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Five degrees of randomness

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ABSTRACT

Randomness is omnipresent, and hence the quantification of randomness is a fundamental necessity across the sciences. As "necessity is the mother of invention", scientists devised various approaches to quantify randomness: statistics uses standard deviation; statistical physics and information theory use entropies (e.g. Shannon); socioeconomics uses inequality indices (e.g. Gini); and ecology uses diversity indices (e.g. Simpson). Alternative to these approaches – which are all continuous quantifications – Mandelbrot suggested a radically different approach: a digital categorization of randomness. Inspired by Mandelbrot, here we showcase a digital categorization comprising five degrees of randomness — á la the Saffir–Simpson hurricane scale, and á la the DEFCON states of defense readiness. Using the reliability-engineering notion of hazard rates, we present a comprehensive study of the digital categorization. From a scholarly viewpoint, we unveil the categorization's profound connections to Gibbs measures in statistical physics, and to the following probability-theory notions: heavy tails, long tails, slow variation, regular variation, and rapid variation. From an applicative viewpoint, we demonstrate the categorization's potency and usability. This paper is relevant to wide audiences: theoreticians and practitioners that are tackling random systems and processes.

1. Introduction

From the tiniest quantum scales to the grandest cosmic scales, from the Sun's solar activities to Earth's geologic activities, from the fluctuations of financial markets to the fluctuations of traffic — *randomness* is everywhere. Due to the ubiquity of randomness, statistics is a critical tool in each and every scientific and engineering field that involves measurements and data. The building blocks of statistics are random variables [1,2], e.g.: the time it will take us to drive from home to work tomorrow; the amount of rainfall we will have this winter; the yield of our investment portfolio this year; etc. Addressing a given random variable X, perhaps the most fundamental question is: *how random is this random variable?*

Arguably, in the context of real quantities, the most common answer to the fundamental question is given by the standard deviation [1,2]. Based on Euclidean geometry, the standard deviation of a random variable manifests its average distance from its mean. The standard deviation is a particular case of a general randomness measure – $m(X) \geq 0$, where X is a real random variable – that exhibits the three following properties. (I) It vanishes if and only if X is deterministic: $m(X) = 0 \Leftrightarrow X = const$. (II) It is shift-invariant: $m(s + X) = m(X)$, where s is a real shift parameter. (III) It responds linearly to scale transformations: $m(s \cdot X) = s \cdot m(X)$, where s is a positive scale parameter.

Another randomness measure that exhibits these three properties is the exponentiation of the Rényi entropy of a real random variable X [3,4]. Special cases of this randomness measure include: the perplexity [5], which is the exponentiation of the Shannon entropy [6,7]; and the inverse Simpson index[1] [13], which is the exponentiation of the collision entropy [4].

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1 Simpson's index coincides with Hirschman's index [8,9], with the participation ratio [10,11], and with Blau's index [12].

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An alternative answer to the fundamental question, in the context of non-negative quantities, is given by inequality indices [14–17] . Designed to measure the socioeconomic inequality of income and wealth distributions, inequality indices are widely applied in economics and in the social sciences. Inequality indices are particular cases of a general randomness measure – $0 \leq m(X) \leq 1$, where X is a non-negative random variable – that exhibits the three following properties. (I) As the standard deviation, it vanishes if and only if X is deterministic: $m(X) = 0 \Leftrightarrow X = const$. (II) It is scale-invariant: $m(s \cdot X) = m(X)$, where s is a positive scale parameter. (III) It is monotone: see [18,19], or [20], for the details of the monotonicity property. The quintessential example of an inequality index that is used across the sciences is the Gini index [21–24].

On the one hand, the former general randomness measure, $m(X) \geq 0$, can be perceived as a "thermometer" of randomness — with its "temperature" taking values in the non-negative half-line $[0, \infty)$. On the other hand, the latter general randomness measure, $0 \leq m(X) \leq 1$, can be perceived as a "score" of randomness that take values in the unit interval $[0, 1]$. While these measures of randomness have markedly different ranges and properties, they share a common key feature: they yield *continuous* values — be it "temperature" in the former, or "score" in the latter.

Rather than quantifying randomness continuously, a radically different approach is to *categorize* it *digitally*. Namely, classifying randomness via a small discrete set of "degrees of randomness" — similarly to the Saffir–Simpson classification of hurricanes, and to the DEFCON classification of defense-readiness states. Mandelbrot pioneered this approach, proposing a digital categorization comprising seven degrees of randomness [25]. Alternative digital categorizations, comprising five degrees of randomness, were introduced and investigated in [26–29].

In this paper we follow the digital categorization of [26–28] , which – for a random variable under consideration – is based on the random-variable's moments and moment generating function. This digital categorization was addressed via the following perspectives: the Langevin equation [26]; diffusion processes [27]; and the Geometric Langevin equation [28]. Here we address the digital categorization of [26–28] via an altogether different perspective: *hazard rates*.

The hazard rate is a principal notion in survival analysis [30–32] and in reliability analysis [33–35]. To describe the hazard-rate notion, consider a system that started operating at time 0, and that did not fail up to the positive time t. The system's hazard rate, at time t, is the likelihood that it will fail immediately after time t.

Addressing the digital categorization of [26–28] via the hazard perspective has two main merits. On the one hand, the mathematical machinery required for the hazard perspective is elemental[2]: basic calculus and basic statistics. On the other hand, the hazard perspective provides a host of novel results and novel insights regarding the digital categorization. Using a financial jargon, the hazard-rate perspective yields a high "return on investment". In bottom line, this paper advances significantly the scientific understanding of the digital categorization of [26–28].

The reminder of this paper is organized as follows. Based on two thresholds (Section 2), a five-degrees categorization of randomness is devised (Section 3), and is then analyzed via the hazard perspective (Section 4). Facilitated by the hazard results, the digital categorization is further analyzed via a Gibbs perspective (Section 5), and via an overshoot perspective (Section 6). Additional aspects of the digital categorization are addressed in the Appendix.

Setting. Along this paper we consider a general non-negative random variable X. The statistical distribution of X is governed by its cumulative distribution function, $F(x) = P(X \leq x)$ $(x \geq 0)$, and by its survival function, $\bar{F}(x) = P(X > x)$ $(x \geq 0)$. Evidently, the cumulative distribution function and the survival function are coupled by a unit sum, $F(x) + \bar{F}(x) = 1$. If these functions are differentiable, then the random variable X has a density function: $f(x) = \frac{d}{dx}F(x) = -\frac{d}{dx}\bar{F}(x)$ $(x > 0)$. Throughout this paper: $\mathbf{E}[\cdot]$ denotes expectation; e is Euler's number; the sign $\approx$ denotes asymptotic equivalence in the limit $x \to \infty$[3]; and the shorthand notation $\varphi(\infty) = \lim_{x\to\infty} \varphi(x)$ is used.[4]

2. Two thresholds

With regard to the general non-negative random variable X, in this section we introduce two thresholds: one induced by the random-variable's moment generating function (MGF), and one induced by its moments. The digital categorization of randomness, to be presented in the next section, is based on these two thresholds.

2.1. MGF threshold

The *moment generating function* (MGF) of the random variable X is given by:

$$\begin{aligned} \mathbf{E}[\exp(qX)] &= \int_0^\infty \exp(qx)\, F(dx) \\ &= 1 + q \int_0^\infty \exp(qx)\, \bar{F}(x)\, dx \,, \end{aligned} \tag{1}$$

2 This is in sharp contrast to [26–28], where the required mathematical machinery is that of stochastic differential equations: Langevin equation, Fokker–Planck equation, Ito diffusions, and Ito calculus.

