Convex sets

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1 Affine and convex sets

1.1 Lines and line segments

Suppose $x_1 \neq x_2$ are two points in $\mathbf{R}^n$. Points of the form

$$y = \theta x_1 + (1-\theta)x_2$$

where $\theta \in \mathbf{R}$, form the line passing through x_1 and x_2.

1.2 Affine sets

A set $C \in \mathbf{R}^n$ is affine if the line through any two distinct points in C lies in C, i.e., if for any x_1, $x_2 \in C$ and $\theta \in \mathbf{R}$, we have $\theta x_1 + (1-\theta)x_2 \in C$. In other words, C contains the linear combination of any two points in C, provided the coefficients in the linear combination sum to one.

This idea can be generalized to more than two points. We refer to a point of the form $\theta_1 x_1 + \cdots + \theta_k x_k$, where $\theta_1 + \cdots + \theta_k = 1$, as an affine combination of the points $x_1, \cdots, x_k$.

If C is an affine set, $x_1, \cdots, x_k \in C$, and $\theta_1 + \cdots + \theta_k = 1$, then the point $\theta_1 x_1 + \cdots + \theta_k x_k$ also belongs to C.

If C is an affine set and $x_0 \in C$, then the set

$$V = C - x_0 = \{x - x_0 | x \in C\}$$

is a subspace, i.e., closed under sums and scalar multiplication.

Thus, the affine set C can be expressed as

$$C = V + x_0 = \{v + x_0 | v \in V\}$$

i.e., as a subspace plus an offset. The subspace V associated with the affine set C does not depend on the choice of x_0, so x_0 can be chosen as any point in C. We define the dimension of an affine set C as the dimension of the subspace $V = C - x_0$, where x_0 is any element of C.

The solution set of a system of linear equations, $C = \{x | Ax = b\}$, where $A \in \mathbf{R}^{m\times n}$ and $b \in \mathbf{R}^m$, is an affine set. To show this, suppose $x_1, x_2 \in C$, i.e.,$Ax_1 = b, Ax_2 = b$. Then for any θ, we have

$$\begin{aligned} A(\theta x_1 + (1-\theta)x_2) &= \theta A x_1 + (1-\theta)Ax_2 \\ &= \theta b + (1-\theta)b \\ &= b \end{aligned}$$

which shows that the affine combination $\theta x_1 + (1-\theta)x_2$ is also in C. The subspace associated with the affine set C is the nullspace of A.

We also have a converse: every affine set can be expressed as the solution set of a system of linear equations.

The set of all affine combinations of points in some set $C \subseteq \mathbf{R}^n$ is called the *affine hull of C*, and denoted $\mathbf{aff}C$:

$$\mathbf{aff}C = \{\theta_1 x_1 + \cdots + \theta_k x_k | x_1 \cdots x_k \in C, \theta_1 + \cdots + \theta_k = 1\}$$

The affine hull is the smallest affine set that contains C, in the following sense: if S is any affine set with $C \subseteq S$, then $\mathbf{aff}C \subseteq S$.

1.3 Affine dimension and relative interior

We define the *affine dimension* of a set C as the dimension of its affine hull.

If the affine dimension of a set C Rn is less than n, then the set lies in the affine set $\mathbf{aff}c \neq \mathbf{R}^n$. We define the relative interior of the set C, denoted **relintC**, as its interior relative to $\mathbf{aff}C$:

$$relint\ \ C = \{x \in C | B(x,r) \cap \mathbf{aff}C\}$$

where $B(x,r) = \{y|\ \ ||y-x|| \leq r\}$, the ball of radius r and center x in the norm $||\cdot||$. We can then define the *relative boundary* of a set C as **clC** $\setminus$ **relintC**, where **cl**C is the closure of C.

