Computação Gráfica
Unidade 2

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DSC - Departamento de Sistemas e Computação
Grupo de Pesquisa em Computação Gráfica, Processamento de Imagens e Entretenimento Digital
http://www.inf.furb.br/gcg/

Unidade 02

- Conceitos básicos de computação gráfica
  - Estruturas de dados para geometria
  - Sistemas de coordenadas no JOGL
  - Primitivas básicas (vértices, linhas, polígonos)

  - Objetivos Específicos
    - Aplicar os conceitos básicos de sistemas de referências e modelagem geométrica em computação gráfica 2D
  - Procedimentos Metodológicos
    - Aula expositiva dialogadaMaterial programado
    - Atividades em grupo (laboratório)
  - Instrumentos e Critérios de Avaliação
    - Trabalhos práticos (avaliação 2)

1. A GeForce GTX 1080Ti é a "palavra final" em placas de vídeo

Placa de Vídeo NVIDIA GeForce GTX 1080 Ti 11 GB GDDR5X 352 Bits Asus ROG-STRIX-GTX1080TI-O11G-GAMING

KaBuM! — **R$ 4.880,79** ou 10x de R$ 574,21 — Ir à loja

Submarino — **R$ 5.290,00** ou 10x de R$ 529,00 — Ir à loja

shoptime — **R$ 5.290,00** ou 10x de R$ 529,00 — Ir à loja

Ver mais sobre este produto

E se você quer investir pesado e está procurando a melhor placa de vídeo do momento, a GeForce GTX 1080Ti oferece a tecnologia mais avançada, com 11GB de memória dedicada e um desempenho fora de série. Para suportar todo esse poder de processamento, é essencial que ela seja combinada com outros componentes de ponta, como os processadores **i7 7700K** ou **Ryzen 7 1800X**, uma combinação que permite fazer modelagens em 3D com uma performance até 20% superior em relação à GTX 1080, o que é um resultado surpreendente, considerando o alto poder de processamento dessas unidades gráficas. Ela também é a única placa de vídeo que consegue manter taxas próximas a 60FPS para quem é alucinado por gráficos e quer jogar em 4K.

Características da placa de vídeo:

- Memória dedicada: 11GB GDDR5X
- Conexões: DisplayPort, DVI e HDMI
- Compatível com *G-Sync*
- Ótima performance em jogos "Triplo A" (4K) e em Realidade Virtual

GEFORCE GTX 1080 Ti

Especificações do mecanismo da placa de vídeo:

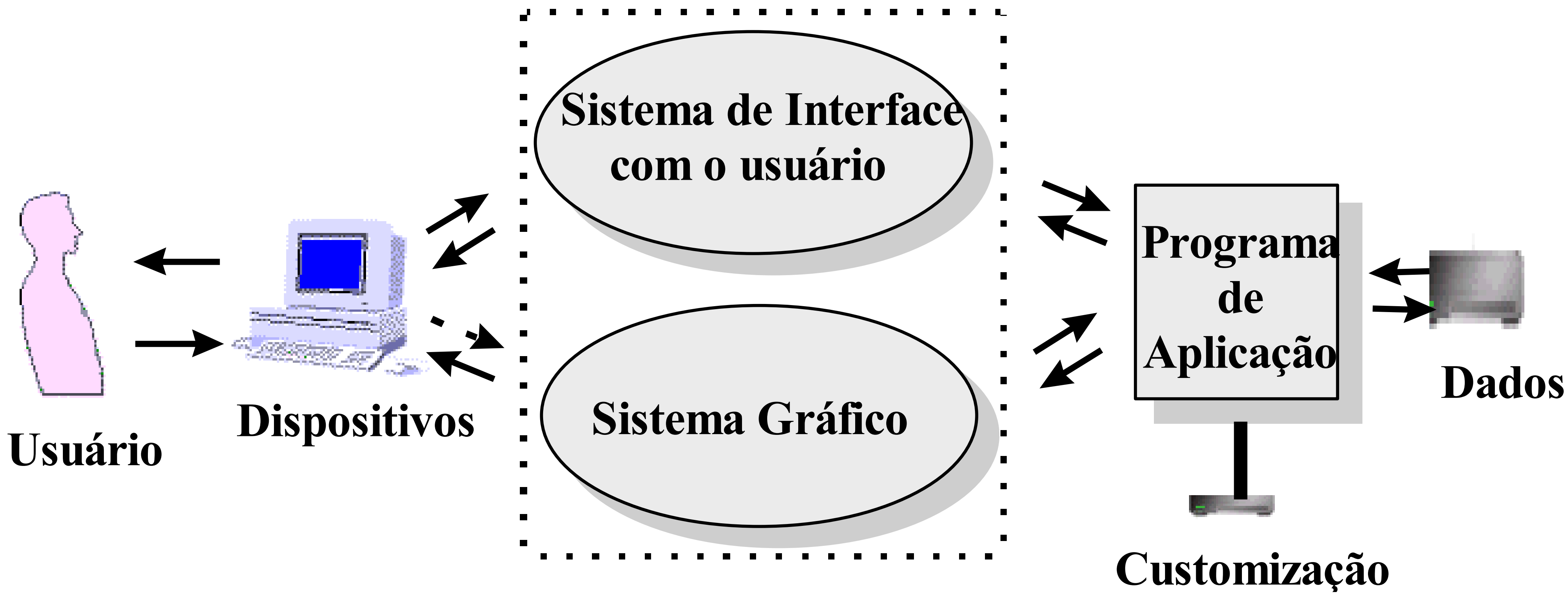
NVIDIA CUDA® Cores	$\boxed{3584}$
Clock básico (MHz)	1582

Especificações de memória:

velocidade da memória	11 Gbps
Configuração de memória padrão	11 GB GDDR5X
Largura da interface de memória	352-bit
Largura de banda de memória (GB/s)	484

https://www.nvidia.com/pt-br/geforce/products/10series/geforce-gtx-1080-ti/

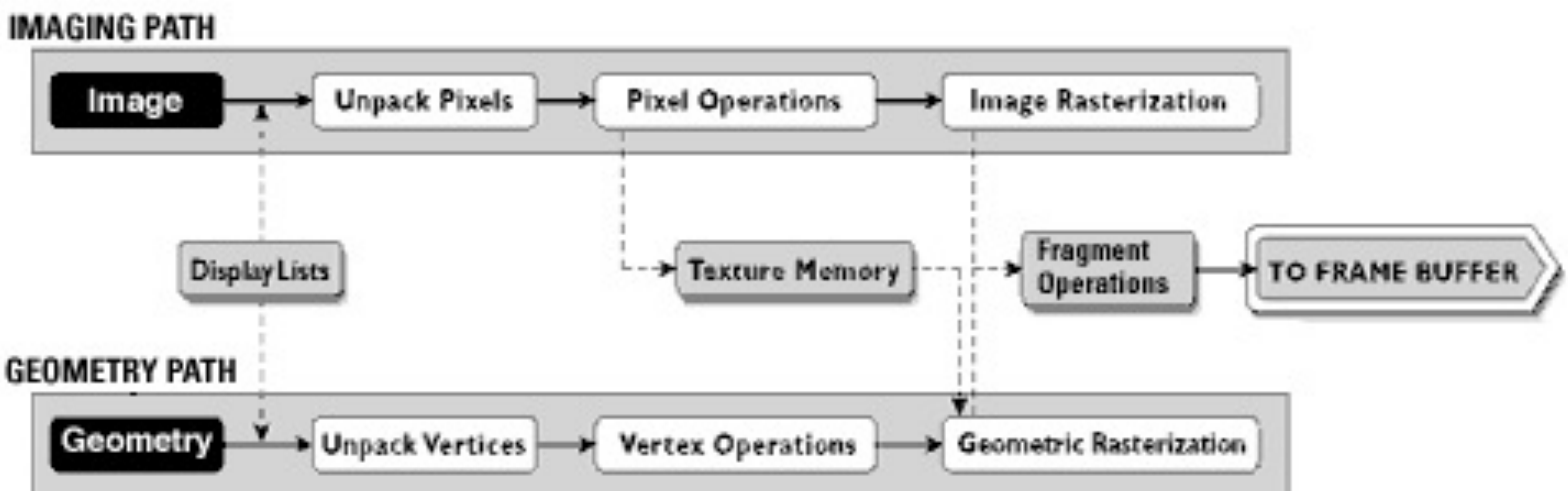
Software de interface para o hardware gráfico



OpenGL - Open Graphics Library

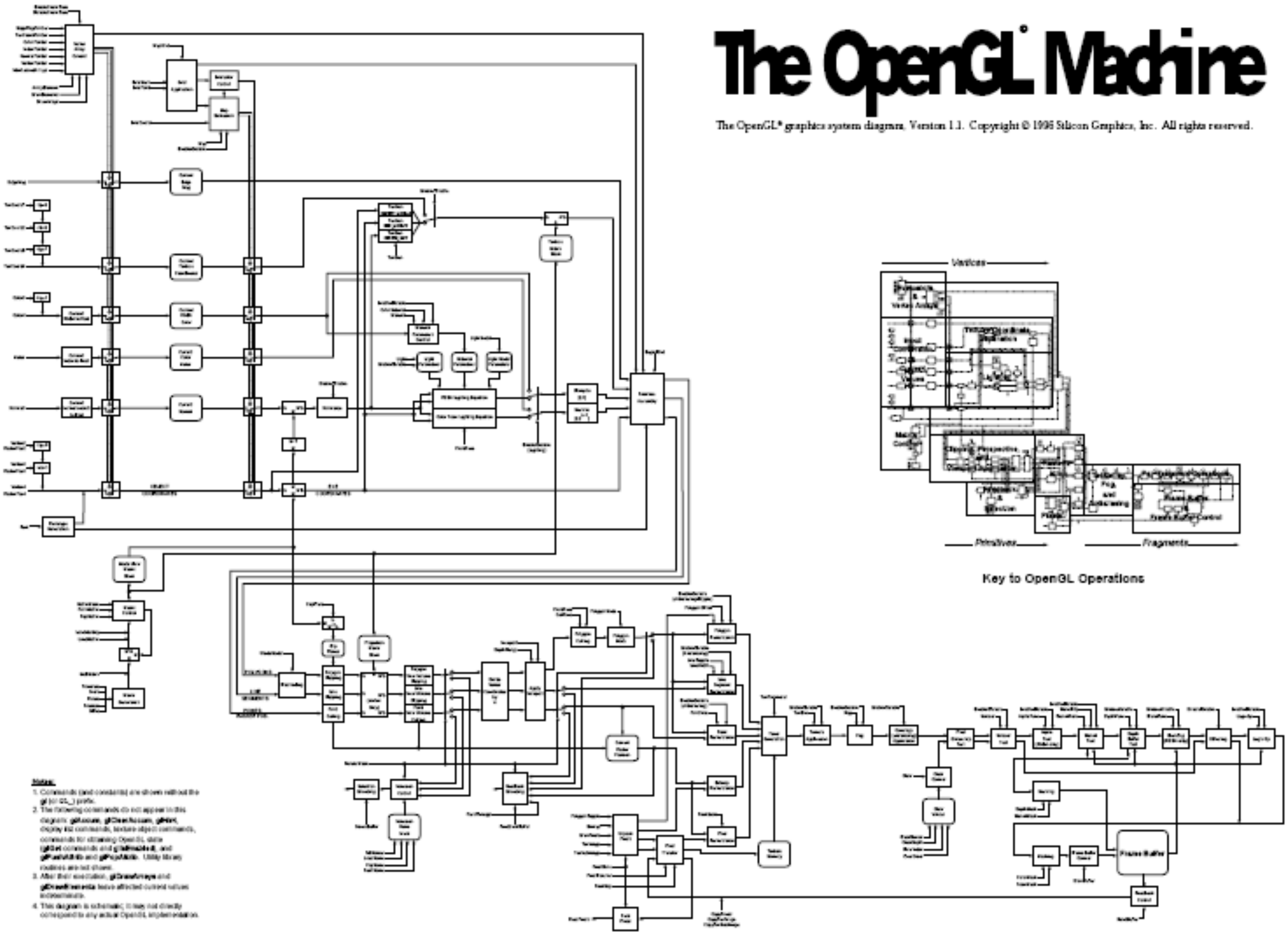
- **Interface**: aplicações de “renderização” gráfica
  - imagens coloridas de alta qualidade
    - primitivas geométricas (2D e 3D) e
    - por imagens
  - independência de sistemas de janelas
  - independência de sistemas operacionais
  - compatível com quase todas as arquiteturas
  - interface gráfica dominante

OpenGL - Open Graphics Library



http://www.opengl.org/about/overview/

– renderização
  - primitivas geométricas (2D e 3D) e
  - por imagens


The OpenGL® Machine
The OpenGL® graphics system diagram, Version 1.1. Copyright © 1996 Silicon Graphics, Inc. All rights reserved.
Vertices
Primitives
Fragments
Key to OpenGL Operations

OpenGL – “Renderizador”

- Primitivas geométricas
  - pontos, linhas e polígonos
- Primitivas de imagens
  - imagens e *bitmaps*
  - canais independentes: geometria e imagem
    - ligação via **mapeamento de textura**
- “Renderização” dependente do estado
  - cores, materiais, fontes de luz, etc.

OpenGL - Sistema de Janelas

- Trata apenas de “renderização”
  - independente do sistema de janelas
    - X, Win32, Mac O/S
  - não possui funções de entrada
- Necessita interagir com o sistema operacional e o sistema de janelas
  - interface dependente do sistema é mínima
    - realizada através de bibliotecas adicionais : GLX, AGL, WGL

OpenGL - GLU, OpenGL Utility Library

- Funções para auxiliar a tarefa de produzir imagens complexas
  - manipulação de imagens
  - polígonos não-convexos
  - curvas
  - superfícies
  - esferas
  - etc.

OpenGL - GLUT, OpenGL Utility Toolkit

- API de janelas para o OpenGL
  - independente do sistema de janelas
  - indicado para programas:
    - pequeno e médio porte
  - processamento orientado à chamada de eventos (*callbacks*)
  - dispositivos de entrada
  - não pertence oficialmente ao OpenGL

API: Interface para Programação de Aplicações

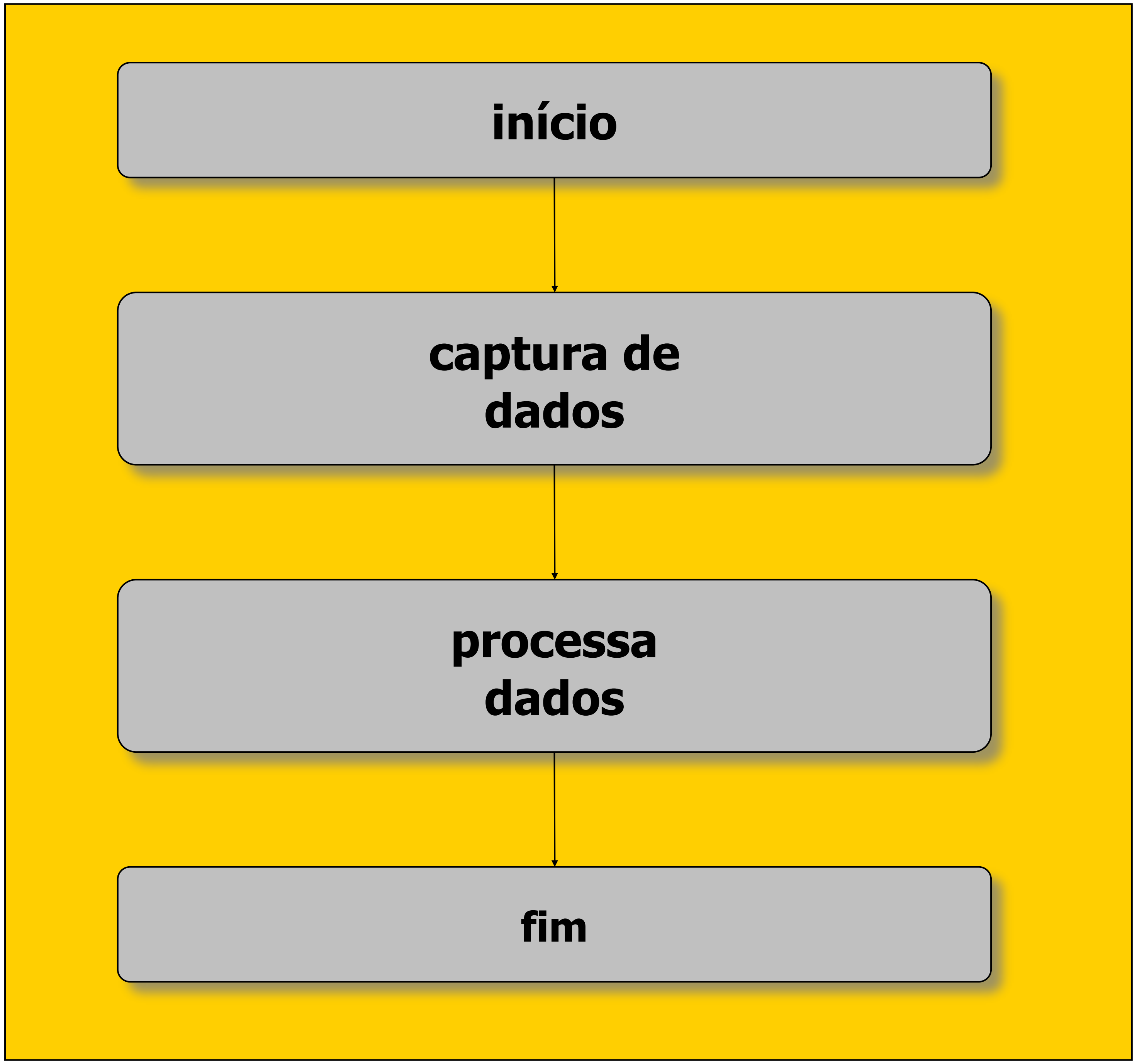
OpenGL - Prefixos

- OpenGL
  - gl, GL, GL_
    - para comandos, tipos e constantes, respectivamente
- GLU
  - glu, GLU, GLU_
- GLUT
  - glut, GLUT, GLUT_

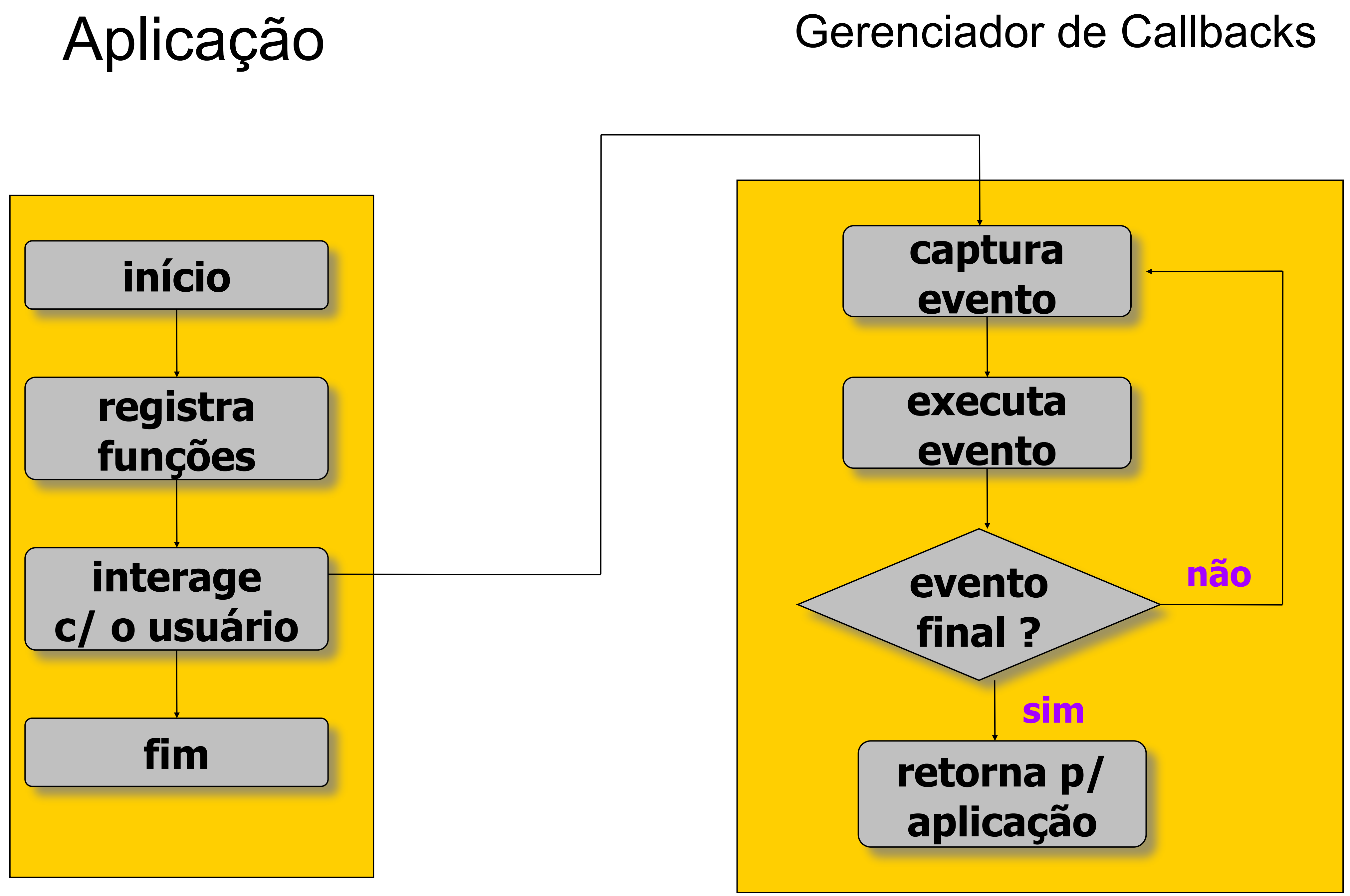
OpenGL -, Passos Básicos

- Configurar e abrir janela (*canvas*)
- Inicializar o estado do OpenGL
- Registrar funções de entrada de *callback*
  - desenho ("renderização")
  - redimensionamento do *canvas*
  - entrada : mouse, teclado, etc.

