

CS 480

Introduction to Artificial Intelligence

February 17, 2022

Announcements / Reminders

- **Midterm: February 24th!**
 - **Online section:** please make arrangements. Contact Mr. Charles Scott (scott@iit.edu) if in doubt
- **Written Assignment #01:**
 - **due: TODAY!, 11:00 PM CST**
- **Written Assignment #02:** will be posted this weekend
- **Programming Assignment #01:**
 - **due: March 6th, 11:00 PM CST**
- **Grading TA assignment:**

https://docs.google.com/spreadsheets/d/1avK4P4MDjKZQceG82mSZd0wkYEDH07_DpQqYJHDQctw/edit?usp=sharing

Plan for Today

- **Propositional logic**

Propositional Logic: Laws/Theorems

Equivalence	Law / Theorems
$p \vee q \Leftrightarrow q \vee p$ $p \wedge q \Leftrightarrow q \wedge p$	Commutative laws
$(p \vee q) \vee r \Leftrightarrow p \vee (q \vee r)$ $(p \wedge q) \wedge r \Leftrightarrow p \wedge (q \wedge r)$	Associative laws
$p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$ $p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$	Distributive laws
$\neg (p \wedge q) \Leftrightarrow \neg p \vee \neg q$ $\neg (p \vee q) \Leftrightarrow \neg p \wedge \neg q$	De Morgan's laws
$p \wedge (p \vee q) \Leftrightarrow p$ $p \vee (p \wedge q) \Leftrightarrow p$	Absorption laws
$\neg (\neg p) \Leftrightarrow p$	Double Negation law (involution)
$p \wedge p \Leftrightarrow p$ $p \vee p \Leftrightarrow p$	Idempotent laws
$p \vee \neg p \Leftrightarrow T$	Law of Excluded Middle (Negation law)
$p \wedge \neg p \Leftrightarrow \perp$	Contradiction (Negation law)
$p \wedge T \Leftrightarrow p$ $p \vee \perp \Leftrightarrow p$	Identity laws
$p \wedge \perp \Leftrightarrow \perp$ $p \vee T \Leftrightarrow T$	Domination laws
$\neg p \vee q \Leftrightarrow p \Rightarrow q$	Implication law
$p \Rightarrow q \Leftrightarrow \neg q \Rightarrow \neg p$	Contraposition law
$(p \wedge q) \vee (\neg q \wedge \neg p) \Leftrightarrow (p \Leftrightarrow q)$ $(p \Rightarrow q) \wedge (q \Rightarrow p) \Leftrightarrow (p \Leftrightarrow q)$	Equivalence law

Deduction

Laws/theorems in propositional logic can be used to prove additional theorems through a process known as **deduction**:

Prove that $((\neg m \vee n) \wedge \neg n) \Rightarrow \neg m$ is a tautology:

$$((\neg m \wedge \neg n) \vee (n \wedge \neg n)) \Rightarrow \neg m$$

$$((\neg m \wedge \neg n) \vee \perp) \Rightarrow \neg m$$

$$(\neg m \wedge \neg n) \Rightarrow \neg m$$

$$\neg(\neg m \wedge \neg n) \vee \neg m$$

$$(\neg\neg m \vee \neg\neg n) \vee \neg m$$

$$(m \vee n) \vee \neg m$$

$$m \vee (n \vee \neg m)$$

$$m \vee (\neg m \vee n)$$

$$(m \vee \neg m) \vee n$$

$$\top \vee n$$

$$n \vee \top$$

$$\top$$

by Distributive law $p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$

by Negation law (contradiction) $p \wedge \neg p \Leftrightarrow \perp$

by Identity law $p \vee \perp \Leftrightarrow p$

by Implication law $\neg p \vee q \Leftrightarrow p \Rightarrow q$

by De Morgan's law $\neg(p \wedge q) \Leftrightarrow \neg q \vee \neg p$

by Double Negation law $\neg(\neg p) \Leftrightarrow p$

by Associative law $(p \vee q) \vee r \Leftrightarrow p \vee (q \vee r)$

by Commutative law $p \vee q \Leftrightarrow q \vee p$

by Associative law $(p \vee q) \vee r \Leftrightarrow p \vee (q \vee r)$

by Law of Excluded Middle $p \vee \neg p \Leftrightarrow \top$

by Commutative law $p \vee q \Leftrightarrow q \vee p$

by Domination Law $p \vee \top \Leftrightarrow \top$

Deduction

Laws/theorems in propositional logic can be used to prove additional theorems through a process known as **deduction**:

Note that we only manipulated symbols at the syntactic level!

Prove that $((\neg m \vee n) \wedge \neg n) \Rightarrow \neg m$ is a tautology:

$$((\neg m \wedge \neg n) \vee (n \wedge \neg n)) \Rightarrow \neg m$$

$$((\neg m \wedge \neg n) \vee \perp) \Rightarrow \neg m$$

$$(\neg m \wedge \neg n) \Rightarrow \neg m$$

$$\neg(\neg m \wedge \neg n) \vee \neg m$$

$$(\neg\neg m \vee \neg\neg n) \vee \neg m$$

$$(m \vee n) \vee \neg m$$

$$m \vee (n \vee \neg m)$$

$$m \vee (\neg m \vee n)$$

$$(m \vee \neg m) \vee n$$

$$\top \vee n$$

$$n \vee \top$$

$$\top$$

by Distributive law $p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$

by Negation law (contradiction) $p \wedge \neg p \Leftrightarrow \perp$

by Identity law $p \vee \perp \Leftrightarrow p$

by Implication law $\neg p \vee q \Leftrightarrow p \Rightarrow q$

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by Double Negation law $\neg(\neg p) \Leftrightarrow p$

by Associative law $(p \vee q) \vee r \Leftrightarrow p \vee (q \vee r)$

by Commutative law $p \vee q \Leftrightarrow q \vee p$

by Associative law $(p \vee q) \vee r \Leftrightarrow p \vee (q \vee r)$

by Law of Excluded Middle $p \vee \neg p \Leftrightarrow \top$

by Commutative law $p \vee q \Leftrightarrow q \vee p$

by Domination Law $p \vee \top \Leftrightarrow \top$

Propositional Logic and KB-Agents

**Propositional
Logic:
Syntax**

**Propositional
Logic:
Semantics**

**Propositional
Logic:
Inference and
Proof Systems**

**KB-Agents:
Inference
algorithms**

Interpretation

The truth value assignment to propositional sentences is called an **interpretation** (an assertion about their truth **in some possible world / model**).

Definition: A mapping $I : \Sigma \rightarrow \{\text{true}, \text{false}\}$, which assigns a truth value to **every proposition variable**, is called an **interpretation**.