1.4 Convex sets

A set C is convex if the line segment between any two points in C lies in C, i.e., if for any $x_1, x_2 \in C$ and any θ with $0 \leq \theta \leq 1$, we have

$$\theta x_1 + (1-\theta)x_2 \in C$$

We call a point of the form $\theta_1 x_1 + \cdots + \theta_k x_k$, where $\theta_1 + \cdots + \theta_k = 1$ and i 0, i = 1, . . . , k, a convex combination of the points x1, . . . , xk. As with affine sets, it can be shown that a set is convex if and only if it contains every convex combination of its points.

The *convex hull* of a set C, denoted **conv** C, is the set of all convex combinations of points in C:

$$conv\ \ C = \{\theta_1 x_1 + \cdots + \theta_k x_k | x_i \in C, \theta_i \geq 0, i = 1, \cdots, k, \theta_1 + \cdots + \theta_k = 1\}$$

As the name suggests, the convex hull *conv* C is always convex. It is the smallest convex set that contains C: If B is any convex set that contains C, then conv $C \subseteq B$.

1.5 Cones

A set C is called a cone, or *nonnegative homogeneous*, if for every $x \in C$ and $\theta \geq 0$ we have $\theta x \in C$. A set C is a *convex cone* if it is convex and a cone, which means that for any $x_1, x_2 \in C$ and $\theta_1, \theta_2 \geq 0$, we have

$$\theta_1 x_1 + \theta_2 x_2 \in C$$

A point of the form $\theta_1 x_1 + \cdots + \theta_k x_k$ with $\theta_1, \cdots, \theta_k \geq 0$ is called a conic combination (or a nonnegative linear combination) of $x_1, \cdots, x_k$. If x_i are in a convex cone C, then every conic combination of x_i is in C. Conversely, a set C is a convex cone if and only if it contains all conic combinations of its elements.

The conic hull of a set C is the set of all conic combinations of points in C, i.e.,

$$\{\theta_1 x_1 + \cdots + \theta_k x_k | x_i \in C, \theta_i \geq 0, i = 1, \cdots, k\}$$

which is also the smallest convex cone that contains C.

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1. The empty set $\emptyset$, any single point (i.e., singleton) $\{x_0\}$, and the whole space $\mathbf{R}_n$ are affine (hence, convex) subsets of $\mathbf{R}_n$.

2. Any line is affine. If it passes through zero, it is a subspace, hence also a convex cone.

3. A line segment is convex, but not affine (unless it reduces to a point).

4. A ray, which has the form $\{x_0 + \theta v | \theta \geq 0\}$, where $v \neq 0$, is convex, but not affine. It is a convex cone if its base x_0 is 0.

5. Any subspace is affine, and a convex cone (hence convex).

2.1 Hyperplanes and halfspaces

A hyperplane is a set of the form

$$\{x | a^T x = b\}$$

where $a \in \mathbf{R}^n, a \neq 0$ and $b \in \mathbf{R}$.

Geometrically, the hyperplane $\{x | a^T x = b\}$ can be interpreted as the set of points with a constant inner product to a given vector a, or as a hyperplane with normal vector a; the constant $b \in \mathbf{R}$ determines the offset of the hyperplane from the origin. This geometric interpretation can be understood by expressing the hyperplane in the form

$$\{x | a^T (x - x_0) = 0\}$$

where x_0 is any point in the hyperplane (i.e., any point that satisfies $a^T x_0 = b$). This representation can in turn be expressed as

$$\{x | a^T (x - x_0) = 0\} = x_0 + a^{\perp}$$

where $a^{\perp}$ denotes the orthogonal complement of a, i.e., the set of all vectors orthogonal to it:

$$a^T = \{v | a^T = 0\}$$

This shows that the hyperplane consists of an offset x_0, plus all vectors orthogonal to the (normal) vector a.

A hyperplane divides $\mathbf{R}^n$ into two *halfspaces.* A (closed) halfspace is a set of the form

$$\{x | a^T x \leq b\}$$

where $a \neq 0$.

