baby rudin Ch.3

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CONVERGENT SEQUENCES

Definition 1. A sequence $\{p_n\}$ in a metric space X is said to be converge if there is a point $p \in X$ with the following property: For every ε there is an integer N such that $n \geq N$ implies that $d(p_n, p) < \varepsilon$.

In this case we also say that $\{p_n\}$ converges to p, or that p is the limit of $\{p_n\}$, and we write $p_n \to p$, or

$$\lim_{n\to\infty} p_n = p$$

If $\{p_n\}$ does not converge, it is said to diverge.

We recall that the set of all points $p_n(n = 1, 2, 3, ...)$ is the range of $\{p_n\}$. The range of a sequence may be a finite set, or it may be infinite. The sequence $\{p_n\}$ is said to be bounded if its range is bounded.

Theorem 1. Let $\{p_n\}$ be a sequence in a metric space X.

(a) $\{p_n\}$ converges to $p \in X$ if and only if every neighborhood of p contains p_n for all but finitely many n.

(b) If $p \in X$, $p' \in X$, and if $\{p_n\}$ converges to p and to p' ,then $p = p'$.

(c) If $\{p_n\}$ converges, then $\{p_n\}$ is bounded.

(d) If $E \subset X$ and if p is a limit point of E, then there is a sequence $\{p_n\}$ in E such that $P = \lim_{n\to\infty} p_n$

Theorem 2. Suppose $\{s_n\}$, $\{t_n\}$ are complex sequences, and $\lim_{n\to\infty} s_n = s$, $\lim_{n\to\infty} t_n = t$. Then

(a)

$$\lim_{n\to\infty}(s_n + t_n) = s + t$$

(b)

$$\lim_{n\to\infty} cs_n = cs, \lim_{n\to\infty}(c + s_n) = c + s$$

for any number c;

(c)

$$\lim_{n\to\infty} s_n t_n = st;$$

(d)

$$\lim_{n\to\infty} \frac{1}{s_n} = \frac{1}{s}$$

provided $s_n \neq 0 (n = 1, 2, 3...)$, and $s \neq 0$.

Theorem 3.

(a) Suppose $\mathtt{x}_n \in R^k$(n=1,2,3...) and

$$\mathtt{x}_n = (\alpha_{1,n}, \cdots, \alpha_{k,n})$$

Then $\{\mathtt{x}_n\}$ converges to $\mathtt{x} = (\{\alpha_1, \cdots, \alpha_k\})$ if and only if

$$\lim_{n\to\infty} \alpha_{j,n} = \alpha_j \qquad (1 \leq j \leq k)$$

(b) Suppose $\{\mathtt{x}_n\}$,$\{\mathtt{y}_n\}$ are sequences in R^k, $\{\beta_n\}$ is a sequence of real numbers, and $\mathtt{x}_n \to \mathtt{x}, \mathtt{y}_n \to \mathtt{y}, \beta_n \to \beta$. Then

$$\lim_{n\to\infty} (\mathtt{x}_n + \mathtt{y}_n) = \mathtt{x} + \mathtt{y}, \quad \lim_{n\to\infty} (\mathtt{x}_n \cdot \mathtt{y}_n) = \mathtt{x} \cdot \mathtt{y}, \quad \lim_{n\to\infty} \beta_n \mathtt{x}_n = \beta_n \mathtt{x}$$

SUBSEQUENCES

Definition 2. Given a sequence $\{p_n\}$, consider a sequence$\{n_k\}$ of positive integers, such that $n_1 < n_2 < n_3 < \cdots$. Then the sequence $\{p_{ni}\}$ is called a subsequence of $\{p_n\}$. If $\{p_{ni}\}$ converges, its limit is called a subsequential limit of $\{p_n\}$.

Theorem 4.

(a) If $\{p_n\}$ is a sequence in a compact metric space X, then some subsequence of $\{p_n\}$ converges to a point of X.

