HYPERGEOMETRIC FUNCTIONS COURSE
ROSS 2023
PROBLEM SET

BRIAN GROVE

Week I:

Lecture 1 Problems

1. Recall that

$$ {}_2F_1\left[\begin{matrix} \frac{1}{2} & \frac{1}{2} \\ & 1 \end{matrix} ; z\right] := \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k \left(\frac{1}{2}\right)_k}{(1)_k (1)_k} z^k $$

for $z \in \mathbb{C}$ with $|z| < 1$.

(a) Show

$$ {}_2F_1\left[\begin{matrix} \frac{1}{2} & \frac{1}{2} \\ & 1 \end{matrix} ; z\right] = \sum_{k=0}^{\infty} \binom{2k}{k}^2 \left(\frac{z}{16}\right)^k. $$

Note: In parts $(b) - (d)$ prove, disprove, or salvage if possible.

(b) Use part (a) to show that

$$ {}_2F_1\left[\begin{matrix} \frac{1}{2} & \frac{1}{2} \\ & 1 \end{matrix} ; 4\right] = \sum_{k=0}^{\infty} \frac{1}{(k!)^2}. $$

(c) Find lower and upper bounds for

$$ {}_2F_1\left[\begin{matrix} \frac{1}{2} & \frac{1}{2} \\ & 1 \end{matrix} ; 4\right] $$

using part (b).

(d) Simplify

$$ {}_2F_1\left[\begin{matrix} \frac{1}{2} & \frac{1}{2} \\ & 1 \end{matrix} ; x\right] $$

for $x = \frac{1}{4}, 32, \frac{1}{16}$, and $\frac{4}{3}$ to the best of your ability. Do you notice any patterns? What lower and upper bounds can you prove in these additional cases?

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2. Define

$$ {}_0F_1\left[\begin{matrix} * \\ a \end{matrix} ; z\right] := \sum_{k=0}^{\infty} \frac{z^k}{(a)_k \cdot k!} $$

for $z \in \mathbb{C}$ such that $|z| < 1$.

Let $x \in \mathbb{R}$. Recall the Maclaurian series for cosine and sine,

$$ \cos(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} $$

and

$$ \sin(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}. $$

(a) Show that

$$\cos(x) = {}_0F_1\left[\begin{matrix} * \\ \frac{1}{2} \end{matrix}; -\frac{x^2}{4}\right]$$

and

$$\sin(x) = x \cdot {}_0F_1\left[\begin{matrix} * \\ \frac{3}{2} \end{matrix}; -\frac{x^2}{4}\right]$$

for $x \in \mathbb{R}$ such that $|x| < 1$.

(b) How are the proofs for $\cos(x)$ and $\sin(x)$ related? Does one of the formulas imply the other or are the arguments distinct?

(c) Recall the double angle and Pythagorean identities from trigonometry, $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$ and $\sin^2(\theta) + \cos^2(\theta) = 1$, respectively.

Rewrite these two identities for sine and cosine in terms of the ${}_0F_1$ function using part (a). Then use the ${}_0F_1$ version of the double angle formula to compute $\sin(\frac{\pi}{3})$ in terms of ${}_0F_1$ functions.

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3. The power series representations for $\sin^{-1}(x)$ and $-\ln(1-z)$ are

$$\sin^{-1}(x) = \sum_{k=0}^{\infty} \frac{(2k)!}{2^{2k}(k!)^2} \frac{x^{2k+1}}{2k+1}$$

for $x \in \mathbb{R}$ such that $|x| < 1$ and

$$-\ln(1-z) = \sum_{k=0}^{\infty} \frac{z^{k+1}}{k+1}$$

for $z \in \mathbb{C}$ such that $|z| < 1$.

(a) Show that

$$\sin^{-1}(x) = x \cdot {}_2F_1\left[\begin{matrix} \frac{1}{2} & \frac{1}{2} \\ & \frac{3}{2} \end{matrix}; x^2\right]$$

when $|x| < 1$.

