_*) \Big|_{a \rightarrow \infty} = -g_*^2 a_0^\epsilon. \quad (172)$$

From the point of an IR observer looking upwards in energy from the WF point,  $\lambda_4$  encodes the leading effect of UV physics. Note that  $\lambda_4^{\text{WF}}$  is suppressed by powers of the coupling compared to  $1/\lambda_4^{\text{Gauss}}$ . This parametric separation comes from the fact that we are considering a weakly coupled flow that moves slowly as we descend in energy. In generic strongly-coupled theories, we expect scales of relevant operators in the UV and scales of irrelevant operators in the IR to be related by  $O(1)$  constants.

*End of lecture 7*

### 3.3. The $O(N)$ -model and the large- $N$ expansion<sup>41</sup>

The Wilson-Fisher theory is one way to replace the Ising model with a related theory that has a small parameter. Another modification that is physically interesting in its own right is a generalization of the Ising model to include  $N$  scalars with an  $O(N)$  symmetry. The resulting theory, called the  $O(N)$ -model, has action

$$S = \int d^d x \left( \frac{1}{2} \sum_{i=1}^N \partial_\mu \phi_i \partial^\mu \phi_i + \frac{1}{2} m^2 \sum_{i=1}^N \phi_i^2 + u \left( \sum_{i=1}^N \phi_i^2 \right)^2 \right) \quad (173)$$

In this lecture, we will return to textbook conventions where we associate Gaussian scaling dimensions to the coupling constants and fields. Specifically  $m^2$  has mass-dimension 2 and  $u$  has mass-dimension  $4 - d$ .

Note that the  $O(1)$  model is the Ising model. As discussed in the introduction, the  $O(2)$  model describes XY magnets, and the  $O(3)$  model describes Heisenberg magnets (among many other systems).

Here are some properties of the  $O(N)$ -model:

- It has a Gaussian fixed-point at  $u = 0$ ,  $m^2 = 0$ . At the Gaussian fixed-point,  $\phi_i$  has scaling dimension  $\Delta_{\phi_i} = \frac{d-2}{2}$ , and the dimensions of other operators are determined by naive dimensional analysis. For example,  $\Delta_{\phi^2} = d - 2$ .
- In  $d < 4$ , the theory has another fixed-point generalizing the Wilson-Fisher fixed-point in the case  $N = 1$ . When  $d = 4 - \epsilon$  with  $\epsilon \ll 1$ , this

<sup>41</sup>Sources for this section: Pufu [1].

fixed-point can be studied in an expansion in  $\epsilon$  for any  $N$ . For example, the critical coupling occurs at

$$u_* = \frac{2\pi^2\epsilon}{N+8} + O(\epsilon^2). \quad (174)$$

- At large  $N \gg 1$ , the theory can be solved in any  $d$  in an expansion in  $1/N$ . In particular, it can be solved directly in  $d = 3$ . One can then try to recover information about the cases  $N = 1, 2, 3$  by extrapolating to small- $N$ . This extrapolation is sometimes successful quantitatively, but it can also help build further intuition about the spectrum of the small- $N$  theories.<sup>42</sup>

In the rest of this lecture we will set  $d = 3$  and study the  $O(N)$  model in an expansion in  $1/N$ .

The large- $N$  theory is solvable because it is in some sense close to the Gaussian theory. In particular, all  $O(N)$ -non-singlet operators have scaling dimensions equal to their dimensions at the Gaussian fixed point, plus corrections of  $O(1/N)$ . For example

$$\Delta_{\phi_i} = \frac{1}{2} + O(1/N). \quad (175)$$

However,  $O(N)$ -singlet operators can have dimensions very different from their Gaussian values. For example

$$\Delta_{\phi^2} = 2 + O(1/N), \quad (176)$$

whereas  $\Delta_{\phi^2}^{\text{Gaussian}} = 1$ . (Here,  $\phi^2 = \sum_i \phi_i \phi_i$ .)

It is instructive to see how the dimension of  $\phi^2$  can be different at the two fixed-points. One way is to study the two-point function

$$F(x) = \langle \phi^2(x) \phi^2(0) \rangle, \quad (177)$$

as a function of  $x$  at nonzero coupling  $u$ . We would like to show that

$$\begin{aligned} F(x) &\sim \frac{1}{|x|^2} & (x \rightarrow 0), \\ F(x) &\sim \frac{1}{|x|^4} & (x \rightarrow \infty). \end{aligned} \quad (178)$$

Here, the  $x \rightarrow 0$  limit probes the UV where the theory is described by the Gaussian fixed-point, while  $x \rightarrow \infty$  probes the IR where the theory is described by the nontrivial critical point.

<sup>42</sup>Another important feature of the  $O(N)$  models is that at large- $N$ , they have a known holographic dual in  $\text{AdS}_4$ . The dual theory is due to Vasiliev and the duality was discovered by Klebanov and Polyakov [arXiv](#)..

We will compute  $F(x)$  by summing Feynman diagrams, and this is much easier in momentum space. In momentum space, the conditions (178) become

$$\begin{aligned} F(p) &\sim \frac{1}{|p|} & (p \rightarrow \infty), \\ F(p) &\sim |p| & (p \rightarrow 0), \end{aligned} \quad (179)$$

where

$$F(p) = \int d^3x e^{ip \cdot x} F(x). \quad (180)$$

The reason we can use Feynman diagrams to describe the large- $N$  theory is that only a small subclass of diagrams will be important in the large- $N$  limit.

In general to reach the critical point, we must tune  $m^2$ . In the scheme we will use, the critical value is  $m^2 = 0$ , so let us set  $m^2 = 0$  from the beginning. The propagator is

$$\langle \phi_i(x) \phi_j(0) \rangle = \delta_{ij} G(x), \quad (181)$$

where

$$G(p) = \frac{1}{p^2} \implies G(x) = \frac{1}{4\pi|x|}. \quad (182)$$

[Draw diagrams for the following discussion](#)

The simplest contribution to  $F(x)$  comes from a bubble diagram<sup>43</sup>

$$\langle \phi^2(x) \phi^2(0) \rangle \supset 2NG(x)^2. \quad (183)$$

The overall factor of  $N$  comes from summing over  $i$  indices for the two propagators. These must be equal because of the contractions in  $\phi^2$ , so we get a single sum over  $N$ . In Fourier-space, the bubble diagram becomes

$$2NG(x)^2 = \frac{2N}{16\pi^2|x|^2} \rightarrow \frac{1}{2} \frac{N}{2|p|}. \quad (184)$$

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<sup>43</sup>An even simpler possibility is to draw two disconnected loops, where we contract the two  $\phi$ 's in  $\phi^2(x)$ , and separately contract the two  $\phi$ 's in  $\phi^2(0)$ . These diagrams are divergent, and the divergence comes from the fact that we have taken two operators on top of each other. The correct procedure is to carefully define the  $\phi^2(x)$  operator by subtracting divergences. For example, in this case we can define it using point-splitting as  $\phi^2(x) = \lim_{\delta x \rightarrow 0} [\phi(x)\phi(x+\delta x) - \langle \phi(x)\phi(x+\delta x) \rangle]$ . Ambiguities in the regularization procedure can be removed by demanding that our operator is an eigenvector under rescaling. We will make this notion more precise in a few lectures.

Another contribution to  $F(x)$  comes from a pair of bubbles connected by an interaction vertex. This gives

$$\frac{1}{2}u \left( \frac{N}{2|p|} \right)^2. \quad (185)$$

Similarly, a chain of  $k$  bubbles contributes

$$\frac{1}{2}u^{k-1} \left( \frac{N}{2|p|} \right)^k. \quad (186)$$

We will be interested in a limit  $N \rightarrow \infty$ , with the product  $uN$  held fixed. In this limit, chains of bubbles are the only diagrams that contribute. Any other diagram requires reconnecting different  $O(N)$  indices in a way that lowers the resulting powers of  $N$  relative to the number of vertices. Thus, in this limit, we only need to sum bubble diagrams, which form a geometric series

$$\begin{aligned} F(p) &= \frac{1}{2} \left( \frac{N}{2|p|} + u \left( \frac{N}{2|p|} \right)^2 + u^2 \left( \frac{N}{2|p|} \right)^3 + \dots \right) \\ &= \frac{1}{2} \frac{N}{2|p|} \frac{1}{1 - \frac{uN}{2|p|}}. \end{aligned} \quad (187)$$

Let us check that this satisfies the expected conditions (179). When  $p$  is large, the result is indeed the same as in the Gaussian theory. When  $p$  is small, we have

$$F(p) \approx -\frac{1}{2} - \frac{|p|}{u^2 N} + O(p^2) \quad (p \text{ small}). \quad (188)$$

At first, this looks like the wrong result. However, note that the Fourier transform of a constant is  $\delta(x)$ , which is zero at nonzero separation  $x \neq 0$ . We call  $\delta(x)$  a “contact term” because it is only nonzero when the points in the two-point function coincide. In general, contact terms are not captured by a low-energy effective theory.<sup>44</sup> Their coefficients are determined by UV physics, and we should not expect to be able to predict them from knowledge of IR scaling-dimensions alone.

The next term in (188) is non-analytic in  $p$ , so its Fourier-transform is nonzero at separated points. Indeed, it has the form we expect, so that  $F(x)$  behaves like  $\frac{1}{|x|^4}$  for large  $x$ .

For higher-order computations, we can write a set of effective Feynman rules that summarize our result. We have a  $\phi_i$ - $\phi_j$  propagator of the usual

<sup>44</sup>An exception is when they are fixed by Ward identities, as we will discuss later.



form  $\frac{\delta_{ij}}{p^2}$ , and a dotted line that represents a geometric series of bubble diagrams that behaves like  $|p|$ .

### 3.3.1. The Hubbard-Stratonovich trick

The Hubbard-Stratonovich trick is a more sophisticated approach that lets us re-derive the above results and systematically proceed to higher orders in  $1/N$ . Let us start with the partition function

$$Z = \int D\phi_i e^{-\int d^3x \left( \frac{1}{2} \partial_\mu \phi \cdot \partial^\mu \phi + u(\phi \cdot \phi)^2 \right)}. \quad (189)$$

We can simplify the theory by introducing an auxiliary field  $\sigma$

$$= \int D\phi_i D\sigma e^{-\int d^3x \left( \frac{1}{2} \partial_\mu \phi \cdot \partial^\mu \phi + \frac{1}{2} i\sigma \phi \cdot \phi + \frac{\sigma^2}{16u} \right)}. \quad (190)$$

The  $\sigma$  integral is Gaussian, so it can be performed by completing the square

$$\frac{1}{16u} (\sigma + 4iu\phi \cdot \phi)^2 + u(\phi \cdot \phi)^2 \quad (191)$$

and dropping the quadratic term in  $\sigma$ . This gives back the original action (189).

The equation of motion for  $\sigma$  is

$$\sigma = -4iu\phi \cdot \phi, \quad (192)$$

so  $\sigma$  is like  $\phi \cdot \phi$ . To take the IR limit of  $F(p)$ , we looked at small  $p$ . However, we could have equivalently looked at large  $u$ . Note that at large- $u$ , the two-point function  $\langle \sigma(x)\sigma(0) \rangle$  becomes  $u$ -independent (unlike  $\langle \phi^2(x)\phi^2(0) \rangle$ ), so  $\sigma$  is a better degree of freedom for describing this limit. In fact, the large- $u$ , or equivalently low-energy, limit of the action (190) is extremely simple:

$$Z|_{\text{IR}} = \int D\phi_i D\sigma e^{-\int d^3x \left( \frac{1}{2} \partial_\mu \phi \cdot \partial^\mu \phi + \frac{1}{2} i\sigma \phi \cdot \phi \right)}. \quad (193)$$

We can now reinterpret the fate of the  $\phi^2$  operator in the long-distance limit. The field  $\sigma$  becomes a Lagrange multiplier that naively sets  $\phi^2 = 0$ , removing it from the theory. However  $\sigma$  remains, and because of the equation of motion (192), it takes over for  $\phi^2$ . The scaling dimension of  $\sigma$  at large- $N$  can be determined by dimensional analysis from the action (193). We obtain  $\Delta_\phi = \frac{1}{2}$ ,  $\Delta_\sigma = 2$ .

In more precise language, the IR theory (193) can be analyzed as follows. Note that the integral over  $\phi$  is Gaussian and can be done exactly

$$Z|_{\text{IR}} = \int D\sigma e^{-\frac{N}{2} \text{Tr} \log(-\partial^2 + i\sigma)} \quad (194)$$

The factor of  $N$  in the exponent comes from the fact that we have  $N$  fields and each one gives a determinant of the effective kinetic term  $-\partial^2 + i\sigma$ . So far, the manipulations we've done have been valid at any value of  $N$ . For generic values of  $N$ , the resulting theory (194) is quite complicated and its not clear how to proceed. However, at large  $N$  the path integral over  $\sigma$  can be done via a saddle-point approximation. Expanding out the action to quadratic order in  $\sigma$ , we obtain a kinetic term that gives a  $\sigma$ -propagator proportional to  $|p|$ . From our earlier work, we know that this propagator represents an infinite series of bubble diagrams in the original theory.

The end result of this analysis is a perturbative expansion for various observables, for example<sup>45</sup>

$$\begin{aligned}\Delta_\phi &= \frac{1}{2} + \frac{4}{3\pi^2 N} + O(1/N^2) \\ \Delta_\sigma &= 2 - \frac{32}{3\pi^2 N} + O(1/N^2).\end{aligned}\tag{195}$$

(In practice, these are not as good as the  $\epsilon$ -expansion for estimating the scaling dimensions of the Ising model.)

### 3.4. Banks-Zaks fixed-points<sup>46</sup>

To close this section, let us discuss some very different examples of RG fixed-points that are not related to the Ising universality class (at least not in a simple way). These fixed-points arise in 4d gauge theories with particular matter content. For concreteness, consider a 4d gauge theory with gauge group  $SU(N_c)$  and  $N_f$  massless Dirac fermions transforming in the fundamental representation of  $SU(N_c)$ . The Lagrangian is

$$\mathcal{L} = -\frac{1}{4g^2} F_{\mu\nu}^a F^{a\mu\nu} + \bar{\psi}(i\not{D})\psi.\tag{196}$$

The  $\beta$ -function is

$$\begin{aligned}\beta(g) &= -\beta_0 \frac{g^3}{(4\pi)^2} + \beta_1 \frac{g^5}{(4\pi)^4} + O(g^7) \\ \beta_0 &= \frac{11}{3}N_c - \frac{2}{3}N_f, \quad \beta_1 \sim O(N_c^2, N_c N_f).\end{aligned}\tag{197}$$

This theory exhibits very different behavior, depending on  $N_f$  and  $N_c$ . Famously, when  $N_f = 0$ , the coupling  $g$  becomes large in the IR and the theory is expected to confine (meaning that asymptotic states do not carry

<sup>45</sup>The large- $N$  expansion for  $\Delta_\phi$  is known to order  $1/N^3$  [ ] and the expansion for  $\Delta_\sigma$  is known to order  $1/N^2$  [ ].

<sup>46</sup>Sources for this section: Rychkov [3].

global  $SU(N_c)$  charge). Confinement is similarly expected to happen for nonzero  $N_f$  as long as  $N_f$  is not too large. By contrast, if  $\beta_0 > 0$ , the coupling becomes smaller in the IR and the theory is described by free particles with global  $SU(N_c)$  charge.

However, it is also clear from the  $\beta$ -function (197) that the theory can exhibit very different behavior in the IR. Let us choose  $N_f, N_c \gg 1$  such that  $\beta_0 \sim O(1) > 0$ . The  $\beta$ -function takes the form

$$\beta(g) \propto -\frac{g^3}{(4\pi)^2} \left( 1 - \frac{\beta_1}{\beta_0} \frac{g^2}{(4\pi)^2} + \dots \right). \quad (198)$$

This makes it clear that there is a zero at  $g = g_*$  with  $g_*^2/(4\pi)^2 = O(1/N_c^2)$ . The perturbative expansion can be organized in terms of the 't Hooft coupling

$$\lambda_* = \frac{N_c g_*^2}{(4\pi)^2}, \quad (199)$$

which is small in the parameter range we are studying. Thus, we are justified in ignoring higher-order terms in (198), and our calculations are under perturbative control. This fixed-point is called a “Banks-Zaks” fixed-point after the physicists who first understood that it defines a nontrivial conformal field theory.

We performed the above analysis assuming  $N_f, N_c \gg 1$  with  $\beta_0 = O(1)$  in order to ensure perturbative control over the resulting fixed-point. However, like the Wilson-Fisher theory, the Banks-Zaks fixed-point is expected to persist at least some distance outside of the perturbative regime. In supersymmetric versions of QCD, we can actually compute nonperturbatively the allowed range of  $N_f, N_c$  for a conformal fixed-point — the so-called “conformal window.” For example, with  $\mathcal{N} = 1$  SUSY, the conformal window is  $\frac{3}{2}N_c \leq N_f \leq 3N_c$ , where  $N_f$  counts the number of chiral multiplets in the fundamental representation of  $SU(N_c)$ . In non-supersymmetric gauge theories, the complete conditions defining the conformal window are unknown. The upper end of the conformal window  $N_f < \frac{11}{2}N_c$  follows from the requirement that the theory not be IR free. However, the lower end cannot be computed perturbatively and is a topic of heated debate.<sup>47</sup>

*End of lecture 8*

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<sup>47</sup>Currently, the lattice QCD community is divided over whether the lower end of the conformal window for  $N_c = 3$  is  $N_f = 12$ ,  $N_f = 11$ , or perhaps somewhat lower.

#### 4. The bootstrap approach

This concludes the first part of the course. We have seen why QFT is the correct language for both quantum and statistical systems, how critical points arise in several concrete examples, and how observables like critical exponents are related to scaling dimensions of operators. We have also studied these concepts in a perturbative setting. However, our main interest will be in nonperturbative theories. To describe these we will need a new set of tools. (The new tools will also help us better understand perturbative theories.)<sup>48</sup>

Our approach for the next part of the course is based on the idea of the *conformal bootstrap*. The conformal bootstrap philosophy is to:

0. focus on the CFT itself and not a specific microscopic realization,
1. determine the full consequences of symmetries,
2. impose consistency conditions,
3. combine (1) and (2) to constrain or even solve the theory.

This strategy was first articulated by Ferrara, Gatto, and Grillo [5] and Polyakov [6] in the 70's. Importantly, it only uses nonperturbative structures, and thus has a hope of working for strongly-coupled theories. Its effectiveness for studying the Ising model will become clear during this course. In addition, sometimes bootstrapping is the *only* known strategy for understanding the full dynamics of a theory. An example is the 6d  $\mathcal{N} = (2, 0)$  supersymmetric CFT describing the IR limit of a stack of M5 branes in M-theory. This theory has no known Lagrangian description, but is amenable to bootstrap analysis [7].<sup>49</sup>

A beautiful and ambitious goal of the bootstrap program is to eventually provide a fully nonperturbative formulation of quantum field theory, removing the need for a Lagrangian.

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<sup>48</sup>For the next several sections, the course will closely follow my TASI Lectures []. The text below is a lightly edited version of those notes, possibly with a few examples and explicit calculations thrown in. If you want a more compact presentation, you can refer to the original set of lectures.

<sup>49</sup>At large central charge, this theory is solved by the AdS/CFT correspondence [8]. Supersymmetry also lets one compute a variety of protected quantities (at any central charge). However, the bootstrap is currently the only known tool for studying non-protected quantities at small central charge.

## 5. Symmetries in quantum field theory

The first step of the conformal bootstrap is to determine the full consequences of symmetries. In this section, we review symmetries in quantum field theory, phrasing the discussion in language that will be useful later. We work in Euclidean signature throughout.

### 5.1. The Stress Tensor

A local quantum field theory has a conserved stress tensor,

$$\partial_\mu T^{\mu\nu}(x) = 0 \quad (\text{operator equation}). \quad (200)$$

This holds as an “operator equation,” meaning it is true away from other operator insertions. In the presence of other operators, (200) gets modified to include contact terms on the right-hand side,

$$\begin{aligned} \partial_\mu \langle T^{\mu\nu}(x) \mathcal{O}_1(x_1) \dots \mathcal{O}_n(x_n) \rangle = & - \sum_i \delta(x - x_i) \partial_i^\nu \langle \mathcal{O}_1(x_1) \dots \mathcal{O}_n(x_n) \rangle \\ & + \frac{\partial}{\partial x^\rho} (\text{other contact terms})^\rho. \end{aligned} \quad (201)$$

This modified conservation equation is called a “Ward identity.”<sup>50</sup>

In practice, the contact terms come about because correlators have singularities as  $T^{\mu\nu}$  approaches other operators of the form<sup>51</sup>

$$\langle T^{\mu\nu}(x) \mathcal{O}(y) \dots \rangle \sim \partial_x^\mu \partial_x^\nu \frac{1}{(x - y)^{d-2}} \langle \mathcal{O}(y) \dots \rangle + \dots \quad (202)$$

Taking the divergence  $\frac{\partial}{\partial x^\mu}$  on the right-hand side we get

$$\partial_x^\nu \partial_x^2 \frac{1}{(x - y)^{d-2}} \propto \partial_x^\nu \delta^d(x - y). \quad (203)$$

A stress tensor is always present in a continuum QFT. However, in a lattice model, a stress tensor must be emergent (if it exists).

Let us prove the Ward identity for a Lagrangian theory with a scalar field  $\phi$  and action  $S[\phi]$ . The stress tensor is the Noether current for translation symmetry, so we can obtain it from the Noether procedure. We consider

<sup>50</sup>Unlike the contact term discussed in the  $O(N)$ -model section, contact terms in Ward identities are fixed by symmetries and do not depend on details of UV physics.

<sup>51</sup>There may be other singularities present, but they don’t contribute to nontrivial contact terms.

an infinitesimal position-dependent translation parameterized by a vector field  $\epsilon_\mu(x)$ . It acts on  $\phi$  as

$$\begin{aligned}\phi(x) &\rightarrow \phi(x + \epsilon(x)) = \phi(x) + \delta_\epsilon \phi(x), \\ \delta_\epsilon \phi &= \epsilon^\mu \partial_\mu \phi(x).\end{aligned}\tag{204}$$

Because  $S[\phi]$  must be invariant when  $\epsilon$  is constant, the shift in  $S[\phi]$  must be the integral of  $\epsilon$  times a total derivative

$$\delta_\epsilon S[\phi] = - \int d^d x \epsilon_\nu(x) \partial_\mu T^{\mu\nu}(x),\tag{205}$$

for some operator  $T^{\mu\nu}(x)$  — the stress tensor.

For example, consider the free scalar theory in  $d$ -dimensions. The action is given by

$$S_{\text{free}} = \int d^d x \frac{1}{2} (\partial\phi)^2.\tag{206}$$

Plugging in the transformation (204), we find

$$\begin{aligned}\delta_\epsilon S_{\text{free}} &= \int d^d x \partial_\mu (\epsilon_\nu \partial^\nu \phi) \partial^\mu \phi \\ &= \int d^d x \left( \partial_\mu \epsilon_\nu \partial^\nu \phi \partial^\mu \phi + \epsilon_\nu \partial^\nu \left( \frac{1}{2} (\partial\phi)^2 \right) \right) \\ &= - \int d^d x \epsilon_\nu \partial_\mu \left( \partial^\nu \phi \partial^\mu \phi - \frac{1}{2} (\partial\phi)^2 \delta^{\mu\nu} \right),\end{aligned}\tag{207}$$

where we integrated by parts to get the last line. We might be tempted to identify the quantity in parentheses as the stress tensor, but the Noether procedure is actually ambiguous: it only determines the stress tensor up to a manifestly conserved term called an “improvement term”<sup>52</sup>

$$T_{\text{free}}^{\mu\nu} = \partial^\nu \phi \partial^\mu \phi - \frac{1}{2} (\partial\phi)^2 \delta^{\mu\nu} + \underbrace{(\partial^\mu \partial^\nu - \delta^{\mu\nu} \partial^2) \Phi}_{\text{improvement term}}.\tag{208}$$

Here,  $\Phi$  can be any scalar (possibly composite) operator. Later, we will impose a physical principle that will pick out a particular  $\Phi$ . But for now any  $\Phi$  will do.

Having determined the symmetry current  $T^{\mu\nu}$ , we can derive the Ward identity. Consider a correlation function of local operators

$$\langle \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) \rangle = \int D\phi \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) e^{-S[\phi]}.\tag{209}$$

<sup>52</sup>We can also add more complicated improvement terms, for example  $\partial_\rho B^{\rho\mu\nu}$  where  $B$  is antisymmetric in  $\rho \leftrightarrow \mu$ . We will remark briefly on such terms in a moment.

