Dhirubhai Ambani Institute of Information and Communication Technology

Introduction to Communication Systems (CT216)

Polar codes

Group - 6(2)

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- We have not copied the work that someone else has done.
- Concepts, understanding, and insights we will be describing are our own.
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1 Implementation of Polar Code

1.1 Channel polarization

Channel polarization (also known as polarization transform) includes 2 steps: **(i) Channel Combining** and **(ii) Channel Splitting**

Channel Combining

Channel combining is a recursive process. Suppose we have N copies of channel W with inputs u_0, u_1, u_{N-1}.

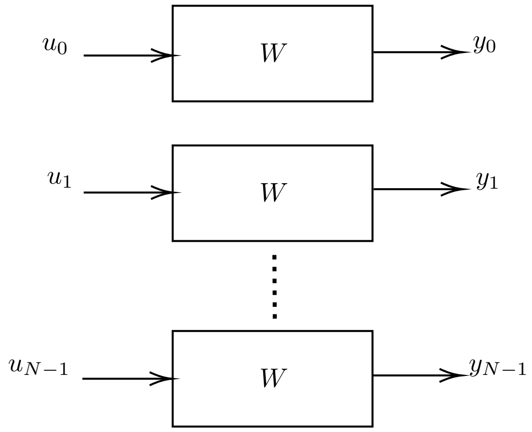


Figure 1: N identical copies of channel W

At any stage of combining, we take 2 channels with inputs $\boldsymbol{x_1}$ and $\boldsymbol{x_2}$, then perform following transformation:

$$(\boldsymbol{x_1}, \boldsymbol{x_2}) \mapsto (\boldsymbol{x_1} \oplus \boldsymbol{x_2}, \boldsymbol{x_2})$$

It is possible that inputs x_1 and x_2 have inputs of more than one bits. So, the inputs with more than one bits are displayed as bold characters. If there are multiple bits in input, we do element wise summation (mod 2) operation to transform it into the form displayed above.

We perform this operation until we get combined channel W_N.

Let us see an example of combining channel W into W_2 and then W_4 to gain more understanding of how the process of channel combining works.

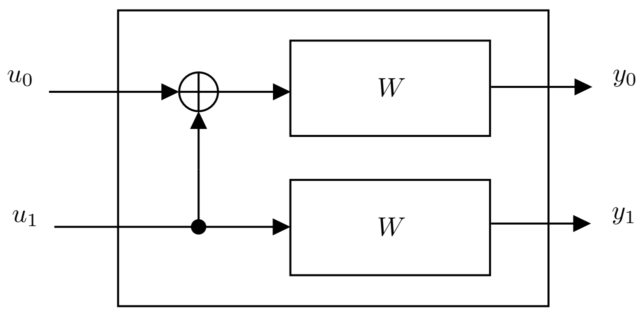


Figure 2: Constructing W_2 channel

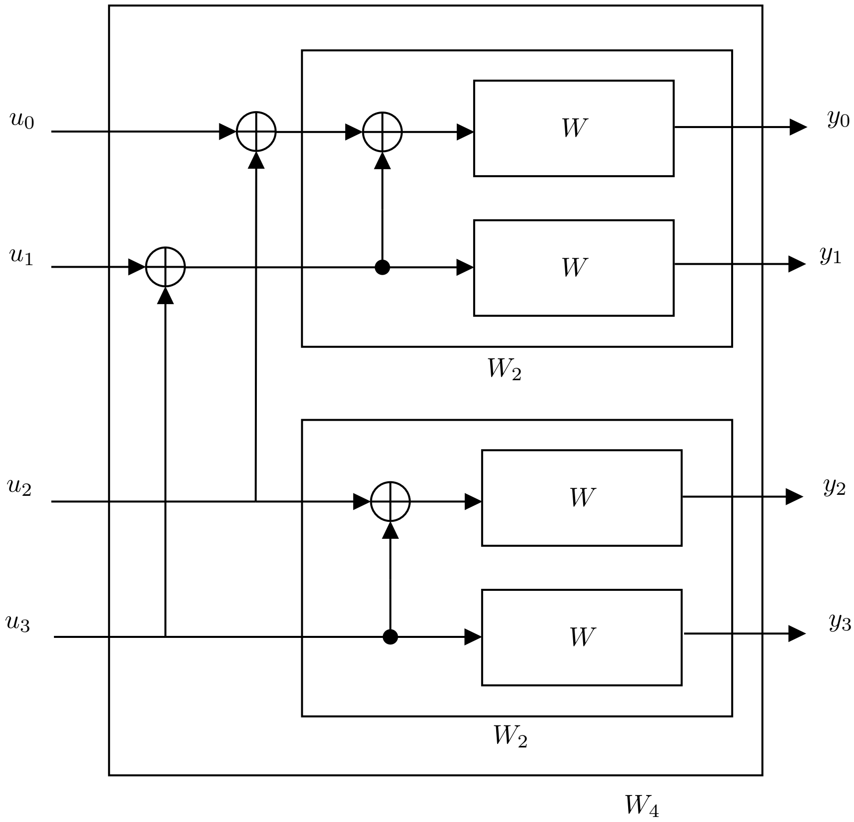


Figure 3: Constructing W_4 channel

Channel Splitting

After the process of channel combination, we split obtained multi-input channel W_N back into set of N channels. This process is shown in figure below:

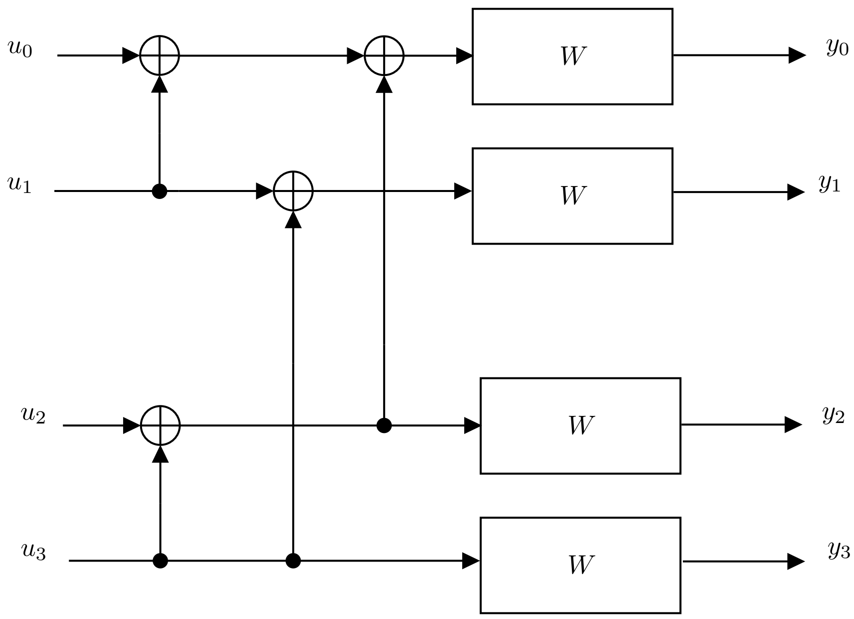


Figure 4: Splitting the W_4 channel

Let us demonstrate a specific example of Binary Erasure Channel with bit erasure probability p and analyze how the process of channel polarization affects capacity of channel W.

A BEC(p) is defined as a channel with an erasure probability p. This channel have a probability 1-p to arrive correctly at the receiver and probability p to be erased.

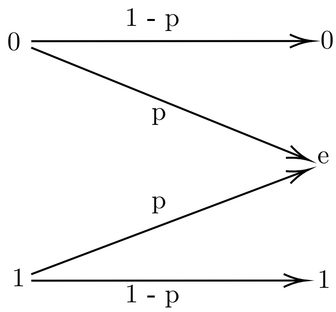


Figure 5: BEC(p) channel

Sending each bit separately on a BEC(p) channel would actually yield two identical channels through which both bits are sent. This means they both experience a probability of error of p. So for decoding the both bit, there is (1-p) probability for get each decoded bit correct. In a simple version of polar code with only two bits, instead of sending the bits, the XOR of the bits and the second bit are transmitted. This is called as polar transform by which eventually we get one channel as noiseless and the other very noisy.

The Polar transform creates two new channels W- and W+. Y_1 and Y_2 are used to decode U_1 under W- channel and the value for U_1 is assumed to be correct. Using U_1, Y_1 and Y_2; U_2 is decoded under the W+ channel.

Figure 6: Polarized BEC channel diagram

$$2I(W) = I(U_1U_2; Y_1Y_2) \quad \text{where } Y_1 = U_1 \oplus U_2 \text{ and } Y_2 = U_2$$
$$2I(W) = I(U_1; Y_1Y_1) + I(U_2; Y_1Y_2/U_1)$$
$$2I(W) = I(U_1; Y_1Y_1) + I(U_2; Y_1Y_2U_1)$$
$$2I(W) = I(W-) + I(W+)$$

1.2 Analyzing W- Channel

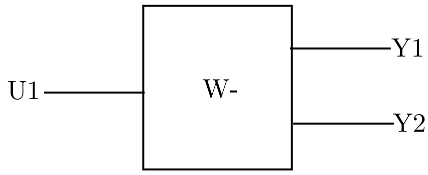


Figure 7: W- Channel

$$
\begin{aligned}
U_1 &: -(Y_1, Y_2) \quad \text{probability} = (1-p)^2 \\
U_1 &: -(?, Y_2) \quad \text{probability} = (1-p) \cdot p \\
U_1 &: -(Y_1, ?) \quad \text{probability} = (1-p) \cdot p \\
U_1 &: -(?, ?) \quad \text{probability} = p^2
\end{aligned}
$$

It is observed that if Y_1 is correctly received then $U_1 \oplus U_2$ is known. If Y_2 is correct, U_2 is known, thus both of them have to be received correctly in order to decode U_1, because $U_1 = Y_1 \oplus Y_2$. The probability for this is $1 - p^2$. So BEC(p) made W_- has an erasure probability of $1 - (1-p)^2$.