3 More specifically, in the context of non-negative functions that are defined on (l, ∞) (where $-\infty \leq l < \infty$): $\varphi_1(x) \approx \varphi_2(x)$ means that $\lim_{x\to\infty} [\varphi_1(x)/\varphi_2(x)] = c$, where c is a positive constant.

4 As with asymptotic equivalence, the notation $\varphi(\infty) = \lim_{x\to\infty} \varphi(x)$ is used in the context of non-negative functions that are defined on (l, ∞) (where $-\infty \leq l < \infty$), and whenever this limit exists in the wide sense $(0 \leq \varphi(\infty) \leq \infty)$.

where the exponent q is a positive parameter ($q > 0$). Integration by parts leads from the integral appearing on the top line of Eq. (1) to the integral appearing on the bottom line. If the random variable X has a density function then, in the top line of Eq. (1), we can replace $F(dx)$ by $f(x)\,dx$.

For a given exponent q, the MGF can either converge ($\mathbf{E}[\exp(qX)] < \infty$), or diverge ($\mathbf{E}[\exp(qX)] = \infty$). Note that, over the positive half-line $(0, \infty)$, the function $\exp(qx)$ is monotone with respect to the exponent q, i.e.: if $q_1 < q_2$ then $\exp(q_1 x) < \exp(q_2 x)$ for all $x > 0$. Consequently, the convergence/divergence of the MGF is determined by the threshold

$$q_*(X) = \sup\{q \mid \mathbf{E}[\exp(qX)] < \infty\} \ . \tag{2}$$

This threshold takes values in the range $0 \le q_*(X) \le \infty$. The convergence/divergence of the MGF is as follows: if $q_*(X) = 0$ then the MGF diverges over the entire positive half-line $(0, \infty)$; if $0 < q_*(X) < \infty$ then the MGF converges over the interval $(0, q_*(X))$, and diverges over the ray $(q_*(X), \infty)$; and if $q_*(X) = \infty$ then the MGF converges over the entire positive half-line $(0, \infty)$.

2.2. Moments threshold

The *moments* of the random variable X are given by:

$$\begin{aligned} \mathbf{E}[X^p] &= \int_0^\infty x^p F(dx) \\ &= p\int_0^\infty x^{p-1}\bar{F}(x)\,dx \ , \end{aligned} \tag{3}$$

where the power p is a positive parameter ($p > 0$). Integration by parts leads from the integral appearing on the top line of Eq. (3) to the integral appearing on the bottom line. If the random variable X has a density function then, in the top line of Eq. (3), we can replace $F(dx)$ by $f(x)\,dx$.

For a given power p, the moment can either converge ($\mathbf{E}[X^p] < \infty$), or diverge ($\mathbf{E}[X^p] = \infty$). Note that, over the ray $(1, \infty)$, the function x^p is monotone with respect to the power p, i.e.: if $p_1 < p_2$ then $x^{p_1} < x^{p_2}$ for all $x > 1$. Consequently, the convergence/divergence of the moments is determined by the threshold

$$p_*(X) = \sup\left\{p \mid \mathbf{E}\left[X^p\right] < \infty\right\} \ . \tag{4}$$

This threshold takes values in the range $0 \le p_*(X) \le \infty$. The convergence/divergence of the moments is as follows: if $p_*(X) = 0$ then all the moments diverge; if $0 < p_*(X) < \infty$ then moments with powers smaller than the threshold $(p < p_*(X))$ converge, and moments with powers larger than the threshold $(p > p_*(X))$ diverge; and if $p_*(X) = \infty$ then all the moments converge.

2.3. Properties

We now describe several key properties of the MGF and moments thresholds, $q_*(X)$ and $p_*(X)$. These properties will serve us in the sections to come.

(I) The MGF threshold is shift-invariant:

$$\begin{aligned} q_*(s+X) &= \sup\{q \mid \mathbf{E}[\exp(q(s+X))] < \infty\} \\ &= \sup\{q \mid \ \exp(qs)\,\mathbf{E}[\exp(qX)] < \infty\} \\ &= \sup\{q \mid \mathbf{E}[\exp(qX)] < \infty\} = q_*(X) \ , \end{aligned} \tag{5}$$

where s is a positive shift parameter.

(II) The MGF threshold responds harmonically to scale transformations:

$$\begin{aligned} q_*(s\cdot X) &= \sup\{q \mid \mathbf{E}[\exp(q(s\cdot X))] < \infty\} \\ &= \tfrac{1}{s}\sup\{qs \mid \mathbf{E}[\exp((qs)X)] < \infty\} = \tfrac{1}{s}q_*(X) \ , \end{aligned} \tag{6}$$

where s is a positive scale parameter.

(III) The MGF threshold of the sum of independent random variables equals the minimum of the random-variables' MGF thresholds:

$$q_*(X_1 + \cdots + X_n) = \min\{q_*(X_1), \ldots, q_*(X_n)\} \ , \tag{7}$$

where $\{X_1, \ldots, X_n\}$ are independent non-negative random variables. Eq. (7) follows straightforwardly from the fact that the MGF of the random-variables' sum equals the product of the random-variables' MGFs: $\mathbf{E}[\exp(q(X_1 + \cdots + X_n))] = \mathbf{E}[\exp(qX_1)] \cdots \mathbf{E}[\exp(qX_n)]$.

Table 1
Five degrees of randomness. The digital categorization of randomness, $\#(X) \in \{1, 2, 3, 4, 5\}$, regarding a non-negative random variable X, is based on the random-variable's MGF threshold $q_*(X)$ (Eq. (2)), and on the random-variable's moments threshold $p_*(X)$ (Eq. (4)). Degree (1): fully convergent MGF. Degree (2): partially convergent/divergent MGF. Degree (3): fully divergent MGF, but yet fully convergent moments. Degree (4): partially convergent/divergent moments. Degree (5): fully divergent moments.

Degree	Randomness	Characterization
$\#(X) = 1$	Infra Mild	$q_*(X) = \infty$
$\#(X) = 2$	Mild	$0 < q_*(X) < \infty$
$\#(X) = 3$	Borderline	$q_*(X) = 0$ & $p_*(X) = \infty$
$\#(X) = 4$	Wild	$0 < p_*(X) < \infty$
$\#(X) = 5$	Ultra Wild	$p_*(X) = 0$

(IV) The moments threshold is scale-invariant:

$$\begin{aligned} p_*(s \cdot X) &= \sup\left\{p \mid \mathbf{E}\left[(s \cdot X)^p\right] < \infty\right\} \\ &= \sup\left\{p \mid s^p \mathbf{E}\left[X^p\right] < \infty\right\} \\ &= \sup\left\{p \mid \mathbf{E}\left[X^p\right] < \infty\right\} = p_*(X) \ , \end{aligned} \tag{8}$$

where s is a positive scale parameter.

(V) The moments threshold responds harmonically to power transformations[5]:

$$\begin{aligned} p_*(X^s) &= \sup\left\{p \mid \mathbf{E}\left[(X^s)^p\right] < \infty\right\} \\ &= \tfrac{1}{s}\sup\left\{sp \mid \mathbf{E}\left[X^{sp}\right] < \infty\right\} = \tfrac{1}{s} p_*(X) \ , \end{aligned} \tag{9}$$

where s is a positive power parameter.