We can think of  $\phi \rightarrow \phi + \delta_\epsilon \phi$  as a change of variables in the path integral, and this change of variables should leave the integral invariant. In other words

$$\begin{aligned} 0 &= \int D\phi \delta_\epsilon \left( \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) e^{-S[\phi]} \right) \\ &= \sum_i \langle \mathcal{O}_1(x_1) \cdots \delta_\epsilon \mathcal{O}_i(x_i) \cdots \mathcal{O}_n(x_n) \rangle \\ &\quad + \langle \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) \int d^d x \epsilon_\nu(x) \partial_\mu T^{\mu\nu}(x) \rangle. \end{aligned} \quad (210)$$

Now note that

$$\delta_\epsilon(\mathcal{O}_i(x_i)) = \epsilon_\nu(x_i) \partial_i^\nu \mathcal{O}_i(x_i) + (\partial\epsilon)(\dots) + (\partial^2\epsilon)(\dots) + \dots, \quad (211)$$

where the first term is proportional to  $\epsilon$  and remaining terms are proportional to derivatives of  $\epsilon$  at  $x_i$ .<sup>53</sup> The Ward identity (201) then follows from taking a functional derivative of (210) with respect to  $\epsilon(x)$ .

There is an alternative formalism that deals more elegantly with improvement terms and ambiguities in the Noether procedure. This is to imagine coupling our QFT to a background metric in a diffeomorphism-invariant way. (Note that without diffeomorphism invariance, the theory would depend on more than just a metric, but also a choice of local coordinates.) The Ward identity can be derived in this formalism as follows.

**Exercise 5.1.** Consider a QFT coupled to a background metric  $g$ . For concreteness, suppose correlators are given by the path integral

$$\langle \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) \rangle_g = \int D\phi \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) e^{-S[g, \phi]}. \quad (212)$$

A stress tensor insertion is the response to a small metric perturbation,

$$\langle T^{\mu\nu}(x) \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) \rangle_g = \frac{2}{\sqrt{g}} \frac{\delta}{\delta g_{\mu\nu}(x)} \langle \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) \rangle_g. \quad (213)$$

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<sup>53</sup>Terms depending on derivatives of  $\epsilon$  would come about if  $\mathcal{O}_i$  depended on derivatives of  $\phi$ , for example.

Derive the Ward identity<sup>54</sup>

$$\partial_\mu \langle T^{\mu\nu}(x) \mathcal{O}_1(x_1) \dots \mathcal{O}_n(x_n) \rangle = - \sum_i \delta(x - x_i) \partial_i^\nu \langle \mathcal{O}_1(x_1) \dots \mathcal{O}_n(x_n) \rangle \quad (214)$$

by demanding that  $S[g, \phi]$  be diffeomorphism invariant near flat space. Assume that the  $\mathcal{O}_i$  are defined so that they transform like scalars under diffeomorphisms. Find how to modify (201) when the  $\mathcal{O}_i$  have spin.

The nice thing about this formalism is that there are no ambiguities in the definition of the stress tensor: it is always the response of the theory to a metric perturbation. However, now we must make an explicit choice of how to couple our theory to a background metric. In particular, there can be new couplings in the action that aren't present in flat space. For example, the action for a free-scalar in curved space admits a new curvature-dependent mass term

$$S_{\text{free}}[g, \phi] = \int d^d x \sqrt{g} \left( \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + \frac{1}{2} \xi R \phi^2 \right) \quad (215)$$

**Exercise 5.2.** Using the definition  $T^{\mu\nu} = -\frac{2}{\sqrt{g}} \frac{\delta S}{\delta g_{\mu\nu}}$ , show that the stress tensor in the theory (215), evaluated in flat space, is

$$T^{\mu\nu} = \partial^\mu \phi \partial^\nu \phi - \frac{1}{2} g^{\mu\nu} (\partial\phi)^2 - \xi (\partial^\mu \partial^\nu - g^{\mu\nu} \partial^2) \phi^2. \quad (216)$$

In other words,  $\xi$  determines a choice of improvement term in (208).

A related complication that this formalism avoids is that the Noether procedure doesn't automatically give a symmetric stress tensor (it does for the free scalar, but that was luck), and one must find an improvement term that fixes this. In the background metric formalism, the stress tensor is automatically symmetric.

A disadvantage of the background metric approach is that it might not be obvious how to couple a given UV system (e.g. a lattice model) to a nontrivial metric. For the Ising CFT, we can take advantage of the fact that there exists a continuum field theory (i.e.  $\phi^4$  theory) in its universality class, which manifestly can be coupled to curved space. However, for more

<sup>54</sup>Note that this version of the Ward identity doesn't include terms proportional to  $\partial\epsilon$ , unlike (201). This is because in defining the  $\mathcal{O}_i$  so that they transform like scalars under diffeomorphisms, we must include factors of  $g^{\mu\nu}$ . For example,  $(\partial\phi)^2 = g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi$ . The variation of these factors cancels terms proportional to  $\partial\epsilon$ . Equivalently, the definition (213) has the effect of adding contact terms to  $T_{\mu\nu}$  that simplify the Ward identity.



general CFTs, the assumption that they can be coupled to a background metric is nontrivial.<sup>55</sup>

### 5.2. Topological operators and symmetries

Consider the integral of  $T^{\mu\nu}$  over a closed surface  $\Sigma$ ,<sup>56,57</sup>

$$P^\nu(\Sigma) \equiv - \int_{\Sigma} dS_\mu T^{\mu\nu}(x). \quad (217)$$

The Ward identity (201) implies that a correlator of  $P^\nu(\Sigma)$  with other operators is unchanged as we move  $\Sigma$ , as long as  $\Sigma$  doesn't cross any operator insertions (figure 2). We say that  $P^\nu(\Sigma)$  is a “topological surface operator.”

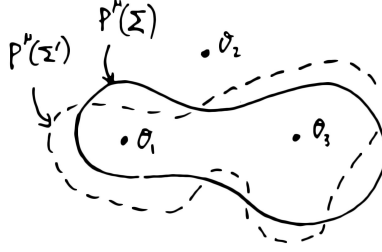


Fig. 2. A surface  $\Sigma$  supporting the operator  $P^\mu(\Sigma)$  can be freely deformed  $\Sigma \rightarrow \Sigma'$  without changing the correlation function, as long as it doesn't cross any operator insertions.

Let  $\Sigma = \partial B$  be the boundary of a ball  $B$  containing  $x$  and no other insertions. Integrating (201) over  $B$  gives

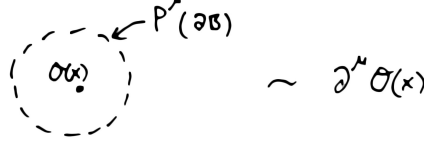
$$\langle P^\mu(\Sigma) \mathcal{O}(x) \dots \rangle = \partial^\mu \langle \mathcal{O}(x) \dots \rangle. \quad (218)$$

In other words, surrounding  $\mathcal{O}(x)$  with the topological surface operator  $P^\mu$  is equivalent to taking a derivative (figure 3).

<sup>55</sup>In fact, some theories have local gravitational anomalies that mean they are not precisely diffeomorphism invariant. However, this is a very mild breaking of diffeomorphism invariance and doesn't affect our analysis.

<sup>56</sup>The word “surface” usually refers to a 2-manifold, but we will abuse terminology and use it to refer to a codimension-1 manifold.

<sup>57</sup>Our definition of  $P^\nu$  differs from the usual one by a factor of  $i$ . This convention is much nicer for Euclidean field theories, but it has the effect of modifying some familiar formulae, and also changing the properties of symmetry generators under Hermitian conjugation. More on this in section 10.1.

Fig. 3. Surrounding  $\mathcal{O}(x)$  with  $P^\mu$  gives a derivative.

In quantum field theory, having a topological codimension-1 operator is the same as having a symmetry.<sup>58,59</sup> This may be unfamiliar language, so to connect to something more familiar, let us revisit the relation between the path integral and Hamiltonian formalisms.

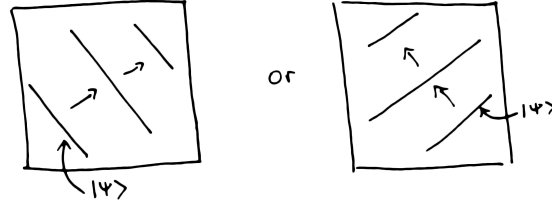


Fig. 4. In a rotationally invariant Euclidean theory, we can choose any direction as time. States live on slices orthogonal to the time direction.

Recall that the path integral can be interpreted in terms of different quantizations. To specify a quantization, we foliate spacetime by slices related by an isometry  $\partial_t$  (figure 4). A slice has an associated Hilbert space of states. A correlation function  $\langle \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) \rangle$  gets interpreted as a time-ordered expectation value

$$\langle \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) \rangle = \langle 0 | T \{ \hat{\mathcal{O}}_1(t_1, \mathbf{x}_1) \cdots \hat{\mathcal{O}}_n(t_n, \mathbf{x}_n) \} | 0 \rangle. \quad (219)$$

Here, the time ordering  $T\{\dots\}$  is with respect to our foliation,  $|0\rangle$  is the vacuum in the Hilbert space  $\mathcal{H}$  living on a spatial slice,<sup>60</sup> and  $\hat{\mathcal{O}}_i(x) : \mathcal{H} \rightarrow \mathcal{H}$  are quantum operators corresponding to the path integral insertions

<sup>58</sup>Topological operators with support on other types of manifolds give “generalized symmetries” [9].

<sup>59</sup>More precisely, to have a symmetry we usually require the topological codimension-1 operators to be invertible. If they are group-like, this means they have a multiplicative inverse, and if they are algebra-like, they have an additive inverse.

<sup>60</sup>Other choices of initial and final state correspond to different boundary conditions for the path integral.

$\mathcal{O}_i(x)$ .

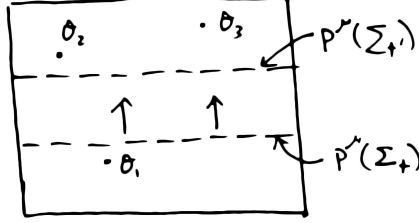


Fig. 5. The charge  $P^\mu(\Sigma_t)$  can be moved from one time to another  $t \rightarrow t'$  without changing the correlation function.

Let  $\Sigma_t$  be a spatial slice at time  $t$  and consider the operator  $P^\mu(\Sigma_t)$ . Because  $P^\mu(\Sigma)$  is topological, we are free to move it forward or backward in time from one spatial slice to another as long as it doesn't cross any operator insertions (figure 5). In fact, we often neglect to specify the surface  $\Sigma_t$  and just write  $P^\mu$  (though we should keep in mind where the surface lives with respect to other operators). We call  $P^\mu$  “momentum,” and the fact that it's topological is the path integral version of the statement that momentum is conserved.

Let us understand what happens when we move  $P^\mu$  past an operator insertion. Consider a local operator  $\mathcal{O}(x)$  at time  $t$ . If  $\Sigma_1, \Sigma_2$  are spatial surfaces at times  $t_1 < t < t_2$ , then when we quantize our theory, the difference  $P^\mu(\Sigma_2) - P^\mu(\Sigma_1)$  becomes a commutator because of time ordering,

$$\langle (P^\mu(\Sigma_2) - P^\mu(\Sigma_1))\mathcal{O}(x) \dots \rangle = \langle 0|T\{[\hat{P}^\mu, \hat{\mathcal{O}}(x)] \dots \}|0\rangle. \quad (220)$$

(We assume that the other insertions “...” are not between times  $t_1$  and  $t_2$ .) Because  $P^\mu$  is topological, we can deform  $\Sigma_2 - \Sigma_1$  to a sphere  $S$  surrounding  $\mathcal{O}(x)$ , as in figure 6. Then using the Ward identity (218), we find

$$\begin{aligned} \langle 0|T\{[\hat{P}^\mu, \hat{\mathcal{O}}(x)] \dots \}|0\rangle &= \langle (P^\mu(\Sigma_2) - P^\mu(\Sigma_1))\mathcal{O}(x) \dots \rangle \\ &= \langle P^\mu(S)\mathcal{O}(x) \dots \rangle \\ &= \partial^\mu \langle \mathcal{O}(x) \dots \rangle \\ &= \partial^\mu \langle 0|T\{\hat{\mathcal{O}}(x) \dots \}|0\rangle, \end{aligned} \quad (221)$$

in other words,

$$[\hat{P}^\mu, \hat{\mathcal{O}}(x)] = \partial^\mu \hat{\mathcal{O}}(x). \quad (222)$$

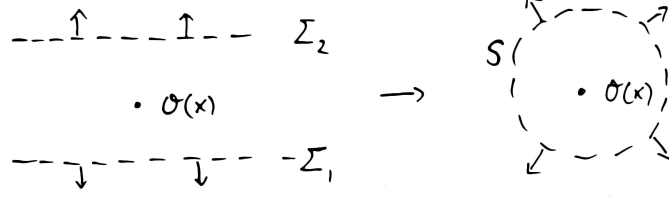


Fig. 6. For any charge  $Q(\Sigma)$ , we can deform  $Q(\Sigma_2) - Q(\Sigma_1) = Q(\Sigma_2 - \Sigma_1)$  to an insertion of  $Q(S)$ . Here, arrows indicate the orientation of the surface.

Figure 6 makes it clear that this result is independent of how we quantize our theory, since we always obtain a sphere surrounding  $\mathcal{O}(x)$  no matter which direction we choose as “time.” Thus, we often write

$$[P^\mu, \mathcal{O}(x)] = \partial^\mu \mathcal{O}(x), \quad (223)$$

without specifying a quantization. In fact, from now on, we will no longer distinguish between path integral insertions  $\mathcal{O}(x)$  and quantum operators  $\hat{\mathcal{O}}(x)$ . The expression  $[Q, \mathcal{O}(x)]$  can be interpreted as either an actual commutator  $[\hat{Q}, \hat{\mathcal{O}}(x)]$  in any quantization of the theory, or in path-integral language as surrounding  $\mathcal{O}(x)$  with a topological surface operator  $Q(S)$ .

Figure 6 also explains why the commutator  $[Q, \mathcal{O}(x)]$  of a charge  $Q$  with a local operator  $\mathcal{O}(x)$  is local, even though  $Q$  is nonlocal (it is the integral of a current). The reason is that the support of  $Q$  can be deformed to an arbitrarily small sphere  $S$  around  $x$ , so that the insertion  $Q(S)\mathcal{O}(x)$  only affects the path integral in an infinitesimal neighborhood of  $x$ . In general, the way local operators transform under symmetry is always insensitive to IR details like spontaneous symmetry breaking or finite temperature. This is because commutators with conserved charges can be computed at short distances.

*End of lecture 9*

Equation (223) integrates to

$$\mathcal{O}(x) = e^{x \cdot P} \mathcal{O}(0) e^{-x \cdot P}. \quad (224)$$

This statement is also true in any quantization of the theory. In path integral language,  $e^{x \cdot P}(\Sigma)$  is another type of topological surface operator. When we surround  $\mathcal{O}(0)$  with  $e^{x \cdot P}(\Sigma)$ , it becomes conjugation  $e^{x \cdot P}(\Sigma)\mathcal{O}(0) \rightarrow e^{\hat{P} \cdot x} \hat{\mathcal{O}}(0) e^{-\hat{P} \cdot x}$  in any quantization.

Consider the time-ordered correlator (219) with  $t_n > \dots > t_1$ . Using

(224), it becomes

$$\begin{aligned}
& \langle \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) \rangle \\
&= \langle 0 | e^{t_n P^0} \mathcal{O}_n(0, \mathbf{x}_n) e^{-t_n P^0} \cdots e^{t_1 P^0} \mathcal{O}_1(0, \mathbf{x}_1) e^{-t_1 P^0} | 0 \rangle \\
&= \langle 0 | \mathcal{O}_n(0, \mathbf{x}_n) e^{-(t_n - t_{n-1}) P^0} \cdots e^{-(t_2 - t_1) P^0} \mathcal{O}_1(0, \mathbf{x}_1) | 0 \rangle. \tag{225}
\end{aligned}$$

In other words, the path integral between spatial slices separated by time  $t$  computes a time evolution operator  $U(t) = e^{-tP^0}$ . In particular,  $P^0$  is the Hamiltonian, as expected.

### 5.3. A topological surface operator on the lattice

Let us give an example of a topological surface operator that is not the integral of a current (or the exponential of the integral of a current). We will again see that it should be thought of as a symmetry.

Consider the lattice Ising model in  $d$ -dimensions with zero applied magnetic field,  $h = 0$ . The operator  $U_{\mathbb{Z}_2}(\Sigma)$  is defined by flipping the signs of all nearest-neighbor bonds that cross the surface  $\Sigma$ . More precisely, let  $\Sigma$  be a closed surface that does not intersect any of the lattice points. For each pair of nearest neighbor sites  $\langle ij \rangle$ , define the sign<sup>61</sup>

$$p_{i,j}(\Sigma) = \begin{cases} -1 & \text{if the edge from } i \text{ to } j \text{ intersects } \Sigma, \\ +1 & \text{otherwise.} \end{cases}$$

An insertion of  $U_{\mathbb{Z}_2}(\Sigma)$  is defined by

$$\langle U_{\mathbb{Z}_2}(\Sigma) \mathcal{O}_1 \cdots \mathcal{O}_n \rangle \equiv \sum_{\{s_i\}} \mathcal{O}_1 \cdots \mathcal{O}_n e^{K \sum_{\langle ij \rangle} p_{i,j}(\Sigma) s_i s_j}. \tag{226}$$

Why is the operator  $U_{\mathbb{Z}_2}(\Sigma)$  topological? Consider a deformation  $\Sigma \rightarrow \Sigma'$ . We can transform  $U_{\mathbb{Z}_2}(\Sigma)$  into  $U_{\mathbb{Z}_2}(\Sigma')$  by doing a change of variables  $s_i \rightarrow -s_i$  for all spins between  $\Sigma$  and  $\Sigma'$  in the path integral (226).

When we flip the spins between  $\Sigma$  and  $\Sigma'$ , any  $\mathbb{Z}_2$ -odd operator in that region will also flip sign. Thus,  $U_{\mathbb{Z}_2}(\Sigma)$  can be deformed freely, but when we move it past an operator, we pick up a factor of the  $\mathbb{Z}_2$  charge of that operator. For example, if  $B$  is a ball containing an operator  $\mathcal{O}$  (and no other operators), we have

$$\langle U_{\mathbb{Z}_2}(\partial B) \mathcal{O} \cdots \rangle = q_{\mathcal{O}} \langle \mathcal{O} \cdots \rangle, \quad q_{\mathcal{O}} = \pm 1. \tag{227}$$

This equation is exactly analogous to (218).

<sup>61</sup>Being more careful to allow for very wiggly surfaces, we should define  $p_{\Sigma}(i, j)$  as  $(-1)^n$ , where  $n$  is the number of times  $\Sigma$  intersects the edge  $\langle ij \rangle$ .

Consider inserting  $U_{\mathbb{Z}_2}$  on a spatial slice  $\Sigma_t$  at fixed time  $t$ .<sup>62</sup> After quantization,  $U_{\mathbb{Z}_2}(\Sigma_t)$  becomes an operator  $\hat{U}_{\mathbb{Z}_2}$  that is independent of  $t$  — in other words it is conserved — because  $U_{\mathbb{Z}_2}$  is topological. The only effect of  $t$  is to determine the ordering of quantum operators. For example, assuming  $t_3 > t_2 > t_1$ , we have<sup>63</sup>

$$\langle \mathcal{O}_3(t_3) U_{\mathbb{Z}_2}(\Sigma_{t_2}) \mathcal{O}_1(t_1) \rangle = \langle 0 | \hat{\mathcal{O}}_3(t_3) \hat{U}_{\mathbb{Z}_2} \hat{\mathcal{O}}_1(t_1) | 0 \rangle. \quad (228)$$

Here  $\hat{\mathcal{O}}_i(t_i)$  are Heisenberg-picture operators as in, e.g. (81). The statement (227) is equivalent to

$$\hat{U}_{\mathbb{Z}_2} \hat{\mathcal{O}} \hat{U}_{\mathbb{Z}_2}^{-1} = q_{\mathcal{O}} \hat{\mathcal{O}}. \quad (229)$$

Concretely, the operator  $\hat{U}_{\mathbb{Z}_2}$  is given by

$$\hat{U}_{\mathbb{Z}_2} = \prod_{i=1}^M \sigma_i^x, \quad \hat{U}_{\mathbb{Z}_2} |\mathbf{s}\rangle = |-\mathbf{s}\rangle. \quad (230)$$

where  $\mathbf{s} = (s_1, \dots, s_M)$  are the spins on a spatial slice.

By now it should be clear that  $\hat{U}_{\mathbb{Z}_2}$  is the generator of a  $\mathbb{Z}_2$  global symmetry acting on the Hilbert space. The path integral perspective shows that we can consider  $U_{\mathbb{Z}_2}$  on surfaces other than just spatial slices.

### 5.3.1. Implementation in the continuum

The operator  $U_{\mathbb{Z}_2}$  can also be defined in continuum  $\phi^4$  theory as follows. Given a surface  $\Sigma$ , let us cover it with small open balls  $B_i$  such that  $\Sigma$  splits each ball into two pieces  $B_i^{\text{left}} \cup B_i^{\text{right}}$ . We do this so we don't have to worry about the global structure of  $\Sigma$ . The action without an insertion of  $U_{\mathbb{Z}_2}(\Sigma)$  is the integral of a Lagrangian density

$$S[\phi] = \sum_i \int_{B_i} d^d x \mathcal{L}[\phi](x),$$

$$\mathcal{L}[\phi](x) = \frac{1}{2}(\partial\phi)^2 + \dots \quad (231)$$

Note that  $S[\phi]$  favors configurations where  $\phi$  is slowly-varying.

<sup>62</sup>Since we are on a lattice,  $t$  is discrete. As usual, we are free to choose any lattice direction as time.

<sup>63</sup>When  $\Sigma$  is inserted between two rows of spins, we evolve from one row to the next using either  $\hat{T}\hat{U}_{\mathbb{Z}_2}$  or  $\hat{U}_{\mathbb{Z}_2}\hat{T}$ , where  $\hat{T}$  is the transfer matrix. The two choices are the same because  $\hat{U}_{\mathbb{Z}_2}$  commutes with  $\hat{T}$ . This is one way of saying that  $\hat{U}_{\mathbb{Z}_2}$  is conserved.

To insert  $U_{\mathbb{Z}_2}(\Sigma)$ , we would like to instead favor configurations where  $\phi$  flips sign as it crosses  $\Sigma$ . Let

$$\phi'_i(x) = \begin{cases} \phi(x) & x \in B_i^{\text{right}} \\ -\phi(x) & x \in B_i^{\text{left}} \end{cases}.$$

The modified action is given by

$$S'[\phi] = \sum_i \int_{B_i} d^d x \mathcal{L}[\phi'_i](x). \quad (232)$$

Inserting  $U_{\mathbb{Z}_2}(\Sigma)$  means performing the path integral with the modified action  $S'$ ,

$$\langle U_{\mathbb{Z}_2}(\Sigma) \mathcal{O}_1 \cdots \mathcal{O}_n \rangle = \int D\phi \mathcal{O}_1 \cdots \mathcal{O}_n e^{-S'[\phi]}. \quad (233)$$

This is the continuum version of flipping the bonds that cross  $\Sigma$ .

In practice, it is difficult to work with  $S'[\phi]$  because it has funny  $\delta$ -functions coming from derivatives of  $\phi'$ . If  $\Sigma = \partial B$  is the boundary of a region  $B$ , we can make a change of variables  $\phi \rightarrow -\phi$  inside  $B$  to get rid of  $U_{\mathbb{Z}_2}(\Sigma)$ . When we do this, any  $\mathbb{Z}_2$ -odd operator inside  $B$  will pick up a sign. This gives the same result (227) that we found on the lattice.

However, if  $\Sigma$  is not the boundary of a region, we cannot make a change of variables to remove  $U_{\mathbb{Z}_2}(\Sigma)$ . As an example, consider the theory on a torus  $T^d$  with  $U_{\mathbb{Z}_2}$  inserted on a spatial slice  $T^{d-1} \subset T^d$ . This slice is homologically nontrivial, so no change of variables will remove  $U_{\mathbb{Z}_2}$ . However, we can simplify the action by defining  $\phi$  to flip sign as we cross  $\Sigma$ ,

$$\phi(t_{\text{below}}) = -\phi(t_{\text{above}}). \quad (234)$$

This turns  $S'$  back into  $S$ , but with the cost of introducing “twisted” boundary conditions for the field  $\phi$

$$\phi(t + \beta) = -\phi(t), \quad (235)$$

where  $\beta$  is the length of the time circle. The path integral with twisted boundary conditions is often more straightforward to evaluate.

