Measure Theory

Dawei Wang

May 22, 2022

Probability Spaces

A probablity space is a triple $(\Omega, \mathcal{F}, P)$, where Ω is a set of "outcomes," $\mathcal{F}$ is a set of "events," and $P : \to$ is a funtion that assigns probabilities to events. We assume that $\mathcal{F}$ is a $\sigma - algebra$.

Without P, $(\Omega, \mathcal{F})$ is called a measurable space, i.e., it is a space on which we can put a measure. A (positive) measure is a nonnegative countably additive set function.

If $\mu(\Omega) = 1$, we call μ a probability measure. In this book, probability measures are usually denoted by P.

Theorem 1.1.1

Let μ be a measure on $(\Omega, \mathcal{F})$

(i) monotonicity. If $A \in B$, then $\mu(A) \leq \mu(B)$.

(ii) subadditivity. If $A \in \cup_{m=1}^{\infty} A_m$, then $\mu(A) \leq \sum_{m=1}^{\infty} \mu(A_m)$.

(iii) continuity from below. If $A_i \uparrow A$(i.e., $A_1 \subset A_2 \subset ...$ and $\cup_i A_i = A$), then $\mu(A_i) \uparrow \mu(A)$.

(iv) continuity from above. If $A_i \downarrow A$(i.e., $A_1 \supset A_2 \supset ...$ and $\cap_i A_i = A$), with $\mu(A_1) < \infty$, then $\mu(A_i) \downarrow \mu(A)$.

Discrete probability spaces

Let $\Omega =$ a countable set. Let $\mathcal{F} =$ the set of all subsets of Ω. Let

$$P(A) = \sum_{w \in A} p(w), \;\; where \;\; p(w) > 0 \;\; and \;\; \sum_{w \in \Omega} p(w) = 1$$

If we are given a set Ω and a collection $\mathcal{A}$ of subsets of Ω, then there is a smallest σ-field containing $\mathcal{A}$. We will call this the σ-field generated by $\mathcal{A}$ and denote it by $\sigma(\mathcal{A})$.

Let $\mathbf{R}^d$ be the set of vectors $(x_1, ...x_d)$ of real numbers and $\mathcal{R}^d$ be the Borel sets.

Measures on the real line

Measures on $(\mathbf{R}, \mathcal{R})$ are defined by giving a **Stieltjes measure function** with the following properties:

(i) F is nodecresing.

(ii) F is right continuous, i.e., $lim_{y \downarrow x} F(y) = F(x)$

Theorem 1.1.4

Associated with each **Stieltjes measure function** F there is a unique measure μ on $(\mathbf{R}, \mathcal{R})$ with $\mu((a, b]) = F(b) - F(a)$
When $F(x) = x$, the resulting measure is called **Lebesgue measure**.

A collection $\mathcal{S}$ of sets is said to be a **semialgebra** if (i) it is closed under intersection, i.e., S, $T \in S$ implies $S \cap T \in S$, and (ii) if $S \in \mathcal{S}$, then S^c is a finite disjoint union of sets in $\mathcal{S}$.

A collection $\mathcal{A}$ of subsets of Ω is called an **algebra** (or **field**) if $A, B \in \mathcal{A}$ implies A^c and $A \cup B$ are in A. Since $A \cap B = (A^c \cup B^c)^c$, it follows that $A \cap B \in A$. An algebra is closed under finite unions.

Lemma 1.1.7

If $\mathcal{S}$ is a semialgebra, then $\overline{\mathcal{S}}$={finite disjoint unions of sets in $\mathcal{S}$} is an algebra, called the algebra generated by $\mathcal{S}$.

By a measure on an algebra $\mathcal{A}$, we mean a set function μ with

(i) $\mu(A) \geq \mu(\emptyset)$ for all $A \in \mathcal{A}$

(ii) if $A_i \in \mathcal{A}$ are disjoint and their union is in $\mathcal{A}$, then

$$\mu(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i)$$

μ is said to be $\sigma-$finite if there is a sequence of sets $A_n \in \mathcal{A}$ so that $\mu(A_n) < \infty$ and $\cup_n A_n = \Omega$.

Theorem 1.1.9

Let $\mathcal{S}$ be a semialgebra and let μ defined on $\mathcal{S}$ have $\mu(\emptyset) = 0$. Suppose (i) if $S \in \mathcal{S}$, is a finite disjoint union of set $S_i \in \mathcal{S}$, then $\mu(S) = \sum_i \mu(S_i)$, and (ii) if $S_i, S \in \mathcal{S}$ with $S = +_{i \geq 1} S_i$, then $\mu(S) \leq \sum_{i \geq 1} \mu(S_i)$. Then has a unique extension $\overline{\mu}$ that is a measure on $\overline{\mathcal{S}}$, the algebra generated by $\mathcal{S}$. If $\overline{\mu}$ is sigma-finite, then there is a unique extension ν that is a measure on $\sigma(\mathcal{S})$.

Lemma 1.1.10

Suppose only that (i) holds

(i) If $A, B^i \in \overline{\mathcal{S}}$ with $A = +_{i=1}^{n} B_i$, then $\overline{\mu}(A) = \sum_i \overline{\mu}(B_i)$

(ii) If $A, B_i \in \overline{\mathcal{S}}$ with $A \in \cup_{i=1}^{n}$, then $\overline{\mu}(A) \leq \sum_i \overline{\mu}(B_i)$