Memristor–based neural circuits

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***Abstract*—Biological neural systems use self–reconfigurable and self–learning primitive elements (synapses) to extract relevant information from complex and noisy environments, to detect specific spatio–temporal patterns in the data of interest and to compute and simultaneously store some significant features. All these desirable attributes may be realized by using two–terminal elements, memristors (memory resistors), which most closely resemble biological synapses.**

This article is organized according to the rule of the ISCAS2013 special session having the same title. We present a short summary of the state–of–the–art of memristor theory and Hodgkin–Huxley neural model. In addition, we briefly introduce a comprehensive nonlinear circuit-theoretic foundation for a novel circuit implementation of the Hodgkin–Huxley neural model with memristors.

I. Introduction

The fundamental unit of living neural tissue is represented by the cell. Cells are specialized in their anatomy and physiology to accomplish different processes. The cell is enclosed by a cell membrane whose thickness is about $7.5 - 10.0\,nm$ [1].

Ionic channels tune the flow of sodium, potassium, and chloride molecules through the cell membrane. This flow controls the voltage gradient (i.e. the difference between the potential of intracellular and extracellular media) across the cell membrane. It is believed that the membrane voltage is important for the execution of cell functions [1]. Thereby the study of ionic channels is of the utmost importance to understand the role of the cell membrane in the neural systems.

Membrane kinetics can be discussed by means of the model proposed by A. L. Hodgkin and A. F. Huxley in 1952 [2]. Hodgkin and Huxley's model is based on giant axons of the squid. The electric behavior of the axon membrane is described by the net ion flow through a great number of ion channels.

L. O. Chua has recently presented a rigorous and comprehensive nonlinear circuit-theoretic foundation for the Hodgkin–Huxley Axon Circuit model [3]. One of the main result in [3] is that the Hodgkin–Huxley Axon circuit comprises a potassium ion-channel memristor and a sodium ion-channel memristor.

The memristor, a passive element with conductivity dependent on the history of voltage across it, was theoretically introduced in 1971 [4] and then classified within the larger class of memristive systems in 1976 [5]. After the first-ever recognition of memristive behavior in a nano-scale double-layer film in 2008 [6], the memristor has received significant attention for the large spectrum of opportunities it opens up in the realization of large-capacity non-volatile memories and neuromorphic systems.

In particular, memristor represents the latest technology breakthrough to build neuromorphic circuits with characteristics that show an intriguing resemblance to the neural cells. With such target in mind, it would be extremely helpful to develop methodologies for the analysis [7] and design of novel nonlinear circuit implementation of the Hodgkin–Huxley neural models.

In [8], it has been recently found that a purely passive circuit, employing an elementary diode bridge and of an RLC series circuit, manifests memristor dynamics.

The aim of this work is twofold:

a) to introduce a novel class of circuits that implement memristor. This class generalizes the circuit proposed in [8];
b) to discuss the implementation of the potassium and sodium ion-channel memristors exploiting this novel class.

As for the structure of the manuscript, Section II briefly summarizes Hodgkin–Huxley and memristor models. Section III presents a novel implementation of memristor based on purely passive circuit elements. This result is used in Section IV to sketch out the circuit implementation of the potassium ion-channel memristor. Finally, Section V outlines the conclusions.

II. Brief review of memristor and Hodgkin–Huxley model

This preliminary section is devoted to the definition of the memristor, a brief summary of the Hodgkin–Huxley model and to the introduction of a proper notation.

A. Definition of the memristor

According to the definition given in [9], a voltage-controlled time-invariant memristor is a nonlinear dynam-

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ical circuit element defined by the following differential-algebraic system of equations:

$$\frac{d\mathbf{x}(t)}{dt} = f(\mathbf{x}(t), v(t)), \tag{1}$$
$$i(t) = g[\mathbf{x}(t), v(t)]v(t), \tag{2}$$

where $\mathbf{x} \in \mathbb{R}^n$ is the state, $v \in \mathbb{R}$ refers to the memristor-voltage, $i \in \mathbb{R}$ describes the memristor-current, $f(\mathbf{x}, v) : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ stands for the *state evolution function*, while $g(\mathbf{x}, v) : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ is the *memductance*[1]. Equation (2) represents a *state/voltage-dependent Ohm's law.* The definition of current–controlled time–invariant memristor can be obtained for duality. In the special case when the memductance is not an explicit function of the voltage $v(t)$ and $\mathbf{x}(t) = \varphi(t) \in \mathbb{R}$, the element is called *ideal memristor*.