One effect of inserting  $U_{\mathbb{Z}_2}$  on a spatial slice  $T^{d-1} \subset T^d$  is that correlators of  $\mathbb{Z}_2$ -odd operators in the presence of  $U_{\mathbb{Z}_2}(T^{d-1})$  become antiperiodic in the time direction. This is because to move them around the time circle, we must cross  $U_{\mathbb{Z}_2}$ , and doing so introduces a sign. This can also be understood in terms of twisted boundary conditions: any operator made of an odd number of  $\phi$ 's will also satisfy the antiperiodicity condition (235).

#### 5.4. More spacetime symmetries

Let us return to discussing symmetries built from the stress tensor. Given a conserved current  $\partial_\mu J^\mu = 0$  (as an operator equation), we can always define a topological surface operator by integration.<sup>64</sup> For  $P^\nu$ , the corresponding currents are  $T^{\mu\nu}(x)$ . More generally, given a vector field  $\epsilon = \epsilon^\mu(x)\partial_\mu$ , the charge

$$Q_\epsilon(\Sigma) = - \int_\Sigma dS_\mu \epsilon_\nu(x) T^{\mu\nu}(x) \quad (236)$$

will be conserved whenever

$$\begin{aligned} 0 &= \partial_\mu(\epsilon_\nu T^{\mu\nu}) \\ &= \partial_\mu \epsilon_\nu T^{\mu\nu} + \epsilon_\nu \partial_\mu T^{\mu\nu} \\ &= \frac{1}{2}(\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu) T^{\mu\nu}, \end{aligned} \quad (237)$$

or

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = 0. \quad (238)$$

This is the Killing equation. It is equivalent to the statement that  $\epsilon$  generates an isometry of the flat metric on  $\mathbb{R}^d$ . It has solutions

$$\begin{aligned} p_\mu &= \partial_\mu && \text{(translations),} \\ m_{\mu\nu} &= x_\nu \partial_\mu - x_\mu \partial_\nu && \text{(rotations).} \end{aligned} \quad (239)$$

The corresponding charges are momentum  $P_\mu = Q_{p_\mu}$  and angular momentum  $M_{\mu\nu} = Q_{m_{\mu\nu}}$ .

*End of lecture 10*

## 6. Conformal symmetry

In a conformal field theory, the stress tensor satisfies an additional condition: it is traceless,

$$T^\mu_\mu(x) = 0 \quad \text{(operator equation).} \quad (240)$$

This is equivalent to the statement that the theory is insensitive to position-dependent rescalings of the metric  $\delta g_{\mu\nu} = 2\omega(x)g_{\mu\nu}$  near flat space.<sup>65</sup>

<sup>64</sup>It is an interesting question whether the converse is true. When a theory has a Lagrangian description, the Noether procedure gives a conserved current for any continuous symmetry (that is manifest in the Lagrangian). Proving Noether's theorem without a Lagrangian is an open problem.

<sup>65</sup>In curved space, there can be Weyl anomalies, which we discuss in section ??.



### 6.1. Scale-invariance plus unitarity implies conformal invariance in 2d

We have seen that critical points exhibit scale-invariance, but it might not be obvious why they should have a traceless stress tensor. Proving that scale-invariance implies  $T_\mu^\mu = 0$  for general  $d$  is an open problem.<sup>66</sup> However, proving it for  $d = 2$  is straightforward.<sup>67</sup>

Consider the two-point function of the stress tensor in momentum space (this will make it easy to study conservation). Symmetry and conservation require it to take the form

$$\begin{aligned} \langle T_{\mu\nu}(q) T_{\rho\sigma}(-q) \rangle = & \frac{1}{2} ((q_\mu q_\rho - q^2 \delta_{\mu\rho})(q_\nu q_\sigma - q^2 \delta_{\nu\sigma}) + \rho \leftrightarrow \sigma) f(q^2) \\ & + (q_\mu q_\nu - q^2 \delta_{\mu\nu})(q_\rho q_\sigma - q^2 \delta_{\rho\sigma}) g(q^2), \end{aligned} \quad (241)$$

where  $f(q^2), g(q^2)$  are general functions of  $q^2$ . Here, we have Fourier transformed the two operators with momenta  $p, q$ , and stripped off an overall momentum-conserving  $\delta^d(p+q)$ .

The form (241) holds in any  $d$ . However in  $d = 2$ , something special happens: the two structures above are linearly-dependent.

**Exercise 6.1.** Let  $d = 2$ , and consider  $\tilde{q}_\mu = \epsilon_{\mu\nu} q^\nu$ , where  $\epsilon_{\mu\nu}$  is the totally antisymmetric symbol:

$$\epsilon_{01} = 1, \quad \epsilon_{10} = -1, \quad \epsilon_{00} = \epsilon_{11} = 0. \quad (242)$$

Simplify the product

$$\tilde{q}_\mu \tilde{q}_\nu \tilde{q}_\rho \tilde{q}_\sigma \quad (243)$$

in two different ways to obtain the structures multiplying  $f(q^2)$  and  $g(q^2)$ .

Thus, in two dimensions, we have simply

$$\langle T_{\mu\nu}(q) T_{\rho\sigma}(-q) \rangle = (q_\mu q_\nu - q^2 \delta_{\mu\nu})(q_\rho q_\sigma - q^2 \delta_{\rho\sigma}) h(q^2) \quad (244)$$

<sup>66</sup>An almost-complete argument exists in 4-dimensions [1]. Part of the challenge is identifying the appropriate assumptions to make. There exist boring/pathological counterexamples in 3d (for example free electromagnetism), and the right formulation of the theorem should exclude these cases.

<sup>67</sup>This argument is from Zohar Komargodski's notes [10]. See [11] for a similar proof in position-space.

for a single function  $h(q^2)$ . Now suppose that the theory is also scale-invariant. This implies that  $h(q^2)$  should be a pure power of  $q$ ,<sup>68</sup>

$$h(q^2) = \frac{b}{q^2}. \quad (245)$$

The dimension of  $h(q^2)$  comes from the fact that  $T_{\mu\nu}(q)$  is dimensionless (it is the  $d$ -dimensional Fourier transform of a quantity with dimension  $d$ ), but we have also stripped off a dimensionful  $\delta^d(p+q)$  to define (244). Taking the trace, we find

$$\langle T(q)T(-q) \rangle = bq^2, \quad (246)$$

where  $T \equiv T_\mu^\mu$ . In position-space, this becomes a contact term

$$\langle T(x)T(y) \rangle \propto b\partial^2\delta^2(x-y). \quad (247)$$

In a unitary quantum field theory, if the two point function of an operator vanishes at separated points, *any* correlator involving that operator vanishes at separated points. Roughly-speaking, unitary means the theory that can be quantized in such a way that the space of states has a positive-definite inner product and the Hamiltonian is Hermitian and bounded from below. In a Euclidean theory, a more correct term is “reflection positivity.” We will discuss unitarity/reflection positivity in more detail later.

The argument for vanishing of  $T(x)$  is as follows. As we discussed in the introduction, in a unitary theory, inserting a complete set of states into a two-point function gives a sum of positive terms (see equation (3))

$$\langle T(x^0, \mathbf{0})T(0) \rangle = \sum_{\psi} |\langle 0|T(0)|\psi \rangle|^2 e^{-E_\psi x^0}. \quad (248)$$

If the left-hand side vanishes, each term on the right-hand side must vanish as well:  $\langle 0|T(0)|\psi \rangle = 0$ , in other words  $T(x)$  must annihilate the vacuum. Now consider a general correlation function of  $T(x)$  with other operators,

$$\langle T(x)\mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) \rangle. \quad (249)$$

This correlator vanishes if  $x$  is in the past or future of all the other  $x_i$ , since then  $T(x)$  acts on the vacuum on either the right or left. However, correlation functions in QFT are analytic in their arguments, and the analytic continuation of zero is zero. Thus, the entire correlator (249) vanishes.

<sup>68</sup>Nontrivial scale-dependence is allowed in correlation functions if it multiplies a contact term (i.e. the violation of scaling is invisible at separated points). In momentum space, this means we can have terms proportional to  $\log(q^2/\mu^2)$ , provided they multiply an analytic function of  $q$  (whose Fourier transform is a contact term). In the case of  $h(q^2)$ , something like  $q^{-2} \log(q^2/\mu^2)$  is disallowed because  $\mu^{\frac{d}{d\mu}} q^{-2} \log(q^2/\mu^2)$  is not analytic in  $q$ .

### 6.1.1. Scale without conformal invariance

Scale without conformal invariance is possible in nonunitary theories. Perhaps the simplest example is the theory of elasticity in 2-dimensions [12]. This is the theory of a vector  $u_\mu$  with action

$$S = \int d^2x \left( g u_{\mu\nu} u^{\mu\nu} + \frac{k}{2} (u_\mu{}^\mu)^2 \right), \quad (250)$$

where  $u_{\mu\nu} = \frac{1}{2}(\partial_\mu u_\nu + \partial_\nu u_\mu)$ . The field  $u_\mu$  is called the “displacement field” and  $u_{\mu\nu}$  is called the “strain tensor.” The coefficient  $g$  represents the shear modulus, and  $k + g$  is the bulk modulus of the described material.

The action (250) is quadratic in  $u$ , so this theory is easy to quantize. Note that  $u_\mu$  describes a massless vector field without a gauge redundancy. Thus, there are negative-norm states, and the theory is nonunitary. Concretely, one can define a conserved stress tensor  $T^{\mu\nu}$  that satisfies the correct Ward identities, and such that  $\langle T_\mu^\mu T_\nu^\nu \rangle = 0$ , but  $T_\mu^\mu$  is not zero. For example, it has non-vanishing correlation functions with other operators:

$$\langle T_\mu^\mu(z) : \partial u \partial u : (0) \rangle = -\frac{k + g}{2\pi^2 g(k + 2g)} \frac{1}{z^4}. \quad (251)$$

(Here, we use complex coordinates  $z = x + iy$  and  $u = u_1 - iu_2$ .)

This example shows that the positive sum (248) for  $\langle TT \rangle$  is crucial for completing the argument in section 6.1. The proof that scale implies conformal invariance in 4d [ ] also uses unitarity in a fundamental way. However, though unitarity may be a sufficient condition for conformal invariance, it is not necessary. We will encounter some interesting nonunitary CFTs (like the Lee-Yang theory) later in the course.

[Discuss virial current?](#)

## 6.2. Symmetries from tracelessness

Recall that the charge  $Q_\epsilon(\Sigma)$  is conserved if

$$\frac{1}{2}(\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu) T^{\mu\nu} = 0. \quad (252)$$

When the stress tensor is traceless, this condition can be solved by demanding a less constraining version of the Killing equation (238),

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = c(x) \delta_{\mu\nu}, \quad (253)$$

where  $c(x)$  is any scalar function. Contracting both sides with  $\delta^{\mu\nu}$  determines  $c$  in terms of  $\epsilon$ ,

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = \frac{2}{d}(\partial \cdot \epsilon)\delta_{\mu\nu}. \quad (254)$$

Equation (254) is the *conformal* Killing equation. Let us find its solutions in  $\mathbb{R}^d$ . Taking  $\partial^\nu \partial^\mu$  of both sides, we find

$$\left(2 - \frac{2}{d}\right) \partial^2(\partial \cdot \epsilon) = 0 \implies \partial^2(\partial \cdot \epsilon) = 0, \quad (255)$$

where we have assumed that  $d \neq 1$ . Now taking  $\partial^\nu \partial_\rho$  and symmetrizing in  $(\rho\mu)$ , we find

$$\left(1 - \frac{2}{d}\right) \partial_\rho \partial_\mu (\partial \cdot \epsilon) = 0 \quad (256)$$

For the moment, let us focus on the case  $d \neq 2$ , so we can conclude that  $\partial_\rho \partial_\mu (\partial \cdot \epsilon) = 0$ . Taking two derivatives of the conformal Killing equation and using (256), we find

$$\begin{aligned} \partial_\alpha (\partial_\beta \partial_\gamma \epsilon_\delta + \partial_\beta \partial_\delta \epsilon_\gamma) &= 0, \\ \partial_\alpha (\partial_\gamma \partial_\beta \epsilon_\delta + \partial_\gamma \partial_\delta \epsilon_\beta) &= 0, \\ \partial_\alpha (\partial_\delta \partial_\gamma \epsilon_\beta + \partial_\delta \partial_\beta \epsilon_\gamma) &= 0. \end{aligned} \quad (257)$$

These are three equations for three unknowns, and they imply

$$\partial_\alpha \partial_\beta \partial_\gamma \epsilon_\delta = 0. \quad (258)$$

Thus  $\epsilon_\delta$  is at most a quadratic polynomial in  $x$ .

From here the analysis is straightforward. Together with  $p_\mu, m_{\mu\nu}$ , the space of solutions is spanned by

$$\begin{aligned} d &= x^\mu \partial_\mu && \text{(dilatations),} \\ k_\mu &= 2x_\mu(x \cdot \partial) - x^2 \partial_\mu && \text{(special conformal transformations).} \end{aligned} \quad (259)$$

The corresponding symmetry charges are  $D = Q_d$  and  $K_\mu = Q_{k_\mu}$ . Overall, there are

$$\underbrace{d}_{p_\mu} + \underbrace{\frac{d(d-1)}{2}}_{m_{\mu\nu}} + \underbrace{1}_d + \underbrace{d}_{k_\mu} = \frac{(d+2)(d+1)}{2} \quad (260)$$

solutions to the conformal Killing equation in  $d > 2$ , each with an associated conserved charge.

Now let us return to the case  $d = 2$ . It is natural to write the conformal Killing equation in terms of holomorphic and antiholomorphic coordinates

$\epsilon = \epsilon \partial_z + \bar{\epsilon} \partial_{\bar{z}}$ , where  $\epsilon, \bar{\epsilon}$  are initially independent functions of both  $z$  and  $\bar{z}$ . In this language, the conformal Killing equation becomes

$$\partial_z \bar{\epsilon} = 0, \quad \partial_{\bar{z}} \epsilon = 0. \quad (261)$$

(The equation involving  $\partial_z \epsilon + \partial_{\bar{z}} \bar{\epsilon}$  is tautological.) A general solution is given by choosing  $\epsilon$  to be a function of  $z$  alone, and  $\bar{\epsilon}$  to be a function of  $\bar{z}$  alone,

$$\epsilon = \epsilon(z) \partial_z + \bar{\epsilon}(\bar{z}) \partial_{\bar{z}}. \quad (262)$$

Note that this includes the solutions  $p_\mu, m_{\mu\nu}, k_\mu, d$  along with many others. Consequently, 2d CFTs have an infinite set of conserved charges given by the integral of the stress tensor against a holomorphic or antiholomorphic vector field. We will study the implications of this infinite set of charges in section ???. For now we focus on the charges that are present in any  $d$ .

[Discuss 1 dimension](#)

[End of lecture 11](#)

### 6.3. Finite conformal transformations

Before discussing the charges  $P_\mu, M_{\mu\nu}, D, K$ , let us take a moment to understand the geometrical meaning of the conformal Killing vectors (239) and (259). Consider an infinitesimal transformation  $x^\mu \rightarrow x'^\mu = x^\mu + \epsilon^\mu(x)$ . If  $\epsilon^\mu$  satisfies the conformal Killing equation, then

$$\frac{\partial x'^\mu}{\partial x^\nu} = \delta_\nu^\mu + \partial_\nu \epsilon^\mu = \left(1 + \frac{1}{d}(\partial \cdot \epsilon)\right) \left(\delta_\nu^\mu + \frac{1}{2}(\partial_\nu \epsilon^\mu - \partial^\mu \epsilon_\nu)\right). \quad (263)$$

The right-hand side is an infinitesimal rescaling times an infinitesimal rotation. Exponentiating gives a coordinate transformation  $x \rightarrow x'$  such that

$$\frac{\partial x'^\mu}{\partial x^\nu} = \Omega(x) R^\mu{}_\nu(x), \quad R^T R = I_{d \times d}, \quad (264)$$

where  $\Omega(x)$  and  $R^\mu{}_\nu(x)$  are finite position-dependent rescalings and rotations. Equivalently, the transformation  $x \rightarrow x'$  rescales the metric by a scalar factor,

$$\delta_{\mu\nu} \frac{\partial x'^\mu}{\partial x^\alpha} \frac{\partial x'^\nu}{\partial x^\beta} = \Omega(x)^2 \delta_{\alpha\beta}. \quad (265)$$

Such transformations are called *conformal*. They comprise the conformal group, a finite-dimensional subgroup of the diffeomorphism group of  $\mathbb{R}^d$ .

The exponentials of  $p_\mu$  and  $m_{\mu\nu}$  are translations and rotations. Exponentiating  $d$  gives a scale transformation  $x \rightarrow \lambda x$ . We can understand the exponential of  $k_\mu$  by first considering an inversion

$$I : x^\mu \rightarrow \frac{x^\mu}{x^2}. \quad (266)$$

$I$  is a conformal transformation, but it is not continuously connected to the identity, so it can't be obtained by exponentiating a conformal Killing vector. This means that a CFT need not have  $I$  as a symmetry.

**Exercise 6.2.** Show that  $I$  is continuously connected to a reflection  $x^0 \rightarrow -x^0$ . Conclude that a CFT is invariant under  $I$  if and only if it is invariant under reflections.

**Exercise 6.3.** Show that  $k_\mu = -Ip_\mu I$ . Conclude that  $e^{a \cdot k}$  implements the transformation

$$x \rightarrow x'(x) = \frac{x^\mu - a^\mu x^2}{1 - 2(a \cdot x) + a^2 x^2}. \quad (267)$$

We can think of  $k_\mu$  as a “translation that moves infinity and fixes the origin” in the same sense that the usual translations move the origin and fix infinity, see figure 7.

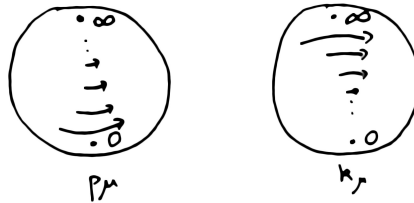


Fig. 7.  $k_\mu$  is analogous to  $p_\mu$ , with the origin and the point at infinity swapped by an inversion.

#### 6.4. The conformal algebra

The charges  $Q_\epsilon$  give a representation of the conformal algebra

$$[Q_{\epsilon_1}, Q_{\epsilon_2}] = Q_{-[\epsilon_1, \epsilon_2]}, \quad (268)$$

where  $[\epsilon_1, \epsilon_2]$  is a commutator of vector fields.<sup>69</sup> This is not obvious and deserves proof. In fact, it is *not true* in 2-dimensional CFTs, where the algebra of charges is a central extension of the algebra of conformal killing vectors.

**Exercise 6.4.** Show that in  $d \geq 3$ ,

$$[Q_\epsilon, T^{\mu\nu}] = \epsilon^\rho \partial_\rho T^{\mu\nu} + (\partial \cdot \epsilon) T^{\mu\nu} - \partial_\rho \epsilon^\mu T^{\rho\nu} + \partial^\nu \epsilon_\rho T^{\rho\mu}. \quad (269)$$

Argue as follows. Assume that only the stress tensor appears on the right-hand side.<sup>70</sup> Using linearity in  $\epsilon$ , dimensional analysis, and the conformal Killing equation, show that (269) contains all terms that could possibly appear.<sup>71</sup> Fix the relative coefficients using conservation, tracelessness, and symmetry under  $\mu \leftrightarrow \nu$ . Fix the overall coefficient by matching with (223).

**Exercise 6.5.** Using (269), prove the commutation relation (268).

**Exercise 6.6.** When  $d = 2$ , it's possible to add an extra term in (269) proportional to the unit operator that is consistent with dimensional analysis, conservation, and tracelessness. Find this term (up to an overall coefficient),<sup>72</sup> and show how it modifies the commutation relations (268). This is the Virasoro algebra!

Write out more of this in detail.

As usual, (268) is true in any quantization of the theory. In path integral language, it tells us how to move the topological surface operators  $Q_\epsilon(\Sigma)$

<sup>69</sup>The minus sign in (268) comes from the fact that when charges  $Q_i$  are represented by differential operators  $\mathcal{D}_i$ , repeated action reverses the order  $[Q_1, [Q_2, \mathcal{O}]] = \mathcal{D}_2 \mathcal{D}_1 \mathcal{O}$ . Alternatively, we could have introduced an extra minus sign in the  $Q$ 's so that  $[Q, \mathcal{O}] = -\mathcal{D}$  and then  $Q, \mathcal{D}$  would have the same commutation relations.

<sup>70</sup>Bonus exercise: can other operators appear?

<sup>71</sup>The terms on the right-hand side are local in  $\epsilon$  because we can evaluate  $[Q_\epsilon, T^{\mu\nu}(x)]$  in an arbitrarily small neighborhood of  $x$ . Assuming the singularity as two  $T^{\mu\nu}$ 's coincide is bounded, we can then replace  $\epsilon$  by its Taylor expansion around  $x$ .

<sup>72</sup>The coefficient can be fixed by comparing with the OPE, see e.g. [13]. It is proportional to the central charge  $c$ .

through each other.

**Exercise 6.7.** *Show that*

$$[M_{\mu\nu}, P_\rho] = \delta_{\nu\rho}P_\mu - \delta_{\mu\rho}P_\nu, \quad (270)$$

$$[M_{\mu\nu}, K_\rho] = \delta_{\nu\rho}K_\mu - \delta_{\mu\rho}K_\nu, \quad (271)$$

$$[M_{\mu\nu}, M_{\rho\sigma}] = \delta_{\nu\rho}M_{\mu\sigma} - \delta_{\mu\rho}M_{\nu\sigma} + \delta_{\nu\sigma}M_{\rho\mu} - \delta_{\mu\sigma}M_{\rho\nu}, \quad (272)$$

$$[D, P_\mu] = P_\mu, \quad (273)$$

$$[D, K_\mu] = -K_\mu, \quad (274)$$

$$[K_\mu, P_\nu] = 2\delta_{\mu\nu}D - 2M_{\mu\nu}, \quad (275)$$

and all other commutators vanish.

The first three commutation relations say that  $M_{\mu\nu}$  generates the algebra of Euclidean rotations  $\text{SO}(d)$  and that  $P_\mu, K_\mu$  transform as vectors. The last three are more interesting. Equations (273) and (274) say that  $P_\mu$  and  $K_\mu$  can be thought of as raising and lowering operators for  $D$ . We will return to this idea shortly.

**Exercise 6.8.** *Define the generators*

$$\begin{aligned} L_{\mu\nu} &= M_{\mu\nu}, \\ L_{-1,0} &= D, \\ L_{0,\mu} &= \frac{1}{2}(P_\mu + K_\mu), \\ L_{-1,\mu} &= \frac{1}{2}(P_\mu - K_\mu), \end{aligned} \quad (276)$$

where  $L_{ab} = -L_{ba}$  and  $a, b \in \{-1, 0, 1, \dots, d\}$ . Show that  $L_{ab}$  satisfy the commutation relations of  $\text{SO}(d+1, 1)$ .

The fact that the conformal algebra is  $\text{SO}(d+1, 1)$  suggests that it might be good to think about its action in terms of  $\mathbb{R}^{d+1,1}$  instead of  $\mathbb{R}^d$ . This is the idea behind the “embedding space formalism” [14–19], which provides a simple and powerful way to understand the constraints of conformal invariance. We will be more pedestrian in this course, but I recommend reading about the embedding space formalism in the lecture notes by Penedones [20] or Rychkov [3].

*End of lecture 12*



## 7. Primaries and descendants

Now that we have our conserved charges, we can classify operators into representations of those charges. We do this in steps. First we classify operators into Poincare representations, then scale+Poincare representations, and finally conformal representations.

### 7.1. Poincare Representations

In a rotationally-invariant QFT, local operators at the origin transform in irreducible representations of the rotation group  $\text{SO}(d)$ ,

$$[M_{\mu\nu}, \mathcal{O}^a(0)] = (\mathcal{S}_{\mu\nu})_b^a \mathcal{O}^b(0), \quad (277)$$

where  $\mathcal{S}_{\mu\nu}$  are matrices satisfying the same algebra as  $M_{\mu\nu}$ , and  $a, b$  are indices for the  $\text{SO}(d)$  representation of  $\mathcal{O}$ .<sup>7374</sup> We often suppress spin indices and write the right-hand side as simply  $\mathcal{S}_{\mu\nu} \mathcal{O}(0)$ . The action (277), together with the commutation relations of the Poincare group, determines how rotations act away from the origin.

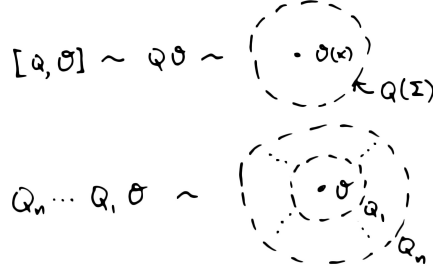


Fig. 8. The shorthand notation  $Q\mathcal{O}$  stands for surrounding  $\mathcal{O}$  with a surface operator  $Q(\Sigma)$ . Equivalently, it stands for  $[Q, \mathcal{O}]$  in any quantization of the theory.