1.3 Analysing W+ channel

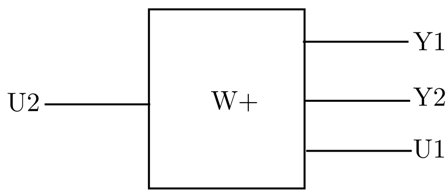


Figure 8: W+ channel

$$
\begin{aligned}
U_2 &: -(Y_1, Y_2, U_1) \quad \text{probability} = (1-p)^2 \\
U_2 &: -(?, Y_2, U_1) \quad \text{probability} = (1-p) \cdot p \\
U_2 &: -(Y_1, ?, U_1) \quad \text{probability} = (1-p) \cdot p \\
U_2 &: -(?, ?, U_1) \quad \text{probability} = p^2
\end{aligned}
$$

It is observed that if Y_2 is correctly received then U_2 is also received correctly. If we assume that U_1 is correct and Y_1 is correctly received, then U_2 can also be received correctly because $U_2 = U_1 \oplus Y_1$, so that the probability of W+ channel is p^2, because this will fail only when both Y_1 and Y_2 have errors.

Using the polar transform, the polarized channels of the two bits become BEC($1 - (1-p)^2$) for the first bit and BEC(p^2) for the second one. W+ is a better channel than the initial channel, as p^2 is a smaller error probability than p, so we can say it became less noisy. On the other hand, W- is a worse channel compared to BEC(p) since $1 - (1-p)^2$ is higher than the initial channel error p, so we can say it became more noisy.

1.4 Analysing 4 bit input channel

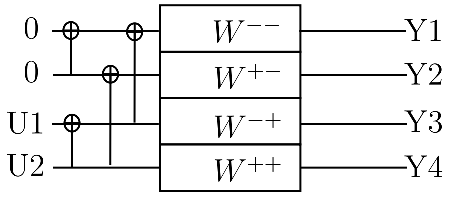


Figure 9: 4 bit input polarized channel

In the above diagram, the 2 bits of input are frozen. So the probability of finding U_1 in the above channel would be $2p^2 - p^4$, and the probability of finding U_2 is p^4. Both of these probabilities are less than the error probability of channel p. So we can say that when the number of channels increases, the error probability decreases.

1.5 Probability diagram of Polar codes

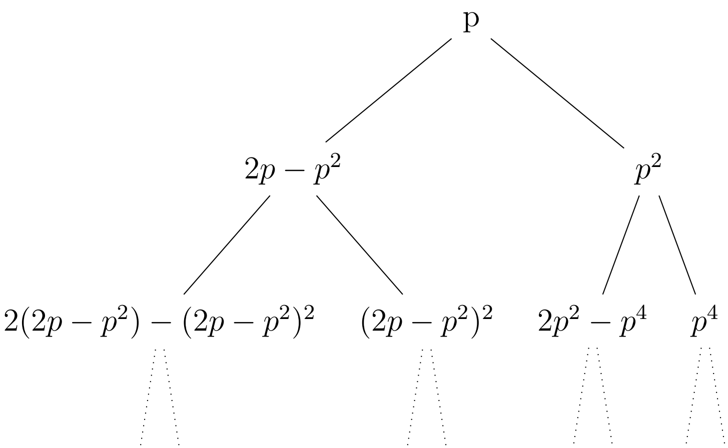


Figure 10: error probability tree of channel polarization

Here, p represents the error probability of the channel. The left side denotes the probability of error occurring during the decoding of the input of the W- channel, while the right side represents the probability of error occurring during the decoding of the input of the W+ channel.

2 Encoding

2.1 Generator Matrix

Polar code encoding involves the use of kernel matrix G2 which is defined as below,

$$G_2 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

For N bit message where N is a power of 2 (N = 2^k where k=1,2,3...) we take kronecker product of G_2 k times. this results in formation of $N \times N$ generator matrix. We multiple message bits with generator matrix G_N to form an encoded message sequence. which is then transmitted over BPSK + AWGN channel. It can be better understood by the binary tree diagram such as below.

$$G_4 = G_2 \bigotimes G_2$$

$$G_4 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \bigotimes \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

In general, for length N codeword, we can write Generator matrix as,

$$G_N = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}^{\otimes n}$$

where $n = log_2(N)$

2.2 Tree representation of Polar codes (polar transformation)

In the above tree we have taken the example for N=8 bits where there are 8 leaf nodes in the graph and have depth 3. Where each leaf is assigned a bit starting from u_1,u_2.... till u_8. At the leaf level two bits are combined to form a 2 bit vector. the resultant 2 bit vector is formed by taking modulo 2 sum (xor) of left node and right node and passing right node as it is. This process is done recursively till the root node. And at the root node we will receive a N bit long encoded message which is the same as we get by multiplying it with the generator matrix G_N.

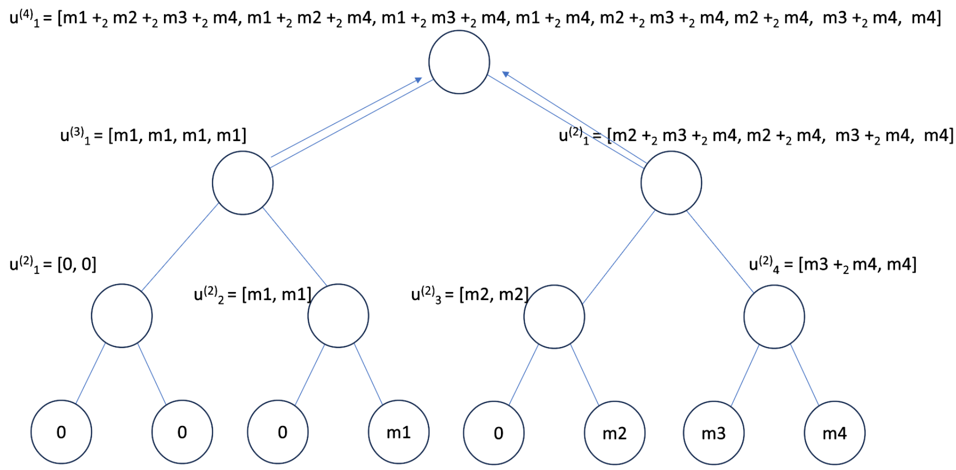


Figure 11: encoding tree of channel polarization

3 Decoding

3.1 Successive Cancellation Decoder (SC)

Because of the recursive structure of channel polarization, we can use a simple decoding scheme called Successive Cancellation (SC) for the decoding purpose.

Let us take an example of length 2 SC decoder to understand how it works.

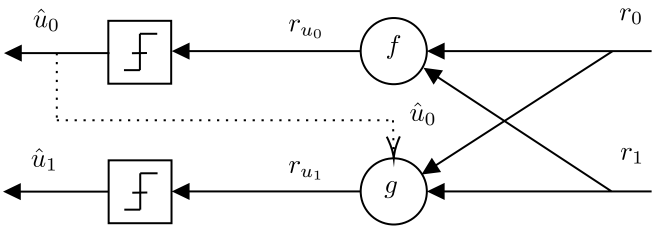


Figure 12: Length 2 SC decoder

As we have seen how to polarize a channel in previous section, we can say that decoding problem reduces to decoding a single parity check code (SPC) to make an estimate for u_0. After making decision for u_0, we can use it to decode u_1, which turns out to decode a repetition code.

As we are transmitting bits in AWGN channel with BPSK modulation, we can assume that we receive LLR (beleif) of transmitted bits.

Let us define LLRs for further calculations.

$$\begin{aligned} L_{r_0} &= log\left(\frac{p\,(r_0=0)}{p\,(r_0=1)}\right) = log\left(\frac{p_{r_0}}{1-p_{r_0}}\right) \\ L_{r_1} &= log\left(\frac{p\,(r_1=0)}{p\,(r_1=1)}\right) = log\left(\frac{p_{r_1}}{1-p_{r_1}}\right) \end{aligned} \tag{1}$$

Where r_0 and r_1 are received bits.

Let us define LLRs for u_0 and u_1 as well.

$$\begin{aligned} L_{u_0} &= log\left(\frac{p\,(u_0=0)}{p\,(u_0=1)}\right) = log\left(\frac{p_{u_0}}{1-p_{u_0}}\right) \\ L_{r_1} &= log\left(\frac{p\,(r_1=0)}{p\,(r_1=1)}\right) = log\left(\frac{p_{u_1}}{1-p_{u_1}}\right) \end{aligned} \tag{2}$$

Now, problem of finding LLR for bit u_0 reduces to single parity check(SPC) decoding of r_0 and r_1.

Transmitted bit u_0 is 0 when both r_1 and r_2 are either 0 or 1. Thus we can write,

$$p_{u_0} = p_{r_0}\,p_{r_1} + (1-p_{r_0})\,(1-p_{r_1}) \tag{3}$$

$$1-p_{u_0} = p_{r_0}\,(1-p_{r_1}) + (1-p_{r_0})\,p_{r_1} \tag{4}$$

Subtracting 4 from 3, we get,

$$
\begin{aligned}
p_{u_0} - (1 - p_{u_0}) &= (p_{r_0} - (1 - p_{r_0}))\,(p_{r_1} - (1 - p_{r_1})) \\
\Rightarrow \frac{p_{u_0} - (1 - p_{u_0})}{p_{u_0} + (1 - p_{u_0})} &= \frac{(p_{r_0} - (1 - p_{r_0}))}{(p_{r_0} + (1 - p_{r_0}))}\,\frac{(p_{r_1} - (1 - p_{r_1}))}{(p_{r_1} + (1 - p_{r_1}))} \\
\Rightarrow \frac{1 - \frac{(1-p_{u_0})}{p_{u_0}}}{1 + \frac{(1-p_{u_0})}{p_{u_0}}} &= \left(\frac{1 - \frac{(1-p_{r_0})}{p_{r_0}}}{1 + \frac{(1-p_{r_0})}{p_{r_0}}}\right)\left(\frac{1 - \frac{(1-p_{r_1})}{p_{r_1}}}{1 + \frac{(1-p_{r_1})}{p_{r_1}}}\right) \\
\Rightarrow \frac{1 - exp(-L_{u_0})}{1 + exp(-L_{u_0})} &= \left(\frac{1 - exp(-L_{r_0})}{1 + exp(-L_{r_0})}\right)\left(\frac{1 - exp(-L_{r_0})}{1 + exp(-L_{r_0})}\right) \\
\Rightarrow tanh\left(\frac{L_{u_0}}{2}\right) &= tanh\left(\frac{L_{r_0}}{2}\right)\,tanh\left(\frac{L_{r_1}}{2}\right) \\
\Rightarrow L_{u_0} &= 2\,tanh^{-1}\left(tanh\left(\frac{L_{r_0}}{2}\right)\,tanh\left(\frac{L_{r_1}}{2}\right)\right)
\end{aligned}
$$

Above function can be approximated as,

$$\Rightarrow L_{u_0} = sgn(L_{r_0})\,sgn(L_{r_1})\,min(|L_{r_0}|, |L_{r_1}|)$$

This function is called minsum function. Let us denote minsum function as

$$f(x_0, x_1) = sgn(x_0)\,sgn(x_1)\,min(|x_0|, |x_1|)$$

Thus, above equation can be rewritten as,

$$L_{u_0} = f(L_{r_0}, L_{r_1})$$

By calculating LLR value for u_0, we can make estimate for u_0 as follow:

$$\hat{u}_0 = \begin{cases} 0, & \text{if } L_{u_0} \geq 0 \\ 1, & \text{if } L_{u_0} < 0 \end{cases}$$

Now, using these estimated value of u_0, we can make estimate for u_1. Suppose $\hat{u}_0 = 0$, then L_{r_0} and L_{r_0} both becomes beleif for u_1. In other case, if $\hat{u}_0 = 1$, then L_{r_0} becomes beleif for and $\bar{u}_1$ and L_{r_0} becomes beleif for u_1.

