

Measure Theory

Dawei Wang

June 9, 2022

Probability Spaces

A probability space is a triple (Ω, \mathcal{F}, P) , where Ω is a set of "outcomes," \mathcal{F} is a set of "events," and $P : \rightarrow$ is a function that assigns probabilities to events. We assume that \mathcal{F} is a σ -algebra.

Without P , (Ω, \mathcal{F}) is called a measurable space, i.e., it is a space on which we can put a measure. A (positive) measure is a nonnegative countably additive set function.

If $\mu(\Omega) = 1$, we call μ a probability measure. In this book, probability measures are usually denoted by P .

Theorem 1.1.1

Let μ be a measure on (Ω, \mathcal{F})

- (i) monotonicity. If $A \subseteq B$, then $\mu(A) \leq \mu(B)$.
- (ii) subadditivity. If $A \subseteq \bigcup_{m=1}^{\infty} A_m$, then $\mu(A) \leq \sum_{m=1}^{\infty} \mu(A_m)$.
- (iii) continuity from below. If $A_i \uparrow A$ (i.e., $A_1 \subset A_2 \subset \dots$ and $\bigcup_i A_i = A$), then $\mu(A_i) \uparrow \mu(A)$.
- (iv) continuity from above. If $A_i \downarrow A$ (i.e., $A_1 \supset A_2 \supset \dots$ and $\bigcap_i A_i = A$), with $\mu(A_1) < \infty$, then $\mu(A_i) \downarrow \mu(A)$.

Discrete probability spaces

Let Ω be a countable set. Let \mathcal{F} be the set of all subsets of Ω . Let

$$P(A) = \sum_{w \in A} p(w), \text{ where } p(w) > 0 \text{ and } \sum_{w \in \Omega} p(w) = 1$$

If we are given a set Ω and a collection \mathcal{A} of subsets of Ω , then there is a smallest σ -field containing \mathcal{A} . We will call this the σ -field generated by \mathcal{A} and denote it by $\sigma(\mathcal{A})$.

Let \mathbf{R}^d be the set of vectors (x_1, \dots, x_d) of real numbers and \mathcal{R}^d be the Borel sets.

Measures on the real line

Measures on $(\mathbf{R}, \mathcal{R})$ are defined by giving a **Stieltjes measure function** with the following properties:

- (i) F is nondecreasing.
- (ii) F is right continuous, i.e., $\lim_{y \downarrow x} F(y) = F(x)$

Theorem 1.1.4

Associated with each **Stieltjes measure function** F there is a unique measure μ on $(\mathbf{R}, \mathcal{R})$ with $\mu((a, b]) = F(b) - F(a)$
When $F(x) = x$, the resulting measure is called **Lebesgue measure**.

A collection \mathcal{S} of sets is said to be a **semialgebra** if (i) it is closed under intersection, i.e., $S, T \in \mathcal{S}$ implies $S \cap T \in \mathcal{S}$, and (ii) if $S \in \mathcal{S}$, then S^c is a finite disjoint union of sets in \mathcal{S} .

A collection \mathcal{A} of subsets of Ω is called an **algebra** (or **field**) if $A, B \in \mathcal{A}$ implies A^c and $A \cup B$ are in \mathcal{A} . Since $A \cap B = (A^c \cup B^c)^c$, it follows that $A \cap B \in \mathcal{A}$. An algebra is closed under finite unions.

Lemma 1.1.7

If \mathcal{S} is a semialgebra, then $\overline{\mathcal{S}} = \{\text{finite disjoint unions of sets in } \mathcal{S}\}$ is an algebra, called the algebra generated by \mathcal{S} .

By a measure on an algebra \mathcal{A} , we mean a set function μ with

- (i) $\mu(A) \geq \mu(\emptyset)$ for all $A \in \mathcal{A}$
- (ii) if $A_i \in \mathcal{A}$ are disjoint and their union is in \mathcal{A} , then

$$\mu(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i)$$

μ is said to be σ -finite if there is a sequence of sets $A_n \in \mathcal{A}$ so that $\mu(A_n) < \infty$ and $\cup_n A_n = \Omega$.

Theorem 1.1.9

Let \mathcal{S} be a semialgebra and let μ defined on \mathcal{S} have $\mu(\emptyset) = 0$. Suppose (i) if $S \in \mathcal{S}$, is a finite disjoint union of set $S_i \in \mathcal{S}$, then $\mu(S) = \sum_i \mu(S_i)$, and (ii) if $S_i, S \in \mathcal{S}$ with $S = \cup_{i \geq 1} S_i$, then $\mu(S) \leq \sum_{i \geq 1} \mu(S_i)$. Then μ has a unique extension $\bar{\mu}$ that is a measure on $\overline{\mathcal{S}}$, the algebra generated by \mathcal{S} . If $\bar{\mu}$ is sigma-finite, then there is a unique extension ν that is a measure on $\sigma(\mathcal{S})$.

Lemma 1.1.10

Suppose only that (i) holds

(i) If $A, B^i \in \overline{\mathcal{S}}$ with $A = +_{i=1}^n B_i$, then $\bar{\mu}(A) = \sum_i \bar{\mu}(B_i)$

(ii) If $A, B_i \in \overline{\mathcal{S}}$ with $A \in \cup_{i=1}^n B_i$, then $\bar{\mu}(A) \leq \sum_i \bar{\mu}(B_i)$

$$\ln(x^a) = \ln(e^{\ln(x)a}) = a\ln(x)$$

$$e^{\ln(x)} = x$$