2.2 Euclidean balls and ellipsoids

A (*Euclidean*) ball (or just ball) in $\mathbf{R}^n$ has the form

$$B(x_c, r) = \{ \; ||x - x_c|| \leq r\} = \{x | (x - x_c)^T (x - x_c) \leq r^2\}$$

Another common representation for the Euclidean ball is

$$B(x_c, r) = \{x_c + ru | \;\; ||u||_2 \leq 1\}$$

A Euclidean ball is a convex set: if $||x_1 - x_c||_2 \leq r$, $||x_2 - x_c||_2 \leq r$, and $0 \leq \theta \leq 1$, Then

$$||\theta x_1 + (1 - \theta) x_2 - x_c||_2 \leq r$$

A related family of convex sets is the ellipsoids, which have the form

$$\varepsilon = \{x | (x - x_c)^T P^{-1} (x - x_c) \leq 1\}$$

where $P = P^T \succ 0$, i.e., P is symmetric and positive definite. The vector $x_c \in \mathbf{R}^n$ is the center of the ellipsoid. The matrix P determines how far the ellipsoid extends in every direction from x_c; the lengths of the semi-axes of E are given by $\sqrt{\lambda_i}$, where λ_i are the eigenvalues of P. A ball is an ellipsoid with $P = r^2 I$.

Another common representation of an ellipsoid is

$$\varepsilon = \{x_c + Au | \;\; ||u||_2 \leq 1\}$$

where A is square and nonsingular. In this representation we can assume without loss of generality that A is symmetric and positive definite. By taking $A = P^{1/2}$, this representation gives the ellipsoid definition. When the matrix A is symmetric positive semidefinite but singular, the set is called a degenerate ellipsoid; its affine dimension is equal to the rank of A. Degenerate ellipsoids are also convex.

2.3 Norm balls and norm cones

Suppose $|| \cdot ||$ is any norm on $\mathbf{R}^n$. From the general properties of norms it can be shown that a norm ball of radius r and center x_c, given by $\{x| \;\; ||x - x_c|| \leq r\}$, is convex. The norm cone associated with the norm $|| \cdot ||$ is the set

$$C = \{(x, t) | \;\; ||x|| \leq t\} \subseteq \mathbf{R}^{n+1}$$

It is (as the name suggests) a convex cone.

2.4 Polyhedra

A polyhedron is defined as the solution set of a finite number of linear equalities and inequalities:

$$\mathcal{P} = \{a_j^T x \leq b_j, j = 1, \cdots, m, c_j^T x = d_j, j = 1, \cdots, p\}$$

A polyhedron is thus the intersection of a finite number of halfspaces and hyperplanes. Affine sets (e.g., subspaces, hyperplanes, lines), rays, line segments, and halfspaces are all polyhedra. It is easily shown that polyhedra are convex sets.

A bounded polyhedron is sometimes called a polytope.

It will be convenient to use the compact notation

$$\mathcal{P} = \{x | Ax \preceq b, Cx = d\}$$

where

$$A = \begin{bmatrix} a_1^T \\ \vdots \\ a_m^T \end{bmatrix} C = \begin{bmatrix} c_1^T \\ \vdots \\ c_p^T \end{bmatrix}$$

and the symbol $\preceq$ denotes *vector inequality* or *componentwise inequality* in $\mathbf{R}^m$: $u \preceq v$ means $u_i \leq v_i$ for $i = 1, \cdots, m$.

Simplexes

Simplexes are another important family of polyhedra. Suppose the $k+1$ points $v_0, \cdots, v_k \in \mathbf{R}^n$ are *affinely independent*, which means $v_1 - v_0, \cdots, v_k - v_0$ are linearly independent. The simplex determined by them is given by

$$C = \mathbf{conv}\{v_0, \cdots, v_k\} = \{\theta_0 v_0 + \cdots + \theta_k v_k | \theta \succeq 0, \mathbf{1}^T \theta = 1\}$$

where $\mathbf{1}$ denotes the vector with all entries one. The affine dimension of this simplex is k, so it is sometimes referred to as a k-dimensional simplex in $\mathbf{R}^n$.