Thus, problem of estimating bit u_1 reduces to make an estimate for repetition code.Assume that $\hat{u}_0 = 0$. For repetition code, we know that,

$$p_{u_1} = \frac{p_{r_0}\,p_{r_1}}{p_{r_0}\,p_{r_1} + (1 - p_{r_0})\,(1 - p_{r_1})} \tag{5}$$

$$1 - p_{u_1} = \frac{(1 - p_{r_0})\,(1 - p_{r_1})}{p_{r_0}\,p_{r_1} + (1 - p_{r_0})\,(1 - p_{r_1})} \tag{6}$$

Dividing 5 by 6, we get,

$$\frac{p_{u_1}}{1 - p_{u_1}} = \left(\frac{p_{r_0}}{1 - p_{r_0}}\right)\left(\frac{p_{r_1}}{1 - p_{r_1}}\right)$$

Taking log on both sides, we get,

$$\log\left(\frac{p_{u_1}}{1-p_{u_1}}\right) = \log\left(\frac{p_{r_0}}{1-p_{r_0}}\right) + \log\left(\frac{p_{r_1}}{1-p_{r_1}}\right)$$

$$\Rightarrow L_{u_1} = L_{r_0} + L_{r_1} \qquad \text{(From 1 and 2)}$$

Similarly, for $\hat{u}_0 = 1$, we get following expression using same steps,

$$L_{u_1} = L_{r_1} - L_{r_0}$$

Combining both the equations above, we get:

$$L_{u_1} = L_{r_1} + (-1)^{\hat{u}_0} L_{r_0}$$

Let us define function $g(x)$ as below:

$$g(x_0, x_1, y) = x_1 + (-1)^y x_0$$

Thus, above equation can be rewritten as,

$$L_{u_1} = g(r_0, r_1, \hat{u}_0)$$

After calculating LLR for u_1, we can make estimate for u_1 as,

$$\hat{u}_1 = \begin{cases} 0, & \text{if } L_{u_1} \geq 0 \\ 1, & \text{if } L_{u_2} < 0 \end{cases}$$

Now, as we have some understanding how decoding works in sequential manner, we can fully understand functioning of SC decoder depicted in ??.

We first pass received LLRs through node f, then we make decision for bit u_0. Using this decision, we pass the values to node g and make decision for bit u_1 and the decoding process is done.

As polar codes have recursive structure, we can apply this algorithm for decoding any length of codeword. Let us take an example of length 4 SC decoder and understand its behaviour.

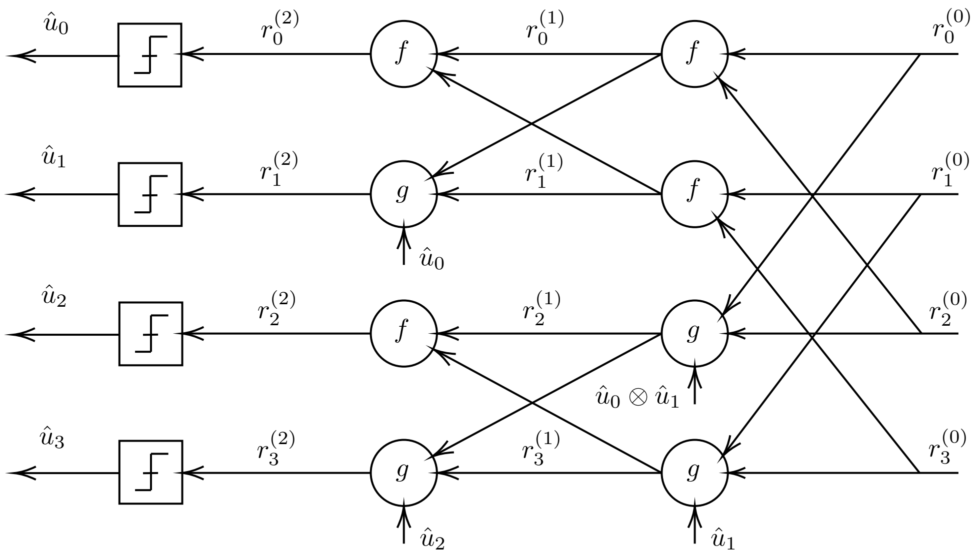


Figure 13: Length 4 SC decoder

Let us see decoding process stepwise with different figures.

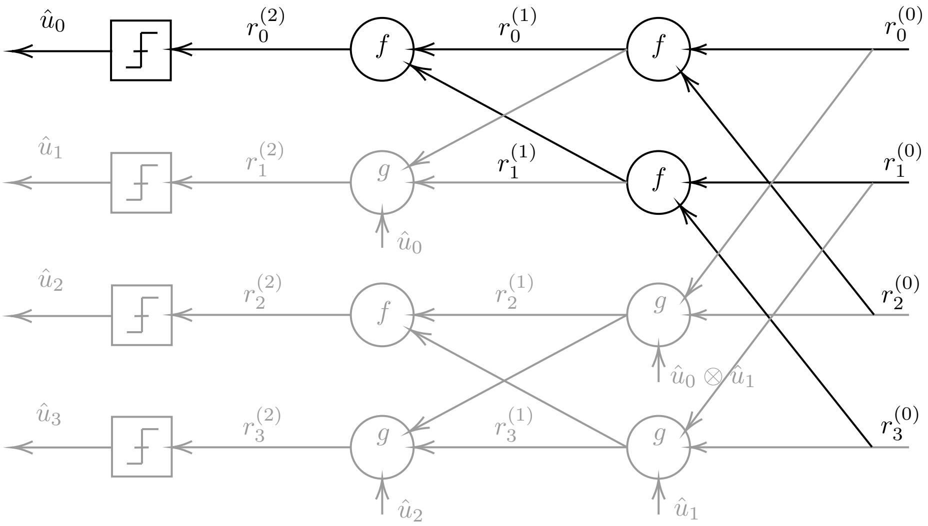


Figure 14: Step 1: We first estimate bit $\hat{u}_0$ by passing LLR values to f nodes.

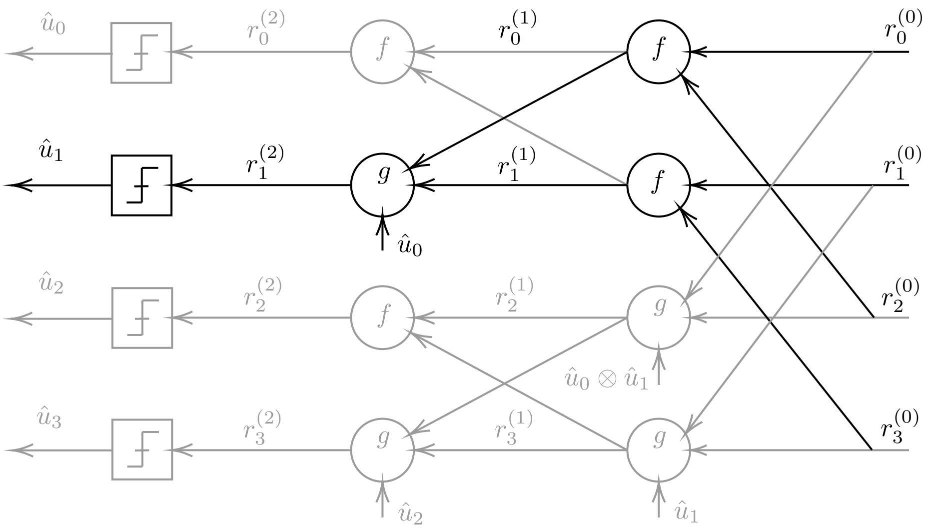


Figure 15: Step 2: Using the estimate for u_0 and calculated LLRs, u1 can be decoded.