(b) Every bounded sequence in R^k contains a convergent subsequence.

Theorem 5. The subsequential limits of a sequence $\{p_n\}$ in a metric space X form a closed subset of X.

Definition 3. A sequence $\{p_n\}$ in a metric space X is said to be a Cauchy sequence if for every $\epsilon > 0$ there is an integer N such that $d(p_n, p_m) < \epsilon$ if $n \geq N$ and $m \geq N$.

Definition 4. Let E be a nonempty subset of a metric space X, and let S be the set of all real numbers of the form $d(p, q)$, with $p \in E$ and $q \in E$. The sup of S is called the diameter of E.

If $\{p_n\}$ is a sequence in X and if E_N consists of the points $p_N, p_{N+1}, , p_{N+2}, \cdots$, it is clear from the two preceding definitions that $\{p_n\}$ is a Cauchy sequence if and only if

$$\lim_{N\to\infty} diam\ E_N = 0.$$

Theorem 6.

(a) If $\overline{E}$ is the closure of a set E in a metric space X, then

$$diam\ \overline{E} = diam\ E$$

(b) If K_n is a sequence of compact sets in X such that $K_n \supset K_{n+1}(n = 1, 2, 3, \cdots)$ and if

$$\lim_{n\to\infty} diam\ K_n = 0,$$

then $\cap_1^\infty K_n$ consists of exactly one point.

Theorem 7.

(a) In any metric space X, every convergent sequence is a Cauchy sequence.

(b) If X is a compact metric space and if $\{p_n\}$ is a Cauchy sequence in X, then $\{p_n\}$ converges some point of X.

(c) In R^k, every Cauchy sequence converges.

The difference between the definition of convergence and the definition of a Cauchy sequence is that the limit is explicitly involved in the former, but not in the latter. Thus Theorem 7(b) may enable us to decide whether or not a given sequence converges without knowledge of the limit to which it may converge.

The fact that a sequence converges in R^k if and only if it is a Cauchy sequence is usually called the Cauchy criterion for convergence.

Definition 5. A metric space in which every Cauchy sequence converges is said to be complete.

Thus theorem 7 says that all compact metric spaces and all Euclidean space are complete. Theorem 7 implies also that every closed subset E of a complete metric space X is complete.

An example of a metric space which is not complete is the space of of all rational numbers, with $d(x, y) = |x - y|$. (Since the existence of Cauchy sequences that do not convergent to any rational number.)

Definition 6. A sequence $\{s_n\}$ of real numnbers is said to be

(a) monotonically increasing if $s_n \leq s_n + 1(n = 1, 2, 3, \cdots)$

(b) monotonically decreasing if $s_n \geq s_n + 1(n = 1, 2, 3, \cdots)$

Theorem 8. Suppose $\{s_n\}$ is monotonic. Then $\{s_n\}$ converges if and only if it is bounded.

UPPER AND LOWER LIMITS

Definition 7. Let $\{s_n\}$ be a sequence of real numbers with the following property: For every real M there is an integer N such that $n \geq N$ implies $s_n \geq M$. We then write

$$s_n \to +\infty$$

Similarly, if for every real M there is an integer N such that $n \geq N$ implies that $s_n \leq M$, we write

$$s_n \to -\infty$$

Definition 8. Let $\{s_n\}$ be a sequence of real numbers. Let E be the set of numbers x (in the extended real number system) such that $s_{nk} \to x$ for some subsequence s_{nk}. This set E contains all subsequential limits as defined in Definition 2, plus the numbers $+\infty, -\infty$.

We now recall the definitions of inf and sup, and put

$$s^* = supE$$

$$s_* = infE$$

The numbers s^*, s_* are called the upper and lower limits of $\{s_n\}$; we use the notaton

$$\lim_{n\to\infty} sup \;\; s_n = s^*, \lim_{n\to\infty} inf \;\; s_n = s_*$$

Theorem 9. Let s_n be a sequence of real numbers. Let E and s^* have the same meaning as in Definition 8. Then s^* has the following two properties:

(a) $s^* \in E$.