(VI) The moments threshold of the product of independent random variables equals the minimum of the random-variables' moments thresholds:

$$p_*(X_1 \cdots X_n) = \min\left\{p_*(X_1), \ldots, p_*(X_n)\right\} \ , \tag{10}$$

where $\{X_1, \ldots, X_n\}$ are independent non-negative random variables. Eq. (10) follows straightforwardly from the fact that the moments of the random-variables' product equal the product of the random-variables' moments: $\mathbf{E}[(X_1 \cdots X_n)^p] = \mathbf{E}\left[X_1^p\right] \cdots \mathbf{E}\left[X_n^p\right]$.

3. Five degrees of randomness

With regard to the general non-negative random variable X, in this section we present its digital categorization to five degrees of randomness. To that end we shall use the two thresholds that were introduced in the previous section: the MGF threshold $q_*(X)$, and the moments threshold $p_*(X)$.

3.1. The digital categorization

If the MGF of the random variable X converges (either over an interval or over the entire positive half-line), then the power-series expansion of the MGF spans the integer moments of the random variable X:

$$\mathbf{E}\left[\exp(qX)\right] = \sum_{i=0}^{\infty} \mathbf{E}\left[X^i\right] \frac{q^i}{i!} \ , \tag{11}$$

where $0 < q < q_*(X)$. Consequently, we obtain the following relations between the MGF threshold $q_*(X)$ and the moments threshold $p_*(X)$. (I) If the MGF converges, then all the moments converge: $q_*(X) > 0 \Rightarrow p_*(X) = \infty$. (II) If there are divergent moments, then the MGF diverges over the entire positive half-line: $p_*(X) < \infty \Rightarrow q_*(X) = 0$.

Based on the MGF threshold $q_*(X)$ and on the moments threshold $p_*(X)$, and exploiting the relations between these two thresholds, we devise a *digital categorization* $\#(X)$ of the random variable X. Comprising *five degrees of randomness*, $\#(X) \in \{1, 2, 3, 4, 5\}$, the digital categorization is presented in Table 1. Illustrative examples, per each degree of randomness, are presented in Table 2; these examples will be explained along the paper.

[5] Power transformations are prevalent in allometry [36].

Table 2
Examples of the five degrees of randomness. One set of examples is manifested in terms of the asymptotic behavior of the survival function $\bar{F}(x)$ of the random variable X. Another set of examples is manifested in terms of the asymptotic behavior of the density function $f(x)$ of the random variable X. In these examples, λ and γ denote positive parameters; the table's right column specifies restrictions, if any, on the parameters' values. A detailed explanation and discussion of these examples is presented in Appendix A.1.

Degree	$\bar{F}(x) \approx$	$f(x) \approx$	Parameter
(1) Infra Mild	$\exp(-\lambda x^{\gamma})$	$\exp(-\lambda x^{\gamma})$	$1 < \gamma < \infty$
(2) Mild	$\exp(-\lambda x)$	$\exp(-\lambda x)$	
(3) Borderline	$\exp(-\lambda x^{\gamma})$	$\exp(-\lambda x^{\gamma})$	$0 < \gamma < 1$
(3) Borderline	$\exp\{-\lambda [\ln(x)]^{\gamma}\}$	$\exp\{-\lambda [\ln(x)]^{\gamma}\}/x$	$1 < \gamma < \infty$
(4) Wild	$1/x^{\lambda}$	$1/x^{\lambda+1}$	
(5) Ultra Wild	$\exp\{-\lambda [\ln(x)]^{\gamma}\}$	$\exp\{-\lambda [\ln(x)]^{\gamma}\}/x$	$0 < \gamma < 1$

3.2. Properties

We now describe four key properties of the digital categorization of randomness. The derivations of some of the categorization's properties use an integrability lemma regarding two integrals, $\int_0^{\infty} \varphi_1(x)\,dx$ and $\int_0^{\infty} \varphi_2(x)\,dx$, with integrands that are integrable at zero. The integrability lemma asserts that: if the two integrands are asymptotically equivalent, $\varphi_1(x) \approx \varphi_2(x)$, then the two integrals converge/diverge jointly.

(I) *Shift invariance*:

$$\#(s + X) = \#(X) \ , \tag{12}$$

where s is a positive shift parameter. The shift-invariance property follows from Eq. (5), and from the observation that: Eq. (3) , together with the integrability lemma, implies that $p_*(s + X) = p_*(X)$. From a socioeconomic perspective – regarding the income of the members of a given human society – the shift-invariance property has the following meaning: the addition of "universal basic income" does not affect the randomness-category of the members' income distribution.

(II) *Scale invariance*:

$$\#(s \cdot X) = \#(X) \ , \tag{13}$$

where s is a positive scale parameter. The scale-invariance property follows from Eqs. (6) and (8). From a socioeconomic perspective – regarding the wealth of the members of a given human society – the scale-invariance property has the following meaning: changing the measurement of wealth from one currency (say USD) to another (say Euro) does not affect the randomness-category of the members' wealth distribution.

(III) *Asymptotic invariance*:

$$\bar{F}_1(x) \approx \bar{F}_2(x) \quad \Rightarrow \quad \#(X_1) = \#(X_2) \ , \tag{14}$$

where X_1 and X_2 are two non-negative random variables, and where $\bar{F}_1(x)$ and $\bar{F}_2(x)$ are their respective survival functions. The asymptotic-invariance property follows from Eqs. (1) and (3), together with the integrability lemma. If the random variables X_1 and X_2 have density functions, $f_1(x)$ and $f_2(x)$, then the asymptotic-invariance property can be expressed in terms of these densities. Indeed, $f_1(x) \approx f_2(x) \Rightarrow \bar{F}_1(x) \approx \bar{F}_2(x)$, and hence we obtain the following asymptotic-invariance corollary: $f_1(x) \approx f_2(x) \Rightarrow \#(X_1) = \#(X_2)$.

(IV) *Monotonicity*:

$$\bar{F}_1(x) \geq \bar{F}_2(x) \quad \Rightarrow \quad \#(X_1) \geq \#(X_2) \ , \tag{15}$$

where X_1 and X_2 are two non-negative random variables, and where $\bar{F}_1(x)$ and $\bar{F}_2(x)$ are their respective survival functions. The monotonicity property follows from Eqs. (1) and (3); for this property to hold, it is sufficient that the inequality $\bar{F}_1(x) \geq \bar{F}_2(x)$ holds over some ray (l, ∞), where l is a positive lower-bound level. Somewhat informally, we can express the monotonicity property as follows: the fatter the tail of the survival function — the higher the degree of randomness.

In the introduction we mentioned two general types of continuous randomness measures: "thermometers", which include the standard deviation, as well as the exponentiation of the Rényi entropy; and "scores", which include inequality indices. On the one hand, similarly to "thermometers", the digital categorization is shift-invariant. On the other hand, similarly to "scores", the digital categorization is scale-invariant and monotone. Hence, the digital categorization of randomness amalgamates properties of both "thermometers" and "scores". A finer 'resolution' of the digital categorization – which is based on a gauge of "mildness" and on a gauge of "wildness" – is presented and discussed in Appendix A.2.