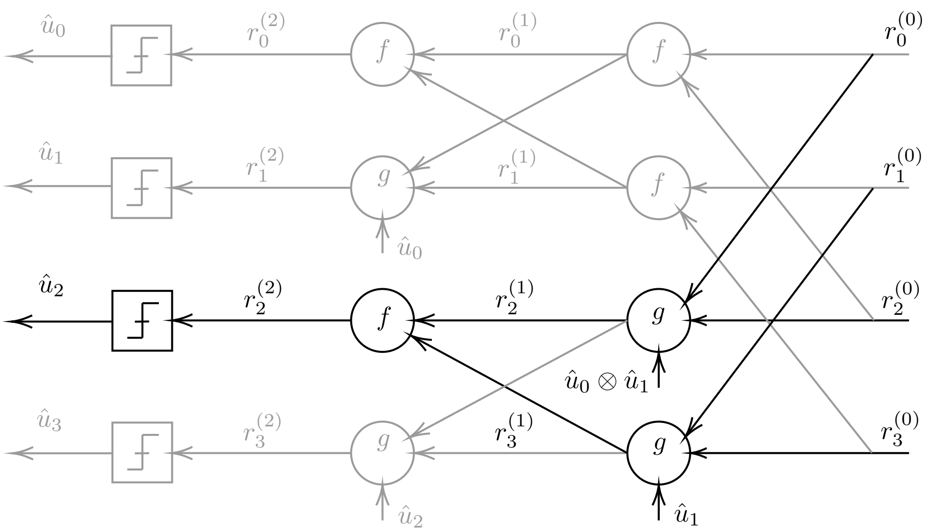


Figure 16: Step 3: Decoding u_2 using u_0 and $u_0 \oplus u_1$ and passing LLRs through shown path

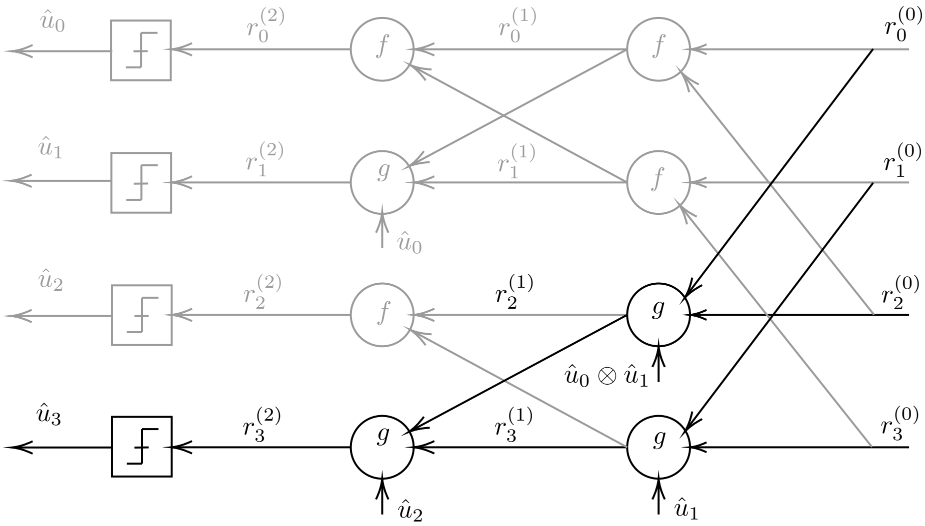


Figure 17: Finaly, using u_2, we can decode u_3 in similar manner.

Let us put some simulation results for SC decoder.

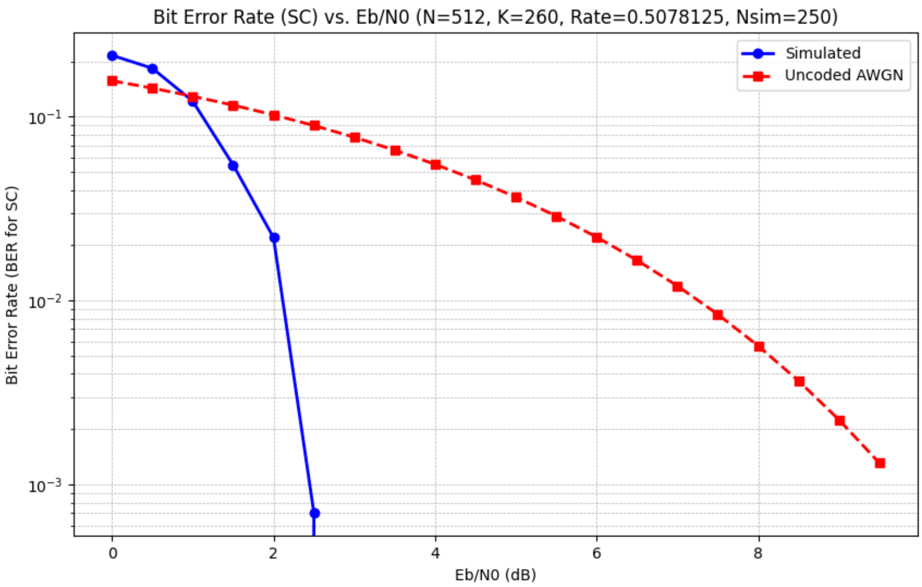


Figure 18: BER plot for SC decoder

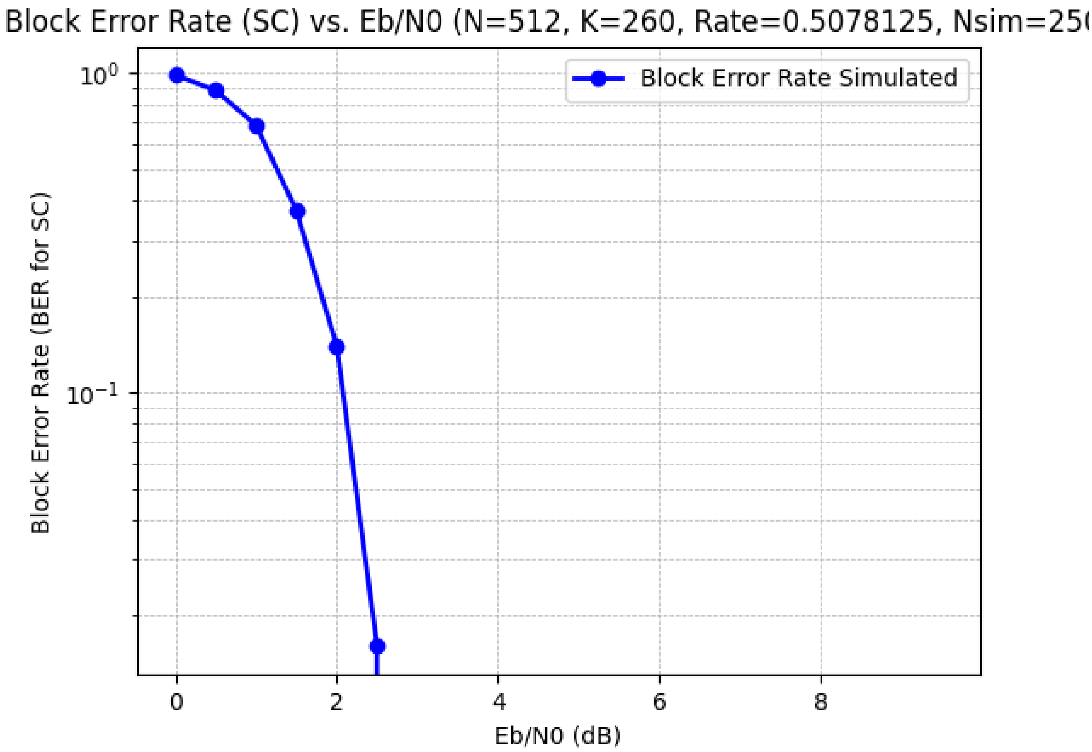


Figure 19: Block Error Rate For SC decoder

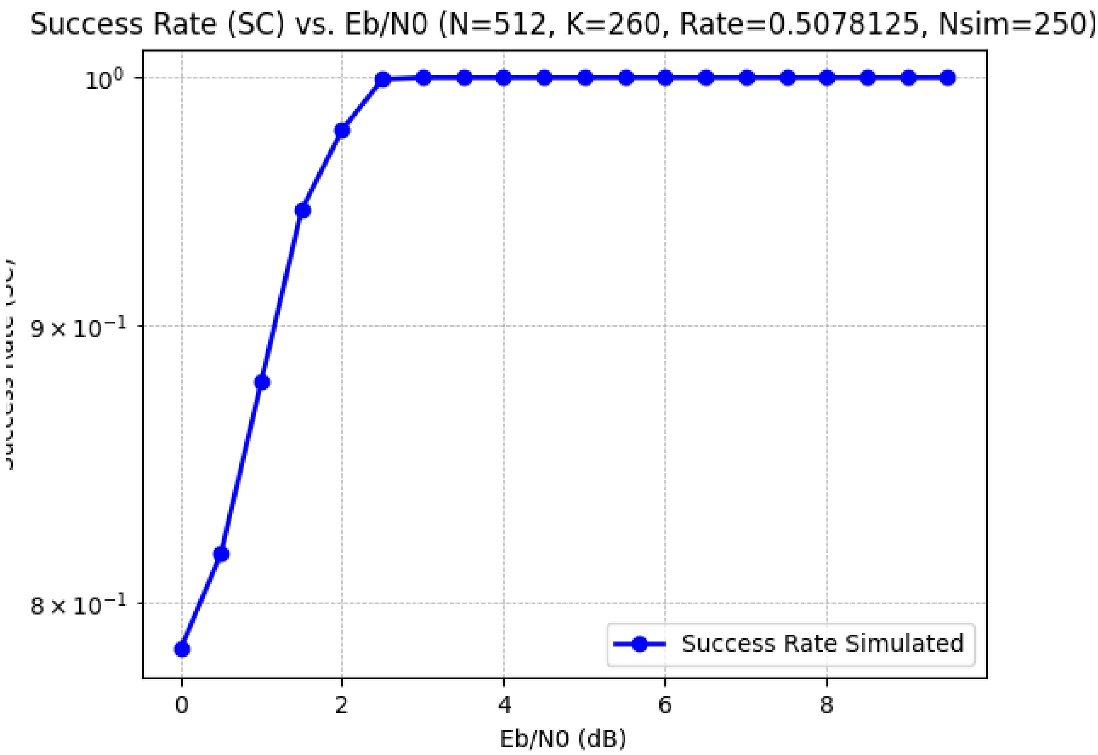


Figure 20: Rate of decoding success for SC decoder

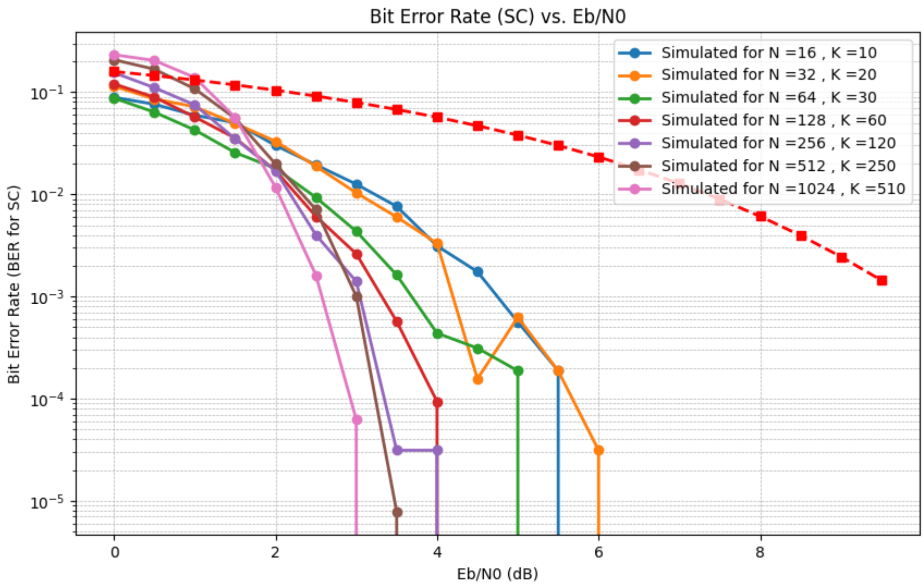


Figure 21: Perfomance of SC decoder for different values of (N, K)