(b) If $x > s^*$, there is an integer N such that $n \geq N$ implies $s_n < x$.

Moreover, s^* is the only number with the properties (a) and (b).

Of course, an analogous result is true for s_*.

Theorem 10. If $s_n \leq t_n$ for $n \geq N$, where N is fixed, then

$$\lim_{n\to\infty} inf \;\; s_n \leq \lim_{n\to\infty} inf \;\; t_n$$

$$\lim_{n\to\infty} sup \;\; s_n \leq \lim_{n\to\infty} sup \;\; t_n$$

SOME SPECIAL SEQUENCES

Theorem 11.

(a) If $P > 0$, then $\lim_{n\to\infty} \frac{1}{n^p} = 0$
(b) If $P > 0$, then $\lim_{n\to\infty} \sqrt[n]{p} = 1$
(c) $\lim_{n\to\infty} \sqrt[n]{n} = 1$
(d) If $p > 0$ and α is real, then $\lim_{n\to\infty} \frac{n^\alpha}{(1+p)^n} = 0$
(e) If $|x| < 1$, then $\lim_{n\to\infty} x^n = 0$

SERIES

In the remainder of this chapter, all sequences and series under consideration will be complex-valued, unless the contrary is explicitly stated.

Definition 9. Given a sequence $\{\alpha_n\}$, we use the notation

$$\sum_{n=p}^{q} \alpha_n \ \ (p \leq q)$$

to denote the sum $a_p + a_{p+1} + \cdots + a_q$. With $\{\alpha_n\}$ we associate a sequence$\{s_n\}$, where

$$s_n = \sum_{k=1}^{n} a_k.$$

For $\{s_n\}$ we also use the symbolic expression

$$a_1 + a_2 + a_3 + \cdots$$

or, more concisely,

$$\sum_{n=1}^{\infty} a_n$$

The above symbol we call an *infinite series*, or just `series`. The numbers s_n are called the partial sums of the series. If $\{s_n\}$ converges to s, we say that the series converges, and we write

$$\sum_{n=1}^{\infty} a_n = s$$

The number s is called the sum of the series; but it should be clearly understood that s is $limit of a sequence of sums$, and is not simply by addition.

If $\{s_n\}$ diverges, the series is said to diverge.

Sometimes, for convenience of notation, we shall consider series of the form

$$\sum_{n=0}^{\infty} a_n$$

And frequently, when there is no possible ambiguity, or when the distinction is immaterial, we shall simply write $\sum a_n$.

It is clear that every theorem about sequences can be stated in terms of series(putting $a_1 = s_1$, and $a_n = s_n - s_{n-1}$ for $n > 1$), and vice versa. But it is nevertheless useful to consider both concepts.

The Cauchy criterion can be restated in the following form:

Theorem 12. $\sum a_n$ converges if and only if for every $\varepsilon > 0$ there is an integer N such that

$$|\sum_{k=n}^{m} a_k| \leq \varepsilon$$

if $m \geq n \geq N$.

In particular, by taking m=n, (6) becomes

$$|a_n| < \varepsilon \ \ (n \geq N)$$

In other words:

Theorem 13. If $\sum a_n$ converges, then $\lim_{n\to\infty} a_n = 0$

The condition $a_n \to 0$ is not, however, sufficient to ensure convergence of $\sum a_n$.

Theorem 8, concerning monotonic sequences, also has an immediate counterpart for series.

Theorem 14. A series of nonnegative terms converges if and only if its partial sums form a bounded sequence.

Theorem 15.

(a) If $|a_n| \leq c_n forn > N_0$, where N_0 is some fixed integer, and if $\sum c_n$ converges, then $\sum a_n$ converges.

(b) If $a_n \geq d_n \geq 0$ for $n \geq N_0$, and if $\sum d_n$ diverges, then $\sum a_n$ diverges.