To see this, it is convenient to adopt shorthand notation where commutators of charges with local operators are implicit,  $[Q, \mathcal{O}] \rightarrow Q\mathcal{O}$ , see figure 8. This notation is valid because of the Jacobi identity (more formally, the fact that adjoint action gives a representation of a Lie algebra). In path integral language,  $Q_n \cdots Q_1 \mathcal{O}(x)$  means surrounding  $\mathcal{O}(x)$  with

<sup>73</sup>The funny index contractions in (277) ensure that  $M_{\mu\nu}$  and  $\mathcal{S}_{\mu\nu}$  have the same commutation relations (exercise!).

<sup>74</sup>Because our commutation relations (272) for  $\text{SO}(d)$  differ from the usual conventions by a factor of  $i$ , the generators  $\mathcal{S}_{\mu\nu}$  will be *anti*-hermitian,  $\mathcal{S}^\dagger = -\mathcal{S}$ .

topological surface operators where  $Q_n$  is the outermost surface and  $Q_1$  is the innermost. The conformal commutation relations tell us how to re-order these surfaces.

Acting with a rotation on  $\mathcal{O}(x)$ , we have

$$\begin{aligned}
 M_{\mu\nu}\mathcal{O}(x) &= M_{\mu\nu}e^{x\cdot P}\mathcal{O}(0) \\
 &= e^{x\cdot P}(e^{-x\cdot P}M_{\mu\nu}e^{x\cdot P})\mathcal{O}(0) \\
 &= e^{x\cdot P}(-x_\mu P_\nu + x_\nu P_\mu + M_{\mu\nu})\mathcal{O}(0) \\
 &= (x_\nu\partial_\mu - x_\mu\partial_\nu + \mathcal{S}_{\mu\nu})e^{x\cdot P}\mathcal{O}(0) \\
 &= (m_{\mu\nu} + \mathcal{S}_{\mu\nu})\mathcal{O}(x).
 \end{aligned} \tag{278}$$

In the third line, we've used the Poincare algebra and the Hausdorff formula

$$e^A B e^{-A} = e^{[A, \cdot]} B = B + [A, B] + \frac{1}{2!}[A, [A, B]] + \dots \tag{279}$$

## 7.2. Scale+Poincare representations

In a scale-invariant theory, it's also natural to diagonalize the dilatation operator acting on operators at the origin,

$$[D, \mathcal{O}(0)] = \Delta \mathcal{O}(0). \tag{280}$$

The eigenvalue  $\Delta$  is the *dimension* of  $\mathcal{O}$ .

**Exercise 7.1.** Mimic the computation (278) to derive the action of dilatation on  $\mathcal{O}(x)$  away from the origin,

$$[D, \mathcal{O}(x)] = (x^\mu \partial_\mu + \Delta)\mathcal{O}(x). \tag{281}$$

Equation (281) is constraining enough to fix two-point functions of scalars up to a constant. Firstly, by rotation and translation invariance, we must have

$$\langle \mathcal{O}_1(x) \mathcal{O}_2(y) \rangle = f(|x - y|), \tag{282}$$

for some function  $f$ .

In a scale-invariant theory with scale-invariant boundary conditions, the simultaneous action of  $D$  on all operators in a correlator must vanish, as illustrated in figure 9. Moving  $D$  to the boundary gives zero.<sup>75</sup> On the

<sup>75</sup>It is also interesting to consider non-scale-invariant boundary conditions. These can be interpreted as having a nontrivial operator at  $\infty$ .

other hand, shrinking  $D$  to the interior gives the sum of its actions on the individual operators. By the Ward identity (281), this is

$$0 = (x^\mu \partial_\mu + \Delta_1 + y^\mu \partial_\mu + \Delta_2) f(|x - y|). \quad (283)$$

We could alternatively derive (283) by working in some quantization, where it follows from trivial algebra and the fact that  $D|0\rangle = 0$ ,

$$\begin{aligned} 0 &= \langle 0 | [D, \mathcal{O}_1(x) \mathcal{O}_2(y)] | 0 \rangle \\ &= \langle 0 | [D, \mathcal{O}_1(x)] \mathcal{O}_2(y) + \mathcal{O}_1(x) [D, \mathcal{O}_2(y)] | 0 \rangle \\ &= (x^\mu \partial_\mu + \Delta_1 + y^\mu \partial_\mu + \Delta_2) \langle 0 | \mathcal{O}_1(x) \mathcal{O}_2(y) | 0 \rangle. \end{aligned} \quad (284)$$

Either way, the solution is

$$f(|x - y|) = \frac{C}{|x - y|^{\Delta_1 + \Delta_2}}. \quad (285)$$

If we had an operator with negative scaling dimension, then its correlators would grow with distance, violating cluster decomposition. This is unphysical, so we expect dimensions  $\Delta$  to be positive. Shortly, we will prove this fact for unitary conformal theories (and derive even stronger constraints on  $\Delta$ ).

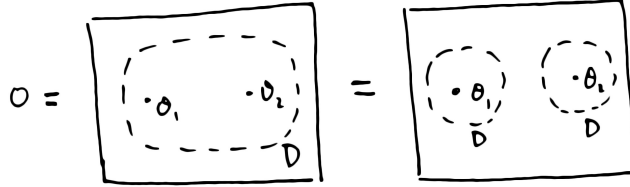


Fig. 9. The Ward identity for scale invariance of a two-point function.

### 7.3. Conformal representations

Note that  $K_\mu$  is a lowering operator for dimension,

$$\begin{aligned} DK_\mu \mathcal{O}(0) &= ([D, K_\mu] + K_\mu D) \mathcal{O}(0) \\ &= (\Delta - 1) K_\mu \mathcal{O}(0). \end{aligned} \quad (286)$$

(Again, we're using shorthand notation  $[Q, \mathcal{O}] \rightarrow Q\mathcal{O}$ .) Thus, given an operator  $\mathcal{O}(0)$ , we can repeatedly act with  $K_\mu$  to obtain operators  $K_{\mu_1} \dots K_{\mu_n} \mathcal{O}(0)$  with arbitrarily low dimension. Because dimensions are

bounded from below in physically sensible theories, this process must eventually terminate. That is, there must exist operators such that

$$[K_\mu, \mathcal{O}(0)] = 0 \quad (\text{primary operator}). \quad (287)$$

Such operators are called “primary.” Given a primary, we can construct operators of higher dimension, called “descendants,” by acting with momentum generators, which act like raising operators for dimension,

$$\begin{aligned} \mathcal{O}(0) &\rightarrow P_{\mu_1} \cdots P_{\mu_n} \mathcal{O}(0) & (\text{descendant operators}) \\ \Delta &\rightarrow \Delta + n. \end{aligned} \quad (288)$$

For example,  $\mathcal{O}(x) = e^{x \cdot P} \mathcal{O}(0)$  is an (infinite) linear combination of descendant operators. The conditions (277, 280, 287) are enough to determine how  $K_\mu$  acts on any descendant using the conformal algebra. For example,

**Exercise 7.2.** Let  $\mathcal{O}(0)$  be a primary operator with rotation representation matrices  $\mathcal{S}_{\mu\nu}$  and dimension  $\Delta$ . Using the conformal algebra, show

$$[K_\mu, \mathcal{O}(x)] = (k_\mu + 2\Delta x_\mu - 2x^\nu \mathcal{S}_{\mu\nu}) \mathcal{O}(x), \quad (289)$$

where  $k_\mu$  is the conformal Killing vector defined in (259).

To summarize, a primary operator satisfies

$$\begin{aligned} [D, \mathcal{O}(0)] &= \Delta \mathcal{O}(0) \\ [M_{\mu\nu}, \mathcal{O}(0)] &= \mathcal{S}_{\mu\nu} \mathcal{O}(0) \\ [K_\mu, \mathcal{O}(0)] &= 0. \end{aligned} \quad (290)$$

From these conditions, we can construct a representation of the conformal algebra out of  $\mathcal{O}(0)$  and its descendants,

operator	dimension	
$\vdots$		
$P_{\mu_1} P_{\mu_2} \mathcal{O}(0)$	$\Delta + 2$	
$\uparrow$		
$P_{\mu_1} \mathcal{O}(0)$	$\Delta + 1$	
$\uparrow$		
$\mathcal{O}(0)$	$\Delta$	(291)

The action of conformal generators on each state follows from the conformal algebra. This should remind you of the construction of irreducible representations of  $SU(2)$  starting from a highest-weight state. In this case, our primary is a *lowest-weight* state of  $D$ , but the representation is built in an

analogous way.<sup>76</sup> It turns out that any local operator in a unitary CFT is a linear combination of primaries and descendants. We will prove this in section 10.4.

**Exercise 7.3.** Show that (223), (278), (281), and (289) can be summarized as

$$[Q_\epsilon, \mathcal{O}(x)] = \left( \epsilon \cdot \partial + \frac{\Delta}{d} (\partial \cdot \epsilon) - \frac{1}{2} (\partial^\mu \epsilon^\nu) \mathcal{S}_{\mu\nu} \right) \mathcal{O}(x). \quad (292)$$

**Exercise 7.4.** Deduce that  $T^{\mu\nu}$  is primary by comparing (292) with (269).

#### 7.4. Finite Conformal Transformations

An exponentiated charge  $U = e^{Q_\epsilon}$  implements a finite conformal transformation. Denote the corresponding diffeomorphism  $e^\epsilon$  by  $x \mapsto x'(x)$ . By comparing with (263) and (264), we find that (292) exponentiates to

$$U \mathcal{O}^a(x) U^{-1} = \Omega(x')^\Delta \rho^a_b(R(x')^{-1}) \mathcal{O}^b(x'), \quad (293)$$

where as before

$$\frac{\partial x'^\mu}{\partial x^\nu} = \Omega(x') R^\mu_\nu(x'), \quad R^\mu_\nu(x') \in \text{SO}(d). \quad (294)$$

Here,  $\rho^a_b(R)$  is a matrix implementing the action of  $R$  in the  $\text{SO}(d)$  representation  $\rho$  of  $\mathcal{O}$ , for example

$$\begin{aligned} \rho(R) &= 1 && \text{(scalar representation),} \\ \rho^\mu_\nu(R) &= R^\mu_\nu && \text{(vector representation),} \\ &\dots && \dots \end{aligned} \quad (295)$$

and so on.

We could have started the whole course by taking (293) as the definition of a primary operator. But the connection to the underlying conformal algebra will be crucial in what follows, so we have chosen to derive it.

**Exercise 7.5.** Show that the transformation (293) composes correctly to give a representation of the conformal group. That is, show

$$U_{g_1} U_{g_2} \mathcal{O}^a(x) U_{g_2}^{-1} U_{g_1}^{-1} = U_{g_1 g_2} \mathcal{O}^a(x) U_{g_1 g_2}^{-1} \quad (296)$$

<sup>76</sup>Generically, the representation (291) is an *induced representation*  $\text{Ind}_H^G(R_H)$ , where  $H$  is the subgroup of the conformal group generated by  $D, M_{\mu\nu}, K_\mu$  (called the isotropy subgroup),  $R_H$  is the finite-dimensional representation of  $H$  defined by (290), and  $G$  is the full conformal group. It is also called a parabolic Verma module. Sometimes the operator  $\mathcal{O}$  satisfies “shortening conditions” where a linear combination of descendants vanishes. (A conserved current is an example.) In this case, the Verma module is reducible and the actual conformal multiplet of  $\mathcal{O}$  is one of the irreducible components.

where  $x \mapsto g_i(x)$  are conformal transformations,  $g_1 g_2$  denotes composition  $x \mapsto g_1(g_2(x))$ , and  $U_g$  is the unitary operator associated to  $g$ .

*End of lecture 13*

## 8. Conformal correlators

### 8.1. Scalar operators

We have already seen that scale invariance fixes two-point functions of scalars up to a constant

$$\langle \mathcal{O}_1(x_1) \mathcal{O}_2(x_2) \rangle = \frac{C}{|x_1 - x_2|^{\Delta_1 + \Delta_2}} \quad (\text{SFT}). \quad (297)$$

For primary scalars in a CFT, the correlators must satisfy a stronger Ward identity,

$$\begin{aligned} \langle \mathcal{O}_1(x_1) \dots \mathcal{O}_n(x_n) \rangle &= \langle (U \mathcal{O}_1(x_1) U^{-1}) \dots (U \mathcal{O}_n(x_n) U^{-1}) \rangle \\ &= \Omega(x'_1)^{\Delta_1} \dots \Omega(x'_n)^{\Delta_n} \langle \mathcal{O}_1(x'_1) \dots \mathcal{O}_n(x'_n) \rangle. \end{aligned} \quad (298)$$

Let us check whether this holds for (297).

**Exercise 8.1.** Show that for a conformal transformation,

$$(x - y)^2 = \frac{(x' - y')^2}{\Omega(x') \Omega(y')}. \quad (299)$$

*Hint: This is obviously true for translations, rotations, and scale transformations. It suffices to check it for inversions  $I : x \rightarrow \frac{x}{x^2}$  (why?).*

Using (299), we find

$$\frac{C}{|x_1 - x_2|^{\Delta_1 + \Delta_2}} = \Omega(x'_1)^{\frac{\Delta_1 + \Delta_2}{2}} \Omega(x'_2)^{\frac{\Delta_1 + \Delta_2}{2}} \frac{C}{|x'_1 - x'_2|^{\Delta_1 + \Delta_2}}. \quad (300)$$

Consistency with (298) then requires  $\Delta_1 = \Delta_2$  or  $C = 0$ . In other words,

$$\langle \mathcal{O}_1(x_1) \mathcal{O}_2(x_2) \rangle = \frac{C \delta_{\Delta_1 \Delta_2}}{x_{12}^{2\Delta_1}} \quad (\text{CFT, primary scalars}), \quad (301)$$

where  $x_{12} \equiv x_1 - x_2$ .

**Exercise 8.2.** Recover the same result using the Ward identity for  $K_\mu$

$$\langle [K_\mu, \mathcal{O}_1(x_1)] \mathcal{O}_2(x_2) \rangle + \langle \mathcal{O}_1(x_1) [K_\mu, \mathcal{O}_2(x_2)] \rangle = 0. \quad (302)$$

Conformal invariance is also powerful enough to fix a three-point function of primary scalars, up to an overall coefficient. Using (299), it's easy to check that the famous formula [21]

$$\langle \mathcal{O}_1(x_1) \mathcal{O}_2(x_2) \mathcal{O}_3(x_3) \rangle = \frac{f_{123}}{x_{12}^{\Delta_1+\Delta_2-\Delta_3} x_{23}^{\Delta_2+\Delta_3-\Delta_1} x_{31}^{\Delta_3+\Delta_1-\Delta_2}}, \quad (303)$$

with  $f_{123}$  constant, satisfies the Ward identity (298). Here, we use the notation  $x_{ij} \equiv x_i - x_j$  and  $x^a = (x^2)^{a/2}$ .

With four points, there are nontrivial conformally invariant combinations of the points called “conformal cross-ratios,”

$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}, \quad v = \frac{x_{23}^2 x_{14}^2}{x_{13}^2 x_{24}^2}. \quad (304)$$

The reason that there are exactly two independent cross-ratios can be understood as follows.

- Using special conformal transformations, we can move  $x_4$  to infinity.
- Using translations, we can move  $x_1$  to zero.
- Using rotations and dilatations, we can move  $x_3$  to  $(1, 0, \dots, 0)$ .
- Using rotations that fix  $x_3$ , we can move  $x_2$  to  $(x, y, 0, \dots, 0)$ .

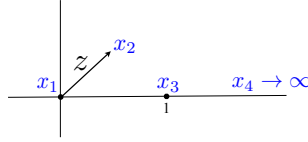


Fig. 10. Using conformal transformations, we can place four points on a plane in the configuration shown above (figure from [22]).

This procedure leaves exactly two undetermined quantities  $x, y$ , giving two independent conformal invariants. Evaluating  $u$  and  $v$  for this special configuration of points (figure 10) gives

$$u = z\bar{z}, \quad v = (1-z)(1-\bar{z}), \quad (305)$$

where  $z \equiv x + iy$ .

Four-point functions can depend nontrivially on the cross-ratios. For a scalar  $\phi$  with dimension  $\Delta_\phi$ , the formula

$$\langle \phi(x_1) \phi(x_2) \phi(x_3) \phi(x_4) \rangle = \frac{g(u, v)}{x_{12}^{2\Delta_\phi} x_{34}^{2\Delta_\phi}} \quad (306)$$

satisfies the Ward identity (298) for any function  $g(u, v)$ .

**Exercise 8.3.** Generalize (306) to the case of non-identical scalars  $\phi_i(x)$  with dimensions  $\Delta_i$ .

The left-hand side of (306) is manifestly invariant under permutations of the points  $x_i$ . This leads to consistency conditions on  $g(u, v)$ ,

$$g(u, v) = g(u/v, 1/v) \quad (\text{from swapping } 1 \leftrightarrow 2 \text{ or } 3 \leftrightarrow 4), \quad (307)$$

$$g(u, v) = \left(\frac{u}{v}\right)^{\Delta_\phi} g(v, u) \quad (\text{from swapping } 1 \leftrightarrow 3 \text{ or } 2 \leftrightarrow 4). \quad (308)$$

All other permutations can be generated from the ones above. We will see shortly that  $g(u, v)$  is actually determined in terms of the dimensions  $\Delta_i$  and three-point coefficients  $f_{ijk}$  of the theory. Equation (307) will be satisfied for trivial reasons. However (308) will lead to powerful constraints on the  $\Delta_i, f_{ijk}$ .

## 8.2. Spinning operators: the conformal frame approach

For operators with spin, the solutions to the conformal Ward identities are more complicated. We will discuss two different approaches, both with advantages and disadvantages. The first approach is to relate an arbitrary configuration of points to some standard configuration (“conformal frame”), where the Ward identities are easier to solve. This is the same type of argument we used above to show that a four-point function can depend on two conformal cross-ratios.

### 8.2.1. Two-point functions

Consider a two-point function of primary operators  $\mathcal{O}_1^a, \mathcal{O}_2^b$  transforming in representations  $\rho_1, \rho_2$  of  $\text{SO}(d)$ , with dimensions  $\Delta_1, \Delta_2$ . Here,  $a, b$  are indices for  $\rho_1, \rho_2$ , respectively. The Ward identity (293) implies

$$\begin{aligned} & \langle \mathcal{O}_1^a(x_1) \mathcal{O}_2^b(x_2) \rangle \\ &= \langle U \mathcal{O}_1^a(x_1) U^{-1} U \mathcal{O}_2^b(x_2) U^{-1} \rangle \\ &= \Omega(x'_1)^{\Delta_1} \Omega(x'_2)^{\Delta_2} (\rho_1)^a{}_c (R(x'_1)^{-1}) (\rho_2)^b{}_d (R(x'_2)^{-1}) \langle \mathcal{O}_1^c(x'_1) \mathcal{O}_2^d(x'_2) \rangle. \end{aligned} \quad (309)$$

Using translations, we have

$$\langle \mathcal{O}_1^a(x_1) \mathcal{O}_2^b(x_2) \rangle = \langle \mathcal{O}_1^a(0) \mathcal{O}_2^b(x_2 - x_1) \rangle. \quad (310)$$



Using the Ward identity for rescaling, we have

$$\langle \mathcal{O}_1^a(0) \mathcal{O}_2^b(x) \rangle = \frac{1}{|x|^{\Delta_1 + \Delta_2}} \langle \mathcal{O}_1^a(0) \mathcal{O}_2^b(\hat{x}) \rangle, \quad \hat{x} \equiv \frac{x}{|x|}. \quad (311)$$

Now fix a unit vector  $e$ , and let  $Q^\mu{}_\nu(\hat{x})$  be any rotation that takes  $e$  to  $\hat{x}$ . ( $Q(\hat{x})$  is not unique, but this won't matter.) Using the Ward identity for rotations, we can relate the correlator at  $\hat{x}$  to the correlator at  $e$ ,

$$\langle \mathcal{O}_1^a(0) \mathcal{O}_2^b(\hat{x}) \rangle = (\rho_1)^a{}_c(Q(\hat{x})) (\rho_2)^b{}_d(Q(\hat{x})) \langle \mathcal{O}_1^c(0) \mathcal{O}_2^d(e) \rangle. \quad (312)$$

Thus, once we know the two-point function for the points  $0, e$ , we know it everywhere.

We cannot specialize the points any further. However, the correlator  $\langle \mathcal{O}_1^c(0) \mathcal{O}_2^d(e) \rangle$  satisfies more constraints because there are conformal transformations that leave the points  $0, e$  invariant but act nontrivially on the correlator. The subgroup of  $\text{SO}(d+1, 1)$  that leaves a set of points invariant is called the “stabilizer group” of those points, and our claim is that the stabilizer group of two points is nontrivial.

The easiest stabilizer group to analyze is when the two points are  $0, \infty$ . In this case, we can apply any rotation and dilatation, so the stabilizer group is  $\mathbb{R}^+ \times \text{SO}(d)$ . It's useful to imagine  $0, \infty$  as the south and north pole of the sphere via stereographic projection. From this point of view, it's clear that dilatations and rotations act in different ways near each point. A dilatation  $x \mapsto \lambda x$  with  $\lambda > 1$  moves points away from  $0$  and towards  $\infty$ . A rotation around the origin gets reflected near  $\infty$ .

To construct the stabilizer group of  $0, e$ , we can conjugate the stabilizer group of  $0, \infty$  by a transformation that maps between the two configurations. In particular, the stabilizer group of any pair of points is  $\mathbb{R}^+ \times \text{SO}(d)$ . Consider an inversion  $I : x \mapsto \frac{x}{x^2}$ . This fixes  $e$ , but moves the origin to  $\infty$ . Now, we can perform an arbitrary rescaling and rotation around  $e$  without moving either  $e$  or  $\infty$ :

$$x^\mu \mapsto e^\mu + \lambda^{-1} h^\mu{}_\nu (x - e)^\nu, \quad \lambda \in \mathbb{R}^+, \quad h \in \text{SO}(d). \quad (313)$$

Finally, we can invert again to map infinity back to the origin. Overall, we have constructed

$$x \mapsto x'(x) = I(e + \lambda^{-1} h(I(x) - e)), \quad I(x) = \frac{x}{x^2}, \quad (314)$$

parameterized by  $\lambda, h$ . Note that because we used  $I$  twice, this map is in the connected component of the identity of  $\text{SO}(d+1, 1)$ , and is thus always a symmetry in a CFT.

Let us study the behavior of (314) near the fixed-points  $0, e$ . An important object will be the derivative of an inversion

$$\frac{\partial I(x)^\mu}{\partial x^\nu} = \frac{1}{x^2} \left( \delta_\nu^\mu - \frac{2x^\mu x_\nu}{x^2} \right) \equiv \frac{1}{x^2} I^\mu{}_\nu(x). \quad (315)$$

Here we have expressed it as an overall rescaling factor  $\frac{1}{x^2}$ , times an orthogonal matrix  $I^\mu{}_\nu(x)$  that represents a reflection in the  $x$  direction.<sup>77</sup> Near the point  $e$ , the map  $x'(x)$  first reflects in the  $e$  direction, then rescales and rotates, then reflects back. Thus, we have

$$x'(x) \sim e^\mu + \lambda^{-1} (I_e h I_e)^\mu{}_\nu (x - e)^\nu + O((x - e)^2), \quad (316)$$

where we're using the shorthand notation  $(I_e)^\mu{}_\nu = I^\mu{}_\nu(e)$ . This looks locally like a rescaling by  $\lambda^{-1}$  and a rotation by  $\tilde{h} = I_e h I_e^{-1}$  (note that  $I_e = I_e^{-1}$  because it is a reflection). Near the point  $0$ , we have

$$x'(x) \sim \lambda h^\mu{}_\nu x^\nu + O(x^2), \quad (317)$$

which looks locally like a rescaling by  $\lambda$  and a rotation by  $h$ . This is consistent with what we saw on the sphere: dilatations move between the two fixed points, and rotations get reflected when we go from one fixed-point to another.

The Ward identity (309) implies

$$\langle \mathcal{O}_1^a(0) \mathcal{O}_2^b(e) \rangle = \lambda^{\Delta_1 - \Delta_2} (\rho_1)^a{}_c (h^{-1})^c{}_d (\tilde{h}^{-1})^d{}_b \langle \mathcal{O}_1^c(0) \mathcal{O}_2^d(0) \rangle. \quad (318)$$

This must hold for all  $\lambda$ , which immediately means  $\Delta_1 = \Delta_2$ , as we concluded before.