Sentence: $(p \vee q) \wedge (\neg q \vee r)$

Interpretation i: $p^i = \text{true}$, $q^i = \text{false}$, $r^i = \text{true}$

Truth Values and Truth Tables

Propositional logic sentences can have a **truth value** assigned to them:

- Atomic sentences:
 - either **true** or **false**
- Compound / complex sentence truth value can be established using a truth table:

p	q	$\neg p$	$p \wedge q$	$p \vee q$	$p \Rightarrow q$	$p \Leftrightarrow q$
true	true	false	true	true	true	true
true	false	false	false	true	false	false
false	true	true	false	true	true	false
false	false	true	false	false	true	true

Evaluation

Evaluation is the process of **determining the truth values of compound/complex sentences** given a truth assignment for the **truth values of proposition constants/atomic sentences**. Consider the following truth assignment i:

$p^i = \text{true}, q^i = \text{false}, r^i = \text{true}$

Assignment

Let's evaluate the following complex sentence $(p \vee q) \wedge (\neg q \vee r)$:

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$p^i = \text{true}, q^i = \text{false}, r^i = \text{true}$ Assignment

Let's evaluate the following complex sentence $(p \vee q) \wedge (\neg q \vee r)$:

$(p \vee q) \wedge (\neg q \vee r) \rightarrow (\text{true} \vee \text{false}) \wedge (\neg \text{false} \vee \text{true})$ Subsitute

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$(\text{true} \vee \text{false}) \wedge (\neg \text{false} \vee \text{true})$ Disjunction

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$\text{true} \wedge (\neg \text{false} \vee \text{true})$ Negation

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$\text{true} \wedge (\neg \text{false} \vee \text{true})$ Negation

$\text{true} \wedge (\text{true} \vee \text{true})$ Disjunction

$\text{true} \wedge \text{true}$ Conjunction

true Interpretation

Complex Sentence: Truth Table

Consider a complex sentence **R** built with **N** propositional variables $p_1, p_2, p_3, \dots, p_{N-1}, p_N$ and logical connectives ($\neg, \vee, \wedge, \Rightarrow, \Leftrightarrow$). Here is a corresponding truth table for sentence **R**.

		N Propositional Variables						Complex sentence R		
		p_1	p_2	p_3	...	p_{N-1}	p_N			
2^N Truth Assignments		true	true	true	...	true	true	false	2^N Interpretations of R	
		true	true	true	...	true	false	true		
		true	true	false	...	false	true	false		
			
		false	false	true	...	true	false	true		
		false	false	true	...	false	true	true		
		false	false	false	...	false	false	false		

Complex Sentence: Truth Table

Consider a complex sentence **R** built with **N** propositional variables $p_1, p_2, p_3, \dots, p_{N-1}, p_N$ and logical connectives ($\neg, \vee, \wedge, \Rightarrow, \Leftrightarrow$). Each truth assignment is a different possible world.

N Propositional Variables						Complex sentence R
p_1	p_2	p_3	...	p_{N-1}	p_N	
true	true	true	...	true	true	false
true	true	true	...	true	false	true
true	true	false	...	false	true	false
...
false	false	true	...	true	false	true
false	false	true	...	false	true	true
false	false	false	...	false	false	false

2^N Possible Worlds (Models)

2^N Interpretations of **R**

Sentence: Syntactic / Semantic Levels

Each propositional logic “exists” on two levels:

- Syntactic: where a sentence is just a legal arrangement of language symbols. Sentence

$$(p \vee q) \wedge (\neg q \vee r)$$

WITHOUT interpretation **HAS NO MEANING**

- we can manipulate symbols, but we CANNOT reason
- Semantic: where a sentence has a truth value is assigned to it (interpretation):

$$(p \vee q) \wedge (\neg q \vee r) \text{ where } p^i = \text{true}, q^i = \text{false}, r^i = \text{true}$$

HAS MEANING (through interpretation) → it is true

Sentence: Syntactic / Semantic Levels

Each propositional logic “exists” on two levels:

- Syntactic: where a sentence is just a legal arrangement of language symbols. Sentence

$$(\text{cool} \vee \text{funny}) \Rightarrow \text{popular}$$

WITHOUT interpretation **HAS NO MEANING**

- we can't tell if a given person is popular here
- Semantic: where a sentence has a truth value is assigned to it (interpretation):

$(\text{cool} \vee \text{funny}) \Rightarrow \text{popular}$ where $\text{cool} = \text{true}$, $\text{funny} = \text{false}$
HAS MEANING → we can deduce that a person is popular

Sentence Semantical Equivalence

Two propositional logic sentences F and G are called **semantically equivalent** if they take on the **same interpretation** for all truth value assignments. If that is the case $F \equiv G$.

Example: sentence $\neg a \vee b$ is equivalent to sentence $a \Rightarrow b$. Proof with a truth table:

a	b	$\neg a$	$\neg a \vee b$	\Leftrightarrow	$a \Rightarrow b$
true	true	false	true	\equiv	true
true	false	false	false		false
false	true	true	true		true
false	false	true	true		true

Sentence Classes

SATISFIABLE

A sentence is **satisfiable** if it is **true for AT LEAST ONE interpretation.**

In plain English:
“You can find **AT LEAST one assignment** of logical values of true and false to individual propositional variables that will make this sentence **true.**”

Example:

$$p \Rightarrow q$$

p	q	$p \Rightarrow q$
true	true	true
true	false	false
false	true	true
false	false	true

(LOGICALLY) VALID/TAUTOLOGY

A sentence is (logically) **valid** if it is **true for ALL interpretations.**
Also called a **tautology.**

In plain English:
“This sentence is **ALWAYS true** regardless of value assignment to individual propositional variables.”

Example:

$$p \vee \neg p$$

p	$\neg p$	$p \vee \neg p$
true	false	true
true	false	true
false	true	true
false	true	true

UNSATISFIABLE/CONTRADICTION

A sentence is **unsatisfiable** if it is **NOT true for ANY interpretation.**
Also called a **contradiction.**

In plain English:
“This sentence is **ALWAYS false** regardless of value assignment to individual propositional variables.”

Example:

$$p \wedge \neg p$$

p	$\neg p$	$p \wedge \neg p$
true	false	false
true	false	false
false	true	false
false	true	false

Propositional Logic and KB-Agents

**Propositional
Logic:
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Inference: The idea

The idea:

Given everything (expressed as sentences) that we know, can we infer if some query (another sentence) is true or not (satisfied or not)?

