ABSTRACT INTEGRATION

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The Concept of Measurability

1.2 Definition

(a) A collection τ of subsets of a set X is said to be a *topology* in X if τ has the following three properties:

(i) $\varnothing \in \tau$ and $X \in \tau$.

(ii) If $V_i \in \tau$ for $i = 1, \cdots, n$, then $V_1 \cap \cdots \cap V_n \in \tau$.

(ii) If V_α is an arbitrary collection of members of τ (finite, countable, or uncountable), then $\bigcup_\alpha V_\alpha \in \tau$.

(b) If τ is a topology in X, then X is called a *topological space*, and the members of τ are called the *open sets* in X.

(c) If X and Y are topological spaces and if f is a mapping of X into Y, then f is said to be *continuous* provided that $f^{-1}(V)$ is an open set in X for every open set V in Y.

1.3 Definition

(a) A collection $\mathfrak{M}$ of subsets of a set X is said to be a σ-algebra in X if $\mathfrak{M}$ has the following properties:

(i) $X \in \mathfrak{M}$.

(ii) If $A \in \mathfrak{M}$, then $A^c \in \mathfrak{M}$, where A^c is the complement of A relative to X.

(iii) If $A = \cup_{n=1}^{\infty} A_n$ and if $A_n \in \mathfrak{M}$ for $n = 1, 2, 3, \cdots$, then $A \in \mathfrak{M}$.

(b) If $\mathfrak{M}$ is a σ-algebra in X is called a *measureable space*, and the members of $\mathfrak{M}$ are called the *measurable sets* in X.

(c) If X is a measurable space, Y is a topological space, and f is a mapping of X into Y, then f is said to be *measurable* provided that $f^{-1}(V)$ is a measurable set in X for every open set V in Y.

1.4 Comments on Definition 1.2

The definition of continuity given in Sec. 1.2(c) is a global one. Frequently it is desirable to define continuity locally: A mapping f of X into Y is said to be *continuous* at the point $x_0 \in X$ if to every neighborhood V of $f(x_0)$ there corresponds a neighborhood W of x_0 such that $f(W) \subset V$.

1.5 Proposition

Let X and Y be topological spaces. A mappinng of X into Y is continuous if and only if f is continuous at every point of X.

1.6 Comments on Definition 1.3

Let $\mathfrak{M}$ be a σ-algebra in a set X. Referring to Properties (i) to (iii) of Definition 1.3(a), we immediately derive the following facts.

(a) Since $\varnothing = X^c$, (i) and (ii) imply that $\varnothing \in \mathfrak{M}$.

(b) Taking $A_{n+1} = A_{n+2} = \cdots = \varnothing$ in (iii), we see that $A_1 \cup A_2 \cup \cdots \cup A_n \in \mathfrak{M}$ if $A_i \in \mathfrak{M}$ for $i = 1, \cdots, n$.

(c) Since

$$\bigcap_{n=1}^{\infty} A_n = (\bigcup_{n=1}^{\infty} A_n^c)^c,$$

$\mathfrak{M}$ is closed under the formation of countable (and also finite) intersections.

(d) Since$A - B = B^c \cap A$, we have $A - B \in \mathfrak{M}$ if $A \in \mathfrak{M}$ and $B \in \mathfrak{M}$.

The prefix σ refers to the fact that (iii) is required to hold for all *countable* unions of members of $\mathfrak{M}$. If (iii) is required for finite unions only, then $\mathfrak{M}$ is called an algebra of sets.

1.7 Theorem

Let Y and Z be topological spaces, and let $g : Y \to Z$ be continuous.

(a) If X is a topological space, if $f : X \to Y$ is continuous, and if $h = g \circ h$, then $h : X \to Z$ is continuous.

(b) If X is a measurable space, if $f : X \to Y$ is continuous, and if $h = g \circ h$, then $h : X \to Z$ is measurable.

Stated informally, continuous functions of continuous functions are continuous; continuous functions of measurable functions are measurable.

