baby rudin Ch.1

Dawei Wang

March 1, 2022

1 INTRODUCTION

The rational number system has certain gaps, in spite of the fact that between any two rationals there is another: If $r < s$ the $r < (r+s)/2 < s$. The real number system fills these gaps.

Definition 1 *If A is any set, we write $x \in A$ to indicate that x is a member(or an element) of A.*

If x is not a member of A, we write: $x \notin A$.

If A and B are sets, and if every element of A is an element of B, we say that A is a subset of B, and write $A \subset B$, or $B \supset A$. If in addition there is an element of B which is not in A, then A is said to be proper subset of B.

If $A \subset B$ and $B \subset A$, we write A=B. Otherwise $A \neq B$.

2 ORDERED SETS

Definition 2 *Let S be a set. An order on S is a relation, denoted by $<$, with the following two properties:*

(i) If $x \in S$ then one and only one of the statements

$$x < y, \qquad x = y, \qquad x > y$$

is true.

(ii) If $x, y, z \in S$, if $x < y$ and $y < z$, then $x < z$.

$x \leq y$ is the negation of $x > y$.

Definition 3 *Suppose S is an ordered set, and $E \subset S$. If there exists a $\beta \in S$ such that $x \leq \beta$ for every $x \in E$, we say that E is bounded above, and call β an upper bound of E.*

Definition 4 *Suppose S is an ordered set, $E \subset S$, and E is bounded above. Suppose there exists an $\alpha \in S$ with the following properties:*

(i) α is an upper bound of E.

(ii) If$\gamma < \alpha$ then γ is not an upper bound of E.

Then α is called the least upper bound of E or the supremum of E, and we write

$$\alpha = sup \quad E$$

The greatest lower bound, or infimum, of a set E which is bounded below is defined in the same manner: The statement

$$\alpha = inf \quad E$$

means that α is a lower bound of E and that no β with $\beta > \alpha$ is a lower bound of E.

Definition 5 *An ordered set S is said to have the least-upper-bound property if the following is true:*

If $E \subset S$, E is not empty, and E is bounded above, then sup E exists in S.

Every ordered set with the least-upper-bound property also has the greatest-lower-bound property.

Q(the set of all rational numbers) does not have the least-upper-bound property.

Theorem 1 *Suppose S is an ordered set with the least-upper-bound property, $B \subset S$, B is not empty, and B is bounded below. Let L be the set of all lower bounds of B. Then*

$$\alpha = sup \quad L$$

exists in S, and $\alpha = inf \quad B$

In particular, inf B exists in S.

3 FIELDS

Definition 6 *A field is a set F with two operations, called addition and multiplication, which satisfy the following so-called "field axioms":*

(A) Axioms for addition

(A1) If $x \in F$ and $y \in F$, then their sum x+y is in F.

(A2) Addition is commutative: x+y=y+xfor all $x, y \in F$.

(A3) Addition is associative: (x+y)+z=x+(y+z) for all $x, y, z \in F$.

(A4) F contains an element 0 such that 0+x=x for every $x \in F$.

(A5) To every $x \in F$ corresponds an element $-x \in F$ such that

$$x + (-x) = 0$$

(M) Axioms for multiplication
(M1) If$x \in F$ and$y \in F$, then their product xy is in F.
(M2) Multiplication is commutative: xy=yx for all $x, y \in F$.
(M3) Multiplication is associative: (xy)z=x(yz) for all $x, y, z \in F$.
(M4) F contains an element $1 \neq 0$ such that 1x=x for every $x \in F$.
(M5) If $x \in F$ and $x \neq 0$ then there exists an element $1/x \in F$ such that

$$x \cdot (1/x) = 1$$

The distributive law

$$x(y + z) = xy + xz$$

holds for all $x, y, z \in F$.

The filed axioms clearly hold in Q, the set of all rational numbers, if addition and multiplication have their customary meaning. Thus Q is a field.

Definition 7 *An ordered field is a field is a field F which is also an ordered set, such that:*

(i)$x + y < x + z$ if $x, y, z \in F$ and $y < z$,
(ii) $xy > 0$ if $x \in F, y \in F$, $x > 0$, and $y > 0$.
If $x > 0$, we call x positive; if $x < 0$, x is negative.

4 THE REAL FIELD

Theorem 2 *There exists an ordered field R which has the least-upper-bound property.*

Moreover, R contains Q as a subfield.

Theorem 3 *(a) If $x \in R, y \in R$, and $x > 0$, then there is a positive integer n such that(Archimedean property)*

$$nx > y$$

(b) If $x \in R, y \in R$, and $x < y$, then there exists a $p \in Q$ such that $x < p < y$.

This means that Q is dense in R: Between ant two real numbers there is a rational number.

Theorem 4 *For every real $x > 0$ and every integer $n > 0$ there is one and only one positive real y such that $y^n = x$.*

This number y is written $\sqrt[n]{x}$ or $x^{1/n}$.