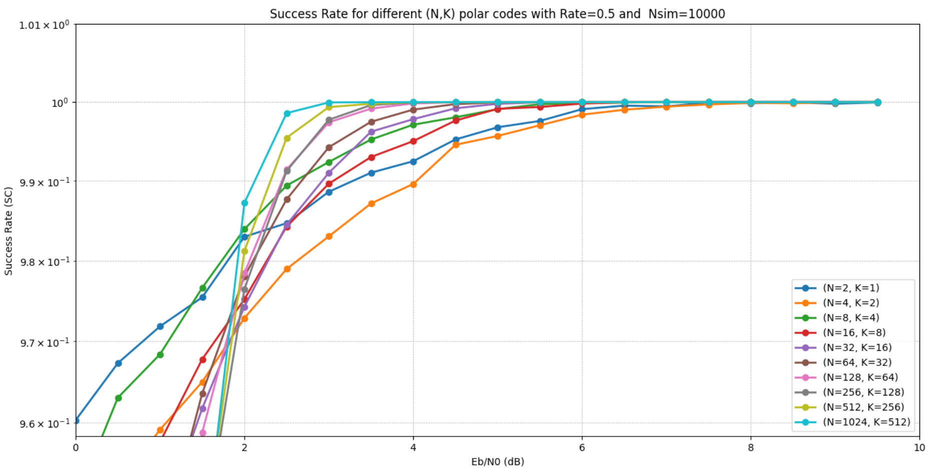


Figure 22: Rate of decoding success for SC decoder for different values of (N, K)

3.2 Successive Cancellation List Decoder (SCL)

This decoding scheme is similar to SC decoding, which we discussed in previous section. The only difference in Successive Cancellation List Decoder is that it maintains a list of possible codewords that are most likely to be transmitted.

Number of codewords that SCL decoder maintains is called the list size.

We have seen in the case of SC decoder that it makes decision about some non-frozen bit u_0, and then uses this bit to make decision for other bit u_1.

Now, in the case of SCL decoder, we take both the possibilities that whether the bits are decoded correctly or not. So, when we make decision for each non-frozen bits, we take two possibilities of either the bit is 0 or 1 irrespective of its LLR. We get the following tree structure when we do this. Doing this for each bit is

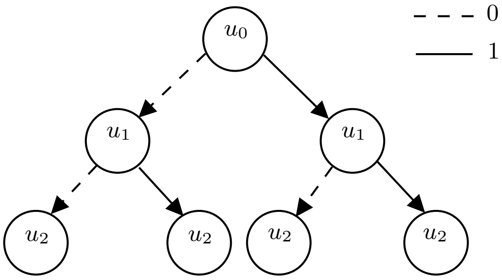


Figure 23: Tree representation for all possible decisions of non-frozen bits

not practical, because number of branches of the tree grows exponentially. That leads to higher complexity decoder, which we want to avoid. Inorder to keep the complexity reasonable, we keep only certain number of branches at any level of this tree. This number is called list size.

To decide which branch to keep and which to discard, we maintain a path metric based on LLRs of the bits for each path.

We continue this process of creation and termination of branches till last bit is decoded. At last, decoding path with lowest path metric is choosen as the decoded codeword. Let us compare the comined results for SC and SCL

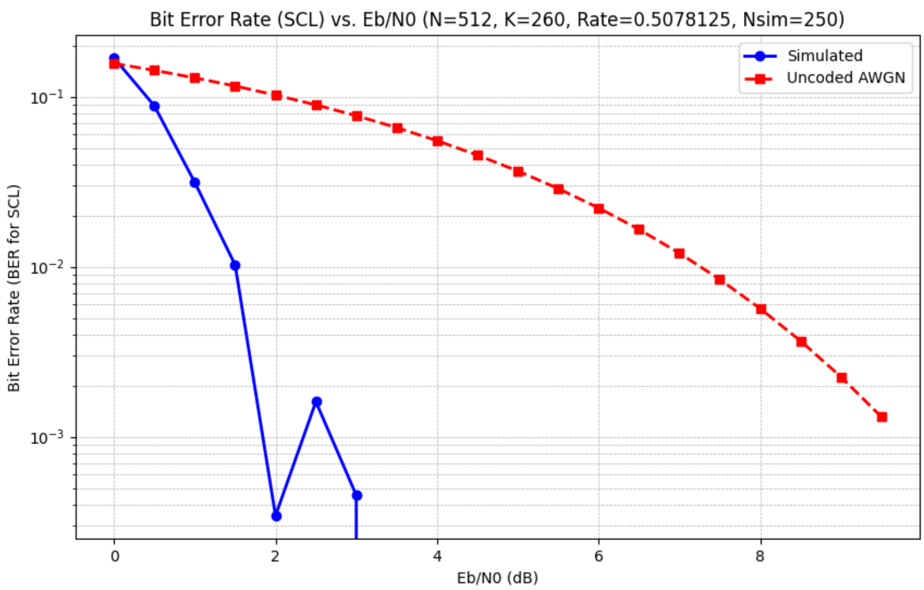


Figure 24: Bit Error Rate for SCL decoder

decoder to understand which decoding scheme is efficient.

