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FINITE, COUNTABLE, AND UNCOUNTABLE SETS

Definition 1. Consider two sets A and B , whose elements may be any objects whatsoever, and suppose that with each element x of A there is associated, in some manner, an element of B , which we denote by $f(x)$. Then set A is called the domain of f (we also say f is defined on A), and elements $f(x)$ are called the values of f . The set of all values of f is called the range of f .

Definition 2. Let A and B be two sets and let f be a mapping of A into B . If $E \subset A$, $f(E)$ is defined to be the set of all elements $f(x)$, for $x \in E$. We call $f(E)$ the image of E under f . In this notation, $f(A)$ is the range of f . It is clear that $f(A) \subset B$. If $f(A) = B$, we say that f maps A onto B .

If $E \subset B$, $f^{-1}(E)$ denotes the set of all $x \in A$ such that $f(x) \in E$. We call $f^{-1}(E)$ the inverse image of E under f . If $y \in B$, $f^{-1}(y)$ is the set of all $x \in A$ such that $f(x) = y$. If, for each $y \in B$, $f^{-1}(y)$ consists of at most one element of A , then f is said to be a 1-1 mapping of A into B . This may also be expressed as follows: f is a 1-1 mapping of A into B provided that $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2, x_1 \in A, x_2 \in A$.

Definition 3. If there exists a 1-1 mapping of A onto B , we say that A and B can be put in 1-1 correspondence, or that A and B have the same cardinal number or, briefly, that A and B are equivalent, and we write $A \sim B$. This relation clearly has the following properties:

It is reflexive: $A \sim A$

It is symmetric: If $A \sim B$, then $B \sim A$

It is transitive: If $A \sim B$ and $B \sim C$, then $A \sim C$

Any relation with these three properties is called an equivalence relation

Definition 4. For any positive integer n , let J_n be the set whose elements are the integers $1, 2, \dots, n$; let J be the set consisting of all positive integers. For any set A , we say:

- (a) A is finite if $A \sim J_n$ (the empty set is also considered to be finite).
- (b) A is infinite if A is not finite
- (c) A is countable if $A \sim J$.
- (d) A is uncountable if A is neither finite nor countable
- (e) A is at most countable if A is finite or countable

Remark 1. A finite set cannot be equivalent to one of its proper subsets. That this is, however, possible for infinite sets.

Definition 5. By a sequence, we mean a function f defined on the set J of all positive integers. If $f(n) = x_n$, for $n \in J$, it is customary denote the sequence f by the symbol x_n , or sometimes by x_1, x_2, x_3, \dots . The values of f , that is, the elements x_n , are called the terms of the sequence. If A is a set and if $x_n \in A$ for all $n \in J$, then x_n is said to be a sequence in A , or a sequence of elements of A .

The terms x_1, x_2, x_3, \dots of a sequence need not be distinct.

Since ever countable set is the range of a 1-1 function defined on J , we may regard every countable set as the range of a sequence of distinct terms. Speaking more loosely, we may say that the elements of any countable set can be "arranged in a sequence."

Some times it is more convenient to replace J in this definition by the set of all nonnegative integers, i.e., to start with 0 rather than with 1.

Theorem 1. Every infinite subset of a countable set A is countable.

The theorem shows that, roughly speaking, countable sets represent the "smallest" infinity: No uncountable set can be a subset of a countable set.

Definition 6. Let A and Ω be sets, and suppose that with each element α of A there is associated a subset of Ω which we denote by E_α .

The set whose elements are the sets E_α will be denoted by E_A . Instead of speaking of sets of sets, we shall sometimes speak of a collection of sets, or a family of sets.

The union of sets E_α is defined to be the set S such that $x \in S$ if and only if $x \in E_\alpha$ for at least one $\alpha \in A$. We use the notation.

$$S = \bigcup_{\alpha \in A} E_\alpha \quad (1)$$

If A consists of the integers $1, 2, \dots, n$, one usually writes

$$S = \bigcup_{m=1}^n E_m \quad (2)$$

or

$$S = E_1 \cup E_2 \cup \dots \cup E_n. \quad (3)$$

If A is the set of all positive integers, the usual notation is

$$S = \bigcup_{m=1}^{\infty} E_m \quad (4)$$

The intersection of the sets E_α is defined to be the set P such that $x \in P$ if and only if $x \in E_\alpha$ for every $\alpha \in A$. We use notation

$$P = \bigcap_{\alpha \in A} E_\alpha \quad (5)$$

or

$$P = \bigcap_{m=1}^n E_m = E_1 \cap E_2 \cap \dots \cap E_n \quad (6)$$

or

$$P = \bigcap_{m=1}^{\infty} E_m \quad (7)$$

as for unions. If $A \cap B$ is not empty, we say that A and B intersect; otherwise they are disjoint.

Remark 2. Many properties of unions and intersections are quite similar to those of sums and products

Theorem 2. Let E_n , $n=1,2,3,\dots$, be a sequence of countable sets, and put

$$S = \bigcup_{n=1}^{\infty} E_n \quad (8)$$

Then S is countable.