5 The extended real number system

Definition 8 *The extended real number system consists of the real field R and two symbols, $+\infty$ and $-\infty$. We preserve the original order in R, and define*

$$-\infty < x < +\infty$$

for every $x \in R$.

It is then clear that $+\infty$ is an upper bound of every subset of the extended real number system, and that every nonempty subset has a least upper bound.

The extended real number system does not form a field.

6 The complex field

Definition 9 *A complex number is an ordered pair (a,b) of real numbers. "Ordered" means that (a,b) and (b,a) are regarded as distinct if $a \neq b$.*

Let x=(a,b),y=(c,d) be two complex numbers. We write x=y if and only if a=c and b=d. We define

$$x + y = (a + c, b + d)$$

$$xy = (ac - bd, ad + bc)$$

Theorem 5 *Theses definitons of addition and multiplication turns the set of all complex numbers into a field, with (0,0) and (1,0) in the role of 0 and 1.*

Theorem 6 *For any real numbers a and b we have*

$$(a, 0) + (b, 0) = (a + b, 0)$$

$$(a, 0)(b, 0) = (ab, 0)$$

the real field is a subfield of the complex field. The notation (a,b) is equivalent to the more customary a+bi;

Definition 10 *i=(0,1)*

Theorem 7 $i^2 = -1$

Theorem 8 *If a and b are real, then (a,b)=a+bi*

Definition 11 *If a, b are real and z=a+bi, then the complex number $\overline{z} = a - bi$ is called the conjugate of z. The numbers a and b are the real part and the imaginary part of z, respectively.*

We shall occasionally write a=Re(z), b=Im(z).

Theorem 9 *If z and w are complex, then*
$\overline{z+w} = \overline{z} + \overline{w}$
$\overline{zw} = \overline{z} \cdot \overline{w}$
$z + \overline{z} = 2Re(z), z - \overline{z} = 2iIm(z)$
$z\overline{z}$ *is real and positive (except when z=0).*

Definition 12 *if z is a complex number, its absolute value* $|z|$ *is the non-negative square root of* $z\overline{z}$*; that is,* $|z| = (z\overline{z})^{1/2}$.

Theorem 10 *Let z and w be complex numbers. Then*
$|z| > 0 unless z = 0, |0| = 0,$
$|\overline{z}| = |z|,$
$|zw| = |z||w|,$
$|Rez| \leq |z|,$
$|z + w| \leq |z| + |w|.$

Theorem 11 *If* $a_1, \cdots, a_n$ *and* $b_1, \cdots, b_n$ *are complex numbers, then*

$$|\sum_{j=1}^{n} a_j \bar{b}_j|^2 \leq \sum_{j=1}^{n} |a_j|^2 \sum_{j=1}^{n} |b_j|^2$$

7 Euclidean spaces

Definition 13 *For each positive integer k, let* R^k *be the set of all ordered k-tuples*

$$\boldsymbol{x} = (x_1, x_2, \cdots x_k),$$

where $x_1, \cdots, x_k$ *are real numbers, called the coorinates of* $\boldsymbol{x}$*. The elements of* R^k *are called points. or vectors, especially when* $k > 1$*. If* $\boldsymbol{y} = (y_1, y_2, \cdots y_k)$ *and if* α *is a real number, put*

$$\boldsymbol{x} + \boldsymbol{y} = (x_1 + y_1, \cdots, x_k + y_k),$$

$$\alpha\boldsymbol{x} = (\alpha x_1, \cdots, \alpha x_k)$$

The zero element of R^k is the point $\mathbf{0}$, all of whose coordinates are 0.

We also define the so-called "inner product"(or scalar product) of $\mathbf{x}$ and $\mathbf{y}$ by

$$\mathbf{x} \cdot \mathbf{y} = \sum_{i=1}^{k} x_i y_i$$

and the norm of $\mathbf{x}$ by

$$|\mathbf{x}| = (\mathbf{x} \cdot \mathbf{x})^{1/2} = (\sum_{1}^{k} x_i^2)^{1/2}$$

The structure now defined is called euclidean k-space.

Theorem 12 *Suppose $x, y, z \in R^k$, and a is real. Then*

(a) $|\boldsymbol{x}| \geq 0$;
(b) $|\boldsymbol{x}| = 0$ *if and only if* $\boldsymbol{x} = \boldsymbol{0}$;
(c) $|\alpha \boldsymbol{x}| = |\alpha||\boldsymbol{x}|$;
(d) $|\boldsymbol{x} \cdot \boldsymbol{y}| \leq |\boldsymbol{x}||\boldsymbol{y}|$;
(e) $|\boldsymbol{x} + \boldsymbol{y}| \leq |\boldsymbol{x}| + |\boldsymbol{y}|$;
(f) $|\boldsymbol{x} - \boldsymbol{z}| \leq |\boldsymbol{x} - \boldsymbol{y}| + |\boldsymbol{y} - \boldsymbol{z}|$.

Theorem (a), (b), (c) and (f) will allow us to regard R^k as a metric space. vc