To understand the  $h$ -dependent part, consider vector operators,

$$\langle V_1^\mu(0) V_2^\nu(e) \rangle \equiv Y^{\mu\nu}. \quad (319)$$

Our Ward identity says (replacing  $h \rightarrow h^{-1}$  for simplicity),

$$h^\mu{}_\rho (I_e h I_e)^\rho{}_\sigma Y^{\rho\sigma} = Y^{\mu\sigma}, \quad (320)$$

or in matrix notation

$$h Y I_e h^T I_e = Y \implies h(Y I_e) h^T = (Y I_e). \quad (321)$$

In other words,  $Y I_e$  must be invariant under conjugation by an arbitrary  $\text{SO}(d)$  transformation, which means  $Y \propto I_e$ . Thus we have

$$\langle V_1^\mu(0) V_2^\nu(e) \rangle = c_{12} I^{\mu\nu}(e) = c_{12} (\delta^{\mu\nu} - 2e^\mu e^\nu), \quad (322)$$

<sup>77</sup>It is not an element of  $\text{SO}(d)$  because  $i$  is not an element of  $\text{SO}(d+1, 1)$ . Instead,  $i \in O(d+1, 1)$  and consequently  $I(x) \in O(d)$ .

for some constant  $c_{12}$ . Applying (311) and (312),<sup>78</sup>

$$\langle V_1^\mu(0)V_2^\nu(x) \rangle = c_{12} \frac{I^{\mu\nu}(x)}{x^{2\Delta}} = c_{12} \frac{\delta^{\mu\nu} - \frac{2x^\mu x^\nu}{x^2}}{x^{2\Delta}}. \quad (323)$$

Note that conformal invariance fixes the relative factor between  $\delta^{\mu\nu}$  and  $\frac{x^\mu x^\nu}{x^2}$  (which is not possible with scale invariance alone).<sup>79</sup>

Here, we see why the choice of  $Q(\hat{x})$  doesn't matter: the correlator  $\langle V_1(0)V_2(e) \rangle$  is invariant under rotations that fix  $e$  (this is a subgroup of the stabilizer group  $\text{SO}(d-1) \subset \mathbb{R}^+ \times \text{SO}(d)$ ), and two different  $Q(\hat{x})$ 's will differ by an  $\text{SO}(d-1)$  rotation. More concretely,  $\langle V_1(0)V_2(e) \rangle$  depends only on  $\delta^{\mu\nu}$  and  $e^\mu$ , so the effect of  $Q(\hat{x})$  is just to replace  $e \rightarrow \hat{x}$ .

*End of lecture 14*

For traceless symmetric tensor operators  $\mathcal{O}_1^{\mu_1 \dots \mu_\ell}$  and  $\mathcal{O}_2^{\mu_1 \dots \mu_{\ell'}}$ , the two-point function can only be nonvanishing if the spins are equal  $\ell = \ell'$ . In that case, the two-point function takes the form

$$\langle \mathcal{O}_1^{\mu_1 \dots \mu_\ell}(0) \mathcal{O}_{2, \nu_1 \dots \nu_\ell}(x) \rangle = c_{12} \frac{I^{(\mu_1}_{\nu_1}(x) \dots I^{\mu_\ell)}_{\nu_\ell}(x) - \text{traces}}{x^{2\Delta}}. \quad (324)$$

Here, parentheses  $(\mu_1 \dots \mu_\ell)$  mean we symmetrize the given indices,<sup>80</sup> and “–traces” means we subtract terms proportional to  $\delta^{\mu_i \mu_j}$  and  $\delta_{\nu_i \nu_j}$  to make the result traceless. For example, for spin-2 operators,

$$\begin{aligned} & \langle \mathcal{O}_1^{\mu_1 \mu_2}(0) \mathcal{O}_{2, \nu_1 \nu_2}(x) \rangle \\ &= c_{12} \frac{\frac{1}{2}(I^{\mu_1}_{\nu_1}(x) I^{\mu_2}_{\nu_2}(x) + I^{\mu_2}_{\nu_1}(x) I^{\mu_1}_{\nu_2}(x)) - \frac{1}{d} \delta^{\mu_1 \mu_2} \delta_{\nu_1 \nu_2}}{x^{2\Delta}}. \end{aligned} \quad (325)$$

One way to think about (324) is that once we've identified a conformally-invariant two-point function for vectors (323), we can take tensor products of it (symmetrized in the appropriate way) to get conformally-invariant two-point functions for other operators.

Here is a more formal way to understand the rotation part of the Ward identity (318). Given a representation  $\rho$  of  $\text{SO}(d)$ , let us define the reflected representation  $\rho^R$  by

$$\rho^R(h) \equiv \rho(I_e h I_e^{-1}), \quad h \in \text{SO}(d). \quad (326)$$

<sup>78</sup>Note that the exact definition of  $Q(\hat{x})$  doesn't matter, since we only needed  $Q(\hat{x})e = \hat{x}$ .

<sup>79</sup>One can give a much simpler derivation of (323) by anticipating the fact that it is actually invariant under an inversion  $I$ , and not just elements of  $\text{SO}(d+1, 1)$ . However,  $I$  is only guaranteed to be a symmetry in parity-invariant theories (exercise ??). The derivation here assumes only  $\text{SO}(d+1, 1)$ -invariance.

<sup>80</sup>it is only necessary to symmetrize in either  $\mu$  or  $\nu$ , but we symmetrize both for clarity

Note that  $I_e h I_e^{-1} \in \text{SO}(d)$ , even though  $I_e$  is not in  $\text{SO}(d)$  (it is a reflection). Thus, the definition (326) makes sense. Note also that a different choice of  $e$  would lead to an equivalent representation because  $I_{e'} = k I_e$  for  $k \in \text{SO}(d)$ , so

$$\rho(I_{e'} h I_{e'}^{-1}) = \rho(k) \rho(I_e h I_e^{-1}) \rho(k)^{-1}. \quad (327)$$

If two representations differ by conjugation by a matrix (in this case  $\rho(k)$ ), they are equivalent.<sup>81</sup>

**Exercise 8.4.** *Show that in four dimensions, the reflection of a left-handed spinor  $\psi_\alpha$  is a right-handed spinor  $\bar{\psi}_{\dot{\alpha}}$ , and the reflection of a self-dual two-form  $F_{\alpha\beta}$  is an anti-self-dual two-form  $F_{\dot{\alpha}\dot{\beta}}$ .*

The Ward identity (318) can be written

$$\langle \mathcal{O}_1^a(0) \mathcal{O}_2^b(e) \rangle = (\rho_1)^a{}_c(h) (\rho_2^R)^b{}_d(h) \langle \mathcal{O}_1^c(0) \mathcal{O}_2^d(0) \rangle. \quad (328)$$

In other words,  $\langle \mathcal{O}_1^a(0) \mathcal{O}_2^b(e) \rangle$  is an invariant-tensor transforming in the tensor product  $\rho_1 \otimes \rho_2^R$ . If  $\rho_1, \rho_2$  are irreducible, then such an invariant exists only if  $\rho_1 = (\rho_2^R)^*$ , where  $\rho^*$  denotes the dual of  $\rho$ . Furthermore, if it exists it is unique.

As an example, traceless symmetric tensors are equivalent to their dual reflected representations, so we have recovered the statement that a two-point function of spin- $\ell$  and spin- $\ell'$  operators can be nonzero only if  $\ell = \ell'$ . As another example, by exercise (8.4) a left-handed spinor  $\psi_\alpha$  in 4d can only have a nonvanishing two-point function with a right-handed spinor  $\bar{\psi}_{\dot{\alpha}}$ .

**Exercise 8.5.** *Find this two-point function of spinors in 4d. Hint:  $\langle \psi_\alpha(0) \bar{\psi}_{\dot{\alpha}}(e) \rangle$  must have the correct indices and be invariant under rotations that fix  $e$ .*

### 8.2.2. Three-point functions

To study a three-point function, we can use the same strategy: we use conformal transformations to fix the three points, and then study the Ward identities of the stabilizer group. For simplicity, consider the case where two of the operators  $\phi_1, \phi_2$  are scalars and the third  $\mathcal{O}_3$  transforms in a nontrivial representation  $\rho$  of  $\text{SO}(d)$ ,

$$\langle \phi_1(x_1) \phi_2(x_2) \mathcal{O}_3^a(x_3) \rangle. \quad (329)$$

<sup>81</sup>The definition (326) is an example of how the outer automorphisms  $\text{Out}(G)$  of a group  $G$  act on the ring of  $G$ -representations. Here, conjugation by  $I_e$  is an outer automorphism of  $\text{SO}(d)$ .

Now, use conformal transformations to place the three operators at  $0, e, \infty$  for some fixed unit vector  $e$ .<sup>82</sup>

To simplify the discussion, we can use a trick. Let us divide by the standard three-point structure (303) for scalars with dimensions  $\Delta_1, \Delta_2, \Delta_3$ ,

$$t^a(x_1, x_2, x_3) \equiv \langle \phi_1(x_1) \phi_2(x_2) \mathcal{O}_3^a(x_3) \rangle x_{12}^{\Delta_1 + \Delta_2 - \Delta_3} x_{23}^{\Delta_2 + \Delta_3 - \Delta_1} x_{31}^{\Delta_3 + \Delta_1 - \Delta_2}. \quad (330)$$

The result  $t^a(x_1, x_2, x_3)$  will behave like a three-point function of dimension-0 operators. We are not claiming that dimension-0 operators actually exist. This is just a trick so that we don't have to worry about  $\Omega(x)$  factors when we use the Ward identities.

Let us send  $x_3 \mapsto \infty$  with an inversion, then translate  $x_2$  to the origin and invert again to send  $x_2 \mapsto \infty$  and  $x_3 \mapsto 0$ . Finally, we rescale so that  $x_1$  gets mapped to a unit vector. Overall, we have

$$x'^\mu(x) = \frac{I(I(x - x_3) - I(x_2 - x_3))^\mu}{|I(I(x_1 - x_3) - I(x_2 - x_3))|}. \quad (331)$$

Near the point  $x_3$ , this is proportional to  $I(I(x - x_3)) = x - x_3$ , so  $R(x'_3)^\mu{}_\nu$  is trivial for this transformation (which is why we chose it). Because of our trick (330), we don't need to keep track of rescaling factors. Thus, we have

$$t^a(x_1, x_2, x_3) = t^a(x'(x_1), \infty, 0) = t^a(\hat{Z}_3, \infty, 0), \quad Z_3 \equiv \frac{x_{13}}{x_{13}^2} - \frac{x_{23}}{x_{23}^2}, \quad \hat{Z}_3 = \frac{Z_3}{|Z_3|}. \quad (332)$$

Finally, let us use an orthogonal transformation  $Q(\hat{Z}_3)$  to relate this to a standard configuration<sup>83</sup>

$$t^a(x_1, x_2, x_3) = \rho^a{}_b(Q(\hat{Z}_3)) t^b(e, \infty, 0). \quad (333)$$

We have fixed the three points using the conformal group. As before, the correlator is still constrained because the stabilizer group of the three points is nontrivial. It consists of  $\text{SO}(d-1)$  rotations that fix the line passing through  $0, e, \infty$ . The Ward identities say that

$$t^a(e, \infty, 0) = \rho^a{}_b(h) t^b(e, \infty, 0), \quad h \in \text{SO}(d-1). \quad (334)$$

which means that  $t^a$  must live in the  $\text{SO}(d-1)$ -invariant subspace of  $\rho$ .

<sup>82</sup>As before, we would like to use inversion an even number of times, so that our transformation is in  $\text{SO}(d+1, 1)$ .

<sup>83</sup>In the case where all three operators are scalars, this verifies that there are no other solutions to the conformal Ward identities besides (303), since  $t(x_1, x_2, x_3) = t(e, \infty, 0)$  is just a constant.

As an example, consider a vector operator  $V^\mu$ . In this case,  $t^\mu(e, \infty, 0)$  carries a single vector index and must be invariant under rotations that fix  $e$ . The only possibility is

$$t^\mu(e, \infty, 0) = f_{123}e^\mu, \quad (335)$$

for some constant  $f_{123}$ . Using (333) and (330), we find

$$\langle \phi_1(x_1)\phi_2(x_2)V_3^\mu(x_3) \rangle = \frac{f_{123}\widehat{Z}_3^\mu}{x_{12}^{\Delta_1+\Delta_2-\Delta_3}x_{23}^{\Delta_2+\Delta_3-\Delta_1}x_{31}^{\Delta_3+\Delta_1-\Delta_2}}. \quad (336)$$

Similarly, for traceless symmetric tensor operators  $\mathcal{O}_3^{\mu_1\cdots\mu_\ell}$ , the only possibility is

$$t^{\mu_1\cdots\mu_\ell}(e, \infty, 0) = f_{123}(e^{\mu_1}\cdots e^{\mu_\ell} - \text{traces}), \quad (337)$$

which leads to

$$\langle \phi_1(x_1)\phi_2(x_2)\mathcal{O}_3^{\mu_1\cdots\mu_\ell}(x_3) \rangle = \frac{f_{123}(\widehat{Z}_3^{\mu_1}\cdots\widehat{Z}_3^{\mu_\ell} - \text{traces})}{x_{12}^{\Delta_1+\Delta_2-\Delta_3}x_{23}^{\Delta_2+\Delta_3-\Delta_1}x_{31}^{\Delta_3+\Delta_1-\Delta_2}}. \quad (338)$$

For any other  $\text{SO}(d)$ -representation  $\rho$ , it's not possible to find a non-vanishing  $t^a$ . The mathematical reason is that only traceless symmetric tensors have an  $\text{SO}(d-1)$ -invariant subspace (and that subspace is one-dimensional). More prosaically,  $e^\mu$  and  $\delta^{\mu\nu}$  are the only  $\text{SO}(d-1)$ -invariant objects we can use to build  $t^a(e, \infty, 0)$ . If we demand irreducibility, at least one term should not contain a  $\delta^{\mu\nu}$  (otherwise we'd have a nonzero trace). Thus, we can only build traceless symmetric tensors.

Note that  $\widehat{Z}_3^\mu$  is odd under  $x_1 \leftrightarrow x_2$ . Consequently, if  $\phi_1 = \phi_2$  are identical operators, the correlator  $\langle \phi_1\phi_1\mathcal{O}^{\mu_1\cdots\mu_\ell} \rangle$  can only be nonzero if  $\ell$  is even. This fact will play an important role later.

When all three operators have nontrivial  $\text{SO}(d)$  representations  $\rho_1, \rho_2, \rho_3$ , the solution of the Ward identities is similar. We again use the conformal group to place all three operators on a line. (However, the relationship between the correlator at  $x_1, x_2, x_3$  and the correlator on the line is more complicated because we must keep track of all three rotations.) We then study the Ward identities for the stabilizer group  $\text{SO}(d-1)$  in this “collinear” configuration. The solution is given by the  $\text{SO}(d-1)$ -invariant subspace of the tensor product of the  $\rho_i$ ,

$$(\rho_1 \otimes \rho_2 \otimes \rho_3)^{\text{SO}(d-1)} = \otimes_{i=1}^3 \text{Res}_{\text{SO}(d-1)}^{\text{SO}(d)} \rho_i. \quad (339)$$

(Here,  $\rho^H$  denotes the  $H$ -invariant subspace of  $\rho$ , and  $\text{Res}_H^G \rho$  denotes the restriction of  $\rho$  from a  $G$ -representation to an  $H$ -representation for  $H \subseteq G$ .)

This means that there can be multiple three-point structures when more than one operator has spin.

For example, consider a three-point function of two vectors and a scalar  $\langle V_1^\mu V_2^\nu f_3 \rangle$ . In the collinear configuration, the correlator must be  $\text{SO}(d-1)$ -invariant. The only possibilities are

$$\langle V_1^\mu(e) V_2^\nu(\infty) f_3(0) \rangle = f_{123}^{(1)} e^\mu e^\nu + f_{123}^{(2)} \delta^{\mu\nu}, \quad (340)$$

where  $f_{123}^{(1)}, f_{123}^{(2)}$  are constants. Abstractly, we can see that there are two structures by counting

$$\begin{aligned} (\square \otimes \square \otimes \mathbf{1})^{\text{SO}(d-1)} &= (\mathbf{1} \oplus \square) \otimes (\mathbf{1} \oplus \square) \otimes \mathbf{1} \\ &= 2\mathbf{1} \oplus 2\square \oplus \square\square \oplus \square, \end{aligned} \quad (341)$$

where we've used  $\text{Res}_{\text{SO}(d-1)}^{\text{SO}(d)} \square = \mathbf{1} \oplus \square$ . This calculation reveals an additional wrinkle: notice that  $\square$  is the trivial representation in  $d=2$ , so there are actually three structures in that case. Indeed, in  $d=2$  there is another term we can include in (340) proportional to  $\epsilon^{\mu\nu}$ .

We can compute the correlator in a general configuration using the Ward identities. Tracing everything through, we find

$$\langle V_1^\mu(x_1) V_2^\nu(x_2) f_3(x_3) \rangle = \frac{a \hat{Z}_1^\mu \hat{Z}_2^\nu + b I^{\mu\nu}(x_{12})}{x_{12}^{\Delta_1+\Delta_2-\Delta_3} x_{23}^{\Delta_2+\Delta_3-\Delta_1} x_{31}^{\Delta_3+\Delta_1-\Delta_2}}, \quad (342)$$

where  $a, b$ , are linear combinations of  $f_{123}^{(1)}, f_{123}^{(2)}$ . Here  $\hat{Z}_1, \hat{Z}_2$  are analogs of  $\hat{Z}_3$  with the points permuted. This result is suggestive that we can think about building conformally-invariant correlators by putting together simple building blocks. The embedding space formalism, which we discuss next, makes this easy to do.

### 8.3. The embedding space

The action of the conformal group on  $\mathbb{R}^d$  is nonlinear, so it takes some work to solve the conformal Ward identities. The embedding space is a way of realizing the conformal group so that it acts *linearly*, which makes it much simpler to solve the conformal Ward identities.

The central observation is that the conformal group is the same as  $\text{SO}(d+1, 1)$  (exercise 6.8), which acts simply by matrix multiplication on  $\mathbb{R}^{d+1,1}$

$$X \mapsto gX, \quad X \in \mathbb{R}^{d+1,1}, \quad g \in \text{SO}(d+1, 1). \quad (343)$$

We will use coordinates  $X = (X^+, X^-, X^1, \dots, X^d)$  on  $\mathbb{R}^{d+1,1}$ , with metric

$$X^2 = -X^+X^- + \sum_{\mu=1}^d X_\mu X^\mu. \quad (344)$$

$\mathbb{R}^{d+1,1}$  is a  $d+2$ -dimensional space, and we would like our theory to live in  $d$  dimensions. We can remove one dimension in a conformally-invariant way by restricting to the null cone

$$X^2 = 0 \quad (\text{null cone}). \quad (345)$$

The null cone is preserved by  $\text{SO}(d+1,1)$ , so we have a  $d+1$ -dimensional space on which the conformal group acts. To finally get to  $d$ -dimensions, we can projectivize, i.e. mod out by rescalings:

$$\text{projective null cone} = \frac{\{X : X^2 = 0\}}{X \sim \lambda X \quad (\lambda \in \mathbb{R})}. \quad (346)$$

*End of lecture 15*

In the projective null cone, each null vector  $X$  represents an equivalence class, and the conformal group maps one equivalence class to another. We can think of rescaling as a “gauge redundancy.” Geometrically, the projective null cone is the space of lines contained in the null cone, and the conformal group maps one line to another.

To be more explicit, let us fix a gauge by choosing a representative for each equivalence class. One simple gauge-fixing is  $X^+ + X^- = 1$ , which is equivalent to cutting the null cone with a plane, with intersection locus  $S^d$ . This shows that the projective null cone is topologically equivalent to  $S^d$ , which we should think of as  $\mathbb{R}^d$  plus the point at infinity.  $S^d$  is called the “conformal compactification” of  $\mathbb{R}^d$ .

The most important gauge fixing for us will be  $X^+ = 1$ , which defines the “Poincare section” of the null cone. Given a null vector  $X = (X^+, X^-, X^\mu)$ , we can rescale by  $1/X^+$  to obtain an equivalent vector  $(1, X^-/X^+, X^\mu/X^+)$  on the Poincare section. The only case in which this doesn’t work is when  $X^+ = 0$ , so that  $X$  is equivalent to  $(0, 1, 0)$ . This corresponds to the point at infinity.

Every null vector on the Poincare section can be written

$$X = (1, x^2, x^\mu), \quad (347)$$

where  $x \in \mathbb{R}^d$  and  $x^2 = x_\mu x^\mu$ .<sup>84</sup> Now consider the action of the conformal group,  $X \mapsto gX$ . Now  $gX$  may not be on the Poincare section anymore, so

<sup>84</sup>Note that the limit of  $(1, x^2, x^\mu)$  as  $x \rightarrow \infty$  is projectively equivalent to  $(0, 1, 0)$ , which justifies the claim that we should think of  $(0, 1, 0)$  as the point at infinity.



we must rescale by  $(gX)^+$  to restore our gauge choice

$$(1, x^2, x) = X \mapsto gX \rightarrow (gX)/(gX)^+ = (1, x'^2, x'^\mu). \quad (348)$$

Following this chain gives a nonlinear map  $x \rightarrow x'$ . We claim that this is exactly the usual action of the conformal group.

To see this, consider some examples. One type of element of  $\text{SO}(d+1, 1)$  are simply  $\text{SO}(d)$  transformations acting on the coordinates  $X^\mu$ ,

$$g_h = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & h^\mu{}_\nu \end{pmatrix}, \quad h^\mu{}_\nu \in \text{SO}(d)$$

$$g_h X = (1, x^2, h^\mu{}_\nu x^\nu). \quad (349)$$

These transformations don't take us off the Poincare section, so they give  $x'^\mu = h^\mu{}_\nu x^\nu$ , which is simply an  $\text{SO}(d)$  rotation.

Another type of  $\text{SO}(d+1, 1)$  transformation is

$$g_a = \begin{pmatrix} 1 & 0 & 0 \\ a^2 & 1 & 2a_\nu \\ a^\mu & 0 & \delta^\mu{}_\nu \end{pmatrix}, \quad a^\mu \in \mathbb{R}^d$$

$$g_a X = (1, x^2 + 2x \cdot a + a^2, x^\mu + a^\mu). \quad (350)$$

Note that we are writing matrices in  $(X^+, X^-, X^\mu)$  coordinates. It may not be immediately obvious that  $g_a$  preserves the metric (344), but it's easy to check that  $g_a X$  is null, which implies that  $g_a \in \text{SO}(d+1, 1)$ . Note that  $g_a$  also does not take us off the Poincare section, so it just gives the transformation  $x^\mu \mapsto x^\mu + a^\mu$ , which is a translation.

A more interesting example is

$$g_\lambda = \begin{pmatrix} \lambda^{-1} & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \delta^\mu{}_\nu \end{pmatrix}, \quad \lambda \in \mathbb{R}^+$$

$$g_\lambda X = (\lambda^{-1}, \lambda x^2, x^\mu). \quad (351)$$

This takes us off the Poincare section, so to restore our gauge choice, we need to divide by  $(g_\lambda X)^+ = \lambda^{-1}$ . This gives

$$(g_\lambda X)/(g_\lambda X)^+ = (1, \lambda^2 x^2, \lambda x^\mu). \quad (352)$$

Thus,  $g_\lambda$  implements a dilatation.

Finally, consider the matrix

$$g_b = \begin{pmatrix} 1 & b^2 & -2b_\nu \\ 0 & 1 & 0 \\ 0 & -b^\mu & \delta^\mu{}_\nu \end{pmatrix}, \quad b^\mu \in \mathbb{R}^d$$

$$g_b X = (1 - 2b \cdot x + b^2 x^2, x^2, x^\mu - b^\mu x^2). \quad (353)$$

Again, we must divide by  $(g_b X)^+$  to restore our gauge-choice, giving

$$(g_b X)/(g_b X)^+ = \left( 1, \frac{x^2}{1 - 2b \cdot x + b^2 x^2}, \frac{x^\mu - b^\mu x^2}{1 - 2b \cdot x + b^2 x^2} \right). \quad (354)$$

This is a special conformal transformation.

Note that for a dilatation  $x \mapsto \lambda x$ , we have  $\Omega(x') = \lambda$ , where  $\Omega(x')$  is the scale factor defined by (264) or (265). For a special-conformal transformation, it's easy to compute

$$\Omega(x') = \frac{1}{1 - 2b \cdot x + b^2 x^2} \quad (\text{special conformal}). \quad (355)$$

In both cases, we have

$$\Omega(x') = 1/(gX)^+. \quad (356)$$

Since this is also trivially true for translations and rotations, it is true for any  $g \in \text{SO}(d+1, 1)$ .