Convex hull description of polyhedra

The convex hull of the finite set $\{v_1, \cdots, v_k\}$ is

$$conv\{v_1, \cdots, v_k\} = \{\theta_1 v_1 + \cdots + \theta_k v_k | \theta \succeq 0, \mathbf{1}^T \theta = 1\}.$$

This set is a polyhedron, and bounded, but it is not simple to express it by a set of linear equalities and inequalities.

A generalization of this convex hull description is

$$\{\theta_1 v_1 + \cdots + \theta_k v_k | \theta_1 + \cdots + \theta_m = 1, \theta_i \geq 0, i = 1, \cdots, k\}$$

where $m \leq k$. Here we consider nonnegative linear combinations of vi, but only the first m coefficients are required to sum to one. Alternatively, we can interpret generalized convex hull as the convex hull of the points $v_1, \cdots, v_m$, plus the conic hull of the points $v_{m+1}, \cdots, v_k$. The above set defines a polyhedron, and conversely, every polyhedron can be represented in this form.

2.5 The positive semidefinite cone

We use the notation $\mathbf{S}^n$ to denote the set of symmetric $n \times n$ matrices,

$$\mathbf{S}^n = \{X \in \mathbf{R}^{n\times n} | X = X^T\}$$

which is a vector space with dimension n(n + 1)/2. We use the notation $\mathbf{S}^n_+$ to denote the set of symmetric positive semidefinite matrices:

$$\mathbf{S}^n_+ = \{X \in \mathbf{R}^{n\times n} | X \succeq 0\}$$

And the notation $\mathbf{S}^n_{++}$ to denote the set of symmetric positive semidefinite matrices:

$$\mathbf{S}^n_{++} = \{X \in \mathbf{R}^{n\times n} | X \succ 0\}$$

The set $\mathbf{S}^n$ is a convex cone: if $\theta_1, \theta_2 \geq 0$ and $A, B \in \mathbf{S}^n_+$, then $\theta_1 A + \theta_2 B \in \mathbf{S}^n_+$.

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3.1 Intersection

Convexity is preserved under intersection: if S_1 and S_2 are convex, then $S_1 cap S_2$ is convex. This property extends to the intersection of an infinite number of sets: if S_α is convex for every $\alpha \in \mathcal{A}$, then $\cap_{\alpha\in\mathcal{A}} S_\alpha$ is convex.

every closed convex set S is a (usually infinite) intersection of halfspaces. In fact, a closed convex set S is the intersection of all halfspaces that contain it:

$$S = \cap\{\mathcal{H} | \mathcal{H} \;\; halfspace, S \subseteq \mathcal{H}\}$$

3.2 Affine functions

Recall that a function $f : \mathbf{R}^n \to \mathbf{R}^m$ is affine if it is a sum of a linear function and a constant, i.e., if it has the form $f(x) = Ax + b$, where $A \in \mathbf{R}^{m\times n}$ and $b \in \mathbf{R}^m$. Suppose $S \subseteq \mathbf{R}^n$ is convex and $f : \mathbf{R}^n \to \mathbf{R}^m$ is an affine function. Then the image of S under f,

$$f(S) = \{f(x) | x \in S\}$$

is convex. Similarly, if $f : \mathbf{R}^k \to \mathbf{R}^n$ is an affine function, the inverse image of S under f,

$$f^{-1}(S) = \{x | f(x) \in S\}$$

is convex.

3.3 Linear-fractional and perspective functions

The perspective function

We define the perspective function $P : \mathbf{R}^{n+1} \to \mathbf{R}^n$, with domain $\mathbf{dom}P = \mathbf{R}^n \times \mathbf{R}_{++}$, as $P(z,t) = z/t$.

If $C \subseteq \mathbf{dom}P$ is convex, then its image

$$P(C) = \{P(x)|x \in C\}$$

is convex.

if $C \subseteq \mathbf{R}^n$ is convex, then

$$P^{-1}(C) = \{(x,t) \in \mathbf{R}^{n+1}|x/t \in C, t > 0\}$$

is convex.