Programação Convencional



Programação por Eventos



OpenGL: exemplos CG-N2

constantes.h	Algumas constantes e rotinas usadas em todos os códigos
CG-N2_HelloWorld	Exemplo simples usando OpenGL para desenhar um segmento de reta e tendo como referência o SRU
CG-N2_Teclado	Exemplo usando o *CallBack* do teclado no OpenGL
CG-N2_Mouse	Exemplo usando o *CallBack* do mouse no OpenGL
CG-N2_OnIdle	Exemplo usando o *CallBack OnIdle* (*thread*) no OpenGL
CG-N2_Point4D	Exemplo usando a classe Point4D (V-ART) para manipular um ponto no espaço 2D
CG-N2_BBox	Exemplo usando a classe BoundingBox (V-ART) para tratar a BBox de um objeto gráfico

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Exemplos Projetos+fontes

http://gcg.inf.furb.br/cg/e2j

__GIT __

https://bitbucket.org/gcgfurb/gcg-cg

OpenGL - Especificação de Primitivas Geométricas

- primitivas são especificadas usando

  glBegin(**tipo_primitiva**);

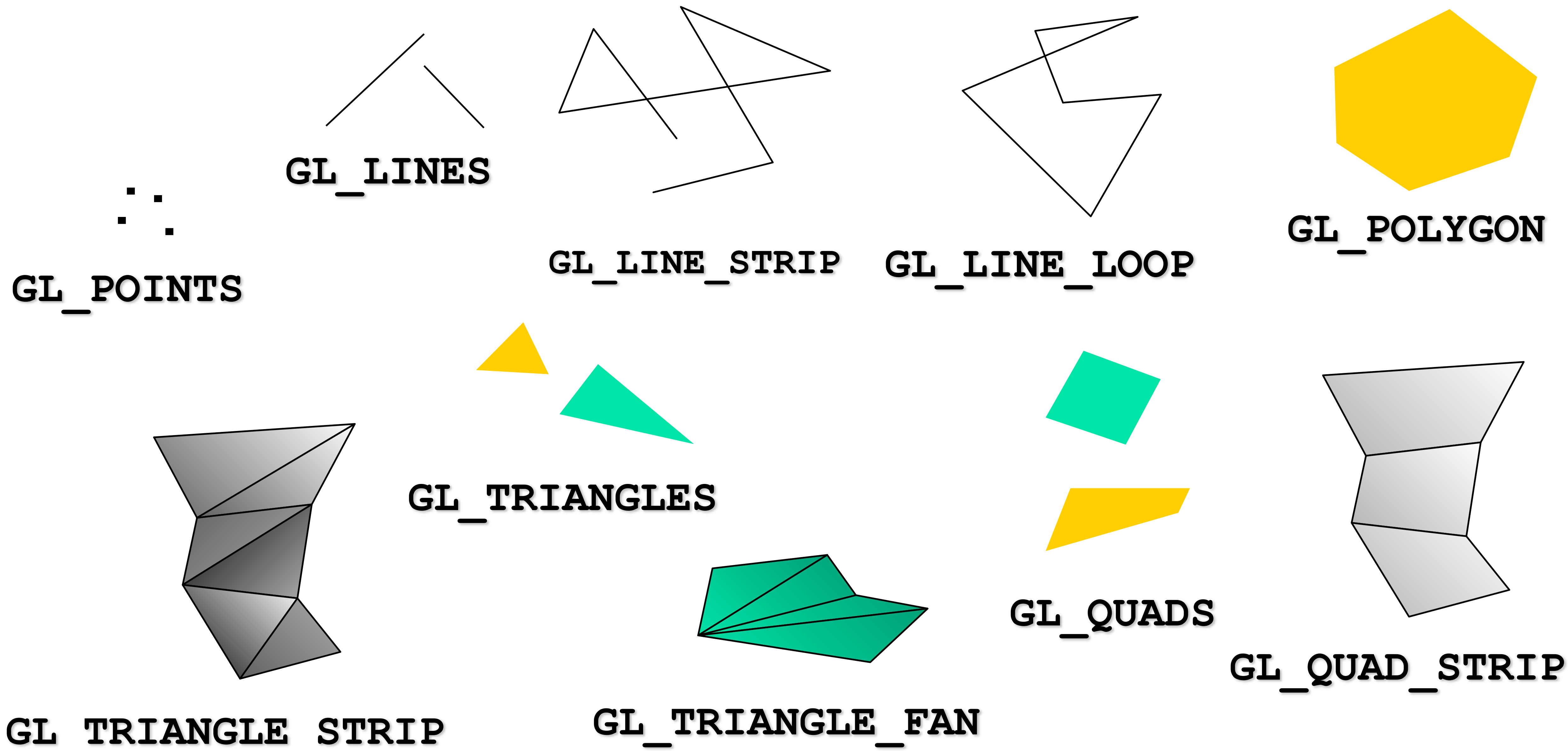
  glEnd( );

  – **tipo_primitiva**: especifica como os vértices serão agrupados

```
gl.glColor3f( 0.0f, 0.0f, 0.0f );
gl.glBegin( GL.GL_LINES );
        gl.glVertex2f(  0.0f,   0.0f );
        gl.glVertex2f( 20.0f, 20.0f );
glEnd();
```

OpenGL - Primitivas Geométricas

Especificadas por vértices



OpenGL - Formato, Especificação do Vértice

glVertex3fv(*v*)

número de componentes

```
2 - (x,y)
3 - (x,y,z)
4 - (x,y,z,w)
```

tipo do dado

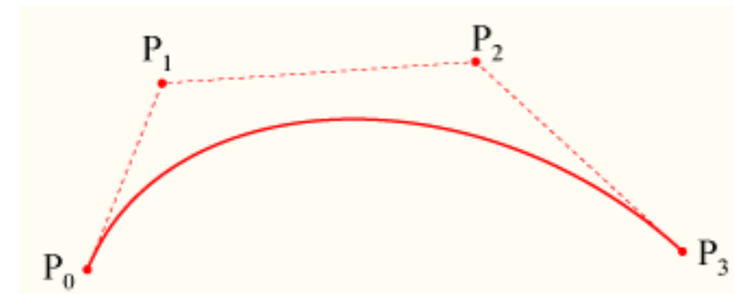
```
b  - byte
ub - unsigned byte
s  - short
us - unsigned short
i  - int
ui - unsigned int
f  - float
d  - double
```

vetor

omitir "v" para forma escalar

```
glVertex2f( x, y )
```

Splines



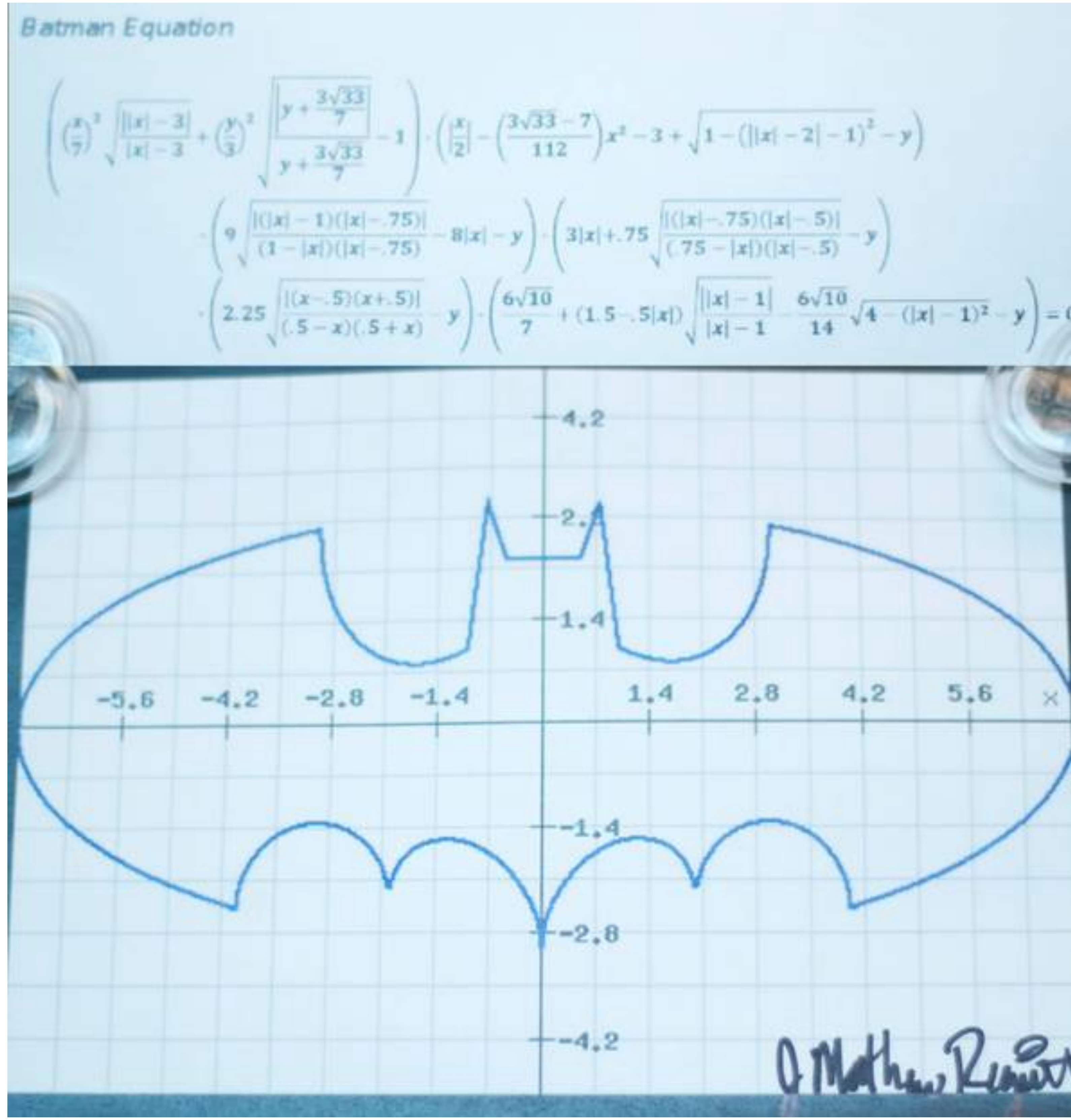
- Splines (ou curva polinomial)
  - origem:
    - desenvolvida: De Casteljau em 1957 (P. De Casteljau, Citroen)
    - formalizado: Bézier 1960 (Pierre Bézier)
    - aplicações CAD/CAM
  - pontos de controle
  - bastante utilizada em modelagem tridimensional

178379
005.1, Z91em, MO (Anote para localizar o material)
Zoz, Jeverson
Estudo de metodos e algoritmos de Splines Bezier, Casteljau e B-Spline /Jeverson Zoz. - 1999. xii, 64p. :il.
Orientador: Dalton Solano dos Reis.

195268
006.6, S586pt, MO (Anote para localizar o material)
Silva, Fernanda Andrade Bordallo da
Prototipo de um ambiente para geracao de superficies 3D com uso de Spline Bezier /Fernanda Andrade Bordallo da Silva. - 2000. ix, 51p. :il.
Orientador: Dalton Solano dos Reis.

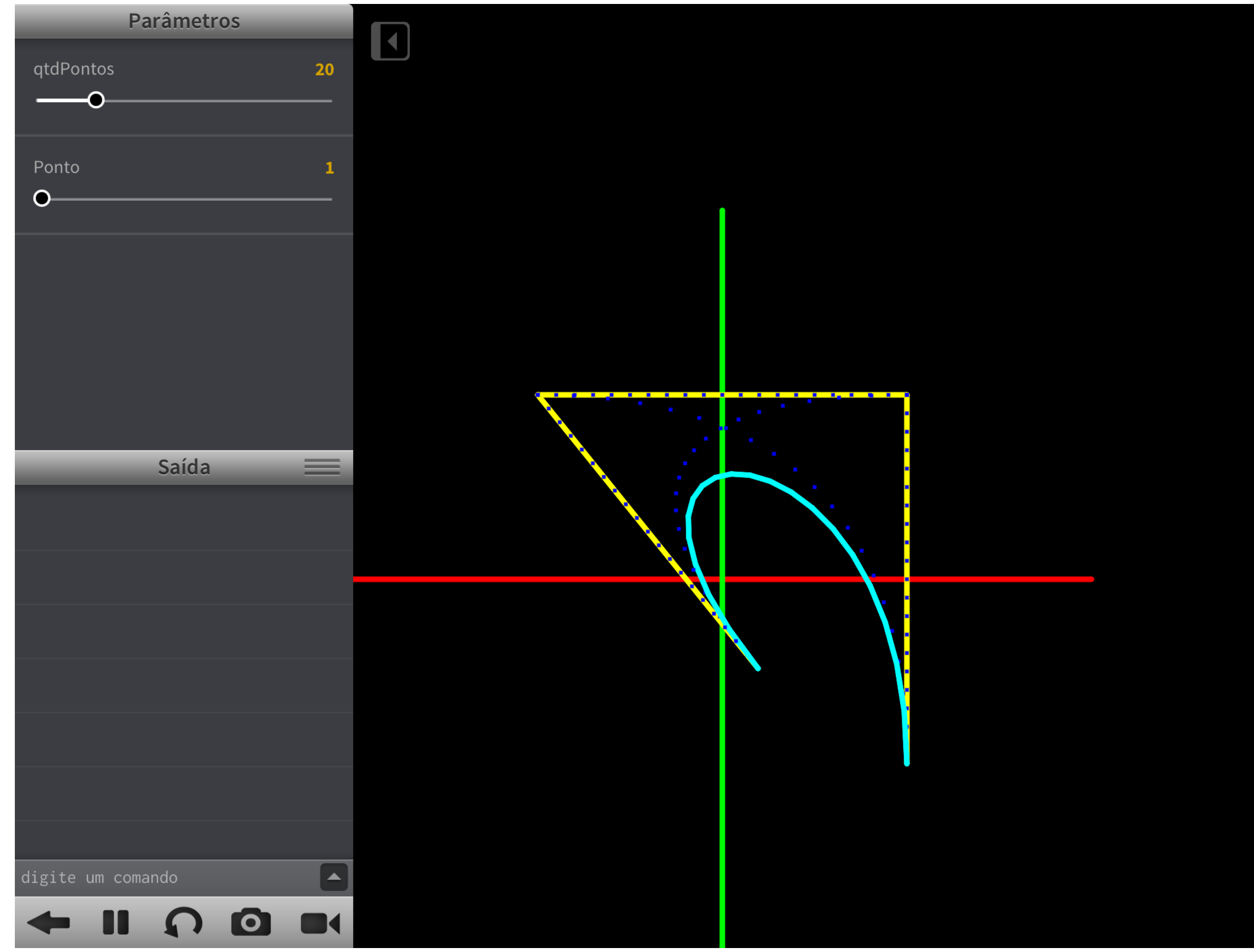
Splines

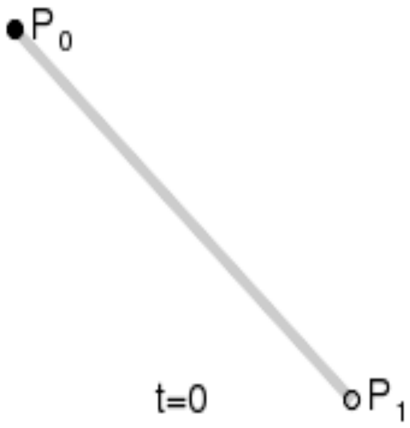
Tudo pode ser modelado por fórmulas, o problema é o custo envolvido

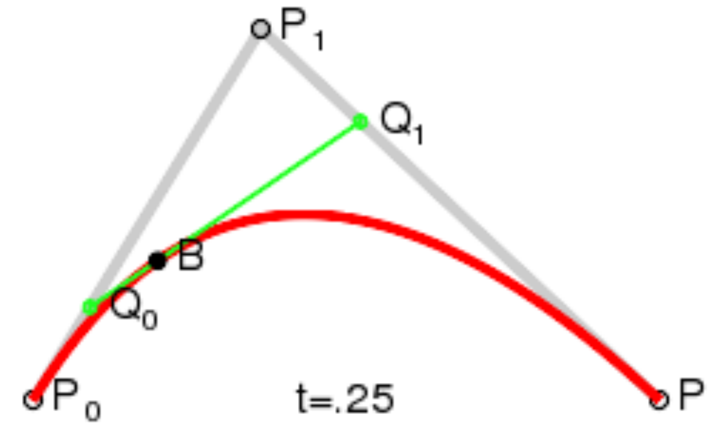


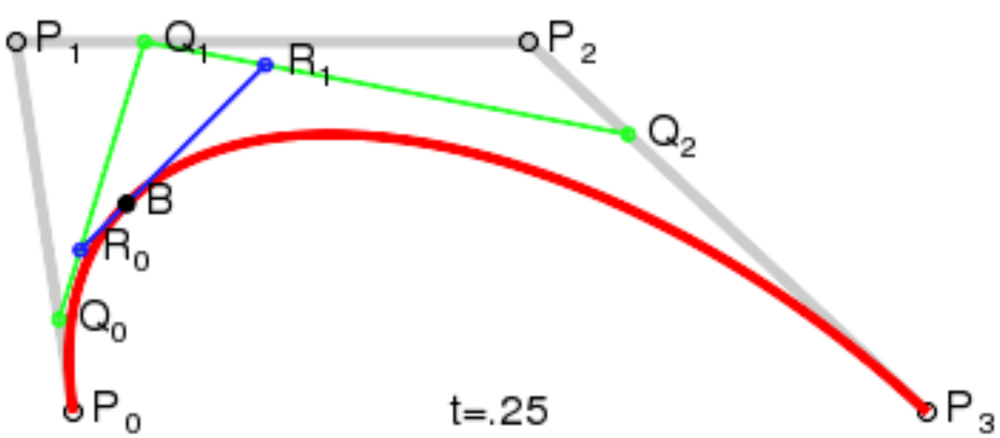
http://blog.wolframalpha.com/data/uploads/2013/07/Batman_lamina_-_Wolfram_Alpha.png

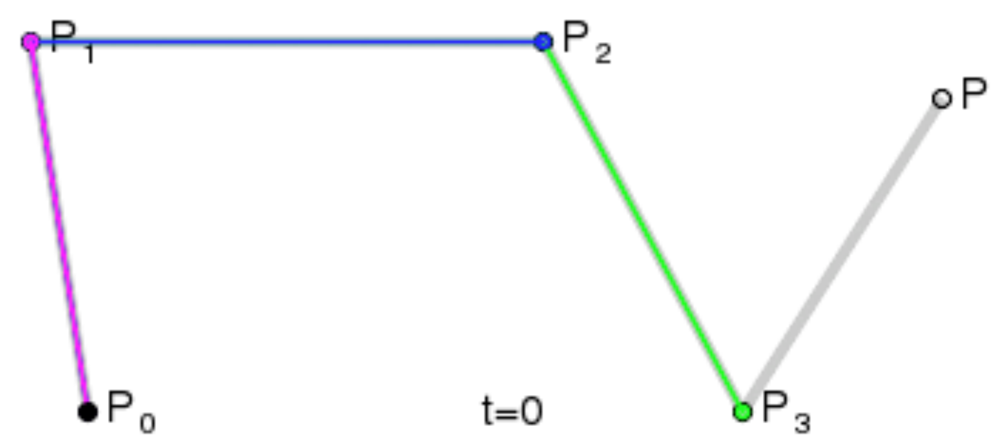
```
end

function SPLINE_Inter(A,B,t,desenha)
    R = vec2(0,0)
    R.x = A.x + (B.x - A.x) * t/qtdPontos
    R.y = A.y + (B.y - A.y) * t/qtdPontos
    if desenha == 1 then
        stroke(0, 0, 255)
        rect(R.x-2,R.y-2,4,4)
    end
    return R
end

function SPLINE_Desenha()
    if CurrentTouch.state == MOVING then
        ListaPtos[Ponto].x = CurrentTouch.x
        ListaPtos[Ponto].y = CurrentTouch.y
    end
    Pant = ListaPtos[1]
    for t = 0, qtdPontos do
        P1P2 = SPLINE_Inter(ListaPtos[1],ListaPtos[2],t,1)
        P2P3 = SPLINE_Inter(ListaPtos[2],ListaPtos[3],t,1)
        P3P4 = SPLINE_Inter(ListaPtos[3],ListaPtos[4],t,1)
        P1P2P3 = SPLINE_Inter(P1P2,P2P3,t,1)
        P2P3P4 = SPLINE_Inter(P2P3,P3P4,t,1)
        stroke(0,255,255)
        P1P2P3P4 = SPLINE_Inter(P1P2P3,P2P3P4,t,0)
        line(Pant.x,Pant.y,P1P2P3P4.x,P1P2P3P4.y)
        Pant = P1P2P3P4
    end

end
```