In the following we consider only voltage-controlled time-invariant memristor (for the sake of brevity named memristor).

B. Memristor–based Hodgkin–Huxley model

The electric behavior of the Hodgkin–Huxley axon membrane is described by the net ion flow through a great number of ion channels. Many experimental results support the assumption that in presence of several permeable ions, the flux trough each ion channel is assumed to be independent of the others (an assumption referred to as the independence principle and formulated by Hodgkin and Huxley [2]).

The Hodgkin–Huxley Axon circuit model (see [2]) takes into account the electric current flowing across the cell membrane during activation due to four current components: *i)* current $i_{Na}(t)$ carried by sodium ions; *ii)* current $i_L(t)$ carried by potassium ions; *iii)* current $i_L(t)$ carried by other ions (designated *leakage current*, mainly due to chloride ions); *(iv)* capacitive current.

As shown in [3], the Hodgkin–Huxley Axon Circuit Model can be replaced by the memristor-based Hodgkin–Huxley Axon Circuit Model because:

- potassium ion-channel, described by a potassium time-varying conductance, $g_K(n)$ is a first-order memristor defined by (denoting $v = v_K + E_K$):

$$\frac{dn}{dt} = f_K(v_K, n) = \frac{0.01(v+10)}{\exp\left(\frac{v+10}{10}\right) - 1}(1-n) - 0.125\exp\left(\frac{v}{80}\right) n \tag{3}$$

$$i_K = g_K[n]v_K = \left(\bar{g}_K\, n^4\right)\, v_K \tag{4}$$

- sodium ion-channel, described by sodium time-varying conductance, $g_{Na}(m, h)$ is a second-order memristor defined by (denoting $v_{Na} = v + E_{Na}$)

[1]According to the definition given in [5], the class of circuit elements defined by (1)–(2) are referred to as memristive circuits. In this manuscript we follows the nomenclature introduced in [9].

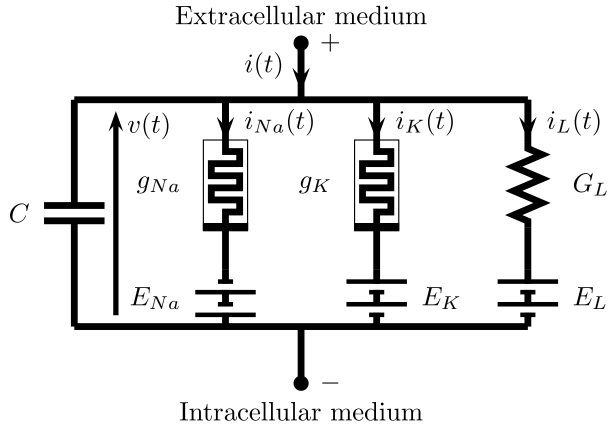

Fig. 1. Memristor-based Hodgkin–Huxley axon membrane circuit model derived in [3]. Circuit parameters are: $C = 1\,\mu F/cm^2$, $G_L = 0.3\,mS/cm^2$, $E_{Na} = 115\,mV$, $E_K = 12\,mV$, $E_L = -10.613\,mV$, $g_K[n(t)] = \bar{g}_K\, n^4(t)$ and $g_{Na}[m(t), h(t)] = \bar{g}_{Na}\, h(t)\, m^3(t)$, where $\bar{g}_K = 36\,mS/cm^2$, $\bar{g}_{Na} = 120\,mS/cm^2$ whereas $n(t)$, $m(t)$ and $h(t)$ are governed by equations (3), (5) and (5), respectively.