Linear-fractional functions

A linear-fractional function is formed by composing the perspective function with an affine function. Suppose $g : \mathbf{R}^n \to \mathbf{R}^{m+1}$ is affine, i.e.,

$$g(x) = \begin{bmatrix} A \\ c^T \end{bmatrix} x + \begin{bmatrix} b \\ d \end{bmatrix}$$

where $A \in \mathbf{R}^{m\times n}$, $b \in \mathbf{R}^m$, $c \in \mathbf{R}^n$, and $d \in \mathbf{R}$. The function $f : \mathbf{R}^n \in \mathbf{R}^m$ given by $f = P \circ g$, i.e.,

$$f(x) = (Ax+b)/(c^Tx+d) \;\; \mathbf{dom}f = \{x|c^Tx+d > 0\}$$

is called a linear-fractional (or projective) function. If $c = 0$ and $d > 0$, the domain of f is $\mathbf{R}^n$, and f is an affine function.

Like the perspective function, linear-fractional functions preserve convexity. If C is convex and lies in the domain of f (i.e., $c^Tx + d > 0$ for $x \in C$), then its image f(C) is convex. Similarly, if $C \subseteq \mathbf{R}^m$ is convex, then the inverse image $f^{-1}(C)$ is convex.

4 Generalized inequalities

4.1 Proper cones and generalized inequalities

A cone $K \subseteq \mathbf{R}^n$ is called a proper cone if it satisfies the following:

1. K is convex
2. K is closed
3. K is solid, which means it has nonempty interior
4. K is is pointed, which means that it contains no line (or equivalently, $x \in K$, $-x \in K \Rightarrow x = 0$).

A proper cone K can be used to define a generalized inequality, which is a partial ordering on $\mathbf{R}^n$ that has many of the properties of the standard ordering on $\mathbf{R}$. We associate with the proper cone K the partial ordering on $\mathbf{R}^n$ defined by

$$x \preceq_k y \Leftrightarrow y - x \in K$$

We also write $x \succeq_K y$ for $y \preceq_K x$. Similarly, we define an associated strict partial ordering by

$$x \prec_K y \Leftrightarrow y - x \in \mathbf{int} K,$$

and write $x \succ_K y$ for $y \prec_K x$.

Properties of generalized inequalities

A generalized inequality $\preceq_K$ satisfies many properties, such as

1. $\preceq_K$ is preserved under addition: if $x \preceq_K y$ and $u \preceq_K v$, then $x + u \preceq_K y + v$.
2. $\preceq_K$ is transitive: if $x \preceq_K y$ and $y \preceq_K z$ then $x \preceq_K z$.
3. $\preceq_K$ is preserved under nonnegative scaling: if $x \preceq_K y$ and $\alpha \geq 0$, then $\alpha x \preceq_K \alpha y$.
4. $\preceq_K$ is reflexive: if $x \preceq_K x$.
5. $\preceq_K$ is antisymmetric: if $x \preceq_K y$, and $y \preceq_K x$, then $x = y$.
6. $\preceq_K$ is preserved under limits: if $x_i \preceq_K y_i$ for $i = 1, 2, \cdots, x_i \to x$ and $y_i \to y$ as $i \to \infty$, then $x \preceq_K y$.

The corresponding strict generalized inequality $\prec K$ satisfies, for example,

1. if $x \prec_K y$ then $x \preceq_K y$.
2. if $x \prec_K y$ and $u \preceq_K v$, then $x + u \prec_K y + v$.
3. if $x \prec_K y$ and $\alpha > 0$, then $\alpha x \prec_K \alpha y$.
4. $x \not\prec x$.
5. if $x \prec_K y$, then for u and v small enough, $x + u \prec_K y + v$.