Last, we note that

$$0 \leq X \leq l \quad \Rightarrow \quad \#(X) = 1 \ , \tag{16}$$

where l is a positive upper-bound level. Namely, Eq. (16) asserts that: if the random variable X is bounded, then its randomness is Infra Mild. Thus, having resolved the case of bounded random variables, we henceforth consider X to be unbounded. Specifically, we henceforth consider the survival function of the random variable X to be positive over the entire positive half-line: $\bar{F}(x) > 0$ $(x > 0)$.

4. Hazard perspective

If the random variable X has a density function, then it also has a *hazard function*:

$$H(x) = -\frac{d}{dx} \ln\left[\bar{F}(x)\right] = \frac{f(x)}{\bar{F}(x)} \tag{17}$$

$(x > 0)$. The hazard function – also know as "hazard rate" and "failure rate" – is a principal tool in survival analysis [30–32] and in reliability analysis [33–35].

To elucidate the meaning of the hazard function, envisage a system that starts operating at time 0, and that fails at the random time X. The system's hazard rate, at the positive time point t, is defined as follows: $H(t) = \lim_{\Delta \to 0} \frac{1}{\Delta} \Pr(X \le t+\Delta \mid X > t)$. Namely, $H(t)$ is the likelihood that the system will fail immediately after time t, provided the information that it did not fail up to time t. This definition yields the hazard function Eq. (17).

In this section we establish hazard formulations for the MGF and moments thresholds, $q_*(X)$ and $p_*(X)$. These hazard formulations will be based on the hazard function $H(x)$, as well as on the function

$$\tilde{H}(x) = xH(x) \tag{18}$$

$(x > 0)$. To establish the hazard formulations we shall use the following hazard-function representation of the corresponding survival function:

$$\bar{F}(x) = \exp\left\{-\int_0^x H(t)\,dt\right\} . \tag{19}$$

Eq. (19) follows from Eq. (17) via integration.

4.1. MGF threshold

Armed with the hazard function $H(x)$, let us address the MGF of the random variable X. Substituting Eq. (19) into the bottom line of Eq. (1), we arrive at the following MGF representation:

$$\mathbf{E}[\exp(qX)] = 1 + q\int_0^\infty \exp\{x[q - A(x)]\}\,dx , \tag{20}$$

where $A(x) = \frac{1}{x}\int_0^x H(t)\,dt$. Assuming that the limit $H(\infty)$ exists, L'Hospital's rule implies that $A(\infty) = H(\infty)$. In turn, Eq. (20) implies that: if $q < H(\infty)$ then $\mathbf{E}[\exp(qX)] < \infty$; and if $q > H(\infty)$ then $\mathbf{E}[\exp(qX)] = \infty$. Consequently, we obtain the following hazard formulation for the MGF threshold:

$$q_*(X) = H(\infty) . \tag{21}$$

4.2. Moments threshold

Armed with the hazard function $H(x)$, as well as with the function $\tilde{H}(x) = xH(x)$, let us address the moments of the random variable X. Substituting Eq. (19) into the bottom line of Eq. (3), and using the change-of-variables $x = \exp(y)$, we arrive at the following moments representation:

$$\mathbf{E}\left[X^p\right] = p\int_{-\infty}^\infty \exp\{y[p - B(y)]\}\,dy , \tag{22}$$

where $B(y) = \frac{1}{y}\int_0^{\exp(y)} H(t)\,dt$. Assuming that the limit $\tilde{H}(\infty)$ exists, L'Hospital's rule implies that $B(\infty) = \tilde{H}(\infty)$. In turn, Eq. (22) implies that: if $p < \tilde{H}(\infty)$ then $\mathbf{E}[X^p] < \infty$; and if $p > \tilde{H}(\infty)$ then $\mathbf{E}[X^p] = \infty$. Consequently, we obtain the following hazard formulation for the moments threshold:

$$p_*(X) = \tilde{H}(\infty) . \tag{23}$$

4.3. Application

To demonstrate the application of the hazard formulations for the MGF and moments thresholds, $q_*(X) = H(\infty)$ and $p_*(X) = \tilde{H}(\infty)$, consider examples that correspond to those appearing in the survival-function column of Table 2. As in Table 2, λ and γ denote positive parameters of the examples' survival functions.

For $\bar{F}(x) = \exp(-\lambda x^{\gamma})$ $(x \geq 0)$ we have $H(x) = \lambda\gamma x^{\gamma-1}$, and hence: the randomness is Infra Mild ($\#(X) = 1$) when $1 < \gamma < \infty$, and is Borderline ($\#(X) = 3$) when $0 < \gamma < 1$. For $\bar{F}(x) = \exp(-\lambda x)$ $(x \geq 0)$ we have $H(x) = \lambda$, and hence: the randomness is Mild ($\#(X) = 2$). For $\exp\{-\lambda[\ln(x)]^{\gamma}\}$ $(x \geq 1)$ we have $H(x) = \lambda\gamma[\ln(x)]^{\gamma-1}/x$ and hence: the randomness is Borderline ($\#(X) = 3$) when $1 < \gamma < \infty$, and is Ultra Wild ($\#(X) = 5$) when $0 < \gamma < 1$. For $\bar{F}(x) = 1/x^{\lambda}$ $(x \geq 1)$ we have $H(x) = \lambda/x$, and hence: the randomness is Wild ($\#(X) = 4$). A discussion of these examples is presented in Appendix A.1.

Yet another example is $\bar{F}(x) = 1/[\ln(x)]^{\lambda}$ $(x \geq e)$. For this example we have $H(x) = \lambda/[x\ln(x)]$, and hence: the randomness is Ultra Wild ($\#(X) = 5$). Statistical distributions whose survival functions admit the asymptotic behavior $\bar{F}(x) \approx 1/[\ln(x)]^{\lambda}$ are intimately related to, so called, ultra-slow diffusions [37–39].

5. Gibbs perspective

In order to apply the hazard formulations, one is required to know the hazard function $H(x)$ of the random variable X — which, in turn, is based on the random-variable's survival function $\bar{F}(x)$. However, it is often the case that the survival function $\bar{F}(x)$ has no closed-form expression. In such cases one can use, analytically, only the density function $f(x)$ of the random variable X.

Considering the density function $f(x)$ to be differentiable, we introduce the function

$$G(x) = -\frac{d}{dx}\ln[f(x)] = -\frac{f'(x)}{f(x)} \tag{24}$$

$(x > 0)$. Up to a scale factor, the function $G(x)$ has a profound meaning [40–42]: it is the derivative of the potential function that underpins the Gibbs representation of the density function $f(x)$. The Gibbs representation arises via two foundational approaches. One approach, which is applied in statistical physics and in information theory, is entropy maximization [43–45]. The other approach, which is applied in statistical physics and in stochastic dynamics, is the Langevin equation [46–48]. In what follows we term $G(x)$ the *Gibbs function* of the random variable X.

In this section we establish Gibbs formulations for the MGF and moments thresholds, $q_*(X)$ and $p_*(X)$. These Gibbs formulations will be based on the Gibbs function $G(x)$, as well as on the function

$$\tilde{G}(x) = xG(x) \tag{25}$$

$(x > 0)$. The Gibbs function $G(x)$ of Eq. (24) is analogous to the hazard function $H(x)$ of Eq. (17), and the function $\tilde{G}(x)$ of Eq. (25) is analogous to the function $\tilde{H}(x)$ of Eq. (18).