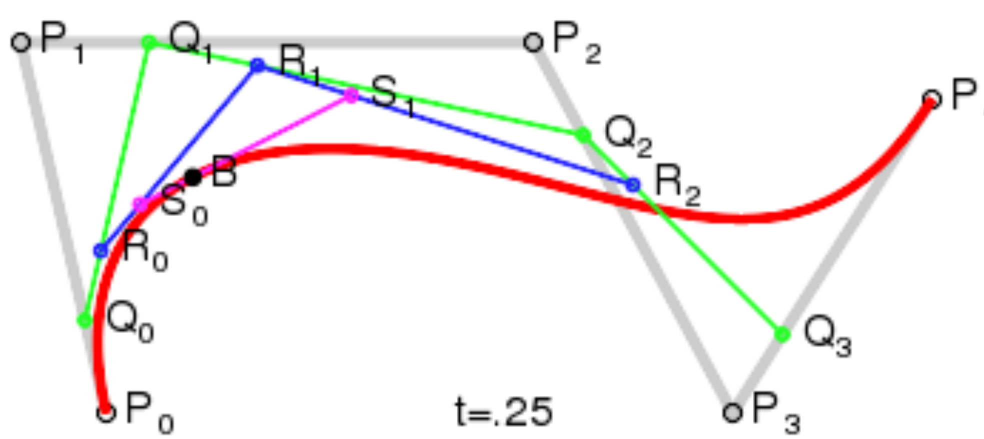






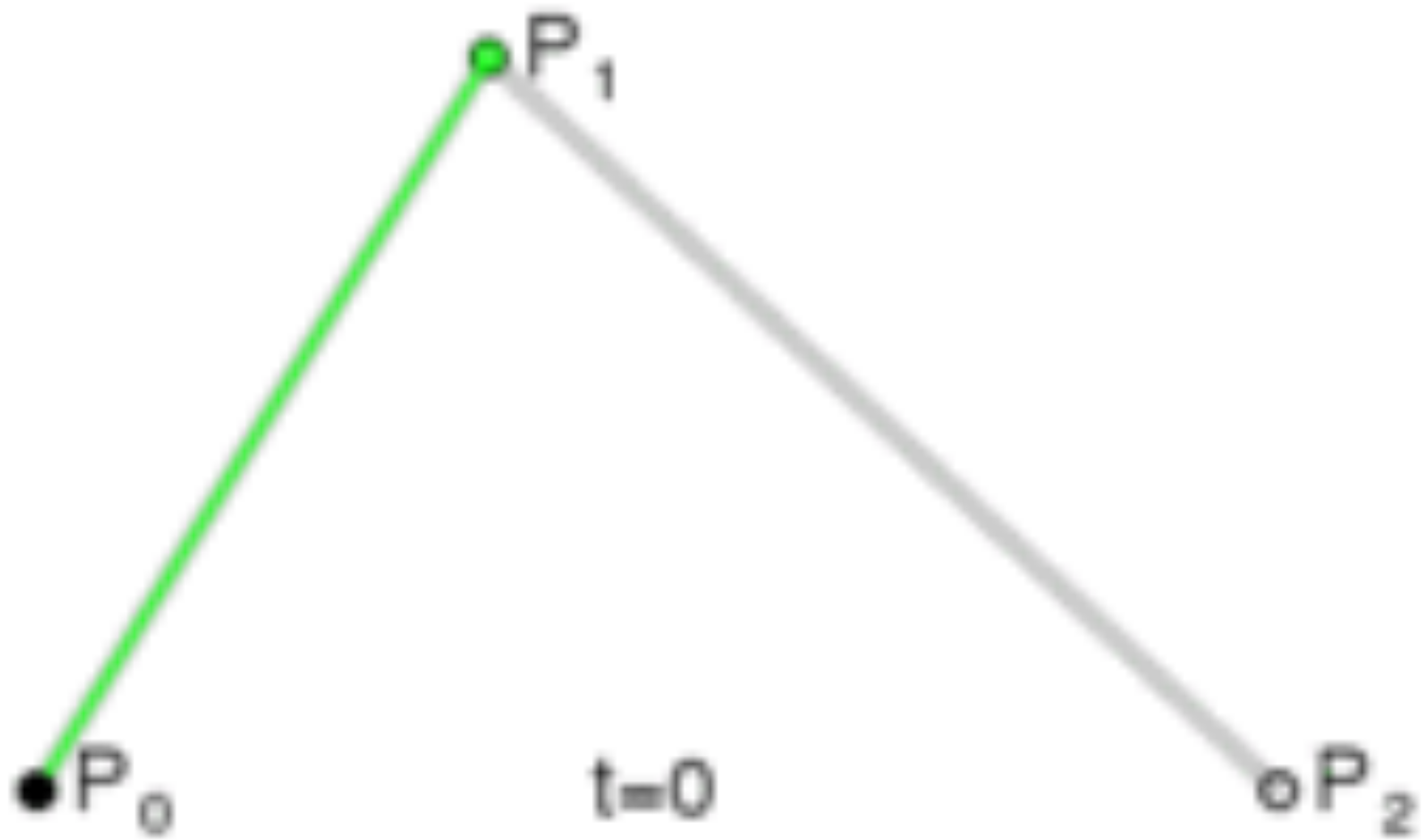




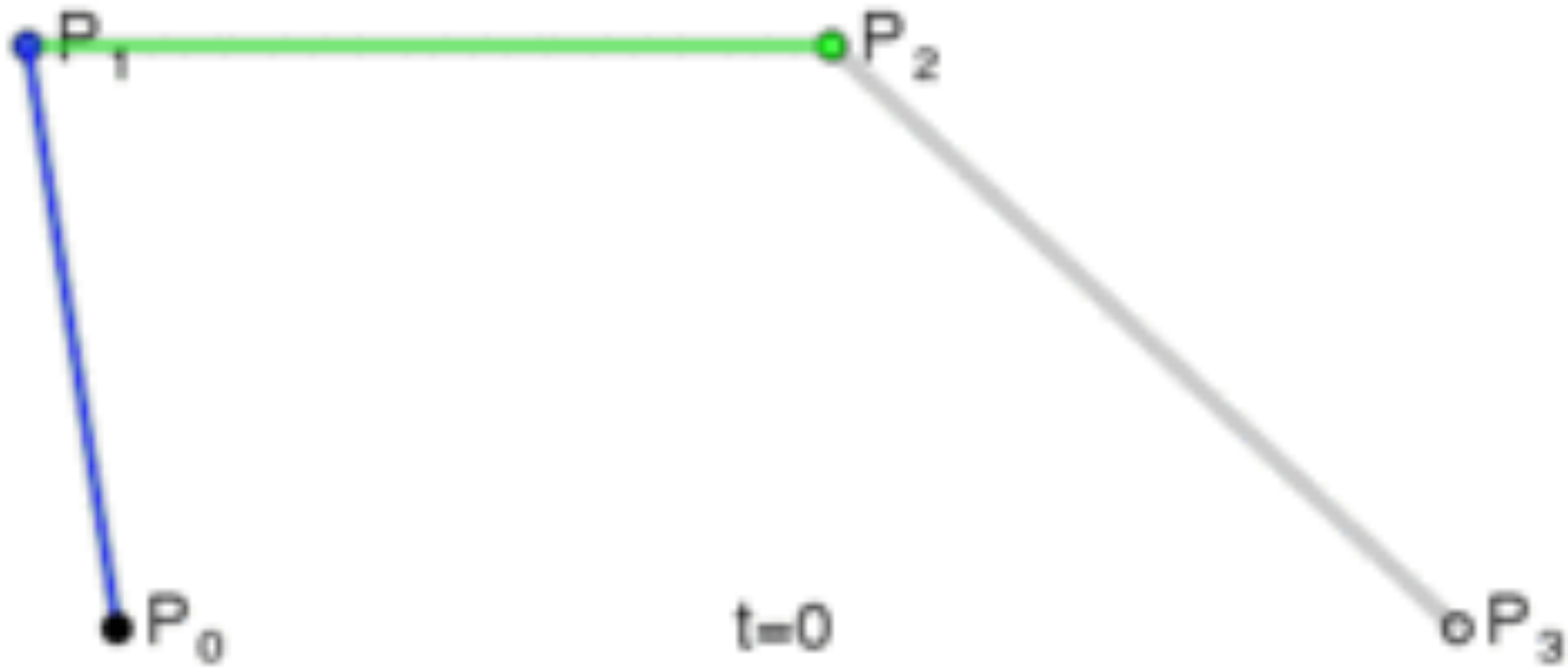


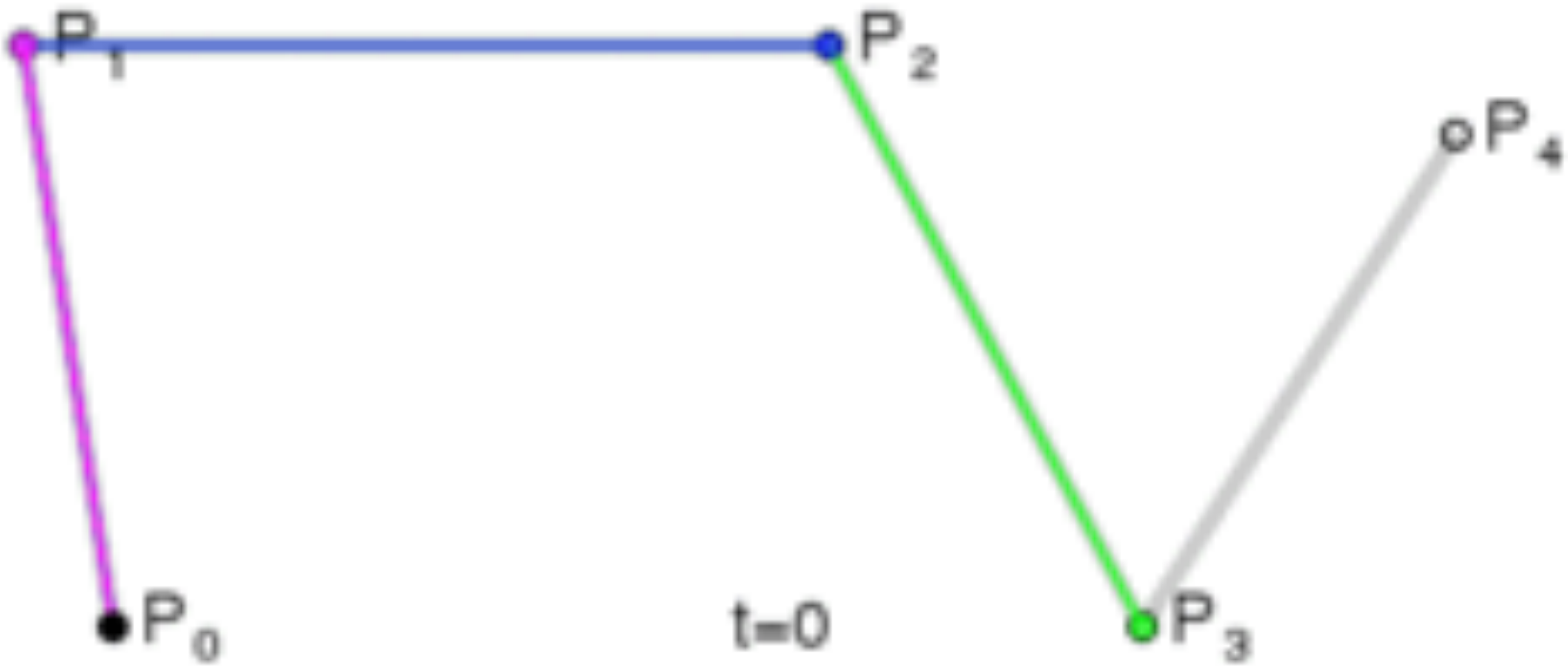
http://www.ibiblio.org/e-notes/Splines/Intro.htm

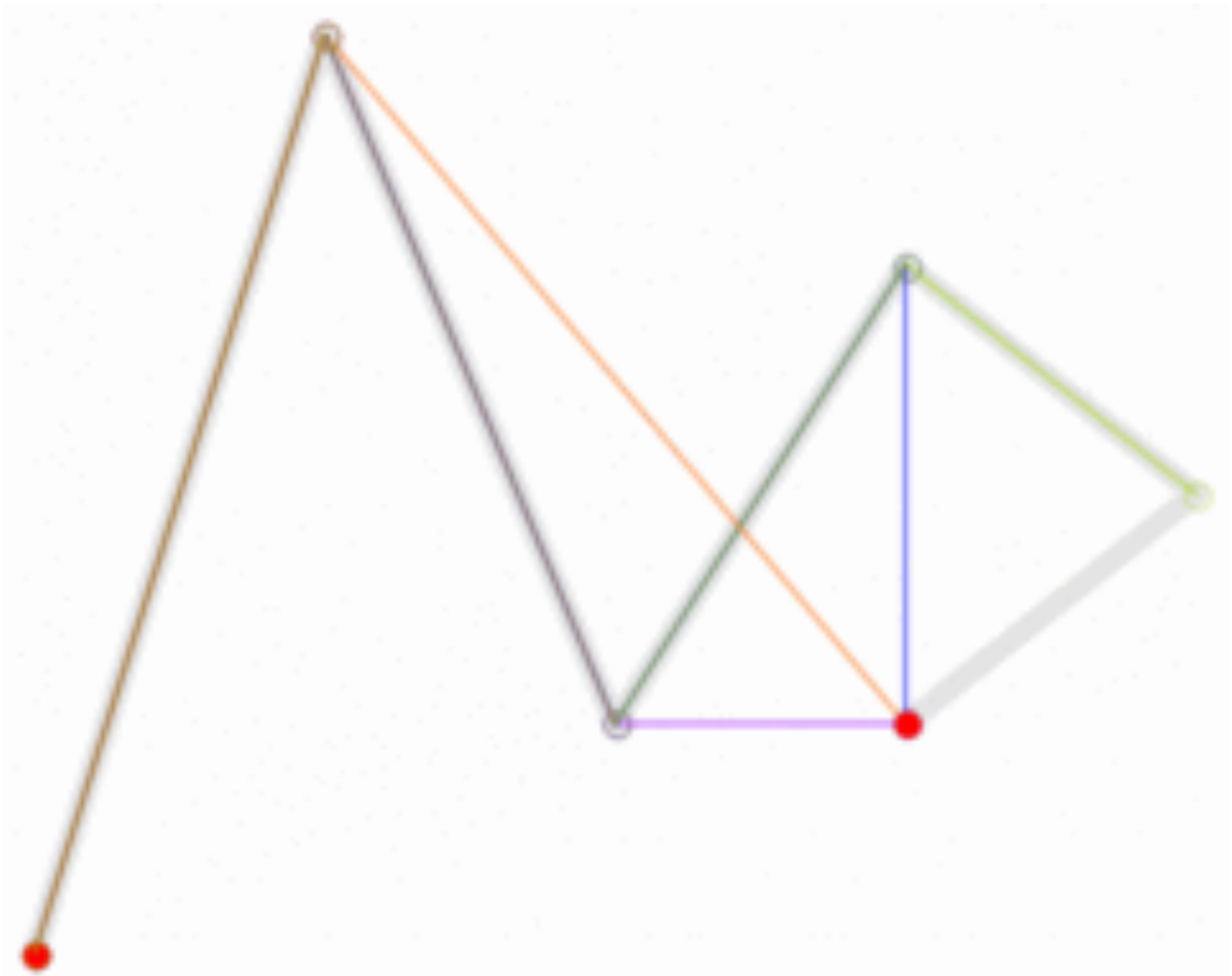
http://en.wikipedia.org/wiki/B%C3%A9zier_curve



P1
P0
t=0
P2

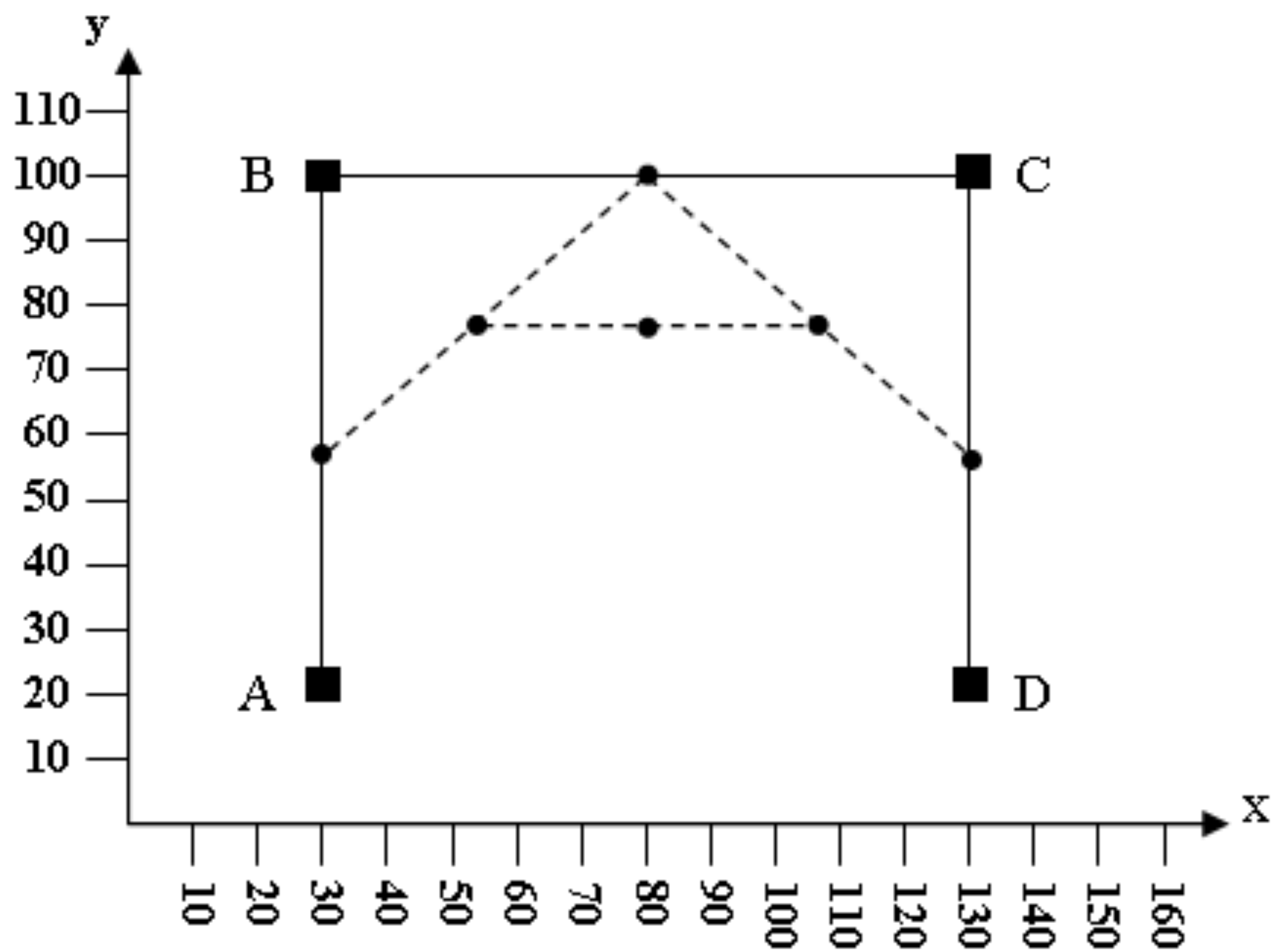

P_1
P_2
P_0
t=0
P_3

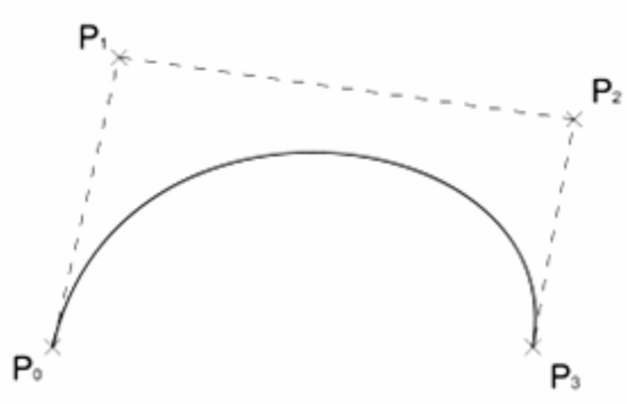

P1
P2
P4
P0
t=0
P3



Splines (Casteljau)

Para o primeiro ponto calculado, t = 0,5: *x=80* e *y=100*





Splines (Casteljau)

- Segue os passos:
  - Inicialmente devem-se definir os pontos de controle (poliedro de controle);
  - Calcular o ponto pertencente à *spline*;
  - Os pontos intermediários são utilizados para definir dois novos poliedros de controle, que deverão ser usados num processo recursivo.
- Expressão de Cálculo:

$$\frac{\dfrac{\dfrac{A_x + B_x}{2} + \dfrac{B_x + C_x}{2}}{2} + \dfrac{\dfrac{B_x + C_x}{2} + \dfrac{C_x + D_x}{2}}{2}}{2} \qquad \frac{\dfrac{\dfrac{A_y + B_y}{2} + \dfrac{B_y + C_y}{2}}{2} + \dfrac{\dfrac{B_y + C_y}{2} + \dfrac{C_y + D_y}{2}}{2}}{2}$$

Splines (Bezier)

$$\mathbf{B}(t) = (1-t)^3\mathbf{P}_0 + 3t(1-t)^2\mathbf{P}_1 + 3t^2(1-t)\mathbf{P}_2 + t^3\mathbf{P}_3 \ , \ t \in [0,1].$$