SERIES OF NONNEGATIVE TERMS

Theorem 16. If $0 \leq x < 1$, then

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

Theorem 17. Suppose $a_1 \geq a_2 \geq a_3 \geq \cdots \geq 0$. Then the series $\sum_{n=1}^{\infty} a_n$ converges if and only if the seires

$$\sum_{k=0}^{\infty} 2^k a_{2^k} = a_1 + 2a_2 + 4a_4 + 8a_8 + \cdots$$

converges

Theorem 18. $\sum \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \leq 1$.

Theorem 19. If $p > 1$,

$$\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}$$

converges; if $p \leq 1$, the series diverges.

This procedure may evidently be continued. For instance,

$$\sum_{n=3}^{\infty} \frac{1}{n \log n \log \log n}$$

diverges, whereas

$$\sum_{n=3}^{\infty} \frac{1}{n \log n (\log \log n)^2}$$

converges.

THE NUMBER e

Definition 10. $e = \sum_{n=0}^{\infty} \frac{1}{n!}$

Theorem 20. $\lim_{n\to\infty}(1+\frac{1}{n})^n = e$

Let $s_n = \sum_{k=0}^{n} \frac{1}{k!}$

$$\begin{aligned} e - s_n &= \frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \frac{1}{(n+3)!} + \cdots \\ &< \frac{1}{(n+1)!}\{1 + \frac{1}{n+1} + \frac{1}{(1+n)^2} + \cdots\} = \frac{1}{n!n} \end{aligned}$$

$$0 < e - s_n < \frac{1}{n!n}$$

Theorem 21. e is irrational.

THE ROOT AND RATIO TESTS

Theorem 22. Given $\sum a_n$, put $\alpha = \lim_{n\to\infty} \sqrt[n]{|\alpha_n|}$. Then
(a) if $\alpha < 1$, $\sum a_n$ converges.
(b) if $\alpha > 1$, $\sum a_n$ diverges.
(c) if $\alpha = 1$, the test gives no information.

Theorem 23. The series $\sum a_n$
(a) converges if $\lim_{n\to\infty} sup|\frac{a_{n+1}}{a_n}| < 1$
(b) diverges if $\lim_{n\to\infty} sup|\frac{a_{n+1}}{a_n}| \geq 1$ for all $n \geq n_0$, where n_0 is some fixed integer.

Note The Knowledge that $\lim a_{n+1}/a_n = 1$ implies nothing about the convergence of $\sum a_n$.

Remark 1. The ratio test is frequently easier to apply than the root test, since it is usually easier to compute ratios than nth roots. However, the root test has wider scope.

Theorem 24. For any sequence $\{c_n\}$ of positive numbers,

$$\lim_{n\to\infty} inf \frac{c_{n+1}}{c_n} \leq \lim_{n\to\infty} inf \sqrt[n]{c_n},$$

$$\lim_{n\to\infty} sup \sqrt[n]{c_n} \leq \lim_{n\to\infty} sup \frac{c_{n+1}}{c_n},$$

POWER SERIES

Given a sequence $\{c_n\}$ of complex numbers, the series

$$\sum_{n=0}^{\infty} c_n z^n$$

is called a power series. The numbers c_n are called the *coefficients* of the series; z is a complex number.

Theorem 25. Given the power series $\sum c_n z^n$, put

$$\alpha = \lim_{n\to\infty} sup \sqrt[n]{c_n}, \quad R = \frac{1}{\alpha}$$

(If $\alpha = 0$, $R = +\infty$; if $\alpha = +\infty$, R=0.) Then $\sum c_n z^n$ converges if $|z| < R$, and diverges if $|z| > R$.

Note: R is called the radius of convergence of $\sum c_n z^n$.