1.8 Theorem

Let u and v be real *measurable* functions on a measurable space X, let Φ be a continuous mapping of the plane into a topological space Y, and define

$$h(x) = \Phi(u(x), v(x))$$

for $x \in X$. Then $h : X \to Y$ is measurable.

1.9 Corollaries of Theorems 1.7 and 1.8

(a) If $f = u + iv$, where u and v are real measurable functions on X, then f is a complex measurable function on X.

(b) If $f = u + iv$ is a complex measurable function on X, then u, v and $|f|$ are real measurable functions on X.

(c) If f and g are complex measurable functions on X, then so are $f + g$ and fg.

(d) If E is a measurable set in X and if

$$\chi_E(x) = \begin{cases} 1 & if\ \ x \in E \\ 0 & if\ \ x \notin E \end{cases}$$

then χ_E is a measurable function.

(e) If f is a complex measurable function on X, there is a complex measurable function α on X such that $|\alpha| = 1$ and $f = \alpha|f|$.

1.0 Theorem

If $\mathscr{F}$ is any collection of subsets of X, there exists a smallest σ-algebra $\mathfrak{M}^*$ in X such that $\mathscr{F} \subset \mathfrak{M}^*$.

1.11 Borel Sets

Let X be a topological space. By Theorem 1.10, there exists a smallest σ-algebra $\mathscr{R}$ in X such that every open set in X belongs to $\mathscr{R}$. The members of $\mathscr{R}$ are called *Borel sets* of X.

Consider the measurable space $(X, \mathscr{R})$. If $f : X \to Y$ is a continuous mapping of X, where Y is any topological sapce, then it is evident from the definitions that $f^{-1}(V) \in \mathscr{R}$ for every open set V in Y. In other words, every continuous mapping of X is *Borel measurable*.

Borel measurable mappings are often called *Borel mappings*, or *Borel functions*.

1.12 Theorem

Suppose $\mathfrak{M}$ is a σ-algebra in X, and Y is a topological space. Let f map X into Y.

(a) If Ω is the collection of all sets $E \subset Y$ such that $f^{-1}(E) \in \mathfrak{M}$, then Ω is a σ-algebra in Y.

(b) If f is measurable and E is a *Borel set* in Y, then $f^{-1}(E) \in \mathfrak{M}$.

(c) If $Y = [-\infty, +\infty]$ and $f^{-1}((a, \infty]) \in \mathfrak{M}$ for every real α, then f is measurable.

(d) If f is measurable, if Z is a topological space, if $g : Y \to Z$ is a *Borel mapping*, and if $h = g \circ f$, then $h : X \to Z$ is measurable.

1.13 Defitnion

Let $\{a_n\}$ be a sequence in $[-\infty, +\infty]$, and put

$$b_k = \sup\{a_k, a_{k+1}, a_{k+2}, \cdots\} \qquad (k = 1, 2, 3, \cdots) \tag{1}$$

and

$$\beta = \inf\{b_1, b_2, b_3, \cdots\}. \tag{2}$$

We call β the *upper limit* of $\{a_n\}$, and write

$$\beta = \lim_{n\to\infty} \sup \ a_n \tag{3}$$

The *lower limit* is defined analogously: simply interchange sup and inf in (1) and (2). Note that

$$\lim_{n\to\infty} \inf \ a_n = - \lim_{n\to\infty} \sup \ (-a_n) \tag{4}$$

If $\{a_n\}$ converges, then evidently

$$\lim_{n\to\infty} \sup \ a_n = \lim_{n\to\infty} \inf \ a_n = \lim_{n\to\infty} a_n \tag{5}$$

Suppose $\{f_n\}$ is a sequence of extended-real functions on a set X. Then $\sup_n f_n$ are the functions defined on X by

$$(\sup_n f_n)(x) = \sup_n (f_n(x)), \tag{6}$$

$$(\lim_{n\to\infty} \sup f_n)(x) = \lim_{n\to\infty} \sup(f_n(x)) \tag{7}$$

If

$$f(x) = \lim_{n\to\infty} f_n(x) \tag{7}$$

the limit being assumed to exist at every $x \in X$, then we call f the *pointwise limit* of the seuqence $\{f_n\}$.