5.1. MGF threshold

Using Eqs. (17) and (24), while assuming that $\lim_{x\to\infty} f(x) = 0$, L'Hospital's rule implies that:

$$\begin{aligned} H(\infty) &= \lim_{x\to\infty} H(x) \\ &= \lim_{x\to\infty}\frac{f(x)}{\bar{F}(x)} = \lim_{x\to\infty}\frac{f'(x)}{-f(x)} \\ &= \lim_{x\to\infty} G(x) = G(\infty)\ . \end{aligned} \tag{26}$$

Hence, assuming that the limit $G(\infty)$ exists, we obtain that $H(\infty) = G(\infty)$. In turn, the hazard formulation for the MGF threshold (Eq. (21)) yields the following Gibbs formulation for this threshold:

$$q_*(X) = G(\infty)\ . \tag{27}$$

5.2. Moments threshold

Using Eqs. (17)–(18) and Eqs. (24)–(25), while assuming that $\lim_{x\to\infty} xf(x) = 0$, L'Hospital's rule implies that:

$$\begin{aligned} \tilde{H}(\infty) &= \lim_{x\to\infty}\tilde{H}(x) \\ &= \lim_{x\to\infty}\frac{xf(x)}{\bar{F}(x)} = \lim_{x\to\infty}\frac{f(x)+xf'(x)}{-f(x)} \\ &= -1 + \lim_{x\to\infty}\tilde{G}(x) = \tilde{G}(\infty) - 1\ . \end{aligned} \tag{28}$$

Hence, assuming that the limit $\tilde{G}(\infty)$ exists, we obtain that $\tilde{H}(\infty) = \tilde{G}(\infty) - 1$. Consequently, the hazard formulation for the moments threshold (Eq. (23)) yields the following Gibbs formulation for this threshold:

$$p_* (X) = \tilde{G}(\infty) - 1 \, . \tag{29}$$

5.3. Application

Albeit for pathological density functions, the aforementioned conditions $\lim_{x\to\infty} f(x) = 0$ and $\lim_{x\to\infty} xf(x) = 0$ are widely met. Consequently, the Gibbs formulations for the MGF and moments thresholds, $q_*(X) = G(\infty)$ and $p_*(X) = \tilde{G}(\infty) - 1$, are widely applicable. To demonstrate the application of these Gibbs formulations, consider examples that correspond to those appearing in the density-function column of Table 2. As in Table 2, λ and γ denote positive parameters of the examples' density functions; the density functions are given up to their normalization constants.

For $f(x) = \exp(-\lambda x^\gamma)$ $(x > 0)$ we have $G(x) = \lambda\gamma x^{\gamma-1}$, and hence: the randomness is Infra Mild ($\#(X) = 1$) when $1 < \gamma < \infty$, and is Borderline ($\#(X) = 3$) when $0 < \gamma < 1$. For $f(x) = \exp(-\lambda x)$ $(x > 0)$ we have $G(x) = \lambda$, and hence: the randomness is Mild ($\#(X) = 2$). For $f(x) = \exp\{-\lambda[\ln(x)]^\gamma\}/x$ $(x > 1)$ we have $G(x) = \{\lambda\gamma[\ln(x)]^{\gamma-1} + 1\}/x$ and hence: the randomness is Borderline ($\#(X) = 3$) when $1 < \gamma < \infty$, and is Ultra Wild ($\#(X) = 5$) when $0 < \gamma < 1$. For $f(x) = 1/x^{\lambda+1}$ $(x > 1)$ we have $G(x) = (\lambda + 1)/x$, and hence: the randomness is Wild ($\#(X) = 4$). A discussion of these examples is presented in Appendix A.1.

Yet another example is $f(x) = \exp(-\lambda x^\gamma)x^{\nu-1}$ $(x > 0)$, where ν is an additional positive parameter. For this example we have $G(x) = \lambda\gamma x^{\gamma-1} + (\nu - 1)/x$, and hence: the randomness is Infra Mild ($\#(X) = 1$) when $1 < \gamma < \infty$, is Mild ($\#(X) = 2$) when $\gamma = 1$, and is Borderline ($\#(X) = 3$) when $0 < \gamma < 1$.

6. Overshoot perspective

So far we presented two sets of formulations – hazard and Gibbs – for the MGF and moments thresholds, $q_*(X)$ and $p_*(X)$. We shall now present a third set of formulations for these two thresholds. The third set is based on the overshoot of the random variable X above the positive level l, provided the information that X exceeds the level l.

6.1. MGF threshold

The *arithmetic overshoot* of the random variable X over the level l – given the information $X > l$ – is the random variable $X - l$. The survival function of the arithmetic overshoot, conditioned on the information, is:

$$\begin{aligned} &\Pr(X - l > y \mid X > l) = \bar{F}(l + y)/\bar{F}(l) \\ &= \exp\left\{-\int_l^{l+y} H(t)\,dt\right\} = \exp\left\{-\int_0^y H(l+u)\,du\right\} , \end{aligned} \tag{30}$$

where $y > 0$. The top line of Eq. (30) holds in general, and the bottom line of Eq. (30) holds whenever the random variable X has a density function. Substituting Eq. (19) into the top line of Eq. (30), and then using the change-of-variables $t = l+u$, yields the bottom line of Eq. (30).

Let us address the conditional survival function of Eq. (30) in the limit $l \to \infty$. Assuming that the limit $H(\infty)$ exists, the integral appearing on the bottom line of Eq. (30) converges to the limit $\int_0^y H(\infty)\,du = H(\infty)\cdot y$. Consequently, setting $y = 1$, and using the hazard formulation for the MGF threshold, $q_*(X) = H(\infty)$, we obtain the following survival and density formulations for this threshold:

$$q_*(X) = -\lim_{l\to\infty} \ln\left[\frac{\bar{F}(l+1)}{\bar{F}(l)}\right] = -\lim_{l\to\infty} \ln\left[\frac{f(l+1)}{f(l)}\right] . \tag{31}$$

In Eq. (31), the density-limit follows from the survival-limit via L'Hospital's rule.

6.2. Moments threshold

The *geometric overshoot* of the random variable X over the level l – given the information $X > l$ – is the random variable X/l. The survival function of the geometric overshoot, conditioned on the information, is:

$$\begin{aligned} &\Pr(X/l > z \mid X > l) = \bar{F}(lz)/\bar{F}(l) \\ &= \exp\left\{-\int_l^{lz} H(t)\,dt\right\} = \exp\left\{-\int_1^z \tilde{H}(lu)\,\frac{du}{u}\right\} , \end{aligned} \tag{32}$$

where $z > 1$, and where $\tilde{H}(x) = xH(x)$ (as in Eq. (18)). The top line of Eq. (32) holds in general, and the bottom line of Eq. (32) holds whenever the random variable X has a density function. Substituting Eq. (19) into the top line of Eq. (32), and then using the change-of-variables $t = lu$, yields the bottom line of Eq. (32).

Let us address the conditional survival function of Eq. (32) in the limit $l \to \infty$. Assuming that the limit $\tilde{H}(\infty)$ exists, the integral appearing on the right-hand side of the bottom line of Eq. (32) converges to the limit $\int_1^z \tilde{H}(\infty) \frac{du}{u} du = \tilde{H}(\infty) \cdot \ln(z)$. Consequently, setting $z = e$, and using the hazard formulation for the moments threshold, $p_*(X) = \tilde{H}(\infty)$, we obtain the following survival and density formulations for this threshold:

$$p_*(X) = -\lim_{l\to\infty} \ln\left[\frac{\bar{F}(le)}{\bar{F}(l)}\right] = -\lim_{l\to\infty} \ln\left[\frac{f(le)}{f(l)}\right] - 1 . \tag{33}$$

In Eq. (33), the density-limit follows from the survival-limit via L'Hospital's rule.