Inversions are also easy to understand in this setting. They are given by the matrix

$$g_I = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & \delta^\mu{}_\nu \end{pmatrix}$$

$$g_I X = (x^2, 1, x^\mu) \rightarrow (1, 1/x^2, x^\mu/x^2). \quad (357)$$

Note that  $\det g_I = -1$ , so  $g_I$  is not an element of  $\text{SO}(d+1, 1)$ , but is instead an element of  $\text{O}(d+1, 1)$ .

### 8.3.1. Scalars in the embedding space

The Ward identity for scalar primary operators is

$$U_g \phi(x) U_g^{-1} = \Omega(x')^\Delta \phi(x'), \quad (358)$$

where  $\Delta$  is the dimension of  $\phi$ . From our discussion in the previous section, we have  $\Omega(x') = 1/(gX)^+$  and  $x'^\mu = (gX)^\mu/(gX)^+$ . Thus, we can introduce the object

$$\Phi(X) \equiv (X^+)^{-\Delta} \phi(X^\mu/X^+), \quad (359)$$

and write the Ward identity (358) as

$$U_g \Phi(X) U_g^{-1} = \Phi(gX). \quad (360)$$

The key point is that conformal transformations now act linearly on the coordinates of  $\Phi$ . Note that  $\Phi(X)$  is just a repackaging of the underlying operator  $\phi(x)$ . It is defined only on the null cone  $X^2 = 0$ . Furthermore, by construction,  $\Phi$  is homogeneous in  $X$  with degree  $-\Delta$ , where  $\Delta$  is the scaling dimension of  $\phi$ ,

$$\Phi(\lambda X) = \lambda^{-\Delta} \Phi(X). \quad (361)$$

Together, these conditions (defined on the null cone and homogeneous) ensure that  $\Phi(X)$  only depends nontrivially on  $d$  coordinates (instead of  $d+2$  coordinates), as we should expect from the fact that it is encoding a  $d$ -dimensional operator.

The original scalar  $\phi(x)$  can be recovered from  $\Phi(X)$  by evaluating it on the Poincare section

$$\phi(x) = \Phi((1, x^2, x)). \quad (362)$$

### 8.3.2. Scalar correlators in the embedding space

Let us now consider a two-point function of scalars  $\langle \Phi_1(X_1) \Phi_2(X_2) \rangle$  with dimensions  $\Delta_1, \Delta_2$ , lifted to the null cone in the embedding space. The two-point function

- must be  $\text{SO}(d+1, 1)$ -invariant

$$\langle \Phi_1(X_1) \Phi_2(X_2) \rangle = \langle U_g (\Phi_1(X_1) \Phi_2(X_2)) U_g^{-1} \rangle = \langle \Phi_1(gX_1) \Phi_2(gX_2) \rangle, \quad (363)$$

which means it can only depend on dot-products  $X_1^2, X_2^2, X_1 \cdot X_2$ ;

- is defined on the null cones  $X_1^2 = X_2^2 = 0$ , so it actually only can depend on  $X_1 \cdot X_2$ ;
- is homogeneous in  $X_i$  of degree  $-\Delta_i$ .

The only possibility consistent with these conditions is

$$\langle \Phi_1(X_1) \Phi_2(X_2) \rangle = \frac{C \delta_{\Delta_1 \Delta_2}}{(-2X_1 \cdot X_2)^{\Delta_1}}, \quad (364)$$

where  $C$  is a constant (we have inserted the factor of  $-2$  in the denominator for convenience). Let us now specialize to the Poincare section to find the

correlator in  $\mathbb{R}^d$ ,

$$\begin{aligned} -2X_1 \cdot X_2 &= -2 \left( -\frac{1}{2}X_1^+ X_2^- - \frac{1}{2}X_1^- X_2^+ + X_1^\mu X_{2\mu} \right) \\ &= -2 \left( -\frac{1}{2}x_1^2 - \frac{1}{2}x_2^2 + x_1 \cdot x_2 \right) \\ &= (x_1 - x_2)^2 \quad (\text{Poincare section}). \end{aligned} \quad (365)$$

Thus, we find

$$\langle \phi_1(x_1) \phi_2(x_2) \rangle = \frac{C \delta_{\Delta_1 \Delta_2}}{(x_1 - x_2)^{2\Delta_1}}, \quad (366)$$

in agreement with (??).

For a three-point function  $\langle \Phi_1(X_1) \Phi_2(X_2) \Phi_3(X_3) \rangle$ , the analysis is similar. The only nontrivial  $\text{SO}(d+1, 1)$ -invariant objects are  $X_1 \cdot X_2$ ,  $X_2 \cdot X_3$ , and  $X_3 \cdot X_1$ . The correlator must have homogeneity  $-\Delta_i$  in  $X_i$ , and the only possibility is

$$\langle \Phi_1(X_1) \Phi_2(X_2) \Phi_3(X_3) \rangle = \frac{f_{123}}{X_{12}^{\Delta_1 + \Delta_2 - \Delta_3} X_{23}^{\Delta_2 + \Delta_3 - \Delta_1} X_{31}^{\Delta_3 + \Delta_1 - \Delta_2}}, \quad (367)$$

where  $X_{ij} \equiv -2X_i \cdot X_j$ . Specializing to the Poincare section  $X_{ij} \rightarrow (x_i - x_j)^2$ , we recover (303).

### 8.3.3. Vector operators in the embedding space

To describe more complicated operators, instead of starting with the conformal Ward identities, let us try to guess what they lift to. Scalar operators lift to scalars on the null cone, so it's natural to guess that a vector  $V^\mu(x)$  on  $\mathbb{R}^d$  lifts to a vector  $\mathcal{V}^A$  ( $A = +, -, 1, \dots, d$ ) in the embedding space,

$$V^\mu(x) \rightarrow \mathcal{V}^A(X). \quad (368)$$

However  $\mathcal{V}^A(X)$  carries  $d+2$  degrees of freedom, whereas  $V^\mu$  only has  $d$  degrees of freedom. Thus, we would like to remove two degrees of freedom from  $\mathcal{V}^A$  in an  $\text{SO}(d+1, 1)$ -invariant way.

The solution to this problem is familiar from first-semester QFT: we can impose a transverseness condition and a gauge-redundancy

$$\begin{aligned} X_A \mathcal{V}^A(X) &= 0 \\ \mathcal{V}^A(X) &\sim \mathcal{V}^A(X) + X^A \lambda(X), \end{aligned} \quad (369)$$

where  $\lambda(X)$  is any scalar function defined on the null cone. Note that the transverseness condition and gauge-redundancy are only consistent if

$X^2 = 0$ . Equivalence classes of  $\mathcal{V}^A(X)$  then carry  $d$  degrees of freedom and transform in a simple way under  $\text{SO}(d+1, 1)$ ,

$$\mathcal{V}^A(X) \rightarrow (g^{-1})^A{}_B \mathcal{V}^B(gX), \quad g \in \text{SO}(d+1, 1). \quad (370)$$

To translate from the embedding space back to  $\mathbb{R}^d$ , we have the following dictionary

$$\begin{aligned} V_\mu(x) &= \frac{\partial X^A(x)}{\partial x^\mu} \mathcal{V}_A(X)|_{X=(1, x^2, x)} \\ &= -x_\mu \mathcal{V}^+(X) + \mathcal{V}_\mu(X)|_{X=(1, x^2, x)}. \end{aligned} \quad (371)$$

Note that

$$\frac{\partial X_A}{\partial x} X^A = \frac{1}{2} \frac{\partial(X^2)}{\partial x} = 0, \quad (372)$$

which means that  $V_\mu(x)$  is invariant under the gauge redundancy  $\mathcal{V}^A \sim \mathcal{V}^A + \lambda X^A$ .

What other conditions should we impose on  $\mathcal{V}$ ? Recall that a dilatation acts via the matrix  $\text{diag}(\lambda^{-1}, \lambda, 1, \dots, 1)$ . In particular, acting on the Poincare section, we have

$$(1, x^2, x) \rightarrow \lambda^{-1}(1, x'^2, x'), \quad x' = \lambda x. \quad (373)$$

Under this transformation, the vector  $V_\mu(x)$  becomes<sup>85</sup>

$$V_\mu(x) \rightarrow -x_\mu(\lambda \mathcal{V}^+(\lambda^{-1} X(x'))) + \mathcal{V}_\mu(\lambda^{-1} X(x')). \quad (374)$$

Thus, if we demand that  $\mathcal{V}^A(X)$  have homogeneity  $-\Delta$ ,

$$\mathcal{V}^A(\lambda X) = \lambda^{-\Delta} \mathcal{V}^A(X), \quad (375)$$

then we obtain

$$V_\mu(x) \rightarrow \lambda^\Delta V_\mu(x'), \quad (376)$$

which is consistent with the Ward identity under dilatation for a vector operator with dimension  $\Delta$ .

This suggests we should try to lift a vector  $V_\mu(x)$  in a CFT to an embedding space object like  $\mathcal{V}^A(X)$  satisfying transverseness, gauge-redundancy, and homogeneity. Such a lift, which is an inverse to (371), is given by

$$\mathcal{V}(X) = (X^+)^{-\Delta} \left( 0, -2 \frac{X^\nu}{X^+} V_\nu \left( \frac{X^\mu}{X^+} \right), V^\mu \left( \frac{X^\mu}{X^+} \right) \right). \quad (377)$$

<sup>85</sup>The term  $\lambda \mathcal{V}^+$  comes because the index of  $\mathcal{V}$  transforms under the inverse matrix  $g^{-1}$  in (370).

Note that this choice of lift is only defined up to gauge redundancy. Here, we have used our gauge freedom to choose  $\mathcal{V}^+ = 0$ . By construction, the above object is homogeneous of degree  $-\Delta$ , and satisfies the transverseness condition  $X \cdot \mathcal{V} = 0$ . We claim that  $V^\mu(x)$  satisfies the conformal Ward identities for a primary vector operator if and only if  $\mathcal{V}(X)$  defined by (377) satisfies

$$U_g \mathcal{V}^a(X) U_g^{-1} = (g^{-1})^A{}_B \mathcal{V}^B(gX), \quad g \in \text{SO}(d+1, 1) \quad (378)$$

(up to gauge redundancy). We have already verified this for dilatations. It is also easy to see for rotations.

**Exercise 8.6.** *Establish this claim by checking the transformation rule for translations and special conformal transformations. Hint: it suffices to check translations and to prove that  $\mathcal{V}^A(X)$  is primary (i.e. zero up to gauge redundancy under an infinitesimal special conformal transformation).*

*End of lecture 16*

#### 8.3.4. Correlators of vectors

Let us now compute a two-point function  $\langle \mathcal{V}_1^A(X_1) \mathcal{V}_2^B(X_2) \rangle$  using this technology. It must be homogeneous of degree  $-\Delta_i$  in  $X_i$ . In addition to depending on  $X_1 \cdot X_2$ , it must also carry two vector indices under the conformal group. The most general such object is

$$\langle \mathcal{V}_1^A(X_1) \mathcal{V}_2^B(X_2) \rangle = \frac{a \delta^{AB} X_1 \cdot X_2 + b X_1^A X_2^B + c X_2^A X_1^B}{X_{12}^{\Delta_1+1}} \quad (379)$$

Using gauge redundancy, we are free to set the term proportional to  $b$  to zero. Another way to say this is that when we use the dictionary (371), this term will give zero because of (372). Thus, let us choose  $b = 0$ .

Finally, we must impose transverseness,

$$0 = X_{1A} (a \delta^{AB} X_1 \cdot X_2 + c X_2^A X_1^B) = (a + c) X_1^B (X_1 \cdot X_2), \quad (380)$$

which implies  $a = -c$ . (Transverseness in  $X_2$  gives the same condition.) Thus, we find

$$\langle \mathcal{V}_1^A(X_1) \mathcal{V}_2^B(X_2) \rangle = c \frac{X_2^A X_1^B - \delta^{AB} X_1 \cdot X_2}{X_{12}^{\Delta_1+1}}. \quad (381)$$

To translate to  $\mathbb{R}^d$ , note that

$$\begin{aligned}\frac{\partial X_1}{\partial x_1^\mu} \cdot X_2 &= (0, 2x_{1\mu}, \delta_{\mu\nu}) \cdot (1, x_2^2, x_2^\nu) = x_{2\mu} - x_{1\mu} \\ \frac{\partial X_1}{\partial x_1^\mu} \frac{\partial X_2}{\partial x_2^\nu} &= (0, 2x_{1\mu}, \delta_{\mu\nu}) \cdot (0, 2x_2^\rho, \delta_\nu^\rho) = \delta_{\mu\nu}.\end{aligned}\quad (382)$$

Thus our dictionary (371) gives

$$\begin{aligned}\langle V_1^\mu(x_1) V_2^\nu(x_2) \rangle &= c \frac{(x_2 - x_1)^\mu (x_1 - x_2)^\nu + \frac{1}{2} \delta^{\mu\nu} (x_1 - x_2)^2}{(x_1 - x_2)^{2\Delta_1+2}} \\ &= \frac{c}{2} \frac{I^{\mu\nu}(x_1 - x_2)}{(x_1 - x_2)^{2\Delta_1}},\end{aligned}\quad (383)$$

in agreement with (323).

Consider now a three-point function of two scalars  $\phi_1, \phi_2$ , and a vector  $\mathcal{V}$ ,

$$\langle \phi_1(X_1) \phi_2(X_2) \mathcal{V}^A(X_3) \rangle = \frac{f_{123} \mathcal{Z}(X_1, X_2, X_3)^A}{X_{12}^{\Delta_1+\Delta_2-\Delta_3} X_{23}^{\Delta_2+\Delta_3-\Delta_1} X_{31}^{\Delta_3+\Delta_1-\Delta_2}}. \quad (384)$$

The numerator should be homogeneous of degree 0 in the  $X_i$  and transverse with respect to  $X_3$ . Because of gauge-redundancy, we can throw away any term proportional to  $X_3^A$ . The only possibility (up to a constant) is

$$\mathcal{Z}(X_1, X_2, X_3)^A = \frac{X_2^A X_{13} - X_1^A X_{23}}{\sqrt{X_{12} X_{23} X_{31}}}. \quad (385)$$

Applying our dictionary, we have

$$\frac{\partial X_{3A}}{\partial x_\mu} \mathcal{Z}(X_1, X_2, X_3)^A \Big|_{X_i=(1, x_i^2, x_i)} = \left( \frac{x_{32}^\mu}{x_{32}^2} - \frac{x_{31}^\mu}{x_{31}^2} \right) \frac{|x_{13}| |x_{23}|}{|x_{12}|} = \widehat{Z}_3^\mu, \quad (386)$$

where  $\widehat{Z}_3$  is defined in (332). Thus, we've managed to solve the conformal Ward identities for two and three-point functions without worrying about writing down specially-designed conformal transformations.

To deal with more complicated correlators, it's extremely useful to use “index-free” notation. Together with the embedding space, this gives a powerful algebraic approach to studying conformal correlators. Versions of this formalism can describe other types of tensors, spinors, superfields, and more in various dimensions. We will not have time to discuss these methods here, but you can read more about them in [1].

## 9. Radial Quantization and the State-Operator Correspondence

So far, we've written lots of commutation relations, and carefully pointed out that they are true in any quantization of the theory. Now we'll really put that idea to use. In general, we should choose quantizations that respect symmetries. In a scale-invariant theory, it's natural to foliate spacetime with spheres around the origin and consider evolving states from smaller spheres to larger spheres using the dilatation operator (figure 11). This is called "radial quantization." The sphere  $S^{d-1}$  has an associated Hilbert space  $\mathcal{H}$ . We can act on  $\mathcal{H}$  by inserting operators on the surface of the sphere. For example, to act with a symmetry generator  $Q$ , we insert the surface operator  $Q(S^{d-1})$  into the path integral (figure 12).

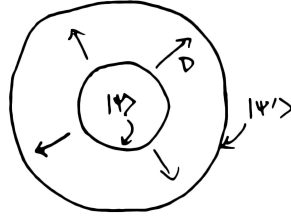


Fig. 11. In radial quantization, states live on spheres, and we evolve from one state to another with the dilatation operator.

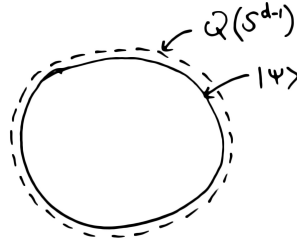


Fig. 12. We act with a charge in radial quantization by inserting  $Q(S^{d-1})$  just outside the sphere on which the state is defined.

In radial quantization, a correlation function gets interpreted as a radi-



ally ordered product,

$$\begin{aligned}
 \langle \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) \rangle &= \langle 0 | \mathcal{R} \{ \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) \} | 0 \rangle \\
 &\equiv \theta(|x_n| - |x_{n-1}|) \cdots \theta(|x_2| - |x_1|) \langle 0 | \mathcal{O}(x_n) \cdots \mathcal{O}(x_1) | 0 \rangle \\
 &\quad + \text{permutations.}
 \end{aligned} \tag{387}$$

Of course, we can perform radial quantization around different points. The same correlation function then gets interpreted as an expectation value of differently ordered operators acting on different states in different (but isomorphic) Hilbert spaces (figure 13). This is completely analogous to changing reference frames in Lorentz invariant theories. The radial ordering prescription is consistent because operators at the same radius but different angles on the sphere commute, just as spacelike-separated operators commute in the usual quantization.

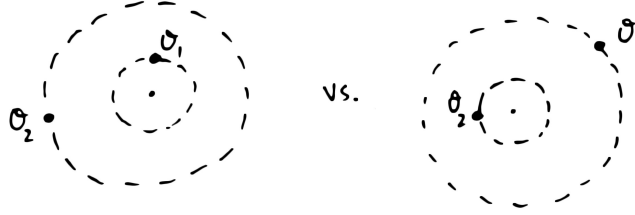


Fig. 13. When we perform radial quantization around different points, the same correlator gets interpreted as a product of operators with different orderings.

### 9.1. *Operator* $\implies$ *State*

The simplest way to prepare a state in radial quantization is to perform the path integral over the interior  $B$  of the sphere, with no operator insertions inside  $B$ . This gives the vacuum state  $|0\rangle$  on the boundary  $\partial B$  (figure 14). It's easy to see that  $|0\rangle$  is invariant under all symmetries because a topological surface on the boundary of  $B$  can be shrunk to zero inside  $B$  (figure 15).

To be explicit, suppose our CFT is given by the path integral over a scalar field  $\phi$ . The Hilbert space in radial quantization is spanned by “field eigenstates”  $|\phi_b\rangle$ , where  $\phi_b(\mathbf{n})$  is a field configuration on the sphere  $\mathbf{n} \in \partial B$ . The subscript “ $b$ ” indicates that  $\phi_b$  is defined only on the boundary  $\partial B$  and not in the interior. A general state is a linear combination of field

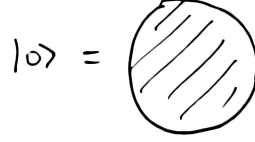


Fig. 14. The vacuum in radial quantization is given by the path integral over the interior of the sphere, with no operator insertions.

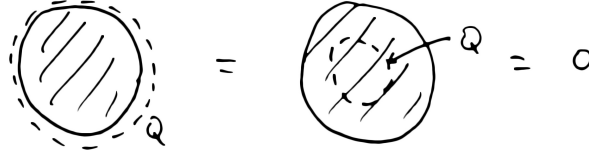


Fig. 15. The vacuum is automatically invariant under all symmetries.

eigenstates

$$|\psi\rangle \equiv \int D\phi_b |\phi_b\rangle \langle \phi_b | \psi \rangle. \quad (388)$$

Here,  $\int D\phi_b$  represents a  $d-1$ -dimensional path integral over fields on  $\partial B$ .

For the vacuum, the coefficients  $\langle \phi_b | 0 \rangle$  are given by the path integral over the interior with boundary conditions  $\phi_b$  and no operator insertions,

$$\langle \phi_b | 0 \rangle = \int_{\substack{\phi(1, \mathbf{n}) = \phi_b(\mathbf{n}) \\ r \leq 1}} D\phi(r, \mathbf{n}) e^{-S[\phi]}. \quad (389)$$

A more exciting possibility is to insert an operator  $\mathcal{O}(x)$  inside  $B$  and then perform the path integral,

$$\langle \phi_b | \mathcal{O}(x) | 0 \rangle = \int_{\substack{\phi(1, \mathbf{n}) = \phi_b(\mathbf{n}) \\ r \leq 1}} D\phi(r, \mathbf{n}) \mathcal{O}(x) e^{-S[\phi]}. \quad (390)$$

This defines a state called  $\mathcal{O}(x)|0\rangle$ , see figure 16. By inserting different operators inside  $B$ , we can prepare a variety of states on the boundary  $\partial B$ . In this language,  $|0\rangle$  is prepared by inserting the unit operator.

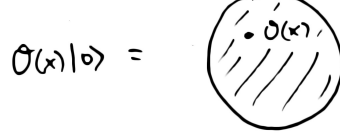


Fig. 16. The state  $\mathcal{O}(x)|0\rangle$  is given by inserting  $\mathcal{O}(x)$  inside the sphere and performing the path integral over the interior.

### 9.2. Operator $\Leftarrow$ State

This construction also works backwards. Let  $|\mathcal{O}_i\rangle$  be eigenstates of the dilatation operator

$$D|\mathcal{O}_i\rangle = \Delta_i|\mathcal{O}_i\rangle. \quad (391)$$

The  $|\mathcal{O}_i\rangle$  can themselves be used as operators: we cut spherical holes  $B_i$  out of the path integral centered around positions  $x_i$  and glue in the states  $|\mathcal{O}_i\rangle$  at the boundary of the holes, as in figure 17. This gives a quantity

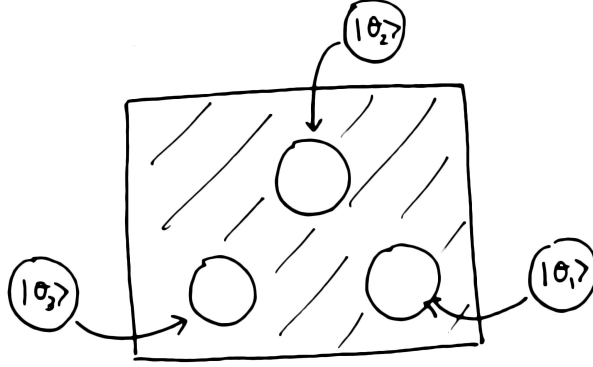


Fig. 17. A correlator of states is defined by cutting holes out of the path integral and gluing states into the holes.

that behaves exactly like a correlator of local operators. In the scalar field example, the gluing procedure gives an object

$$C(|\mathcal{O}_1\rangle, x_1; \dots; |\mathcal{O}_n\rangle, x_n) \equiv \int \prod_i D\phi_{bi} \langle \phi_{bi} | \mathcal{O}_i \rangle \int_{\substack{\phi_{\partial i} = \phi_{bi} \\ x \notin B_i}} D\phi(x) e^{-S[\phi]}, \quad (392)$$

where the path integral  $D\phi(x)$  is performed over the region outside the balls  $B_i$ , and the integrals  $D\phi_{b_i}$  are over field configurations on the boundaries  $\partial B_i$ . Here,  $\phi_{\partial i}$  denotes the restriction of the bulk field  $\phi(x)$  to the  $i$ -th boundary  $\partial B_i$ .

This construction only works when the  $x_i$  are far enough apart that the balls  $B_i$  don't overlap. However, a correlator of local operators should make sense for arbitrary configurations of non-coincident points. To overcome this problem, we can use the fact that we have diagonalized the dilatation operator. Let us define a correlator of the states  $|\mathcal{O}_i\rangle$  by

$$\langle \mathcal{O}(x_1) \cdots \mathcal{O}(x_n) \rangle \equiv \lambda^{\sum_i \Delta_i} C(|\mathcal{O}_1\rangle, \lambda x_1; \dots; |\mathcal{O}_n\rangle, \lambda x_n), \quad (393)$$

where  $\lambda \in \mathbb{R}$  is any number sufficiently large that  $\lambda|x_{ij}| > 1$ . Since the  $x_i$  can now be arbitrarily close together, we have defined local operators.<sup>86</sup>

We should check that (393) doesn't depend on  $\lambda$  (assuming  $\lambda$  satisfies the condition  $\lambda|x_{ij}| > 1$ ). Note that

$$\begin{aligned} & \frac{\partial}{\partial \lambda} \langle \mathcal{O}(x_1) \cdots \mathcal{O}(x_n) \rangle \\ &= \lambda^{\sum_i \Delta_i} \sum_i \left( \Delta_i + x_i \cdot \frac{\partial}{\partial x_i} \right) C(|\mathcal{O}_1\rangle, \lambda x_1; \dots; |\mathcal{O}_n\rangle, \lambda x_n) \end{aligned} \quad (394)$$

Because of (391), the latter expression is the same as inserting a topological surface operator for  $D$  in the path integral (392), which vanishes if we have scale-invariant boundary conditions.