(b) Substitute $x = \frac{1}{\sqrt{2}}$ into the equation in part (a) to express π as a ${}_2F_1$ value.

(c) Compare your answer from part (b) to the formula for π we proved in the lecture,

$$\pi = 4 \cdot {}_2F_1\left[\begin{matrix} \frac{1}{2} & 1 \\ & \frac{3}{2} \end{matrix}; -1\right],$$

to conclude that

$${}_2F_1\left[\begin{matrix} \frac{1}{2} & 1 \\ & \frac{3}{2} \end{matrix}; -1\right] = \frac{1}{\sqrt{2}} \cdot {}_2F_1\left[\begin{matrix} \frac{1}{2} & \frac{1}{2} \\ & \frac{3}{2} \end{matrix}; \frac{1}{2}\right].$$

(d) Show that

$$-\ln(1-z) = z \cdot {}_2F_1\left[\begin{matrix} 1 & 1 \\ & 2 \end{matrix}; z\right]$$

for $|z| < 1$.

(e) Substitute $z = 1 - \frac{e}{4}$ into the result of part (d) to write $\frac{1}{e-4}$ as a ${}_2F_1$ value.

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4. Is the operation $*$, where $P * Q$ means the third point on the line through the points P and Q on an elliptic curve, a group operation? If so, describe the identity and inverse elements.

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5. Prove the addition of two points on an elliptic curve is closed under the elliptic curve addition law.

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6. Prove the associativity of addition for the elliptic curve addition law with a picture.

Lecture 2 Problems

1. Let $s \in \mathbb{C}$. The gamma function is defined as

$$\Gamma(s) := \int_0^\infty t^{s-1}e^{-t}\,dt$$

when $\mathrm{Re}(s) > 0$.

(a) Show that $\Gamma(1) = 1$ and $\Gamma(s+1) = s \cdot \Gamma(s)$ for $s \in \mathbb{C} \setminus \{\mathbb{Z}_{\leq 0}\}$ and $\mathrm{Re}(s) > 0$.

(b) Use part (a) to show that $\Gamma(k) = (k-1)!$, where k is a positive integer.

(c) Another common way to define the gamma function is as

$$\Gamma(s) = \lim_{s\to\infty} \frac{k^{s-1}k!}{(s)_k}$$

for all $s \in \mathbb{C} \setminus \{\mathbb{Z}_{\leq 0}\}$.

Use the above product definition of the gamma function to give another proof of the results in parts (a) and (b).

(d) Why is the restriction that $s \notin \mathbb{Z}_{\leq 0}$ needed in the limit definition of $\Gamma(s)$? Try some examples, if necessary.

(e) Do you think it is easier to prove parts (a) and (b) with the integral or limit definition of $\Gamma(s)$? How are the two proofs similar and how do they differ?

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2. The reflection formula for the gamma function states that

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)}$$

for $s \in \mathbb{C} \setminus \mathbb{Z}$.

(a) Why is the reflection formula **not** valid for integers? Try some examples, if necessary.

(b) Use the reflection formula to show $\Gamma(\frac{1}{2})^2 = \pi$.

(c) Show the gamma function is always positive when $\mathrm{Re}(s) > 0$.

(d) Conclude that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$ from parts (b) and (c).

(e) Use the reflection formula to compute $\Gamma(\frac{n}{2})$ for $n = 3, 5, 7$. Guess a general formula for $\Gamma(\frac{2n+1}{2})$ when $n \geq 1$. Can you prove your general formula?

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3. Show that $\Gamma(s)$ has no zeros. [*Hint: The reflection formula may be useful.*]

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4. Define the Beta function as

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,dt$$

for $\mathrm{Re}(x), \mathrm{Re}(y) > 0$.

Show that

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

when $\mathrm{Re}(x), \mathrm{Re}(y) > 0$.

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5. In this problem prove, disprove, and salvage if possible for each statement.

(a) $\Gamma(0)$ is undefined.

