Phys 229ab Advanced Mathematical Methods: Conformal Field Theory

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Notes for Physics 229, 2017-2018. These notes are in progress. They were last updated on January 4, 2018.

Contents

1. Introduction

Resources

These introductory notes are heavily based on Silviu Pufu's Bootstrap 2017 lectures [] and John McGreevy's lectures on QFT [].

1.1. *QFT and emergent symmetry*

Quantum Field Theory is a universal language for theoretical physics. It shows up in many different settings, for example

- statistical physics,
- condensed matter physics,
- particle physics (SM and beyond),
- string theory/holography.

The microscopic physics in all of these settings can be quite complicated. However, often the macroscopic physics displays extra "emergent" symmetries that can help us do computations.

For example, in condensed matter physics, we are interested in describing a material made up of atoms with some lattice spacing a.[a] We refer to the detailed lattice system as the "microscopic theory." Quantum field theory is a good description at distances much larger than the lattice spacing $x \gg a$, or equivalently energy/momenta much lower than the UV cutoff $\Lambda_{UV} = 1/a$.[b] At these large scales, the discrete translation symmetry of the lattice becomes a continuous symmetry.

Similarly, in statistical physics, particle physics, and string theory, QFT can describe distance scales much larger than the characteristic scales of the microscopic theory.

Statistical systems are described by QFTs in Euclidean signature, e.g. on $\mathbb{R}^d$. Such QFTs capture properties of the equilibrium state. By contrast, condensed matter and particle systems are described by QFTs in Lorentzian signature, e.g. on $\mathbb{R}^{d-1,1}$. Such QFTs encode time-dependent quantum dynamics.

We will be interested in QFTs with rotational symmetry, by which we mean $\mathrm{SO}(d)$ symmetry in Euclidean signature and $\mathrm{SO}(d-1,1)$ symmetry in Lorentzian signature. In particle physics and string theory, this symmetry is built into the microscopic theory. However in lattice systems, rotational

[a] We ignore the possibility of lattice defects for the moment.
[b] QFTs that are valid below a UV cutoff are often called "effective field theories" (EFTs).

symmetry must be emergent. This means that correlation functions become rotationally invariant in the limit of large distances, even though microscopic correlation functions are not rotationally-invariant.[c] In particular, for condensed matter systems, the effective "speed of light" associated with $\mathrm{SO}(d-1,1)$ is an emergent property and has nothing to do with the speed of actual light. (We will see some explicit examples later.)

Under general conditions, $\mathrm{SO}(d)$-invariant Euclidean QFTs are in one-to-one correspondence with $\mathrm{SO}(d-1,1)$-invariant Lorentzian QFTs. The map between them is called Wick rotation, and we will discuss it in detail. Because of this correspondence, we can focus mostly on Euclidean QFTs, and later understand Lorentzian QFTs by Wick rotating what we learned in Euclidean signature.

1.2. *The mass gap and critical points*

So far, we are interested in theories with Poincare symmetry

$$G_{\mathrm{Poincare}} = \mathbb{R}^d \rtimes \mathrm{SO}(d). \tag{1}$$

From the point of view of long-distance physics, the most important property of a Poincare-invariant theory is its mass gap

$$m_{\mathrm{gap}} = E_1 - E_0, \tag{2}$$

where E_0, E_1 are the energies of the ground state and first excited state, respectively.[d] Theories with $m_{\mathrm{gap}} > 0$ are called "gapped."