4.2 Minimum and minimal elements

We say that $x \in S$ is the *minimum* element of S (with respect to the generalized inequality $\preceq_K$) if for every $y \in S$ we have $x \preceq_K y$. We define the maximum element of a set S, with respect to a generalized inequality, in a similar way. If a set has a minimum (maximum) element, then it is unique. A related concept is minimal element. We say that $x \in S$ is a minimal element of S (with respect to the generalized inequalit $y \preceq_K$) if $y \in S$, $y \preceq_K x$ only if y = x. We define maximal element in a similar way. A set can have many different minimal (maximal) elements.

We can describe minimum and minimal elements using simple set notation. A point $x \in S$ is the minimum element of S if and only if

$$S \subseteq x + K$$

Here $x + K$ denotes all the points that are comparable to x and greater than or equal to x (according to $\preceq_K$). A point $x \in S$ is a minimal element if and only if

$$(x - K) \cap S = \{x\}$$

5 Separating and supporting hyperplanes

5.1 Separating hyperplane theorem

The basic result is the separating hyperplane theorem: Suppose C and D are nonempty disjoint convex sets, i.e., $C \cap D = \emptyset$. Then there exist $a \neq 0$ and b such that $a^T x \leq b$ for all $x \in C$ and $a^T x \geq b$ for all $x \in D$. In other words, the affine function $a^T x - b$ is nonpositive on C and nonnegative on D. The hyperplane $\{x | a^T x = b\}$ is called a separating hyperplane for the sets C and D, or is said to separate the sets C and D.

Strict separation

Strict separation of the sets C and D:

$$a^T x < b \text{ for all } x \in C \text{ and } a^T x > b \text{ for all } x \in D$$

Simple examples show that in general, disjoint convex sets need not be strictly separable by a hyperplane.

Strict separation of a point and a closed convex set. Let C be a closed convex set and $x_0 \notin C$. Then there exists a hyperplane that strictly separates x_0 from C.

a closed convex set is the intersection of all halfspaces that contain it. Indeed, let C be closed and convex, and let S be the intersection of all halfspaces containing C. Obviously $x \in C \Rightarrow x \in S$. To show the converse, suppose there exists $x \in S$, $x \notin C$. By the strict separation result there exists a hyperplane that strictly separates x from C, i.e., there is a halfspace containing C but not x. In other words, $x \notin S$.

Converse separating hyperplane theorems

The converse of the separating hyperplane theorem (i.e., existence of a separating hyperplane implies that C and D do not intersect) is not true.

Any two convex sets C and D, at least one of which is open, are disjoint if and only if there exists a separating hyperplane.

Theorem of alternatives for strict linear inequalities.

We derive the necessary and sufficient conditions for solvability of a system of strict linear inequalities

$$Ax \prec b$$

These inequalities are infeasible if and only if the (convex) sets:

$$C = \{b - Ax | x \in \mathbf{R}^n\}, \ \ D = \mathbf{R}^n_{++} = \{y \in \mathbf{R}^m | y \succ 0\}$$

do not intersect.

These two convex sets do not intersect if and only if there exists $\lambda \mathbf{R}^m$ such that

$$\lambda \neq 0 \qquad \lambda \succeq 0 \qquad A^T \lambda = 0 \qquad \lambda^T b \leq 0$$

This is also a system of linear inequalities and linear equations in the variable $\lambda \in \mathbf{R}^m$.

We say that these is a pair of alternatives: for any data A and b, exactly one of them is solvable.

Supporting hyperplanes

Suppose $C \in \mathbf{R}^n$, and x_0 is a point in its boundary **bd**C, i.e.,

$$x_0 \in \mathbf{bd}C = \mathbf{cl}C \backslash \mathbf{int}C.$$

If $a \neq 0$ satisfies $a^T x \leq a^T x_0$ for all $x \in C$, then the hyperplane $\{x | a^T x = a^T x_0\}$ is called a supporting hyperplane to C at the point x_0.

This is equivalent to saying that the point x_0 and the set C are separated by the hyperplane $\{x | a^T x = a^T x_0\}$.