*End of lecture 17*

### 9.3. Operator $\Longleftrightarrow$ State

So far I've been vague about what I mean by a local operator. But now, we can give a rigorous definition: we will simply *define* a local operator to be an eigenstate of  $D$  in radial quantization.<sup>87</sup> With this definition, the two constructions above are inverse to each other, with the identification

$$\mathcal{O}(0) \longleftrightarrow \mathcal{O}(0)|0\rangle \equiv |\mathcal{O}\rangle. \quad (395)$$

This is the “state-operator correspondence.”

<sup>86</sup>A more careful construction of the state  $\implies$  operator map that doesn't require this rescaling trick is given in Polchinski [13] volume 1, chapter 2.

<sup>87</sup>The dilatation operator is diagonalizable in unitary (reflection positive) CFTs. However, there exist interesting non-unitary theories where  $D$  has a nontrivial Jordan block decomposition. In these cases, we define a local operator as a finite-dimensional representation of  $D$ .

It is straightforward to see how the conformal group acts on states in radial quantization. A primary operator creates a state that is killed by  $K_\mu$  and transforms in a finite-dimensional representation of  $D$  and  $M_{\mu\nu}$ ,

$$[K_\mu, \mathcal{O}(0)] = 0 \quad \longleftrightarrow \quad K_\mu |\mathcal{O}\rangle = 0, \quad (396)$$

$$[D, \mathcal{O}(0)] = \Delta \mathcal{O}(0) \quad \longleftrightarrow \quad D |\mathcal{O}\rangle = \Delta |\mathcal{O}\rangle, \quad (397)$$

$$[M_{\mu\nu}, \mathcal{O}(0)] = \mathcal{S}_{\mu\nu} \mathcal{O}(0) \quad \longleftrightarrow \quad M_{\mu\nu} |\mathcal{O}\rangle = \mathcal{S}_{\mu\nu} |\mathcal{O}\rangle. \quad (398)$$

This follows by acting on  $|0\rangle$  with the operator equations above and using the fact that  $|0\rangle$  is killed by  $K, D$ , and  $M$ .

A conformal multiplet in radial quantization is given by acting with momentum generators on a primary state

$$|\mathcal{O}\rangle, P_\mu |\mathcal{O}\rangle, P_\mu P_\nu |\mathcal{O}\rangle, \dots \quad (\text{conformal multiplet}). \quad (399)$$

This is equivalent to acting with derivatives of  $\mathcal{O}(x)$  at the origin, for example

$$\partial_\mu \mathcal{O}(x)|_{x=0}|0\rangle = [P_\mu, \mathcal{O}(0)]|0\rangle = P_\mu |\mathcal{O}\rangle. \quad (400)$$

The operator  $\mathcal{O}(x)$  creates an infinite linear combination of descendants,

$$\mathcal{O}(x)|0\rangle = e^{x \cdot P} \mathcal{O}(0) e^{-x \cdot P} |0\rangle = e^{x \cdot P} |\mathcal{O}\rangle = \sum_{n=0}^{\infty} \frac{1}{n!} (x \cdot P)^n |\mathcal{O}\rangle. \quad (401)$$

As with the classification of operators, the action of the conformal algebra on a multiplet in radial quantization is determined by the commutation relations of the algebra. In fact the required computations look *exactly identical* to the computations we did to determine the action of conformal generators on operators (278, 281, 289). This is because by surrounding operators with charges supported on spheres, we were secretly doing radial quantization all along!

#### 9.4. Another View of Radial Quantization

To study a conformal Killing vector  $\epsilon$ , it is often helpful to perform a Weyl rescaling of the metric  $g \rightarrow \Omega(x)^2 g$  so that  $\epsilon$  becomes a regular Killing vector, i.e. an isometry. We can turn a dilatation into an isometry by performing a Weyl rescaling from  $\mathbb{R}^d$  to the cylinder  $\mathbb{R} \times S^{d-1}$ ,

$$\begin{aligned} ds_{\mathbb{R}^d}^2 &= dr^2 + r^2 ds_{S^{d-1}}^2 \\ &= r^2 \left( \frac{dr^2}{r^2} + ds_{S^{d-1}}^2 \right) \\ &= e^{2\tau} (d\tau^2 + ds_{S^{d-1}}^2) = e^{2\tau} ds_{\mathbb{R} \times S^{d-1}}^2, \end{aligned} \quad (402)$$

where  $r = e^\tau$ .

Dilatations  $r \rightarrow \lambda r$  become shifts of radial time  $\tau \rightarrow \tau + \log \lambda$ . Radial quantization in flat space is equivalent to the usual quantization on the cylinder. States live on spheres and time evolution is generated by acting with  $e^{-D\tau}$ . While the development of radial quantization in the previous sections relied only on scale invariance, the cylinder picture relies on conformal invariance because we have performed a nontrivial Weyl rescaling.

Let us build a more detailed dictionary between the two pictures. Under a Weyl rescaling, correlation functions of local operators transform as<sup>88</sup>

$$\frac{\langle \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) \rangle_g}{\langle 1 \rangle_g} = \left( \prod_i \Omega(x_i)^{\Delta_i} \right) \frac{\langle \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) \rangle_{\Omega^2 g}}{\langle 1 \rangle_{\Omega^2 g}}. \quad (403)$$

This is a nontrivial claim — if we implement the Ising model in flat space, compute expectation values and take the continuum limit, it's not obvious that the answer should be simply related to the same lattice theory on the cylinder.<sup>89</sup> In general it isn't, but at the critical value of the coupling when the theory becomes conformal, tracelessness of the stress tensor implies insensitivity to Weyl rescalings, and the answers become related.

**Exercise 9.1.** By integrating by parts in (281), show that

$$T_\mu^\mu(x) \mathcal{O}(y) = \Delta \delta(x - y) \mathcal{O}(y). \quad (404)$$

An insertion of  $T_\mu^\mu$  is the response of the theory to an infinitesimal Weyl transformation  $g \rightarrow e^{2\delta\omega} g$ . Derive (403) by exponentiation.<sup>90</sup>

Thus, given an operator  $\mathcal{O}(x)$  in  $\mathbb{R}^d$ , it is natural to define a cylinder operator

$$\mathcal{O}_{\text{cyl}}(\tau, \mathbf{n}) \equiv e^{\Delta\tau} \mathcal{O}_{\text{flat}}(x = e^\tau \mathbf{n}). \quad (405)$$

<sup>88</sup>In even dimensions, the partition function itself can transform with a Weyl anomaly  $\langle 1 \rangle_g = \langle 1 \rangle_{\Omega^2 g} e^{S_{\text{Weyl}}[g]}$ . This will not be important for our discussion, so we have divided through by the partition function.

<sup>89</sup>Comparing the flat and cylindrical Ising models is relatively easy in 2d, but harder in 3d since  $S^2$  is curved. See [23] for a recent attempt.

<sup>90</sup>We cheated here by only deriving (404) in flat space. In curved space there is an additional contribution to  $T_\mu^\mu$  coming from the Weyl anomaly. This factor cancels in (403). There could also be modifications to the contact term (404). However, in a conformally flat metric, we can simply define the curved space operator  $\mathcal{O}(x)$  so that it satisfies (404). For instance, we may modify the Weyl factor so that it is constant in a tiny neighborhood of  $\mathcal{O}(x)$  and the flat-space calculation applies. This definition might not be consistent with other independent definitions. For instance, if  $\mathcal{O}(x)$  is the stress tensor, it gives a different answer from the canonical definition (213) because of the Weyl anomaly.

We often omit the subscripts “cyl.” and “flat,” relying on the coordinates to indicate which type of operator we’re discussing.

**Exercise 9.2.** Using (403), compute a two-point function of cylinder operators

$$\langle \mathcal{O}(\tau_1, \mathbf{n}_1) \mathcal{O}(\tau_2, \mathbf{n}_2) \rangle. \quad (406)$$

Verify that it is time-translationally invariant on the cylinder. Show that in the limit of large time separation  $\tau = \tau_2 - \tau_1 \gg 1$ , the two-point function has an expansion in terms of the form  $e^{-(\Delta+n)\tau}$  with integer  $n \geq 0$ . Interpret these as coming from the exchange of states in the conformal multiplet of  $\mathcal{O}$ .

## 10. Reflection Positivity and Unitarity Bounds

### 10.1. Reflection Positivity

In Lorentzian signature, we are interested in unitary theories: theories where the conserved charges (including the Hamiltonian) are Hermitian operators so that they generate unitary transformations. Unitarity in Lorentzian signature is equivalent to a property called “reflection positivity” in Euclidean signature.<sup>91</sup>

Consider a Lorentzian theory with a local operator  $\mathcal{O}_L$  and Hermitian energy-momentum generators  $(H, \mathbf{P}_L)$  ( $L$  is for “Lorentzian”). We have the textbook formula

$$\mathcal{O}_L(t, \mathbf{x}) = e^{iHt - i\mathbf{x} \cdot \mathbf{P}_L} \mathcal{O}_L(0, 0) e^{-iHt + i\mathbf{x} \cdot \mathbf{P}_L}. \quad (407)$$

Let  $\mathcal{O}_L(0, 0)$  be Hermitian. It follows from (407) that  $\mathcal{O}_L(t, \mathbf{x})$  is Hermitian too.

Now, let us Wick-rotate to Euclidean signature,

$$\mathcal{O}_E(t_E, \mathbf{x}) \equiv \mathcal{O}_L(-it_E, \mathbf{x}) = e^{Ht_E - i\mathbf{x} \cdot \mathbf{P}_L} \mathcal{O}_L(0, 0) e^{-Ht_E + i\mathbf{x} \cdot \mathbf{P}_L}. \quad (408)$$

The Euclidean operator satisfies

$$\mathcal{O}_E(t_E, \mathbf{x})^\dagger = \mathcal{O}_E(-t_E, \mathbf{x}). \quad (409)$$

---

<sup>91</sup>We make some brief comments about Euclidean vs. Lorentzian field theory and analytic continuation in appendix ??.

To Wick-rotate an operator with spin, we conventionally add factors of  $-i$  to the time components,<sup>92</sup> e.g. for a vector operator  $\mathcal{O}_L^\mu$ ,

$$\begin{aligned}\mathcal{O}_E^0(t_E, \mathbf{x}) &= -i\mathcal{O}_L^0(-it_E, \mathbf{x}), \\ \mathcal{O}_E^i(t_E, \mathbf{x}) &= \mathcal{O}_L^i(-it_E, \mathbf{x}).\end{aligned}\quad (410)$$

This leads to

$$\mathcal{O}_E^{\mu_1 \dots \mu_\ell}(t_E, \mathbf{x})^\dagger = \Theta^{\mu_1}_{\nu_1} \dots \Theta^{\mu_\ell}_{\nu_\ell} \mathcal{O}_E^{\nu_1 \dots \nu_\ell}(-t_E, \mathbf{x}), \quad (411)$$

where  $\Theta^\mu_\nu = \delta^\mu_\nu - 2\delta^\mu_0 \delta^\nu_0$  is a reflection in the time-direction.

Thus, the way Hermitian conjugation acts on a Euclidean operator depends on which direction we call time. Whether an operator is Hermitian or not depends on how we quantize the theory! This is very different from Lorentzian signature, where the conjugation properties of operators don't depend on a choice of reference frame.

*End of lecture 18*

As an example, consider the momentum generators

$$P^\mu = - \int d^{d-1} \mathbf{x} T^{\mu 0}(0, \mathbf{x}). \quad (412)$$

(From now on, we work in the Euclidean theory and omit the  $E$  subscripts.) Using (411), we have

$$\begin{aligned}T^{i0}(0, \mathbf{x})^\dagger &= -T^{i0}(0, \mathbf{x}), \\ T^{00}(0, \mathbf{x})^\dagger &= T^{00}(0, \mathbf{x}).\end{aligned}\quad (413)$$

It follows that  $P^0$  is Hermitian, and the  $P^i$  are *antihermitian*. We may write

$$P^0 = H, \quad P^j = -iP_L^j, \quad (414)$$

with  $H, P_L$  Hermitian, and then (224) agrees with the formula we got from Wick rotation (408). If we had quantized with a different time direction, say the  $x_1$ -direction, then we would conclude that  $P^1$  is Hermitian, while  $P^0, P^2, \dots, P^{d-1}$  are antihermitian.

To reiterate, *the way conjugation acts on operators depends on how we quantize our theory*. This makes sense, because Hermitian conjugation is something you do to operators on Hilbert spaces, and different quantizations have different Hilbert spaces.

This raises the question: given a Euclidean path integral, how do we know if it computes the Wick-rotation of a unitary Lorentzian theory? One

<sup>92</sup>These factors are needed to make Euclidean correlation functions manifestly covariant under  $\text{SO}(d)$  rotations.



important condition is that norms of states should be positive. Consider some in-state  $|\psi\rangle$  given by acting on the vacuum with a bunch of operators at negative Euclidean time

$$|\psi\rangle = \mathcal{O}(-t_{E1}) \cdots \mathcal{O}(-t_{En})|0\rangle. \quad (415)$$

For brevity, we suppress the spatial positions of the operators. The conjugate state is given by

$$\begin{aligned} \langle\psi| &= (\mathcal{O}(-t_{E1}) \cdots \mathcal{O}(-t_{En})|0\rangle)^\dagger \\ &= \langle 0|\mathcal{O}(t_{En}) \cdots \mathcal{O}(t_{E1}). \end{aligned} \quad (416)$$

That is,  $\langle\psi|$  is given by taking the vacuum in the future and positioning operators in a time-reflected way. Thus, the condition

$$\langle\psi|\psi\rangle \geq 0 \quad (417)$$

says that a time-reflection symmetric configuration should have a positive path integral, see figure 18. This is called “reflection positivity.”

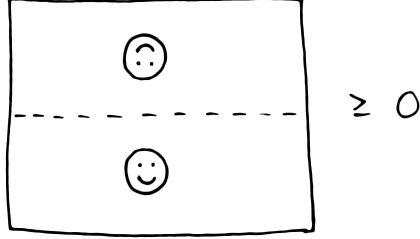


Fig. 18. Reflection positivity.

If a Euclidean theory is the Wick-rotation of a unitary Lorentzian theory, then it will be reflection positive. However, some theories are more naturally defined in Euclidean signature. In this case, reflection positivity must be checked. It often suffices to check it in any microscopic theory in the same universality class as the CFT we’re interested in.

**Exercise 10.1.** Consider the 2d Ising lattice correlator shown in figure 19. Show that it can be written as a sum of squares, and is hence positive. (Hint: first sum over spins off the line  $L$ , and then sum over spins on  $L$ .) Generalize your proof to argue that the 2d Ising model is reflection-positive.

The Osterwalder-Schrader reconstruction theorem says that, given a collection of Euclidean correlators satisfying reflection positivity (and some

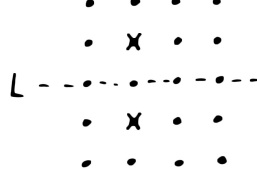


Fig. 19. A two point function on a  $4 \times 5$  Ising lattice with free boundary conditions, with spin operators inserted at the sites marked with an X.

additional technical assumptions), we can reconstruct a unitary Lorentzian quantum field theory by analytic continuation [24]. So reflection positivity in Euclidean signature and unitarity in Lorentzian signature are essentially equivalent, and we will use the terms interchangeably.

#### 10.1.1. Real vs. Complex Operators

Because Hermitian conjugation is tricky in Euclidean signature, it is helpful to introduce some extra terminology. We call a local operator “real” if it is Hermitian in Lorentzian signature. In Euclidean signature, real operators satisfy (411). By contrast, for a complex operator  $\mathcal{O}_L^\dagger = \mathcal{O}_L^*$ , we have

$$\mathcal{O}_E^{\mu_1 \dots \mu_\ell}(t_E, \mathbf{x})^\dagger = \Theta^{\mu_1}_{\nu_1} \dots \Theta^{\mu_\ell}_{\nu_\ell} \mathcal{O}_E^{\nu_1 \dots \nu_\ell*}(-t_E, \mathbf{x}). \quad (418)$$

Later we will need the following result. If  $\phi_1, \phi_2$  are real scalars and  $\mathcal{O}$  is a real operator with spin  $\ell$  in a unitary theory, then the three-point coefficient  $f_{\phi_1 \phi_2 \mathcal{O}}$  is real. This is easiest to see in Lorentzian signature when the operators are spacelike separated  $x_{ij}^2 > 0$ . Because local operators commute at spacelike separation, we have

$$\langle 0 | \phi_1(x_1) \phi_2(x_2) \mathcal{O}^{\mu_1 \dots \mu_\ell}(x_3) | 0 \rangle^* = \langle 0 | \phi_1(x_1) \phi_2(x_2) \mathcal{O}^{\mu_1 \dots \mu_\ell}(x_3) | 0 \rangle. \quad (419)$$

Substituting (338) gives  $f_{\phi_1 \phi_2 \mathcal{O}}^* = f_{\phi_1 \phi_2 \mathcal{O}}$ .

## 10.2. Reflection Positivity on the Cylinder

Reflection-positivity (or unitarity) has interesting consequences for CFTs on the cylinder. The Hermitian conjugate of a real cylinder operator is

$$\mathcal{O}_{\text{cyl.}}(\tau, \mathbf{n})^\dagger_{\text{rad}} = \mathcal{O}_{\text{cyl.}}(-\tau, \mathbf{n}). \quad (420)$$

Using (405), this becomes

$$\mathcal{O}_{\text{flat}}(x)^{\dagger}_{\text{rad}} = x^{-2\Delta} \mathcal{O}_{\text{flat}}\left(\frac{x^\mu}{x^2}\right). \quad (421)$$

Above, we have written  $\dagger_{\text{rad}}$  to emphasize that Hermitian conjugation in radial quantization is different from Hermitian conjugation in the usual  $P^0$  quantization. From now on we write simply  $\dagger$ , and hope that the meaning will be clear from context.

The right-hand side of (421) is just the image of  $\mathcal{O}(x)$  under an inversion  $I : x^\mu \rightarrow \frac{x^\mu}{x^2}$ . The same is true for operators with spin, where the full formula (293) gives

$$\begin{aligned} \mathcal{O}^{\mu_1 \dots \mu_\ell}(x)^{\dagger} &= I^{\mu_1}{}_{\nu_1}(x) \dots I^{\mu_\ell}{}_{\nu_\ell}(x) x^{-2\Delta} \mathcal{O}^{\nu_1 \dots \nu_\ell}\left(\frac{x}{x^2}\right), \\ I^{\mu}{}_{\nu}(x) &= \delta^{\mu}_{\nu} - \frac{2x^{\mu}x_{\nu}}{x^2}. \end{aligned} \quad (422)$$

**Exercise 10.2.** Check that the two-point function of spin-1 operators (323) satisfies reflection-positivity on the cylinder if  $c_{12} > 0$ .

Applying (422) to the stress tensor, we find that the action of conjugation on the conformal charges in radial quantization is

$$Q_{\epsilon}^{\dagger} = -Q_{I\epsilon I}. \quad (423)$$

In particular, we have

$$\begin{aligned} M_{\mu\nu}^{\dagger} &= -M_{\mu\nu}, \\ D^{\dagger} &= D, \\ P_{\mu}^{\dagger} &= K_{\mu}. \end{aligned} \quad (424)$$

These facts let us calculate properties of correlation functions purely algebraically. As an example, consider a two-point function. Letting  $\tilde{y} = y/y^2$ , we have

$$\begin{aligned} \langle \mathcal{O}(y) \mathcal{O}(x) \rangle &= \langle 0 | (y^{-2\Delta} \mathcal{O}(\tilde{y}))^{\dagger} \mathcal{O}(x) | 0 \rangle \\ &= y^{-2\Delta} \langle 0 | (e^{\tilde{y} \cdot P} \mathcal{O}(0) e^{-\tilde{y} \cdot P})^{\dagger} e^{x \cdot P} \mathcal{O}(0) e^{-x \cdot P} | 0 \rangle \\ &= y^{-2\Delta} \langle 0 | e^{-\tilde{y} \cdot K} \mathcal{O}(0)^{\dagger} e^{\tilde{y} \cdot K} e^{x \cdot P} \mathcal{O}(0) e^{-x \cdot P} | 0 \rangle \\ &= y^{-2\Delta} \langle 0 | \mathcal{O}(0)^{\dagger} e^{\tilde{y} \cdot K} e^{x \cdot P} \mathcal{O}(0) | 0 \rangle \\ &= y^{-2\Delta} \langle \mathcal{O} | e^{\tilde{y} \cdot K} e^{x \cdot P} | \mathcal{O} \rangle, \end{aligned} \quad (425)$$

where we've defined

$$\langle \mathcal{O} | \equiv \langle 0 | \mathcal{O}(0)^\dagger = \lim_{y \rightarrow \infty} y^{2\Delta} \langle 0 | \mathcal{O}(y). \quad (426)$$

By expanding the exponentials, we can evaluate (425) using the conformal algebra. For example, the first couple terms are

$$\langle \mathcal{O}(y) \mathcal{O}(x) \rangle = y^{-2\Delta} \left( \langle \mathcal{O} | \mathcal{O} \rangle + \frac{y^\mu}{y^2} x^\nu \langle \mathcal{O} | K_\mu P_\nu | \mathcal{O} \rangle + \dots \right), \quad (427)$$

where we've used that  $K | \mathcal{O} \rangle = \langle \mathcal{O} | P = 0$  because  $\mathcal{O}$  is primary. Using the conformal commutation relations,

$$\begin{aligned} \langle \mathcal{O} | K_\mu P_\nu | \mathcal{O} \rangle &= \langle \mathcal{O} | [K_\mu, P_\nu] | \mathcal{O} \rangle \\ &= \langle \mathcal{O} | (2D\delta_{\mu\nu} - 2M_{\mu\nu}) | \mathcal{O} \rangle \\ &= 2\Delta\delta_{\mu\nu} \langle \mathcal{O} | \mathcal{O} \rangle. \end{aligned} \quad (428)$$

Thus,

$$\langle \mathcal{O}(y) \mathcal{O}(x) \rangle = y^{-2\Delta} \langle \mathcal{O} | \mathcal{O} \rangle \left( 1 + 2\Delta \frac{y \cdot x}{y^2} + \dots \right). \quad (429)$$

This exactly matches the expansion of  $\langle \mathcal{O} | \mathcal{O} \rangle / (x - y)^{2\Delta}$  in small  $|x|/|y|$ . You can imagine computing all the higher terms and matching the whole series expansion.

**Exercise 10.3.** *This computation is not directly relevant to the course, but it is instructive for getting used to radial ordering. Consider a three-point function of scalars*

$$\begin{aligned} \langle \mathcal{O}_i(x_1) \mathcal{O}_j(x_2) \mathcal{O}_k(x_3) \rangle &= \langle 0 | \mathcal{R} \{ \mathcal{O}_i(x_1) \mathcal{O}_j(x_2) \mathcal{O}_k(x_3) \} | 0 \rangle \\ &= \theta(|x_3| \geq |x_2| \geq |x_1|) \langle 0 | \mathcal{O}_k(x_3) \mathcal{O}_j(x_2) \mathcal{O}_i(x_1) | 0 \rangle \\ &\quad + \text{permutations.} \end{aligned} \quad (430)$$

Consider the operator  $e^{2\pi i \mathcal{D}_1}$  where

$$\mathcal{D}_1 = x_1 \cdot \partial_1 + \Delta_1. \quad (431)$$

Using the fact that  $e^{2\pi i \mathcal{D}_1} \mathcal{O}_1(x_1) = e^{2\pi i D} \mathcal{O}_1(x_1) e^{-2\pi i D}$ , compute the action of  $e^{2\pi i \mathcal{D}_1}$  on each of the terms above. You will get different answers for each of the different operator orderings.

Now determine the action of  $e^{2\pi i \mathcal{D}_1}$  on the known answer for the scalar three-point function (303). Check that the two answers agree.

*End of lecture 19*

## 10.2.1. BPZ conjugation

**Note:** *This section is somewhat technical, and can be skipped on a first reading.*

To generalize this computation, we need to define a bra state  $\langle \mathcal{O} |$  in radial quantization for an operator  $\mathcal{O}$  in a general representation  $\rho$  of  $\text{SO}(d)$ . The formula (422) gives the radial conjugate of a tensor operator. In our explanation, the  $I^\mu{}_\nu(x)$ 's came from factors of  $i$  introduced during Wick rotation, e.g. (410). However, there is no natural convention for Wick-rotating other representations, e.g. spinors.