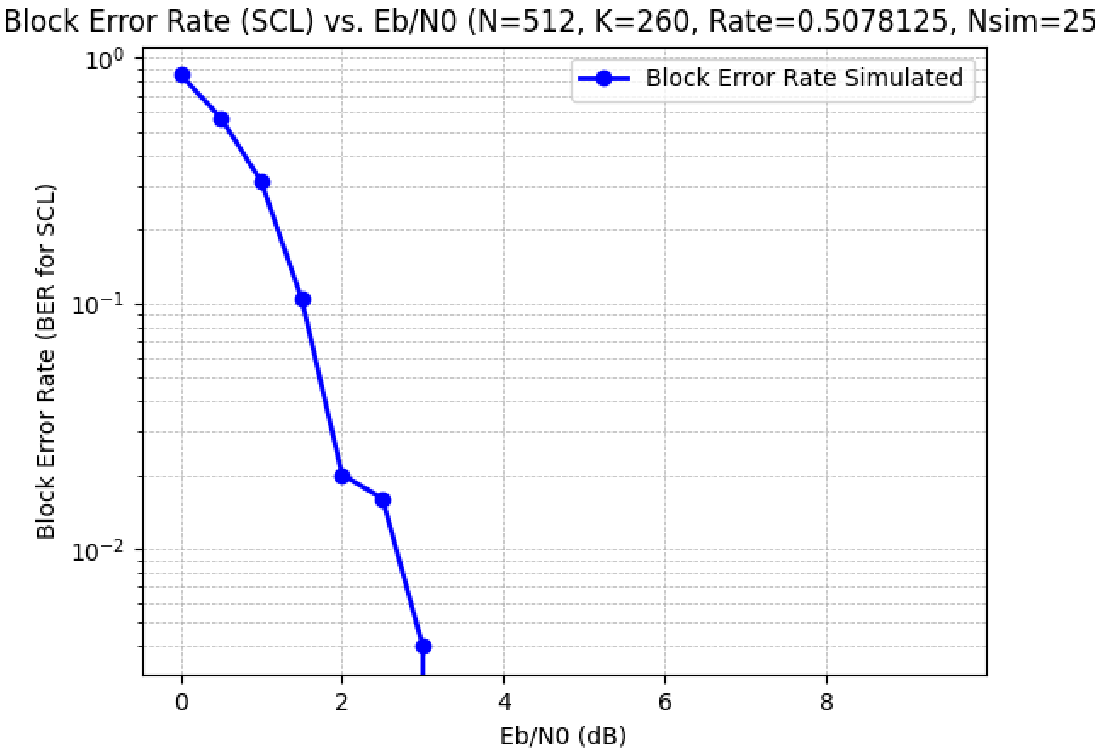


Figure 25: Block Error Rate for SCL decoder

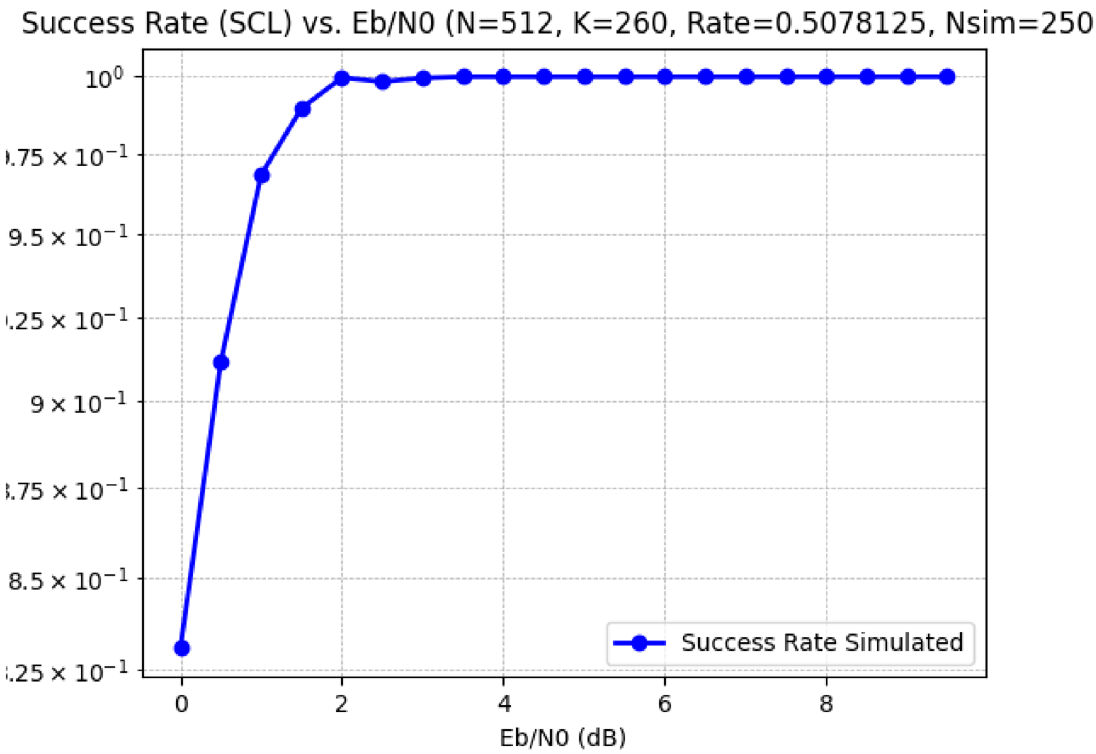


Figure 26: Rate of decoding success for SCL decoder

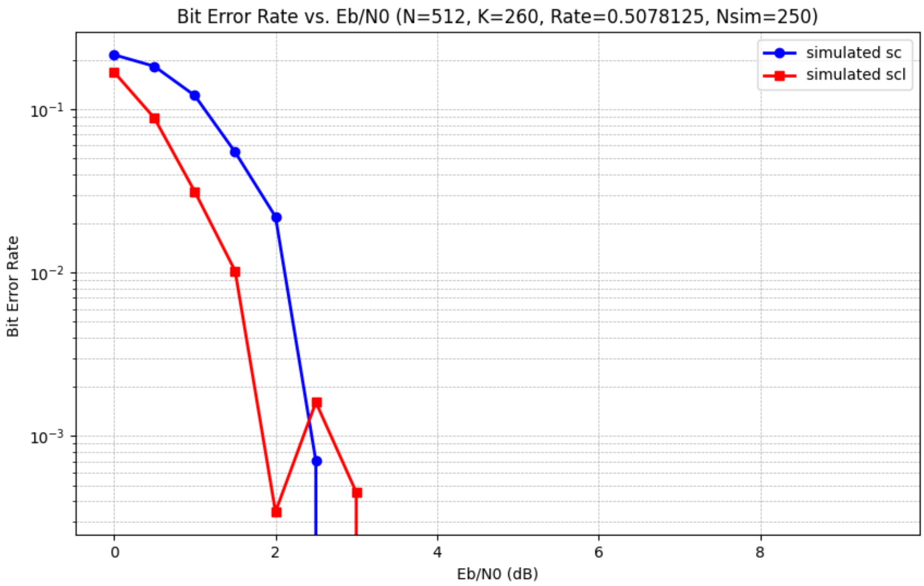


Figure 27: Bit Error Rate Comparison for SC and SCL decoder

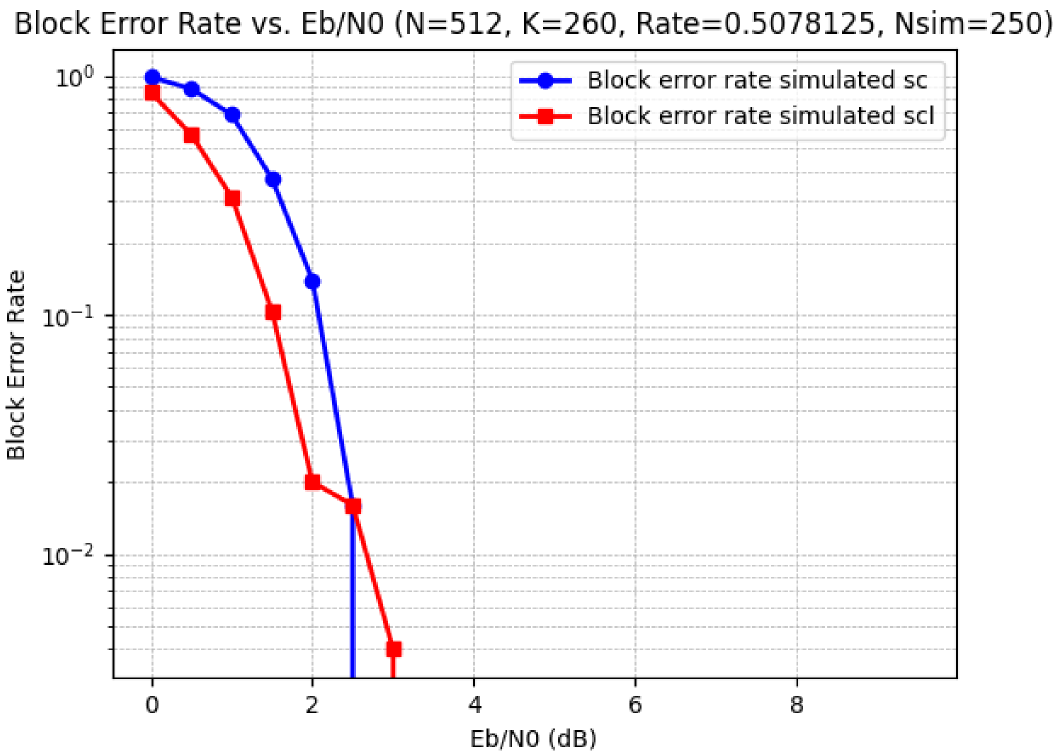


Figure 28: Block Error Rate Comparison for SC and SCL decoder

Efficiency of SCL decoder further increases when we add Cyclic Redundacy Check(CRC) code to our message before transmitting it. Here are some results.

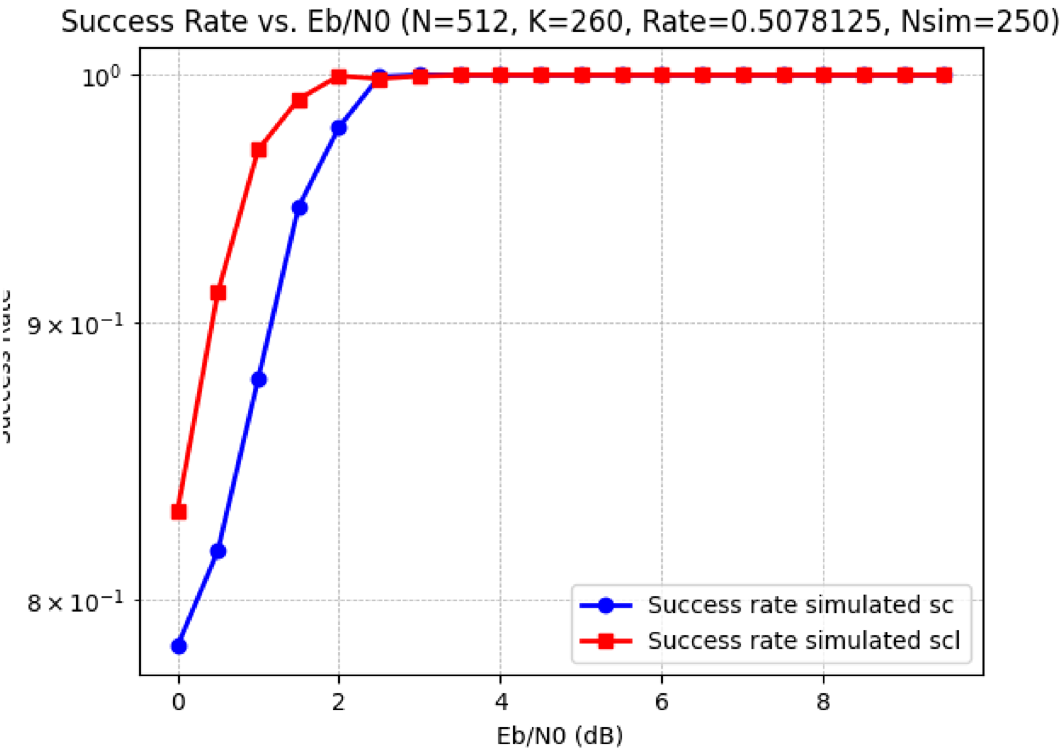


Figure 29: Rate of decoding success for SC and SCL decoder

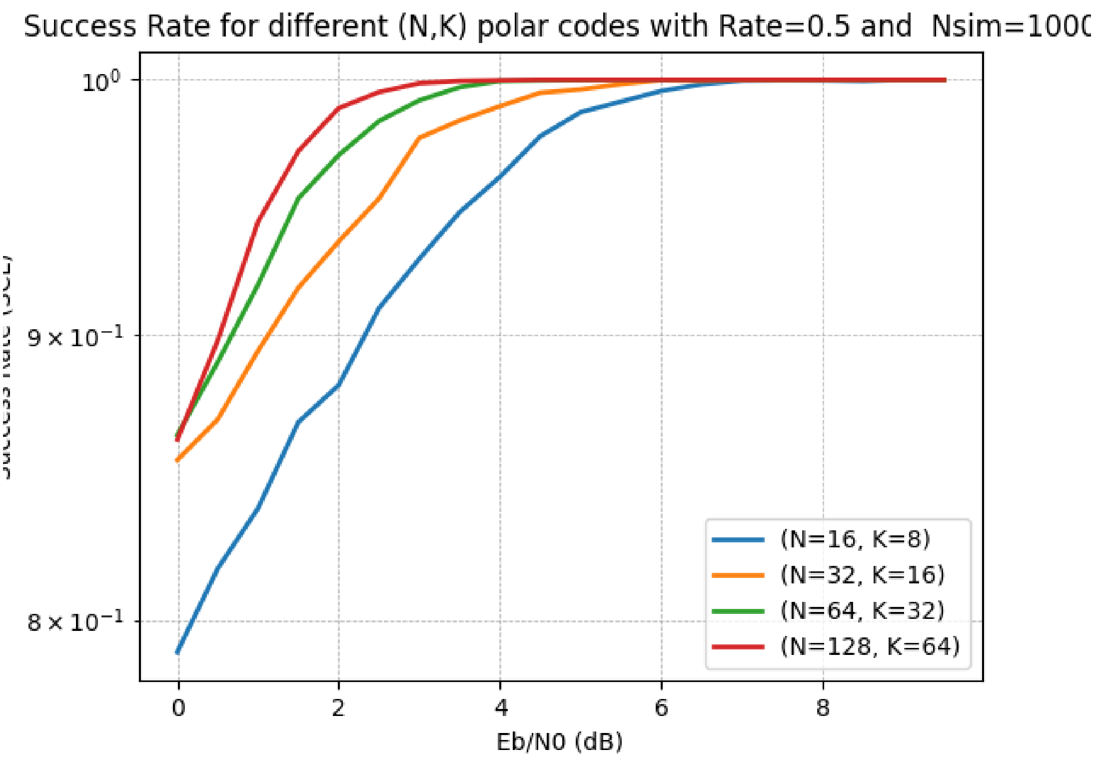


Figure 30: Rate of decoding success with CRC for SCL decoder

4 Constructing Polar Codes using Gaussian Approximation

Construction of polar codes using this method includes approximating Bhattacharya Parameter $Z(W)$ for each channel through distribution of LLR.