To understand why m_{gap} is important, let us study a two-point function of a scalar local operator $\phi(x) = \phi(x^0, \mathbf{x})$. We demand that $\phi(0)$ have vanishing vacuum expectation value by subtracting off an appropriate multiple of the unit operator. By Poincare invariance, it suffices to consider $\langle 0|\phi(x^0, \mathbf{0})\phi(0)|0\rangle$ with $x^0 > 0$. The two-point function is then given by

$$\begin{aligned} \langle 0|\phi(x^0, \mathbf{0})\phi(0)|0\rangle &= \langle 0|\phi(0)e^{-Hx^0}\phi(0)|0\rangle \\ &= \sum_\psi |\langle 0|\phi(0)|\psi\rangle|^2 e^{-E_\psi x^0}. \end{aligned} \tag{3}$$

Here, H is the Hamiltonian with the vacuum energy subtracted off, and ψ runs over an orthonormal basis of eigenstates of H. To get the right-hand

[c] Emergent rotational symmetry is very familiar: we often cannot determine the orientation of a microscopic lattice using macroscopic observations. Some examples of materials *without* emergent $\mathrm{SO}(d)$ symmetry are crystals like salt. A very exotic example is the Haah code [].

[d] Some QFTs with topological order can have multiple degenerate ground states.

side, we have used $\phi(x^0, \mathbf{0}) = e^{Hx^0}\phi(0, \mathbf{0})e^{-Hx^0}$ and $H|0\rangle = 0$.[e]

The key point is that the operator e^{-Hx^0} exponentially damps states with energy $E_\psi \gg 1/x^0$. At large x^0, the correlator is dominated by ψ with the smallest nonzero eigenvalue of H, namely m_{gap}.[f]

Thus, when m_{gap} is nonzero, correlation functions of local operators fall off at least as fast as $e^{-|x|/\xi}$, where $\xi \equiv 1/m_{\text{gap}}$ is called the "correlation length." Generic statistical and condensed matter systems have microscopic correlation lengths $\xi \sim a$, or equivalently $m_{\text{gap}} \sim \Lambda_{UV}$. At long distances, they are described by QFTs whose local correlation functions vanish, called topological quantum field theories (TQFTs).

However, sometimes by tuning parameters in the microscopic Hamiltonian, we can make m_{gap} much smaller than Λ_{UV}, and even arrange for m_{gap} to vanish. Points in parameter space where $m_{\text{gap}} = 0$ are called critical points. At a critical point, the system experiences a phase transition, and develops nonzero correlations at arbitrarily large distances.[g]

Long-distance correlation functions at critical points have no intrinsic length-scale because all memory of dimensionful microscopic quantities (like the lattice spacing a) disappears when distances become arbitrarily large. For example, critical two-point functions behave as pure power laws

$$\langle\phi(x)\phi(0)\rangle = \frac{C}{|x|^{2\Delta}} \qquad \text{(critical point, } x \gg a), \tag{4}$$

where C and Δ are constants depending on ϕ. The quantity Δ is called the scaling dimension of ϕ.

1.3. *Scaling and conformal symmetry*

A more precise way to state the lack of an intrinsic length scale is to say that theories with $m_{\text{gap}} = 0$ have an emergent symmetry under rescaling

$$x^\mu \to \lambda x^\mu \qquad (\lambda > 0). \tag{5}$$

Under very general conditions (that we will discuss), critical points also display less obvious emergent symmetries called conformal transformations.

[e]Note that the Euclidean time-evolution operator is e^{-Hx^0} as opposed to the familiar e^{-iHt} in Lorentzian signature. They are related by Wick rotation $x^0 = it$. We will discuss this in much more detail in later sections.

[f]Note that the vacuum does not contribute as an intermediate state because we have demanded $\langle 0|\phi(0)|0\rangle = 0$.)

[g]If the Standard Model were like a generic condensed matter system, we might expect m_{gap} to be close to the UV cutoff, which is perhaps the GUT scale 10^{15}GeV or Planck scale 10^{18}GeV. The hierarchy problem is the problem of explaining why the Standard Model is so close to a critical point.

