Axioms for the integers

Let

$$\mathbb{Z} = \text{the set of } \textit{integers} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$$

and

$$\begin{aligned} \mathbb{N} &= \text{the set of } \textit{natural numbers} = \{1, 2, 3, \ldots\}, \\ -\mathbb{N} &= \text{the set of } \textit{negative integers} = \{-1, -2, -3, \ldots\}. \end{aligned}$$

- **Addition Properties.** There is an operation $+$, called addition, such that
    - $a + b \in \mathbb{Z}$ for all $a, b \in \mathbb{Z}$;
    - $a + b = b + a$ for all $a, b \in \mathbb{Z}$;
    - $(a + b) + c = a + (b + c)$ for all $a, b, c \in \mathbb{Z}$;
    - $a + 0 = a$ for all $a \in \mathbb{Z}$;
    - for any $a \in \mathbb{Z}$ there exists $-a \in \mathbb{Z}$ such that $a + (-a) = 0$.
- **Multiplication Properties**. There is an operation $\cdot$ (or $\times$), called multiplication, such that
    - $a \cdot b \in \mathbb{Z}$ for all $a, b \in \mathbb{Z}$;
    - $a \cdot b = b \cdot a$ for all $a, b \in \mathbb{Z}$;
    - $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ for all $a, b, c \in \mathbb{Z}$;
    - $a \cdot 1 = a$ for all $a \in \mathbb{Z}$;
- **Distributive Property**. We have $a \cdot (b + c) = a \cdot b + a \cdot c$ for all $a, b, c \in \mathbb{Z}$.
- **Trichotomy Principle**. The set of integers can be partitioned into three disjoint sets: $\mathbb{Z} = -\mathbb{N} \cup \{0\} \cup \mathbb{N}$. We write $a > b$ if $a - b \in \mathbb{N}$, $a = b$ if $a - b = 0$, and $a < b$ if $a - b \in -\mathbb{N}$.
- **Positivity**. If $a, b \in \mathbb{N}$, then $a + b \in \mathbb{N}$ and $a \cdot b \in \mathbb{N}$.
- **Well-Ordering Principle**. Every nonempty subset of $\mathbb{N}$ has a least (smallest) element.

Useful results easily derived from the axioms

- **Relations $\leqslant$ and $\geqslant$.** We write $a \leqslant b$ or $b \geqslant a$ to signify that $a < b$ or $a = b$. The integers $\mathbb{Z}$ are ordered by this relation, and we have
  - $a \leqslant a$ for all $a \in \mathbb{Z}$;
  - if $a \leqslant b$ and $b \leqslant a$, then $a = b$;
  - $a \leqslant b$ and $b \leqslant c$, then $a \leqslant c$;
  - for all $a, b \in \mathbb{Z}$, either $a \leqslant b$ or $b \leqslant a$.
- **Principle of Induction.** Let S be a subset of $\mathbb{N}$ such that

  $$1 \in S \qquad \text{and} \qquad k \in S \implies k+1 \in S.$$

  Then $S = \mathbb{N}$.
- **Cancellation Laws.** If $a + x = a + y$, then $x = y$. If $a \cdot x = a \cdot y$ and $a \neq 0$, then $x = y$.
- **Zero Multiplication.** $a \cdot 0 = 0$ for all $a \in \mathbb{Z}$.
- **Integral Domain Property.** If $a \cdot b = 0$, then $a = 0$ or $b = 0$.
- **Properties of Negatives.** We have $(-a) \cdot b = a \cdot (-b) = -(a \cdot b)$ and $(-a) \cdot (-b) = ab$ for all $a, b \in \mathbb{Z}$. In particular, the product of any two negative numbers is positive.
- **FOIL Law.** $(a+b) \cdot (c+d) = a \cdot c + a \cdot d + b \cdot c + b \cdot d$ for all $a, b, c, d \in \mathbb{Z}$.
- **General Associative-Commutative Laws.**

  ◦ When adding a collection of n integers $a_1 + a_2 + \cdots + a_n$, the numbers may be grouped in any way and added in any order. In particular, the expression $a_1 + a_2 + \cdots + a_n$ is *well-defined* (no parentheses are needed to specify the order of operations).

  ◦ When multiplying a collection of n integers $a_1 \times a_2 \times \cdots \times a_n$, the numbers may be grouped in any way and added in any order. In particular, the expression $a_1 \times a_2 \times \cdots \times a_n$ is *well-defined*.