$$\frac{dm}{dt} = \frac{0.1(v+25)}{\exp\left(\frac{v+25}{10}\right) - 1}(1-m) - 4\exp\left(\frac{v}{18}\right) m$$
$$\frac{dh}{dt} = 0.07\exp\left(\frac{v}{20}\right)(1-h) - \frac{1}{\exp\left(\frac{v+30}{10}\right) + 1} h$$

$$i_{Na} = g_{Na}(m,h)v_{Na} = \left(\bar{g}_{Na}\, h\, m^3\right)\, v_{Na}$$

The memristor–based Hodgkin–Huxley Axon Circuit Model, with time-varying potassium (sodium) conductance replaced by a first–order (second-order) memristor, and the circuit parameters are shown in Fig. 1. Pinched hysteresis curves of potassium and sodium ion-channel memristors are reported in [3].

In the next section, we introduce a simple class of purely–passive electronic circuits with nonlinearity provided by a nonlinear resistive two–port and dynamics supplied by a nonlinear dynamic bipole. This elementary circuit exhibits the fingerprint of memristive systems, i.e. a pinched hysteretic current–voltage loop for any state initial condition and for any amplitude and non-infinite frequency of any periodic sign-varying driving source. This novel memristor implementation, that generalizes the circuit presented in [8], is of utmost importance to build memristor–based Hodgkin–Huxley axon membrane circuit and, in a broader sense, neural circuits with memristor.

III. Novel implementation of a memristor

This section aims at presenting a novel simple class of purely–passive circuits useful to implement potassium and sodium ion-channel memristors of the memristor-based Hodgkin–Huxley Axon Circuit (see Fig. 1).

The realization of memristors in the form of active circuits was presented in the Chua's seminal paper [4]. Such implementation was based on a linear dynamic two-port (mutator) connected to a nonlinear resistive (or dynamic) one-port.

An original realization of memristor having only passive circuits has been recently proposed in [8]. This novel implementation of memristor relies on a special class of (passive) nonlinear resistive two–port connected to a nonlinear dynamic one-port (see Fig. 2). In particular, we consider

- voltage–controlled nonlinear resistive two–ports described as follows[2]:

$$\begin{aligned} i_1(t) &= g_{11}(v_1(t))g_{12}(v_2(t)) &(5)\\ i_2(t) &= g_2(v_1(t), v_2(t)) &(6)\end{aligned}$$

  where $g_{11}(v_1(t))$ is a function such that $g_{11}(0) = 0$. It follows that:

$$g_{11}(v_1(t)) = \tilde{g}_{11}(v_1(t))v_1(t) \quad (7)$$

- nonlinear dynamic one–ports obtained by the parallel connection of a linear dynamic bipole and a voltage–controlled nonlinear resistor, that is (note that $i_2(t)$ is opposite to the associated reference direction):

$$-i_2(t) = Y(D)\, v_2(t) + f_R(v_2(t)) \quad (8)$$

  where $D = d(\cdot)/dt$ the first–order time–derivative operator, $Y(D)$ is the differential operator associated to the admittance of the linear dynamic bipole and $f_R(v_2(t))$ is the nonlinear constitutive equation of the nonlinear resistor.

It turns out that (5)–(8) can be rewritten as follows:

$$\begin{aligned} 0 &= Y(D)\, v_2 + f_R(v_2) + g_2(v_1, v_2) &(9)\\ i_1(t) &= [\tilde{g}_{11}(v_1)g_{12}(v_2)]\, v_1(t) = g(v_1, v_2)\, v_1(t) &(10)\end{aligned}$$

where eq. (9) (eq. (10)) corresponds to eq. (1) (eq. (2)) and the state $\mathbf{x}(t)$ is represented by $v_2(t)$. Thereby, the design of nonlinear resistive two-ports such that i_1 can be factored into the product of $g_{12}(v_2)$ and $g_{11}(v_1(t)) = [\tilde{g}_{11}(v_1)]\, v_1(t)$ is crucial to implement the memristor *state/voltage-dependent Ohm's law*. The next section presents a simple class of nonlinear resistive two–ports that fulfills these properties.