A conformal transformation $x \to x'(x)$ is a map that looks like a rotation and rescaling near each point,

$$\frac{\partial x'^{\mu}}{\partial x^{\nu}} = \Omega(x) R^{\mu}{}_{\nu}(x), \qquad R^{\mu}{}_{\nu} \in \mathrm{SO}(d). \tag{6}$$

An example is a special conformal transformation,

$$x^{\mu} \to \frac{x^{\mu} - b^{\mu} x^2}{1 - 2b \cdot x + b^2 x^2} \qquad (b \in \mathbb{R}^d). \tag{7}$$

In 2-dimensions, there are more exotic examples, like the one pictured in figure **??**.[h]

Here is some rough intuition for why critical points display conformal symmetry. We know that a critical theory is invariant under rescalings and rotations. If the theory is also local, in the sense that degrees of freedom at a point only interact directly with other degrees of freedom at nearby points, then the theory should also be invariant under transformations that locally look like a rescaling and rotation. This is the defining property of a conformal transformation. Turning this rough intuition into a theorem is a difficult problem (that we will discuss in more detail later). However, it seems to be true in a wide class of systems.

QFTs that are invariant under conformal symmetry are called conformal field theories (CFTs). To summarize, CFTs describe critical points where $m_{\text{gap}} = 0$. One can also understand the neighborhood of a critical point where m_{gap} is nonzero (but still $m_{\text{gap}} \ll \Lambda_{UV}$) by studying perturbations of the associated CFT.

2. Examples of critical points

So far our discussion has been very abstract, so let us introduce some examples. One of our goals will be to infer from examples a set of axioms that CFTs should satisfy. We will study these axioms in the next part of the course. Another goal will be to be more precise about how and why statistical and condensed matter systems are described by quantum field theory, and how critical points come about.

2.1. *Magnets*

Our first examples of critical points occur in magnets. Given a magnet with temperature T, we can apply a magnetic field $\vec{H}$ and measure the

[h] We will give a precise definition for what it means for a theory to be invariant under transformations like (5) and (7) later in the course.

magnetization $\vec{M}$. There are three main types of magnets in 3-dimensions, which are distinguished by their symmetries:[i]

- *Uni-axial magnet*: individual magnetic moments $\vec{\mu}$ are confined to lie along a fixed axis. Uni-axial magnets have an $O(1) = \mathbb{Z}_2$ symmetry under which $\vec{H} \to -\vec{H}$ and $\vec{M} \to -\vec{M}$.
- *XY magnet*: magnetic moments $\vec{\mu}$ are oriented in a plane. Such magnets have an $O(2)$ symmetry under which $\vec{H}$ and $\vec{M}$ rotate in the plane.
- *Heisenberg magnet*: magnetic moments $\vec{\mu}$ are unconstrained. Such magnets have $O(3)$ symmetry under which $\vec{H}$ and $\vec{M}$ transform in the vector representation.

For the moment, we will focus on the simplest case of uni-axial magnets. We denote the projections of $\vec{H}, \vec{M}$ onto the appropriate axis by H, M.

In experiments, we observe the following:

- For $T < T_c$ below the "critical temperature", the preferred state of the magnet has nonzero magnetization $M \neq 0$ when $H = 0$. In other words, the $\mathbb{Z}_2$ symmetry is spontaneously broken.
- For $T > T_c$, the magnet has $M = 0$ when $H = 0$, i.e. the $\mathbb{Z}_2$ symmetry is unbroken.

The corresponding phase diagram is pictured in figure **??**. The point

$$H = 0,\ T = T_c \tag{8}$$

is the critical point, and is described by a CFT at long distances. To reach the critical point, we must tune two parameters: H and T. Tuning $H = 0$ is easy because that is where the microscopic theory has $\mathbb{Z}_2$ symmetry. However, the value of T_c depends on the specific material.

In more detail, the behavior of the magnetization in different phases is shown in figure **??**. Close to T_c, observables exhibit so-called "scaling" behavior, characterized by various critical exponents. Let us define the dimensionless couplings

$$t \equiv \frac{T - T_c}{T_c}, \qquad h = \frac{H}{k_B T}. \tag{9}$$

Some examples of critical exponents are

[i] Here, we mean non-spacetime symmetries, usually called "global" or "flavor" symmetries. The emergent spacetime symmetry group is still the Poincare group, or the conformal group at the critical point.