6.3. Probabilistic discussion

With regard to the tails of statistical distributions, probability theory applies five key notions [49–53]: heavy tails, long tails, slow variation, regular variation, and rapid variation. We now turn to discuss the relations between these notions and the digital categorization of randomness.

6.3.1. Heavy tails and long tails

Assuming that it exists, consider the limiting survival function

$$\Phi(y) = \lim_{l\to\infty} \frac{\bar{F}(l+y)}{\bar{F}(l)} \tag{34}$$

($y > 0$). Namely, $\Phi(y)$ is the limit, as $l \to \infty$, of the arithmetic-overshoot survival function of Eq. (30).

It is straightforward to observe that the limiting survival function of Eq. (34) satisfies the following property: $\Phi(a+b) = \Phi(a) \cdot \Phi(b)$, where $0 < a, b < \infty$. In turn, this property implies that the limiting survival function of Eq. (34) admits the following exponential representation:

$$\Phi(y) = \exp(-\epsilon y) \tag{35}$$

($y > 0$), where the exponent ϵ takes values in the range $0 \le \epsilon \le \infty$.

Consider the random variable X to have a density function. Then, Eq. (30) implies that: the limiting survival function $\Phi(y)$ exists if and only if the limit $H(\infty)$ exists — in which case we have

$$\epsilon = H(\infty) = q_*(X) . \tag{36}$$

Namely, Eq. (36) establishes the coincidence of the exponent ϵ and the MGF threshold $q_*(X)$.

The random variable X is termed *heavy-tailed* if its MGF is fully divergent, $q_*(X) = 0$. Also, the random variable X is termed *long-tailed* if the limiting survival function of Eq. (34) exists, and if $\epsilon = 0$. Consequently – considering the random variable X to have a density function – we obtain the following relations between the notion of long tails, the notion of heavy tails, and the digital categorization of randomness:

$$\text{Long Tails} \Rightarrow \text{Heavy Tails} \Leftrightarrow 3 \le \#(X) \le 5 .$$

To elucidate the meaning of the limiting survival function of Eq. (34) , set Y to be a random variable whose statistical distribution is governed by this survival function. Namely, Y is the limiting random variable – in law, as $l \to \infty$ – of the arithmetic overshoot $X - l$, given the information $X > l$. This limiting random variable takes values in the range $0 \le Y \le \infty$. Depending on the value of the exponent ϵ, the limiting random variable Y exhibits three markedly different behaviors. (I) If $\epsilon = 0$ then $\Phi(y) = 1$, and hence: $Y = \infty$ with probability one. (II) If $0 < \epsilon < \infty$ then Y is Exponentially distributed. (III) If $\epsilon = \infty$ then $\Phi(y) = 0$, and hence: $Y = 0$ with probability one.

6.3.2. Slow, regular, and rapid variation

Assuming that it exists, consider the limiting survival function

$$\Psi(z) = \lim_{l\to\infty} \frac{\bar{F}(lz)}{\bar{F}(l)} \tag{37}$$

($z > 1$). Namely, $\Psi(z)$ is the limit, as $l \to \infty$, of the geometric-overshoot survival function of Eq. (32).

It is straightforward to observe that the limiting survival function of Eq. (37) satisfies the following property: $\Psi(a \cdot b) = \Psi(a) \cdot \Psi(b)$, where $1 < a, b < \infty$. In turn, this property implies that the limiting survival function of Eq. (37) admits the following power representation:

$$\Psi(z) = \frac{1}{z^\delta} \tag{38}$$

($z > 1$), where the power δ takes values in the range $0 \le \delta \le \infty$.

Table 3
Thresholds summary. The two thresholds, $q_*(X)$ and $p_*(X)$, of a general non-negative random variable X, are formulated via: the MGF $\mathbf{E}[\exp(qX)]$ and the moments $\mathbf{E}[X^p]$; the survival function $\bar{F}(x)$; the density function $f(x)$; the hazard function $H(x)$ of Eq. (17); and the Gibbs function $G(x)$ of Eq. (24).

	$q_*(X) =$	$p_*(X) =$
MGF/Moments	$\sup\{q \mid \mathbf{E}[\exp(qX)] < \infty\}$	$\sup\{p \mid \mathbf{E}[X^p] < \infty\}$
Survival	$-\lim_{x\to\infty} \ln\left[\frac{\bar{F}(x+1)}{\bar{F}(x)}\right]$	$-\lim_{x\to\infty} \ln\left[\frac{\bar{F}(xe)}{\bar{F}(x)}\right]$
Density	$-\lim_{x\to\infty} \ln\left[\frac{f(x+1)}{f(x)}\right]$	$-1-\lim_{x\to\infty} \ln\left[\frac{f(xe)}{f(x)}\right]$
Hazard	$\lim_{x\to\infty} H(x)$	$\lim_{x\to\infty} [xH(x)]$
Gibbs	$\lim_{x\to\infty} G(x)$	$-1+\lim_{x\to\infty} [xG(x)]$

Consider the random variable X to have a density function. Then, Eq. (32) implies that: the limiting survival function $\Psi(z)$ exists if and only if the limit $\tilde{H}(\infty)$ exists — in which case we have

$$\delta = \tilde{H}(\infty) = p_*(X) \ . \tag{39}$$

Namely, Eq. (39) establishes the coincidence of the power δ and the moments threshold $p_*(X)$.

If the limiting survival function of Eq. (37) exists, then the random variable X is termed: *slowly varying* if $\delta = 0$; *regularly varying* if $0 < \delta < \infty$; and *rapidly varying* if $\delta = \infty$. Consequently – considering the random variable X to have a density function – we obtain the following relations between the notions of rapid/regular/slow variation, and the digital categorization of randomness:

$$\begin{array}{lll} \text{Rapid Variation} & \Rightarrow & 1 \le \#(X) \le 3, \\ \text{Regular Variation} & \Rightarrow & \#(X) = 4, \\ \text{Slow Variation} & \Rightarrow & \#(X) = 5. \end{array}$$

To elucidate the meaning of the limiting survival function of Eq. (37) , set Z to be a random variable whose statistical distribution is governed by this survival function. Namely, Z is the limiting random variable – in law, as $l \to \infty$ – of the geometric overshoot X/l, given the information $X > l$. This limiting random variable takes values in the range $1 \le Z \le \infty$. Depending on the value of the power δ, the limiting random variable Z exhibits three markedly different behaviors. (I) If $\delta = 0$ then $\Psi(z) = 1$, and hence: $Z = \infty$ with probability one. (II) If $0 < \delta < \infty$ then Z is Pareto distributed. (III) If $\delta = \infty$ then $\Psi(z) = 0$, and hence: $Z = 1$ with probability one.