(Automated) Proof System

In AI we are interested in taking existing knowledge (sentences in KB) and from that:

- **deriving new knowledge (new sentences)**
- **answering questions (query sentences)**

In Propositional Logic this means showing that some sentence Q follows from a Knowledge Base KB

where:

- **Q - some query sentence**
- **KB - knowledge base (a sentence made of sentences)**

Inference: Real-life Example

If it is raining, I will need an umbrella. It is raining. Therefore, I will need an umbrella.

Inference: Real-life Example

If it is raining, **then** I will need an umbrella. It is raining. **Therefore**, I will need an umbrella.

Inference: Real-life Example

If it is raining, then I will need an umbrella. It is raining. Therefore, I will need an umbrella.

Propositional Logic: An Argument

An argument A in propositional logic has the following form:

A:	P1	PREMISES
	P2	
	...	
	PN	
	<hr/>	
	∴ C	CONCLUSION

An argument A is said to be **valid** if the implication formed by taking the conjunction of the premiseses (antecedent) and the conclusion C (consequent),

$(P1 \wedge P2 \wedge P3 \wedge \dots \wedge P_N) \Rightarrow C$ is a **tautology**.

Propositional Logic: An Argument

An argument A in propositional logic has the following form:

A:	P1	PREMISES
	P2	
	...	
	PN	
	<hr/>	
	$\therefore C$	CONCLUSION

Premises are taken for granted (assumed to be **true**).

Inference: Real-life Example

If it is raining, then I will need an umbrella.

It is raining.

Therefore, I will need an umbrella.

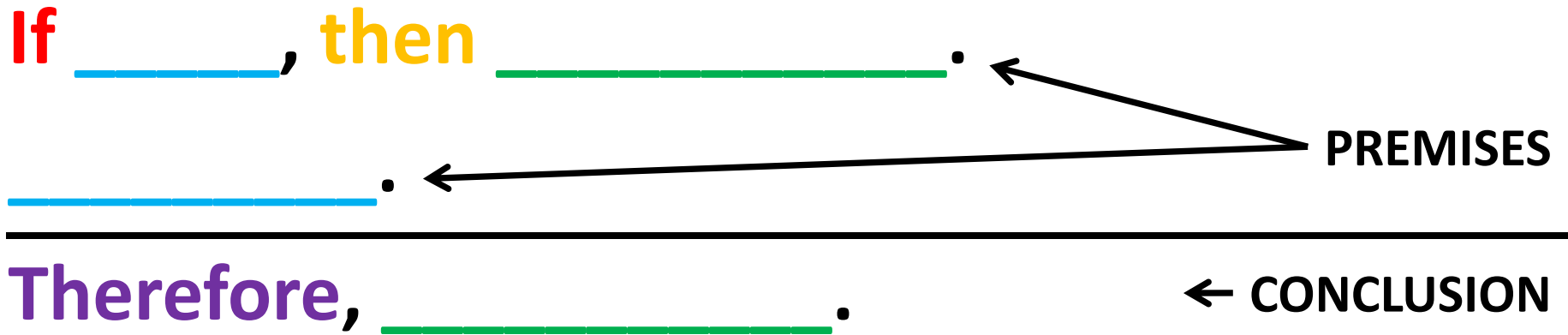
Inference: Real-life Example

If it is raining, then I will need an umbrella.

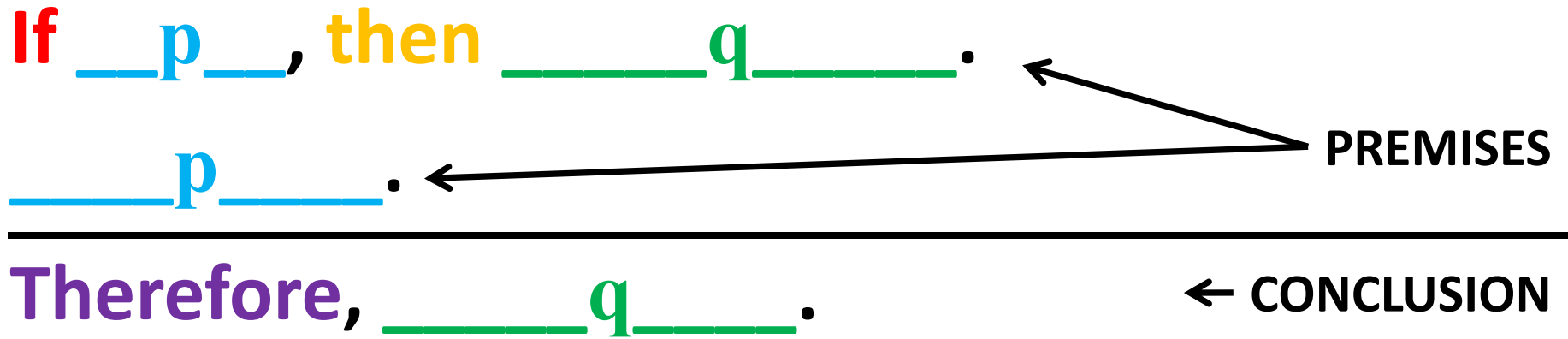
It is raining. ← PREMISES

Therefore, I will need an umbrella. ← CONCLUSION

Inference: Real-life Example



Inference: Real-life Example



p = "It is raining."

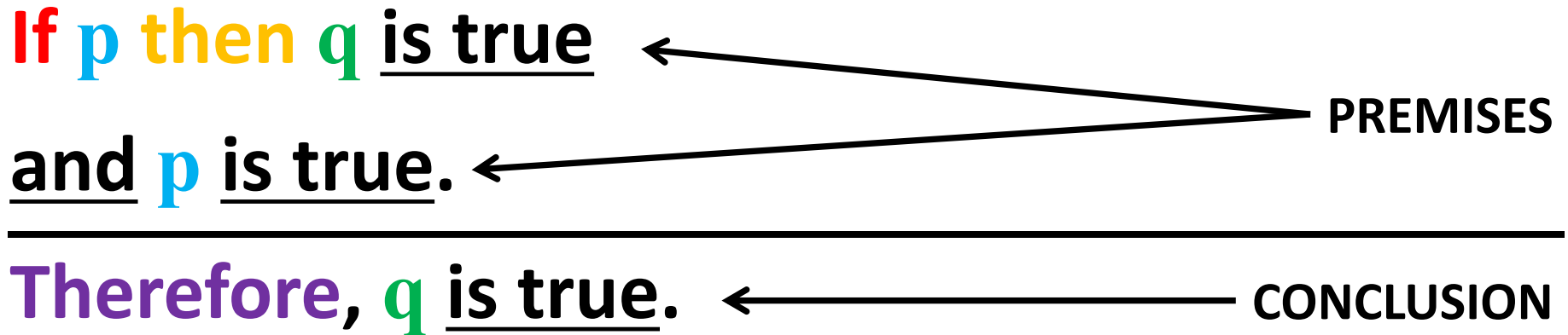
q = "I will need an umbrella."