Let

$$y = x + n$$

where,
y = received signal
x = transmitted signal
n = normally distributed noise with mean 0 and variance σ^2

LLR for original channel W can be represented as,

$$\begin{aligned}
LLR &= log\left(\frac{P(y|x=1)}{P(y|x=-1)}\right) \\
&= log\left(\frac{\frac{1}{\sqrt{2\,\pi\,\sigma^2}}\ exp\left(-\frac{(y-1)^2}{2\,\sigma^2}\right)}{\frac{1}{\sqrt{2\,\pi\,\sigma^2}}exp\left(-\frac{(y+1)^2}{2\,\sigma^2}\right)}\right) \\
&= log\left(exp\left(\frac{1}{2\,\sigma^2}(\ (y+1)^2 - (y-1)^2)\right)\right) \\
&= log\left(exp\left(\frac{2\,y}{\sigma^2}\right)\right) \\
\Rightarrow LLR &= \frac{2\,y}{\sigma^2}
\end{aligned}$$

Now, we find distribution of LLR by assuming that bit 0 or symbol 1 was transmitted. In this case, we have $y = 1 + n$. Thus, y follows gaussian distribution with mean 1 and variance σ^2.

By looking an expression for LLR, we can say that it follows gaussian distribution with mean $\frac{2}{\sigma^2}$ and variance $\frac{4}{\sigma^2}$.

Based on these results, we can conclude that, given the mean of LLR, we can obtain complete information on the LLR distribution of an AWGN channel (The variance is twice the mean).

When we do channel polarization(process of channel combinig and splitting), we get various channels with different reliabilities. We can get mean of LLR for these various channel through given recursive formula:[6][11]

$$m_N^{(2\,j-1)} = f^{-1}(1-(1-f(m_{N/2}^{(j)}))^2)$$
$$m_N^{(2\,j)} = 2\,m_{N/2}^{(j)}$$

where $m_1^{(1)} = \frac{2}{\sigma^2}$.

Function f is defined as,

$$f(x) = \begin{cases} 1 - \frac{1}{\sqrt{4\pi x}} \int_R tanh\left(\frac{u}{2}\right) exp(-\frac{(u-x)^2}{4x}) du, & \text{if } x > 0 \\ 1, & \text{if } x = 0 \end{cases}$$

This function f can be approximated as,

$$f(x) = \begin{cases} exp(\alpha\, x^{\gamma} + \beta), & \text{if } x < 10 \\ \sqrt{\frac{\pi}{x}} \left(1 - \frac{10}{7x}\right)\ exp(-\frac{x}{4}), & \text{if } x > 10 \end{cases}$$

Finding inverse of this f function is difficult. One possible way to find inverse of f is to store value of $f(x)$ for different x in table, and then using this table, we can approximate $f^{-1}(x)$.[14]

Knowing the values of mean of LLR for rach channel, we can find variance of noise corresponding to the channel $W_N^{(i)}$ using following relation,

$$(\sigma_N^{(i)})^2 = \frac{2}{m_N^{(i)}} \tag{7}$$

Now, we can compute Bhattacharya Parameter $Z(W)$ as

$$\begin{aligned} Z(W) &= \int_{-\infty}^{\infty} \sqrt{p(y|0)\ p(y|1)}\, dy \\ &= \int_{-\infty}^{\infty} \sqrt{\frac{1}{\sqrt{2\,\pi\,\sigma^2}}\, exp\left(-\frac{(y-1)^2}{2\,\sigma^2}\right) \frac{1}{\sqrt{2\,\pi\,\sigma^2}}\, exp\left(-\frac{(y+1)^2}{2\,\sigma^2}\right)}\, dy \\ Z(W) &= exp\left(-\frac{1}{2\,\sigma^2}\right) \end{aligned}$$

With the knowledge of mean of LLR for each channels $W_N^{(i)}$, Bhattacharya Parameter $Z_N^{(i)}$ can be calculated as,

$$Z_N^{(i)} = exp\left(-\frac{1}{2\,\left(\sigma_N^{(i)}\right)^2}\right)$$

Using equation 7, we get,

$$Z_N^{(i)} = exp\left(-\frac{m_N^{(i)}}{4}\right)$$

We know that higher the Bhattacharya Parameter of channel, lesser the reliability of channel. After approximating $Z_N^{(i)}$ for each channels, we can sort them in increasing order and get channels in decreasing order of reliability.

After this process, We can select first k channels to transmit information bits and we can send foreknown bits (usually we take them as 0) through remaining $N-k$ channels for (N,k) Polar codes.

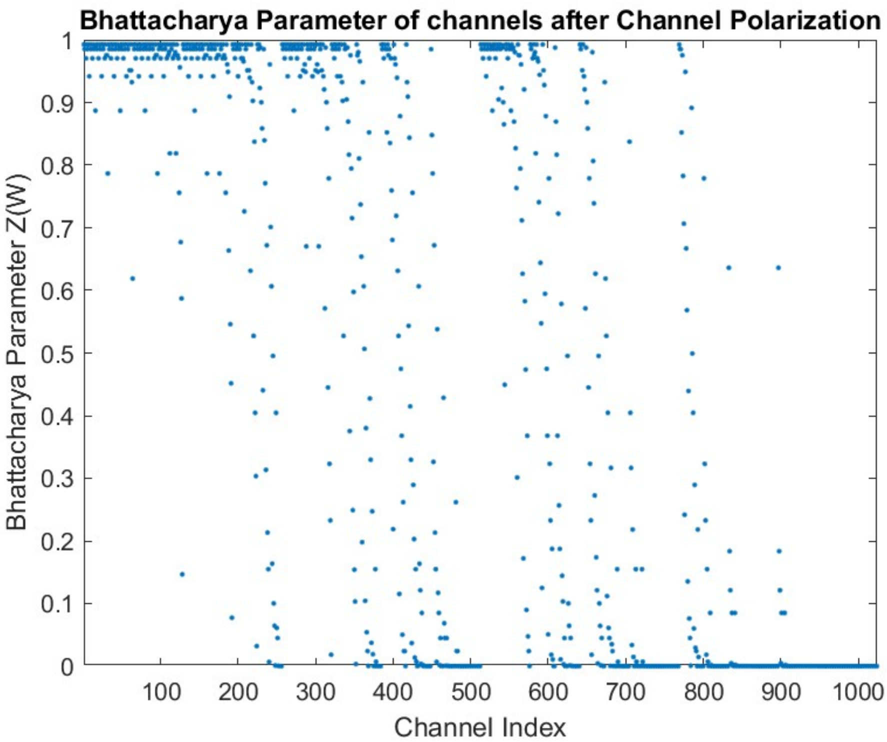


Figure 31: Bhattacharya Parameter for polarized channels using Gaussian Approximation method

After sorting these values, we get following graph.

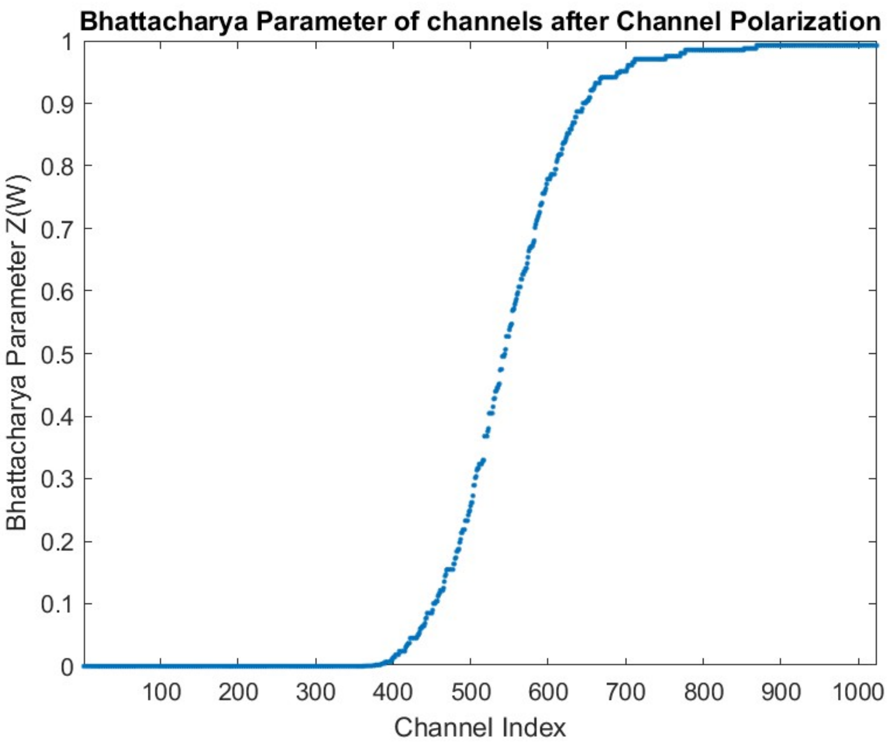
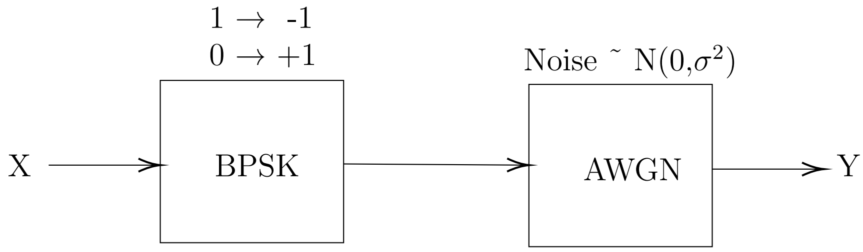