7. Conclusion

The Gauss bell curve – which is the density function of the normal distribution – is perhaps the most iconic shape in probability theory and statistics. With regard to a quantity whose statistical distribution is governed by the Gauss bell curve, the curve's bell shape is interpreted as follows [42]: there are a few small values, there are a few large values, and there is a massive bulk of near-average values. This interpretation is correct for the Gauss bell curve; however, the interpretation is not correct due to the bell shape. Indeed, there are many statistical distributions whose density functions display a bell shape that is identical to that of the Gauss bell curve, but yet for which the aforementioned interpretation does not hold — e.g. the Cauchy distribution.[6]

"God is in the details", and in the case of statistical distributions important details hide in their tails. While it is informative to observe the shape of the density function of a statistical distribution of interest, it is absolutely critical to address the quantitative behavior of the statistical-distribution's tails. As pointed out by Mandelbrot and Taleb [54], failing to understand the criticality of the tails can lead to severe underestimation of the inherent randomness, and hence to severe consequences. Being hit by a category-four hurricane, while preparing for a category-one hurricane, can result in devastating damages. Similarly, operating in a complex system that produces Wild randomness (e.g., a Cauchy-distributed output), while risk-managing for Infra Mild randomness (e.g., a normally-distributed output), can be catastrophic.

The *digital categorization* of randomness that was presented and explored in this paper comprises – á la the Saffir–Simpson scale for hurricanes, and á la the DEFCON states of defense readiness – *five degrees of randomness*: (1) Infra Mild; (2) Mild; (3) Borderline; (4) Wild; (5) Ultra Wild. From a human perspective, it very easy to grasp compact digital scales, and hence such scales constitute a highly effective method of communicating complex information to policymakers and to the public. From a computation perspective, as demonstrated along the paper, the digital categorization of randomness is very easy to apply, and has a host appealing analytic properties.

Table 3 summarizes the various formulations, established here, for the two thresholds on which the digital categorization of randomness is based. Additional aspects of the digital categorization are addressed in the Appendix below.

[6] In the case of the Cauchy distribution the very notion of average is not defined.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Appendix

A.1. Weibull And subbotin

Underpinning the examples of Table 2 are four statistical distributions: Weibull, Subbotin, log-Weibull, and log-Subbotin. In this subsection we discuss these distributions. As in Table 2, λ and γ denote the distributions' positive parameters. As the Subbotin distribution is defined over the real line $(-\infty, \infty)$, we begin with extending the digital categorization of randomness from the realm of non-negative random variables to the realm of real random variables.

A.1.1. Real random variables

While defined for non-negative random variables, the digital categorization of randomness is well applicable also for real random variables. Indeed, given a general real random variable R, we apply the digital categorization to its positive and negative parts. Specifically, the positive part is the non-negative random variable $R_+ = \max\{R, 0\}$; the survival function of this part satisfies $\Pr(R_+ > x) = \Pr(R > x)$ $(x > 0)$. And, the negative part is the non-negative random variable $R_- = -\min\{R, 0\}$; the survival function of this part satisfies $\Pr(R_- > x) = \Pr(R < -x)$ $(x > 0)$.

A real random variable R is termed symmetric if: $R = -R$, where the equality is in law. The digital categorization of randomness of a symmetric random variable R is given by

$$\#(R_+) = \#(R_-) = \#(|R|) \ . \tag{40}$$

Note that the survival function of the absolute-value random variable $|R|$ satisfies $\Pr(|R| > x) = 2\Pr(R > x)$ $(x > 0)$.

As an illustrative example of the application of Eq. (40), consider the Student's t-distribution [55]. Up to a normalization constant, the t-distribution's density function is given by $1/(1 + \lambda x^2)^\gamma$ $(-\infty < x < \infty)$, where λ and γ are positive parameters. Evidently, this density is a symmetric function, and hence Eq. (40) can be applied indeed. The density function of the absolute-value random variable that corresponds to the t-distribution satisfies $f(x) \approx 1/x^{2\gamma}$, and hence the randomness of the t-distribution is Wild. As a special case, the Student's t-distribution includes the Cauchy distribution $(\gamma = 1)$.

A.1.2. Weibull and Subbotin

The top three examples that appear in the survival-function column of Table 2 are asymptotically Weibull. The Weibull distribution [56,57] is characterized by the survival function $\bar{F}(x) = \exp(-\lambda x^\gamma)$ $(x \geq 0)$. This distribution has numerous uses in science and engineering [58–60], and it is one of the three universal extreme-value laws [61–63].

Determined by the parameter γ, the Weibull distribution has three regimes. (I) $\gamma < 1$, which characterizes the Stretched Exponential distribution [64–66], and whose randomness is Borderline. (II) $\gamma = 1$, which characterizes the Exponential distribution [67], and whose randomness is Mild. (III) $\gamma > 1$, which includes the Rayleigh distribution [68] $(\gamma = 2)$, and whose randomness is Infra Mild.

The top three examples that appear in the density-function column of Table 2 are asymptotically Subbotin [69–72]. Up to its normalization constant, the Subbotin-distribution's density function is given by $\exp(-\lambda|x|^\gamma)$ $(-\infty < x < \infty)$. Evidently, this density is a symmetric function, and hence Eq. (40) can be applied in order to categorize the Subbotin-distribution's randomness.

Determined by the parameter γ, the Subbotin distribution has three regimes. (I) $\gamma < 1$, whose randomness is Borderline. (II) $\gamma = 1$, which characterizes the Laplace distribution [73], and whose randomness is Mild. (III) $\gamma > 1$, which includes the normal (Gauss) distribution [74] $(\gamma = 2)$, and whose randomness is Infra Mild. The normal (Gauss) distribution is the universal law of the Central Limit Theorem [75,76].

A.1.3. Log-Weibull and log-Subbotin

The notion of log-distributions is defined as follows: if the statistical distribution of a real random variable R is $\mathcal{D}$, then the statistical distribution of the random variable $\exp(R)$ is termed "log-$\mathcal{D}$". Log-distributions are induced by multiplicative processes, and we shall address general log-distributions in Appendix A.3 below.

The bottom three examples that appear in the survival-function column of Table 2 are asymptotically log-Weibull, and the bottom three examples that appear in the density-function column of Table 2 are asymptotically log-Subbotin. Similarly to the Weibull and Subbotin distributions, and determined by the parameter γ, the log-Weibull and log-Subbotin distributions have three regimes. (I) $\gamma < 1$, whose randomness is Ultra Wild. (II) $\gamma = 1$, whose randomness is Wild. (III) $\gamma > 1$, whose randomness is Borderline.

As noted above, the Weibull distribution with $\gamma = 1$ yields the Exponential distribution. The log-Exponential distribution coincides with the Pareto distribution [77], which is characterized by the survival function $1/x^\lambda$ $(x \geq 1)$.

The Pareto distribution is the keystone model for statistical distributions with power-law tails [78–82], as well as the keystone model for 'fractal' statistical distributions [25,83,84].

As noted above, the Subbotin distribution with $\gamma = 2$ yields the normal (Gauss) distribution. The log-normal (log-Gauss) distribution [85] arises universally via the combination of multiplicative schemes and the Central Limit Theorem, and hence its profound importance [86–90]. An important example of a multiplicative scheme that leads to the log-normal distribution is Gibrat's "law of proportionate effects" [91]. Gibrat's law establishes the paradigmatic role of the log-normal distribution in economics and finance [92–94].

A.2. Mildness and wildness

In Section 3 we devised a digital categorization of randomness. In this subsection we discuss a finer 'resolution' of the digital categorization. With regard to the general non-negative random variable X, this resolution is attained via two gauges: one quantifying the "mildness" of X, and one quantifying the "wildness" of X. These two gauges mimic the coefficient of variation.