1.14 Theorem

If $f_n : X \to [-\infty, +\infty]$ is measurable, for $n = 1, 2, 3, \cdots$, and

$$g = \sup_{n \geq 1} f_n, h = \lim_{n \to \infty} \sup f_n,$$

then g and h are measurable.

Corollaries

(a) The limit of every pointwise converge sequence of complex measurable functions is measurable.

(b) If f and g are measurable (with range in $[-\infty, +\infty]$), then so are max $\{f, g\}$ and min $\{f, g\}$. In particular, this is true of the functions

$$f^+ = \max\{f, 0\} \text{ and } f^- = -\min\{f, 0\}.$$

Proposition

If $f = g - h$, $g \geq 0$, and $h \geq 0$, then $f^+ \leq g$ and $f^- \leq h$.

Simple Functions

1.16 Definition

A complex function s on a measurable space X whose range consists of only finitely many points will be called a *simple function*. Among these are the nonnegative simple functions, whose range is a finite subset of $[0, \infty)$. Note that we explicitly exclude ∞ from the values of a simple function.

If $\alpha_1, \cdots, \alpha_n$ are the distinct values of a simple function s, and if we set $A_i = x : s(x) = \alpha_i$, then clearly

$$s = \sum_{i=1}^{n} \alpha_i \chi_{A_i},$$

where χ_{A_i} is the characteristic function of A_i, as defined in Sec. 1.9(d).

It is also clear that s is measurable if and only if each of the sets A_i is measurable.

1.17 Theorem

Let $f : X \to [0, \infty]$ be *measurable*. There exist *simple measurable functions* s_n on X such that

(a) $0 \leq s_1 \leq s_2 \leq \cdots \leq f$.

(b) $s_n(x) \to f(x)$ as $n \to \infty$, for every $x \in X$.

Elementary Properties of Measures

1.18 Definition

(a) A positive measure is a function μ, defined on a σ-algebra $\mathfrak{M}$, whose range is in $[0, \infty]$ and which is countably additive. This means that if A_i is a *disjoint* countable collection of members of $\mathfrak{M}$, then

$$\mu(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i).$$

(b) A *measure space* is a measurable space which has a positive measure defined on the σ-algebra of its measurable sets.

(c) A *complex measure* is a complex-valued countably additive function defined on a σ-algebra.

1.19 Theorem

Let μ be a positive measure on a σ-algebra $\mathfrak{M}$. Then,

(a) $\mu(\varnothing) = 0$

(b) $\mu(A_1 \cup \cdots \cup A_n) = \mu(A_1) + \cdots + \mu(A_n)$ if $A_i, \cdots, A_n$ are pairwise disjoint members of $\mathfrak{M}$.

(c) $A \subset B$ implies $\mu(A) \leq \mu(B)$ if $A \in \mathfrak{M}$, $B \in \mathfrak{M}$

(d) $\mu(A_n) \to \mu(A)$ as $n \to \infty$ if $A = \bigcup_{n=1}^{\infty} A_n, A_n \in \mathfrak{M}$, and

$$A_1 \subset A_2 \subset A_3 \cdots .$$

(e) $\mu(A_n) \to \mu(A)$ as $n \to \infty$ if $A = \cap_{n=1}^{\infty} A_n,\ A_n \in \mathfrak{M}$,

$$A_1 \supset A_2 \supset A_3 \supset \cdots ,$$

and $\mu(A_1)$ is finite.

Arithmetic in $[0, \infty]$

Sums and products of measurable functions into $[0, \infty]$ are measurale.

Integration of Positive Functions

In this section, $\mathfrak{M}$ will be a σ-algebra in a set X and μ will be a positive measure on $\mathfrak{M}$.

1.23 Definition

If $s : X \to [0, \infty)$ is a measurable simple function, of the form

$$s = \sum_{i=1}^{n} \alpha_i \chi_{A_i}, \tag{1}$$

where $\alpha_1, \cdots, \alpha_n$ are the distinct values of s (compare Definition 1.16), and if $E \in \mathfrak{M}$, we define

$$\int_E s d\mu = \sum_{i=1}^{n} \alpha_i \mu(A_i \cap E) \tag{2}$$

The convention $0 \cdot \infty = 0$ is used here; it may happen that $\alpha_i = 0$ for some i and that $\mu(A_i \cap E) = \infty$.