(b) $B(-\frac{1}{2}, -\frac{1}{2}) = \frac{\Gamma(-\frac{1}{2})\Gamma(-\frac{1}{2})}{\Gamma(-1)} = 4\pi$.

(c) ${}_2F_1\left[\begin{matrix}\frac{1}{3} & 4 \\ & \frac{3}{2}\end{matrix};1\right]$ converges with the sum definition of the ${}_2F_1$ function.

(d)

$$B\left(\frac{1}{3},\frac{2}{3}\right)=\frac{2\pi}{\sqrt{3}}.$$

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6. Let $a\in\mathbb{Q}$ and recall the binomial series $(1-z)^{-a}$, where $z\in\mathbb{C}$. Show that

$$(1-z)^{-a}=\sum_{k=0}^{\infty}\frac{(a)_k}{k!}z^k$$

when $|z|<1$, the Maclaurian series.

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7. In the lecture, we showed that the real period of a Legendre elliptic curve is related to a ${}_2F_1$ value. This elliptic integral is a special case of more a general phenomenon called periods, introduced by Kontsevich and Zagier. Periods show up in many parts of number theory, algebraic geometry, topology, and beyond. Read Periods if you want to learn more about periods.

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8. Define the Fibonacci sequence as $F_0=0$, $F_1=1$, with $F_{n+2}=F_n+F_{n+1}$ for all integers n. Similarly, define the Lucas sequence as $L_0=2$, $L_1=1$, and $L_{n+1}=L_n+L_{n-1}$ for all integers n. All statements below are valid for all integers n.

(a) Show that $F_{-n}=(-1)^{n-1}F_n$ and $L_{-n}=(-1)^nL_n$.

(b) Show that $L_n=F_{n+1}+F_{n-1}$ and $F_n=\frac{1}{5}(L_{n+1}+L_{n-1})$.

(c) In the lecture we proved the Binet formula for F_n. Prove the Binet formula for L_n,

$$L_n=\left(\frac{1+\sqrt{5}}{2}\right)^n+\left(\frac{1-\sqrt{5}}{2}\right)^n.$$

(d) Let p be an odd prime. Show that

$$L_p\equiv 1 \mod p$$

and

$$F_p\equiv\left(\frac{5}{p}\right) \mod p,$$

where $\left(\frac{a}{p}\right)$ denotes the Legendre symbol.

Lecture 3 Problems

1. Use the Pfaff transformation for the ${}_2F_1$ function to prove the Euler transformation for the ${}_2F_1$ function.

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2. Recall the Gauss evaluation formula for the ${}_2F_1$ function at $z=1$ from the lecture. (put the formula here)

Use this formula to prove, disprove, and salvage if possible for the following statements.

(a)

$${}_2F_1\left[\begin{matrix}\frac{5}{2} & \frac{3}{2} \\ & \frac{1}{2}\end{matrix};1\right]=\frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(-2)\Gamma(-1)}=2\pi.$$

(b)

$${}_2F_1\left[\begin{matrix}\frac{1}{2} & \frac{3}{2} \\ & \frac{7}{2}\end{matrix};1\right]=\frac{15\pi}{32}.$$

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3. Let B_k denote the k-th Bernoulli number. Show that $B_{2k+1} = 0$ for $k \geq 1$.

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4. Let $s \in \mathbb{C}$. Recall the Riemann zeta function is defined as

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

when $\mathrm{Re}(s) > 1$.

(a) Let p be a prime. Show that

$$\zeta(s) = \prod_p \frac{1}{1 - \frac{1}{p^s}}$$

when $\mathrm{Re}(s) > 1$, where $\prod_p$ denotes the product over all primes p.

(b) Use the functional equation for $\zeta(s)$,

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s)\zeta(1-s)$$

to show that $\zeta(-2k) = 0$ for all positive integers k. Why is $\zeta(2k)$ nonzero for all positive integers k?