Remark 1: The purely-passive electronic circuit with nonlinearity provided by a simple network of 4 diodes arranged in a Graëtz bridge configuration and dynamics supplied by inductor and capacitor of a RLC series filter (see details in [8]) is a special case of (9)–(10) where the nonlinear resistive one–port, described by $f_R(v_2)$, is replaced by an open–circuit.

[2]Current–controlled time-invariant memristor can be obtained, *mutatis mutandis*, considering current–controlled nonlinear resistive two-port connected to a nonlinear dynamic one-port.

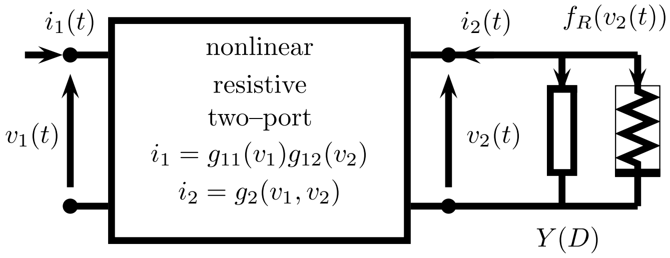

Fig. 2. Novel implementation of a memristor based on a special class of (passive) nonlinear resistive two-port connected to a nonlinear dynamic one-port.

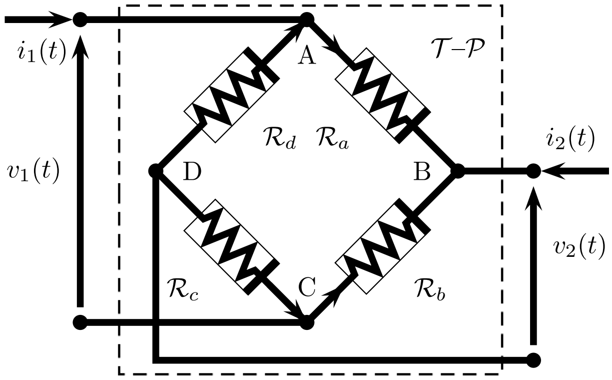

Fig. 3. Nonlinear resistive two-port connected so as to satisfy equations (5)–(7). The current in each bipole has the direction specified by the arrow and voltage is defined by the associated reference direction.

A. Nonlinear resistive two–port for memristor implementation

Let us consider a nonlinear resistive two–port composed of four identical voltage–controlled nonlinear resistor ($\mathcal{R}_k$, with $k = \{a, b, c, d\}$) connected as shown in Fig. 3. The current i_k in each bipole has the direction specified by the arrow. The direction of voltage v_k across each bipole is defined by the associated reference direction. Let us assume that the characteristic equation of $\mathcal{R}_k$ is defined as follows:

$$i_k = I\, [\alpha \exp(\gamma\, v_k) + \beta \exp(-\gamma v_k) - (\alpha + \beta)] \quad (11)$$

where $I \in \mathbf{R}$ is a current, $\gamma^{-1} \in \mathbf{R}$ is a voltage, α and β are real constants. It is worth noting that (11) defines the ideal diode characteristics for $(\alpha, \beta, \gamma) = (1, 0, 1/V_T)$ (with V_T denoting the thermal voltage).

The four-terminal multipole with nodes $\{A, B, C, D\}$ can be seen as a nonlinear resistive two–port by considering the pairs (A, C) and (B, D) as the input and the output port, respectively. Current and voltage at the input (output) port are denoted by i_1 and v_1 (i_2 and v_2), respectively.

Following the procedure presented in [8], it can be readily derived that, under the condition[3] $i_1(t) \neq 0$, Kirchhoff's Voltage Law equation $v_a + v_d = v_b + v_c$, Kirchhoff's Current Law equation $i_a + i_b = i_c + i_d$ and (11) imply the constrains

[3]It turns out that $i_1(t)$ given in (5) has to be different from zero.

$v_a = v_c$ and $v_b = v_d$ from which it follows that $i_a = i_c$ and $i_b = i_d$.