PREMISE1 = "If it is raining, then I will need an umbrella."

PREMISE2 = "It is raining."

CONCLUSION = "I will need an umbrella."

Inference: Real-life Example



p = "It is raining."

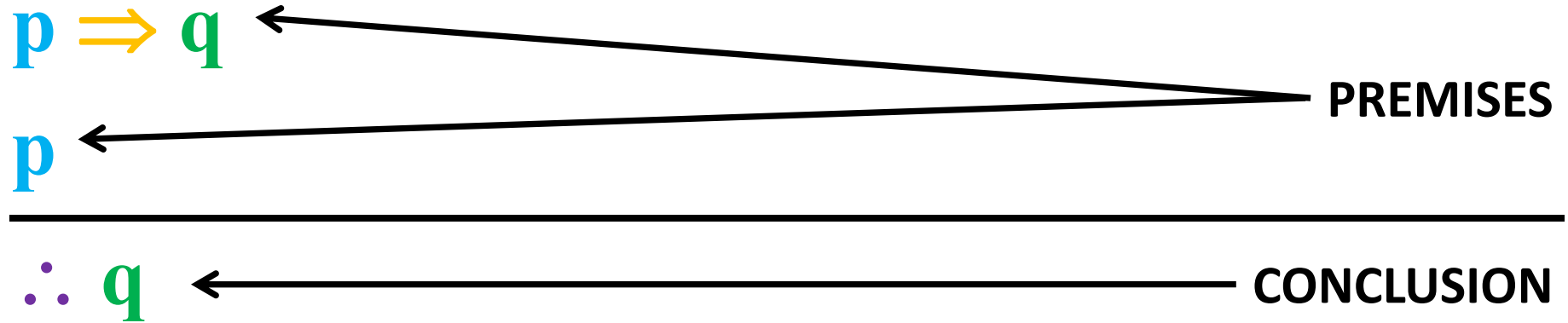
q = "I will need an umbrella."

PREMISE1 = "If it is raining, then I will need an umbrella."

PREMISE2 = "It is raining."

CONCLUSION = "I will need an umbrella."

Inference Rules: Modus Ponens



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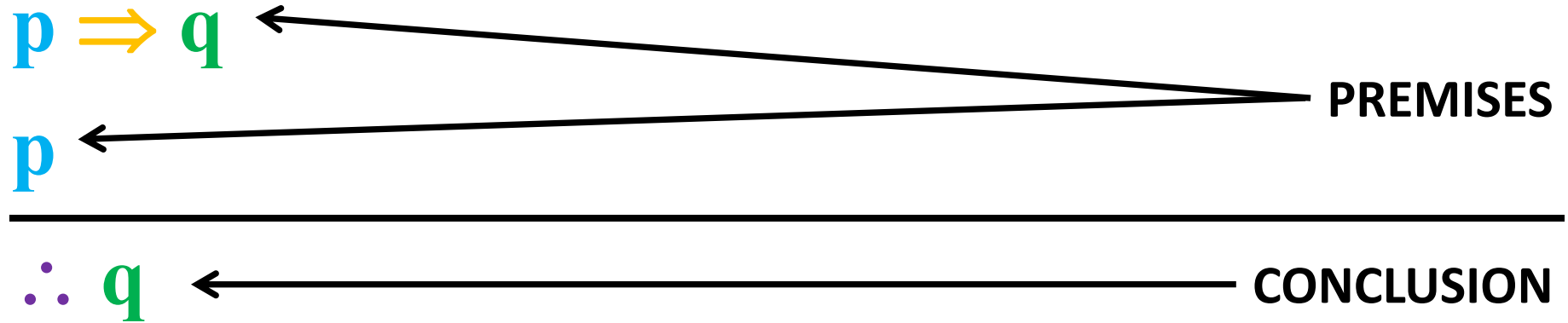
q = "I will need an umbrella."

PREMISE1 = $p \Rightarrow q$

PREMISE2 = p

CONCLUSION = q

Inference Rules: Modus Ponens



p = "It is raining."

q = "I will need an umbrella."

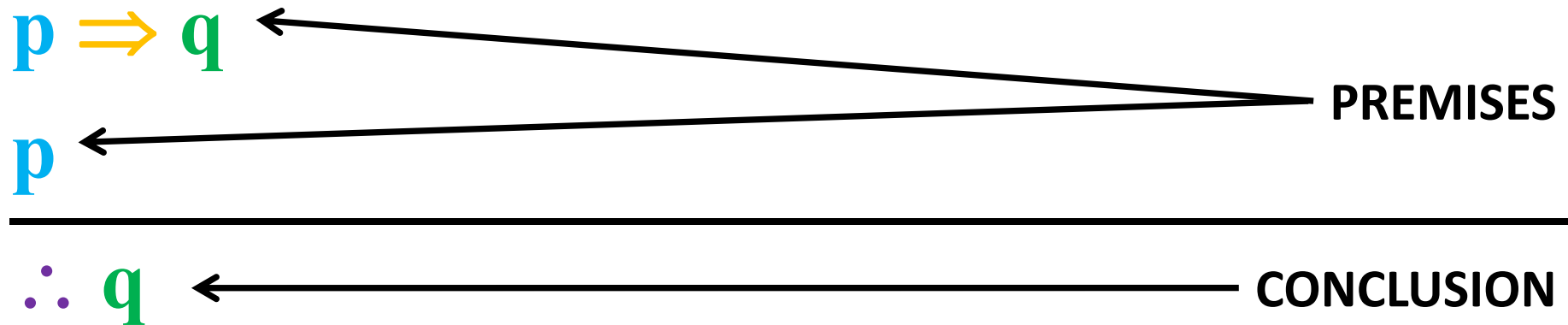
PREMISE1 = $p \Rightarrow q$

PREMISE2 = p

CONCLUSION = q

IF PREMISES ARE TRUE,
THEREFORE THE
CONCLUSION MUST
ALSO BE TRUE

Inference: Modus Ponens



p = "It is raining."

q = "I will need an umbrella."