If $f : X \to [0, \infty]$ is measurable, and $E \in \mathfrak{M}$, we define

$$\int_E f d\mu = \sup \int_E s d\mu, \tag{3}$$

the supremum being taken over all simple measurable functions s such that $0 \leq s \leq f$.

The left member of (3) is called the *Lebesgue integral* of f over E, with respect to the measure μ. It is a number in $[0, \infty]$.

The following propositions are immediate consequence of the definitions. The functions and sets occuring in them are assumed to be measurable:

(a) If $0 \leq f \leq g$, then $\int_E f d\mu \leq \int_E g d\mu$.

(b) If $A \subset B$ and $f \geq 0$, then $\int_A f d\mu \leq \int_B f d\mu$.

(c) If $f \geq 0$ and c is a constant, $0 \leq c < \infty$, then

$$\int_E cf d\mu = c \int_E f d\mu.$$

(d) If $f(x) = 0$ for all $x \in E$, then $\int_E f d\mu = 0$, even if $\mu(E) = \infty$.

(e) If $\mu(E) = 0$, then $\int_E f d\mu = 0$, even if $f(x) = \infty$ for every $x \in E$.

(f) If $f \geq 0$, then $\int_E f d\mu = \int_X \chi_E f d\mu$.

1.25 Proposition

Let s and t be nonnegative measurable simple functions on X. For $E \in \mathfrak{M}$, define

$$\varphi(E) = \int_E s d\mu. \tag{1}$$

Then φ is a measure on $\mathfrak{M}$. Also

$$\int_X (s+t) d\mu = \int_X s d\mu + \int_X t d\mu. \tag{2}$$

1.26 Lebesgue's Monotone Convergence Theorem

Let $\{f_n\}$ be a sequence of measurable functions on X, and suppose that

(a) $0 \leq f_1(x) \leq f_2(x) \leq \cdots \leq \infty$ for every $x \in X$,

(b) $f_n(x) \to f(x)$ as $n \to \infty$, for every $x \in X$.

Then f is measurable, and

$$\int_X f_n d\mu \to \int_X f d\mu \qquad as \;\; n \to \infty.$$

1.27 Theorem

If $f_n : X \to [0, \infty]$ is measurable, for $n = 1, 2, 3, \cdots$, and

$$f(x) = \sum_{n=1}^{\infty} f_n(x) \qquad (x \in X),$$

then

$$\int_X f d\mu = \sum_{n=1}^{n} \int_X f_n d\mu.$$

Corollary

If $a_{ij} \geq 0$ for i and $j = 1, 2, 3, \cdots$, then

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij} = \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} a_{ij}.$$

1.28 Fatou's Lemma

If $f_n : X \to [0, \infty]$ is measurable, for each positive integer n, then

$$\int_X (\lim_{n\to\infty} \inf f_n) d\mu \leq \lim_{n\to\infty} \inf \int_X f_n d\mu.$$

1.29 Theorem

Suppose $f : X \to [0, \infty]$ is measurable, and

$$\varphi(E) = \int_E f d\mu \qquad (E \in \mathfrak{M}).$$

Then φ is a measure on $\mathfrak{M}$, and

$$\int_X g d\varphi = \int_X g f d\mu$$

for every measurable g on X with range in $[0, \infty]$.

Integration of Complex Functions

As before, μ will in this section be a positive measure on an arbitary measurable space X.

1.30 Definition

We define $L^1(\mu)$ to be the collection of all complex measurable functions f on X for which

$$\int_X |f| d\mu < \infty.$$

The members of $L^1(\mu)$ are called *Lebesgue integrable* functions (with respect to μ) or *summable functions*.

1.31 Definition

If $f = u + iv$, where u and v are real measurable functions on X, and if $f \in L^1(\mu)$, we define

$$\int_E f d\mu = \int_E u^+ d\mu - \int_E u^- d\mu + i \int_E v^+ d\mu - i \int_E v^- d\mu$$

for every measurable set E.