These constrains allow us to write i_1 and i_2 in terms of v_1 and v_2, i.e the voltage–controlled characteristic of the nonlinear resistive two–port $\mathcal{T}$–$\mathcal{P}$. By observing that $v_1 = v_a - v_b$ and $v_2 = -v_a - v_b$, it is readily derived that:

$$v_a = 0.5\,(v_1 - v_2) \tag{12}$$
$$v_b = -0.5\,(v_1 + v_2) \tag{13}$$

Using (12) and (13), some algebraic manipulations yield:

$$\begin{aligned} i_1 &= i_a - i_b = I\,\{\alpha[\exp(\gamma\,v_a) - \exp(\gamma\,v_b)] + \\ &\qquad \beta[\exp(-\gamma\,v_a) - \exp(-\gamma\,v_b)]\} \\ &= 2I\,\sinh\left(\frac{\gamma\,v_1}{2}\right)\left[\alpha\,\exp\left(-\frac{\gamma v_2}{2}\right) - \beta\,\exp\left(\frac{\gamma v_2}{2}\right)\right] \end{aligned} \tag{14}$$

$$\begin{aligned} i_2 &= -i_a - i_b = -I\,\{\alpha[\exp(\gamma\,v_a) + \exp(\gamma\,v_b)] + \\ &\qquad \beta[\exp(-\gamma\,v_a) + \exp(-\gamma\,v_b)] - 2(\alpha+\beta)\} \\ &= -2I\,\cosh\left(\frac{\gamma\,v_1}{2}\right)\left[\alpha\,\exp\left(-\frac{\gamma v_2}{2}\right) + \beta\,\exp\left(\frac{\gamma v_2}{2}\right)\right] \\ &\quad +2I(\alpha+\beta) \end{aligned} \tag{15}$$

It turns out that eq. (15) corresponds to eq. (6) and eq. (14) is in the form given in (10). In particular, we can identify (such that γI is a conductance)

$$\tilde{g}_{11}(v_1) = \gamma I \sum_{q=0}^{+\infty} \left(\frac{1}{2}\right)^{2q} \frac{(\gamma v_1)^{2q}}{(2q+1)!} \tag{16}$$

$$g_{12}(v_2) = \left[\alpha\,\exp\left(-\frac{\gamma v_2}{2}\right) - \beta\,\exp\left(\frac{\gamma v_2}{2}\right)\right] \tag{17}$$

$$\begin{aligned} g_2(v_1, v_2) &= 2I(\alpha+\beta) - 2I\,\cosh\left(\frac{\gamma\,v_1}{2}\right)\cdot \\ &\qquad \left[\alpha\,\exp\left(-\frac{\gamma v_2}{2}\right) + \beta\,\exp\left(\frac{\gamma v_2}{2}\right)\right] \end{aligned} \tag{18}$$

Thereby, if the output port of the nonlinear resistive two-port shown in Fig. 3 is terminated with a nonlinear dynamic bipole defined by (8) then the circuit at the input port is equivalent to a memristor described by (9)–(10) with $\tilde{g}_{11}(v_1)$, $g_{12}(v_2)$ and $g_2(v_1, v_2)$ given in (16)–(18). The special case $(\alpha, \beta, \gamma) = (1, 0, 1/V_T)$ is reported in [8] where it is shown that the resulting circuit manifests the typical pinched hysteretic current–voltage loop characterizing a memristor.

IV. Circuit implementation of the potassium ion–channel memristor

Due to the lack of space, this section just sketches out the implementation of the first–order potassium ion–channel memristor based on the nonlinear resistive two–port $\mathcal{T}$–$\mathcal{P}$ terminated with a linear dynamic bipole. Similar thinking can be used to implement the sodium ion–channel memristor.

With reference to (3)–(4), let us consider $v_1 \leftrightarrow v_K$, $v_2 \leftrightarrow n$ and $i_1 \leftrightarrow i_K$. Under the assumption that $\gamma v_1 \ll 1$, equation (16) results in $\tilde{g}_{11}(v_1) \cong \gamma I$. It follows that I can be set such that $\gamma I = \bar{g}_K$ (see (4)). A suitable choice of parameters α and β in (17) allows us to approximate equation (4) of the first-order potassium ion–channel memristor.