$B_x(0,5) = 0,125 * 30 + 0,375 * 30 + 0,375 * 130 + 0,125 * 130 = 80$
$B_y(0,5) = 0,125 * 20 + 0,375 * 100 + 0,375 * 130 + 0,125 * 20 = 100$

Pesos	**0,000**	0,100	0,200	0,300	0,400	**0,500**	0,600	0,700	0,800	0,900	**1,000**
P0	**1,000**	0,729	0,512	0,343	0,216	**0,125**	0,064	0,027	0,008	0,001	**0,000**
P1	**0,000**	0,243	0,384	0,441	0,432	**0,375**	0,288	0,189	0,096	0,027	**0,000**
P2	**0,000**	0,027	0,096	0,189	0,288	**0,375**	0,432	0,441	0,384	0,243	**0,000**
P3	**0,000**	0,001	0,008	0,027	0,064	**0,125**	0,216	0,343	0,512	0,729	**1,000**
Soma	1,000	1,000	1,000	1,000	1,000	1,000	1,000	1,000	1,000	1,000	1,000

Pesos	0,000	0,100	0,200	0,300	0,400	0,500	0,600	0,700	0,800	0,900	1,000
P0	1,000	0,729	0,512	0,343	0,216	0,125	0,064	0,027	0,008	0,001	0,000
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P2	0,000	0,027	0,096	0,189	0,288	0,375	0,432	0,441	0,384	0,243	0,000
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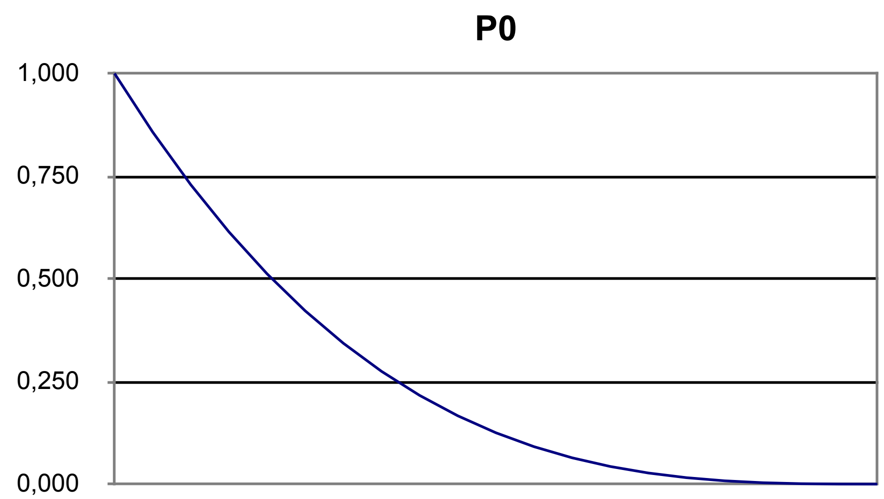
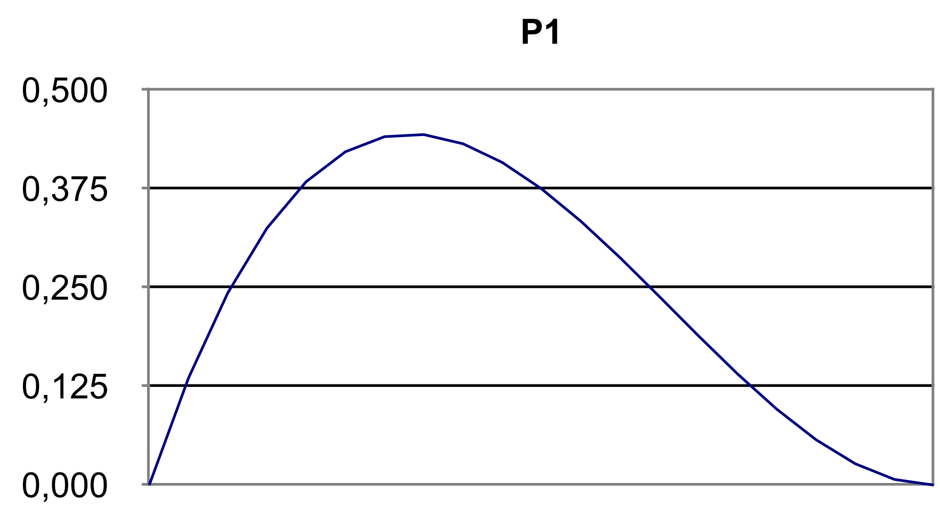
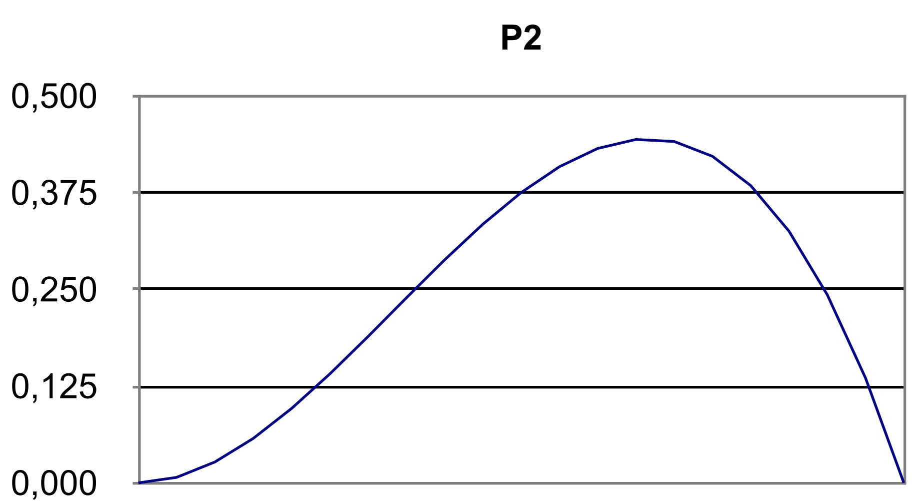
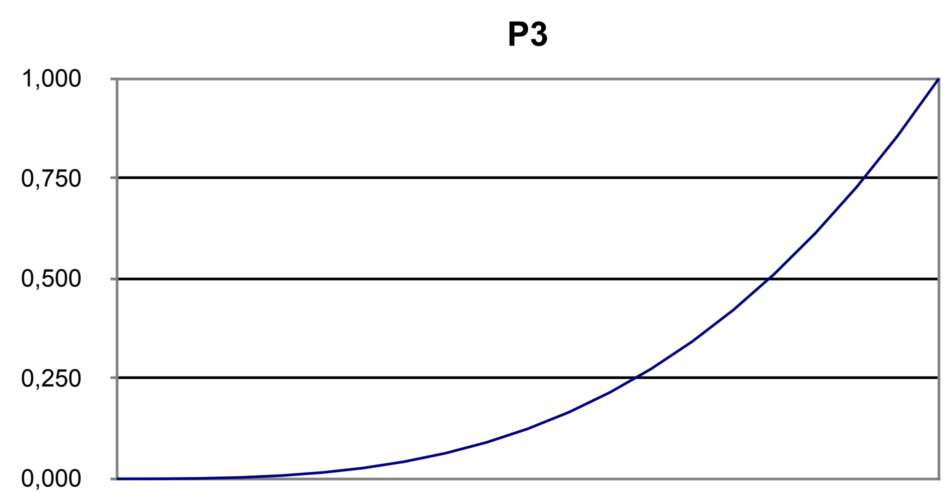
Soma	1,000	1,000	1,000	1,000	1,000	1,000	1,000	1,000	1,000	1,000	1,000









X1
X2
X3
X4

Xr1 = x1 + (x2 - x1)t
Xr2 = x2 + (x3 - x2)t
Xr3 = x3 + (x4 - x3)t

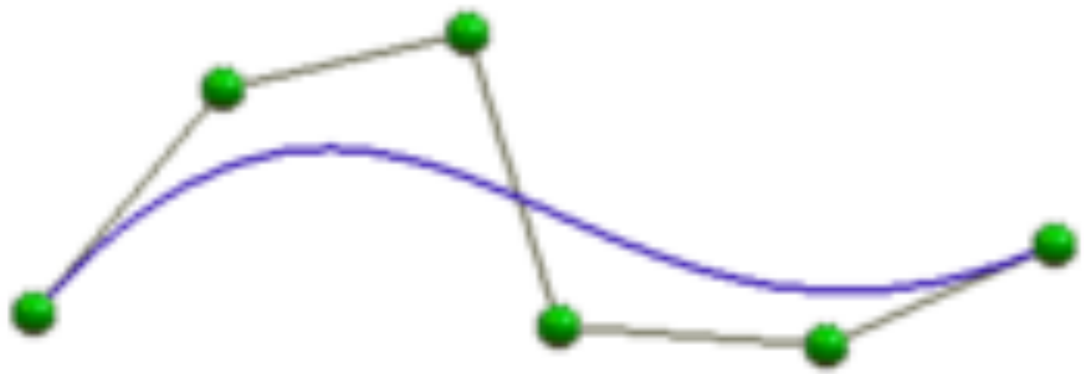
Xrr1 = Xr1 + (Xr2 - Xr1)t

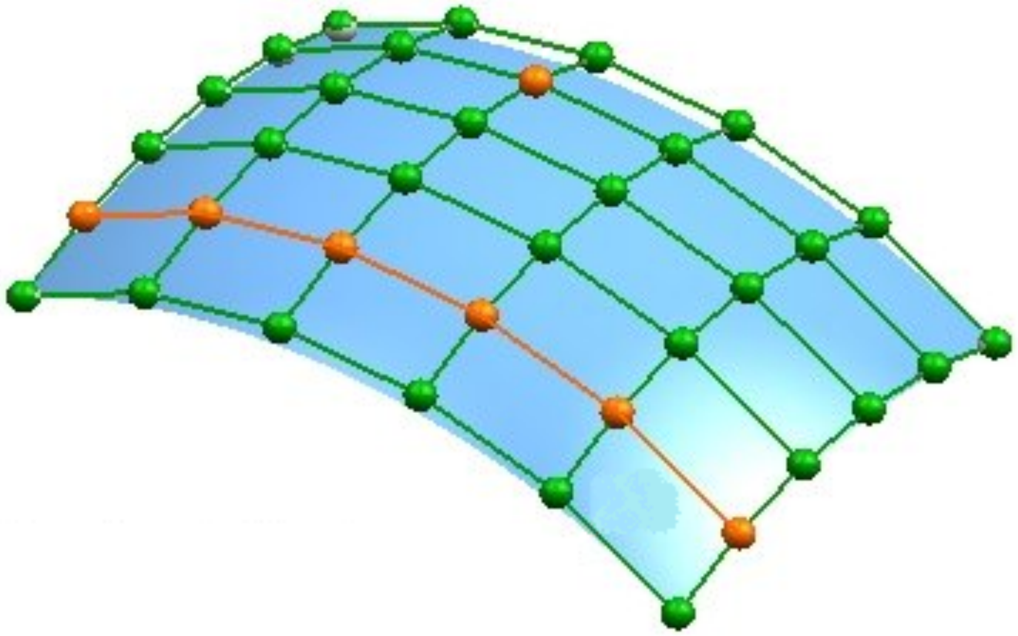
Xrr1 = (x1 + (x2 - x1)t) + ((x2 + (x3 - x2)t) - (x1 + (x2 - x1)t))t
Xrr1 = (x1 + x2t - x1t) + (x2 + x3t - x2t)t + (-x1 - x2t + x1t)t
Xrr1 = x1 + x2t - x1t + x2t + x3t≤ - x2t≤ - x1t - x2t≤ + x1t≤
Xrr1 = x1 + 2(x2 - x1)t + (x3 - 2x2 + x1)t≤

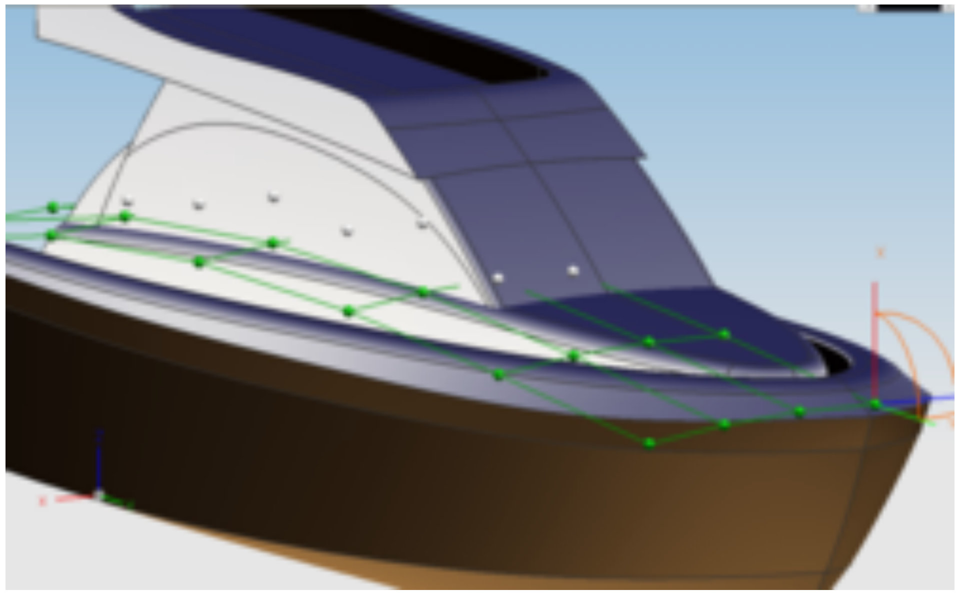
Xrr2 = Xr2 + (Xr3 - Xr2)t
Xrr2 = x2 + 2(x3 - x2)t + (x4 - 2x3 + x2)t≤

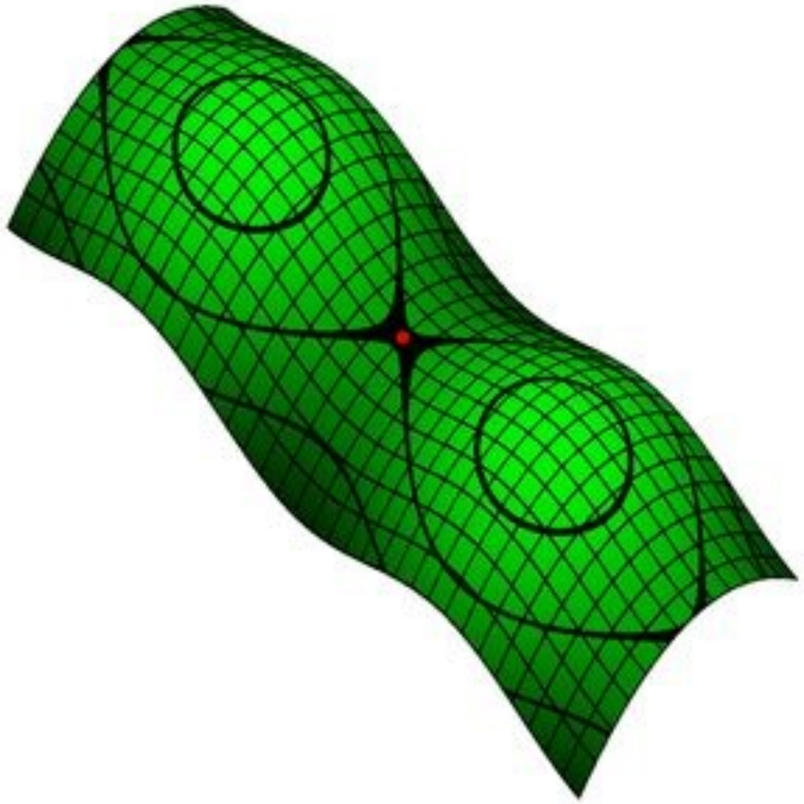
Xrrr = Xrr1 + (Xrr2 - Xrr1)t
Xrrr = (x1 + 2(x2 - x1)t + (x3 - 2x2 + x1)t≤) + ((x2 + 2(x3 - x2)t + (x4 - 2x3 + x2)t≤) - (x1 + 2(x2 - x1)t + (x3 - 2x2 + x1)t≤))t
Xrrr = x1 + 2(x2 - x1)t + (x3 - 2x2 + x1)t≤ + (x2 + 2(x3 - x2)t + (x4 - 2x3 + x2)t≤)t - (x1 + 2(x2 - x1)t + (x3 - 2x2 + x1)t≤)t
Xrrr = x1 + 2(x2 - x1)t + (x3 - 2x2 + x1)t≤ + (x2 + 2x3t - 2x2t + x4t≤ - 2x3t≤ + x2t≤)t - (x1 + 2x2t - 2x1t + x3t≤ - 2x2t≤ + x1t≤)t
Xrrr = x1 + 2x2t - 2x1t + x3t≤ - 2x2t≤ + x1t≤ + x2t + 2x3t≤ - 2x2t≤ + x4t≥ - 2x3t≥ + x2t≥ - (x1t + 2x2t≤ - 2x1t≤ + x3t≥ - 2x2t≥ + x1t≥)
Xrrr = x1 + 2x2t - 2x1t + x3t≤ - 2x2t≤ + x1t≤ + x2t + 2x3t≤ - 2x2t≤ + x4t≥ - 2x3t≥ + x2t≥ + (-x1t - 2x2t≤ + 2x1t≤ - x3t≥ + 2x2t≥ - x1t≥)
Xrrr = x1 + 2x2t - 2x1t + x3t≤ - 2x2t≤ + x1t≤ + x2t + 2x3t≤ - 2x2t≤ + x4t≥ - 2x3t≥ + x2t≥ - x1t - 2x2t≤ + 2x1t≤ - x3t≥ + 2x2t≥ - x1t≥
Xrrr = x1 - 3x1t + 3x1t≤ - x1t≥ + 3x2t - 6x2t≤ + 3x2t≥ + 3x3t≤ - 3x3t≥ + x4t≥
Xrrr = x1(1 - 3t + 3t≤ - t≥) + x2(3t - 6t≤ + 3t≥) + x3(3t≤ - 3t≥) + x4t≥
Xrrr = x1(1 - 3t + 3t≤ - t≥) + 3x2t(1 - 2t + t≤) + 3x3t≤(1-t) + x4t≥
Xrrr = x1((1-t)(1-t)(1-t)) + 3x2t((1-t)(1-t)) + 3x3t≤(1-t) + x4t≥
Xrrr = x1((1 - t)(1 - t)(1 - t)) + 3x2t((1 - t)(1 - t)) + 3x3t≤(1 - t) + x4t≥
Xrrr = (1 - t)≥x1 + 3t(1 - t)≤x2 + 3t≤(1 - t)x3 + t≥x4

Splines









Splines

Ver exemplo: http://www.ibiblio.org/e-notes/Splines/
http://www.ibiblio.org/e-notes/Splines/animation.html