Corollary 1. Suppose A is at most countable, and, for every $\alpha \in A$, B_α is at most countable. Put

$$T = \bigcup_{\alpha \in A} B_\alpha \quad (9)$$

Then T is at most countable.

For T is equivalent to a subset of (15).

Theorem 3. Let A be a countable set, and let B_n be the set of all n-tuples (a_1, \dots, a_n) , where $a_k \in A (k = 1, \dots, n)$, and the elements a_1, \dots, a_n need not be distinct. Then B_n is countable.

Corollary 2. The set of all rational numbers is countable.

In fact, even the set of all algebraic numbers is countable.

Theorem 4. Let A be the set of all sequences whose elements are the digits 0 and 1. This set A is uncountable.

Theorem 4 implies that the set of all real numbers is uncountable.

METRIC SPACE

Definition 7. A set X , whose elements we shall call points, is said to be a metric space if with any two points p and q of X there is associated a real number $d(p,q)$, called the distance from p to q , such that

- (a) $d(p,q) > 0$ if $p \neq q$; $d(p,q) = 0$;
- (b) $d(p,q) = d(q,p)$;
- (c) $d(p,q) \leq d(p,r) + d(r,q)$, for any $r \in X$.

Any function with these three properties is called a distance function, or a metric.

The most important examples of metric spaces, from our standpoint, are the euclidean spaces R^k , especially R^1 and R^2 ; the distance in R^k is defined by

$$d(\mathbf{x}, \mathbf{y}) = |\mathbf{x} - \mathbf{y}| \quad (\mathbf{x}, \mathbf{y} \in R^k) \quad (10)$$

It is important that every subset Y of a metric space X is a metric space in its own right, with the same distance function.

Thus every subset of a euclidean space is a metric space.

Definition 8. By the segment (a,b) we mean the set of all real numbers x such that $a < x < b$

By the interval $[a,b]$ we mean the set of all real numbers x such that $a \leq x \leq b$

If $a_i < b_i$ for $i=1, \dots, k$, the set of all points $\mathbf{x} = (x_1, \dots, x_k)$ in R^k whose coordinates satisfy the inequalities $a_i \leq x_i \leq b_i$ ($1 \leq i \leq k$) is called a k -cell.

If $\mathbf{x} \in R^k$ and $r > 0$, then open (or closed) ball B with center at \mathbf{x} and radius r is defined to be the set of all $\mathbf{y} \in R^k$ such that $|\mathbf{y} - \mathbf{x}| < r$ (or $|\mathbf{y} - \mathbf{x}| \leq r$)

We call a set $E \subset R^k$ convex if

$$\lambda \mathbf{x} + (1 - \lambda) \mathbf{y} \in E$$

whenever $\mathbf{x} \in E, \mathbf{y} \in E$, and $0 < \lambda < 1$.

Definition 9. Let X be a metric space. All points and sets mentioned below are understood to be elements and subsets of X .

(a) A neighborhood of p is a set $N_r(p)$ consisting of all q such that $d(p, q) < r$. The number r is called the radius of $N_r(p)$.

(b) A point p is a limit point of the set E if every neighborhood of p contains a point $q \neq p$ such that $q \in E$.

(c) If $p \in E$ and p is not a limit point of E , then p is called an isolated point of E .

(d) E is closed if every limit point of E is a point of E .

(e) A point p is an interior point of E if there is a neighborhood N of p such that $N \subset E$.

(f) E is open if every point of E is an interior point of E .

(g) The complement of E (denoted by E^c) is the set of all points $p \in X$ such that $p \notin E$.

(h) E is perfect if E is closed and if every point of E is a limit point of E .

(i) E is bounded if there is a real number M and a point $q \in \mathbf{x}$ such that $d(p, q) < M$ for all $p \in E$.

(j) E is dense in X if every point of X is a limit point of E , or a point of E (or both).

Let us note that in R^1 neighborhoods are segments, whereas in R^2 neighborhoods are interiors of circles.

Theorem 5. Every neighborhood is an open set.

Theorem 6. If p is a limit point of a set E , then every neighborhood of p contains infinitely many points of E .

Corollary 3. A finite point set has no limit points.

Theorem 7. Let E_α be a (finite or infinite) collection of sets E_α . Then

$$\left(\bigcup_{\alpha} E_{\alpha}\right)^c = \bigcap_{\alpha} (E_{\alpha}^c) \quad (11)$$

Theorem 8. A set E is open if and only if its complement is closed.

Corollary 4. A set F is closed if and only if its complement is open

Theorem 9. (a) For any collection G_α of open sets, $\bigcup_{\alpha} G_{\alpha}$ is open.

(b) For any collection F_α of closed sets, $\bigcap_{\alpha} F_{\alpha}$ is closed.

(c) For any finite collection G_1, \dots, G_n of open sets, $\bigcap_{i=1}^n G_i$ is open.

(d) For any finite collection F_1, \dots, F_n of closed sets, $\bigcup_{i=1}^n F_i$ is closed.