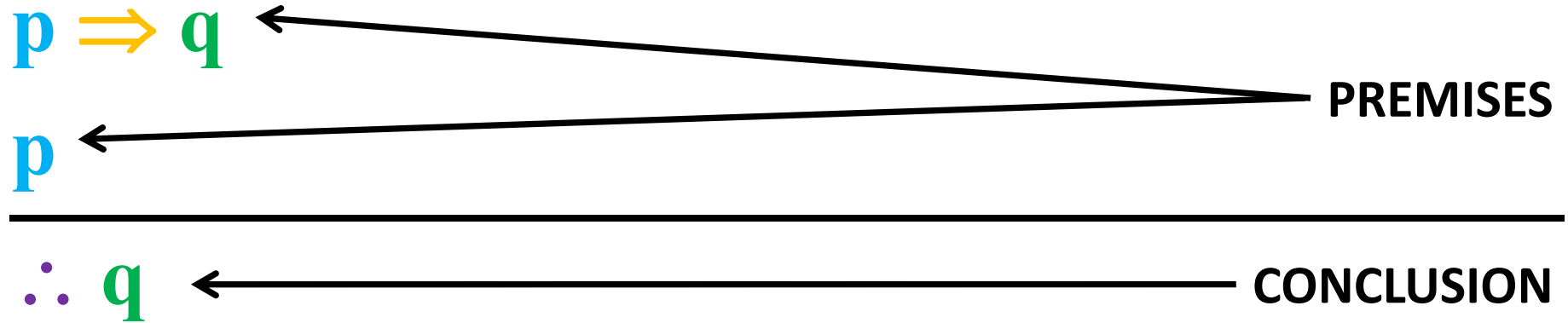
PREMISE1 = $p \Rightarrow q$

PREMISE2 = p

CONCLUSION = q

PROPOSITIONAL VARIABLES		IMPLICATION
p	q	$p \Rightarrow q$
true	true	true
true	false	false
false	true	true
false	false	true

Inference Rules: Modus Ponens



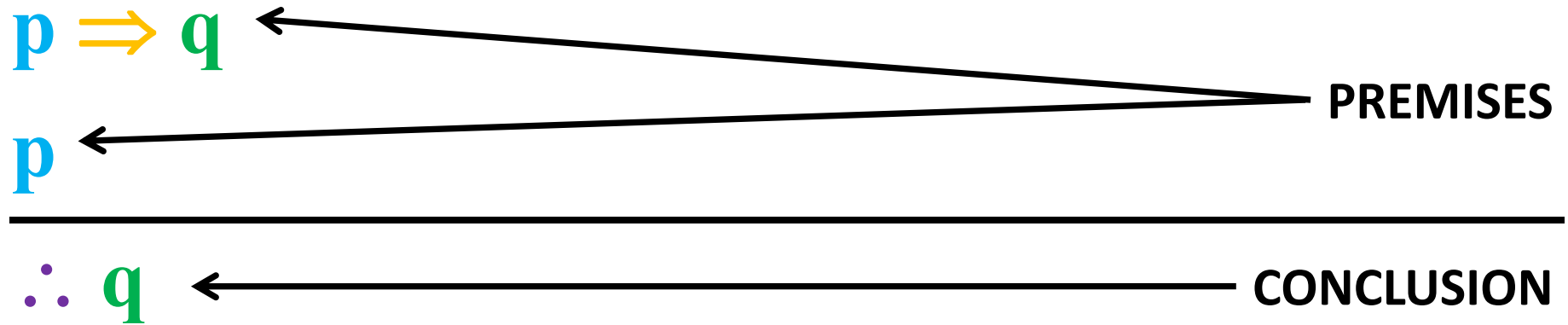
p = "It is raining."

q = "I will need an umbrella."

PREMISES = PREMISE1 AND PREMISE2 = $(p \Rightarrow q) \wedge p$

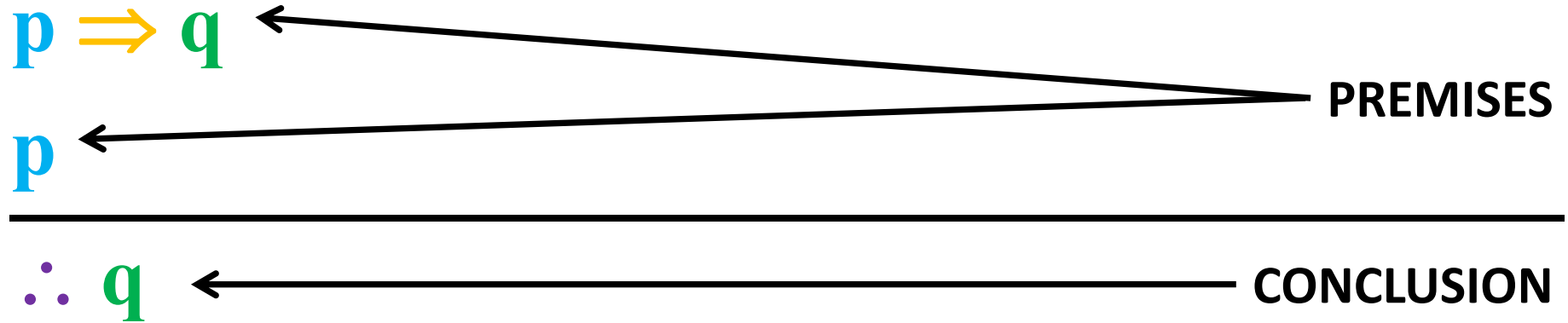
CONCLUSION = q

Inference Rules: Modus Ponens



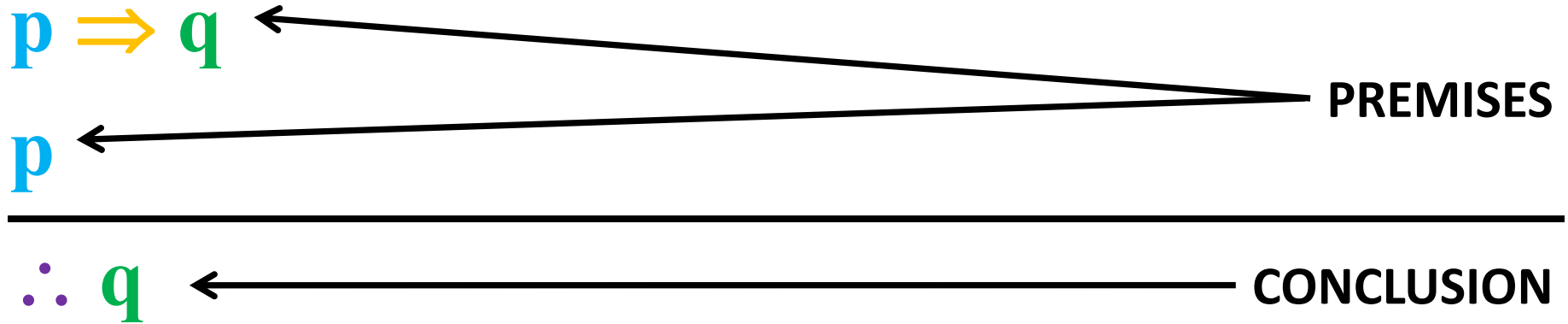
PROPOSITIONAL VARIABLES		INDIVIDUAL PREMISE		PREMISES	CONCLUSION
p	q	P1: $p \Rightarrow q$	P2: p	$P1 \wedge P2: (p \Rightarrow q) \wedge p$	q
true	true	true	true	true	true
true	false	false	true	false	false
false	true	true	false	false	true
false	false	true	false	false	false