Splines

WireFrame bordas ocultas

Face *WireFrame*

Shaded

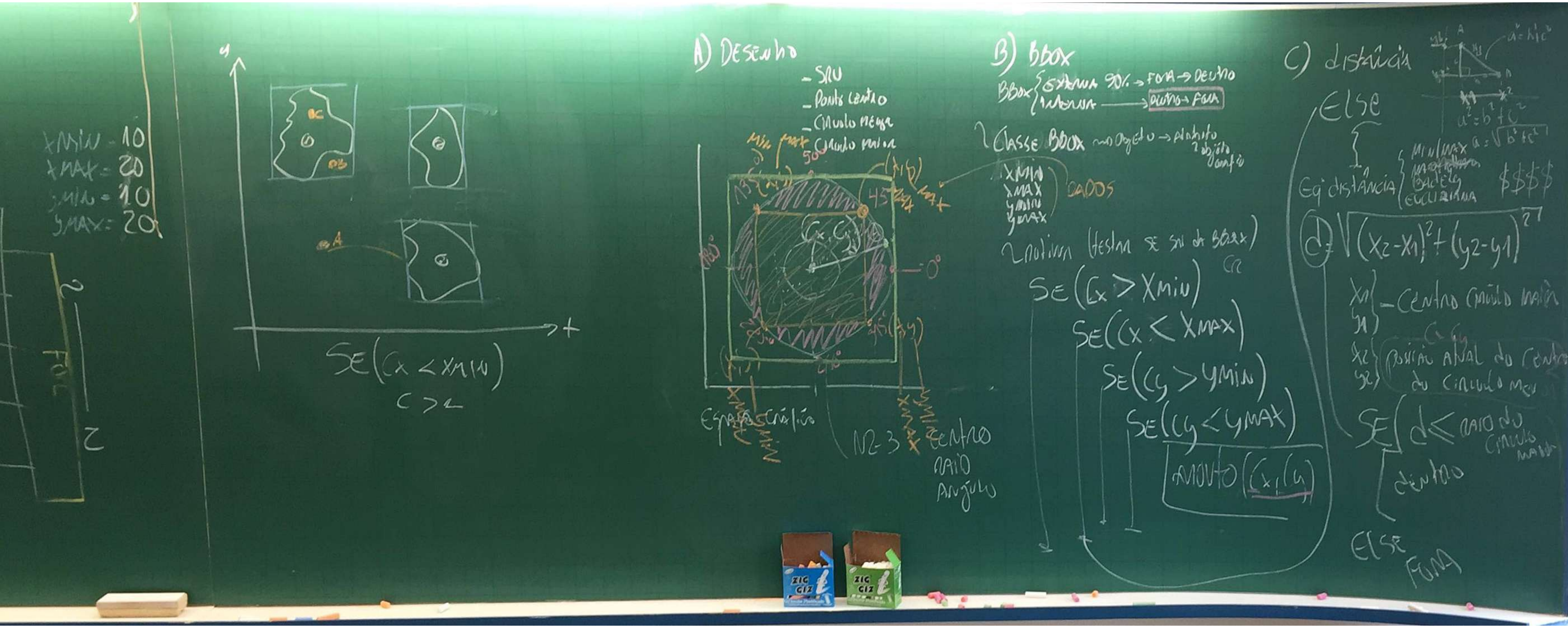
WireFrame uv isolinhas

Face *Shaded*

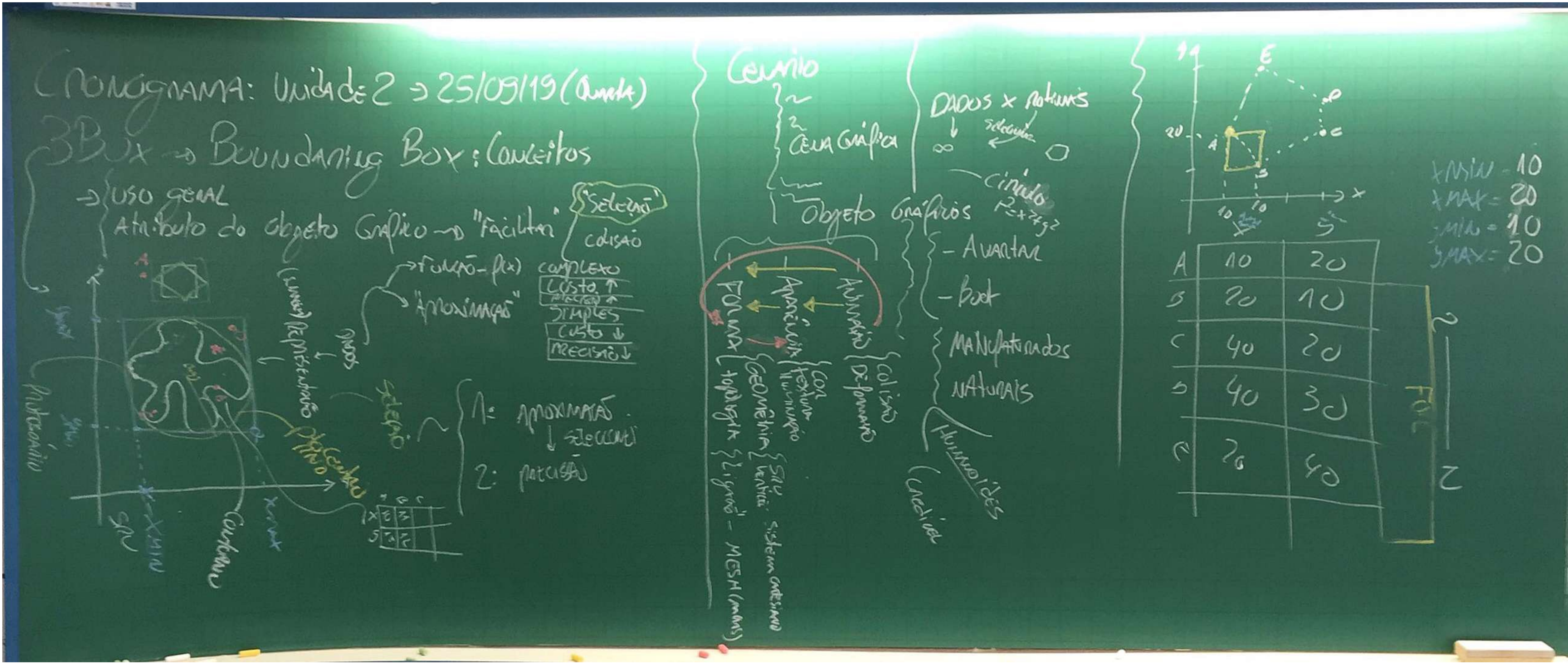
Linhas de reflexão

Imagem refletida

Box



Box


Cronograma: Unidade 2 → 25/09/19 (Quarta)
BBox → Bounding Box: Conceitos
Uso geral
Atributo do objeto Gráfico → "Facilitar"
Seleção
Colisão
Cenário
Cena Gráfica
Objeto Gráficos
Dados x Rotinas
Avatar
Bot
Manufaturados
Naturais
Humanoides
xmin = 10
xmax = 20
ymin = 10
ymax = 20
Forma
Aparência
Animação
Colisão
Deformação
Cor
Textura
Iluminação
Geometria
Topologia
Sistema cartesiano
Mesh (malhas)
Aproximação
Seleção
Precisão
Completo
Custo ↑
Simples
Custo ↓
Precisão ↓

Box

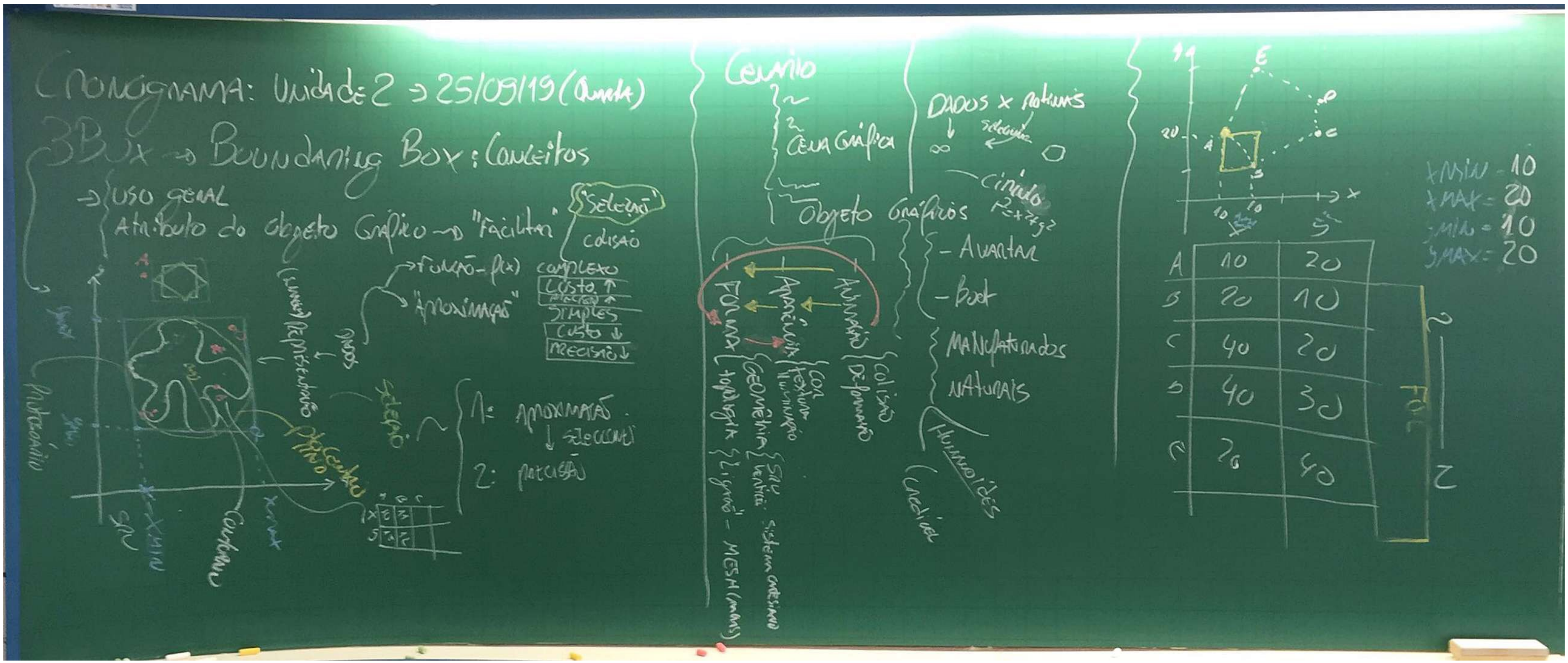

Cronograma: Unidade 2 → 25/09/19 (Quarta)
BBox → Bounding Box: Conceitos
Uso geral
Atributo do objeto Gráfico → "Facilitar"
Seleção
Colisão
Aproximação
Cenário
Cena Gráfica
Objeto Gráficos
Manufaturados
Naturais
xmin = 10
xmax = 20
ymin = 10
ymax = 20

Tabela senos/cosenos e Teorema de Pitágoras

$$a^2 = b^2 + c^2$$

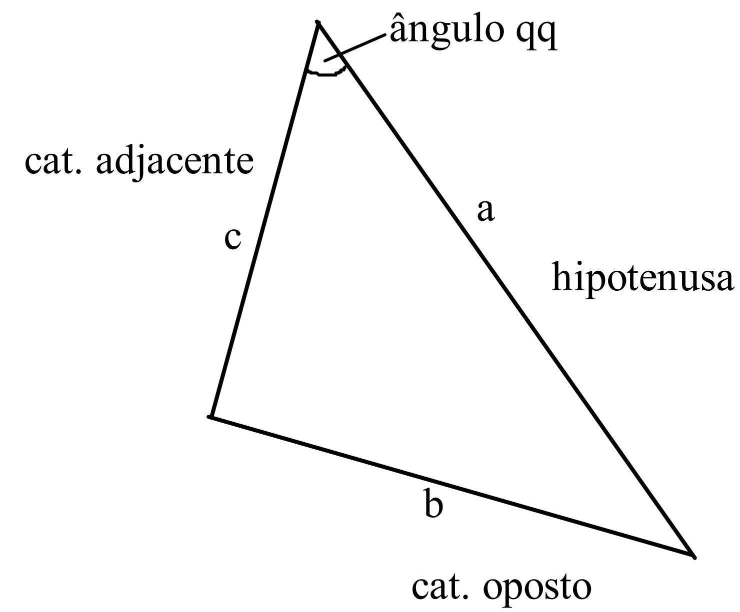
SEN	COS	$grau$
$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$30°$
$\frac{\sqrt{2}}{2}$	$\sqrt{2}/2$	$45°$
$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$60°$
SEN	COS	$grau$
1	0	$90°$
0	1	$0°$

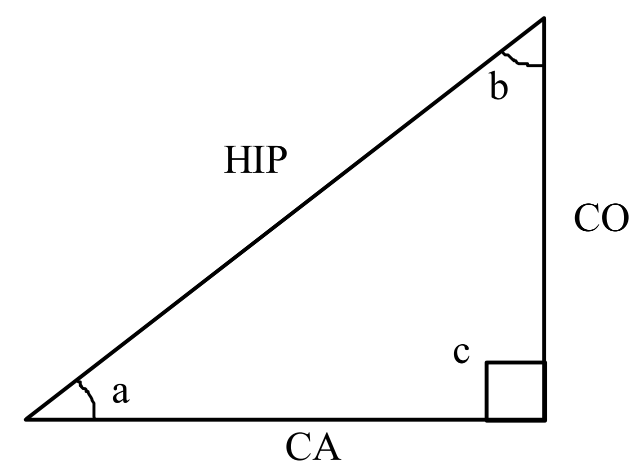
$$\operatorname{sen}\alpha = \frac{CO}{HIP} \qquad \cos\alpha = \frac{CA}{HIP} \qquad \hat{a} + \hat{b} + \hat{c} = 180^\circ$$

$$\operatorname{sen}\alpha = 1 - \cos\alpha \qquad \tan\alpha = \frac{\operatorname{sen}\alpha}{\cos\alpha}$$

$$\cos\theta = \frac{ca}{h} \qquad \cos(\alpha \pm \theta) = \cos\alpha \times \cos\theta \mp \operatorname{sen}\alpha \times \operatorname{sen}\theta$$

$$\operatorname{sen}\theta = \frac{co}{h} \qquad \operatorname{sen}(\alpha \pm \theta) = \operatorname{sen}\alpha \times \cos\theta \pm \cos\alpha \times \operatorname{sen}\theta$$





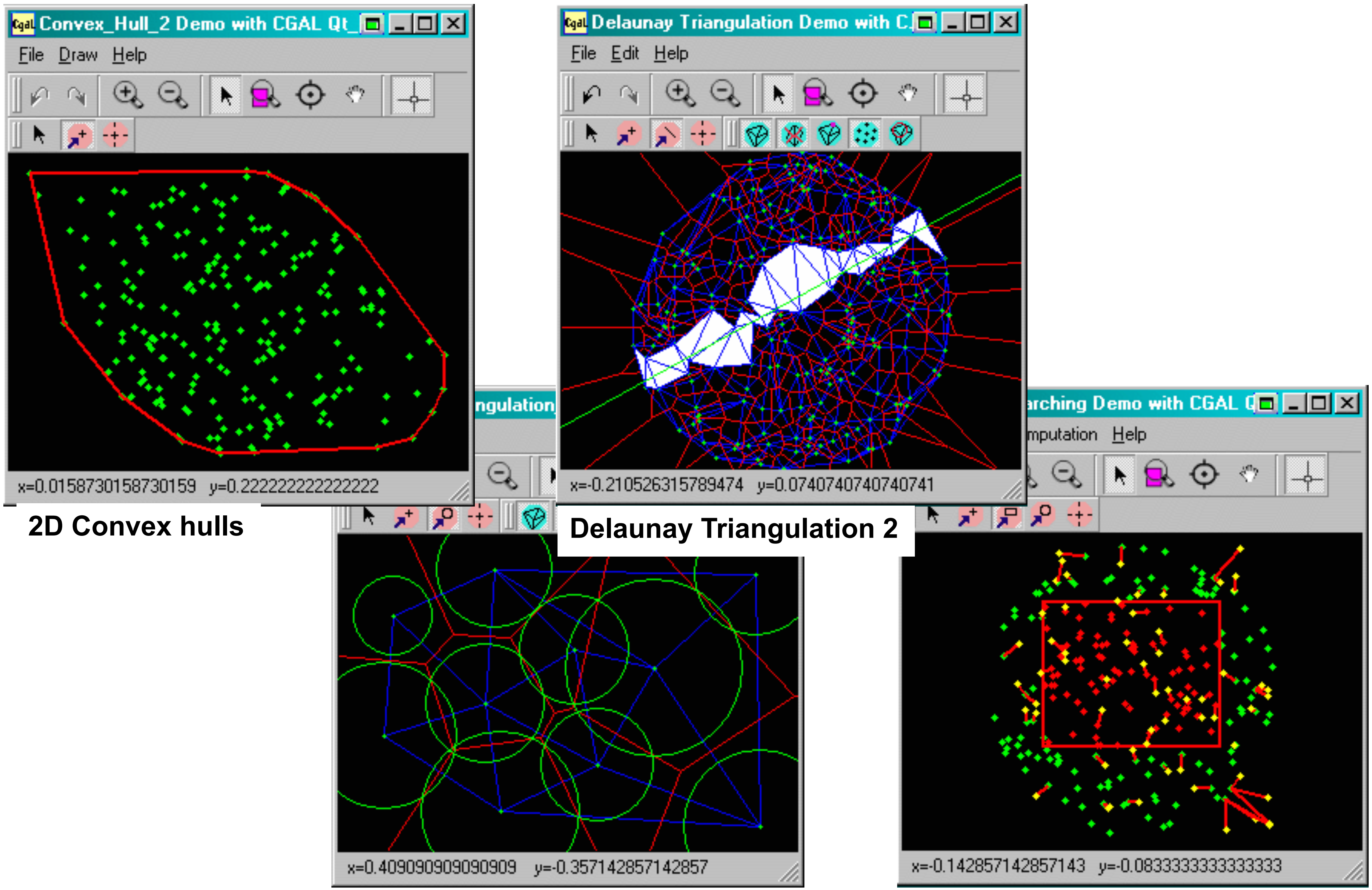
radiano:=grau * PI / 180;

```
public double RetornaX(double a){
        return (5 * Math.cos(Math.PI * a / 180.0));
}

public double RetornaY(double a){
        return (5 * Math.sin(Math.PI * a / 180.0));
}
```

Computational Geometry Algorithms Library - CGAL

http://www.cgal.org/



2D Convex hulls

Delaunay Triangulation 2

Regular Triangulations

Spatial Searching

Theoretical Computer Science Cheat Sheet

Definitions

$f(n) = O(g(n))$	iff $\exists$ positive c, n_0 such that $0 \le f(n) \le cg(n)$ $\forall n \ge n_0$.
$f(n) = \Omega(g(n))$	iff $\exists$ positive c, n_0 such that $f(n) \ge cg(n) \ge 0$ $\forall n \ge n_0$.
$f(n) = \Theta(g(n))$	iff $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$.
$f(n) = o(g(n))$	iff $\lim_{n\to\infty} f(n)/g(n) = 0$.
$\lim_{n\to\infty} a_n = a$	iff $\forall \epsilon > 0$, $\exists n_0$ such that $\lvert a_n - a\rvert < \epsilon$, $\forall n \ge n_0$.
$\sup S$	least $b \in \mathbb{R}$ such that $b \ge s$, $\forall s \in S$.
$\inf S$	greatest $b \in \mathbb{R}$ such that $b \le s$, $\forall s \in S$.
$\liminf_{n\to\infty} a_n$	$\lim_{n\to\infty} \inf\{a_i \mid i \ge n, i \in \mathbb{N}\}$.
$\limsup_{n\to\infty} a_n$	$\lim_{n\to\infty} \sup\{a_i \mid i \ge n, i \in \mathbb{N}\}$.
$\binom{n}{k}$	Combinations: Size k subsets of a size n set.
$\left[{n \atop k}\right]$	Stirling numbers (1st kind): Arrangements of an n element set into k cycles.
$\left\{{n \atop k}\right\}$	Stirling numbers (2nd kind): Partitions of an n element set into k non-empty sets.
$\left\langle{n \atop k}\right\rangle$	1st order Eulerian numbers: Permutations $\pi_1\pi_2\dots\pi_n$ on $\{1,2,\dots,n\}$ with k ascents.
$\left\langle\!\left\langle{n \atop k}\right\rangle\!\right\rangle$	2nd order Eulerian numbers.
C_n	Catalan Numbers: Binary trees with $n+1$ vertices.

Series

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}, \quad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}, \quad \sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}.$$

In general:

$$\sum_{i=1}^{n} i^m = \frac{1}{m+1}\left[(n+1)^{m+1} - 1 - \sum_{i=1}^{n}\left((i+1)^{m+1} - i^{m+1} - (m+1)i^m\right)\right]$$

$$\sum_{i=1}^{n-1} i^m = \frac{1}{m+1}\sum_{k=0}^{m}\binom{m+1}{k} B_k n^{m+1-k}.$$

Geometric series:

$$\sum_{i=0}^{n} c^i = \frac{c^{n+1}-1}{c-1}, \quad c \ne 1, \quad \sum_{i=0}^{\infty} c^i = \frac{1}{1-c}, \quad \sum_{i=1}^{\infty} c^i = \frac{c}{1-c}, \quad |c| < 1,$$

$$\sum_{i=0}^{n} ic^i = \frac{nc^{n+2} - (n+1)c^{n+1} + c}{(c-1)^2}, \quad c \ne 1, \quad \sum_{i=0}^{\infty} ic^i = \frac{c}{(1-c)^2}, \quad |c| < 1.$$

Harmonic series:

$$H_n = \sum_{i=1}^{n} \frac{1}{i}, \qquad \sum_{i=1}^{n} iH_i = \frac{n(n+1)}{2}H_n - \frac{n(n-1)}{4}.$$

$$\sum_{i=1}^{n} H_i = (n+1)H_n - n, \quad \sum_{i=1}^{n}\binom{i}{m}H_i = \binom{n+1}{m+1}\left(H_{n+1} - \frac{1}{m+1}\right).$$

Identities

1. $\binom{n}{k} = \frac{n!}{(n-k)!k!}$, 2. $\sum_{k=0}^{n}\binom{n}{k} = 2^n$, 3. $\binom{n}{k} = \binom{n}{n-k}$,

4. $\binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1}$, 5. $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$,

6. $\binom{n}{m}\binom{m}{k} = \binom{n}{k}\binom{n-k}{m-k}$, 7. $\sum_{k\le n}\binom{r+k}{k} = \binom{r+n+1}{n}$,

8. $\sum_{k=0}^{n}\binom{k}{m} = \binom{n+1}{m+1}$, 9. $\sum_{k=0}^{n}\binom{r}{k}\binom{s}{n-k} = \binom{r+s}{n}$,

10. $\binom{n}{k} = (-1)^k\binom{k-n-1}{k}$, 11. $\left\{{n \atop 1}\right\} = \left\{{n \atop n}\right\} = 1$,

12. $\left\{{n \atop 2}\right\} = 2^{n-1} - 1$, 13. $\left\{{n \atop k}\right\} = k\left\{{n-1 \atop k}\right\} + \left\{{n-1 \atop k-1}\right\}$,

14. $\left[{n \atop 1}\right] = (n-1)!$, 15. $\left[{n \atop 2}\right] = (n-1)!H_{n-1}$, 16. $\left[{n \atop n}\right] = 1$, 17. $\left[{n \atop k}\right] \ge \left\{{n \atop k}\right\}$,

18. $\left[{n \atop k}\right] = (n-1)\left[{n-1 \atop k}\right] + \left[{n-1 \atop k-1}\right]$, 19. $\left\{{n \atop n-1}\right\} = \left[{n \atop n-1}\right] = \binom{n}{2}$, 20. $\sum_{k=0}^{n}\left[{n \atop k}\right] = n!$, 21. $C_n = \frac{1}{n+1}\binom{2n}{n}$,

22. $\left\langle{n \atop 0}\right\rangle = \left\langle{n \atop n-1}\right\rangle = 1$, 23. $\left\langle{n \atop k}\right\rangle = \left\langle{n \atop n-1-k}\right\rangle$, 24. $\left\langle{n \atop k}\right\rangle = (k+1)\left\langle{n-1 \atop k}\right\rangle + (n-k)\left\langle{n-1 \atop k-1}\right\rangle$,

25. $\left\langle{0 \atop k}\right\rangle = \begin{cases}1 & \text{if } k = 0,\\ 0 & \text{otherwise}\end{cases}$ 26. $\left\langle{n \atop 1}\right\rangle = 2^n - n - 1$, 27. $\left\langle{n \atop 2}\right\rangle = 3^n - (n+1)2^n + \binom{n+1}{2}$,

28. $x^n = \sum_{k=0}^{n}\left\langle{n \atop k}\right\rangle\binom{x+k}{n}$, 29. $\left\langle{n \atop m}\right\rangle = \sum_{k=0}^{m}\binom{n+1}{k}(m+1-k)^n(-1)^k$, 30. $m!\left\{{n \atop m}\right\} = \sum_{k=0}^{n}\left\langle{n \atop k}\right\rangle\binom{k}{n-m}$,

31. $\left\langle{n \atop m}\right\rangle = \sum_{k=0}^{n}\left\{{n \atop k}\right\}\binom{n-k}{m}(-1)^{n-k-m}k!$, 32. $\left\langle\!\left\langle{n \atop 0}\right\rangle\!\right\rangle = 1$, 33. $\left\langle\!\left\langle{n \atop n}\right\rangle\!\right\rangle = 0$ for $n \ne 0$,

34. $\left\langle\!\left\langle{n \atop k}\right\rangle\!\right\rangle = (k+1)\left\langle\!\left\langle{n-1 \atop k}\right\rangle\!\right\rangle + (2n-1-k)\left\langle\!\left\langle{n-1 \atop k-1}\right\rangle\!\right\rangle$, 35. $\sum_{k=0}^{n}\left\langle\!\left\langle{n \atop k}\right\rangle\!\right\rangle = \frac{(2n)^n}{2^n}$,

36. $\left\{{x \atop x-n}\right\} = \sum_{k=0}^{n}\left\langle\!\left\langle{n \atop k}\right\rangle\!\right\rangle\binom{x+n-1-k}{2n}$, 37. $\left\{{n+1 \atop m+1}\right\} = \sum_{k}\binom{n}{k}\left\{{k \atop m}\right\} = \sum_{k=0}^{n}\left\{{k \atop m}\right\}(m+1)^{n-k}$,

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Identities Cont.