Considering the random variable X to have a positive mean ($0 < \mathbf{E}[X] < \infty$), the random-variable's coefficient of variation is the ratio of its standard deviation to its mean: $C(X) = \sigma(X)/\mathbf{E}[X]$. The coefficient of variation is a 'normalized' version of the standard deviation, it takes values in the range $0 \le C(X) \le \infty$, and it is scale-invariant: $C(s \cdot X) = C(X)$, where s is a positive scale parameter.

For a random variable X whose randomness is between Infra Mild to Borderline, $1 \le \#(X) \le 3$, we introduce the following *coefficient of mildness*:

$$C_{Mild}(X) = \frac{1}{\mathbf{E}(X) \cdot q_*(X)} . \tag{41}$$

Analogously to the coefficient of variation, the coefficient of mildness takes values in the range $0 \le C_{Mild}(X) \le \infty$, and it is scale-invariant. This coefficient gauges the mildness of the random variable X continuously: ranging from the lower-bound mildness level $C_{Mild}(X) = 0$ (Infra Mild randomness), via the intermediate mildness levels $0 < C_{Mild}(X) < \infty$ (Mild randomness), to the upper-bound mildness level $C_{Mild}(X) = \infty$ (Borderline randomness).

For a random variable X whose randomness is between Borderline to Ultra Wild, $3 \le \#(X) \le 5$, we introduce the following *coefficient of wildness*:

$$C_{Wild}(X) = \frac{1}{p_*(X)} . \tag{42}$$

Analogously to the coefficient of variation, the coefficient of wildness takes values in the range $0 \le C_{Wild}(X) \le \infty$, and it is scale-invariant. This coefficient gauges the wildness of the random variable X continuously: ranging from the lower-bound wildness level $C_{Wild}(X) = 0$ (Borderline randomness), via the intermediate wildness levels $0 < C_{Wild}(X) < \infty$ (Wild randomness), to the upper-bound wildness level $C_{Wild}(X) = \infty$ (Ultra Wild randomness).

In probability theory, substantial focus is set on the arithmetic averages of independent and identically valued (IID) random variables [1,2]. In the context of the random variable X, the corresponding arithmetic average is: $(X_1 + \cdots + X_n)/n$, where the summed elements are IID copies of X, and where n is the average's integer 'dimension'. Combined together, Eq. (6), Eq. (7), and Eq. (41) imply that

$$C_{Mild}\left(\frac{X_1 + \cdots + X_n}{n}\right) = \frac{1}{n} \cdot C_{Mild}(X) . \tag{43}$$

Hence – for a random variable X whose randomness is Mild, $\#(X) = 2$ – we obtain that: the mildness of the arithmetic average is inversely proportional to its dimension n.

Two major results of probability theory address the asymptotic behavior, in the limit $n \to \infty$, of the arithmetic average $(X_1 + \cdots + X_n)/n$. The Law of Large Numbers (LLN) applies whenever the mean of the random variable X is convergent ($\mathbf{E}[X] < \infty$), and it describes the arithmetic-average's limit [1,2]. The Central Limit Theorem (CLT) applies whenever the second-order moment of the random variable X is convergent ($\mathbf{E}[X^2] < \infty$), and it describes the arithmetic-average's fluctuations about its limit [1,2]. Hence – within the category of Wild randomness – the LLN and the CLT induce three tranches: (I) divergent mean; (II) convergent mean, but yet divergent second-order moment; (III) convergent second-order moment.

Last, let us shift from arithmetic averages to geometric averages. In the context of the random variable X, the corresponding geometric average is: $\sqrt[n]{X_1 \cdots X_n}$, where the multiplied elements are IID copies of X, and where n is the average's integer 'dimension'. Combined together, Eq. (9), Eq. (10), and Eq. (42) imply that

$$C_{Wild}\left(\sqrt[n]{X_1 \cdot \cdots \cdot X_n}\right) = \frac{1}{n} \cdot C_{Wild}(X) . \tag{44}$$

Hence – for a random variable X whose randomness is Wild, $\#(X) = 4$ – we obtain that: the wildness of the geometric average is inversely proportional to its dimension n.

A.3. Exponentiation

Multiplicative processes are prevalent in science and engineering [95–100]. In the context of the random variable X, multiplicative processes induce the exponentiation $X \mapsto \exp(X)$. The notion of log-distributions, which was described in Appendix A.1 above, is founded on the exponentiation $X \mapsto \exp(X)$. Using the hazard and Gibbs results of Sections 4 and 5 , in this subsection we discuss the relation between the digital categorizations of randomness of the random variables X and $\exp(X)$.

It is evident from Eqs. (2) and (4) that the moments threshold of the random variable $\exp(X)$ coincides with the MGF threshold of the random variable X:

$$p_*[\exp(X)] = q_*(X) \ . \tag{45}$$

In turn, combined together with Eqs. (21) and (27), Eq. (45) yields the following hazard and Gibbs formulations for the moments threshold of the random variable $\exp(X)$:

$$p_*[\exp(X)] = H(\infty) = G(\infty) \ . \tag{46}$$

The exponentiation $X \mapsto \exp(X)$ straightforwardly implies that the survival function and the density function of the random variable $\exp(X)$ are, respectively, $\bar{F}[\ln(x)]$ and $f[\ln(x)]/x$. Setting the functions

$$\hat{H}(x) = \frac{H(x)}{\exp(x)} \ \ \& \ \ \hat{G}(x) = \frac{G(x)}{\exp(x)} \ , \tag{47}$$

simple calculations imply that the hazard function and the Gibbs function of the random variable $\exp(X)$ are, respectively, $\hat{H}[\ln(x)]$ and $\hat{G}[\ln(x)] + \frac{1}{x}$. Consequently, applying Eqs. (21) and (27) with respect to these hazard and Gibbs functions, we obtain the following hazard and Gibbs formulations for the MGF threshold of the random variable $\exp(X)$:

$$q_*[\exp(X)] = \hat{H}(\infty) = \hat{G}(\infty) \ . \tag{48}$$

We underscore the fact that Eqs. (46) and (48) formulate the moments threshold and the MGF threshold of the random variable $\exp(X)$ in terms of the hazard and Gibbs functions of the random variable X. Hence, Eqs. (46) and (48) facilitate the computation of $\#[\exp(X)]$ – the digital categorization of randomness of the random variable $\exp(X)$ – directly from the random variable X. For example, using Eqs. (46) and (48) : the aforementioned hazard function of the Weibull distribution, $H(x) = \lambda\gamma x^{\gamma-1}$, yields directly the log-Weibull categories of randomness (noted in Appendix A.1); and the aforementioned Gibbs function of the Subbotin distribution, $G(x) = \lambda\gamma x^{\gamma-1}$, yields directly the log-Subbotin categories of randomness (noted in Appendix A.1).

Last, we note that Eq. (45) implies the following relations between the randomness-categories of the random variable X on the one hand, and the randomness-categories of the random variable $\exp(X)$ on the other hand:

$$\begin{array}{lcl} \#(X) = 1 & \Leftrightarrow & 1 \le \#[\exp(X)] \le 3, \\ \#(X) = 2 & \Leftrightarrow & \#[\exp(X)] = 4, \\ 3 \le \#(X) \le 5 & \Leftrightarrow & \#[\exp(X)] = 5. \end{array}$$

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