Inference Rules: Modus Ponens



PROPOSITIONAL VARIABLES		INDIVIDUAL PREMISE		PREMISES	CONCLUSION
p	q	P1: $p \Rightarrow q$	P2: p	$P1 \wedge P2: (p \Rightarrow q) \wedge p$	q
true	true	true	true	true	true
true	false	false	true	false	false
false	true	true	false	false	true
false	false	true	false	false	false

Inference Rules: Modus Ponens



IF **PREMISES** ARE **TRUE**,
THEREFORE THE
CONCLUSION MUST
ALSO BE TRUE

INDIVIDUAL PREMISE		PREMISES	CONCLUSION
P1: $p \Rightarrow q$	P2: p	$P1 \wedge P2: (p \Rightarrow q) \wedge p$	q
true	true	true	true
false	true	false	false
true	false	false	true
true	false	false	false

Inference Rules: Summary

Rules of Inference:

Modus Ponens $P \Rightarrow Q$ P <hr/> $\therefore Q$	Modus Tollens $P \Rightarrow Q$ $\neg Q$ <hr/> $\therefore \neg P$	Hypothetical Syllogism (Transitivity) $P \Rightarrow Q$ $Q \Rightarrow R$ <hr/> $\therefore P \Rightarrow R$	Conjunction P Q <hr/> $\therefore P \wedge Q$
Addition P <hr/> $\therefore P \vee Q$	Simplification $P \wedge Q$ <hr/> $\therefore P$	Disjunctive Syllogism $P \vee Q$ $\neg P$ <hr/> $\therefore Q$	Resolution $P \vee Q$ $\neg P \vee R$ <hr/> $\therefore Q \vee R$

Tautological forms:

Modus Ponens: $((P \Rightarrow Q) \wedge P) \Rightarrow Q$ | **Modus Tollens:** $((P \Rightarrow Q) \wedge \neg Q) \Rightarrow \neg P$

Hypothetical Syllogism: $((P \Rightarrow Q) \wedge (Q \Rightarrow R)) \Rightarrow (P \Rightarrow R)$

Disjunctive Syllogism: $((P \vee Q) \wedge \neg P) \Rightarrow Q$

Addition: $P \Rightarrow P \vee Q$ | **Simplification:** $(P \wedge Q) \Rightarrow P$

Conjunction: $(P) \wedge (Q) \Rightarrow (P \wedge Q)$ | **Resolution:** $((P \vee Q) \wedge (\neg P \vee R)) \Rightarrow (Q \vee R)$

Argument Validity: Truth Table Proof

$$p \Rightarrow q$$

$$p \Rightarrow \neg r$$

$$\neg p \Rightarrow \neg r$$

$$\therefore \neg r$$

p	q	r	P1:p \Rightarrow q	P2:q \Rightarrow \neg r	P3: \neg p \Rightarrow \neg r	P1 \wedge P2 \wedge P3	(P1 \wedge P2 \wedge P3) \Rightarrow \neg r
true	true	true	true	false	true	false	true
true	true	false	true	true	true	true	true
true	false	true	false	true	true	false	true
true	false	false	false	true	true	false	true
false	true	true	true	false	false	false	true
false	true	false	true	true	true	true	true
false	false	true	true	true	false	false	true
false	false	false	true	true	true	true	true

Argument Validity: Truth Table Proof

$$p \Rightarrow q$$

$$A \Leftrightarrow ((p \Rightarrow q) \wedge (p \Rightarrow \neg r) \wedge (\neg p \Rightarrow \neg r) \Rightarrow \neg r)$$

$$p \Rightarrow \neg r$$

An argument A is **valid** if it is a **tautology**.

$$\neg p \Rightarrow \neg r$$

$$\therefore \neg r$$

p	q	r	P1:p \Rightarrow q	P2:q \Rightarrow \neg r	P3: \neg p \Rightarrow \neg r	P1 \wedge P2 \wedge P3	(P1 \wedge P2 \wedge P3) \Rightarrow \neg r
true	true	true	true	false	true	false	true
true	true	false	true	true	true	true	true
true	false	true	false	true	true	false	true
true	false	false	false	true	true	false	true
false	true	true	true	false	false	false	true
false	true	false	true	true	true	true	true
false	false	true	true	true	false	false	true
false	false	false	true	true	true	true	true

Argument Validity: Truth Table Proof

$$p \Rightarrow q$$

$$A \Leftrightarrow ((P1) \wedge (P2) \wedge (P3) \Rightarrow \neg r)$$

$$p \Rightarrow \neg r$$

An argument A is **valid** if it is a **tautology**.

$$\neg p \Rightarrow \neg r$$

Argument A is valid, because it is a tautology

$$\therefore \neg r$$

(always true regardless of **p**, **q**, **r** truth assignments)

p	q	r	P1: $p \Rightarrow q$	P2: $q \Rightarrow \neg r$	P3: $\neg p \Rightarrow \neg r$	$P1 \wedge P2 \wedge P3$	A
true	true	true	true	false	true	false	true
true	true	false	true	true	true	true	true
true	false	true	false	true	true	false	true
true	false	false	false	true	true	false	true
false	true	true	true	false	false	false	true
false	true	false	true	true	true	true	true
false	false	true	true	true	false	false	true
false	false	false	true	true	true	true	true

Logical Entailment

A set of sentences (called **premises**) logically **entails** a sentence (called a **conclusion**) if and only if **every truth assignment that satisfies the premises also satisfies the conclusion.**

$$\text{PREMISES} \models \text{CONCLUSION}$$

Logical Entailment

Definition: A sentence KB entails a sentence Q (or Q follows from KB) if every model of KB is also a model of Q. We write:

$$KB \models Q$$

In other words:

- For every interpretation in which KB is **true**, Q is also **true**
- “Whenever KB is **true**, Q is also **true**”

Entailment: Deriving Conclusions

You can prove if:

$$KB \models Q$$

is **true** in a number of ways:

- Model checking (enumeration)
- Truth table proof (enumeration)
- Proof by resolution

You can also prove related sentences:

- prove that $KB \wedge \neg Q$ is **unsatisfiable** (by contradiction)
- prove that $KB \Rightarrow Q$ is a **tautology**

Model / “Possible World”

A **model** (a “possible world”) is a single truth assignment / interpretation.