38. $\left[{n+1 \atop m+1}\right] = \sum_{k}\left[{n \atop k}\right]\binom{k}{m} = \sum_{k=0}^{n}\left[{k \atop m}\right]n^{\underline{n-k}} = n!\sum_{k=0}^{n}\frac{1}{k!}\left[{k \atop m}\right]$, 39. $\left[{x \atop x-n}\right] = \sum_{k=0}^{n}\left\langle\!\left\langle{n \atop k}\right\rangle\!\right\rangle\binom{x+k}{2n}$,

40. $\left\{{n \atop m}\right\} = \sum_{k}\binom{n}{k}\left\{{k+1 \atop m+1}\right\}(-1)^{n-k}$, 41. $\left[{n \atop m}\right] = \sum_{k}\left[{n+1 \atop k+1}\right]\binom{k}{m}(-1)^{m-k}$,

42. $\left\{{m+n+1 \atop m}\right\} = \sum_{k=0}^{m}k\left\{{n+k \atop k}\right\}$, 43. $\left[{m+n+1 \atop m}\right] = \sum_{k=0}^{m}k(n+k)\left[{n+k \atop k}\right]$,

44. $\binom{n}{m} = \sum_{k}\left\{{n+1 \atop k+1}\right\}\left[{k \atop m}\right](-1)^{m-k}$, 45. $(n-m)!\binom{n}{m} = \sum_{k}\left[{n+1 \atop k+1}\right]\left\{{k \atop m}\right\}(-1)^{m-k}$, for $n \ge m$,

46. $\left\{{n \atop n-m}\right\} = \sum_{k}\binom{m-n}{m+k}\binom{m+n}{n+k}\left[{m+k \atop k}\right]$, 47. $\left[{n \atop n-m}\right] = \sum_{k}\binom{m-n}{m+k}\binom{m+n}{n+k}\left\{{m+k \atop k}\right\}$,

48. $\left\{{n \atop \ell+m}\right\}\binom{\ell+m}{\ell} = \sum_{k}\left\{{k \atop \ell}\right\}\left\{{n-k \atop m}\right\}\binom{n}{k}$, 49. $\left[{n \atop \ell+m}\right]\binom{\ell+m}{\ell} = \sum_{k}\left[{k \atop \ell}\right]\left[{n-k \atop m}\right]\binom{n}{k}$.

Trees

Every tree with n vertices has $n-1$ edges.

Kraft inequality: If the depths of the leaves of a binary tree are $d_1, \dots, d_n$:

$$\sum_{i=1}^{n} 2^{-d_i} \le 1,$$

and equality holds only if every internal node has 2 sons.

Recurrences

Master method:

$$T(n) = aT(n/b) + f(n), \quad a \ge 1, b > 1$$

If $\exists \epsilon > 0$ such that $f(n) = O(n^{\log_b a - \epsilon})$ then

$$T(n) = \Theta(n^{\log_b a}).$$

If $f(n) = \Theta(n^{\log_b a})$ then $T(n) = \Theta(n^{\log_b a}\log_2 n)$.

If $\exists \epsilon > 0$ such that $f(n) = \Omega(n^{\log_b a + \epsilon})$, and $\exists c < 1$ such that $af(n/b) \le cf(n)$ for large n, then

$$T(n) = \Theta(f(n)).$$

Substitution (example): Consider the following recurrence

$$T_{i+1} = 2^{2^i} \cdot T_i^2, \quad T_1 = 2.$$

Note that T_i is always a power of two. Let $t_i = \log_2 T_i$. Then we have

$$t_{i+1} = 2^i + 2t_i, \quad t_1 = 1.$$

Let $u_i = t_i/2^i$. Dividing both sides of the previous equation by 2^{i+1} we get

$$\frac{t_{i+1}}{2^{i+1}} = \frac{2^i}{2^{i+1}} + \frac{t_i}{2^i}.$$

Substituting we find

$$u_{i+1} = \tfrac{1}{2} + u_i, \quad u_1 = \tfrac{1}{2},$$

which is simply $u_i = i/2$. So we find that T_i has the closed form $T_i = 2^{i2^{i-1}}$.

Summing factors (example): Consider the following recurrence

$$T(n) = 3T(n/2) + n, \quad T(1) = 1.$$

Rewrite so that all terms involving T are on the left side

$$T(n) - 3T(n/2) = n.$$

Now expand the recurrence, and choose a factor which makes the left side "telescope"

$$1\bigl(T(n) - 3T(n/2) = n\bigr)$$
$$3\bigl(T(n/2) - 3T(n/4) = n/2\bigr)$$
$$\vdots \quad \vdots \quad \vdots$$
$$3^{\log_2 n - 1}\bigl(T(2) - 3T(1) = 2\bigr)$$

Let $m = \log_2 n$. Summing the left side we get $T(n) - 3^mT(1) = T(n) - 3^m = T(n) - n^k$ where $k = \log_2 3 \approx 1.58496$. Summing the right side we get

$$\sum_{i=0}^{m-1}\frac{n}{2^i}3^i = n\sum_{i=0}^{m-1}\left(\tfrac{3}{2}\right)^i.$$

Let $c = \frac{3}{2}$. Then we have

$$n\sum_{i=0}^{m-1}c^i = n\left(\frac{c^m - 1}{c-1}\right)$$
$$= 2n(c^{\log_2 n} - 1)$$
$$= 2n(c^{(k-1)\log_c n} - 1)$$
$$= 2n^k - 2n,$$

and so $T(n) = 3n^k - 2n$. Full history recurrences can often be changed to limited history ones (example): Consider

$$T_i = 1 + \sum_{j=0}^{i-1}T_j, \quad T_0 = 1.$$

Note that

$$T_{i+1} = 1 + \sum_{j=0}^{i}T_j.$$

Subtracting we find

$$T_{i+1} - T_i = 1 + \sum_{j=0}^{i}T_j - 1 - \sum_{j=0}^{i-1}T_j$$
$$= T_i.$$

And so $T_{i+1} = 2T_i = 2^{i+1}$.

Generating functions:

1. Multiply both sides of the equation by x^i.
2. Sum both sides over all i for which the equation is valid.
3. Choose a generating function $G(x)$. Usually $G(x) = \sum_{i=0}^{\infty}x^ig_i$.
3. Rewrite the equation in terms of the generating function $G(x)$.
4. Solve for $G(x)$.
5. The coefficient of x^i in $G(x)$ is g_i.

Example:

$$g_{i+1} = 2g_i + 1, \quad g_0 = 0.$$

Multiply and sum:

$$\sum_{i\ge0}g_{i+1}x^i = \sum_{i\ge0}2g_ix^i + \sum_{i\ge0}x^i.$$

We choose $G(x) = \sum_{i\ge0}x^ig_i$. Rewrite in terms of $G(x)$:

$$\frac{G(x) - g_0}{x} = 2G(x) + \sum_{i\ge0}x^i.$$

Simplify:

$$\frac{G(x)}{x} = 2G(x) + \frac{1}{1-x}.$$

Solve for $G(x)$:

$$G(x) = \frac{x}{(1-x)(1-2x)}.$$

Expand this using partial fractions:

$$G(x) = x\left(\frac{2}{1-2x} - \frac{1}{1-x}\right)$$
$$= x\left(2\sum_{i\ge0}2^ix^i - \sum_{i\ge0}x^i\right)$$
$$= \sum_{i\ge0}(2^{i+1} - 1)x^{i+1}.$$

So $g_i = 2^i - 1$.

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$\pi \approx 3.14159$, $e \approx 2.71828$, $\gamma \approx 0.57721$, $\phi = \frac{1+\sqrt{5}}{2} \approx 1.61803$, $\hat{\phi} = \frac{1-\sqrt{5}}{2} \approx -.61803$

i	2^i	p_i
1	2	2
2	4	3
3	8	5
4	16	7
5	32	11
6	64	13
7	128	17
8	256	19
9	512	23
10	1,024	29
11	2,048	31
12	4,096	37
13	8,192	41
14	16,384	43
15	32,768	47
16	65,536	53
17	131,072	59
18	262,144	61
19	524,288	67
20	1,048,576	71
21	2,097,152	73
22	4,194,304	79
23	8,388,608	83
24	16,777,216	89
25	33,554,432	97
26	67,108,864	101
27	134,217,728	103
28	268,435,456	107
29	536,870,912	109
30	1,073,741,824	113
31	2,147,483,648	127
32	4,294,967,296	131

Pascal's Triangle

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1
1 9 36 84 126 126 84 36 9 1
1 10 45 120 210 252 210 120 45 10 1

General

Bernoulli Numbers ($B_i = 0$, odd $i \neq 1$):

$$B_0 = 1,\ B_1 = -\tfrac{1}{2},\ B_2 = \tfrac{1}{6},\ B_4 = -\tfrac{1}{30},\ B_6 = \tfrac{1}{42},\ B_8 = -\tfrac{1}{30},\ B_{10} = \tfrac{5}{66}.$$

Change of base, quadratic formula:

$$\log_b x = \frac{\log_a x}{\log_a b}, \qquad \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Euler's number e:

$$e = 1 + \tfrac{1}{2} + \tfrac{1}{6} + \tfrac{1}{24} + \tfrac{1}{120} + \cdots$$

$$\lim_{n\to\infty}\left(1 + \frac{x}{n}\right)^n = e^x.$$

$$\left(1 + \tfrac{1}{n}\right)^n < e < \left(1 + \tfrac{1}{n}\right)^{n+1}.$$

$$\left(1 + \tfrac{1}{n}\right)^n = e - \frac{e}{2n} + \frac{11e}{24n^2} - O\left(\frac{1}{n^3}\right).$$

Harmonic numbers:

$$1, \tfrac{3}{2}, \tfrac{11}{6}, \tfrac{25}{12}, \tfrac{137}{60}, \tfrac{49}{20}, \tfrac{363}{140}, \tfrac{761}{280}, \tfrac{7129}{2520}, \ldots$$

$$\ln n < H_n < \ln n + 1,$$

$$H_n = \ln n + \gamma + O\left(\frac{1}{n}\right).$$

Factorial, Stirling's approximation:

1, 2, 6, 24, 120, 720, 5040, 40320, 362880, ...

$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n\left(1 + \Theta\left(\frac{1}{n}\right)\right).$$

Ackermann's function and inverse:

$$a(i,j) = \begin{cases} 2^j & i = 1 \\ a(i-1, 2) & j = 1 \\ a(i-1, a(i, j-1)) & i, j \geq 2 \end{cases}$$

$$\alpha(i) = \min(j \mid a(j,j) \geq i).$$

Binomial distribution:

$$\Pr[X = k] = \binom{n}{k} p^k q^{n-k}, \qquad q = 1 - p,$$

$$E[X] = \sum_{k=1}^{n} k \binom{n}{k} p^k q^{n-k} = np.$$

Poisson distribution:

$$\Pr[X = k] = \frac{e^{-\lambda}\lambda^k}{k!}, \quad E[X] = \lambda.$$

Normal (Gaussian) distribution:

$$p(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2}, \quad E[X] = \mu.$$

The "coupon collector": We are given a random coupon each day, and there are n different types of coupons. The distribution of coupons is uniform. The expected number of days to pass before we to collect all n types is

$$nH_n.$$

Probability

Continuous distributions: If

$$\Pr[a < X < b] = \int_a^b p(x)\,dx,$$

then p is the probability density function of X. If

$$\Pr[X < a] = P(a),$$

then P is the distribution function of X. If P and p both exist then

$$P(a) = \int_{-\infty}^{a} p(x)\,dx.$$

Expectation: If X is discrete

$$E[g(X)] = \sum_x g(x)\Pr[X = x].$$

If X continuous then

$$E[g(X)] = \int_{-\infty}^{\infty} g(x)p(x)\,dx = \int_{-\infty}^{\infty} g(x)\,dP(x).$$

Variance, standard deviation:

$$\mathrm{VAR}[X] = E[X^2] - E[X]^2,$$

$$\sigma = \sqrt{\mathrm{VAR}[X]}.$$

For events A and B:

$$\Pr[A \vee B] = \Pr[A] + \Pr[B] - \Pr[A \wedge B]$$

$$\Pr[A \wedge B] = \Pr[A] \cdot \Pr[B],$$

iff A and B are independent.

$$\Pr[A|B] = \frac{\Pr[A \wedge B]}{\Pr[B]}$$

For random variables X and Y:

$$E[X \cdot Y] = E[X] \cdot E[Y],$$

if X and Y are independent.

$$E[X + Y] = E[X] + E[Y],$$

$$E[cX] = c\,E[X].$$

Bayes' theorem:

$$\Pr[A_i|B] = \frac{\Pr[B|A_i]\Pr[A_i]}{\sum_{j=1}^{n}\Pr[A_j]\Pr[B|A_j]}.$$

Inclusion-exclusion:

$$\Pr\left[\bigvee_{i=1}^{n} X_i\right] = \sum_{i=1}^{n}\Pr[X_i] + \sum_{k=2}^{n}(-1)^{k+1}\sum_{i_1 < \cdots < i_k}\Pr\left[\bigwedge_{j=1}^{k} X_{i_j}\right].$$

Moment inequalities:

$$\Pr\left[|X| \geq \lambda E[X]\right] \leq \frac{1}{\lambda},$$

$$\Pr\left[\left|X - E[X]\right| \geq \lambda \cdot \sigma\right] \leq \frac{1}{\lambda^2}.$$

Geometric distribution:

$$\Pr[X = k] = pq^{k-1}, \qquad q = 1 - p,$$

$$E[X] = \sum_{k=1}^{\infty} kpq^{k-1} = \frac{1}{p}.$$

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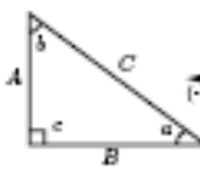
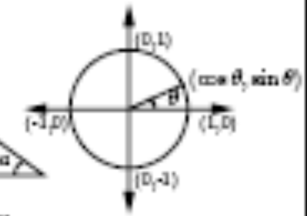
Trigonometry





Pythagorean theorem:

$$C^2 = A^2 + B^2.$$

Definitions:

$$\sin a = A/C, \quad \cos a = B/C,$$

$$\csc a = C/A, \quad \sec a = C/B,$$

$$\tan a = \frac{\sin a}{\cos a} = \frac{A}{B}, \quad \cot a = \frac{\cos a}{\sin a} = \frac{B}{A}.$$

Area, radius of inscribed circle:

$$\tfrac{1}{2}AB, \qquad \frac{AB}{A + B + C}.$$

Identities:

$$\sin x = \frac{1}{\csc x}, \qquad \cos x = \frac{1}{\sec x},$$

$$\tan x = \frac{1}{\cot x}, \qquad \sin^2 x + \cos^2 x = 1,$$

$$1 + \tan^2 x = \sec^2 x, \qquad 1 + \cot^2 x = \csc^2 x,$$

$$\sin x = \cos\left(\tfrac{\pi}{2} - x\right), \qquad \sin x = \sin(\pi - x),$$

$$\cos x = -\cos(\pi - x), \qquad \tan x = \cot\left(\tfrac{\pi}{2} - x\right),$$

$$\cot x = -\cot(\pi - x), \qquad \csc x = \cot\tfrac{x}{2} - \cot x,$$

$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y,$$

$$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y,$$

$$\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y},$$

$$\cot(x \pm y) = \frac{\cot x \cot y \mp 1}{\cot x \pm \cot y},$$

$$\sin 2x = 2\sin x \cos x, \qquad \sin 2x = \frac{2\tan x}{1 + \tan^2 x},$$

$$\cos 2x = \cos^2 x - \sin^2 x, \qquad \cos 2x = 2\cos^2 x - 1,$$

$$\cos 2x = 1 - 2\sin^2 x, \qquad \cos 2x = \frac{1 - \tan^2 x}{1 + \tan^2 x},$$

$$\tan 2x = \frac{2\tan x}{1 - \tan^2 x}, \qquad \cot 2x = \frac{\cot^2 x - 1}{2\cot x},$$

$$\sin(x + y)\sin(x - y) = \sin^2 x - \sin^2 y,$$

$$\cos(x + y)\cos(x - y) = \cos^2 x - \sin^2 y.$$

Euler's equation:

$$e^{ix} = \cos x + i\sin x, \qquad e^{i\pi} = -1.$$

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Matrices

Multiplication:

$$C = A \cdot B, \quad c_{i,j} = \sum_{k=1}^{n} a_{i,k}b_{k,j}.$$

Determinants: $\det A \neq 0$ iff A is non-singular.

$$\det A \cdot B = \det A \cdot \det B,$$

$$\det A = \sum_{\pi}\prod_{i=1}^{n}\operatorname{sign}(\pi)a_{i,\pi(i)}.$$

2×2 and 3×3 determinant:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc,$$

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = g\begin{vmatrix} b & c \\ e & f \end{vmatrix} - h\begin{vmatrix} a & c \\ d & f \end{vmatrix} + i\begin{vmatrix} a & b \\ d & e \end{vmatrix}$$

$$= \begin{matrix} aei + bfg + cdh \\ -ceg - fha - ibd. \end{matrix}$$

Permanents:

$$\operatorname{perm} A = \sum_{\pi}\prod_{i=1}^{n} a_{i,\pi(i)}.$$

Hyperbolic Functions

Definitions:

$$\sinh x = \frac{e^x - e^{-x}}{2}, \qquad \cosh x = \frac{e^x + e^{-x}}{2},$$

$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}, \qquad \operatorname{csch} x = \frac{1}{\sinh x},$$

$$\operatorname{sech} x = \frac{1}{\cosh x}, \qquad \coth x = \frac{1}{\tanh x}.$$

Identities:

$$\cosh^2 x - \sinh^2 x = 1, \qquad \tanh^2 x + \operatorname{sech}^2 x = 1,$$

$$\coth^2 x - \operatorname{csch}^2 x = 1, \qquad \sinh(-x) = -\sinh x,$$

$$\cosh(-x) = \cosh x, \qquad \tanh(-x) = -\tanh x,$$

$$\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y,$$

$$\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y,$$

$$\sinh 2x = 2\sinh x \cosh x,$$

$$\cosh 2x = \cosh^2 x + \sinh^2 x,$$

$$\cosh x + \sinh x = e^x, \qquad \cosh x - \sinh x = e^{-x},$$

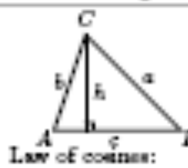
$$(\cosh x + \sinh x)^n = \cosh nx + \sinh nx, \quad n \in \mathbb{Z},$$

$$2\sinh^2\tfrac{x}{2} = \cosh x - 1, \qquad 2\cosh^2\tfrac{x}{2} = \cosh x + 1.$$

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$
0	0	1	0
$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
$\frac{\pi}{2}$	1	0	∞

... in mathematics you don't understand things, you just get used to them.
– J. von Neumann

More Trig.