The approximation of (3) by means of (9) and (18) is more difficult. It requires to replace $f_R(v_2)$ with a voltage-controlled nonlinear resistive two-port $\mathcal{T}$–$\mathcal{P}_2$ connected in parallel to $\mathcal{T}$–$\mathcal{P}$ and terminated with a linear bipole.

By assuming that $\mathcal{T}$–$\mathcal{P}_2$ is defined by:

$$\hat{i}_1(t) = 0$$
$$\hat{i}_2(t) = \hat{g}(v_1, v_2)$$

it is easily realized that (10) is not influenced by $\mathcal{T}$–$\mathcal{P}_2$, whereas (9) becomes

$$D\,v_2 = -\hat{g}(v_1, v_2) - g_2(v_1, v_2) \tag{19}$$

where $Y(D) = D$ in order to realize a first–order (potassium ion–channel) memristor and $g_2(v_1, v_2)$ given in(18).

The assumption $\gamma v_1 \ll 1$ and the comparison between the r.h.s. of (3) and (19) permit us to identify

$$\begin{aligned} \hat{g}(v_1, v_2) &= 2I\left\{ \left[\alpha\,\exp\left(-\frac{\gamma v_2}{2}\right) + \beta\,\exp\left(\frac{\gamma v_2}{2}\right)\right] \right. \\ &\qquad \left. -(\alpha+\beta) \right\} - f_K(v_1, v_2) \end{aligned} \tag{20}$$

A Piecewise-Linear approximation [10] of (20) leads to standard electronic implementations of $\mathcal{T}$–$\mathcal{P}_2$ and an accurate approximation of the nonlinear dynamics of (3) by means of (19). More details and numerical simulations will be shown at the conference venue.

V. Conclusions

This manuscript presents a novel class of circuits useful to a implement memristor. This class generalizes the circuit proposed in [8]. In addition, the implementation of the potassium and sodium ion–channel memristors in the Hodgkin–Huxley Axon circuit model is briefly outlined.

References

[1] E. R. Kandel, J. H. Schwartz, and T. Jessell, *Principles of Neural Science*. New York: McGraw-Hill, 2000.

[2] A. L. Hodgkin and A. F. Huxley, "A quantitative description of membrane current and its application to conduction and excitation in nerve," *J. Physiol.*, vol. 117, pp. 500–544, 1952.

[3] L. O. Chua, V. Sbitnevn, and H. Kim, "Hodgkin–huxley axon is made of memristors," *Int. J. of Bifurcation and Chaos*, vol. 22, no. 3, pp. 1 230 011–1–48, 2012.

[4] L. O. Chua, "Memristor: the missing circuit element," *IEEE Trans. on Circuit Theory*, vol. 18, no. 5, pp. 507–519, 1971.

[5] L. O. Chua and S. M. Kang, "Memristive devices and systems," *Proc. IEEE*, vol. 64, no. 2, pp. 209–223, 1976.

[6] D. B. Strukov, G. S. Snider, D. R. Stewart, and R. S. Williams, "The missing memristor found," *Nature*, vol. 453, pp. 80–83, 2008.

[7] F. Corinto, A. Ascoli, and M. Gilli, "Nonlinear dynamics of memristor oscillators," *IEEE Trans. on Circuits and Systems – I*, vol. 58, no. 6, pp. 1323–1336, 2011.

[8] F. Corinto and A. Ascoli, "Memristive diode bridge with lcr filter," *Electronics Letters*, vol. 48, no. 14, pp. 824–825, 2012.

[9] L. O. Chua, "Resistance switching memories are memristors," *Applied Physics A*, vol. 102, no. 4, pp. 765–783, 2011.

[10] L. O. Chua and A. Deng, "Canonical piecewise-linear modeling," *IEEE Trans. on Circuits and Systems*, vol. 33, no. 5, pp. 511–525, 1986.