- α: the heat capacity at $h = 0$ behaves as
$$C = \frac{\partial^2 F}{\partial T^2} \propto |t|^{-\alpha}. \tag{10}$$
(Here F is the free-energy.)
- β: the spontaneous magnetization behaves as
$$\lim_{H \to 0^+} M \propto (-t)^{\beta}. \tag{11}$$
- γ: the zero-field susceptibility behaves as
$$\chi = \left.\frac{\partial M}{\partial H}\right|_{H=0} \propto |t|^{-\gamma}. \tag{12}$$
- δ: the magnetization at $T = T_c$ behaves as
$$|M| \propto |h|^{1/\delta}. \tag{13}$$
- ν and η: the correlation length ξ can be measured by studying a two-point correlation functions of spins
$$G(x) = \langle s(x)s(0)\rangle - \langle s(0)\rangle^2. \tag{14}$$
Away from the critical point, $G(x) \sim e^{-|x|/\xi}$ decays exponentially. However, as $t \to 0$, the correlation length diverges as
$$\xi \propto |t|^{-\nu}. \tag{15}$$
Equivalently, the mass-gap goes to zero as $m_{\text{gap}} \propto |t|^{\nu}$. Precisely at $t = 0$, the two-point function takes the form
$$G(x) \propto \frac{1}{|x|^{d-2+\eta}}, \tag{16}$$
i.e. the spin operator has dimension $\Delta_s = \frac{d-2}{2} + \frac{\eta}{2}$.

Don't worry, I can't keep track of all these critical exponents either. We will see shortly that all of this behavior can be explained using effective field theory, scaling symmetry, and dimensional analysis.

Now, here is an amazing fact:

> **We find the same critical exponents in many different uni-axial magnets, regardless of what material they're made of.**

In fact, critical uni-axial magnets are all described by the same scale-invariant QFT at long distances. This phenomenon is called "critical universality."

2.2. *Liquid-vapor transitions*

Other critical points appear in liquid-vapor transitions. For example, the phase-diagram of water is pictured in figure **??**. Near room temperatures and pressures, there is a sharp distinction between the liquid and gas phases. However, at higher temperatures and pressures, the distinction between liquid and gas disappears at a critical point (T_c, P_c). For example, in water $T_c = 647\,\mathrm{K}$, $P_c = 374\,\mathrm{Atm}$.

Note that the critical points of magnets and water are both obtained by tuning two parameters. Comparing neighborhoods of the critical points in figures **??** and **??**, we can make the following rough analogy between water and magnets:

$$\begin{aligned} P - P_c &\sim H, \\ \rho - \rho_c &\sim M. \end{aligned} \tag{17}$$

where ρ is the density and ρ_c is the critical density.[j]

In measurements of critical water we again observe scaling behavior, for example the heat capacity behaves as

$$C \sim |t|^{-\alpha}, \tag{18}$$

and the difference in density between the liquid and gas phases behaves as

$$\rho_{\text{liquid}} - \rho_{\text{gas}} \sim (-t)^{\beta}, \tag{19}$$

where t is again given by (9). Amazingly,

> **Water and other liquid-vapor transitions have precisely the same critical exponents as uni-axial magnets.**

We say that liquid-vapor transitions are in the same "universality class" as uni-axial magnets.

[j] Water does not have a microscopic $\mathbb{Z}_2$ symmetry, but it turns out that one emerges near the critical point. Roughly speaking, the $\mathbb{Z}_2$ switches the liquid (high-density) and gas (low-density) phases. To make a more precise analogy, we should identify M with the combination of ρ and P that flips sign under the emergent $\mathbb{Z}_2$.