Law of cosines:

$$c^2 = a^2 + b^2 - 2ab\cos C.$$

Area:

$$A = \tfrac{1}{2}hc,$$

$$= \tfrac{1}{2}ab\sin C,$$

$$= \frac{c^2 \sin A \sin B}{2\sin C}.$$

Heron's formula:

$$A = \sqrt{s \cdot s_a \cdot s_b \cdot s_c},$$

$$s = \tfrac{1}{2}(a + b + c),$$

$$s_a = s - a,$$

$$s_b = s - b,$$

$$s_c = s - c.$$

More identities:

$$\sin\tfrac{x}{2} = \sqrt{\frac{1 - \cos x}{2}},$$

$$\cos\tfrac{x}{2} = \sqrt{\frac{1 + \cos x}{2}},$$

$$\tan\tfrac{x}{2} = \sqrt{\frac{1 - \cos x}{1 + \cos x}},$$

$$= \frac{1 - \cos x}{\sin x},$$

$$= \frac{\sin x}{1 + \cos x},$$

$$\cot\tfrac{x}{2} = \sqrt{\frac{1 + \cos x}{1 - \cos x}},$$

$$= \frac{1 + \cos x}{\sin x},$$

$$= \frac{\sin x}{1 - \cos x},$$

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i},$$

$$\cos x = \frac{e^{ix} + e^{-ix}}{2},$$

$$\tan x = -i\frac{e^{ix} - e^{-ix}}{e^{ix} + e^{-ix}},$$

$$= -i\frac{e^{2ix} - 1}{e^{2ix} + 1},$$

$$\sin x = \frac{\sinh ix}{i},$$

$$\cos x = \cosh ix,$$

$$\tan x = \frac{\tanh ix}{i}.$$

Theoretical Computer Science Cheat Sheet

Number Theory

The Chinese remainder theorem: There exists a number C such that:

$$C \equiv r_1 \bmod m_1$$
$$\vdots \quad \vdots \quad \vdots$$
$$C \equiv r_n \bmod m_n$$

if m_i and m_j are relatively prime for $i \neq j$.

Euler's function: $\phi(x)$ is the number of positive integers less than x relatively prime to x. If $\prod_{i=1}^{n} p_i^{e_i}$ is the prime factorization of x then

$$\phi(x) = \prod_{i=1}^{n} p_i^{e_i-1}(p_i - 1).$$

Euler's theorem: If a and b are relatively prime then

$$1 \equiv a^{\phi(b)} \bmod b.$$

Fermat's theorem:

$$1 \equiv a^{p-1} \bmod p.$$

The Euclidean algorithm: if $a > b$ are integers then

$$\gcd(a, b) = \gcd(a \bmod b, b).$$

If $\prod_{i=1}^{n} p_i^{e_i}$ is the prime factorization of x then

$$S(x) = \sum_{d|x} d = \prod_{i=1}^{n} \frac{p_i^{e_i+1} - 1}{p_i - 1}.$$

Perfect Numbers: x is an even perfect number iff $x = 2^{n-1}(2^n - 1)$ and $2^n - 1$ is prime.

Wilson's theorem: n is a prime iff

$$(n-1)! \equiv -1 \bmod n.$$

Möbius inversion:

$$\mu(i) = \begin{cases} 1 & \text{if } i = 1. \\ 0 & \text{if } i \text{ is not square-free.} \\ (-1)^r & \text{if } i \text{ is the product of } r \text{ distinct primes.} \end{cases}$$

If

$$G(a) = \sum_{d|a} F(d),$$

then

$$F(a) = \sum_{d|a} \mu(d) G\left(\frac{a}{d}\right).$$

Prime numbers:

$$p_n = n \ln n + n \ln \ln n - n + n \frac{\ln \ln n}{\ln n} + O\left(\frac{n}{\ln n}\right),$$

$$\pi(n) = \frac{n}{\ln n} + \frac{n}{(\ln n)^2} + \frac{2!n}{(\ln n)^3} + O\left(\frac{n}{(\ln n)^4}\right).$$

Graph Theory

Definitions:

Loop	An edge connecting a vertex to itself.
Directed	Each edge has a direction.
Simple	Graph with no loops or multi-edges.
Walk	A sequence $v_0 e_1 v_1 \ldots e_\ell v_\ell$.
Trail	A walk with distinct edges.
Path	A trail with distinct vertices.
Connected	A graph where there exists a path between any two vertices.
Component	A maximal connected subgraph.
Tree	A connected acyclic graph.
Free tree	A tree with no root.
DAG	Directed acyclic graph.
Eulerian	Graph with a trail visiting each edge exactly once.
Hamiltonian	Graph with a cycle visiting each vertex exactly once.
Cut	A set of edges whose removal increases the number of components.
Cut-set	A minimal cut.
Cut edge	A size 1 cut.
k-Connected	A graph connected with the removal of any $k-1$ vertices.
k-Tough	$\forall S \subseteq V, S \neq \emptyset$ we have $k \cdot c(G - S) \leq \lvert S \rvert$.
k-Regular	A graph where all vertices have degree k.
k-Factor	A k-regular spanning subgraph.
Matching	A set of edges, no two of which are adjacent.
Clique	A set of vertices, all of which are adjacent.
Ind. set	A set of vertices, none of which are adjacent.
Vertex cover	A set of vertices which cover all edges.
Planar graph	A graph which can be embedded in the plane.
Plane graph	An embedding of a planar graph.

$$\sum_{v \in V} \deg(v) = 2m.$$

If G is planar then $n - m + f = 2$, so

$$f \leq 2n - 4, \quad m \leq 3n - 6.$$

Any planar graph has a vertex with degree ≤ 5.

Notation:

$E(G)$	Edge set
$V(G)$	Vertex set
$c(G)$	Number of components
$G[S]$	Induced subgraph
$\deg(v)$	Degree of v
$\Delta(G)$	Maximum degree
$\delta(G)$	Minimum degree
$\chi(G)$	Chromatic number
$\chi_E(G)$	Edge chromatic number
G^c	Complement graph
K_n	Complete graph
K_{n_1,n_2}	Complete bipartite graph
$r(k, \ell)$	Ramsey number

Geometry

Projective coordinates: triples (x, y, z), not all x, y and z zero.

$$(x, y, z) = (cx, cy, cz) \quad \forall c \neq 0.$$

Cartesian	Projective
(x, y)	$(x, y, 1)$
$y = mx + b$	$(m, -1, b)$
$x = c$	$(1, 0, -c)$

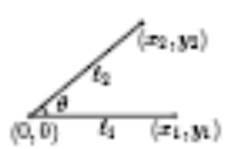
Distance formula, L_p and L_∞ metric:

$$\sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2},$$
$$\left[\lvert x_1 - x_0 \rvert^p + \lvert y_1 - y_0 \rvert^p\right]^{1/p},$$
$$\lim_{p \to \infty} \left[\lvert x_1 - x_0 \rvert^p + \lvert y_1 - y_0 \rvert^p\right]^{1/p}.$$

Area of triangle (x_0, y_0), (x_1, y_1) and (x_2, y_2):

$$\frac{1}{2} \operatorname{abs} \begin{vmatrix} x_1 - x_0 & y_1 - y_0 \\ x_2 - x_0 & y_2 - y_0 \end{vmatrix}.$$

Angle formed by three points:



$$\cos\theta = \frac{(x_1, y_1) \cdot (x_2, y_2)}{\ell_1 \ell_2}.$$

Line through two points (x_0, y_0) and (x_1, y_1):

$$\begin{vmatrix} x & y & 1 \\ x_0 & y_0 & 1 \\ x_1 & y_1 & 1 \end{vmatrix} = 0.$$

Area of circle, volume of sphere:

$$A = \pi r^2, \qquad V = \frac{4}{3}\pi r^3.$$

If I have seen farther than others, it is because I have stood on the shoulders of giants.
– Isaac Newton

Theoretical Computer Science Cheat Sheet

π

Wallis' identity:

$$\pi = 2 \cdot \frac{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdots}{1 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \cdots}$$

Brouncker's continued fraction expansion:

$$\frac{\pi}{4} = 1 + \cfrac{1^2}{2 + \cfrac{3^2}{2 + \cfrac{5^2}{2 + \cfrac{7^2}{2 + \cdots}}}}$$

Gregrory's series:

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots$$

Newton's series:

$$\frac{\pi}{6} = \frac{1}{2} + \frac{1}{2 \cdot 3 \cdot 2^3} + \frac{1 \cdot 3}{2 \cdot 4 \cdot 5 \cdot 2^5} + \cdots$$

Sharp's series:

$$\frac{\pi}{6} = \frac{1}{\sqrt{3}}\left(1 - \frac{1}{3^1 \cdot 3} + \frac{1}{3^2 \cdot 5} - \frac{1}{3^3 \cdot 7} + \cdots\right)$$

Euler's series:

$$\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots$$
$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \cdots$$
$$\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} - \cdots$$

Partial Fractions

Let $N(x)$ and $D(x)$ be polynomial functions of x. We can break down $N(x)/D(x)$ using partial fraction expansion. First, if the degree of N is greater than or equal to the degree of D, divide N by D, obtaining

$$\frac{N(x)}{D(x)} = Q(x) + \frac{N'(x)}{D(x)},$$

where the degree of N' is less than that of D. Second, factor $D(x)$. Use the following rules: For a non-repeated factor:

$$\frac{N(x)}{(x-a)D(x)} = \frac{A}{x-a} + \frac{N'(x)}{D(x)},$$

where

$$A = \left[\frac{N(x)}{D(x)}\right]_{x=a}.$$

For a repeated factor:

$$\frac{N(x)}{(x-a)^m D(x)} = \sum_{k=0}^{m-1} \frac{A_k}{(x-a)^{m-k}} + \frac{N'(x)}{D(x)},$$

where

$$A_k = \frac{1}{k!}\left[\frac{d^k}{dx^k}\left(\frac{N(x)}{D(x)}\right)\right]_{x=a}.$$

The reasonable man adapts himself to the world; the unreasonable persists in trying to adapt the world to himself. Therefore all progress depends on the unreasonable.
– George Bernard Shaw

Calculus

Derivatives:

1. $\frac{d(cu)}{dx} = c\frac{du}{dx}$,
2. $\frac{d(u+v)}{dx} = \frac{du}{dx} + \frac{dv}{dx}$,
3. $\frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$,
4. $\frac{d(u^n)}{dx} = nu^{n-1}\frac{du}{dx}$,
5. $\frac{d(u/v)}{dx} = \frac{v\left(\frac{du}{dx}\right) - u\left(\frac{dv}{dx}\right)}{v^2}$,
6. $\frac{d(e^{cu})}{dx} = ce^{cu}\frac{du}{dx}$,
7. $\frac{d(c^u)}{dx} = (\ln c)c^u\frac{du}{dx}$,
8. $\frac{d(\ln u)}{dx} = \frac{1}{u}\frac{du}{dx}$,
9. $\frac{d(\sin u)}{dx} = \cos u\frac{du}{dx}$,
10. $\frac{d(\cos u)}{dx} = -\sin u\frac{du}{dx}$,
11. $\frac{d(\tan u)}{dx} = \sec^2 u\frac{du}{dx}$,
12. $\frac{d(\cot u)}{dx} = \csc^2 u\frac{du}{dx}$,
13. $\frac{d(\sec u)}{dx} = \tan u \sec u\frac{du}{dx}$,
14. $\frac{d(\csc u)}{dx} = -\cot u \csc u\frac{du}{dx}$,
15. $\frac{d(\arcsin u)}{dx} = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$,
16. $\frac{d(\arccos u)}{dx} = \frac{-1}{\sqrt{1-u^2}}\frac{du}{dx}$,
17. $\frac{d(\arctan u)}{dx} = \frac{1}{1+u^2}\frac{du}{dx}$,
18. $\frac{d(\operatorname{arccot} u)}{dx} = \frac{-1}{1+u^2}\frac{du}{dx}$,
19. $\frac{d(\operatorname{arcsec} u)}{dx} = \frac{1}{u\sqrt{1-u^2}}\frac{du}{dx}$,
20. $\frac{d(\operatorname{arccsc} u)}{dx} = \frac{-1}{u\sqrt{1-u^2}}\frac{du}{dx}$,
21. $\frac{d(\sinh u)}{dx} = \cosh u\frac{du}{dx}$,
22. $\frac{d(\cosh u)}{dx} = \sinh u\frac{du}{dx}$,
23. $\frac{d(\tanh u)}{dx} = \operatorname{sech}^2 u\frac{du}{dx}$,
24. $\frac{d(\coth u)}{dx} = -\operatorname{csch}^2 u\frac{du}{dx}$,
25. $\frac{d(\operatorname{sech} u)}{dx} = -\operatorname{sech} u \tanh u\frac{du}{dx}$,
26. $\frac{d(\operatorname{csch} u)}{dx} = -\operatorname{csch} u \coth u\frac{du}{dx}$,
27. $\frac{d(\operatorname{arcsinh} u)}{dx} = \frac{1}{\sqrt{1+u^2}}\frac{du}{dx}$,
28. $\frac{d(\operatorname{arccosh} u)}{dx} = \frac{1}{\sqrt{u^2-1}}\frac{du}{dx}$,
29. $\frac{d(\operatorname{arctanh} u)}{dx} = \frac{1}{1-u^2}\frac{du}{dx}$,
30. $\frac{d(\operatorname{arccoth} u)}{dx} = \frac{1}{u^2-1}\frac{du}{dx}$,
31. $\frac{d(\operatorname{arcsech} u)}{dx} = \frac{-1}{u\sqrt{1-u^2}}\frac{du}{dx}$,
32. $\frac{d(\operatorname{arccsch} u)}{dx} = \frac{-1}{\lvert u \rvert\sqrt{1+u^2}}\frac{du}{dx}$.

Integrals:

1. $\int cu\,dx = c\int u\,dx$,
2. $\int (u+v)\,dx = \int u\,dx + \int v\,dx$,
3. $\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1$,
4. $\int \frac{1}{x}\,dx = \ln x$,
5. $\int e^x\,dx = e^x$,
6. $\int \frac{dx}{1+x^2} = \arctan x$,
7. $\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$,
8. $\int \sin x\,dx = -\cos x$,
9. $\int \cos x\,dx = \sin x$,
10. $\int \tan x\,dx = -\ln\lvert\cos x\rvert$,
11. $\int \cot x\,dx = \ln\lvert\cos x\rvert$,
12. $\int \sec x\,dx = \ln\lvert\sec x + \tan x\rvert$,
13. $\int \csc x\,dx = \ln\lvert\csc x + \cot x\rvert$,
14. $\int \arcsin\frac{x}{a}\,dx = \arcsin\frac{x}{a} + \sqrt{a^2 - x^2}, \quad a > 0$,

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Calculus Cont.

15. $\int \arccos\frac{x}{a}\,dx = x\arccos\frac{x}{a} - \sqrt{a^2-x^2}$, $a>0$,

16. $\int \arctan\frac{x}{a}\,dx = x\arctan\frac{x}{a} - \frac{a}{2}\ln(a^2+x^2)$, $a>0$,

17. $\int \sin^2(ax)\,dx = \frac{1}{2a}\big(ax - \sin(ax)\cos(ax)\big)$,

18. $\int \cos^2(ax)\,dx = \frac{1}{2a}\big(ax + \sin(ax)\cos(ax)\big)$,

19. $\int \sec^2 x\,dx = \tan x$,

20. $\int \csc^2 x\,dx = -\cot x$,

21. $\int \sin^n x\,dx = -\frac{\sin^{n-1}x\cos x}{n} + \frac{n-1}{n}\int \sin^{n-2}x\,dx$,

22. $\int \cos^n x\,dx = \frac{\cos^{n-1}x\sin x}{n} + \frac{n-1}{n}\int \cos^{n-2}x\,dx$,

23. $\int \tan^n x\,dx = \frac{\tan^{n-1}x}{n-1} - \int \tan^{n-2}x\,dx$, $n\neq 1$,

24. $\int \cot^n x\,dx = -\frac{\cot^{n-1}x}{n-1} - \int \cot^{n-2}x\,dx$, $n\neq 1$,

25. $\int \sec^n x\,dx = \frac{\tan x\sec^{n-1}x}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2}x\,dx$, $n\neq 1$,

26. $\int \csc^n x\,dx = -\frac{\cot x\csc^{n-1}x}{n-1} + \frac{n-2}{n-1}\int \csc^{n-2}x\,dx$, $n\neq 1$,

27. $\int \sinh x\,dx = \cosh x$,

28. $\int \cosh x\,dx = \sinh x$,

29. $\int \tanh x\,dx = \ln|\cosh x|$,

30. $\int \coth x\,dx = \ln|\sinh x|$,

31. $\int \operatorname{sech} x\,dx = \arctan\sinh x$,

32. $\int \operatorname{csch} x\,dx = \ln\left|\tanh\frac{x}{2}\right|$,

33. $\int \sinh^2 x\,dx = \frac14\sinh(2x) - \frac12 x$,

34. $\int \cosh^2 x\,dx = \frac14\sinh(2x) + \frac12 x$,

35. $\int \operatorname{sech}^2 x\,dx = \tanh x$,

36. $\int \operatorname{arcsinh}\frac{x}{a}\,dx = x\operatorname{arcsinh}\frac{x}{a} - \sqrt{x^2+a^2}$, $a>0$,

37. $\int \operatorname{arctanh}\frac{x}{a}\,dx = x\operatorname{arctanh}\frac{x}{a} + \frac{a}{2}\ln|a^2-x^2|$,

38. $\int \operatorname{arccosh}\frac{x}{a}\,dx = \begin{cases} x\operatorname{arccosh}\frac{x}{a} - \sqrt{x^2+a^2}, & \text{if } \operatorname{arccosh}\frac{x}{a} > 0 \text{ and } a>0, \\ x\operatorname{arccosh}\frac{x}{a} + \sqrt{x^2+a^2}, & \text{if } \operatorname{arccosh}\frac{x}{a} < 0 \text{ and } a>0, \end{cases}$

39. $\int \frac{dx}{\sqrt{a^2+x^2}} = \ln\left(x+\sqrt{a^2+x^2}\right)$, $a>0$,

40. $\int \frac{dx}{a^2+x^2} = \frac1a\arctan\frac{x}{a}$, $a>0$,

41. $\int \sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\arcsin\frac{x}{a}$, $a>0$,

42. $\int (a^2-x^2)^{3/2}dx = \frac{x}{8}(5a^2-2x^2)\sqrt{a^2-x^2} + \frac{3a^4}{8}\arcsin\frac{x}{a}$, $a>0$,

43. $\int \frac{dx}{\sqrt{a^2-x^2}} = \arcsin\frac{x}{a}$, $a>0$,

44. $\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right|$,

45. $\int \frac{dx}{(a^2-x^2)^{3/2}} = \frac{x}{a^2\sqrt{a^2-x^2}}$,

46. $\int \sqrt{a^2\pm x^2}\,dx = \frac{x}{2}\sqrt{a^2\pm x^2} \pm \frac{a^2}{2}\ln\left|x+\sqrt{a^2\pm x^2}\right|$,

47. $\int \frac{dx}{\sqrt{x^2-a^2}} = \ln\left|x+\sqrt{x^2-a^2}\right|$, $a>0$,

48. $\int \frac{dx}{ax^2+bx} = \frac1a\ln\left|\frac{x}{a+bx}\right|$,

49. $\int x\sqrt{a+bx}\,dx = \frac{2(3bx-2a)(a+bx)^{3/2}}{15b^2}$,

50. $\int \frac{\sqrt{a+bx}}{x}\,dx = 2\sqrt{a+bx} + a\int\frac{1}{x\sqrt{a+bx}}\,dx$,

51. $\int \frac{x}{\sqrt{a+bx}}\,dx = \frac{1}{\sqrt2}\ln\left|\frac{\sqrt{a+bx}-\sqrt a}{\sqrt{a+bx}+\sqrt a}\right|$, $a>0$,

52. $\int \frac{\sqrt{a^2-x^2}}{x}\,dx = \sqrt{a^2-x^2} - a\ln\left|\frac{a+\sqrt{a^2-x^2}}{x}\right|$,

53. $\int x\sqrt{a^2-x^2}\,dx = -\frac13(a^2-x^2)^{3/2}$,

54. $\int x^2\sqrt{a^2-x^2}\,dx = \frac{x}{8}(2x^2-a^2)\sqrt{a^2-x^2} + \frac{a^4}{8}\arcsin\frac{x}{a}$, $a>0$,

55. $\int \frac{dx}{\sqrt{a^2-x^2}} = -\frac1a\ln\left|\frac{a+\sqrt{a^2-x^2}}{x}\right|$,

56. $\int \frac{x\,dx}{\sqrt{a^2-x^2}} = -\sqrt{a^2-x^2}$,

57. $\int \frac{x^2\,dx}{\sqrt{a^2-x^2}} = -\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\arcsin\frac{x}{a}$, $a>0$,

58. $\int \frac{\sqrt{a^2+x^2}}{x}\,dx = \sqrt{a^2+x^2} - a\ln\left|\frac{a+\sqrt{a^2+x^2}}{x}\right|$,

59. $\int \frac{\sqrt{x^2-a^2}}{x}\,dx = \sqrt{x^2-a^2} - a\arccos\frac{a}{|x|}$, $a>0$,

60. $\int x\sqrt{x^2\pm a^2}\,dx = \frac13(x^2\pm a^2)^{3/2}$,

61. $\int \frac{dx}{x\sqrt{x^2+a^2}} = \frac1a\ln\left|\frac{x}{a+\sqrt{a^2+x^2}}\right|$,

Theoretical Computer Science Cheat Sheet

Calculus Cont.