If a sentence U is **true** in model K , K satisfies U .

$M(U)$: set of **ALL** models of U (that satisfy U)

Now:

$KB \models Q$ if and only if $M(KB) \subseteq M(Q)$

$KB \models Q$ is **true** if and only if **in EVERY model** in which KB is **true**, Q is also **true**.

Logical Entailment with Truth Table

$$p \Rightarrow q \quad \text{KB}$$

$$p \Rightarrow \neg r$$

$$\neg p \Rightarrow \neg r$$

$$\text{KB} \Leftrightarrow (p \Rightarrow q) \wedge (p \Rightarrow \neg r) \wedge (\neg p \Rightarrow \neg r)$$

$$Q \Leftrightarrow \neg r$$

$$\therefore \neg r \quad Q$$

Model	p	q	r	P1:p \Rightarrow q	P2:q \Rightarrow \neg r	P3: \neg p \Rightarrow \neg r	KB	Q
M1	true	true	true	true	false	true	false	false
M2	true	true	false	true	true	true	true	true
M3	true	false	true	false	true	true	false	false
M4	true	false	false	false	true	true	false	true
M5	false	true	true	true	false	false	false	false
M6	false	true	false	true	true	true	true	true
M7	false	false	true	true	true	false	false	false
M8	false	false	false	true	true	true	true	true

Entailment: Model Checking

$$p \Rightarrow q \quad \text{KB}$$

$$p \Rightarrow \neg r$$

$$\neg p \Rightarrow \neg r$$

$$\therefore \neg r \quad Q$$

$$\text{KB} \Leftrightarrow (p \Rightarrow q) \wedge (p \Rightarrow \neg r) \wedge (\neg p \Rightarrow \neg r) \mid Q \Leftrightarrow \neg r$$

Models where KB is true: $M(\text{KB}) = \{M2, M6, M8\}$

Models where Q is true: $M(Q) = \{M2, M4, M6, M8\}$

Model	p	q	r	P1: $p \Rightarrow q$	P2: $q \Rightarrow \neg r$	P3: $\neg p \Rightarrow \neg r$	KB	Q
M1	true	true	true	true	false	true	false	false
M2	true	true	false	true	true	true	true	true
M3	true	false	true	false	true	true	false	false
M4	true	false	false	false	true	true	false	true
M5	false	true	true	true	false	false	false	false
M6	false	true	false	true	true	true	true	true
M7	false	false	true	true	true	false	false	false
M8	false	false	false	true	true	true	true	true

Entailment: Model Checking

$$p \Rightarrow q \quad \text{KB}$$

$$p \Rightarrow \neg r$$

$$\neg p \Rightarrow \neg r$$

$$\therefore \neg r \quad Q$$

$$\text{KB} \Leftrightarrow (p \Rightarrow q) \wedge (p \Rightarrow \neg r) \wedge (\neg p \Rightarrow \neg r) \mid Q \Leftrightarrow \neg r$$

Models where KB is true: $M(\text{KB}) = \{M2, M6, M8\}$

Models where Q is true: $M(Q) = \{M2, M4, M6, M8\}$

$M(\text{KB}) \subseteq M(Q)$ so Q follows KB

Model	p	q	r	P1: $p \Rightarrow q$	P2: $q \Rightarrow \neg r$	P3: $\neg p \Rightarrow \neg r$	KB	Q
M1	true	true	true	true	false	true	false	false
M2	true	true	false	true	true	true	true	true
M3	true	false	true	false	true	true	false	false
M4	true	false	false	false	true	true	false	true
M5	false	true	true	true	false	false	false	false
M6	false	true	false	true	true	true	true	true
M7	false	false	true	true	true	false	false	false
M8	false	false	false	true	true	true	true	true

KB \Rightarrow Q is a **Tautology** Proof

$$p \Rightarrow q$$

$$p \Rightarrow \neg r$$

$$\neg p \Rightarrow \neg r$$

$$\therefore \neg r$$

KB \Rightarrow Q is **true** for ALL models / interpretations

KB \Rightarrow Q is a **tautology**



p	q	r	P1:p \Rightarrow q	P2:q \Rightarrow \neg r	P3: \neg p \Rightarrow \neg r	KB	KB \Rightarrow Q
true	true	true	true	false	true	false	true
true	true	false	true	true	true	true	true
true	false	true	false	true	true	false	true
true	false	false	false	true	true	false	true
false	true	true	true	false	false	false	true
false	true	false	true	true	true	true	true
false	false	true	true	true	false	false	true
false	false	false	true	true	true	true	true

Enumeration: Issues

Consider a complex sentence R built with N propositional variables $p_1, p_2, p_3, \dots, p_{N-1}, p_N$ and logical connectives ($\neg, \vee, \wedge, \Rightarrow, \Leftrightarrow$). Each truth assignment is a different possible world.

N Propositional Variables						Complex sentence R
p_1	p_2	p_3	...	p_{N-1}	p_N	
true	true	true	...	true	true	false
true	true	true	...	true	false	true
true	true	false	...	false	true	false
...
false	false	true	...	true	false	true
false	false	true	...	false	true	true
false	false	false	...	false	false	false

2^N Possible Worlds (Models)

2^N Interpretations of R

Logical Entailment

Definition: A sentence **KB** entails sentence **Q** (or **Q** follows from **KB**) if every model of **KB** is also a model of **Q**. We write:

$$\text{KB} \models \text{Q}$$

One more way to look at it:

If **KB** entails **Q**,

- “the truth of **KB** guarantees truth of **Q**”
- “the falsity of **KB** guarantees falsity of **Q**”

Implication | Equivalence | Entailment

IMPLICATION

A sentence is **satisfiable** if it is **true for AT LEAST ONE interpretation.**

In plain English:
true implies true
true DOES NOT imply false
false implies true
false implies false

Notation:
 $KB \Rightarrow Q$

KB	Q	$KB \Rightarrow Q$
true	true	true
true	false	false
false	true	true
false	false	true

EQUIVALENCE

A sentence is (logically) **valid** if it is **true for ALL interpretations.**
Also called a **tautology.**

In plain English:
true equivalent to true
true NOT equivalent to false
false NOT equivalent to true
false equivalent to false

Notation:
 $KB \Leftrightarrow Q$

KB	Q	$KB \Leftrightarrow Q$
true	true	true
true	false	false
false	true	false
false	false	true

ENTAILMENT

A sentence is **unsatisfiable** if it is **NOT true for ANY interpretation.**
Also called a **contradiction.**

In plain English:
true follows from true
false DOES NOT follow from true
true DOES NOT follow from false
false DOES NOT follow from false