(c) A known formula for $\zeta(s)$ at positive even integers is

$$\zeta(2k) = \frac{(-1)^{k+1} B_{2k} 2^{2k-1} \pi^{2k}}{(2k)!}.$$

Use the functional equation for $\zeta(s)$ and the $\zeta(2k)$ formula to show that

$$\zeta(1-2k) = (-1)^{2k-1} \frac{B_{2k}}{2k}$$

for all positive integers k.

Week II:

Lecture 4 Problems

1. Show that $\mathbb{F}_p^\times$ and $\widehat{\mathbb{F}_p^\times}$ are isomorphic as groups.

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2. Prove both versions of the orthogonality of characters on $\mathbb{F}_p^\times$ or look up the proofs.

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3. Let $A, B \in \widehat{\mathbb{F}_p^\times}$ and recall the Gauss and Jacobi sums,

$$g(A) = \sum_{x \in \mathbb{F}_p^\times} A(x) \zeta_p^x$$

and

$$J(A, B) = \sum_{x \in \mathbb{F}_p^\times} A(x) B(1-x),$$

respectively.

(a) Show that

$$J(A, B) = \frac{g(A) g(B)}{g(AB)}$$

if $AB \neq \varepsilon$.

(b) Salvage the result in part (a) when $AB = \varepsilon$.

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4. (a) Show that $J(A, \overline{A}) = -A(-1)$ if $A \neq \varepsilon$, where $\overline{A}$ denotes the conjugate of the character A.

(b) Show that $J(A, \varepsilon) = -1$ if $A \neq \varepsilon$.

(c) Salvage the results in parts (a) and (b) when $A = \varepsilon$.

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5. The *double-angle formula* for Gauss sums says that

$$g(A)g(\phi A) = g(A^2)g(\phi)\overline{A}(4)$$

for every multiplicative character $A \in \widehat{\mathbb{F}_p^\times}$.

(a) Use the double-angle formula to show

$$J(A, A) = \overline{A}(4)J(A, \phi).$$

(b) Use part (a) to determine $J(\varepsilon, \varepsilon)$ and $J(\varepsilon, \phi)$ in another way.

(c) Suppose $p \equiv 1 \mod 4$. Substitute $A = \eta_4\chi$ into the double angle formula and solve for $g(\phi\chi^2)$.

Remark: The substitution in part (c) is a commonly used tool for simplifying Gauss sums inside the H_p function.

Lecture 5 Problems

1. Compute

(a)

$$H_p\left[\begin{matrix} \frac{1}{3} & \frac{2}{3} \\ & 1 \end{matrix} ; 1\right]$$

for primes $p \equiv 1 \mod 3$.

(b)

$$H_p\left[\begin{matrix} \frac{1}{4} & \frac{3}{4} \\ & 1 \end{matrix} ; 1\right]$$

for primes $p \equiv 1 \mod 4$.

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2. Let $\alpha = \{a_1, \dots, a_n\}$ and $\beta = \{1, b_2, \dots, b_n\}$ be the first and second rows of the hypergeometric data, respectively, with all $a_i, b_i \in \mathbb{Q}$.

The hypergeometric data $\{\alpha, \beta\}$ is <u>primitive</u> if $a_i - b_j \notin \mathbb{Z}$ for all $i, j \in [1, n]$.

Another property of hypergeometric data is to be defined over $\mathbb{Q}$. We say a multiset, say α, is defined over $\mathbb{Q}$ if

$$\prod_{j=1}^{n}(X - e^{2\pi i a_j}) \in \mathbb{Z}[x],$$

where the $a_j \in \alpha$.

The hypergeometric data $\{\alpha, \beta\}$ is <u>defined over $\mathbb{Q}$</u> if both multisets, α and β, are defined over $\mathbb{Q}$.

In each case determine if the hypergeometric data is algebraic, primitive, and defined over $\mathbb{Q}$.

(a) $\alpha = \{\frac{1}{2}, \frac{1}{4}, \frac{3}{4}, \frac{1}{6}\}$ and $\beta = \{1, \frac{3}{2}, \frac{1}{5}, \frac{4}{5}\}$.