1.32 Theorem

Suppose f and $g \in L^-1(\mu)$ and α and β are complex numbers. Then $\alpha f + \beta g \in L^1(\mu)$, and

$$\int_X (\alpha f + \beta g) d\mu = \alpha \int_X f d\mu + \beta \int_X g d\mu.$$

1.33 Theorem

If $f \in L^1(\mu)$, then

$$|\int_X f d\mu| \leq \int_X |f| d\mu.$$

1.34 Lebesgue's Dominated Convergence Theorem

Suppose $\{f_n\}$ is a sequence of complex measurable functions on X such that

$$f(x) = \lim_{n\to\infty} f_n(x)$$

exists for every $x \in X$. If there is a function $g \in L^1(\mu)$ such that

$$|f_n(x)| \leq g(x) \qquad (n = 1, 2, 3, \cdots; x \in X),$$

then $f \in L^1(\mu)$,

$$\lim_{n\to\infty} \int_X |f_n - f| d\mu = 0,$$

and

$$\lim_{n\to\infty} \int_X f_n d\mu = \int_X f d\mu$$

1.35 Definition

If μ is a measure on a σ-algebra $\mathfrak{M}$ and if $E \in \mathfrak{M}$, the statement "P holds almost everywhere on E" (abbreviated to "P holds a.e. on E") means that there exists an $N \in \mathfrak{M}$ such that $\mu(N) = 0, N \subset E$, and P holds at every point of $E - N$. This concept of a.e. depends of course very strongly on the given measure, and we shall write "a.e. [μ]" whenever clarity requires that the measure be indicated.

The transitivity ($f \sim g$ and $g \sim h$ implies $f \sim h$) is a consequence of the fact that the union of two sets of measure 0 has measure 0.

1.36 Theorem

Let $(X, \mathfrak{M}, \mu)$ be a measure space, let $\mathfrak{M}^*$ be the collection of all $E \subset X$ for which there exist sets A and $B \in \mathfrak{M}$ such that $A \subset E \subset B$ and $\mu(B - A) = 0$, and define $\mu(E) = \mu(A)$ in this situation. Then $\mathfrak{M}^*$ is a σ-algebra, and μ is a measure on $\mathfrak{M}^*$.

1.38 Theorem

Suppose $\{f_n\}$ is a sequence of complex measurable functions defined a.e. on X such that

$$\sum_{n=1}^{\infty} \int_X |f_n| d\mu < \infty.$$

Then the series

$$f(x) = \sum_{n=1}^{\infty} f_n(x)$$

converges for almost all x, $f \in L^1(\mu)$, and

$$\int_X f d\mu = \sum_{n=1}^{\infty} \int_X f_n d\mu$$

1.39 Theorem

(a) Suppose $f : X \to [0, +\infty]$ is measurable, $E \in \mathfrak{M}$, and $\int_E f d\mu = 0$. Then $f = 0$ a.e. on E.

(b) Suppose $f \in L^1(\mu)$ and $\int_E f d\mu = 0$ for every $E \in \mathfrak{M}$. Then $f = 0$ a.e. on X.

(c) Suppose $f \in L^1(\mu)$ and

$$|\int_X f d\mu| = \int_X |f| d\mu.$$

Then there is a constant α such that $\alpha f = |f|$ a.e. on X.

1.40 Theorem

Suppose $\mu(X) < \infty$, $f \in L^1(\mu)$, S is a closed set in the complex plane, and the averages

$$A_E(f) = \frac{1}{\mu(E)} \int_E f d\mu$$

lie in S for every $E \in \mathfrak{M}$ with $\mu(E) > 0$. Then $f(x) \in S$ for almost all $x \in X$.

1.41 Theorem

Let $\{E_k\}$ be a sequence of measurable sets in X, such that

$$\sum_{k=1}^{\infty} \mu(E_k) < \infty.$$

Then almost all $x \in X$ lie in at most finitely many of the sets E_k.