A basic result, called the supporting hyperplane theorem, states that for any nonempty convex set C, and any $x_0 \in \mathbf{bd}C$, there exists a supporting hyperplane to C at x_0.

There is also a partial converse of the supporting hyperplane theorem: If a set is closed, has nonempty interior, and has a supporting hyperplane at every point in its boundary, then it is convex.

6 Dual cones and generalized inequalities

6.1 Dual cones

Let K be a cone. The set

$$K^* = \{y | x^T y \geq 0 \text{ for all } x \in K\}$$

is called the dual cone of K. As the name suggests, K^* is a cone, and is always convex, even when the original cone K is not. Geometrically, $y \in K^*$ if and only if $-y$ is the normal of a hyperplane that supports K at the origin.

Dual cones satisfy several properties, such as:

1. K^* is closed and convex
2. $K_1 \subseteq K_2$ implies $K_2^* \subset K_1^*$
3. If K has nonempty interior, then K^* is pointerd.
4. If the closure of K is pointed then K^* has nonempty interior.
5. K^{**} is the closure of the convex hull of K. (Hence if K is convex and closed, $K^{**} = K$.)

These properties show that if K is a proper cone, then so is its dual K^*, and moreover, that $K^{**} = K$.

6.2 Dual generalized inequalities

Now suppose that the convex cone K is proper, so it induces a generalized inequality K. Then its dual cone K^* is also proper, and therefore induces a generalized inequality. We refer to the generalized inequality K as the dual of the generalized inequality K. Some important properties relating a generalized inequality and its dual are:

1. $x \preceq_K y$ if and only if $\lambda^T x \leq \lambda^T y$ for all $\lambda \succeq_{K^*} 0$.
2. $x \prec_K y$ if and only if $\lambda^T x < \lambda^T y$ for all $\lambda \succeq_{K^*} 0$, $\lambda \neq 0$.

Since $K = K^{**}$, the dual generalized inequality associated with $\preceq_{K^*}$ is $\preceq_K$, so these properties hold if the generalized inequality and its dual are swapped.

Theorem of alternatives for linear strict generalized inequalities.
Suppose $K \subseteq \mathbf{R}^m$ is a proper cone. Consider the strict generalized inequality

$$Ax \prec_K b$$

where $x \in \mathbf{R}^n$.
if above inequality is infeasible, then there exists λ such that

$$\lambda \neq 0, \qquad \lambda \succeq_{K^*} 0, \qquad A^T\lambda = 0, \qquad \lambda^T b \leq 0$$

Thus, the inequality systems are alternatives: for any data A, b, exactly one of them is feasible.

6.3 Minimum and minimal elements via dual inequalities

We can use dual generalized inequalities to characterize minimum and minimal elements of a (possibly nonconvex) set $S \subseteq \mathbf{R}^m$ with respect to the generalized inequality induced by a proper cone $\mathbf{K}$.

Dual characterization of minimum element

We first consider a characterization of the minimum element: x is the minimum element of S, with respect to the generalized inequality $\preceq K$, if and only if for all $\lambda \succ_{K^*} 0$, x is the unique minimizer of $\lambda^T z$ over $z \in S$. Geometrically, this means that for any $\lambda \succ_{K^*} 0$, the hyperplane

$$\{z | \lambda^T(z - x) = 0\}$$

is a strict supporting hyperplane to S at x. (By strict supporting hyperplane, we mean that the hyperplane intersects S only at the point x.) Note that convexity of the set S is not required.

Dual characterization of minimal elements

We now turn to a similar characterization of minimal elements. Here there is a gap between the necessary and sufficient conditions. If $\lambda \succ_{K^*} 0$ and x minimizes $\lambda^T z$ over $z \in S$, then x is minimal.
Provided the set S is convex, we can say that for any minimal element x there exists a nonzero $\lambda \succeq 0$ such that x minimizes $\lambda^T z$ over $z \in S$.