(b) $\alpha = \{\frac{1}{2}, \frac{1}{6}, \frac{5}{6}\}$ and $\beta = \{1, \frac{1}{3}, \frac{2}{3}\}$.

(c) $\alpha = \{\frac{1}{8}, \frac{3}{8}, \frac{5}{8}, \frac{7}{8}\}$ and $\beta = \{1, \frac{1}{2}, \frac{1}{4}, \frac{3}{4}\}$.

(d) $\alpha = \{\frac{1}{2}, \frac{1}{3}, \frac{2}{3}\}$ and $\beta = \{1, 1, 1\}$.

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3. Recall the algebraic formula from the lecture,

$$
{}_2F_1\left[\begin{matrix} a & a+\frac{1}{2} \\ & \frac{1}{2} \end{matrix} ; z\right] = \frac{1}{2}\left[(1+\sqrt{z})^{-2a} + (1-\sqrt{z})^{-2a}\right]
$$

where $0 < a < \frac{1}{2}$ and $z \in \mathbb{C}$ with $|z| < 1$.

(a) Let $a = \frac{1}{3}$ in the algebraic formula above and think about what the finite field analog of this formula should be.

(b) Show that if $p \equiv 1 \mod 6$ and $\lambda \in \mathbb{F}_p^\times$ then

$$
H_p\left[\begin{matrix} \frac{1}{3} & \frac{5}{6} \\ & \frac{1}{2} \end{matrix} ; \lambda\right] = \left(\frac{1+\phi(\lambda)}{2}\right)\left[\eta_3(1+\sqrt{\lambda}) + \eta_3(1-\sqrt{\lambda})\right].
$$

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4. In each part simplify your answer to the best of your ability.

(a) Compute

$$
H_p\left[\begin{matrix} \frac{1}{3} & \frac{2}{3} \\ & \frac{1}{2} \end{matrix} ; 2\right]
$$

for the primes $p = 7, 13$, and 19.

(b) Compute

$$
H_p\left[\begin{matrix} \frac{1}{6} & \frac{5}{6} \\ & \frac{1}{2} \end{matrix} ; 4\right]
$$

for the primes $p = 13, 17$, and 61.

(c) Compute

$$
H_5\left[\begin{matrix} \frac{1}{4} & \frac{3}{4} \\ & \frac{1}{2} \end{matrix} ; \lambda\right]
$$

for all $\lambda \in \mathbb{F}_5^\times$.

Lecture 6 Problems

1. Let $\tilde{E}_\lambda$ denote the mod p reduction of the Legendre elliptic curve

$$
E_\lambda : y^2 = x(1-x)(1-\lambda x)
$$

where $\lambda \in \mathbb{Z} \setminus \{0, 1\}$ and p is a prime of good reduction. Recall that $|\tilde{E}_\lambda(\mathbb{F}_p)|$ is the number of solutions to $\tilde{E}_\lambda$ over $\mathbb{F}_p$ plus one, for the point at infinity.

(a) Compute $\tilde{E}_\lambda(\mathbb{F}_{11})$ for all $\lambda \in \mathbb{F}_{11} \backslash \{0, 1\}$. Do you see any patterns? Any conjectures?

(b) Use your computations in part (a) to determine the values of

$$
H_{11}\left[\begin{matrix} \frac{1}{2} & \frac{1}{2} \\ & 1 \end{matrix} ; \lambda\right]
$$

for all $\lambda \in \mathbb{F}_{11} \setminus \{0, 1\}$. Any patterns or conjectures for these H_{11} values?

(c) Repeat parts (a) and (b) with $p = 17$ and use your computations to refine your conjectures from parts (a) and (b).

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2. Read Mazur's article to learn more about the Sato-Tate conjecture and related equidistribution problems.

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3. Read this summary on the introduction of hypergeometric moments by Ono, Saad, and Saikia.