62. $\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac1a\arccos\frac{a}{|x|}$, $a>0$,

63. $\int \frac{dx}{x^2\sqrt{x^2\pm a^2}} = \mp\frac{\sqrt{x^2\pm a^2}}{a^2x}$,

64. $\int \frac{x\,dx}{\sqrt{x^2\pm a^2}} = \sqrt{x^2\pm a^2}$,

65. $\int \frac{\sqrt{x^2\pm a^2}}{x^4}\,dx = \mp\frac{(x^2+a^2)^{3/2}}{3a^2x^3}$,

66. $\int \frac{dx}{ax^2+bx+c} = \begin{cases} \frac{1}{\sqrt{b^2-4ac}}\ln\left|\frac{2ax+b-\sqrt{b^2-4ac}}{2ax+b+\sqrt{b^2-4ac}}\right|, & \text{if } b^2>4ac, \\ \frac{2}{\sqrt{4ac-b^2}}\arctan\frac{2ax+b}{\sqrt{4ac-b^2}}, & \text{if } b^2<4ac, \end{cases}$

67. $\int \frac{dx}{\sqrt{ax^2+bx+c}} = \begin{cases} \frac{1}{\sqrt a}\ln\left|2ax+b+2\sqrt a\sqrt{ax^2+bx+c}\right|, & \text{if } a>0, \\ \frac{1}{\sqrt{-a}}\arcsin\frac{-2ax-b}{\sqrt{b^2-4ac}}, & \text{if } a<0, \end{cases}$

68. $\int \sqrt{ax^2+bx+c}\,dx = \frac{2ax+b}{4a}\sqrt{ax^2+bx+c} + \frac{4ac-b^2}{8a}\int\frac{dx}{\sqrt{ax^2+bx+c}}$,

69. $\int \frac{x\,dx}{\sqrt{ax^2+bx+c}} = \frac{\sqrt{ax^2+bx+c}}{a} - \frac{b}{2a}\int\frac{dx}{\sqrt{ax^2+bx+c}}$,

70. $\int \frac{dx}{x\sqrt{ax^2+bx+c}} = \begin{cases} \frac{-1}{\sqrt c}\ln\left|\frac{2\sqrt c\sqrt{ax^2+bx+c}+bx+2c}{x}\right|, & \text{if } c>0, \\ \frac{1}{\sqrt{-c}}\arcsin\frac{bx+2c}{|x|\sqrt{b^2-4ac}}, & \text{if } c<0, \end{cases}$

71. $\int x^3\sqrt{x^2+a^2}\,dx = \left(\frac13x^2-\frac{2}{15}a^2\right)(x^2+a^2)^{3/2}$,

72. $\int x^n\sin(ax)\,dx = -\frac1a x^n\cos(ax) + \frac{n}{a}\int x^{n-1}\cos(ax)\,dx$,

73. $\int x^n\cos(ax)\,dx = \frac1a x^n\sin(ax) - \frac{n}{a}\int x^{n-1}\sin(ax)\,dx$,

74. $\int x^ne^{ax}\,dx = \frac{x^ne^{ax}}{a} - \frac{n}{a}\int x^{n-1}e^{ax}\,dx$,

75. $\int x^n\ln(ax)\,dx = x^{n+1}\left(\frac{\ln(ax)}{n+1} - \frac{1}{(n+1)^2}\right)$,

76. $\int x^n(\ln ax)^m\,dx = \frac{x^{n+1}}{n+1}(\ln ax)^m - \frac{m}{n+1}\int x^n(\ln ax)^{m-1}\,dx$.

$x^1 = x^{\underline{1}} = x^{\overline{1}}$

$x^2 = x^{\underline{2}} + x^{\underline{1}} = x^{\overline{2}} - x^{\overline{1}}$

$x^3 = x^{\underline{3}} + 3x^{\underline{2}} + x^{\underline{1}} = x^{\overline{3}} - 3x^{\overline{2}} + x^{\overline{1}}$

$x^4 = x^{\underline{4}} + 6x^{\underline{3}} + 7x^{\underline{2}} + x^{\underline{1}} = x^{\overline{4}} - 6x^{\overline{3}} + 7x^{\overline{2}} - x^{\overline{1}}$

$x^5 = x^{\underline{5}} + 15x^{\underline{4}} + 25x^{\underline{3}} + 10x^{\underline{2}} + x^{\underline{1}} = x^{\overline{5}} - 15x^{\overline{4}} + 25x^{\overline{3}} - 10x^{\overline{2}} + x^{\overline{1}}$

$x^{\overline{1}} = x^1 \qquad x^{\underline{1}} = x^1$

$x^{\overline{2}} = x^2 + x^1 \qquad x^{\underline{2}} = x^2 - x^1$

$x^{\overline{3}} = x^3 + 3x^2 + 2x^1 \qquad x^{\underline{3}} = x^3 - 3x^2 + 2x^1$

$x^{\overline{4}} = x^4 + 6x^3 + 11x^2 + 6x^1 \qquad x^{\underline{4}} = x^4 - 6x^3 + 11x^2 - 6x^1$

$x^{\overline{5}} = x^5 + 10x^4 + 35x^3 + 50x^2 + 24x^1 \qquad x^{\underline{5}} = x^5 - 10x^4 + 35x^3 - 50x^2 + 24x^1$

Finite Calculus

Difference, shift operators:

$\Delta f(x) = f(x+1) - f(x)$,

$E\,f(x) = f(x+1)$.

Fundamental Theorem:

$f(x) = \Delta F(x) \Leftrightarrow \sum f(x)\delta x = F(x) + C$.

$\sum_a^b f(x)\delta x = \sum_{i=a}^{b-1} f(i)$.

Differences:

$\Delta(cu) = c\Delta u$, $\quad \Delta(u+v) = \Delta u + \Delta v$,

$\Delta(uv) = u\Delta v + E\,v\Delta u$,

$\Delta(x^{\underline{n}}) = nx^{\underline{n-1}}$,

$\Delta(H_x) = x^{\underline{-1}}$, $\quad \Delta(2^x) = 2^x$,

$\Delta(c^x) = (c-1)c^x$, $\quad \Delta\binom{x}{m} = \binom{x}{m-1}$.

Sums:

$\sum cu\,\delta x = c\sum u\,\delta x$,

$\sum(u+v)\,\delta x = \sum u\,\delta x + \sum v\,\delta x$,

$\sum u\Delta v\,\delta x = uv - \sum E\,v\Delta u\,\delta x$,

$\sum x^{\underline{n}}\,\delta x = \frac{x^{\underline{n+1}}}{m+1}$, $\quad \sum x^{\underline{-1}}\,\delta x = H_x$,

$\sum c^x\,\delta x = \frac{c^x}{c-1}$, $\quad \sum\binom{x}{m}\delta x = \binom{x}{m+1}$.

Falling Factorial Powers:

$x^{\underline{n}} = x(x-1)\cdots(x-n+1)$, $n>0$,

$x^{\underline{0}} = 1$,

$x^{\underline{n}} = \frac{1}{(x+1)\cdots(x+|n|)}$, $n<0$,

$x^{\underline{n+m}} = x^{\underline{m}}(x-m)^{\underline{n}}$.

Rising Factorial Powers:

$x^{\overline{n}} = x(x+1)\cdots(x+n-1)$, $n>0$,

$x^{\overline{0}} = 1$,

$x^{\overline{n}} = \frac{1}{(x-1)\cdots(x-|n|)}$, $n<0$,

$x^{\overline{n+m}} = x^{\overline{m}}(x+m)^{\overline{n}}$.

Conversion:

$x^{\underline{n}} = (-1)^n(-x)^{\overline{n}} = (x-n+1)^{\overline{n}}$
$= 1/(x+1)^{\overline{-n}}$,

$x^{\overline{n}} = (-1)^n(-x)^{\underline{n}} = (x+n-1)^{\underline{n}}$
$= 1/(x-1)^{\underline{-n}}$,

$x^n = \sum_{k=1}^{n} {n \brace k} x^{\underline{k}} = \sum_{k=1}^{n} {n \brace k} (-1)^{n-k} x^{\overline{k}}$,

$x^{\underline{n}} = \sum_{k=1}^{n} \left[{n \atop k}\right] (-1)^{n-k} x^k$,

$x^{\overline{n}} = \sum_{k=1}^{n} \left[{n \atop k}\right] x^k$.

Theoretical Computer Science Cheat Sheet

Series

Taylor's series:

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2}f''(a) + \cdots = \sum_{i=0}^{\infty} \frac{(x-a)^i}{i!} f^{(i)}(a).$$

Expansions:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots = \sum_{i=0}^{\infty} x^i,$$

$$\frac{1}{1-cx} = 1 + cx + c^2x^2 + c^3x^3 + \cdots = \sum_{i=0}^{\infty} c^i x^i,$$

$$\frac{1}{1-x^n} = 1 + x^n + x^{2n} + x^{3n} + \cdots = \sum_{i=0}^{\infty} x^{ni},$$

$$\frac{x}{(1-x)^2} = x + 2x^2 + 3x^3 + 4x^4 + \cdots = \sum_{i=0}^{\infty} i x^i,$$

$$x^k \frac{d^n}{dx^n}\left(\frac{1}{1-x}\right) = x + 2^n x^2 + 3^n x^3 + 4^n x^4 + \cdots = \sum_{i=0}^{\infty} i^n x^i,$$

$$e^x = 1 + x + \tfrac{1}{2}x^2 + \tfrac{1}{6}x^3 + \cdots = \sum_{i=0}^{\infty} \frac{x^i}{i!},$$

$$\ln(1+x) = x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4 - \cdots = \sum_{i=1}^{\infty} (-1)^{i+1}\frac{x^i}{i},$$

$$\ln\frac{1}{1-x} = x + \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 + \tfrac{1}{4}x^4 + \cdots = \sum_{i=1}^{\infty} \frac{x^i}{i},$$

$$\sin x = x - \tfrac{1}{3!}x^3 + \tfrac{1}{5!}x^5 - \tfrac{1}{7!}x^7 + \cdots = \sum_{i=0}^{\infty} (-1)^i \frac{x^{2i+1}}{(2i+1)!},$$

$$\cos x = 1 - \tfrac{1}{2!}x^2 + \tfrac{1}{4!}x^4 - \tfrac{1}{6!}x^6 + \cdots = \sum_{i=0}^{\infty} (-1)^i \frac{x^{2i}}{(2i)!},$$

$$\tan^{-1} x = x - \tfrac{1}{3}x^3 + \tfrac{1}{5}x^5 - \tfrac{1}{7}x^7 + \cdots = \sum_{i=0}^{\infty} (-1)^i \frac{x^{2i+1}}{(2i+1)},$$

$$(1+x)^n = 1 + nx + \tfrac{n(n-1)}{2}x^2 + \cdots = \sum_{i=0}^{\infty} \binom{n}{i} x^i,$$

$$\frac{1}{(1-x)^{n+1}} = 1 + (n+1)x + \tbinom{n+2}{2}x^2 + \cdots = \sum_{i=0}^{\infty} \binom{i+n}{i} x^i,$$

$$\frac{x}{e^x - 1} = 1 - \tfrac{1}{2}x + \tfrac{1}{12}x^2 - \tfrac{1}{720}x^4 + \cdots = \sum_{i=0}^{\infty} \frac{B_i x^i}{i!},$$

$$\frac{1}{2x}(1 - \sqrt{1-4x}) = 1 + x + 2x^2 + 5x^3 + \cdots = \sum_{i=0}^{\infty} \frac{1}{i+1}\binom{2i}{i} x^i,$$

$$\frac{1}{\sqrt{1-4x}} = 1 + x + 2x^2 + 6x^3 + \cdots = \sum_{i=0}^{\infty} \binom{2i}{i} x^i,$$

$$\frac{1}{\sqrt{1-4x}}\left(\frac{1-\sqrt{1-4x}}{2x}\right)^n = 1 + (2+n)x + \tbinom{4+n}{2}x^2 + \cdots = \sum_{i=0}^{\infty} \binom{2i+n}{i} x^i,$$

$$\frac{1}{1-x}\ln\frac{1}{1-x} = x + \tfrac{3}{2}x^2 + \tfrac{11}{6}x^3 + \tfrac{25}{12}x^4 + \cdots = \sum_{i=1}^{\infty} H_i x^i,$$

$$\frac{1}{2}\left(\ln\frac{1}{1-x}\right)^2 = \tfrac{1}{2}x^2 + \tfrac{3}{4}x^3 + \tfrac{11}{24}x^4 + \cdots = \sum_{i=2}^{\infty} \frac{H_{i-1}x^i}{i},$$

$$\frac{x}{1-x-x^2} = x + x^2 + 2x^3 + 3x^4 + \cdots = \sum_{i=0}^{\infty} F_i x^i,$$

$$\frac{F_n x}{1-(F_{n-1}+F_{n+1})x-(-1)^n x^2} = F_n x + F_{2n}x^2 + F_{3n}x^3 + \cdots = \sum_{i=0}^{\infty} F_{ni} x^i.$$

Ordinary power series:

$$A(x) = \sum_{i=0}^{\infty} a_i x^i.$$

Exponential power series:

$$A(x) = \sum_{i=0}^{\infty} a_i \frac{x^i}{i!}.$$

Dirichlet power series:

$$A(x) = \sum_{i=1}^{\infty} \frac{a_i}{i^x}.$$

Binomial theorem:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k.$$

Difference of like powers:

$$x^n - y^n = (x-y)\sum_{k=0}^{n-1} x^{n-1-k} y^k.$$

For ordinary power series:

$$\alpha A(x) + \beta B(x) = \sum_{i=0}^{\infty} (\alpha a_i + \beta b_i) x^i,$$

$$x^k A(x) = \sum_{i=k}^{\infty} a_{i-k} x^i,$$

$$\frac{A(x) - \sum_{i=0}^{k-1} a_i x^i}{x^k} = \sum_{i=0}^{\infty} a_{i+k} x^i,$$

$$A(cx) = \sum_{i=0}^{\infty} c^i a_i x^i,$$

$$A'(x) = \sum_{i=0}^{\infty} (i+1) a_{i+1} x^i,$$

$$xA'(x) = \sum_{i=1}^{\infty} i a_i x^i,$$

$$\int A(x)\,dx = \sum_{i=1}^{\infty} \frac{a_{i-1}}{i} x^i,$$

$$\frac{A(x) + A(-x)}{2} = \sum_{i=0}^{\infty} a_{2i} x^{2i},$$

$$\frac{A(x) - A(-x)}{2} = \sum_{i=0}^{\infty} a_{2i+1} x^{2i+1}.$$

Summation: If $b_i = \sum_{j=0}^{i} a_i$ then

$$B(x) = \frac{1}{1-x} A(x).$$

Convolution:

$$A(x)B(x) = \sum_{i=0}^{\infty}\left(\sum_{j=0}^{i} a_j b_{i-j}\right) x^i.$$

God made the natural numbers;
all the rest is the work of man.
– Leopold Kronecker

Theoretical Computer Science Cheat Sheet

Series

Expansions:

$$\frac{1}{(1-x)^{n+1}}\ln\frac{1}{1-x} = \sum_{i=0}^{\infty} (H_{n+i} - H_n)\binom{n+i}{i} x^i,$$

$$x^{\overline{n}} = \sum_{i=0}^{\infty} \begin{bmatrix} n \\ i \end{bmatrix} x^i,$$

$$\left(\ln\frac{1}{1-x}\right)^n = \sum_{i=0}^{\infty} \begin{bmatrix} i \\ n \end{bmatrix} \frac{n!x^i}{i!},$$

$$\tan x = \sum_{i=1}^{\infty} (-1)^{i-1} \frac{2^{2i}(2^{2i}-1)B_{2i}x^{2i-1}}{(2i)!},$$

$$\frac{1}{\zeta(x)} = \sum_{i=1}^{\infty} \frac{\mu(i)}{i^x},$$

$$\zeta(x) = \prod_p \frac{1}{1-p^{-x}},$$

$$\zeta^2(x) = \sum_{i=1}^{\infty} \frac{d(i)}{x^i} \quad \text{where } d(n) = \textstyle\sum_{d|n} 1,$$

$$\zeta(x)\zeta(x-1) = \sum_{i=1}^{\infty} \frac{S(i)}{x^i} \quad \text{where } S(n) = \textstyle\sum_{d|n} d,$$

$$\zeta(2n) = \frac{2^{2n-1}|B_{2n}|}{(2n)!}\pi^{2n}, \quad n \in \mathbb{N},$$

$$\frac{x}{\sin x} = \sum_{i=0}^{\infty} (-1)^{i-1} \frac{(4^i - 2)B_{2i}x^{2i}}{(2i)!},$$

$$\left(\frac{1-\sqrt{1-4x}}{2x}\right)^n = \sum_{i=0}^{\infty} \frac{n(2i+n-1)!}{i!(n+i)!} x^i,$$

$$e^x \sin x = \sum_{i=1}^{\infty} \frac{2^{i/2}\sin\frac{i\pi}{4}}{i!} x^i,$$

$$\sqrt{\frac{1-\sqrt{1-x}}{x}} = \sum_{i=0}^{\infty} \frac{(4i)!}{16^i\sqrt{2}(2i)!(2i+1)!} x^i,$$

$$\left(\frac{\arcsin x}{x}\right)^2 = \sum_{i=0}^{\infty} \frac{4^i i!^2}{(i+1)(2i+1)!} x^{2i}.$$

$$\left(\frac{1}{x}\right)^{\overline{-n}} = \sum_{i=0}^{\infty} \begin{Bmatrix} i \\ n \end{Bmatrix} x^i,$$

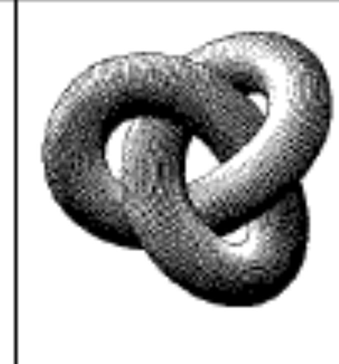
$$(e^x - 1)^n = \sum_{i=0}^{\infty} \begin{Bmatrix} i \\ n \end{Bmatrix} \frac{n!x^i}{i!},$$

$$x \cot x = \sum_{i=0}^{\infty} \frac{(-4)^i B_{2i} x^{2i}}{(2i)!},$$

$$\zeta(x) = \sum_{i=1}^{\infty} \frac{1}{i^x},$$

$$\frac{\zeta(x-1)}{\zeta(x)} = \sum_{i=1}^{\infty} \frac{\phi(i)}{i^x},$$

Escher's Knot



Stieltjes Integration

If G is continuous in the interval $[a,b]$ and F is nondecreasing then

$$\int_a^b G(x)\,dF(x)$$

exists. If $a \le b \le c$ then

$$\int_a^c G(x)\,dF(x) = \int_a^b G(x)\,dF(x) + \int_b^c G(x)\,dF(x).$$

If the integrals involved exist

$$\int_a^b \big(G(x) + H(x)\big)\,dF(x) = \int_a^b G(x)\,dF(x) + \int_a^b H(x)\,dF(x),$$

$$\int_a^b G(x)\,d\big(F(x) + H(x)\big) = \int_a^b G(x)\,dF(x) + \int_a^b G(x)\,dH(x),$$

$$\int_a^b c \cdot G(x)\,dF(x) = \int_a^b G(x)\,d\big(c \cdot F(x)\big) = c\int_a^b G(x)\,dF(x),$$

$$\int_a^b G(x)\,dF(x) = G(b)F(b) - G(a)F(a) - \int_a^b F(x)\,dG(x).$$

If the integrals involved exist, and F possesses a derivative F' at every point in $[a,b]$ then

$$\int_a^b G(x)\,dF(x) = \int_a^b G(x)F'(x)\,dx.$$

Cramer's Rule

If we have equations:

$$\begin{aligned} a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n &= b_1 \\ a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,n}x_n &= b_2 \\ \vdots \qquad\quad \vdots \qquad\qquad & \vdots \\ a_{n,1}x_1 + a_{n,2}x_2 + \cdots + a_{n,n}x_n &= b_n \end{aligned}$$

Let $A = (a_{i,j})$ and B be the column matrix (b_i). Then there is a unique solution iff $\det A \neq 0$. Let A_i be A with column i replaced by B. Then

$$x_i = \frac{\det A_i}{\det A}.$$

Improvement makes strait roads, but the crooked roads without Improvement, are roads of Genius.
– William Blake (The Marriage of Heaven and Hell)

00	47	18	76	29	93	85	34	61	52
86	11	57	28	70	39	94	45	02	63
95	80	22	67	38	71	49	56	13	04
59	96	81	33	07	48	72	60	24	15
73	69	90	82	44	17	58	01	35	26
68	74	09	91	83	55	27	12	46	30
37	08	75	19	92	84	66	23	50	41
14	25	36	40	51	62	03	77	88	99
21	32	43	54	65	06	10	89	97	78
42	53	64	05	16	20	31	98	79	87

The Fibonacci number system: Every integer n has a unique representation

$$n = F_{k_1} + F_{k_2} + \cdots + F_{k_m},$$

where $k_i \ge k_{i+1} + 2$ for all i, $1 \le i < m$ and $k_m \ge 2$.

Fibonacci Numbers

$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \ldots$

Definitions:

$$F_i = F_{i-1} + F_{i-2}, \quad F_0 = F_1 = 1,$$

$$F_{-i} = (-1)^{i-1}F_i,$$

$$F_i = \frac{1}{\sqrt{5}}\left(\phi^i - \hat{\phi}^i\right),$$

Cassini's identity: for $i > 0$:

$$F_{i+1}F_{i-1} - F_i^2 = (-1)^i.$$

Additive rule:

$$F_{n+k} = F_k F_{n+1} + F_{k-1}F_n,$$

$$F_{2n} = F_n F_{n+1} + F_{n-1}F_n.$$

Calculation by matrices:

$$\begin{pmatrix} F_{n-2} & F_{n-1} \\ F_{n-1} & F_n \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^n.$$

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