The intersection of an infinite collection of open sets need not be open. Similarly, the union of an infinite collection of closed sets need not be closed.

Definition 10. If X is a metric space, if $E \subset X$, and if E' denotes the set of all limit points of E in X , then the closure of E is the set $\overline{E} = E \cup E'$.

Theorem 10. If X is a metric space and $E \subset X$, then

(a) \overline{E} is closed,

(b) $E = \overline{E}$ if and only if E is closed,

(c) $\overline{E} \subset F$ for every closed set $F \subset X$ such that $E \subset F$.

By (a) and (c), \overline{E} is the smallest closed subset of X that contains E .

Theorem 11. Let E be a nonempty set of real numbers which is bounded above. Let $y = \sup E$. Then $y \in \overline{E}$. Hence $y \in E$ if E is closed.

Remark 3. Suppose $E \subset Y \subset X$, where X is a metric space. To say that E is an open subset of X means that to each point $p \in E$ there is associated with a positive number r such that the conditions $d(p, q) < r, q \in X$ imply that $q \in E$. But we have already observed that Y is also a metric space, so that our definitions may equally well be made within Y . To be quite explicit, let us say that E is open relative to Y if to each $P \in E$ there is associated an $r > 0$ such that $q \in E$ whenever $d(p, q) < r$ and $q \in Y$. A set may be open relative to Y without being an open subset of X .

Theorem 12. Suppose $Y \subset X$. A subset E of Y is open relative to Y if and only if $E = Y \cap G$ for some open subset G of X .

COMPACT SETS

Definition 11. By an open cover of a set E in a metric space X we mean a collection G_α of X such that $E \subset \bigcup_\alpha G_\alpha$

Definition 12. A subset K of a metric space X is said to be compact if every open cover of K contains a finite subcover.

More explicitly, the requirement is that if G_α is an open cover of K , then there are finitely many indices $\alpha_1, \dots, \alpha_n$ such that

$$K \subset G_{\alpha_1} \cup \dots \cup G_{\alpha_n}$$

It is clear that every finite set is compact.

The property of being open depends on the space in which E is embedded. The same is true of the property of being closed.

Theorem 13. Suppose $K \subset Y \subset X$. Then K is compact relative to X if and only if K is compact relative to Y .

By virtue of this theorem we are able, in many situations, to regard compact sets as metric spaces in their own right, without paying any attention to any embedding space. In particular, although it makes little sense to talk of open spaces, or of closed spaces (every metric space X is an open subset of itself, and is a closed subset of itself), it does make sense to talk of compact metric spaces.

Theorem 14. Compact subsets of metric spaces are closed.

Theorem 15. Closed subsets of compact sets are compact

Corollary 5. If F is closed and K is compact, then $F \cap K$ is compact.

Theorem 16. If $\{K_\alpha\}$ is a collection of compact subsets of a metric space X such that the intersection of every finite subcollection of $\{K_\alpha\}$ is nonempty, then $\bigcap K_\alpha$ is nonempty.

Corollary 6. If $\{K_n\}$ is a sequence of nonempty compact sets such that $K_n \supset K_{n+1}$ ($n=1,2,3,\dots$), then $\bigcap_1^\infty K_n$ is not empty.

Theorem 17. If E is an infinite subset of a compact set K , then E has a limit point in K .

Theorem 18. If $\{I_n\}$ is a sequence of intervals in R^1 , such that $I_n \supset I_{n+1}$ ($n=1,2,3,\dots$), then $\bigcap_1^\infty I_n$ is not empty.

Theorem 19. Let k be a positive integer. If $\{I_n\}$ is a sequence of k -cells such that $I_n \supset I_{n+1}$ ($n=1,2,3,\dots$), then $\bigcap_1^\infty I_n$ is not empty.

Theorem 20. Every k -cell is compact.

Theorem 21. If a set E in R^k has one of the following three properties, then it has the other two:

- (a) E is closed and bounded.
- (b) E is compact
- (c) Every infinite subset of E has a limit point in E

Theorem 22. Every bounded infinite subset of R^k has a limit point in R^k .

PERFECT SETS

Theorem 23. Let P be a nonempty perfect set in R^k . Then P is uncountable.

Corollary 7. Every interval $[a, b]$ ($a < b$) is uncountable. In particular, the set of all real numbers is uncountable.

The Cantor set. The Cantor set shows that there exist perfect sets in R^1 which contain no segment.

One of the most interesting properties of the Cantor set is that it provides us with an example of an uncountable set of measure zero.

CONNECTED SETS

Definition 13. Two subsets A and B of a metric space X are said to be separated if both $A \cap \overline{B}$ and $\overline{A} \cap B$ are empty, i.e., if no point of A lies in the closure of B and no point of B lies in the closure of A .

A set $E \subset X$ is said to be connected if E is not a union of two nonempty separated sets.

Remark 4. Separated sets are of course disjoint, but disjoint sets need not be separated.

Theorem 24. A subset E of the real line R^1 is connected if and only if it has the following property: If $x \in E, y \in E$, and $x < y < z$, then $z \in E$.