SUMMATION BY PARTS

Theorem 26. Given two sequences $\{a_n\}$,$\{b_n\}$, put

$$A_n = \sum_{k=0}^{n} a_k$$

if $n \geq 0$; put $A_{-1} = 0$. Then if $0 \leq p \leq q$, we have

$$\sum_{n=p}^{q} a_n b_n = \sum_{n=p}^{q-1} A_n(b_n - b_{n+1}) + A_q b_q - A_{p-1} b_p$$

Theorem 27. Suppose
(a) the partial sum A_n of $\sum a_n$ form a bounded sequence;
(b) $b_0 \geq b_1 \geq b_2 \geq \cdots$;
(c) $\lim_{n\to\infty} b_n = 0$.
Then $\sum a_n b_n$ converges.

Theorem 28. Suppose
(a) $|c_1| \geq |c_2| \geq |c_3| \geq \cdots$;
(b) $c_{2m-1} \geq 0, c_{2m} \leq 0$ (m=1,2,3,$\cdots$);
(c) $\lim_{n\to\infty} c_n = 0$.
Then $\sum c_n$ converges.

Theorem 29. Suppose the radius of convergence of $\sum c_n z^n$ is 1, and suppose $c_0 \geq c_1 \geq c_2 \geq \cdots$, $\lim_{n\to\infty} c_n = 0$. Then $\sum c_n z^n$ converges at every point on the circle $|z| = 1$, except possibly at z=1.

ABSOLUTE CONVERGENCE

The series $\sum a_n$ is said to be converge absolutely if the series $\sum |a_n|$ converges

Theorem 30. If $\sum a_n$ converges absolutely, then $\sum a_n$ converges.

Remark 2. For series of positive terms, absolute convergence is the same as convergence.

If $\sum a_n$ converges, but $\sum |a_n|$ diverges, we say that $\sum a_n$ converges nonabsolutely.

We shall see that we may operate with absolutely convergent series very much as with finite sums. We may multiply them term by term and we may change the order in which the additions are carried out, without affecting the sum of the series. But for nonabsolutely convergent series this is no longer true, and more care has to be taken when dealing with them.

ADDITION AND MULTIPLICATION OF SERIES

Theorem 31. If $\sum a_n = A$, and $\sum b_n = B$, then $\sum(a_n + b_n) = A + B$, and $\sum ca_n = cA$, for any fixed c.

Definition 11. Given $\sum a_n$ and $\sum b_n$, we put

$$c_n = \sum_{k=0}^{n} a_k b_{n-k} \quad (n = 0, 1, 2, ...)$$

and call $\sum c_n$ the product of the two given series.

Theorem 32. Suppose
(a) $\sum_{n=0}^{\infty} a_n$ converges absolutely,
(b) $\sum_{n=0}^{\infty} a_n = A$,
(c) $\sum_{n=0}^{\infty} b_n = B$,
(d) $c_n = \sum_{k=0}^{n} a_k b_n - k \quad (n = 0, 1, 2, \cdots)$.
Then

$$\sum_{n=0}^{\infty} c_n = AB.$$

That is the product of two convergent series converges, and to the right value, if at least one of the two series converges absolutely.

Theorem 33. If the series $\sum a_n$, $\sum b_n$, $\sum c_n$ converge to A, B, C, and $c_n = a_0 b_n + \cdots + a_n b_0$, then $C = AB$.

REARRANGEMENTS

Definition 12. Let $\{k_n\}$, $n = 1, 2, 3, \cdots$, be a sequence in which every positive integer appears once and only once (that is, $\{k_n\}$ is a 1-1 function from J onto J). Putting

$$a'_n = a_{kn} \qquad (n = 1, 2, 3, \cdots)$$

We say that $\sum a'_n$ is a *rearrangement* of $\sum a_n$

Theorem 34. Let $\sum a_n$ be a series of real numbers which converges, but not absolutely. Suppose

$$-\infty \leq \alpha \leq \beta \leq \infty$$

Theorem 35. If $\sum a_n$ is a series of complex numbers which converges absolutely, then every rearrangement of $\sum a_n$ converges, and they all converge to the same sum.