The correct generalization of (422) to arbitrary  $\rho$  is

$$\mathcal{O}^a(x)^{\dagger_{\text{rad}}} = \frac{g_{a\bar{a}}(x)}{x^{2\Delta}} \mathcal{O}^{\dagger\bar{a}}\left(\frac{x}{x^2}\right). \quad (432)$$

Here,  $\mathcal{O}^{\dagger\bar{a}}(y)$  denotes the non-radial Hermitian conjugate of  $\mathcal{O}$  at the origin,<sup>93</sup> translated to the point  $y$

$$\mathcal{O}^{\dagger\bar{a}}(y) = e^{y \cdot P} \mathcal{O}^{\dagger\bar{a}}(0) e^{-y \cdot P}. \quad (433)$$

The tensor  $g_{a\bar{a}}(x)$  is the inverse of the numerator of a two-point function

$$\langle \mathcal{O}^{\dagger\bar{a}}(x_2) \mathcal{O}^b(x_1) \rangle = c_{\mathcal{O}} \frac{g^{\bar{a}b}(x_{12})}{x_{12}^{2\Delta}}, \quad (434)$$

$$g_{a\bar{a}}(x) g^{\bar{a}b}(x) = \delta_a^b, \quad (435)$$

where  $c_{\mathcal{O}}$  is a positive operator-dependent constant. Note that (432) reduces to (422) when  $\mathcal{O}$  is a real, traceless symmetric tensor.<sup>94</sup> The definition (432) is called Belavin-Polyakov-Zamolodchikov (BPZ) conjugation.

Let us make some comments about (434). Recall from section ?? that  $\mathcal{O}(x)$  can only have a nonzero two-point function with an operator in the dual reflected representation  $(\rho^R)^*$  of  $\text{SO}(d)$ . It turns out that  $\mathcal{O}^{\dagger}(x)$  always transforms in this representation. In other-words, we can always find a nonzero conformally-invariant two-point structure (434). This must be the case because  $\langle 0 | \mathcal{O}^{\dagger} \mathcal{O} | 0 \rangle$  is the norm of a state, so it had better be nonzero.

<sup>93</sup>Note that different choices of time direction at the origin may give different definitions of  $\mathcal{O}^{\dagger\bar{a}}$ . However, these choices will cancel out in the combination  $g_{a\bar{a}} \mathcal{O}^{\dagger\bar{a}}$ . Also, the origin is not important here: we could have chosen any reference point  $y_0$  and defined  $\mathcal{O}^{\dagger}(y_0)$  in any quantization such that  $y_0$  is at zero time.

<sup>94</sup>We have written  $\dagger_{\text{rad}}$  in (432) to emphasize that  $\mathcal{O}^a(x)^{\dagger_{\text{rad}}}$  and  $\mathcal{O}^{\dagger b}(x)$  are different. You will often see people drop the “rad” and simply distinguish between the two objects by whether  $\dagger$  is on the outside or inside of the parentheses. This notation can be very confusing, but is sometimes convenient.

We can also understand this more explicitly. Given a representation  $\rho$  of  $\text{SO}(d)$ , we can form a representation  $\rho_L$  of  $\text{SO}(d-1, 1)$  by

$$\rho_L(M_{\mu\nu}^L) = i^{\delta_{\mu 0} + \delta_{\nu 0}} \rho(M_{\mu\nu}). \quad (436)$$

Here  $M_{\mu\nu}$  are the generators of  $\text{SO}(d)$  and  $M_{\mu\nu}^L$  are the generators of  $\text{SO}(d-1, 1)$ . The factors of  $i$  are because of Wick rotation.<sup>95</sup> They ensure that if  $\rho(M_{\mu\nu})$  satisfy the algebra of  $\text{SO}(d)$ , then  $\rho_L(M_{\mu\nu}^L)$  satisfy the algebra of  $\text{SO}(d-1, 1)$ . Taking the complex conjugate of both sides, we find

$$\begin{aligned} \rho_L(M_{\mu\nu}^L)^* &= (-i)^{\delta_{\mu 0} + \delta_{\nu 0}} \rho(M_{\mu\nu})^* \\ &= i^{\delta_{\mu 0} + \delta_{\nu 0}} \rho(I_\mu{}^\rho I_\nu{}^\sigma M_{\rho\sigma})^*, \end{aligned} \quad (437)$$

where  $I_\mu{}^\nu$  is a reflection in the 0 direction. On the left-hand side are matrix elements acting on  $\mathcal{O}^\dagger$  in Lorentzian signature, so  $\rho(I_\mu{}^\rho I_\nu{}^\sigma M_{\rho\sigma})^*$  are matrix elements acting on  $\mathcal{O}^\dagger$  in Euclidean signature. Because  $\text{SO}(d)$  is compact, complex conjugation is the same as taking the dual. Recalling the definition (326), we see that  $\rho(I_\mu{}^\rho I_\nu{}^\sigma M_{\rho\sigma})^*$  are matrix elements of the dual reflected representation  $(\rho^R)^*$ .

As an example, in exercise 8.4 we saw that the dual reflected representation of a left-handed spinor  $\psi_\alpha$  in 4d is a right-handed spinor  $\bar{\psi}_{\dot{\alpha}}$  (reflected is the same as dual reflected in this case, since each type of spinor in 4d is self-dual). In Lorentzian signature, these representations are indeed complex conjugates of each other.

We are now ready to define the dual state to  $|\mathcal{O}^a\rangle$  in radial quantization:

$$\begin{aligned} \langle \mathcal{O}_a | &= (\mathcal{O}^a(0)|0\rangle)^\dagger_{\text{rad}} \\ &= \lim_{x \rightarrow 0} (\mathcal{O}^a(x)|0\rangle)^\dagger_{\text{rad}} \\ &= \lim_{x \rightarrow 0} \langle 0 | \mathcal{O}^a(x)^\dagger_{\text{rad}} \\ &= \lim_{y \rightarrow \infty} y^{2\Delta} g_{ab}(y) \langle 0 | \mathcal{O}^{\dagger b}(y), \end{aligned} \quad (438)$$

where  $y = x/x^2$ . We have given  $\langle \mathcal{O}_a |$  a lowered index because it transforms in the dual representation  $\rho^*$ . One very nice property of BPZ conjugation is that the limit  $\langle \mathcal{O}_a |$  is independent of the direction along which we take

<sup>95</sup>Note that we can naturally Wick-rotate the matrix elements but there is no natural way to Wick-rotate the vectors that  $\rho(M)$  acts on, in general.

$\mathcal{O}^\dagger$  to infinity. We can see this by computing an inner product

$$\begin{aligned}\langle \mathcal{O}_b | \mathcal{O}^a \rangle &= \lim_{y \rightarrow \infty} y^{2\Delta} g_{b\bar{a}}(y) \langle 0 | \mathcal{O}^{\dagger\bar{a}}(y) \mathcal{O}^a(0) | 0 \rangle \\ &= \lim_{y \rightarrow \infty} y^{2\Delta} g_{b\bar{a}}(y) c_{\mathcal{O}} \frac{g^{\bar{a}a}(y)}{y^{2\Delta}} \\ &= c_{\mathcal{O}} \delta_b^a.\end{aligned}\tag{439}$$

**Exercise 10.4. Warning: trick question!** Let  $\mathcal{O}(x)$  be a Hermitian operator. Recall that

$$[D, \mathcal{O}(0)] = \Delta \mathcal{O}(0),\tag{440}$$

with  $\Delta$  real. Furthermore, in radial quantization, we have  $D^\dagger = D$ . Now note that the commutator of two Hermitian operators is anti-hermitian:

$$[A, B]^\dagger = [B, A] = -[A, B].\tag{441}$$

How can this be consistent with (440)? Hint: see footnote 94.

### 10.3. Unitarity Bounds

Thinking about the theory on the cylinder gives a natural inner product on states in radial quantization. Unitarity (or reflection positivity) implies that the norms of states must be nonnegative. By demanding positivity for every state in a conformal multiplet, we obtain bounds on dimensions of primary operators [25–27]. We have already seen an example in (428). We found

$$|P_0|\mathcal{O}\rangle|^2 = \langle \mathcal{O} | K_0 P_0 | \mathcal{O} \rangle = 2\Delta \langle \mathcal{O} | \mathcal{O} \rangle.\tag{442}$$

Unitarity implies  $\Delta \geq 0$ .

Let us generalize this argument to the case where  $\mathcal{O}^a$  transforms in a nontrivial representation of  $\text{SO}(d)$ . We can normalize  $\mathcal{O}$  so that

$$\langle \mathcal{O}_b | \mathcal{O}^a \rangle = \delta_b^a.\tag{443}$$

Taking inner products between first-level descendants and using the conformal algebra, we find

$$(P_\mu |\mathcal{O}^a\rangle)^\dagger P^\nu |\mathcal{O}^b\rangle = \langle \mathcal{O}_a | K_\mu P^\nu | \mathcal{O}^b \rangle = 2\Delta \delta_\mu^\nu \delta_a^b - 2(\mathcal{S}_\mu^\nu)_a{}^b.\tag{444}$$

The state  $P^\nu |\mathcal{O}^b\rangle$  lives in the representation  $\square \otimes \rho$  of  $\text{SO}(d)$ , where  $\square$  is the vector representation. Unitarity implies that (444) must be positive-definite as a matrix acting on this space. This implies

$$\Delta \geq \text{max-eigenvalue}((\mathcal{S}_\mu^\nu)_a{}^b).\tag{445}$$

Let us write

$$\begin{aligned}
 (\mathcal{S}_\mu^\nu)_a{}^b &= \frac{1}{2} \sum_{\alpha, \beta} (L^{\alpha\beta})_\mu{}^\nu (\mathcal{S}_{\alpha\beta})_a{}^b \\
 &= \sum_{\substack{A=\alpha\beta \\ \alpha < \beta}} (L^A)_\mu{}^\nu (\mathcal{S}_A)_a{}^b, \\
 (L^{\alpha\beta})_{\mu\nu} &\equiv \delta_\mu^\alpha \delta_\nu^\beta - \delta_\nu^\alpha \delta_\mu^\beta,
 \end{aligned} \tag{446}$$

where  $(L^{\alpha\beta})_\mu{}^\nu$  is the generator of rotations in the vector representation  $\square$ . In the second line, we have written  $A = \alpha\beta$  with  $\alpha < \beta$ , which we should think of as an adjoint index of  $\text{SO}(d)$ . Thinking of  $L^A, \mathcal{S}_A$  as operators on  $\square \otimes \rho$ , this becomes

$$\begin{aligned}
 \sum_A L^A \otimes \mathcal{S}_A &= \frac{1}{2} ((L \otimes 1 + 1 \otimes \mathcal{S})^2 - (L \otimes 1)^2 - (1 \otimes \mathcal{S})^2) \\
 &= \frac{1}{2} (-C_2(\square \otimes \rho) + C_2(\square) \otimes 1 + 1 \otimes C_2(\rho)),
 \end{aligned} \tag{447}$$

where we've used the familiar trick from basic quantum mechanics to get a linear combination of quadratic Casimir operators. (The quadratic Casimir is  $-L^2 = -\sum_A L^A L_A$  because our generators are antihermitian and differ from the conventional ones by a factor of  $i$ .)

Let's specialize to the case where  $\rho = V_\ell$  is the spin- $\ell$  traceless symmetric tensor representation.  $V_\ell$  has Casimir

$$C_2(V_\ell) = \ell(\ell + d - 2). \tag{448}$$

To get the maximal eigenvalue of  $L \cdot \mathcal{S}$ , we need the minimal Casimir of

$$\square \otimes V_\ell = V_{\ell-1} \oplus \dots \quad (\ell > 0). \tag{449}$$

Here the " $\dots$ " are irreducible representations with larger Casimirs. Thus,

$$\begin{aligned}
 \Delta &\geq \frac{1}{2} (-C_2(V_{\ell-1}) + C_2(\square) + C_2(V_\ell)) \\
 &= \ell + d - 2.
 \end{aligned} \tag{450}$$

This computation was valid only for  $\ell > 0$ , since for scalars  $\square \otimes V_{\ell=0} = V$ .

One can also consider more complicated descendants.

**Exercise 10.5.** Compute the norm of  $P_\mu P^\mu |\mathcal{O}\rangle$ , where  $\mathcal{O}$  is a scalar. Show that unitarity implies either  $\Delta = 0$  or  $\Delta \geq \frac{d-2}{2}$ . This gives a stronger condition than what we derived above ( $\Delta \geq 0$ ) for scalars.



For traceless symmetric tensors in general conformal field theories, these inequalities are the best you can do (other descendants give no new information).<sup>96</sup> In theories with more symmetry, like supersymmetric theories or 2d CFTs, unitarity bounds can be more interesting. A classic reference for unitarity bound computations is [27]. In the math literature, unitarity bounds for higher-dimensional CFTs were essentially computed long ago by Jantzen [26], though the relevance of that work for physics has only been emphasized recently [28, 29].

In summary, we have the unitarity bounds

$$\begin{aligned} \Delta &= 0 \text{ (unit operator), or} \\ \Delta &\geq \begin{cases} \frac{d-2}{2} & \ell = 0, \\ \ell + d - 2 & \ell > 0. \end{cases} \end{aligned} \quad (451)$$

### 10.3.1. Null States and Conserved Currents

If  $\Delta$  saturates the bounds (451), the conformal multiplet will have a null state. For the unit operator, all descendants are null. For a scalar with dimension  $\frac{d-2}{2}$ , the null state is

$$P^2|\mathcal{O}\rangle = 0. \quad (452)$$

In operator language, this says  $\partial^2\mathcal{O}(x) = 0$ , which means  $\mathcal{O}$  satisfies the Klein-Gordon equation, and is thus a free scalar that decouples from the rest of the CFT.

For a spin- $\ell$  operator, the null state is<sup>97</sup>

$$P_\mu|\mathcal{O}^{\mu\mu_2\cdots\mu_\ell}\rangle = 0. \quad (453)$$

In operator language, this becomes the equation for a conserved current

$$\partial_\mu\mathcal{O}^{\mu\mu_2\cdots\mu_\ell}(x) = 0. \quad (454)$$

The implication also works the other way:

$$\Delta = \ell + d - 2 \quad \text{if and only if} \quad \mathcal{O} \text{ is a conserved current.} \quad (455)$$

<sup>96</sup>An elegant way to study all possible descendants at once is to go to momentum space in Lorentzian signature []. In fact, this is how unitarity bounds were originally derived by Mack. Unfortunately, this direction would take us too far afield from our discussion.

<sup>97</sup>The null state has spin  $\ell - 1$  because the unitarity bound came from  $V_{\ell-1} \subset V \otimes V_\ell$ . Something special happens for vectors in 2d, where  $V \otimes V = \mathbf{1} \oplus \mathbf{1} \oplus V_2$ , with the extra  $\mathbf{1}$  appearing because of the antisymmetric  $\epsilon_{\mu\nu}$  symbol. Unitarity then implies that  $J^\mu$  and  $\epsilon^{\mu\nu}J_\nu$  are each separately conserved.

Some important examples are global symmetry currents ( $\ell = 1$ ,  $\Delta = d - 1$ ) and the stress tensor ( $\ell = 2$ ,  $\Delta = d$ ). For CFTs in  $d \geq 3$ , the presence of currents with spin  $\ell \geq 3$  implies that the theory is free [30–32].<sup>98</sup>

#### 10.4. Only Primaries and Descendants

With a positive-definite inner product, we can now prove that all operators in unitary CFTs are linear combinations of primaries and descendants. We will use one additional physical assumption: that the partition function of the theory on  $S^{d-1} \times S^1_\beta$  is finite,

$$\mathcal{Z}_{S^{d-1} \times S^1_\beta} = \text{Tr}(e^{-\beta D}) < \infty. \quad (456)$$

This means that  $e^{-\beta D}$  is trace-class, and hence diagonalizable with a discrete spectrum (by the spectral theorem).<sup>99</sup> It follows that  $D$  is also diagonalizable, with real eigenvalues because  $D$  is Hermitian.

Now consider a local operator  $\mathcal{O}$ , and assume for simplicity it is an eigenvector of dilatation with dimension  $\Delta$ . By finiteness of the partition function, there are a finite number of primary operators  $\mathcal{O}_p$  with dimension less than or equal to  $\Delta$ . Using the inner product, we may subtract off the projections of  $\mathcal{O}$  onto the conformal multiplets of  $\mathcal{O}_p$  to get  $\mathcal{O}'$ . Now suppose (for a contradiction) that  $\mathcal{O}' \neq 0$ . Acting on it with  $K_\mu$ 's, we must eventually get zero (again by finiteness of the partition function), which means there is a new primary with dimension below  $\Delta$ , a contradiction. Thus  $\mathcal{O}' = 0$ , and  $\mathcal{O}$  is a linear combination of states in the multiplets  $\mathcal{O}_p$ .

### 11. The Operator Product Expansion

If we insert two operators  $\mathcal{O}_i(x)\mathcal{O}_j(0)$  inside a ball and perform the path integral over the interior, we get some state on the boundary. Because every state is a linear combination of primaries and descendants, we can decompose this state as

$$\mathcal{O}_i(x)\mathcal{O}_j(0)|0\rangle = \sum_k C_{ijk}(x, P)\mathcal{O}_k(0)|0\rangle, \quad (457)$$

<sup>98</sup>One must also assume the existence of exactly one stress tensor, since otherwise the theory could contain a free subsector, decoupled from the rest.

<sup>99</sup>Assuming  $e^{-\beta D}$  is trace-class may be too strong for some applications. Boundedness of  $e^{-\beta D}$  suffices for  $D$  to be diagonalizable (with a possibly continuous spectrum). An interesting example is Liouville theory, which has a divergent partition function and continuous spectrum, but still has many properties of a sensible CFT, like an OPE.

where  $k$  runs over primary operators and  $C_{ijk}(x, P)$  is an operator that packages together primaries and descendants in the  $k$ -th conformal multiplet (figure 20).

$$\text{Circle with } \mathcal{O}_i, \mathcal{O}_j = \sum_k C_{ijk} \text{Circle with } \mathcal{O}_k$$

Fig. 20. A state created by two operator insertions can be expanded as a sum of primary and descendant states.

Eq. (457) is an exact equation that can be used in the path integral, as long as all other operators are outside the sphere with radius  $|x|$ . Using the state-operator correspondence, we can write

$$\mathcal{O}_i(x_1)\mathcal{O}_j(x_2) = \sum_k C_{ijk}(x_{12}, \partial_2)\mathcal{O}_k(x_2), \quad (\text{OPE}) \quad (458)$$

where it is understood that (458) is valid inside any correlation function where the other operators  $\mathcal{O}_n(x_n)$  have  $|x_{2n}| \geq |x_{12}|$ . Eq. (458) is called the Operator Product Expansion (OPE).

We could alternatively perform radial quantization around a different point  $x_3$ , giving

$$\mathcal{O}_i(x_1)\mathcal{O}_j(x_2) = \sum_k C'_{ijk}(x_{13}, x_{23}, \partial_3)\mathcal{O}_k(x_3), \quad (459)$$

where  $C'_{ijk}(x_{13}, x_{23}, \partial_3)$  is some other differential operator (figure 21). The form (458) is usually more convenient for computations, but the existence of (459) is important. It shows that we can do the OPE between two operators whenever it's possible to draw any sphere that separates the two operators from all the others.

$$\text{Circle with } \mathcal{O}_i, \mathcal{O}_j = \sum_k C'_{ijk} \text{Circle with } \mathcal{O}_k$$

Fig. 21. It isn't necessary for one of the operators to be at the origin.

We are being a bit schematic in writing the above equations. It's possible for all the operators to have spin. In this case, the OPE looks like

$$\mathcal{O}_i^a(x_1)\mathcal{O}_j^b(x_2) = \sum_k C_{ijk}^{ab}(x_{12}, \partial_2)\mathcal{O}_k^c(x_2), \quad (460)$$

where  $a, b, c$  are indices for (possibly different) representations of  $\text{SO}(d)$ .

### 11.1. Consistency with Conformal Invariance

Conformal invariance strongly restricts the form of the OPE. For simplicity, suppose  $\mathcal{O}_i$ ,  $\mathcal{O}_j$ , and  $\mathcal{O}_k$  are scalars.

**Exercise 11.1.** By acting on both sides of (457) with  $D$ , prove that  $C_{ijk}(x, \partial)$  has an expansion of the form

$$C_{ijk}(x, \partial) \propto |x|^{\Delta_k - \Delta_i - \Delta_j} (1 + \#x^\mu \partial_\mu + \#x^\mu x^\nu \partial_\mu \partial_\nu + \#x^2 \partial^2 + \dots). \quad (461)$$

This is just a fancy way of saying we can do dimensional analysis and that  $\mathcal{O}_i$  has length-dimension  $-\Delta_i$ . We're also implicitly using rotational invariance by contracting all the indices appropriately. We could have proved this too by acting with  $M_{\mu\nu}$ .

We get a more interesting constraint by acting with  $K_\mu$ . In fact, consistency with  $K_\mu$  completely fixes  $C_{ijk}$  up to an overall coefficient. In this way, we can determine the coefficients in (461).

This computation is a little annoying (exercise!), so here's a simpler way to see why the form of the OPE is fixed, and to get the coefficients in (461). Take the correlation function of both sides of (458) with a third operator  $\mathcal{O}_k(x_3)$  (we will assume  $|x_{23}| \geq |x_{12}|$ , so that the OPE is valid),

$$\langle \mathcal{O}_i(x_1)\mathcal{O}_j(x_2)\mathcal{O}_k(x_3) \rangle = \sum_{k'} C_{ijk'}(x_{12}, \partial_2) \langle \mathcal{O}_{k'}(x_2)\mathcal{O}_k(x_3) \rangle. \quad (462)$$

The three-point function on the left-hand side is fixed by conformal invariance, and is given in (303). We can choose an orthonormal basis of primary operators, so that  $\langle \mathcal{O}_k(x_2)\mathcal{O}_{k'}(x_3) \rangle = \delta_{kk'} x_{23}^{-2\Delta_k}$ . The sum then collapses to a single term, giving

$$\frac{f_{ijk}}{x_{12}^{\Delta_i + \Delta_j - \Delta_k} x_{23}^{\Delta_j + \Delta_k - \Delta_i} x_{31}^{\Delta_k + \Delta_i - \Delta_j}} = C_{ijk}(x_{12}, \partial_2) x_{23}^{-2\Delta_k}. \quad (463)$$

This determines  $C_{ijk}$  to be proportional to  $f_{ijk}$ , times a differential operator that depends only on the  $\Delta_i$ 's. The operator can be obtained by matching the small  $|x_{12}|/|x_{23}|$  expansion of both sides of (463).

**Exercise 11.2.** Consider the special case  $\Delta_i = \Delta_j = \Delta_\phi$ , and  $\Delta_k = \Delta$ . Show

$$C_{ijk}(x, \partial) = f_{ijk} x^{\Delta-2\Delta_\phi} \left( 1 + \frac{1}{2} x \cdot \partial + \alpha x^\mu x^\nu \partial_\mu \partial_\nu + \beta x^2 \partial^2 + \dots \right), \quad (464)$$

where

$$\alpha = \frac{\Delta + 2}{8(\Delta + 1)}, \quad \text{and} \quad \beta = -\frac{\Delta}{16(\Delta - \frac{d-2}{2})(\Delta + 1)}. \quad (465)$$

### 11.2. Computing Correlators with the OPE

Equation (462) gives an example of using the OPE to reduce a three-point function to a sum of two-point functions. In general, we can use the OPE to reduce any  $n$ -point function to a sum of  $n - 1$ -point functions,

$$\langle \mathcal{O}_1(x_1) \mathcal{O}_2(x_2) \cdots \mathcal{O}_n(x_n) \rangle = \sum_k C_{12k}(x_{12}, \partial_2) \langle \mathcal{O}_k(x_2) \cdots \mathcal{O}_n(x_n) \rangle. \quad (466)$$

Recurring, we reduce everything to a sum of one-point functions, which are fixed by dimensional analysis,

$$\langle \mathcal{O}(x) \rangle = \begin{cases} 1 & \text{if } \mathcal{O} \text{ is the unit operator,} \\ 0 & \text{otherwise.} \end{cases} \quad (467)$$

This gives an algorithm for computing any flat-space correlation function using the OPE. It shows that all these correlators are determined by dimensions  $\Delta_i$ , spins, and OPE coefficients  $f_{ijk}$ .<sup>100</sup>

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<sup>100</sup>The OPE is also valid on any conformally flat manifold. The difference is that on nontrivial manifolds, non-unit operators can have nonzero one-point functions. An example is  $\mathbb{R}^{d-1} \times S^1_\beta$ , which has the interpretation as a CFT at finite temperature. By dimensional analysis, we have  $\langle \mathcal{O} \rangle_{\mathbb{R}^{d-1} \times S^1_\beta} \propto \beta^{-\Delta_\mathcal{O}} \propto T^{\Delta_\mathcal{O}}$ .

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