Figure 32: Sorted Values for Bhattacharya Parameter for polarized channels using Gaussian Approximation method

4.1 Analysis of uncoded BPSK+AWGN Channel and reliabilty sequence



Error: $P(X = 0\,\text{and}\,n < -1) + P(X = 1\,\text{and}\,n > 1)$
X and n are independent, so

$$P(X = 0)P(n < -1) + P(X = 1)P(n > 1)$$

Given $P(X = 0) = P(X = 1) = 0.5$ and $P(n < -1) = P(n > 1)$, we have

$$Error : 2P(X = 0)P(n < -1)$$

For AWGN with a normal distribution $N(0, \sigma^2)$, we have:
Error Probability Calculation for AWGN Channel:
Given a standard normal random variable n with mean 0 and variance σ^2, we have:

$$2 \cdot P(n < -1) = 2\int_1^\infty \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}}\,dx$$

Using the substitution $\frac{x^2}{2\sigma^2} = t^2$, we get:

$$\frac{x}{\sqrt{2}\sigma} = t \quad \text{and} \quad \frac{dx}{\sqrt{2}\sigma} = dt$$

This simplifies the integral to:

$$\frac{2}{\sqrt{\pi}} \int_{\frac{1}{\sqrt{2}\sigma}}^\infty e^{-t^2}\,dt$$

The error probability is then given by:

$$1 - \frac{2}{\sqrt{\pi}} \int_0^{\frac{1}{\sqrt{2}\sigma}} e^{-t^2}\,dt$$

Where the complementary error function $\text{erfc}(x)$ is defined as:

$$\text{erfc}(x) = 1 - \text{erf}(x)$$

and the error function $\text{erf}(x)$ is given by:

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-x^2}\,dx$$

Thus, the error probability is:

$$\text{Error} = 0.5 \cdot \text{erfc}\left(\frac{1}{\sqrt{2}\sigma}\right)$$

And in terms of the Q function (the tail probability of a standard normal distribution):

$$Q(x) = 0.5 \cdot \operatorname{erfc}(x)$$

The error probability can be expressed as:

$$\text{Error} = Q\left(\sqrt{\frac{\text{SNR}}{2}}\right)$$

In sections 1.2 and 1.3, the error probability of the W^- and W^+ channels is determined as $2p - p^2$ and p^2 respectively. This is established as a recursive algorithm for subsequent polarization, as detailed in section 1.5 and illustrated in Figure 9. It follows that obtaining the reliability sequence from this recurrence relation is straight-forward.

Algorithm for obtaining the reliability sequence:

global var index = 1

define N, map(double, int) m, i = 0
$n = \log_2 N$
$p = Q\left(\sqrt{\frac{\text{SNR}}{2}}\right)$
split(m, p, index, i)
reliability sequence = m.values

define function split(m, p, index, i)
    if (i == n $\vee$ $index == N$)
        $m[p]$ = index
        index = index + 1
        return
split(m,$2p - p^2$,index,i+1)
split(m,p^2,index,i+1)
return

5 Analysis :

5.1 Shannon's channel capacity for polar codes

The message of length K passes through the Binary Phase Shift Keying (BPSK) modulation scheme in conjunction with an Additive White Gaussian Noise (AWGN) channel. The channel capacity is denoted by $I(W)$, where W represents the channel.

Now, for a binary sequence $\{0,1\}^k$ and $\forall \epsilon > 0$ to achieve a capacity of $I(W) - \epsilon$, there exist constants $a_w < \infty$ and $b_w > 0$ such that the required block length for this rate is given by:

$$N \geq \frac{a_w}{\epsilon^2} \tag{8}$$

and the error of miscommunication is given by:

$$\text{error} \geq 2^{-b\epsilon^2 N} \tag{9}$$

Now, as shown above,

$$N \geq \frac{a_w}{\epsilon^2} \Rightarrow \epsilon^2 \geq \frac{a_w}{N} \Rightarrow \epsilon \geq \sqrt{\frac{a_w}{N}}$$

Now as $N \to \infty$, then $\epsilon \to 0$, so that

$$I(W) - \epsilon \to I(W)$$

That shows as N goes large the channel capacity tends to Shannon's channel capacity.

Now for get the block length we have to do n steps polarization, where $n = \log_2 N$

5.2 Capacity Conservation

Figure 33: basic structure of channel polarization

Now the mutual capacity of the channel is:

$$\begin{aligned}
I(W) + I(W) &= I(X_1; Y_1) + I(X_2; Y_2) \\
2I(W) &= I(X_1 X_2; Y_1 Y_2) \\
2I(W) &= I(U_1 U_2; Y_1 Y_2) \qquad \text{(Process of combining using the chain rule)} \\
2I(W) &= I(U_1; Y_1 Y_1) + I(U_2; Y_1 Y_2 / U_1) \qquad \text{(Process of splitting)}
\end{aligned}$$

Since after result Y_1, Y_2 are independent of U_1

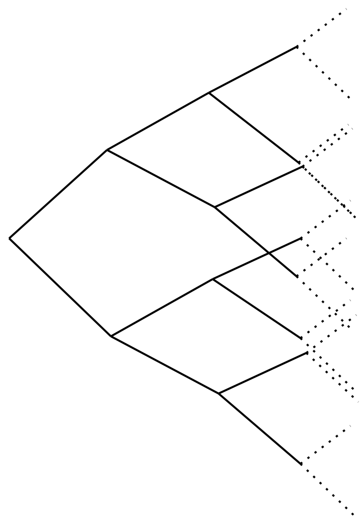
$$2I(W) = I(U_1; Y_1 Y_1) + I(U_2; Y_1 Y_2 U_1)$$

$$2I(W) = I(W-) + I(W+) \tag{10}$$

Above equation represents the law of capacity conservation.

5.3 Intermediate channels

5.3.1 level graph of channel polarization



	0		1		2		3	
I(W) =	α_0	$=$	α_1	$=$	α_2	$=$	α_3	:average of capacity at each level
$I(W)^2 =$	β_0	$\leq$	β_1	$\leq$	β_2	$\leq$	β_3	:average of square capacity at each level
	γ_0		γ_1		γ_2		γ_3	:fraction of intermediate channel at each level

Figure 34: channel polarization sectional graph

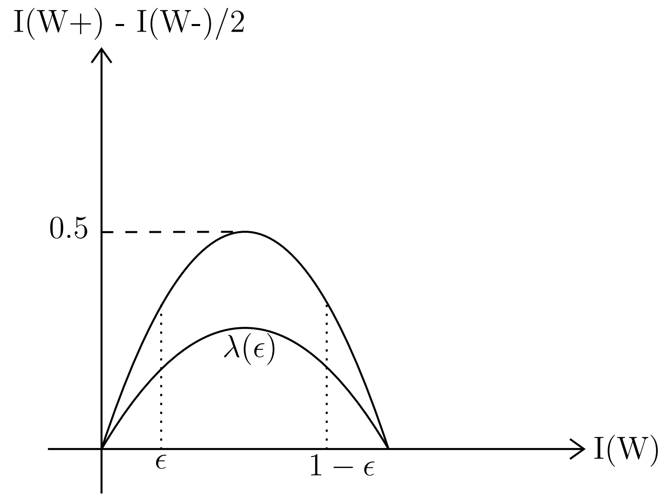


Figure 35: capacity of intermediate channel

Here, $\lambda(\epsilon)$ determines the lower bound of the capacity difference,
Now, let's determine the $\kappa(\epsilon) = min(\lambda(\epsilon), \lambda(1-\epsilon))$
Now, $\beta_{n+1} = \beta_n + (\kappa(\epsilon))^2\gamma_n$
As $n \to \infty$, The $\beta_n \to \beta_{n+1}$
$\Rightarrow (\kappa(\epsilon))^2\gamma_n \to 0$

$\Rightarrow \gamma_n \to 0$

That shows that as $n \to \infty$ the fraction of the intermediate channel goes to 0, that means at very large n every channel become either ideal or pure noisy.

5.3.2 Intermediate channels by Martingale graph

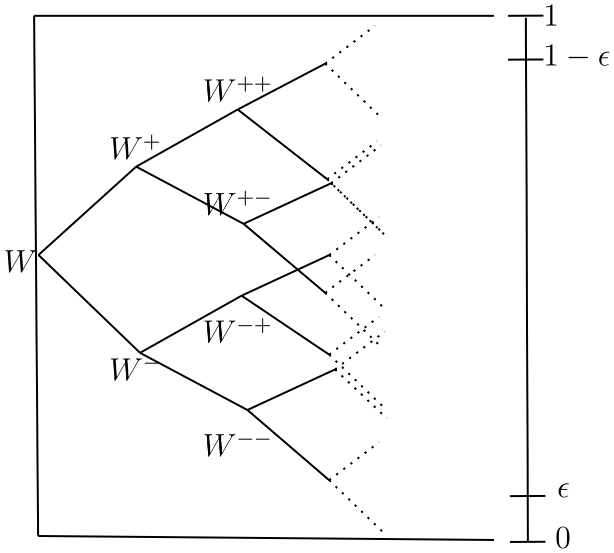


Figure 36: Bounded Martingale graph

Above figure depicts a bounded Martingale graph. When the channel approaches ideality, its polarization also tends towards ideal channels. This assertion stems from the law of capacity conservation. According to this law, if one of the polarized channels attains an intermediate state, the other channel would necessarily need to traverse the boundary of the Martingale graph. However, such a scenario is not feasible. Consequently, it indicates that the polarization of a purely noisy channel yields noisy channels, whereas that of ideal channels yields ideal channels.

$$\lim_{n\to\infty} \sum_{i=1}^{n} \epsilon < I(W_i) < 1-\epsilon \to 0 \qquad \epsilon > 0$$

It shows that as n increases, the fraction of intermediate channels tends towards 0. Consequently, the message passing through these channels approaches ideal channels, while the frozen bit passes through the noisy channels. The characteristics of these ideal and noisy channels depend on the reliability sequence.

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