Notation:
 $KB \models Q$

KB	Q	$KB \models Q$
true	true	true
true	false	false
false	true	false
false	false	false

Entailment: Deriving Conclusions

You can prove that:

$$KB \models Q$$

is **true** in a number of ways:

- Model checking (enumeration)
- Truth table proof (enumeration)
- Proof by resolution

You can also prove related sentences:

- prove that $KB \wedge \neg Q$ is **unsatisfiable** (by contradiction)
- prove that $KB \Rightarrow Q$ is a **tautology**

Entailment Proofs

$$KB \equiv (p \Rightarrow q) \wedge (p \Rightarrow \neg r) \wedge (\neg p \Rightarrow \neg r) \quad | \quad Q \equiv \neg r$$

Prove that $(KB \Rightarrow Q) \Leftrightarrow T$
 (Show that $KB \Rightarrow Q$ is a
tautology)

Prove that $(KB \wedge \neg Q) \Leftrightarrow \perp$
 (Show that $KB \wedge \neg Q$ is a
contradiction)

Proof by model checking
 Show that all models that are **true**
 for Q are also **true** for KB

Model	p	q	r	$p \Rightarrow q$	$q \Rightarrow \neg r$	$\neg p \Rightarrow \neg r$	KB	Q	$KB \Rightarrow Q$	$KB \wedge \neg Q$
M1	true	true	true	true	false	true	false	false	true	false
M2	true	true	false	true	true	true	true	true	true	false
M3	true	false	true	false	true	true	false	false	true	false
M4	true	false	false	false	true	true	false	true	true	false
M5	false	true	true	true	false	false	false	false	true	false
M6	false	true	false	true	true	true	true	true	true	false
M7	false	false	true	true	true	false	false	false	true	false
M8	false	false	false	true	true	true	true	true	true	false

Entailment Proofs

$$KB \equiv (p \Rightarrow q) \wedge (p \Rightarrow \neg r) \wedge (\neg p \Rightarrow \neg r) \quad | \quad Q \equiv \neg r$$

Prove that $(KB \Rightarrow Q) \Leftrightarrow T$
(Show that $KB \Rightarrow Q$ is a
tautology)

Prove that $(KB \wedge \neg Q) \Leftrightarrow \perp$
(Show that $KB \wedge \neg Q$ is a
contradiction)

Proof by model checking
Show that all models that are **true**
for Q are also **true** for KB

$KB \Rightarrow Q$ is **true** for all models,
so KB entails Q

Model	p	q	r	$p \Rightarrow q$	$q \Rightarrow \neg r$	$\neg p \Rightarrow \neg r$	KB	Q	$KB \Rightarrow Q$	$KB \wedge \neg Q$
M1	true	true	true	true	false	true	false	false	true	false
M2	true	true	false	true	true	true	true	true	true	false
M3	true	false	true	false	true	true	false	false	true	false
M4	true	false	false	false	true	true	false	true	true	false
M5	false	true	true	true	false	false	false	false	true	false
M6	false	true	false	true	true	true	true	true	true	false
M7	false	false	true	true	true	false	false	false	true	false
M8	false	false	false	true	true	true	true	true	true	false

Entailment Proofs

$$KB \equiv (p \Rightarrow q) \wedge (p \Rightarrow \neg r) \wedge (\neg p \Rightarrow \neg r) \quad | \quad Q \equiv \neg r$$

Prove that $(KB \Rightarrow Q) \Leftrightarrow T$
(Show that $KB \Rightarrow Q$ is a
tautology)

Prove that $(KB \wedge \neg Q) \Leftrightarrow \perp$
(Show that $KB \wedge \neg Q$ is a
contradiction)

Proof by model checking
Show that all models that are **true**
for Q are also **true** for KB

$KB \Rightarrow Q$ is **true** for all models,
so KB entails Q

$KB \wedge \neg Q$ is **false** for all models,
so KB entails Q

Model	p	q	r	$p \Rightarrow q$	$q \Rightarrow \neg r$	$\neg p \Rightarrow \neg r$	KB	Q	$KB \Rightarrow Q$	$KB \wedge \neg Q$
M1	true	true	true	true	false	true	false	false	true	false
M2	true	true	false	true	true	true	true	true	true	false
M3	true	false	true	false	true	true	false	false	true	false
M4	true	false	false	false	true	true	false	true	true	false
M5	false	true	true	true	false	false	false	false	true	false
M6	false	true	false	true	true	true	true	true	true	false
M7	false	false	true	true	true	false	false	false	true	false
M8	false	false	false	true	true	true	true	true	true	false

Entailment Proofs

$$KB \equiv (p \Rightarrow q) \wedge (p \Rightarrow \neg r) \wedge (\neg p \Rightarrow \neg r) \quad | \quad Q \equiv \neg r$$

Prove that $(KB \Rightarrow Q) \Leftrightarrow T$
(Show that $KB \Rightarrow Q$ is a
tautology)

$KB \Rightarrow Q$ is **true** for all models,
so KB entails Q

Prove that $(KB \wedge \neg Q) \Leftrightarrow \perp$
(Show that $KB \wedge \neg Q$ is a
contradiction)

$KB \wedge \neg Q$ is **false** for all models,
so KB entails Q

Proof by model checking
Show that all models that are **true**
for Q are also **true** for KB



$M(KB) \subseteq M(Q)$ so KB entails Q

Model	p	q	r	$p \Rightarrow q$	$q \Rightarrow \neg r$	$\neg p \Rightarrow \neg r$	KB	Q	$KB \Rightarrow Q$	$KB \wedge \neg Q$
M1	true	true	true	true	false	true	false	false	true	false
M2	true	true	false	true	true	true	true	true	true	false
M3	true	false	true	false	true	true	false	false	true	false
M4	true	false	false	false	true	true	false	true	true	false
M5	false	true	true	true	false	false	false	false	true	false
M6	false	true	false	true	true	true	true	true	true	false
M7	false	false	true	true	true	false	false	false	true	false
M8	false	false	false	true	true	true	true	true	true	false

Model Checking as a Search Problem

Model checking can be considered a search problem. Searching a truth table for models in which **KB** entails **Q** (**Q** follows from **KB**). It is a $O(2^N)$ problem.

N Propositional Variables							KB ⊨ Q
p ₁	p ₂	p ₃	...	p _{N-1}	p _N		
true	true	true	...	true	true	false	
true	true	true	...	true	false	true	
true	true	false	...	false	true	false	
...	
false	false	true	...	true	false	true	
false	false	true	...	false	true	true	
false	false	false	...	false	false	false	

2^N Possible Worlds (Models)

2^N Interpretations